INS Error Models: Phi vs. Psi

Part 2 of the series on Fault Tolerant Low-Cost Inertial Navigation Systems

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I. Introduction

An Inertial Navigation System (INS) algorithm generates velocity, position and attitude information based on the inputs of measured accelerometer and gyro outputs. Attitude is the result of integration of the IMU measured angular rate, while velocity and position are the result of transformation and integration of the IMU specific force acceleration, Coriolis acceleration and modeled gravity relative to the desired navigation frame.

Due to uncertainties in the sensors of an INS and the gravity field, the navigation parameters obtained from the INS mechanization equation contain errors. Many models have been developed to describe the time-dependent behavior of these errors, the choice of which is largely dependent on the application. This paper will summarize several error models of importance, and discuss the implications on low-cost inertial navigation.

Furthermore, because the velocity, position and attitude values are integrated from initial conditions, it is important for the initial alignment errors to be small. A brief treatment of the subject on initial conditions and heading uncertainty is included in the last section of this paper.

II. Phi-Angle Error Model

Perturbation analysis is a classical approach to INS error analysis, where the navigation parameters are perturbed with respect to the true navigation-frame (n-frame hereafter). In continuous-time, the navigation parameters are defined as

$$\begin{aligned}
\dot{\mathbf{v}}^n &= \mathbf{C}_b^n \mathbf{f}^b + \mathbf{g}^n - (2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n) \times v^n, && (1)\\
\dot{\mathbf{C}}_n^e &= \mathbf{C}_n^e (\boldsymbol{\omega}_{en}^n \times), && (2)\\
\dot{\mathbf{h}} &= -v_D, && (3)\\
\dot{\mathbf{C}}_b^n &= \mathbf{C}_b^n (\boldsymbol{\omega}_{ib}^b \times) - (\boldsymbol{\omega}_{in}^n \times)\mathbf{C}_b^n. && (4)
\end{aligned}$$

The perturbation model can be obtained by perturbing all of the discrete-time navigation parameters from their continuous time counterparts in eqns. (1-4), which is equivalent to applying the Taylor series expansion, and neglecting the higher order terms [1]. This assumes that all errors are small, particularly in heading. The derivation of the perturbation model is well understood in the literature [2], [3], [4], [5].

First let us define the position error vector as

$$\delta \mathbf{r}^n = [\delta r_N,\ \delta r_E,\ \delta r_D]^\top , \tag{5}$$

where δ denotes errors and $\delta r_D = -\delta h$ for the NED coordinate frame. Permutations on the other parameters can be written as follows:

$$\begin{aligned}
\hat{\mathbf{f}}^b &= \mathbf{f}^b + \delta \mathbf{f}^b, && (6)\\
\hat{\boldsymbol{\omega}}_{ib}^b &= \boldsymbol{\omega}_{ib}^b + \delta \boldsymbol{\omega}_{ib}^b, && (7)\\
\hat{\mathbf{v}}^n &= \mathbf{v}^n + \delta \mathbf{v}^n, && (8)\\
\hat{\mathbf{C}}_b^n &= [\mathbf{I} - (\boldsymbol{\phi} \times)]\mathbf{C}_b^n, && (9)\\
\hat{\boldsymbol{\omega}}_{ie}^n &= \boldsymbol{\omega}_{ie}^n + \delta \boldsymbol{\omega}_{ie}^n, && (10)\\
\hat{\boldsymbol{\omega}}_{in}^n &= \boldsymbol{\omega}_{in}^n + \delta \boldsymbol{\omega}_{in}^n, && (11)\\
\hat{\mathbf{g}}^n &= \mathbf{g}^n + \delta \mathbf{g}^n. && (12)
\end{aligned}$$

