Technical Note: Hairspring Calculations

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I. Introduction

According to several watchmaking books and industry literature, the material selection and length of a hairspring is determined *experimentally*, i.e., by trial and error. However, we can use *mathematics* to calculate the *theoretical* length within 0.2 mm of the *experimental* length, and measuring instrument tolerances limit further precision.

For convenience, this technical report is split into two parts.

- In *part one* (Section II) the calculation of hairspring length and coil spacing is demonstrated using several examples, using Sections III and IV equations like a recipe.
- In *part two* (Sections III and IV) the necessary equations are derived from "first principles" with comments to aid the reader's understanding of the underlying algebra and calculus. Expertise in mathematics is not necessary.

Because equation manipulations can be difficult and error-prone, this technical note illustrates step-by-step unit conversions and equation manipulations.

The Archimedean spiral arc length in Cartesian coordinates, Eqn. 19, is the key equation of this technical note. **To the author's knowledge, this equation does not exist anywhere in the watchmaking literature**. Because this equation is unique, the complete derivation is provided in Sections IV-B - IV-E.

II. Examples

Before we look at examples we first define the steps necessary to calculate the length and geometry of a hairspring.

We require a few known values:

- *Material properties*: modulus of elasticity of the steel to be used in the hairspring.
- *Dimensions of the hairspring*: inner diameter, outer diameter, height and thickness.
- *Properties of the balance*: oscillating frequency and moment of inertia.

In the examples we will show a tabular calculation method if some values are unknown.

The calculation of hairspring length requires three steps:

1) *Material property calculation* of length based on the elastic torque of the hairspring and balance wheel inertia.
2) *Geometry calculation* of length based on the geometry of the spiral; this is also called the "arc length of an Archimedian spiral".
3) *Comparison and selection*.

As we will see, these steps each require a few sub-steps, e.g. Step 1a, 1b, etc.

Finally, we will use Table I for reference [1], where h is the height of the hairspring wire, e is the thickness, L_1 is the length from the pinning point of the collet to the pins at the vibrating point, L_2 is the length from the pins at the vibrating point to the stud, and c_n is the number of coils.

TABLE I

Hairspring Examples with a beat-rate of 2.5 Hz

Make & Model	h (mm)	e (mm)	L_1 (mm)	L_2 (mm)	D_o (mm)	D_i (mm)	c_n (#)
ETA 6498-1	0.190	0.040	163	4	7.18	1.31	13
ETA 6497-1	0.180	0.020		13	6.91	1.31	12
Eterna 1145	0.155	0.029					
Citizen 1161	0.130	0.033					
Roysdon 1	0.220	0.030	162	4	7.15	1.35	13
Shapiro X	0.196	0.032	166	4	7.12	1.41	13

A. Step 1a: Elastic Torque Calculation

If the numerical values for the parameters of Eqn. 1 are known, then the elastic torque is calculated directly. Let the moment of inertia of the balance wheel $I = 25\ [mg \cdot cm^2]$ and the frequency $f = 2.5\ Hz$, and reference Table III,

$$
\begin{aligned}
M &= I \cdot 4\pi^2 \cdot f^2 \\
&= 25\ [mg \cdot cm^2] \cdot 4\pi^2 \cdot \left(2.5 \left[\frac{1}{s}\right]\right)^2 \\
&= 25\ [mg \cdot cm^2] \cdot 4\pi^2 \cdot 6.25 \left[\frac{1}{s^2}\right] \\
&= 6168.5\ \frac{mg \cdot cm^2}{s^2} \\
&= 6.1685 \times 10^{-7}\ \frac{kg \cdot m^2}{s^2} \\
&= 6.1685 \times 10^{-4}\ \frac{N \cdot mm}{rad}.
\end{aligned}
$$

B. Step 1b: Length from Elastic Torque

If the numerical values for the parameters of Eqn. 3 are known, then the length is calculated directly. Let the modulus of elasticity in tension (Young's Modulus) $E = 27.79 \times 10^3\ ksi$, $h = 0.19\ mm$, and the wire thickness

[1] Measurements collected by M. Rose and P. Roysdon on 11/10/2022. *Shapiro X* are calculated values from this technical note.

