A Comparison of Two Approaches to Pure-Inertial and Doppler-Inertial Error Analysis

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Abstract

Error equations for inertial navigation systems are derived using a perturbation (or true frame) approach and a psi angle (or computer frame) approach in a manner which shows the underlying as sumptions and allows direct comparison of the two methods. The comparison is general since the analysis is not associated with any particular mechanization. Different definitions of velocity errors and misalignment angles result from the two methods of error analysis, and, consequently, have significance in testing and analysis of pure-inertial systems, Doppler-inertial systems, and inertially aided weapon delivery systems. Examples and numerical results are presented for a local-level north-pointing mechanization.

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I. Introduction

Error equations for inertial navigation systems have been derived in many references using a perturbation (or true frame) approach [1], [2] and a psi angle (or computer frame) approach [3], [4]. A direct comparison showing the equivalence of the two methods does not appear to be available in the literture. The purpose of this paper then is to derive both forms of the error equations in such a way as to clearly define assumptions and compare the perturbation approach with the psi angle approach. Many references use one or the other of these approaches to analyze particular systems. For example, [6] derives error equations for a space-stable system by perturbation analysis and compares the error propagation to local-level systems in [7]. Important consequences of comparing the two error analysis approaches are the relations between the variables when either of the approaches is used in practice. Such useful applications are testing and simulation of high-speed inertial systems, evaluation of inertially aided weapon delivery systems, and the expression for velocity divergence when, for example, Doppler velocity measurements are combined with the inertial system indication of velocity.

Inertial navigation systems basically solve Newton's force equations from measurements (accelerometer outputs) of specific force (nongravitational acceleration) coordinatized in a frame whose orientation with respect to an inertial frame is controlled or known via gyroscopes. For terrestrial navigation, the equations in terms of ground velocity in an arbitrary rotating frame are

$$dV/dt\big|_r + (\omega_{ie} + \omega_{ir}) \times V - g = f \tag{1}$$

and

$$V = dR/dt\big|_e = dR/dt\big|_r + \omega_{er} \times R \tag{2}$$

where

V = velocity with respect to Earth
f = specific force
g = plumb-bob gravity
ω_{ie} = Earth rate
ω_{ir} = angular velocity of the r frame with respect to an Earth-centered inertial frame[1]
ω_{er} = angular velocity of the r frame with respect to an Earth-fixed frame
R = position vector from the center of the Earth, as shown in Fig. 1
$d/dt|_{e;\,r}$ = derivative in an Earth-fixed e frame or an arbitrary rotating r frame, respectively.

[1] Definitions of suitable inertial frames are found in [5].

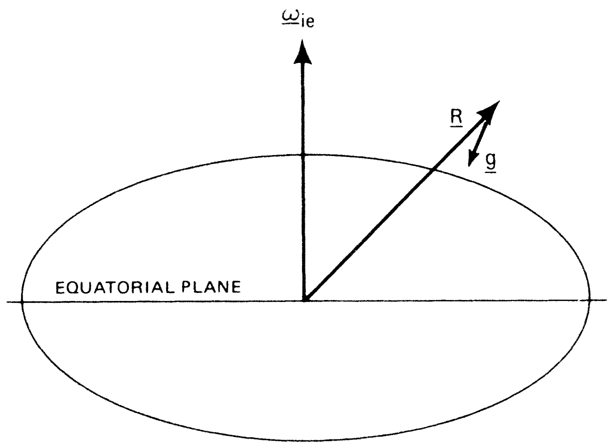


Fig. 1. Vector relations with respect to the Earth (oblateness exaggerated). Symbols with underbars correspond to boldface symbols in the text.

The navigation system solves the scalar component representation of (1) and (2).[2] Conceptually, for the purposes of error analysis, the navigation system can be viewed as attempting solutions of (1) and (2) in either a computer c frame or a true t frame. Appropriate definitions of the various coordinate frames are found in, for example, [3] and [4].

II. Psi Angle Error Analysis

A. Velocity Error Equations

In the psi angle error analysis, the navigation equations are assumed to be solved in the computer c frame, and the error equations are derived from perturbation of the computer frame solution. The velocity error equations first are derived, while the position error equations are deferred until Section V.

