Technical Note: INS State Error Model

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***Abstract*— Due to space limitations in [1] and [2], this Technical Note is supplied to explain why the dimension of the state vector $\boldsymbol{x} \in \mathbb{R}^{16}$ and the dimension of error state vector $\delta\boldsymbol{x} \in \mathbb{R}^{15}$. This Technical Note also describes the additive and multiplicative operations required to use $\delta\boldsymbol{x}$ to correct $\boldsymbol{x}$.**

I. Introduction

Let $\boldsymbol{x} \in \mathbb{R}^{n_s}$ denote the rover state vector:

$$\boldsymbol{x}(t) = [\mathbf{p}^\intercal(t), \mathbf{v}^\intercal(t), \mathbf{q}^\intercal(t), \mathbf{b}_a^\intercal(t), \mathbf{b}_g^\intercal(t)]^\intercal \in \mathbb{R}^{n_s},$$

where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{b}_a$, $\mathbf{b}_g$ each in $\mathbb{R}^3$ represent the position, velocity, accelerometer bias and gyro bias vectors, respectively, and $\mathbf{q} \in \mathbb{R}^4$ represents the attitude quaternion ($n_s = 16$), each at time t. Let $\hat{\boldsymbol{x}} \in \mathbb{R}^{n_s}$ denote the estimate of the rover state vector:

$$\hat{\boldsymbol{x}}(t) = [\hat{\mathbf{p}}^\intercal(t), \hat{\mathbf{v}}^\intercal(t), \hat{\mathbf{q}}^\intercal(t), \hat{\mathbf{b}}_a^\intercal(t), \hat{\mathbf{b}}_g^\intercal(t)]^\intercal \in \mathbb{R}^{n_s}.$$

The error between $\boldsymbol{x}(t)$ and $\hat{\boldsymbol{x}}(t)$ is denoted as $\delta\boldsymbol{x}$. The error vector is

$$\delta\boldsymbol{x} = [\delta\mathbf{p}^\intercal, \delta\mathbf{v}^\intercal, \boldsymbol{\rho}^\intercal, \delta\mathbf{b}_a^\intercal, \delta\mathbf{b}_g^\intercal]^\intercal \in \mathbb{R}^{n_e},$$

where $\delta\mathbf{p}$ and $\delta\mathbf{v}$, each in $\mathbb{R}^3$, represent the error between the true and computed position and velocity, respectively. The small-angle error state, denoted as $\boldsymbol{\rho} \in \mathbb{R}^{3\times 1}$, is defined in Section 2.5.5 of [3], and discussed in Section II-B. The errors $\delta\mathbf{b}_a$ and $\delta\mathbf{b}_g$, each in $\mathbb{R}^3$, represent the accelerometer bias, and gyro bias errors, respectively. Therefore $\delta\boldsymbol{x} \in \mathbb{R}^{15}$ (i.e. $n_e = 15$). The fact that $n_s = 16$ and $n_e = 15$ is discussed in Section II-B.

II. State Correction

Let $\delta\hat{\boldsymbol{x}}$ denote an estimate of $\delta\boldsymbol{x}$. The state correction to the state vector $\hat{\boldsymbol{x}}$ is denoted as

$$\hat{\boldsymbol{x}}^+ = \hat{\boldsymbol{x}}^- \oplus \delta\hat{\boldsymbol{x}}.$$

The symbol $(-)$ denotes the prior estimate, whereas $(+)$ is the updated estimate. The symbol $\oplus$ is discussed in Sections II-A and II-B

A. Position, Velocity, and Bias Updates

Position, velocity, accelerometer bias and gyro bias, each have corrections which are additive. The state correction step is

$$\begin{aligned}
\hat{\mathbf{p}}^+ &= \hat{\mathbf{p}}^- + \delta\mathbf{p} \\
\hat{\mathbf{v}}^+ &= \hat{\mathbf{v}}^- + \delta\mathbf{v} \\
\hat{\mathbf{b}}_a^+ &= \hat{\mathbf{b}}_a^- + \delta\mathbf{b}_a \\
\hat{\mathbf{b}}_g^+ &= \hat{\mathbf{b}}_g^- + \delta\mathbf{b}_g.
\end{aligned}$$

B. Attitude Update

When the attitude error is sufficiently small (see Section 2.5.5 of [3]), the attitude can be represented as a set of small-angle planar rotations $\{\rho_x, \rho_y, \rho_z\}$ about three orthogonal axes $\{x, y, z\}$, thus the attitude error can be defined in $\mathbb{R}^3$.