The simplified inverse gravity model from [6] can be written as

$$\delta g^n = [0,\ 0,\ 2g\delta r_D/(R+h)]^\top , \tag{13}$$

where $R = \sqrt{R_M R_N}$ is the Earth's mean radius of curvature. Thus the ϕ-angle perturbation model can be written as follows

$$\begin{aligned}
\delta \dot{\mathbf{r}}^n =& -\boldsymbol{\omega}_{en}^n \times \delta \mathbf{r}^n + \delta \boldsymbol{\theta} \times \mathbf{v}^n + \delta \mathbf{v}^n, && (14)\\
\delta \dot{\mathbf{v}}^n =&\ \mathbf{C}_b^n \delta \mathbf{f}^b + \mathbf{C}_b^n \mathbf{f}^b \times \boldsymbol{\phi} + \delta \mathbf{g}^n - (\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{in}^n) \\
&\times \delta \mathbf{v}^n - (\delta \boldsymbol{\omega}_{ie}^n + \delta \boldsymbol{\omega}_{in}^n) \times \mathbf{v}^n, && (15)\\
\dot{\boldsymbol{\phi}} =& -\boldsymbol{\omega}_{in}^n \times \boldsymbol{\phi} + \delta \boldsymbol{\omega}_{in}^n - \mathbf{C}_b^n \delta \boldsymbol{\omega}_{ib}^b. && (16)
\end{aligned}$$

where $\delta\boldsymbol{\theta}$ is defined as

$$\delta \boldsymbol{\theta} = \begin{bmatrix} \delta r_E/(R_N+h) \\ \delta r_N/(R_M+h) \\ \delta r_E \tan\phi/(R_N+h) \end{bmatrix}. \tag{17}$$

Since the attitude errors are expressed in terms of the ϕ-angle, this model is commonly referred to as the ϕ-angle error model.

III. Psi-Angle Error Model

The perturbation analysis can also be performed on the computer-frame (c-frame hereafter), resulting in the ψ-angle model. The computed navigation parameters may have one of the two representations listed below:

$$\hat{\mathbf{v}}^n = \mathbf{v}^n + \delta\mathbf{v}_1^n$$
$$= \mathbf{v}^c + \delta\mathbf{v}_2^c, \tag{18}$$
$$\hat{\mathbf{g}}^n = \mathbf{g}^n + \delta\mathbf{g}_1^n$$
$$= \mathbf{g}^c + \delta\mathbf{g}_2^c, \tag{19}$$
$$\hat{\boldsymbol{\omega}}_{ie}^n = \boldsymbol{\omega}_{ie}^n + \delta\boldsymbol{\omega}_{ie}^n$$
$$= \boldsymbol{\omega}_{ie}^c, \tag{20}$$
$$\hat{\boldsymbol{\omega}}_{in}^n = \boldsymbol{\omega}_{in}^n + \delta\boldsymbol{\omega}_{in}^n$$
$$= \boldsymbol{\omega}_{ie}^c, \tag{21}$$

In the c-frame, $\mathbf{C}_c^e$, $\boldsymbol{\omega}_{ie}^c$, and $\boldsymbol{\omega}_{ic}^c$ are known without error because we know the position and transport rate of the c-frame from the navigation computer. In [5], it was shown that the navigation parameter errors between the two representations have the following relationships:

$$\delta\mathbf{v}_1^n = \delta\mathbf{v}_2^c - \delta\boldsymbol{\theta} \times \mathbf{v}^c, \tag{22}$$
$$\delta\mathbf{g}_1^n = \delta\mathbf{g}_2^c - \delta\boldsymbol{\theta} \times \mathbf{g}^c, \tag{23}$$
$$\delta\boldsymbol{\omega}_{ie}^n = -\delta\boldsymbol{\theta} \times \delta\boldsymbol{\omega}_{ie}^c, \tag{24}$$

From eqns. (13, 17 & 23) we can write the gravity error in the c-frame as

$$\delta\mathbf{g}_2^c = \left[\frac{-g\delta r_N}{R_M + h}, \frac{-g\delta r_E}{R_N + h}, \frac{2g\delta r_D}{R + h}\right]^\top,$$
$$\approx \left[-\omega_s^2\delta r_N, \; -\omega_s^2\delta r_E, \; 2\omega_s^2\delta r_D\right]^\top, \tag{25}$$

where ω_s is the Schuler frequency. The c-frame analysis results in the following error model

$$\delta\dot{\mathbf{r}}^c = -\boldsymbol{\omega}_{ec}^c \times \delta\mathbf{r}^c + \delta\mathbf{v}^c, \tag{26}$$
$$\delta\dot{\mathbf{v}}^c = \mathbf{f}^c \times \boldsymbol{\psi} - (2\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \times \delta\mathbf{v}^c + \delta\mathbf{g}^c + \mathbf{C}_b^p\delta\mathbf{f}^b, \tag{27}$$
$$\dot{\boldsymbol{\psi}} = -(\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \times \boldsymbol{\psi} - \mathbf{C}_b^n\delta\boldsymbol{\omega}_{ib}^b. \tag{28}$$