The equations for true velocity when the rotating frame is the computer frame and the components are also coordinatized in the computer frame are

$$\dot{V}_t^c + (\omega_{ie}^c + \omega_{ic}^c) \times V_t^c - g_t^c = f_t^c \tag{3}$$

where V_t^c is true velocity with respect to the ground coordinatized in the c frame[3] and $\omega_{ic}^c \times$ and $\omega_{ie}^c \times$ are cross product matrixes, and where

[2] In many mechanizations, (2) is not solved explicitly. For example, in local-level systems, two components of position are determined by angles (latitude and longitude) from appropriately integrating angular velocity (by direction cosine or Euler angle differential equations).

[3] Vectors are represented by components with the superscript indicating the frame in which they are coordinatized.

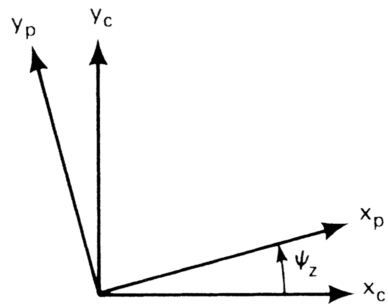


Fig. 2. Relation between computer frame and platform frame.

$$\omega_{ic}^c \times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{4}$$

Because of errors in the navigation system, the following variables appear in (3):

$$(V)_i = V_t^c + \delta V_1^c \tag{5}$$

$$(g)_i = g_t^c + \delta g_1^c \tag{6}$$

where subscript 1 refers to a psi angle analysis error variable and

$(V)_i$ = three velocity components indicated by the navigation system
δV_1^c = velocity errors of the psi angle analysis
$(g)_i$ = computed value of gravity

and where ω_{ic}^c and ω_{ie}^c are known without error by the definition of the computer frame.

Instead of f_t^c, only $f_t^p + \delta f^p$ are available since the accelerometers are in platform axes and they have errors δf^p (bias, scale factor, etc). Thus, the equations that the navigation system solves are

$$(\dot{V})_i + (\omega_{ie}^c + \omega_{ic}^c) \times (V)_i - (g)_i = f_{it}^p + \delta f^p. \tag{7}$$

If the platform p frame is rotated from the computer frame by small angles, as shown in Fig. 2, then the relation between the two frames is given by the direction cosine matrix

$$C_c^p = \begin{bmatrix} 1 & \psi_z & -\psi_y \\ -\psi_z & 1 & \psi_x \\ \psi_y & -\psi_x & 1 \end{bmatrix} = I - \psi \times \tag{8}$$

and the components of vectors transform as

$$f^p = C_c^p f^c = f^c - \psi \times f^c. \tag{9}$$

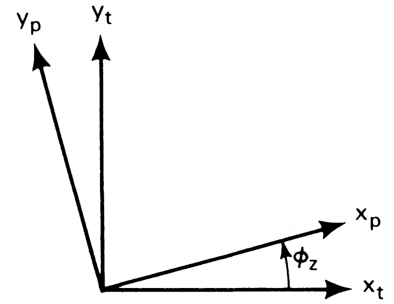


Fig. 3. Relation between platform frame and true frame.

Substituting (5), (6), and (9) into (7) and subtracting (3) from the result yields

$$\delta \dot{V}_1^c + (\omega_{ie}^c + \omega_{ic}^c) \times \delta V_1^c - \delta g_1^c = -\psi \times f^c + \delta f^c. \quad (10)$$

B. Platform Frame to Computer Frame Error Equations

Since the navigation system assumes that the platform axes are coincident with the computer axes, the gyros are torqued with ω_{ic}^c. However, the gyros control the platform along platform axes (except for nonorthogonalities) and, further, the gyros have a drift rate ϵ^p. Thus, the platform rate is

$$\omega_{ip}^p = \omega_{ic}^c + \epsilon^p. \quad (11)$$

An alternative explanation of (11) is that the platform rate ω_{ip}^p is equal to the torquing rates applied to the gyros plus the gyro drift rate ϵ^p. The torquing rate numbers are the desired angular velocity of the platform. Since the navigation system assumes the platform frame is always coincident with the computer frame, this angular velocity is ω_{ic}^c.