1) Rotation Matrix: Let $\mathbf{R}_b^n \in \mathbb{R}^{3\times 3}$ represent the true rotation from body-frame (b) to navigation-frame (n) that is equivalent to $\mathbf{q}(t)$ (see eqn. D.13 in [3]). Let $\hat{\mathbf{R}}_b^n \in \mathbb{R}^{3\times 3}$ represent the computed rotation that is equivalent to $\hat{\mathbf{q}}(t)$. The error between the true and computed rotation is

$$\mathbf{R}_{\hat{n}}^n = (\mathbf{R}_b^n)(\hat{\mathbf{R}}_n^b),$$

where $\mathbf{R}_{\hat{n}}^n$ represents the rotation matrix from the computed to actual navigation frame. When the error between the true and computed rotation is zero, then $\mathbf{R}_{\hat{n}}^n = \mathbf{I}$. Otherwise, as discussed in Section 2.6.1 of [3],

$$\mathbf{R}_{\hat{n}}^n = [\mathbf{I} - \mathbf{P}]$$

where $\mathbf{P} = [\boldsymbol{\rho}\times]$, and $\boldsymbol{\rho} = [\rho_x, \rho_y, \rho_z]^\intercal \in \mathbb{R}^3$ (see eqn. 10.28 of [3]).

Using this notation, the attitude update (as defined in eqn. 10.29 of [3]) is

$$(\mathbf{R}_b^n)^+ = [\mathbf{I} - \mathbf{P}](\hat{\mathbf{R}}_b^n)^-.$$

Note that the attitude correction is multiplicative.

2) Quaternion: A similar approach to Section II-B.1 is valid when the attitude error is represented by a quaternion. Let $\mathbf{q}_b^n$ represent the true quaternion from b-frame to n-frame. Let $\hat{\mathbf{q}}_b^n$ represent the computed quaternion. The error may be represented as

$$\mathbf{q}_{\hat{n}}^n = \mathbf{q}_b^n \otimes \hat{\mathbf{q}}_n^b$$

where $\mathbf{q}_{\hat{n}}^n$ represents the quaternion from the computed to actual navigation frame. The symbol $\otimes$ represents the quaternion multiplication operation defined in Section D of [3]. When the error between the true and computed rotation is zero, then $\mathbf{q}_{\hat{n}}^n = [1, 0, 0, 0]^\intercal$, otherwise $\mathbf{q}_{\hat{n}}^n$ may be represented as

$$\mathbf{q}_{\hat{n}}^n = \begin{bmatrix} \hat{\mathbf{q}}_s \\ \hat{\mathbf{q}}_v \end{bmatrix} = \begin{bmatrix} \sqrt{1 - \left\|\frac{1}{2}\boldsymbol{\rho}\right\|_2^2} \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix}, \quad (1)$$

where the scalar part of the quaternion is $\hat{\mathbf{q}}_s = 1$, and the vector part is $\hat{\mathbf{q}}_v = \boldsymbol{\rho}$. The approximation on the right-hand side of eqn. (1) is shown in Appendix I.

Using this notation, the multiplicative quaternion update is

$$\hat{\mathbf{q}}_b^{n\,+} = \mathbf{q}_{\hat{n}}^n \otimes \hat{\mathbf{q}}_b^{n\,-}.$$

Quaternion operations are defined in Section D of [3].