$e = 0.040\ mm$. First convert the units of E, where 1 $ksi = 6.895\ \frac{N}{mm^2}$ (see Table III), then

$$E = (27.79 \times 10^3) \times 6.895$$
$$= 191,605\ \frac{N}{mm^2}.$$

Now calculate L using Eqn. 3,

$$L = \frac{E \cdot h \cdot e^3}{12 \cdot M}$$

$$= \frac{191,605\ \frac{N}{mm^2} \cdot 0.190\ mm \cdot (0.040\ mm)^3}{12 \cdot 6.1685 \times 10^{-4}\ N \cdot mm}$$

$$= 314.8\ mm$$

Given the length of the ETA 6497-1 hairspring (see Table I), and our input values above, this result seems questionable.

C. Example: Length Calculation with Multiple Unknowns

If some values of Eqn. 3 are not known, or the watchmaker has options from a material supplier, e.g., height, thickness, and elastic modulus, then a table can be produced. In the example below, Eqn. 3 is calculated for each combination of height and thickness using the E and M values from the example in Sections II-A and II-B to produce Table II.

TABLE II
LENGTH CALCULATION WITH MULTIPLE UNKNOWNS

		h (mm)			
		0.150	**0.175**	**0.200**	**0.225**
e (mm)	**0.030**	104.84	122.31	139.78	157.26
	0.035	166.48	194.22	221.97	249.72
	0.040	248.50	289.92	331.34	372.76
	0.045	353.83	412.80	471.77	530.74
	0.050	485.36	566.25	647.15	728.04

MatLab code and a spreadsheet calculator accompany this technical report. Several values of L are calculated **automatically** for a sequence of values h and e. The watchmaker is only required to enter the values for I, f, and E.

D. Step 2a: Spiral Spacing

Using the values for the ETA 6498-1 in Table I, outer diameter $D_o = 7.18\ mm$ and inner diameter $D_i = 1.31\ mm$, spiral spacing is easily calculated using Eqns. 5 - 7,

$$\Delta r = \frac{D_o}{2} - \frac{D_i}{2} = \frac{7.18}{2} - \frac{1.31}{2} = 2.935\ mm.$$

The spacing between coils is

$$b = \left(\frac{\Delta r}{c_n} - \frac{e}{2}\right) = \left(\frac{2.935}{13} - \frac{0.040}{2}\right) = 0.206\ mm.$$

The distance between the start of the spiral and the origin is

$$a = \frac{D_i}{2} = \frac{1.31}{2} = 0.655\ mm.$$

E. Step 2b: Archimedean Spiral Arc Length

To calculate the Archimedean spiral arc length in Cartesian coordinates, Eqn. 19, let the start point $c = 0$, the end point $d = \theta = 2\pi c_n = 81.681$, and the use results for a and $\frac{b}{2\pi}$ in Section II-D. Then,

$$L = b\Bigg(\frac{(a+bd)\sqrt{b^2+(a+bd)^2}}{2b^2} - \frac{(a+bc)\sqrt{b^2+(a+bc)^2}}{2b^2} + \frac{1}{2}\Bigg(\ln\Bigg(\frac{|(a+bd)\sqrt{b^2+(a+bd)^2}|}{|b|}\Bigg) + \ln\Bigg(\frac{|(a+bc)\sqrt{b^2+(a+bc)^2}|}{|b|}\Bigg)\Bigg)\Bigg)$$

$$= 162.8\ mm$$

Comparing the ETA 6498-1 (measured) length in Table I to the calculated (theoretical) result above, we see the calculated length is accurate to a tenth of a millimeter. Again note that the length measurement instrument for Table I was accurate to the nearest whole millimeter. **This comparison result is rather shocking, and demonstrates the accuracy of this mathematical approach to determine hairspring length!**

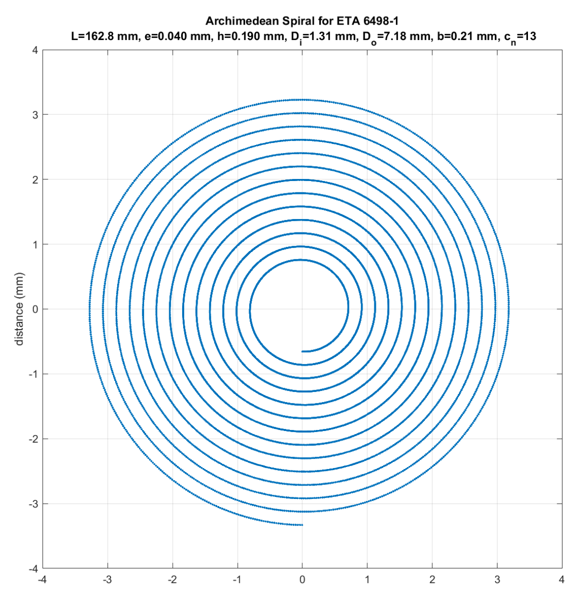

Fig. 1. Archimedes Spiral (hairspring) for ETA 6498-1.