Since the attitude errors are expressed in terms of the ψ-angle, this model is commonly referred to as the ψ-angle error model.

A. *Discussion*

Benson [7] showed the basic equivalence of the ϕ-angle and ψ-angle error models. However, [1] discussed the following distinctions in their implementation:

- The ψ-angle dynamics, eqn. (28), are rate-stable, independent of all other INS errors, and driven only by the gyro biases.
- The position and velocity error dynamics, eqns. (26 & 27), are independent of angular rate errors that are due to the Earth rate and transport rate mis-resolved by the c-frame misalignment.
- If the INS is aided by GNSS, the position errors become small, and the c-frame misalignment $\delta\boldsymbol{\theta}$ becomes very small. Hence, the ϕ-angle converges to the ψ-angle.
- The ψ-angle error model contains fewer terms and hence is more easily implemented in a Stochastic filter, such as an EKF.
- Finally, because the ψ-angle error model contains fewer terms there are fewer floating point operations, resulting in lower likelihood of round-off errors, and increased computational efficiency.

IV. Modified Error Models

Both ψ-angle and ϕ-angle error models contain mis-resolved specific force terms, $\mathbf{f}^n \times \boldsymbol{\phi}$ and $\mathbf{f}^c \times \boldsymbol{\psi}$, respectively. Therefore the specific force terms in the transition matrix for a discrete-time Kalman filter. Consequently, a high-speed integration must be applied in the transition matrix computation if the vehicle is moving with high dynamics (e.g. quad-copters, aerobatic planes, etc.). For low-cost IMUs, the specific force outputs are usually corrupted by large biases and sources of noise, which can lead to composing an erroneous transition matrix and therefore distorting the estimates. [8] developed modified error models to solve these problems, which cancel the specific force terms by introducing the following velocity transformations

$$\Delta\mathbf{v}_1^n = \delta\mathbf{v}_1^n - \hat{\mathbf{v}}^n \times \boldsymbol{\phi}, \tag{29}$$
$$\Delta\mathbf{v}_2^c = \delta\mathbf{v}_2^c - \hat{\mathbf{v}}^n \times \boldsymbol{\psi}. \tag{30}$$

The modified ϕ-angle error model is written as follows:

$$\delta\dot{\mathbf{r}}^n = -\boldsymbol{\omega}_{en}^n \times \delta\mathbf{r}^n + \Delta\mathbf{v}_1^n + (\delta\boldsymbol{\theta} - \boldsymbol{\phi}) \times \mathbf{v}^n, \tag{31}$$
$$\Delta\dot{\mathbf{v}}_1^n = \mathbf{C}_b^n\delta\mathbf{f}^b + \delta\mathbf{g}_1^n - \hat{\mathbf{g}}^n \times \boldsymbol{\phi} - (\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ic}^c) \times \Delta\mathbf{v}_1^n - \hat{\mathbf{v}}^n \times (\boldsymbol{\omega}_{ie}^c \times \boldsymbol{\phi}) + \hat{\mathbf{v}}^n \times \delta\boldsymbol{\omega}_{ie}^n + \hat{\mathbf{v}}^n \times \mathbf{C}_b^n\delta\boldsymbol{\omega}_{ib}^b, \tag{32}$$
$$\dot{\boldsymbol{\phi}} = -\boldsymbol{\omega}_{in}^n \times \boldsymbol{\phi} + \delta\boldsymbol{\omega}_{in}^n - \mathbf{C}_b^n\delta\boldsymbol{\omega}_{ib}^b. \tag{33}$$

