Hairspring Calculations - Technical Note

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I. Introduction

Occasionally in watchmaking, a watchmaker desires to make or repair a hairspring. While there exists literature with equations on the subject of hairsprings, the **derivations** of the equations are not provided. Because manipulation of equations can be difficult, this short report illustrates the method to calculate a watch hairspring length, as well as the various unit conversions and equation manipulations. Section II explains the derivations of values and units, Section III provide specific examples, and Section IV provides common unit conversions.

Note: You can skip to Section III if you are not concerned with understanding the conversions of units and equation manipulation.

II. Equation Manipulation

In this section we will use some basic algebra to convert equations provided in [1] and [2], and data found in [3], into equations we can use for the calculation of the hairspring length. After converting the equations we will check our work to make sure the units are still correct (the units will not match if we made an algebraic error). We will use [4] and [5] as reference for the algebraic manipulations.

It is possible to calculate a parameter, e.g., Elastic Torque, M, using multiple approaches (equations) as we will see in the following sections. **Do not be confused by multiple equations of the same parameter.** This is equivalent to the calculation of the number 4 by two methods: $4 = 1 \times 4$ and $4 = 2 \times 2$, both are valid methods that arrive at the same answer. In Sections II-A and II-B we will use two different equations for M to calculate other parameters of interest.

A. Reformulating Frequency to Define Elastic Torque

The elastic torque of the hairspring is derived from the frequency equation

$$f = \frac{1}{2\pi} \cdot \sqrt{\frac{M}{I}},$$

where f is the oscillating frequency in $[Hz] = [\frac{1}{s}]$, M is the elastic constant of the hairspring with units $[\frac{N \cdot mm}{rad}]$, and I is the moment of inertia of the balance in units $[mg \cdot cm^2]$.

We can reformulate f to define M, such that

$$f = \frac{1}{2\pi} \cdot \sqrt{\frac{M}{I}}$$

$$(2\pi) \cdot f = \frac{1}{\cancel{2\pi}} \cdot \sqrt{\frac{M}{I}} \cdot (\cancel{2\pi})$$

$$(2\pi \cdot f)^2 = \left(\sqrt{\frac{M}{I}}\right)^2$$

$$(I) \cdot 4 \cdot \pi^2 \cdot f^2 = \frac{M}{\cancel{I}} \cdot (\cancel{I})$$

$$I \cdot 4 \cdot \pi^2 \cdot f^2 = M$$

$$M = I \cdot 4 \cdot \pi^2 \cdot f^2. \tag{1}$$

In the first step, both sides are multiplied by 2π, canceling terms on the right hand side. The square root is canceled by squaring both sides in the second step. In the third step, I is multiplied on both sides to remove I from the right hand side. The final step is just a rearrangement of terms.

Using Eqn. 1 we can check the units, canceling terms just as we did before and converting units using Table III.

$$[N \cdot mm] = [mg \cdot cm^2] \cdot \left[\frac{1}{s}\right]^2$$

$$= [mg \cdot cm^2] \cdot \left[\frac{1}{s^2}\right]$$

$$= \frac{[mg \cdot cm^2]}{[s^2]}$$

$$= 1 \times 10^{-7} \frac{[kg \cdot m^2]}{[s^2]}$$

$$[N \cdot mm] = 1 \times 10^{-4} [N \cdot mm] \quad \checkmark$$

Note in the final step the units are the same but we have a factor of 1×10^{-4} as a result of converting units.

B. Reformulating Elastic Torque Equation to Define Length

The elastic torque of the hairspring is defined as

$$M = \frac{E \cdot h \cdot e^3}{12 \cdot L} \tag{2}$$

where M is the elastic constant of the hairspring with units $[\frac{N \cdot mm}{rad}]$, E is the modulus of elasticity for the hairspring in $[\frac{N}{mm^2}]$, h is the height $[mm]$, e is the thickness $[mm]$, and L is the length with units $[mm]$.

Using algebra, we can rearrange Eqn. 2 to define the length, L:

$$M = \frac{E \cdot h \cdot e^3}{12 \cdot L}$$

$$(L) \cdot M = \frac{E \cdot h \cdot e^3}{12 \cdot \cancel{L}} \cdot (\cancel{L})$$

$$\left(\frac{1}{\cancel{M}}\right) \cdot L \cdot \cancel{M} = \frac{E \cdot h \cdot e^3}{12} \cdot \left(\frac{1}{M}\right)$$

$$L = \frac{E \cdot h \cdot e^3}{12 \cdot M}. \tag{3}$$

In the first step we multiply both sides by L, canceling terms on the right hand side. In the second step we multiply both sides by $\frac{1}{M}$, canceling terms in the left hand side. The result is the equation for L.

