OFDM and DMT: System Model

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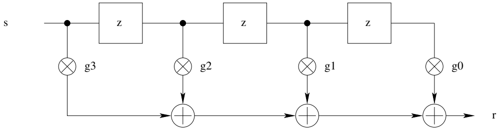
Review: LTI channel and dispersion

- Time-dispersive LTI channel



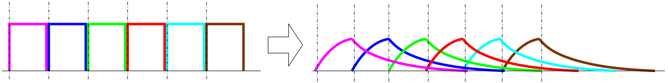
- Dispersion M = length of impulse response − 1 = order of channel filter
- LTI channel performs linear convolution

$$r(n) = \sum_k s(k)h(n-k)$$

- Length-N input $s(n), n = 0, \ldots, N-1$ yields length-$(N+M)$ output $r(n), n = 0, \ldots, N+M-1$
- Input block is "smeared out" (dispersed) over M additional samples

Review: Dispersion can cause unacceptable ISI/ICI

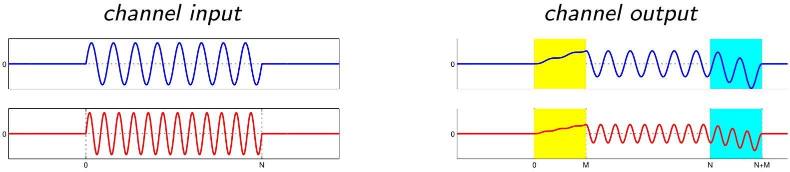
- High datarate $\longrightarrow$ high symbolrate $\longrightarrow$ severe ISI



high data rate

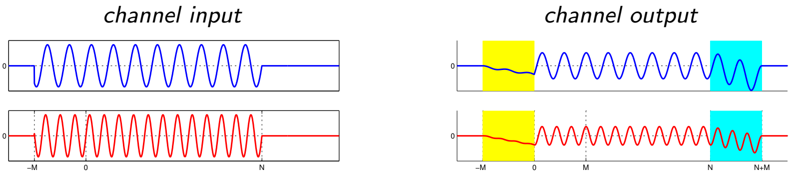
severe ISI!

- Dispersion causes transients: severe ICI $\longleftrightarrow$ loss of orthogonality



Review: Ideas of multicarrier transmission

- Use several, mutually orthogonal, longer, sinusoidal transmit waveforms
    - "several": compensate for the "longer" in terms of data rate
    - "mutually orthogonal": avoid ICI
    - "sinusoidal": **eigenfunctions of LTI systems**, efficient implementation (FFT)
- Cyclic extension avoids ISI and ICI in time-dispersive channels



- System design limits
    - $T_{\mathrm{MC}} \gg \tau_{\max}$
    - $T_{\mathrm{MC}} \ll \min\{T_{\mathrm{coh}}, \text{latency limit, carrier-spacing limit}\}$

CP: linear convolution $\longrightarrow$ circular convolution

$$\xrightarrow{\tilde{s}(n)} \boxed{\mathrm{CP}} \xrightarrow{s(n)} \boxed{h(n)} \xrightarrow{\tilde{r}(n)} \boxed{\cancel{\mathrm{CP}}} \xrightarrow{r(n)}$$

- CP-add block: $s(n) = \begin{cases} \tilde{s}(n), & n = 0, \ldots, N-1 \\ \tilde{s}(N+n), & n = -L, \ldots, -1 \end{cases}$
- Channel: linear convolution $\tilde{r}(n) = \sum_k s(k)h(n-k)$
- CP-remove block: $r(n) = \tilde{r}(n), n = 0, \ldots, N-1$

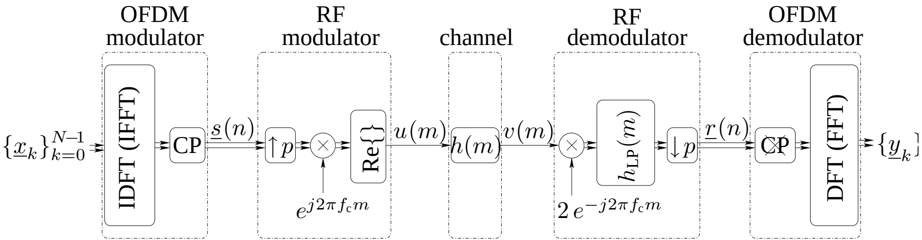
For $L \geq M$: CP-add + channel + CP-remove yield circular convolution

$$r(n) = \tilde{r}(n) = \sum_k \tilde{s}((n-k) \bmod N)h(k), \qquad n = 0, \ldots, N-1$$

and consequently $R[k] = \tilde{S}[k]H[k], k = 0, \ldots, N-1$ holds.