The modified ψ-angle error model is written as follows:

$$\delta\dot{\mathbf{r}}^c = -\boldsymbol{\omega}_{ec}^c \times \delta\mathbf{r}^c + \Delta\mathbf{v}^c + \hat{\mathbf{v}}^n \times \boldsymbol{\psi}, \tag{34}$$
$$\delta\dot{\mathbf{v}}_2^c = \mathbf{C}_b^p\delta\mathbf{f}^b + \delta\mathbf{g}_2^c - \hat{\mathbf{g}}^n \times \boldsymbol{\psi} - (2\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \times \Delta\mathbf{v}_2^c - \hat{\mathbf{v}}^n \times (\boldsymbol{\omega}_{ie}^c \times \boldsymbol{\psi}) + \hat{\mathbf{v}}^n \times \mathbf{C}_b^p\delta\boldsymbol{\omega}_{ib}^b, \tag{35}$$
$$\dot{\boldsymbol{\psi}} = -(\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \times \boldsymbol{\psi} - \mathbf{C}_b^p\delta\boldsymbol{\omega}_{ib}^b. \tag{36}$$

The attitude error dynamics equations are identical in both modified error models.

V. Large Heading Uncertainty Models

Consider, for example, the initial alignment equations for pitch and heading:

$$\theta = \tan^{-1}(-v_D/\sqrt{v_N^2 + v_E^2}), \tag{37}$$
$$\psi = \tan^{-1}(v_E/v_N), \tag{38}$$

where v_N, v_E and v_D are the north, east and down velocities, respectively.

The basic assumption about the error models discussed so far is that all of the attitude errors are small. However, the initial heading may have large uncertainty or be completely unknown. For instance, if the heading is obtained from eqn. (38) while the vehicle is moving backward, then the heading can have an error of $\pm 180°$. Furthermore, if we apply this to helicopters and quad-copters, which can move through the air in any direction, the heading cannot be derived from GNSS velocity. Hence, Large Heading Uncertainty (LHU) models have been developed primarily for in-motion or in-air alignment until the heading error becomes small enough (typically a few degrees) for the fine alignment routine to be activated using Small Heading Uncertainty (SHU) models. Unlike high-quality IMUs which are capable of gyro-compassing down to tenths of a degree, if a low-cost IMU is used, the heading can be completely unknown even in stationary mode because of large gyro-compassing errors. Therefore a LHU model can also be applied for alignment of a low-cost INS.

There are three dominant approaches in the development of LHU models. Rogers [9] used errors of trigonometric functions of the wander azimuth angle as a part of the state vector, Scherzinger [10] used trigonometric functions of the heading error based on the ψ-angle model, and Pham [11] used a similar approach based on the ϕ-angle model. In [9] a completely different error model is required as the heading uncertainty goes below a certain threshold. However in [10], although still requiring a switch in the attitude error dynamics model, provides continuous transition. If a low-quality IMU is used, the heading error can grow quickly in a very short amount of time in the absence of aiding information. This is due to uncertainties in the z-gyroscope which drive the heading error, and the unaided INS cannot estimate and regulate its heading error. This situation can also happen when the vehicle is driven (or flown) with a constant velocity, due to the poor observability of the heading. In this case, the error model switch can be done in both directions and therefore the latter approach will be more appropriate. The latter approach results in developing the modified geographic consistent (GC) model and the modified platform consistent (PC) model. The only limitation of the latter approach is that the model selection is dependent upon the type of aiding sensors. If a GNSS is used, then the modified GC model is indicated. On the other hand, if a body-referenced velocity sensor is aiding the INS, the modified PC model must be chosen. Since the modified PC model has implementation difficulties due to nonlinearity and the GNSS is assumed to be the main aiding sensor in our application, the modified GC model will be discussed here.