Substituting (8) into (11) yields, to first order in ω and ϵ^p,

$$\omega_{cp}^p = \psi \times \omega_{ip}^p + \epsilon^p. \quad (12)$$

ω_{cp}^p is the angular velocity of the platform with respect to the computer frame and, for small angles, is equal to $\dot{\psi}^p$ (see Fig. 2),[4] consequently giving the familiar ψ equation

$$\dot{\psi}^p = -\omega_{ip}^p \times \psi^p + \epsilon^p. \quad (13)$$

III. Perturbation Error Analysis

A. Velocity Error Equations

In the perturbation analysis, the frame in which the navigation equations are assumed to be solved is the true t frame and in this frame the ideal solution is

$$\dot{V}_t^t + (\omega_{ie}^t + \omega_{it}^t) \times V_t^t - g_t^t = f_t^t. \quad (14)$$

However, because of errors in the navigation system, the following variables appear in the above equation:

$$(V)_i = V_t^t + \delta V_2^t \quad (15)$$

$$(g)_i = g_t^t + \delta g_2^t \quad (16)$$

$$(\omega_{it}^t)_i = \omega_{it}^t + \delta \omega^t \quad (17)$$

$$(\omega_{ie}^t)_i = \omega_{ie}^t + \delta \omega_{ie}^t \quad (18)$$

where subscript 2 refers to a perturbation analysis error variable. Also, instead of f_t^t, only $f_t^p + \delta f^p$ are available from the accelerometers. Thus, the equations that the navigation system solves are

$$(\dot{V})_i + [(\omega_{it}^t)_i + (\omega_{ie}^t)_i] \times (V)_i - (g)_i = f^p + \delta f^p. \quad (19)$$

An angular relation between the platform frame and the true frame is defined by the angles shown in Fig. 3. For small angles, the direction cosine matrix relating the two frames is

$$C_t^p = \begin{bmatrix} 1 & \phi_z & -\phi_y \\ -\phi_z & 1 & \phi_x \\ \phi_y & -\phi_x & 1 \end{bmatrix} = I - \phi \times \quad (20)$$

and the components of vectors transform as

$$f^p = C_t^p f^t = f^t - \phi \times f^t. \quad (21)$$

Substituting (17), (18), and (21) into (19) yields, after subtracting (14),

$$\delta \dot{V}_2^t + (\omega_{ie}^t + \omega_{it}^t) \times \delta V_2^t + (\delta\omega_{ie}^t + \delta\omega^t) \times V_t^t$$
$$- \delta g_2^t = -\phi \times f^t + \delta f^t. \quad (22)$$

B. Platform Frame to True Frame Error Equations

The navigation system assumes that the platform axes are coincident with the true frame. However, ω_{it}^t is not available for gryo torquing, but only $\omega_{it}^t + \delta\omega^t$. The gyros control the platform along platform axes and, further, they have a drift rate ϵ^p.

Consequently, the platform rate is

$$\omega_{ip}^p = \omega_{it}^t + \delta\omega^t + \epsilon^p. \quad (23)$$

Substituting (20) into (23) yields

$$\omega_{tp}^p = \phi \times \omega_{ip}^p + \delta\omega^p + \epsilon^p. \quad (24)$$

[4] The relation $\omega_{cp}^p = \dot{\psi}^p$ can also be derived easily by keeping first-order terms in the direction cosine differential equation $\dot{C}_p^c = C_p^c \, \omega_{cp}^p \times$. The superscripts are dropped from the angle errors, since to first order in error quantities $\psi^p = \psi^c = \psi^t$ and $\dot{\psi}^p = \dot{\psi}^c = \dot{\psi}^t$. Also, for any error quantity, say δx, $\delta x^p = \delta x^c = \delta x^t$. These relations are used throughout the error analysis.

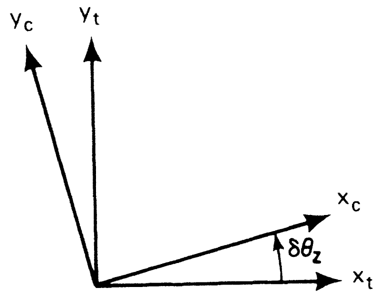


Fig. 4. Relation between true frame and computer frame.

