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OFDM Tutorial

version 0.1

Contents

1 Scope

The document reviews some very basic features of an OFDM transmission system. The need of such a document may be questionable since there is already a wealth of introductory texts on this topic. However the main motivation to start such a document was to familiarise myself with the principles of OFDM transmission systems and to document my "learning progress".

The basic signal processing steps found in OFDM systems are provided both in the analogue domain and as a time discrete description which is thought to be more suitable for simulation projects. From the time discrete formulation of the signal processing a simple simulation tool is derived. There will be two versions of a basic simulator:

- Python + several extension modules (required for MATLAB style array processing and plotting capabilities comparable to MATLAB)
- MATLAB

Choosing two different programming environments is mainly for historical reasons. I started using MATLAB around 1993 for almost any scientific programming task. Other programming tasks however lead me to use Python and its extension modules as well. Overtime it became apparent that Python can be a very good substitute for MATLAB in numerous projects of scientific programming. Considering the price tag attached to MATLAB and its toolboxes, the existence of free software providing equivalent functionality should trigger some experiments with these tools anyway. This becomes even more important when you start to do some programming for self-education at home where access to a MATLAB license is more or less unlikely.

As further versions of this document evolve the focus will be more on how to write a fully fledged simulation tool covering more advanced topics such as synchronisation issues of the OFDM receiver and the impact of various imperfections commonly found in any practical implementation of the transmitter and the receiver. Programming of such tools will use mainly Python + numeric extension modules + C/C++ programs in those cases where computational speed is of paramount importance.

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2 Signal description

2.1 analogue signal processing

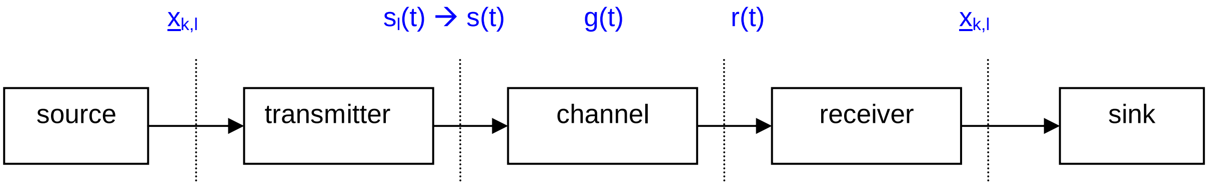

fig. 2-1) model of signal transmission

2.2 source

The source provides complex data symbols $\underline{x}_{k,l}$.

k is an index in the range $[0, N-1]$

l is the block index ; used to number OFDM symbols.

2.2.1 analogue transmitter

An OFDM transmitter uses a set of waveforms $\Phi_k(t)$ for data transmission. Moreover each OFDM symbol comprises N subcarriers with center frequencies $k \cdot \Delta f$. Index k denotes the sub-carrier index . The range of k is in :

$k \in [0, N-1]$

$\Delta f = 1/T_p$

Furthermore we define a duration T_{cp} (duration of the *cyclic prefix*) and the duration T of an OFDM symbol.

$T = T_p + T_{cp}$

With these definitions waveforms $\Phi_k(t)$ are expressed as :

$$\phi_k(t) = \begin{cases} e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t - T_{cp})} & for \quad t \in [0, T] \\ 0 & otherwise \end{cases}$$

Sometimes it is more convenient to use this definition

$$\phi_k(t) = \begin{cases} e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t - T_{cp})} \cdot p(t) & p(t) = 1 \quad for \quad t \in [0, T] \\ 0 & otherwise \end{cases}$$

Note that for $t \in [0, T_{cp}]$ we have

$$\phi_k(t) = \phi_k(t + T_p) \quad for \quad 0 \le t \le T_{cp}$$

(due to this property the time interval $[0, T_{cp}]$ is called the *cyclic prefix*) .

Each OFDM symbol is a weighted superposition of these N waveforms. Specifically the l'th OFDM symbol is

$$s_l(t) = \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \phi_k(t - l \cdot T)$$

The weights $x_{k,l}$ are complex numbers representing data for transmission (e.g. a set of M-QAM symbols).

From the definition of waveforms the l'th OFDM symbol is defined on the interval

$l \cdot T \leq t \leq [l+1] \cdot T$

The sub interval

$l \cdot T \leq t \leq l \cdot T + T_{cp}$

belongs to the cyclic prefix part of the l'th OFDM symbol .