$R[k]$, $\tilde{S}[k]$, $H[k]$ are the N-point FFTs of $r(n)$, $\tilde{s}(n)$, $h(n)$, respectively.

Orthogonal Frequency Division Multiplex (OFDM) system



$$
\begin{aligned}
\underline{s}(n) &= \mathrm{IDFT}_N\,\{\underline{x}_k\}_{k=0}^{N-1} && n = 0, \ldots, N-1 \\
\underline{s}(n) &= \underline{s}(n+N), && n = -L, \ldots, -1 \\
\underline{s}(m) &= \{\underline{s}(n)\}_{\uparrow p}\,, && m = -Lp, \ldots, Np-1 \\
u(m) &= \mathrm{Re}\left\{\underline{s}(m)e^{j2\pi f_c m}\right\} && \\
v(m) &= u(m) * h(m) && \\
\underline{r}(m) &= \left(v(m)\,2e^{-j2\pi f_c m}\right) * h_{\mathrm{LP}}(m) && \\
\underline{r}(n) &= \{\underline{r}(m)\}_{\downarrow p}\,, && n = 0, \ldots, N-1 \\
\underline{y}_k &= \mathrm{DFT}_N\{\underline{r}(n)\}_{n=0}^{N-1} && k = 0, \ldots, N-1
\end{aligned}
$$

Receive signal components (passband)

Real-valued passband receive signal components[1]:

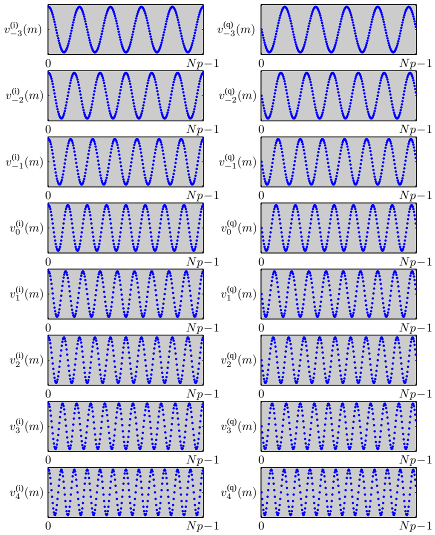
$$
\left.
\begin{aligned}
v_k^{(\mathrm{i})}(m) &= \frac{1}{\sqrt{N}} \cos\left(2\pi \left(f_{\mathrm{c}} + \frac{k}{Np}\right) m\right),\\
v_k^{(\mathrm{q})}(m) &= -\frac{1}{\sqrt{N}} \sin\left(2\pi \left(f_{\mathrm{c}} + \frac{k}{Np}\right) m\right),
\end{aligned}
\right.
\qquad m = 0, \ldots, Np-1
$$

for $k = -N/2+1, \ldots, N/2$.

- f_{c} is a positive *carrier frequency*
- p is the oversampling factor and sufficiently large to ensure that $f_{\mathrm{c}} + \frac{k}{pN} < \frac{1}{2}$, $k = -N/2+1, \ldots, N/2$ holds for all the N normalised frequencies

[1] Throughout the lecture, we assume that N is even.

Receive signal components (passband) in time-domain



- $v_k^{(i)}(m)$ (cosine signals, left column) and $v_k^{(q)}(m)$ (sine signals, right column); parameters: $N = 8$, $p = 32$, $f_c = 1/32$, $k = -3, \dots, 4$
- There are $2N$ mutually orthogonal real-valued length-pN discrete-time sinusoidal waveforms with an integer number of periods

Transmit signal components (passband)

Similarly, we consider the transmit signals of length $(N+L)p$ samples

$$u_k^{(\mathrm{i})}(m) = \frac{1}{\sqrt{N}} \cos(2\pi \left(f_\mathrm{c} + \frac{k}{Np} \right) m), \qquad m = -Lp, \ldots, Np-1$$
$$u_k^{(\mathrm{q})}(m) = -\frac{1}{\sqrt{N}} \sin(2\pi \left(f_\mathrm{c} + \frac{k}{Np} \right) m),$$

where $k = -N/2+1, \ldots, N/2$.