Note: length is defined as the distance between the pinning point on the collet to the vibrating point at the regulating pin. The result calculated using Eqn. 19 is the **theoretical length**. The **actual length will vary slightly**, depending on physical factors of pinning, etc.

F. Step 3: Comparison and Selection

Comparing results for the length calculation in Section II-B, $L = 314.8\ mm$, with Section II-E, $L = 162.8\ mm$, we see a discrepancy. We know the geometric length (Section II-E) has fixed parameters, e.g., D_i and D_o, based on the diameter of the balance wheel. Therefore, changing this length will not suffice. However, we also know that the length calculated from elastic torque (Section II-B) is based on the modulus of elasticity E. Therefore to reduce the value calculated in Section II-B we must use a material with a lower modulus of elasticity or modify h or e.

Using Eqn. 20, where $h = 0.192\ mm$, and $e = 0.032\ mm$, we can calculate the elastic modulus E,

$$E = \frac{12 \cdot M \cdot L}{h \cdot e^3}$$

$$= \frac{12 \cdot 6168.5 \times 10^{-4}\ \frac{N}{mm^2} \cdot 162.8\ mm}{0.192\ mm \cdot (0.032\ mm)^3} \times \frac{1}{6.895}$$

$$= 27.78 \times 10^3\ ksi.$$

Referencing [3], this is an acceptable value. The final results are shown in Fig. 1 with the values listed in the title.

This comparison illustrates the necessity of both length calculation methods. **If E is known, use Eqn. 3 to calculate L. If E is unknown, or to be selected, use Eqn.19 to calculate L then substitute Eqn. 19 into Eqn. 20 to calculate E.**

III. Equation Derivations - Elastic Torque

In this section we will use some basic algebra to convert equations provided in [1] and [2], and data found in [3], into equations we can use for the calculation of the hairspring length. After converting the equations we will check our work to make sure the units are still correct (the units will not match if we made an algebraic error). For algebraic manipulations we reference [4], [5], unit conversions in Table III, and common Young's Modulus values in Table IV.

It is possible to calculate a parameter, e.g., Elastic Torque, M, using multiple approaches (equations) as we will see in the following sections. **Do not be confused by multiple equations of the same parameter.** This is equivalent to the calculation of the number 4 by two methods: $4 = 1 \times 4$ and $4 = 2 \times 2$, both are valid methods that arrive at the same answer. In Sections III-A and III-B we will use two different equations for M to calculate other parameters of interest.

A. Reformulating Frequency to Define Elastic Torque

The elastic torque of the hairspring is derived from the frequency equation

$$f = \frac{1}{2\pi} \cdot \sqrt{\frac{M}{I}},$$

where f is the oscillating frequency in $[Hz] = [\frac{1}{s}]$, M is the elastic constant of the hairspring with units $[\frac{N \cdot mm}{rad}]$, and I is the moment of inertia of the balance in units $[mg \cdot cm^2]$.

We can reformulate f to define M, such that

$$f = \frac{1}{2\pi} \cdot \sqrt{\frac{M}{I}}$$

$$(2\pi) \cdot f = \frac{1}{\cancel{2\pi}} \cdot \sqrt{\frac{M}{I}} \cdot (\cancel{2\pi})$$

$$(2\pi \cdot f)^2 = \left(\sqrt{\frac{M}{I}}\right)^2$$

$$(I) \cdot 4 \cdot \pi^2 \cdot f^2 = \frac{M}{\cancel{I}} \cdot (\cancel{I})$$

$$I \cdot 4 \cdot \pi^2 \cdot f^2 = M$$

$$M = I \cdot 4 \cdot \pi^2 \cdot f^2. \tag{1}$$

In the first step, both sides are multiplied by 2π, canceling terms on the right hand side. The square root is canceled by squaring both sides in the second step. In the third step, I is multiplied on both sides to remove I from the right hand side. The final step is just a rearrangement of terms.