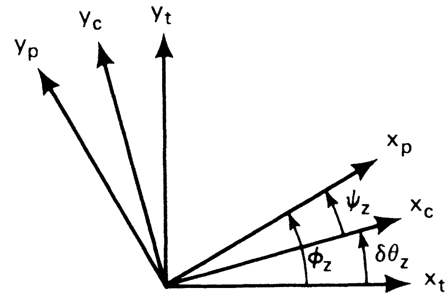


Fig. 5. Relation between the three coordinate frames.

ω^p_{tp} is the angular velocity of the platform frame with respect to the true frame and, for small angle, is $\dot{\phi}$; thus,

$$\omega^p_{tp} = \dot{\phi} = -\,\omega^p_{ip} \times \phi + \delta\omega^p + \epsilon^p. \tag{25}$$

IV. Comparison of Perturbation Analysis with Psi Angle Analysis

Conceptually, the navigation system can be viewed as attempting to mechanize a solution of Newton's force equation in the computer frame or a solution in the true frame. Error analysis of each point of view leads to different definitions of the error variables. However, the error variables in each analysis are related by observing that the navigation system solves only one set of scalar equations. Consequently, (7) is the same equation as (19) and the corresponding variables are equivalent. Identifying the equivalent variables by comparing (7), (19), (15) through (18), (5), and (6) gives

$$(V)_i = V^c_t + \delta V^c_1 = V^t_t + \delta V^t_2 \tag{26}$$

$$(g)_i = g^c_t + \delta g^c_1 = g^t_t + \delta g^t_2 \tag{27}$$

$$\omega^c_{ie} = (\omega^t_{ie})_i = \omega^t_{ie} + \delta\omega^t_{ie} \tag{28}$$

$$\omega^c_{ic} = (\omega^t_{it})_i = \omega^t_{it} + \delta\omega^t. \tag{29}$$

The relation between the variables can be completed by defining the angles between the true frame and the computer frame, as shown in Fig. 4. For small angles, the direction cosine matrix relating the two frames is

$$C^c_t = \begin{bmatrix} 1 & \delta\theta_z & -\delta\theta_y \\ -\delta\theta_z & 1 & \delta\theta_x \\ \delta\theta_y & -\delta\theta_x & 1 \end{bmatrix} = I - \delta\theta \times. \tag{30}$$

Substituting (30) into (26) through (29) gives

$$\delta V^t_2 = -\,\delta\theta \times V^c_t + \delta V^c_1 \tag{31}$$

$$\delta g^t_2 = -\,\delta\theta \times g^c_t + \delta g^c_1 \tag{32}$$

$$\delta\omega^t_{ie} = -\,\delta\theta \times \omega^c_{ie}. \tag{33}$$

The relation between velocity error along computer frame axes and along true frame axes, as given by (31), has been noted in [7] for space-stable inertial systems. Also,

$$\omega^c_{tc} = \delta\theta \times \omega^c_{it} + \delta\omega^t \tag{34}$$

where ω^c_{tc} is the angular velocity of the computer frame with respect to the true frame and, to first order in small quantities, is equal to $\delta\dot{\theta}$. Thus,

$$\omega^c_{tc} = \delta\dot{\theta} = -\omega^c_{it} \times \delta\theta + \delta\omega^t. \tag{35}$$

The psi angle error equations (10) and (13) are derived from the perturbation equations using the three coordinate frames shown in Fig. 5. For small angles, the relation between the angles becomes

$$\phi = \psi + \delta\theta. \tag{36}$$

If (35) and (25) are coordinatized in the same frame, then subtracting (35) from (25) yields, to first order in small quantities,

$$\dot{\phi} - \delta\dot{\theta} = -\,\omega^p_{ip} \times (\phi - \delta\theta) + \epsilon^p. \tag{37}$$

Substituting (36) in the above equation gives the familiar ψ equation. Similarly, (10) can be derived from (22) with some manipulation when (26) through (29) and (35) are substituted into (22). The perturbation error equations can be derived from the psi angle equations by reversing the procedure.

V. Position Error Equations

For terrestrial navigation position is determined relative to an Earth-fixed frame as in (2). As previously discussed, this equation is not always solved explicitly. However, for the purposes of error analysis an explicit form of (2) is used.