The entire signal $s(t)$ is just the sequence of OFDM symbols.

$$s(t) = \sum_{l=-\infty}^{\infty} s_l(t) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \phi_k(t - l \cdot T)$$

2.2.2 spectral properties

The power spectral density is defined as the Fourier transformation of the autocorrelation function $R(\tau)$.

$$R(\tau) = E\{s(t) \cdot s^*(t-\tau)\}$$

$$S(f) = \int_{-\infty}^{\infty} E\{s(t) \cdot s^*(t-\tau)\} \cdot e^{-j \cdot 2\pi \cdot f \cdot \tau} \cdot d\tau$$

$$s(t) \cdot s^*(t-\tau) = \sum_{l=-\infty}^{\infty} \sum_{l'=-\infty}^{\infty} s_l(t) \cdot s_l^*(t-\tau)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \sum_{l'=-\infty}^{\infty} \sum_{k'=0}^{N-1} \underline{x}_{k,l} \cdot \underline{x}^*_{k',l'} \cdot \phi_k(t - l \cdot T) \cdot \phi^*_{k'}(t - \tau - l' T) \cdot p(t - l \cdot T) \cdot p(t - \tau - l' T)$$

When taking expected the expected value

$$E\{s(t) \cdot s^*(t-\tau)\}$$

non-zero contribution occur only if we have simultaneously $k = k'$ and $l = l'$.

$$E\{s(t) \cdot s^*(t-\tau)\} = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} E\{|\underline{x}_{k,l}|^2 \cdot \phi_k(t - l \cdot T) \cdot \phi_k^*(t - \tau - l \cdot T) \cdot p(t - l \cdot T) \cdot p(t - \tau - l \cdot T)\}$$

with

$$\phi_k(t - l \cdot T) \cdot \phi_k^*(t - \tau - l \cdot T) \cdot p(t - l \cdot T) \cdot p(t - \tau - l \cdot T) = e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot \tau} \cdot p(t - l \cdot T) \cdot p(t - \tau - l \cdot T)$$

$$E\left\{\left|\underline{x}_{k,l}\right|^2\right\} = \sigma_k^2$$

and finally

$$\sum_{k=0}^{N-1} E\left\{\left|\underline{x}_{k,l}\right|^2 \cdot \phi_k(t - l \cdot T) \cdot \phi_k^*(t - \tau - l \cdot T) \cdot p(t - l \cdot T) \cdot p(t - \tau - l \cdot T)\right\}$$
$$= \sigma_k^2 \cdot \frac{1}{T} \cdot e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot \tau} \cdot \left(1 - \left|\frac{\tau}{|T|}\right|\right) \quad for \quad -T \le \tau \le T$$

Hence the power spectral density is expressed as

$$S(f) = \sum_{k=0}^{N-1} \sigma_k^2 \cdot \frac{1}{T} \cdot \int_{-T}^{T} \left(1 - \left|\frac{\tau}{|T|}\right|\right) \cdot e^{-j \cdot 2\pi \cdot [f - \Delta f \cdot k] \cdot \tau} \cdot d\tau$$

$$S(f) = \sum_{k=0}^{N-1} \sigma_k^2 \cdot \frac{1}{T} \cdot 2 \cdot \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) \cdot \cos(2\pi \cdot [f - \Delta f \cdot k] \cdot \tau) \cdot d\tau$$

$$S(f) = \sum_{k=0}^{N-1} \sigma_k^2 \cdot \frac{1}{T} \cdot 2 \cdot \int_{0}^{T} \cos(2\pi \cdot [f - \Delta f \cdot k] \cdot \tau) \cdot d\tau$$
$$- \sum_{k=0}^{N-1} \sigma_k^2 \cdot \frac{1}{T^2} \cdot 2 \cdot \int_{0}^{T} \tau \cdot \cos(2\pi \cdot [f - \Delta f \cdot k] \cdot \tau) \cdot d\tau$$

$$S(f) = -\sum_{k=0}^{N-1} \sigma_k^2 \cdot \frac{1}{T^2} \cdot 2 \cdot \frac{\cos(2\pi \cdot [f - \Delta f \cdot k] \cdot T) - 1}{(2\pi \cdot [f - \Delta f \cdot k])^2} = \left(\frac{1}{4}\right) \cdot \sum_{k=0}^{N-1} \sigma_k^2 \cdot \left[\frac{\sin(\pi \cdot [f - \Delta f \cdot k] \cdot T)}{(\pi \cdot [f - \Delta f \cdot k] \cdot T)}\right]^2$$

In an OFDM transmission system the sub-carrier spacing Δf is as

$\Delta f = 1/T$

The maximum spectral contribution of each sub-carrier occurs at frequencies where the spectral components of all other sub-carriers is zero.