Using Eqn. 1 we can check the units, canceling terms just as we did before and converting units using Table III.

$$[N \cdot mm] = [mg \cdot cm^2] \cdot \left[\frac{1}{s}\right]^2$$

$$= [mg \cdot cm^2] \cdot \left[\frac{1}{s^2}\right]$$

$$= \frac{[mg \cdot cm^2]}{[s^2]}$$

$$= 1 \times 10^{-7} \frac{[kg \cdot m^2]}{[s^2]}$$

$$[N \cdot mm] = 1 \times 10^{-4} [N \cdot mm] \quad \checkmark$$

Note in the final step the units are the same but we have a factor of 1×10^{-4} as a result of converting units.

B. Reformulating Elastic Torque Equation to Define Length

The elastic torque of the hairspring is defined as

$$M = \frac{E \cdot h \cdot e^3}{12 \cdot L}, \tag{2}$$

where M is the elastic constant of the hairspring with units $[\frac{N \cdot mm}{rad}]$, E is the modulus of elasticity for the hairspring in $[\frac{N}{mm^2}]$, h is the height $[mm]$, e is the thickness $[mm]$, and L is the length with units $[mm]$.

Using algebra, we can rearrange Eqn. 2 to define the length,

$$M = \frac{E \cdot h \cdot e^3}{12 \cdot L}$$

$$(L) \cdot M = \frac{E \cdot h \cdot e^3}{12 \cdot \cancel{L}} \cdot (\cancel{L})$$

$$\left(\frac{1}{\cancel{M}}\right) \cdot L \cdot \cancel{M} = \frac{E \cdot h \cdot e^3}{12} \cdot \left(\frac{1}{M}\right)$$

$$L = \frac{E \cdot h \cdot e^3}{12 \cdot M}. \quad (3)$$

In the first step we multiply both sides by L, canceling terms on the right hand side. In the second step we multiply both sides by $\frac{1}{M}$, canceling terms in the left hand side. The result is the equation for L.

Now that we have an equation for L, let's look at the units to verify that all of the units properly cancel. L is in units of $[mm]$ so all units on the right hand side should cancel, with only $[mm]$ remaining.

$$[mm] = \frac{[\frac{N}{mm^2}] \cdot [mm] \cdot [mm^3]}{[N \cdot mm]}$$

$$= \frac{[\frac{N}{mm^2}] \cdot [mm^4]}{[N \cdot mm]}$$

$$= \frac{[\frac{N}{\cancel{mm^2}}] \cdot [\cancelto{mm^2}{mm^4}]}{[N \cdot mm]}$$

$$= \frac{[\cancel{N} \cdot mm^2]}{[\cancel{N} \cdot mm]}$$

$$= \frac{[\cancelto{mm}{mm^2}]}{[\cancel{mm}]}$$

$$[mm] = [mm] \quad \checkmark$$

In the first step we combine terms; recall that multiplication of the base unit requires addition of the exponent by the algebra rule $x^a \cdot x^b = x^{a+b}$, see [5]. The next step we cancel terms, replacing the numerator with $[mm^2]$ using the rule $\frac{x^a}{x^b} = x^{a-b}$, see [5]. The next step we cancel the $[N]$ in both numerator and denominator. Finally we cancel terms, this time canceling the denominator with the exponent of the numerator, again see [5].

IV. Equation Derivations - Arc Length

According to Section 7.5 of [2], the hairspring is an Archimedean spiral. In mathematics, an Archimedean spiral is defined in polar coordinates, and many derivations and algebraic manipulations are necessary to calculate arc length (i.e., hairspring length) in a coordinate frame we can use in practice, e.g. Cartesian coordinates.

1) First, the coil spacing is derived in Section IV-A.
2) Then, in Section IV-B, we define a general equation to calculate the *arc length* for any curve.
3) In Section IV-C, we derive a general *parametric equation* for arc length.
4) In Section IV-D we convert the general parametric equation into a parametric equation for arc length in *polar coordinates*.
5) Finally, in Section IV-E we combine the coil spacing values and the polar coordinate parametric equation to derive the parametric equation for an Archimedean spiral, and then transform it into Cartesian coordinates.

As we will see, the hairspring length equation is complex. However, the example in Section II-E, and the MatLab code and spreadsheet calculator that accompany this technical report should alleviate the reader's concerns.