A. Psi Angle Error Analysis

The errorless position at time τ is given by (except for initial position)

$$R^e_t = \int_0^T C^e_c V^c_t. \tag{38}$$

However, the navigation system uses $(V)_i$ in place of V^c_t to generate $(R)_i$, and, by the definition of the computer frame, C^e_c is known without error; thus,

$$(R)_i = \int_0^T C^e_c (V)_i. \tag{39}$$

$(R)_i$ is in error from R^e_t by δR^e:

$$(R)_i = R^e_t + \delta R^e. \tag{40}$$

Using (5) and the above equations,

$$\delta R^e = \int_0^T C^e_c \delta V^c_1 \tag{41}$$

$$\delta R^t = C^t_e \, \delta R^e = C^t_e \int_0^T C^e_c \delta V^c_1. \tag{42}$$

Differentiating the above equation gives, to first order in error quantities,

$$\delta \dot{R}^t = -\omega^t_{et} \times \delta R^t + \delta V^t_1. \tag{43}$$

B. Perturbation Error Analysis

The errorless position is given by (except for initial position)

$$R^e_t = \int_0^T C^e_t V^t_t, \tag{44}$$

but the navigation system generates

$$(R)_i = \int_0^T (C^e_t)_i (V)_i \tag{45}$$

where

$$(C^e_t)_i = C^e_t (I + \delta\theta_1 \times). \tag{46}$$

Since

$$(C^e_t)_i = C^e_c = C^e_t C^t_c = C^e_t [I + \delta\theta \times], \tag{47}$$

then

$$\delta\theta_1 \times = \delta\theta \times. \tag{48}$$

Hence,

$$\delta \dot{R}^t = C^t_e \int_0^T (C^e_t \delta\theta \times V^t_t + C^e_t \delta V^t_2) \tag{49}$$

$$\delta \dot{R}^t = -\omega^t_{et} \times \delta R^t + \delta\theta \times V^t_t + \delta V^t_2. \tag{50}$$

The equivalence of the two equations is seen by substituting (31) into (50).

VI. Doppler-Inertial Velocity Divergence Error Expressions

Important consequence of understanding the two error analysis approaches are expressions for velocity divergence when, for example, Doppler velocity measurements are combined with the inertial indication of velocity. The velocity difference or divergence V_d is used to settle the inertial system errors.

The error analysis discussed here applies to the case where the inertial system is used as an attitude reference for the external velocity measurements. The reference velocity components are then along platform axes.

A. Psi Angle Error Analysis

The velocity divergence is

$$V_d = V_{\text{inertial}} - V_{\text{reference}}. \tag{51}$$

But, in terms of psi angle analysis variables, the right-hand side becomes

$$= V^c_t + \delta V^c_1 - (V^p_t + \delta V^p_r) \tag{52}$$

where V_d is the velocity divergence and δV^p_r are Doppler velocity errors due to bias, boresight, scale factor, gimbal angle errors, etc.

Substituting (8) into (52) yields, to first order in error quantities,

$$V_d = \delta V^c_1 + \psi \times V^c_t - \delta V^c_r. \tag{53}$$

B. Perturbation Error Analysis

The expression for the velocity divergence in terms of the indicated variables is, of course, the same as (51); viz.,

$$V_d = V_{\text{inertial}} - V_{\text{reference}}.$$

But, in terms of perturbation analysis variables, the right-hand side becomes

$$= V^t_t + \delta V^t_2 - V^p_t + \delta V^p_r. \tag{54}$$

Substituting (20) into (54) gives, to first order in error quantities,

$$V_d = \delta V^t_2 + \phi \times V^t_t - \delta V^t_r. \tag{55}$$

Equations (53) and (55) are shown to be equivalent by substituting (31) and (26) into (55).