- Cyclic prefix of length pL (passband)

RF modulator

We write the passband transmit signal as

$$
\begin{aligned}
u(m) &= \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} \left(x^{(\mathrm{i})}_{[k]_N} \cos\left(2\pi \left(f_c + \frac{k}{Np}\right) m\right) - x^{(\mathrm{q})}_{[k]_N} \sin\left(2\pi \left(f_c + \frac{k}{Np}\right) m\right) \right) \\
&= \mathrm{Re}\left\{ \underbrace{\frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} \underline{x}_{[k]_N} e^{j2\pi \frac{k}{Np} m}}_{\underline{s}(m)} \, e^{j2\pi f_c m} \right\},
\end{aligned}
$$

where $[k]_N$ denotes the modulo-N operation applied to k.

- This operation corresponds to a cos/sin modulator structure
- Perfect interpolation by factor p of $\underline{s}(n)$ yields $\underline{s}(m)$
- Note that $\underline{s}(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} \underline{x}_{[k]_N} e^{j2\pi \frac{k}{N} n} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \underline{x}_k e^{j2\pi \frac{k}{N} n}$

Baseband transmit signal processing (OFDM modulator)

$$
\begin{aligned}
\underline{s}(n) &= \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} \underline{x}_{[k]_N} e^{j2\pi \frac{k}{N} n} \\
&= \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{N/2} \underline{x}_k e^{j2\pi \frac{k}{N} n} + \sum_{k=-N/2+1}^{-1} \underline{x}_{(\underbrace{N+k}_{\triangleq i, \rightarrow k = i-N})} e^{j2\pi \frac{k}{N} n} \right) \\
&= \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{N/2} \underline{x}_k e^{j2\pi \frac{k}{N} n} + \sum_{i=N/2+1}^{N-1} \underline{x}_i \underbrace{e^{j2\pi \frac{i-N}{N} n}}_{e^{j2\pi \frac{i}{N} n} e^{-j2\pi \frac{N}{N} n} = e^{j2\pi \frac{i}{N} n}} \right) \\
&= \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{N/2} \underline{x}_k e^{j2\pi \frac{k}{N} n} + \sum_{k=N/2+1}^{N-1} \underline{x}_k e^{j2\pi \frac{k}{N} n} \right) \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \underline{x}_k e^{j2\pi \frac{k}{N} n}, \qquad \text{which is the IDFT of } \underline{x}_k
\end{aligned}
$$

Channel output

The channel coefficients are given by

$$\underline{H}_{[k]_N}=A_{[k]_N}+jB_{[k]_N}=\sum_{m=0}^{(M+1)p-1} h(m)e^{-j2\pi(f_c+\frac{k}{pN})m},\ k=-N/2+1,\ldots,N/2.$$

The passband transmit signal $u(m)$ yields the channel output

$$v(m)=\frac{1}{\sqrt{N}}\sum_{k=-N/2+1}^{N/2}\Bigg(\big(A_{[k]_N}x^{(i)}_{[k]_N}-B_{[k]_N}x^{(q)}_{[k]_N}\big)\cos\Big(2\pi\left(f_c+\frac{k}{pN}\right)m\Big)-$$

$$\big(B_{[k]_N}x^{(i)}_{[k]_N}+A_{[k]_N}x^{(q)}_{[k]_N}\big)\sin\Big(2\pi\left(f_c+\frac{k}{pN}\right)m\Big)\Bigg)$$

RF demodulator

Receiver performs the cosine-sine demodulation, low pass filtering and downsampling by factor p of the receive signal, which yields

$$\underline{r}(n)=\frac{1}{\sqrt{N}}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\Big((A_{[k]_N}x^{(\mathrm{i})}_{[k]_N}-B_{[k]_N}x^{(\mathrm{q})}_{[k]_N})+j(B_{[k]_N}x^{(\mathrm{i})}_{[k]_N}+A_{[k]_N}x^{(\mathrm{q})}_{[k]_N}\Big)\,e^{j2\pi\frac{k}{N}n},$$

which can be written as

$$\underline{r}(n)=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\Big((A_k x^{(\mathrm{i})}_k-B_k x^{(\mathrm{q})}_k)+j(B_k x^{(\mathrm{i})}_k+A_k x^{(\mathrm{q})}_k)\Big)\,e^{j2\pi\frac{k}{N}n} \tag{1}$$