Now that we have an equation for L, let's look at the units to verify that all of the units properly cancel. L is in units of $[mm]$ so all units on the right hand side should cancel, with only $[mm]$ remaining.

$$[mm] = \frac{[\frac{N}{mm^2}] \cdot [mm] \cdot [mm^3]}{[N \cdot mm]}$$

$$= \frac{[\frac{N}{mm^2}] \cdot [mm^4]}{[N \cdot mm]}$$

$$= \frac{[\frac{N}{\cancel{mm^2}}] \cdot [\cancelto{mm^2}{mm^4}]}{[N \cdot mm]}$$

$$= \frac{[\cancel{N} \cdot mm^2]}{[\cancel{N} \cdot mm]}$$

$$= \frac{[\cancelto{mm}{mm^2}]}{[\cancel{mm}]}$$

$$[mm] = [mm] \quad \checkmark$$

In the first step we combine terms; recall that multiplication of the base unit requires addition of the exponent by the algebra rule $x^a \cdot x^b = x^{a+b}$, see [5]. The next step we cancel terms, replacing the numerator with $[mm^2]$ using the rule $\frac{x^a}{x^b} = x^{a-b}$, see [5]. The next step we cancel the $[N]$ in both numerator and denominator. Finally we cancel terms, this time canceling the denominator with the exponent of the numerator, again see [5].

III. Examples

Now that we have the equations we need, let's look at a few examples.

A. Example: Hairsprings

First, let's look at a few hairspring examples[1] for watches with a beat-rate of 2.5 Hz.

TABLE I
HAIRSPRING EXAMPLES

Make/ Model	**Thickness** (mm)	**Height** (mm)	**Length** (mm)
ETA 6498-1	0.19	0.04	
ETA 6497-1	0.222	0.044	450
Hamilton 945	0.21	0.04	
Eterna 1145	0.155	0.029	
Citizen 1161	0.130	0.033	

B. Example: Elastic Torque Calculation

If the numerical values for the parameters of Eqn. 1 are known, then the elastic torque is calculated directly. Let $I = 25\ [mg \cdot cm^2]$ and $f = 2.5\ Hz$.

$$\begin{aligned} M &= I \cdot 4\pi^2 \cdot f^2 \\ &= 25\ [mg \cdot cm^2] \cdot 4\pi^2 \cdot \left(2.5 \left[\frac{1}{s}\right]\right)^2 \\ &= 25\ [mg \cdot cm^2] \cdot 4\pi^2 \cdot 6.25 \left[\frac{1}{s^2}\right] \\ &= 6168.5\ \frac{mg \cdot cm^2}{s^2} \\ &= 6.1685 \times 10^{-7}\ \frac{kg \cdot m^2}{s^2} \\ &= 6.1685 \times 10^{-4}\ \frac{N \cdot mm}{rad}. \end{aligned}$$

C. Example: Length Calculation

If the numerical values for the parameters of Eqn. 3 are known, then the length is calculated directly. Let $E = 27.79 \times 10^3\ ksi$, $h = 0.19\ mm$, and $e = 0.040\ mm$. First convert the units of E, where $1ksi = 6.895\frac{N}{mm^2}$ (see Table III), then

$$\begin{aligned} E &= (27.79 \times 10^3) \times 6.895 \\ &= 191,605\ \frac{N}{mm^2}. \end{aligned}$$

[1]Measurements collected by Michael Rose and Paul Roysdon on 11/10/2022.

Now calculate L using Eqn. 3,

$$L = \frac{E \cdot h \cdot e^3}{12 \cdot M}$$

$$= \frac{191,605 \ \frac{N}{mm^2} \cdot 0.19 \ mm \cdot (0.040 \ mm)^3}{12 \cdot 6.1685 \times 10^{-4} \ N \cdot mm}$$

$$= 314.8 \ mm$$

Given the length of the ETA 6497-1 hairspring, see Table I, and our input values above, this result seems reasonable.

Note: length is defined as the distance between the pinning point on the collet to the vibrating point at the stud (or regulating pin). The result calculated using Eqn. 3 is the **theoretical length**. The **actual length will vary slightly**, depending on physical factors of pinning, etc.