VII. Local-Level North-Pointing Error Equations

In order to simplify the error analysis a spherical Earth is assumed. The error equations will represent the equations for a mechanization using an elliptical Earth model if products involving the ellipticity e and the error quantities (e.g., δV) are negligible compared to other error sources. Also, if n is the local-level north-pointing frame as shown in

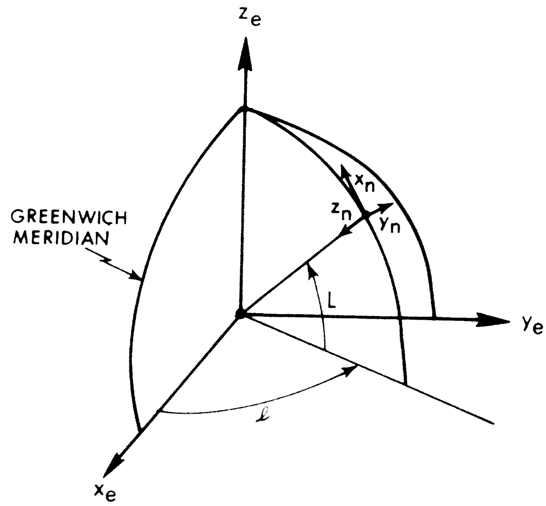


Fig. 6. Relation of local-level north-pointing frame to the Earth.

Fig. 6, plumb-bob gravity will be modeled as

$$g_t^n = \begin{bmatrix} -\xi g \\ -\eta g \\ g + \Delta g \end{bmatrix} \tag{56}$$

where

$$g = g_0 R^2/(R+h)^2 \tag{57}$$

and where

g_0 = plumb-bob gravity at the Earth's surface
ξ, η = vertical deflections
Δg = gravity anomaly
h = altitude
R = Earth's radius
L = latitude
l = longitude.

From Fig. 6,

$$(C_n^e)_i = C_c^e = \begin{bmatrix} -\cos l_c \sin L_c & -\sin l_c & -\cos l_c \cos L_c \\ -\sin l_c \sin L_c & \cos l_c & -\sin l_c \cos L_c \\ \cos L_c & 0 & -\sin L_c \end{bmatrix}$$

where

$$L_c = L + \delta L$$

$$l_c = l + \delta l$$

$$C_c^e = C_n^e C_c^n = C_n^e\,[I + \delta\theta\,\times].$$

Thus,

$$\delta\theta\,\times = \begin{bmatrix} 0 & \delta l \sin L & -\delta L \\ -\delta l \sin L & 0 & -\delta l \cos L \\ \delta L & \delta l \cos L & 0 \end{bmatrix} \tag{58}$$

Also from Fig. 6,

$$\delta R_N = (R+h)\,\delta L \tag{59}$$

$$\delta R_E = \delta l (R+h) \cos L. \tag{60}$$

Substituting (59) and (60) into (58) yields

$$\delta\theta\,\times = \begin{bmatrix} 0 & \delta R_E \tan L/(R+h) & -\delta R_N/(R+h) \\ -\delta R_E \tan L/(R+h) & 0 & -\delta R_E/(R+h) \\ \delta R_N/(R+h) & \delta R_E/(R+h) & 0 \end{bmatrix}. \tag{61}$$

A. Psi Angle Error Equations

From Fig. 6,

$$\omega_{in}^n = \begin{bmatrix} [V_E/(R+h)] + \omega_{ie} \cos L \\ -V_N/(R+h) \\ [-V_E \tan L/(R+h)] - \omega_{ie} \sin L \end{bmatrix} \tag{62}$$

$$\omega_{ie}^n = \begin{bmatrix} \omega_{ie} \cos L \\ 0 \\ -\omega_{ie} \sin L \end{bmatrix} \tag{63}$$

From (56) and (61),

$$g_t^c = C_n^c g_t^n = g_t^n - \delta\theta \times g_t^n = \begin{bmatrix} -\xi g + [\delta R_N g/(R+h)] \\ -\eta g + [\delta R_E g/(R+h)] \\ g + \Delta g \end{bmatrix} \tag{64}$$

For this mechanization, the computed value of gravity is

$$(g)_i = \begin{bmatrix} 0 \\ 0 \\ g_0 R^2/(R+h_c)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g - [2g\delta h/(R+h)] \end{bmatrix} \tag{65}$$

where

$$h_c = h + \delta h. \tag{66}$$

Thus,

$$\delta g_1^c = (g)_i - g_t^c = \begin{bmatrix} \xi g - \delta R_N [g/(R+h)] \\ \eta g - \delta R_E [g/(R+h)] \\ -\Delta g - [2g/(R+h)]\,\delta h \end{bmatrix}. \tag{67}$$

Substituting (62), (63), and (67) into (10), (13), and (43) yields the psi angle set of error equations.