We can interpret (1) as receive multiplex using the receive signal components

$$\underline{r}_k(n)=\frac{1}{\sqrt{N}}e^{j2\pi\frac{kn}{N}},\qquad n=0,\ldots,N-1$$

Baseband receive signal processing (OFDM demodulator)

Finally, we pass the complex-valued receive multiplex

$$\underline{r}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left((A_k x_k^{(\mathrm{i})} - B_k x_k^{(\mathrm{q})}) + j(B_k x_k^{(\mathrm{i})} + A_k x_k^{(\mathrm{q})}) \right) e^{+j2\pi \frac{k}{N} n}$$

through a bank of correlators, which yields

$$\langle \underline{r}, \underline{r}_k \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \underline{r}(n) e^{-j2\pi \frac{k}{N} n} = (A_k x_k^{(\mathrm{i})} - B_k x_k^{(\mathrm{q})}) + j(B_k x_k^{(\mathrm{i})} + A_k x_k^{(\mathrm{q})})$$

where $k = 0, \ldots, N-1$. The operation performed by these correlators is equivalent to the scaled N-point Discrete Fourier Transform (DFT) of the complex-valued baseband receive multiplex $\underline{r}(n)$

$$\underline{y}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \underline{r}(n) e^{-j2\pi \frac{k}{N} n}.$$

Complex baseband representation

We now introduce a model of a fictitious system which has the same *behaviour* as the real-world, implementable passband system above. The received *complex* signal can be written as

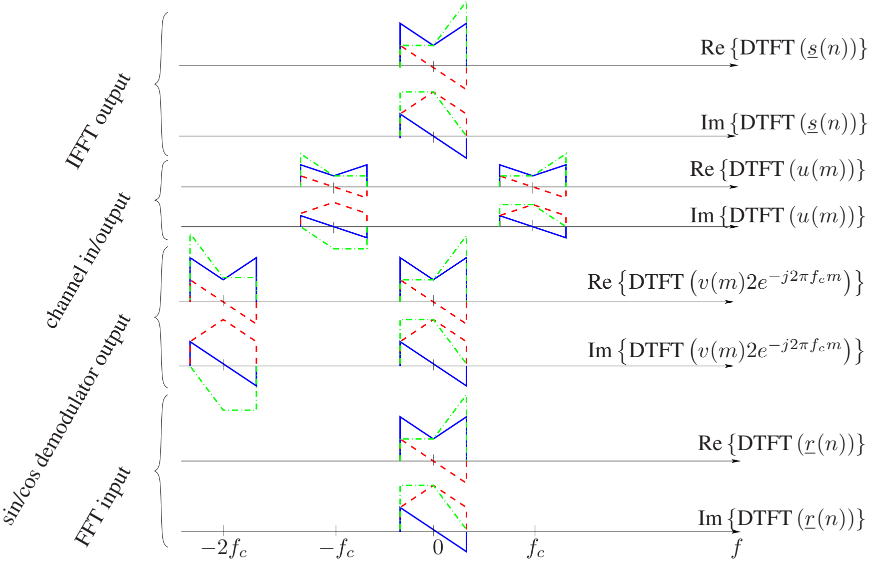
$$\underline{r}(n) = \underline{h}(n) * \underline{s}(n),$$

where the *complex* channel impulse response $\underline{h}(n)$ is given by

$$\underline{h}(n) = \mathrm{IDFT}_N \{\underline{H}_k\} = \sum_{k=0}^{N-1} \underline{H}_k e^{j2\pi \frac{kn}{N}}.$$

- We model the (real-valued) passband signals as (complex-valued) baseband signals
- In essence, we omit RF modulator and RF demodulator and transform the passband channel into an equivalent baseband channel

OFDM spectra (flat channel)



solid blue: DTFT of $\mathrm{Re}\{x\}$. dashed red: DTFT of $\mathrm{Im}\{x\}$. dashed-dotted green: DTFT of $x = \mathrm{Re}\{x\} + j\mathrm{Im}\{x\}$.