D. Example: Length Calculation with Multiple Unknowns

If some values of Eqn. 3 are not known, or the watchmaker has options from a material supplier, e.g., height and thickness, then a table can be produced. In the example below, Eqn. 3 is calculated for each combination of thickness and height using the E and M values from the example in Sections III-B and III-C.

TABLE II
LENGTH CALCULATION WITH MULTIPLE UNKNOWNS

		height [mm]			
		0.150	**0.175**	**0.200**	**0.225**
thickness [mm]	**0.030**	104.84	122.31	139.78	157.26
	0.035	166.48	194.22	221.97	249.72
	0.040	248.50	289.92	331.34	372.76
	0.045	353.83	412.80	471.77	530.74
	0.050	485.36	566.25	647.15	728.04

A spreadsheet calculator accompanies this report. Several values of L are calculated **automatically** for a sequence of values h and e. The watchmaker is only required to enter the values for I, f, E and M.

IV. COMMON UNIT CONVERSIONS

A. Beat-rate to Vibrations per Hour

The conversion of vibrations per hour from beat-rate (in units of Hz or $\frac{beats}{s}$) is straightforward. A watch operating at $4Hz$ implies the balance oscillates at 4 oscillations per second, or 8 semi-oscillations (or vibrations) per second. There are 3600 seconds in an hour (60 seconds in a minute and 60 minutes in an hour). Multiplying the number of oscillations per second, times two (for semi-oscillations), times the number of seconds per hour, the total vibrations per hour result:

$$4Hz \times 2 \times 3600 = 28,800vph$$

Let's check the units as we did in previous examples.

$$\left[\frac{beat}{s}\right] \times \left[\frac{vib}{beat}\right] \times \left[\frac{s}{hr}\right] = \left[\frac{vib}{hr}\right]$$

$$\left[\frac{\cancel{beat}}{\cancel{s}}\right] \times \left[\frac{vib}{\cancel{beat}}\right] \times \left[\frac{\cancel{s}}{hr}\right] = \left[\frac{vib}{hr}\right]$$

$$\left[\frac{vib}{hr}\right] = \left[\frac{vib}{hr}\right] \quad \checkmark$$

This method is valid for any beat rate, e.g., $2.5 \ Hz = 18,000 \ vph$, $3 \ Hz = 21,600 \ vph$, etc.

B. Vibrations per Hour to Beat-rate

Using algebra, we rearrange the terms above to calculate the beat-rate from vibrations per hour. Again we use the $28,800vph$ example.

$$4Hz = \frac{28,800vph}{2 \times 3600}$$

C. Chronograph Timing Accuracy

Using the semi-oscillations per second, we can calculate the accuracy of a chronograph, or stopwatch, for a sporting event. Again using the $4Hz$ example, the chronograph beats at 8 semi-oscillations per second (i.e., $8\frac{semi-oscillations}{s}$), the timing accuracy is the inverse of the semi-oscillations per second, i.e., $\frac{1}{8}^{th}$ of a second.

Using the above results, a watch with a rate of $2.5 \ Hz$ can time events to the nearest $\frac{1}{5}^{th}$ of a second, a rate of 3 Hz equates to $\frac{1}{6}^{th}$ of a second, and a rate of 5 Hz to the nearest $\frac{1}{10}^{th}$ of a second.

D. Other Conversions

Table III lists unit conversions that are used in this report.

TABLE III
COMMON UNIT CONVERSIONS

conversion	notes
$1ksi = 6.895\frac{N}{mm^2}$	units needed to calculate M
$1Hz = \frac{1}{s}$	units needed to calculate M
$1Hz = 7200vph$	beat-rate to vibrations per hour
$\frac{mg \cdot cm^2}{s^2} = 1 \times 10^{-4} N \cdot mm$	units needed to calculate M

REFERENCES

[1] (various), "Spiraux - Numerotation CGS," *NIHSG 35-10, Schweizer Guideline 283510*, 2022-03.
[2] C.-E. Reymondin, *The Theoru of Horology*. The Swiss Federation of Technical Colleges, 2003.
[3] Special Metals Corporation, "NI-SPAN-C Alloy 902," vol. SMC-086, 2004-09.
[4] P. F. Roysdon, *Math Refresher for Machine Learning*. Fibonacci Press, 2022.
[5] ——, *Math Handbook for Machine Learning*. Fibonacci Press, 2022.