B. Perturbation Error Equations

Subtracting (56) from (65) gives

$$\delta g_2^n = \begin{bmatrix} \xi g \\ \eta g \\ -\Delta g - [2g/(R+h)]\,\delta h \end{bmatrix}. \tag{68}$$

Equations (59), (60), (62), and (63) yield

$$\delta\omega^n = \begin{bmatrix} [\delta V_E/(R+h)] - [V_E/(R+h)^2]\,\delta h - [\omega_{ie} \sin L/(R+h)]\,\delta R_N \\ [-\delta V_N/(R+h)] + [V_N/(R+h)^2]\,\delta h \\ [-\delta V_E \tan L/(R+h)] + [\delta h V_E \tan L/(R+h)^2] - \{[V_E/(R+h)\cos^2 L] + \omega_{ie}\cos L\}\,\delta R_N/(R+h) \end{bmatrix} \tag{69}$$

$$\delta\omega_{ie}^n = \begin{bmatrix} [-\omega_{ie} \sin L/(R+h)]\,\delta R_N \\ 0 \\ [-\omega_{ie} \cos L/(R+h)]\,\delta R_N \end{bmatrix}. \tag{70}$$

Substituting (61) through (63) and (68) through (70) into (22), (25), and (50) yields the perturbation set of error equations.

C. Numerical Results

Numerical results from the psi angle set of error equations and the perturbation set of equations are computed for the following conditions:

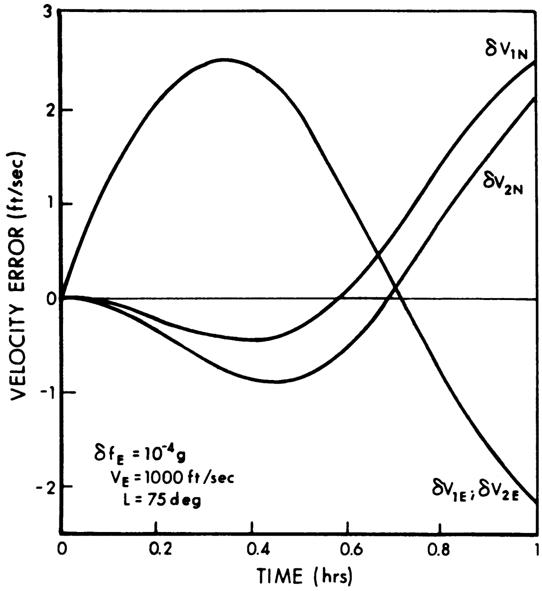


Fig. 7. Velocity error response from east accelerometer bias for east vehicle velocity of 1000 ft/s.

$$V_N = 0$$

$$V_E = \pm\ 1000 \text{ ft/s}$$

$$V_Z = 0$$

$$L = 75°.$$

In Figs. 7 and 8 an east accelerometer bias of 10^{-4} g causes north velocity errors δV_{1N} and δV_{2N} to differ by as much as 0.8 ft/s over one hour. Similarly, Figs. 9 and 10 show that δV_{1N} and δV_{2N} differ by 0.7 ft/s over one hour when a north gyro bias of 0.01°/h forces the system.

The results are shown for the one-hour interval since the undamped vertical channel (unstable mechanization) is included. The purpose of leaving the undamped loop in is to show that the results of the perturbation set can be computed from the psi angle set of equations via (31) and (36), and vice versa. The solutions agree to better than one part in 10^4.