Appendix I
Quaternion Update Approximation

Let $\boldsymbol{f}(\boldsymbol{\rho}) = \sqrt{1 - \left\|\frac{1}{2}\boldsymbol{\rho}\right\|_2^2} \in \mathbb{R}^1$, and $\delta\boldsymbol{\rho} = \boldsymbol{\rho} - \mathbf{0} \in \mathbb{R}^{3\times 1}$. By first-order Taylor series expansion of $\boldsymbol{f}(\boldsymbol{\rho})$, assuming small-angle $\boldsymbol{\rho}$, the quantity $\mathbf{q}_{\hat{n}}^n$ is

$$
\begin{aligned}
\mathbf{q}_{\hat{n}}^n &= \begin{bmatrix} \boldsymbol{f}(\boldsymbol{\rho}) \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{f}(\mathbf{0}) + \frac{\partial \boldsymbol{f}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\Big|_{\boldsymbol{\rho}=\mathbf{0}} \delta\boldsymbol{\rho} + \delta\boldsymbol{\rho}^\intercal \frac{\partial^2 \boldsymbol{f}(\boldsymbol{\rho})}{2\,\partial \boldsymbol{\rho}^2}\Big|_{\boldsymbol{\rho}=\mathbf{0}} \delta\boldsymbol{\rho} + \cdots \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix} \\
&= \begin{bmatrix} 1 + \left[\frac{-\boldsymbol{\rho}}{2\sqrt{1-\left\|\frac{1}{2}\boldsymbol{\rho}\right\|_2^2}}\right]\Bigg|_{\boldsymbol{\rho}=\mathbf{0}} \delta\boldsymbol{\rho} \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\rho} \end{bmatrix},
\end{aligned}
$$

where the derivation of the gradient and Hessian is provided in Appendix II. Note:

- $\boldsymbol{f}(\mathbf{0}) = \sqrt{1 - \left\|\frac{1}{2}\mathbf{0}\right\|_2^2} = 1.$
- For $\boldsymbol{\rho} \ll 1$, then $\delta\boldsymbol{\rho}^\intercal \frac{\partial^2 \boldsymbol{f}(\boldsymbol{\rho})}{2\,\partial \boldsymbol{\rho}^2}\Big|_{\boldsymbol{\rho}=\mathbf{0}} \delta\boldsymbol{\rho} \approxeq 0.$
- $\hat{\mathbf{q}}_v$ is linear already.

Appendix II
Gradient and Hessian Derivation

Let $\mathbf{h}(\boldsymbol{x}) = (\boldsymbol{x}^\intercal\boldsymbol{x})^{1/2}$ where $\boldsymbol{x} \in \mathbb{R}^{3\times 1}$. The Jacobian of $\mathbf{h}(\boldsymbol{x})$ is

$$\frac{\partial \mathbf{h}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \frac{\boldsymbol{x}^\intercal}{(\boldsymbol{x}^\intercal\boldsymbol{x})^{1/2}}.$$

The Hessian of $\mathbf{h}(\boldsymbol{x})$ is

$$
\begin{aligned}
\frac{\partial^2 \mathbf{h}(\boldsymbol{x})}{\partial \boldsymbol{x}^2} &= \frac{\mathbf{I}}{(\boldsymbol{x}^\intercal\boldsymbol{x})^{1/2}} - \left(\frac{1}{2}\right)(2)\frac{\boldsymbol{x}\boldsymbol{x}^\intercal}{(\boldsymbol{x}^\intercal\boldsymbol{x})^{3/2}} \\
&= \frac{\boldsymbol{x}^\intercal\boldsymbol{x}\ \mathbf{I} - \boldsymbol{x}\boldsymbol{x}^\intercal}{(\boldsymbol{x}^\intercal\boldsymbol{x})^{3/2}}.
\end{aligned}
$$

References

[1] P. F. Roysdon and J. A. Farrell, "GPS-INS Outlier Detection and Elimination using a Sliding Window Filter," *American Control Conference, In Presc.*, 2017.

[2] ——, "Robust GPS-INS Outlier Accomodation using a Sliding Window Filter," *22th IFAC World Congress*, 2017.

[3] J. A. Farrell, *Aided Navigation: GPS with High Rate Sensors*. McGraw Hill, 2008.