Scherzinger [10] first defined the extended misalignment vector $\boldsymbol{\psi}_e$ as follows:

$$\delta\boldsymbol{\psi}' \equiv \begin{bmatrix} \psi_x \\ \psi_y \\ \sin\psi_z \end{bmatrix}, \tag{39}$$

$$\delta\boldsymbol{\psi}_e \equiv \begin{bmatrix} \boldsymbol{\psi}' \\ \cos\psi_z - 1 \end{bmatrix}, \tag{40}$$

where $\sin\psi_z$ and $\cos\psi_z - 1$ are treated as random constants when the heading error is large [10]. For small heading errors $\boldsymbol{\psi}'$ converges to $\boldsymbol{\psi}$. The matrix representation of the extended cross-product type $(+)$ operator was defined as

$$(\mathbf{a}\times)_{e+} \equiv \left[\begin{array}{c|c} (\mathbf{a}\times) & \begin{matrix} a_x \\ a_y \\ a_x \end{matrix} \end{array} \right], \tag{41}$$

where $\mathbf{a}$ is an arbitrary 3×1 vector, $\mathbf{a} = [a_x \ a_y \ a_z]^\top$. The GC LHU model based on the ψ-angle error model

$$\delta\dot{\mathbf{r}}^c = -\boldsymbol{\omega}_{ec}^c \times \delta\mathbf{r}^c + \Delta\mathbf{v}^c, \tag{42}$$

$$\begin{aligned} \delta\dot{\mathbf{v}}^c = & (\mathbf{f}^c\times)_{e+}\boldsymbol{\psi}_e + \mathbf{f}_z(\mathbf{f}^c, \boldsymbol{\psi}) - (2\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \\ & \times \delta\mathbf{v}^c + \delta\mathbf{g}^c + \mathbf{C}_b^p\delta\mathbf{f}^b, \end{aligned} \tag{43}$$

$$\begin{aligned} \dot{\boldsymbol{\psi}}_e = & \begin{bmatrix} -(\boldsymbol{\omega}_{ic}^c\times)_{e+} \\ \mathbf{0}_{1\times4} \end{bmatrix} \boldsymbol{\psi}_e + \begin{bmatrix} \mathbf{0}_{2\times1} \\ \omega_x\psi_y - \omega_y\psi_x \\ 0 \end{bmatrix} \\ & - \begin{bmatrix} \mathbf{C}_b^p \\ \mathbf{0}_{1\times3} \end{bmatrix} \delta\boldsymbol{\omega}_{ib}^b, \end{aligned} \tag{44}$$

where $\boldsymbol{\omega}_{ic}^c = [\omega_x \ \omega_y \ \omega_z]^\top$, $\mathbf{f}^c = [f_x \ f_y \ f_z]^\top$, $\mathbf{f}_z(\mathbf{f}^c, \boldsymbol{\psi})$ in eqn. (44) is defined as

$$\mathbf{f}_z(\mathbf{f}^c, \boldsymbol{\psi}) \equiv \begin{bmatrix} 0 \\ 0 \\ (f_x(\psi_x \sin\psi_z + \psi_y(\cos\psi_z - 1)) \\ +f_y(\psi_y \sin\psi_z + \psi_x(\cos\psi_z - 1))) \end{bmatrix}, \tag{45}$$

and contains nonlinear terms in the vertical channel, which can be considered as either negligible or approximately random with respect to larger long-term vertical acceleration errors [10]. If the heading uncertainty becomes small, eqn. (44) can be replaced by

$$\dot{\boldsymbol{\psi}}_e = \begin{bmatrix} -(\boldsymbol{\omega}_{ic}^c\times)_{e+} \\ \mathbf{0}_{1\times4} \end{bmatrix} \boldsymbol{\psi}_e - \begin{bmatrix} \mathbf{C}_b^p \\ \mathbf{0}_{1\times3} \end{bmatrix} \delta\boldsymbol{\omega}_{ib}^b, \tag{46}$$

which is equavalent to the attitude dynamics of the SHU ψ-angle model, eqn. (28).