Power spectral density (PSD)

- Average measure for spectral allocation of transmit power
- Stationary signals: $\mathcal{PSD}(f) = \mathcal{F}_m\left(r(m)\right)$
  - PSD is Fourier transform of transmit signal's autocorrelation function $r(m)$
- Cyclostationary signals: $\mathcal{PSD}(f) = \mathcal{F}_m\left(\frac{1}{N'}\sum_n r(m,n)\right)$
  - PSD is Fourier transform of time-averaged transmit signal's autocorrelation function $r(m,n)$. $(r(m,n) = r(m,n+N'))$
- For uncorrelated data (over both time and subchannels):

$$\mathcal{PSD}(f) = \sum_{k \in \text{set of used subchannels}} P_k \left|\text{DTFT of } k\text{th basis function}\right|^2$$

where P_k is the power on the kth subchannel.

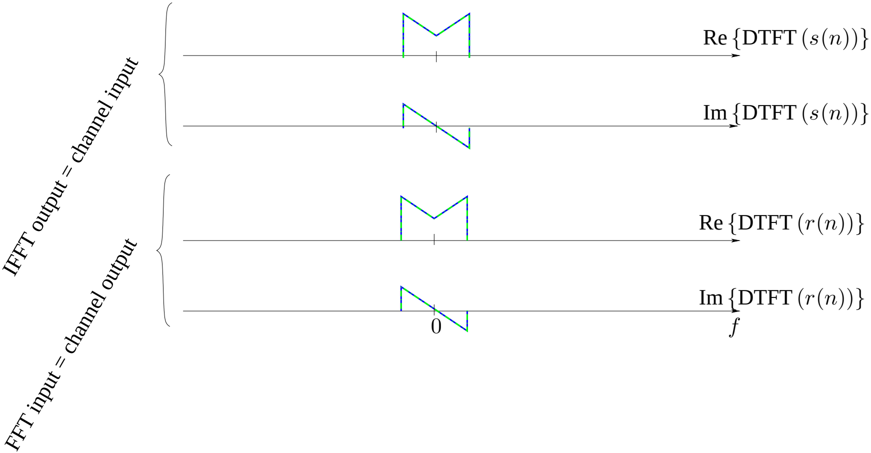
- cf. `psdmc.m`

$\mathcal{F}_\cdot(r(\cdot,\ldots))$ is the Fourier transform (DTFT) of the continuous(discrete)-time signal r with respect to $\cdot$

Discrete Multi-Tone (DMT) spectra (flat channel)

In case we have a baseband channel, we

- do not need the RF modulator/demodulator
- need a real-valued transmit multiplex



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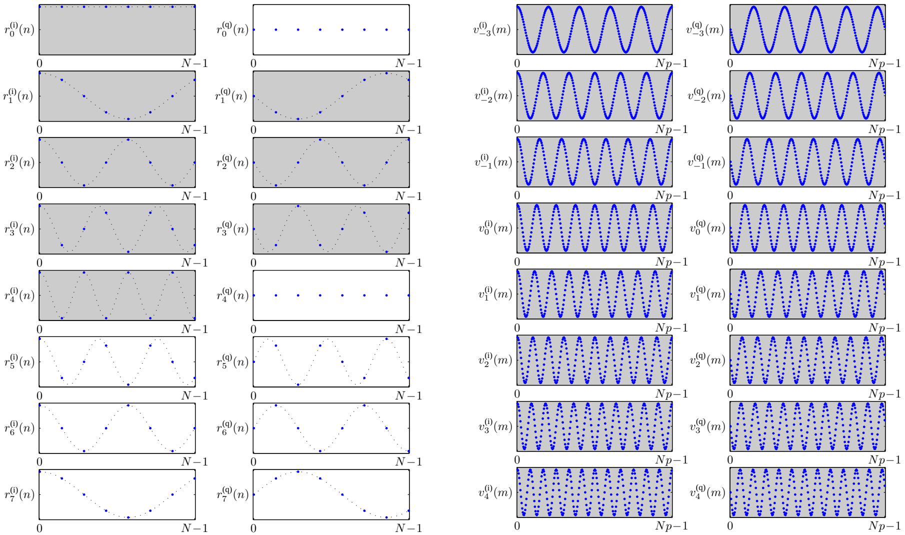
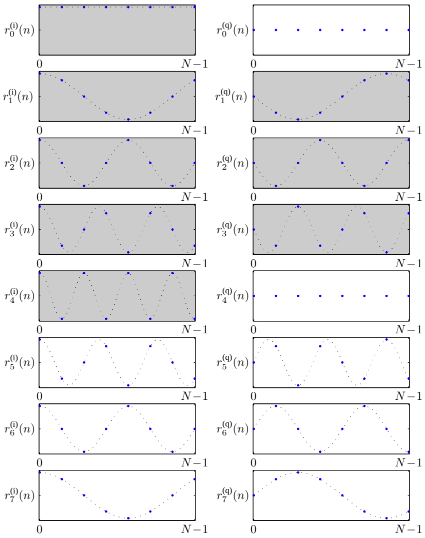
solid blue: DTFT of $\mathrm{Re}\{x\}$. dashed red: DTFT of $\mathrm{Im}\{x\}$. dashed-dotted green: DTFT of $x = \mathrm{Re}\{x\} + j\mathrm{Im}\{x\}$.