A. Coil Spacing

The Archimedean spiral, in polar coordinates, is defined as

$$r = a + b\theta. \quad (4)$$

where r is the distance from the origin, a is the distance from the starting point to the origin, b is the distance between turns of the spiral (coil separation), and θ is the angle. We can solve the distance between the coils as the distance between two points

$$\theta_1 = 0 \implies r_1 = a$$
$$\theta_2 = 2\pi \implies r_2 = a + b \cdot 2\pi,$$

then,

$$\begin{aligned} \theta_2 - \theta_1 &= r_2 - r_1 \\ &= \cancel{a} + 2\pi b - \cancel{a} \\ &= 2\pi b. \end{aligned}$$

Using simple geometry, we can define the distance between the coils in Cartesian coordinates. First define the delta radius, Δr, of coils between the inner diameter, D_i, and the outer diameter, D_o as

$$\Delta r = \frac{D_o}{2} - \frac{D_i}{2}. \quad (5)$$

From Eqn. 5, the spacing between coils, b, is

$$b = \frac{\Delta r}{c_n} - \frac{e}{2}, \quad (6)$$

where c_n is the number of coils, and e is the wire thickness [2]. Finally, the distance between the start of the spiral and the origin is

$$a = \frac{D_i}{2}. \quad (7)$$

[2]If the coils are unknown, b can be reformulated to find $c_n = \frac{\Delta r}{b + \frac{e}{2}}$, however this is not recommended.

Using Eqns. 6 and 7 we can calculate the arc length of the Archimedean spiral; see Section IV-E.

B. Arc Length

To derive an equation for the arc length of an Archimedean spiral, we will start from "first principles" and define the arc length of a general curve. Note, material in this section is often covered in Calculus I; see [4] and [5]. This equation, or function, will apply to any continuous function $y = f(x)$ on the interval $[a, b]$. We also require a "smooth curve", meaning the derivative of the function exists everywhere and continuous on $[a, b]$.

For our general function, we need to first estimate the length of the curve. The curve in Fig. 2 is divided in n equal length subintervals (here $n = 9$) each of width Δx with intersection points denoted as P_i. We can approximate the curve with the straight lines connecting the points.

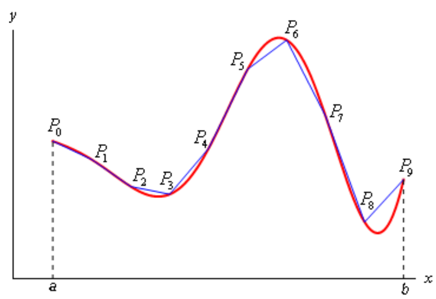

Fig. 2. Graph of an unknown function on the domain $a \le x \le b$, split into 9 equal subintervals. A line connects the points approximating the curve.

The length of each line segment is defined as $|P_{i-1}\ P_i|$ and the total length is approximated by

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|.$$

The approximation improves as n increases in the limit, resulting in the exact length

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|.$$

Let's further refine the line segments such that

$$\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$$

then,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \Delta y_i^2}. \end{aligned}$$

By the *Mean Value Theorem* [5] we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Therefore, the length can be written as

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + [f'(x_i^*)]^2 \Delta x^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\ \ \Delta x, \end{aligned}$$

and the exact length of the curve is

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \ \Delta x. \end{aligned}$$

Applying the definition of the *definite integral* [5],

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned} \tag{8}$$

To illustrate that we can derive a function for any domain, instead of defining the function with respect to the x-axis direction, let's define a new function with respect to the y-axis. In this new axis we still use the points P_i, but the integral changes. We can define a new domain $[c, d]$ with a function $x = h(y)$, such that

$$\begin{aligned} L &= \int_c^d \sqrt{1 + [h'(y)]^2}\, dy \\ &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \end{aligned} \tag{9}$$

Notice Eqns. 8 and 9 are *equivalent* (not equal), Eqn. 8 is defined for x and Eqn. 9 for y.

As we have shown, we can define many functions for the length of a curve. We can simplify to a single general equation,

$$L = \int ds \tag{10}$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if, } y = f(x)\,,\ a \le x \le b \tag{11}$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if, } x = h(y)\,,\ c \le y \le d \tag{12}$$

Note that Eqn. 10 does not have limits because the limits depend on ds. Using the first ds will require x limits of integration and using the second ds will require y limits of integration.