VIII. Conclusion

The inertial navigation error equations have been derived in this paper by a perturbation (or true frame) approach and

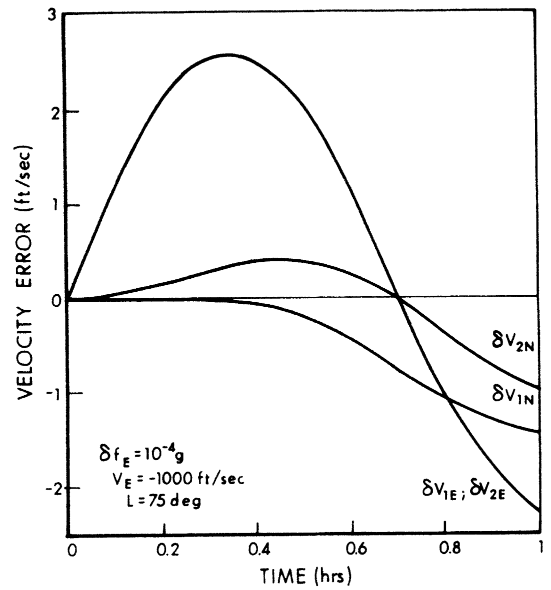


Fig. 8. Velocity error response from east accelerometer bias for west vehicle velocity of 1000 ft/s.

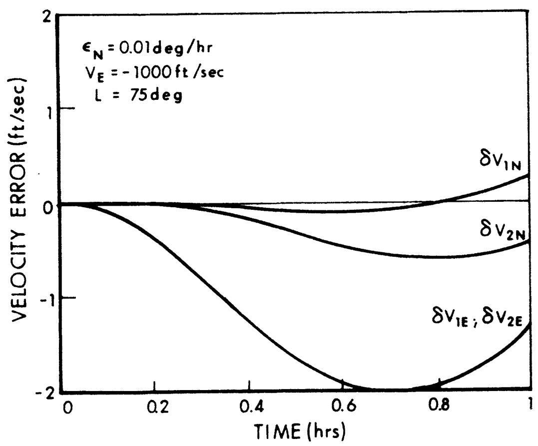
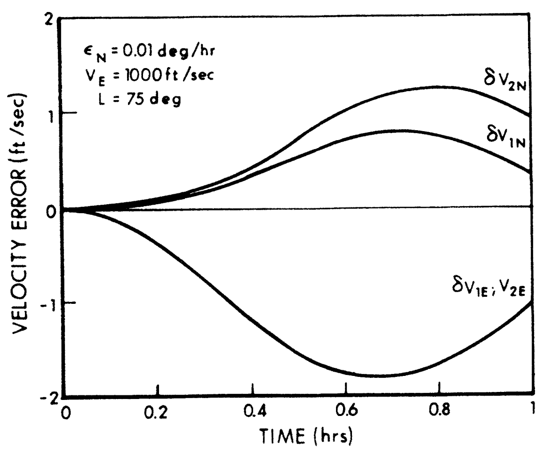


Fig. 10. Velocity error response from north gyro bias for west vehicle velocity of 1000 ft/s.

Fig. 9. Velocity error response from north gyro bias for east vehicle velocity of 1000 ft/s.



a psi angle (or computer frame) approach in such a manner as to compare and show the equivalence between the two approaches. Different definitions of velocity errors (δV_1 or δV_2) and misalignment angles (ψ angles or ϕ angles) are natural consequences of viewing the navigation system as attempting to solve the force equations in either a computer frame or a true frame, respectively. Even though the results presented here have been derived for systems with commanded torques to the gyros, the results apply to strapdown and electrostatic gyro (ESG) navigation systems if the gyro and accelerometer errors along the instrument sensitive axes are projected into the true or computer frames.

A facility in manipulation of both sets of equations has applications in filter design and evaluation of high-speed aircraft Doppler-inertial systems and weapon delivery analysis since the definition of velocity error is different in each set of equations. An example for a pure inertial local-level north-pointing mechanization with typical aircraft system gyro and accelerometer errors shows that because of the two definitions of velocity error, results can differ by as much as 0.8 ft/s over a one-hour flight.

The perturbation set of equations perhaps is most useful to help interpret simulation and test results of high-speed inertial systems. If the simulated actual system velocity is differenced with the simulated true velocity in true frame axes, then the velocity error is δV_2 directly. To obtain δV_1 the true velocity in the true frame must be transformed to components along computer frame axes and then differenced. Similarly, for tests in which highly accurate tracking instrumentation is used to determine the true velocity at the true position, the velocity error again would be δV_2. On the other hand, the psi angle set of equations is simpler in form since there is no coupling from the position and velocity errors to the psi misalignment angle equations. Also, this set of equations is generally easier to use for long-term error analysis and filter design for marine inertial navigation systems.

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