Receive signal components: baseband $\leftrightarrow$ passband



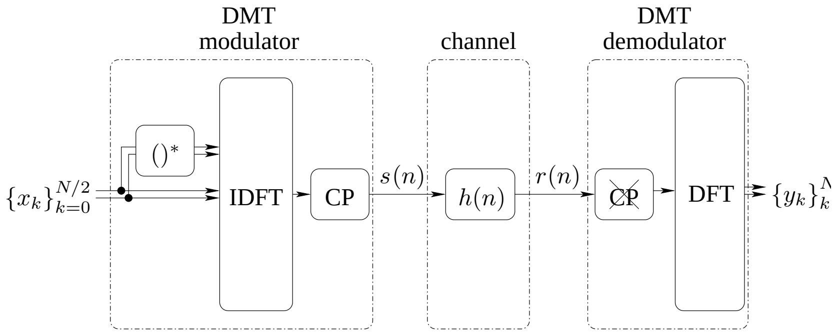
Sinusoidal receive signal components



- There are N mutually orthogonal real-valued length-N discrete-time sinusoidal waveforms with an integer number of periods ($N = 8$ in this example).
- For $k \in \{\lfloor N/2 \rfloor + 1, \ldots, N-1\}$, the number of oscillations per N samples decreases with increasing k.
- In fact, it is easy to verify that

$$\cos(2\pi \frac{k}{N} n) = \cos(2\pi \frac{(N-k)}{N} n),$$
$$\sin(2\pi \frac{k}{N} n) = -\sin(2\pi \frac{(N-k)}{N} n) \quad (2)$$

Discrete Multi-Tone (DMT) system



$$
\begin{aligned}
\underline{x}_{N-k} &= \underline{x}_k^* && k = 0, \ldots, N/2 \\
s(n) &= \mathrm{IDFT}_N \{\underline{x}_k\}_{k=0}^{N-1} && n = 0, \ldots, N-1 \\
s(n) &= s(n+N), && n = -L, \ldots, -1 \\
r(n) &= h(n) * s(n), && n = 0, \ldots, N-1 \\
\underline{y}_k &= \mathrm{DFT}_N \{r(n)\}_{n=0}^{N-1} && k = 0, \ldots, N/2
\end{aligned}
$$

DMT: Hermitian symmetry of transmit symbols

We make the following choice for the transmit symbols

$$x_k^{(\mathrm{i})} = x_{N-k}^{(\mathrm{i})} \quad \text{and} \quad x_k^{(\mathrm{q})} = -x_{N-k}^{(\mathrm{q})}, \tag{3}$$

which implies $x_0^{(\mathrm{q})} = x_{N/2}^{(\mathrm{q})} = 0$. (3) ensures Hermitian symmetry of the transmit symbol blocks. With (2), we obtain for $n = -L, \dots, N-1$

$$\begin{aligned} s(n) = {} & \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(x_k^{(\mathrm{i})} \cos(2\pi \frac{k}{N} n) - x_k^{(\mathrm{q})} \sin(2\pi \frac{k}{N} n) \right) \\ & + j \frac{1}{\sqrt{N}} \underbrace{\sum_{k=0}^{N-1} \left(x_k^{(\mathrm{i})} \sin(2\pi \frac{k}{N} n) + x_k^{(\mathrm{q})} \cos(2\pi \frac{k}{N} n) \right)}_{=0} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \underline{x}_k e^{j2\pi \frac{k}{N} n}. \end{aligned}$$

- $\rightarrow$ On the interval $0, \dots, N-1$, the transmit signal multiplex $s(n)$ is generated by the scaled N-point Inverse Discrete Fourier Transform (IDFT) of the data symbols $\underline{x}_k$.