C. Arc Length with Parametric Equations

Now that we have a general function for the arc length, Eqn. 10, we can further parameterize the function. Note, material in this section is often covered in Calculus I; see [4] and [5].

Consider the arc length of the parametric curve given by,

$$x = f\left(t\right), \qquad y = g\left(t\right), \qquad \alpha \le t \le \beta.$$

Assume the curve is traced out exactly once as t increases from α to β, from left to right, such that

$$\frac{dx}{dt} \ge 0, \qquad \text{for } \alpha \le t \le \beta.$$

Let

$$dx = f'(t)\ dt = \frac{dx}{dt}\ dt. \tag{13}$$

Inserting Eqn. 13 into Eqn. 11, the arc length is

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\ \frac{dx}{dt}dt \\ &= \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\ \frac{dx}{dt}dt \\ &= \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ \frac{dx}{dt}dt\,. \end{aligned}$$

The last step is obtained by factoring out the denominator from the square root.

Because we assumed the curve is traced out from left to right we can drop the absolute value bars, and thereby cancel terms. The *parametric equation for arc length* is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,. \tag{14}$$

Notice that we could have used Eqn. 12 with the assumption

$$\frac{dy}{dt} \ge 0, \qquad \text{for } \alpha \le t \le \beta,$$

and the same equation (Eqn. 14) results.

D. Arc Length with Polar Coordinates

Because the Archimedean sprial is defined in polar coordinates, we need to transform our parametric equation for arc length, Eqn. 14, into a parametric equation for arc length in polar coordinates. Note, material in this section is often covered in Calculus II; see [4] and [5].

The general equation for the arc length of a curve in polar coordinates is given by,

$$r = f\left(\theta\right), \qquad \alpha \le \theta \le \beta$$

Again assume the curve is traced out exactly once.

By definition

$$x = r\cos\theta = f\left(\theta\right)\cos\theta$$

$$y = r\sin\theta = f\left(\theta\right)\sin\theta,$$

and the derivatives with respect to θ are

$$\begin{aligned} \frac{dx}{d\theta} &= f'\left(\theta\right)\cos\theta - f\left(\theta\right)\sin\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\theta} &= f'\left(\theta\right)\sin\theta + f\left(\theta\right)\cos\theta \\ &= \frac{dr}{d\theta}\sin\theta + r\cos\theta. \end{aligned}$$

We can do some preliminary work by deriving the terms under the square root in Eqn. 14, such that

$$\begin{aligned} &\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \\ &\quad = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\ &\quad = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\cos^2\theta \\ &\quad = \left(\frac{dr}{d\theta}\right)^2 \cancelto{1}{\left(\cos^2\theta + \sin^2\theta\right)} + r^2 \cancelto{1}{\left(\cos^2\theta + \sin^2\theta\right)} \\ &\quad = r^2 + \left(\frac{dr}{d\theta}\right)^2. \end{aligned}$$

In the first step we input the definition of the derivative, then square the expression in the parenthesis, then cancel terms and combine like terms, then again cancel terms to arrive at the result.

The *parametric equation for arc length in polar coordinates* is,

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{15}$$

E. Archimedean Spiral Arc Length

The Archimedean spiral is the spiral created by an object which moves away from a fixed point at a constant speed while rotating with a constant velocity; see Eqn. 4.

The theoretical Archimedean spiral defines arc length as a function of π radians, assumes the coil thickness is infinitely thin, i.e., zero thickness, and zero inner diameter, i.e., a point; see Fig. 3. We need an arc in engineering units (mm), real thickness in (mm), and a defined inner diameter and outer diameter both in (mm); see Fig. 1.

To calculate the length of a hairspring, we need the length of the Archimedean spiral from one angle c to another angle d. Using Eqn. 15, where r is the radius and θ is the angle, define L such that,

$$
\begin{aligned}
L &= \int_c^d \sqrt{r^2 + \left(\frac{d}{d\theta} r\right)^2} \quad d\theta \\
&= \int_c^d \sqrt{(a+b\theta)^2 + \left(\frac{d}{d\theta} a + b\theta\right)^2} \quad d\theta \\
&= \int_c^d \sqrt{(a+b\theta)^2 + b^2} \quad d\theta \\
&= \int_{a+bc}^{a+bd} \frac{\sqrt{u^2+b^2}}{b} \quad du \\
&= \frac{1}{b} \int_{a+bc}^{a+bd} \sqrt{u^2+b^2} \quad du \\
&= \frac{1}{b} \int_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} b^2 \sec^3(v) \quad dv \\
&= b\left(\left.\frac{\sec^2(v)\sin(v)}{2}\right|_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} \right. \\
&\qquad \left. + \frac{1}{2} \int_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} \sec(v) \quad dv \right).
\end{aligned} \tag{16}
$$