Finally, with the velocity transformation

$$\Delta\mathbf{v} = \delta\mathbf{v}^c - (\mathbf{v}^c\times)_{e+}\boldsymbol{\psi}_e, \tag{47}$$

the modified GC LHU model, according to [10], is derived as follows,

$$
\begin{aligned}
\delta\dot{\mathbf{r}}^c =& -\boldsymbol{\omega}_{ec}^c \times \delta\mathbf{r}^c + \Delta\mathbf{v}^c + (\mathbf{v}^c\times)_{e+}\boldsymbol{\psi}_e, \qquad (48)\\
\delta\dot{\mathbf{v}}^c =& -\mathbf{g}^c \times \boldsymbol{\psi}' - (2\boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{ec}^c) \times \Delta\mathbf{v}^c + \delta\mathbf{g}^c \\
&+ \mathbf{C}_b^p \delta\mathbf{f}^b - \mathbf{v}^c \times (\boldsymbol{\omega}_{ie}^c \times \boldsymbol{\psi}') + (\mathbf{v}^c\times)\mathbf{C}_b^p \delta\boldsymbol{\omega}_{ib}^b \\
&+ \mathbf{f}_z(\mathbf{f}^c, \boldsymbol{\psi}) + \mathbf{f}_c(\boldsymbol{\psi}', \mathbf{v}^c) \\
&+ \begin{bmatrix} \Omega_E^c v_D^c \\ \Omega_N^c v_D^c \\ (-(3\Omega_N^c + 2\rho_N^c)v_E^c \\ +(3\Omega_E^c + 2\rho_E^c)v_N^c) \end{bmatrix} (\cos\psi_z - 1), \qquad (49)\\
\dot{\boldsymbol{\psi}}_e =& \begin{bmatrix} -(\boldsymbol{\omega}_{ic}^c\times)_{e+} \\ \mathbf{0}_{1\times 4} \end{bmatrix} \boldsymbol{\psi}_e + \begin{bmatrix} \mathbf{0}_{2\times 1} \\ \omega_x\psi_y - \omega_y\psi_x \\ 0 \end{bmatrix} \\
&- \begin{bmatrix} \mathbf{C}_b^p \\ \mathbf{0}_{1\times 3} \end{bmatrix} \delta\boldsymbol{\omega}_{ib}^b, \qquad (50)
\end{aligned}
$$

where $\mathbf{v}^c = [v_N^c\ v_E^c\ v_D^c]^\top$, $\boldsymbol{\omega}_{ie}^c = [\Omega_N^c\ \Omega_E^c\ \Omega_D^c]^\top$, and $\boldsymbol{\omega}_{ec}^c = [\rho_N^c\ \rho_E^c\ \rho_D^c]^\top$. In eqn. (50) $\mathbf{f}_c(\boldsymbol{\psi}, \mathbf{v}^c)$ is a correction function to account for the choice of the misalignment error model eqns. (46) or (50), [10].

When the SHU misalignment dynamics are assumed, eqn. (46) is used, and $\mathbf{f}_c(\boldsymbol{\psi}, \mathbf{v}^c) = \mathbf{0}$.

When the LHU misalignment dynamics are assumed, eqn. (50) is used, and $\mathbf{f}_c(\boldsymbol{\psi}, \mathbf{v}^c)$ is defined as

$$
\mathbf{f}^c(\boldsymbol{\psi}, \mathbf{v}^c) = \begin{bmatrix} (\Omega_E^c + \rho_E^c)v_E^c & -(\Omega_N^c + \rho_N^c)v_E^c & 0 \\ -(\Omega_E^c + \rho_E^c)v_N^c & (\Omega_N^c + \rho_N^c)v_N^c & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\psi}'. \qquad (51)
$$

VI. Conclusion

While the ψ-angle model is common in the literature because it does not perturb the transport rate and requires fewer floating-point operations, the ϕ-angle model is preferable on low-cost Inertial Navigation Systems for the following reason:

When considering large angle error, it was noted that a high-grade INS can perform gyro-compassing to reduce the initial attitude uncertainty to the levels suitable for the small angle assumption. However, in low-cost systems, we almost always have to perform an ad-hoc or in-motion alignment starting with a large heading uncertainty. Recall the definition of the c-frame, the effect of large heading uncertainty manifest itself on the velocity errors in the ψ-models. Therefore both the position and the velocity errors are affected by the nonlinearity of the large attitude errors in the ψ-models. On the other hand, the large heading uncertainty only affects the position errors in the ϕ-models. Therefore, ϕ-models behave better than ψ-models under large azimuth error conditions.

Part 3 in this series will cover INS alignment methods for low-cost Inertial Navigation Systems.

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