DMT and OFDM

- DMT: real-valued baseband multiplex $s(n)$, OFDM: complex-valued baseband multiplex $\underline{s}(n)$
- DMT: wireline communications (lowpass channel), OFDM: wireless communications (passband channel)
- DMT: channel known at the transmitter, OFDM: channel unknown at transmitter

Examples of OFDM/DMT systems and parameters

	HiperLAN/2	DVB-T 8k	ADSL DS	VDSL DS	DAB
multiplex[2]:	complex	complex	real	real	complex
N:	64	8192	512	8192	2048
L:	16	256	32 or[3] 40	640	504
bandwidth:	20 MHz	7.61 MHz	966 kHz	$\approx$ 8 MHz	1.536 MHz
band:	5.15-5.35 GHz	UHF	0-1.1 MHz	0-12 MHz	III
$\frac{F_s}{N}$:	0.3125 MHz	1.116 kHz	4.3125 kHz	4.3125 kHz	1 kHz
datarate:	6-54 Mbps	5-30 Mbps	0.5-8 Mbps	$\leq$ 54 Mbps	$\leq$ 348 kbps
modulation:	B/Q/8PSK 16/64QAM	4/16/64QAM	BPSK 4-2^{15}QAM	BPSK 4-2^{15}QAM	diff. QPSK
mobility:	pedestrian	vehicular	none	none	vehicular
channel:	wireless	wireless	wire	wire	wireless
reach:	$\leq$ 150 m	$\approx$ km	$\leq$ 5 km	$\leq$ 1.5 km	$\approx$ km

III band: 174MHz - 240MHz; UHF band: 790MHz - 806MHz

Note that many standards include various modes with different parameter sets.

[2] complex $\hat{=}$ OFDM. real $\hat{=}$ DMT.

[3] $L = 32$ in case a synch-symbol is used, otherwise $L = 40$

Matrix representation of DMT and OFDM

- Comes in handy since OFDM/DMT is a block transmission scheme
- More compact and less painful than scalar description
- Yields another interpretation of cyclic prefixing, ISI-free and ICI-free transmission
- Notation:
  - lowercase boldface: (usually a column[4]) vector (*e.g.* $\mathbf{a} \in \mathbb{R}^N$)
  - uppercase boldface: matrix (*e.g.* $\mathbf{A} \in \mathbb{C}^{N \times (N+L)}$)
  - $\mathbf{A}^{\mathrm{T}}$: transpose. $\mathbf{A}^{\mathrm{H}}$: Hermitian transpose (transposed conjugate).

[4] In channel coding, they like to use **a** as row vector.

Channel: linear convolution matrix

We describe the channel dispersion using a convolution matrix $\mathbf{H}$, where we assume $L = M$:

$$
\begin{bmatrix} r(0) \\ r(1) \\ \vdots \\ r(N-1) \end{bmatrix} =
\begin{bmatrix}
{\color{red}h(L)} & {\color{red}\cdots} & {\color{red}h(1)} & h(0) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & {\color{red}\ddots} & & h(1) & h(0) & \ddots & & & & \vdots \\
\vdots & & {\color{red}h(L)} & \vdots & h(1) & \ddots & & & & \vdots \\
\vdots & & & h(L) & \vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & 0 & h(L) & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \cdots & 0 & h(L) & \cdots & h(1) & h(0)
\end{bmatrix}
\begin{bmatrix} {\color{red}s(-L)} \\ {\color{red}\vdots} \\ {\color{red}s(-1)} \\ s(0) \\ s(1) \\ \vdots \\ s(N-1) \end{bmatrix}
\tag{4}
$$

- Both DMT ($\mathbf{H}$ real-valued) or OFDM ($\mathbf{H}$ complex-valued) can be represented
- The first L received samples $r(n), n = -L, \ldots, -1$ are implicitly ignored