The steps are as follows: first solve by substituting $r = a + b\theta$ for r, then simplify using $\frac{d}{d\theta}(a+b\theta) = b$, then apply u-substitution where $u = f(x)$ and $du = \frac{f(x)}{dx}\, dx$, then factor out the constant by the definition $\int a \cdot f(x)dx = a \int f(x)dx$, then use trigonometric substitution, then again factor the constant out, and finally perform integral reduction and simplify terms.

We can solve Eqn. 16 in two parts. The left side is simplified as follows,

$$
\begin{aligned}
&\left.\frac{\sec^2(v)\sin(v)}{2}\right|_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} \\
&= \left.\frac{1}{2}\sec(v)\tan(v)\right|_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} \\
&= \frac{(a+bd)\sqrt{b^2+(a+bd)^2}}{2b^2} - \frac{(a+bc)\sqrt{b^2+(a+bc)^2}}{2b^2}.
\end{aligned} \tag{17}
$$

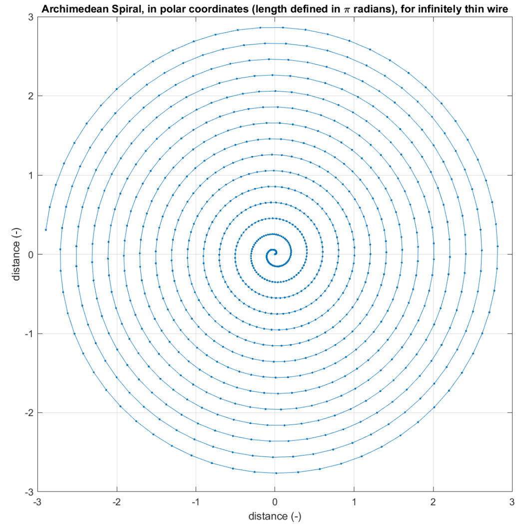

Fig. 3. Theoretical Archimedean spiral, continuous from the origin, with an infinitely thin arc.

The right side is

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathrm{atan}\left(\frac{a+bc}{b}\right)}^{\mathrm{atan}\left(\frac{a+bd}{b}\right)} \sec(v) \quad dv \\
&= \frac{1}{2}\left(\ln\left(\frac{|(a+bd)\sqrt{b^2+(a+bd)^2}|}{|b|} \right) \right. \\
&\qquad \left. + \ln\left(\frac{|(a+bc)\sqrt{b^2+(a+bc)^2}|}{|b|} \right) \right).
\end{aligned} \tag{18}
$$

Substituting Eqns.17 and 18 into Eqn. 16,

$$
\begin{aligned}
L = b\Bigg(&\frac{(a+bd)\sqrt{b^2+(a+bd)^2}}{2b^2} \\
&- \frac{(a+bc)\sqrt{b^2+(a+bc)^2}}{2b^2} \\
&+ \frac{1}{2}\left(\ln\left(\frac{|(a+bd)\sqrt{b^2+(a+bd)^2}|}{|b|} \right) \right. \\
&\left. + \ln\left(\frac{|(a+bc)\sqrt{b^2+(a+bc)^2}|}{|b|} \right) \right) \Bigg).
\end{aligned} \tag{19}
$$

Finally, the *arc length of the Archimedean spiral in Cartesian coordinates* is calculated using Eqns. 6, 7, and 19.

V. Equation Derivations - Arc Length to E

If the type of hairspring material is known, with known material properties, then the Sections III and IV will calculate the correct length. If material unknown and must be selected, then we must first use Eqn. 19 to determine the geometry of the hairspring, then derive the elastic modulus E and possibly iterate to find e and h. To find the elastic modulus, we can reformulate Eqn. 2,

$$M = \frac{E \cdot h \cdot e^3}{12 \cdot L} \implies E = \frac{12 \cdot M \cdot L}{h \cdot e^3}. \qquad (20)$$

VI. Appendix

A. Beat-rate to Vibrations per Hour

The conversion of vibrations per hour from beat-rate (in units of Hz or $\frac{beats}{s}$) is straightforward. A watch operating at $4\ Hz$ implies the balance oscillates at 4 oscillations per second, or 8 semi-oscillations (or vibrations) per second. There are 3600 seconds in an hour (60 seconds in a minute and 60 minutes in an hour). Multiplying the number of oscillations per second, times two (for semi-oscillations), times the number of seconds per hour, the total vibrations per hour result,

$$4\ Hz \times 2 \times 3600 = 28,800\ vph.$$

Let's check the units as we did in previous examples.

$$\left[\frac{beat}{s}\right] \times \left[\frac{vib}{beat}\right] \times \left[\frac{s}{hr}\right] = \left[\frac{vib}{hr}\right]$$

$$\left[\frac{\cancel{beat}}{\cancel{s}}\right] \times \left[\frac{vib}{\cancel{beat}}\right] \times \left[\frac{\cancel{s}}{hr}\right] = \left[\frac{vib}{hr}\right]$$

$$\left[\frac{vib}{hr}\right] = \left[\frac{vib}{hr}\right] \qquad \checkmark$$

This method is valid for any beat rate, e.g., $2.5\ Hz = 18,000\ vph$, $3\ Hz = 21,600\ vph$, etc.

B. Vibrations per Hour to Beat-rate

Using algebra, we rearrange the terms above to calculate the beat-rate from vibrations per hour. Again we use the $28,800 vph$ example.

$$\frac{28,800\ vph}{2 \times 3600} = \frac{28,800\ vph}{7200} = 4\ Hz$$

C. Chronograph Timing Accuracy

For precise and accurate timing of events, the Nyquist theorem [6], [7], [8] states that to avoid aliasing of the frequency f, the sample frequency f_s must be at least twice the rate of f,

$$f \geq 2 f_s.$$

In practice f_s is often ten or more times greater than f. Thus, modern precision timing is performed using digital systems, e.g., lasers and optical transducers.

In watchmaking, we can approximate this accuracy. Using the semi-oscillations per second, we can calculate the accuracy of a chronograph, or stopwatch, for a sporting event. Again using the $4\ Hz$ example, the chronograph beats at 8 semi-oscillations per second (i.e., $8\ \frac{semi-oscillations}{s}$), the timing accuracy is the inverse of the semi-oscillations per second, i.e., $\frac{1}{8}^{th}$ of a second.

Using the above results, a watch with a rate of $2.5\ Hz$ can time events to the nearest $\frac{1}{5}^{th}$ of a second, a rate of $3\ Hz$ equates to $\frac{1}{6}^{th}$ of a second, and a rate of $5\ Hz$ to the nearest $\frac{1}{10}^{th}$ of a second.

D. Common Conversions & Materials

TABLE III
Common Unit Conversions

$$1\ Hz = \frac{1}{s} = 7200\ vph$$

$$1\ \frac{mg \cdot cm^2}{s^2} = 1 \times 10^{-4}\ N \cdot mm$$

$$1\ Pa = 1\ \frac{N}{m^2} = 1 \times 10^{-6}\ \frac{N}{mm^2} = 1.4504 \times 10^{-4}\ psi$$

$$1\ MPa = 10^6\ Pa = 0.145 \times 10^3\ psi = 0.145\ ksi$$

$$1\ GPa = 10^9\ \frac{N}{m^2} = 10^6\ \frac{N}{cm^2} = 10^3\ \frac{N}{mm^2} = 0.145 \times 10^6\ psi$$

$$1\ Mpsi = 10^6\ psi = 10^3\ ksi$$

$$1\ psf = 1\ \frac{lbf}{ft^2}$$

$$1\ psi = 1\ \frac{lb}{in^2} = 144\ \frac{lbf}{ft^2} = 6,894.8 \frac{N}{m^2} = 6.895 \times 10^{-3} \frac{N}{mm^2}$$

TABLE IV
Young's Modulus for Common Materials

Material	Young's Modulus ($Mpsi$)
6061-T6 Aluminum	10.1
G5 Titanium	17.4
Beryllium Copper	18.0
360 Brass	18.1
304 Stainless Steel	26.1
Med. Carbon Steel	30.2
Inconel	31.1
Rhodium	42.1
Sapphire	63.1

VII. Acknowledgments

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