Channel + CP: circulant convolution matrix

Exploiting the fact that $s(n) = s(N-n)$, $n = -L, \ldots, -1$, we can rewrite (4) as

$$\underbrace{\begin{bmatrix} r(0) \\ r(1) \\ \vdots \\ r(N-1) \end{bmatrix}}_{\mathbf{r}} = \underbrace{\begin{bmatrix} h(0) & 0 & \cdots & 0 & h(L) & \cdots & h(1) \\ h(1) & h(0) & \ddots & & 0 & \ddots & \vdots \\ \vdots & h(1) & \ddots & & \ddots & \ddots & h(L) \\ h(L) & \vdots & \ddots & \ddots & & & 0 \\ 0 & h(L) & & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h(L) & & h(1) & h(0) \end{bmatrix}}_{\tilde{\mathbf{H}}} \underbrace{\begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ s(N-1) \end{bmatrix}}_{\mathbf{s}}.$$

- Only possible for $L \geq M$
- Addition of the CP and removal of the CP are implicitly included

Receive signal processing

$$\underbrace{\begin{bmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \vdots \\ \underline{y}_{N-1} \end{bmatrix}}_{\mathbf{y}} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} r(0) \\ r(1) \\ \vdots \\ r(N-1) \end{bmatrix}}_{\mathbf{r}}$$

where $w = e^{-j2\pi \frac{1}{N}}$.

- $\mathbf{R}$ is the normalised DFT matrix
- Rows of $\mathbf{R}$ contain the receive signal components
- Multiplication of $\mathbf{R}$ with $\mathbf{r}$ corresponds to computing N correlations (one per subcarrier/subchannel)
- Can be implemented in $\mathcal{O}(N \log N)$ using the FFT

Transmit signal processing

$$\underbrace{\begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ s(N-1) \end{bmatrix}}_{\mathbf{s}} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^{-1} & w^{-2} & \dots & w^{-(N-1)} \\ 1 & w^{-2} & w^{-4} & \dots & w^{-2(N-1)} \\ & \vdots & \vdots & & \vdots \\ 1 & w^{-(N-1)} & w^{-2(N-1)} & \dots & w^{-(N-1)^2} \end{bmatrix}}_{\mathbf{R}^{\mathrm{H}}} \underbrace{\begin{bmatrix} \underline{x}_0 \\ \underline{x}_1 \\ \vdots \\ \underline{x}_{N-1} \end{bmatrix}}_{\mathbf{x}}$$

- $\mathbf{R}^{\mathrm{H}}$ is the normalised IDFT matrix
- The columns of $\mathbf{R}^{\mathrm{H}}$ are the transmit signal components
- kth transmit symbol ($\underline{x}_{k-1}$) scales the kth column of $\mathbf{R}^{\mathrm{H}}$
- Transmit multiplex is the sum of N scaled complex exponentials (scaled by the data)
- Can be implemented in $\mathcal{O}(N \log N)$ using the IFFT

"Diagonalisation" of the circulant channel matrix

- Note:
  - $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{H}}$ ($\mathbf{R}$ is unitary matrix) $\longrightarrow \mathbf{R}\mathbf{R}^{\mathrm{H}} = \mathbf{R}^{\mathrm{H}}\mathbf{R} = \mathbf{I}$
  - *Any circulant matrix* $\tilde{\mathbf{H}}$ can be written as $\tilde{\mathbf{H}} = \mathbf{R}^{\mathrm{H}}\boldsymbol{\Lambda}\mathbf{R}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix
- Consequently, we can write

$$\mathbf{y} \triangleq \mathbf{R}\mathbf{r} = \mathbf{R}\tilde{\mathbf{H}}\mathbf{s} = \mathbf{R}\tilde{\mathbf{H}}\mathbf{R}^{\mathrm{H}}\mathbf{x} = \mathbf{R}\underbrace{\tilde{\mathbf{H}}}_{\mathbf{R}^{\mathrm{H}}\boldsymbol{\Lambda}\mathbf{R}}\mathbf{R}^{\mathrm{H}}\mathbf{x} = \boldsymbol{\Lambda}\mathbf{x}.$$

- The DFT/IDFT matrix-pair diagonalises any circulant channel matrix
- As long as $L \geq M$ (proper cyclic prefix), we obtain parallel, independent subchannels (no ISI, no ICI)

Summary

1. Transmit signal processing: linear combination of transmit signal components (complex exponentials) $\longrightarrow$ IFFT
2. Receive signal processing: correlation with complex exponentials $\longrightarrow$ FFT
3. OFDM vs DMT
4. Three interpretations of cyclic prefixing
5. Matrix notation