example

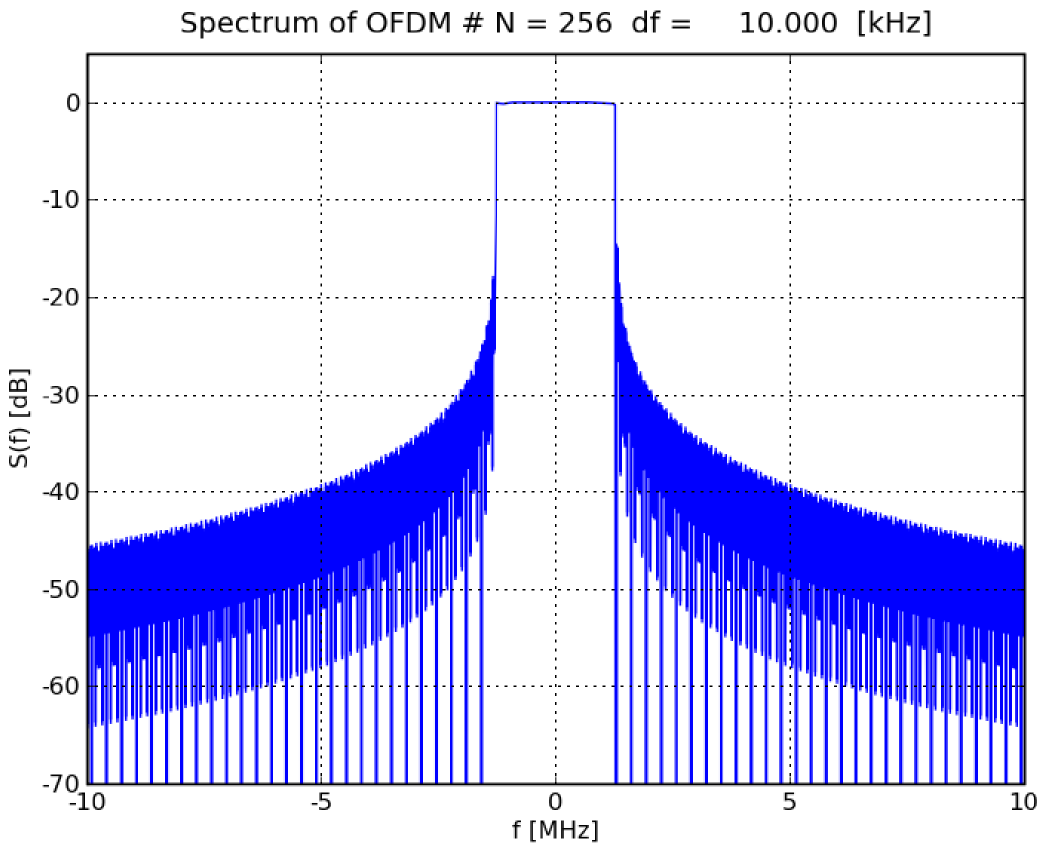

fig. 2-2) normalised OFDM spectrum

2.2.3 physical channel # analogue

The physical channel is either characterised by its channel impulse response $\underline{g}(t)$ or in the frequency domain by its Fourier transform $\underline{G}(f)$. Initially we assume a fixed channel impulse response. Later this simplification may be dropped. Furthermore a finite duration of $\underline{g}(t)$ is assumed.

$g(t)$ defined for $0 \le t \le T_{cp}$

2.2.4 analogue receiver

The input signal of the receiver is denoted $\underline{r}(t)$.

$$\underline{r}(t) = \underline{g}(t) * \underline{s}(t) + \underline{n}(t) = \int_0^{T_{cp}} \underline{g}(\tau) \cdot \underline{s}(t-\tau) \cdot d\tau + \underline{n}(t)$$

$\underline{n}(t)$ is the noise contribution of the receiver .

$$\underline{r}(t) = \sum_{l=-\infty}^{\infty} \int_0^{T_{cp}} \underline{g}(\tau) \cdot \underline{s}_l(t-\tau) \cdot d\tau + \underline{n}(t)$$

Defining the l'th filtered OFDM symbol by $\underline{u}_l(t)$

$$\underline{u}_l(t) = \int_0^{T_{cp}} \underline{g}(\tau) \cdot \underline{s}_l(t-\tau) \cdot d\tau$$

we observe that $\underline{u}_l(t)$ is defined on the finite time interval

$$\underline{u}_l(t) = \begin{cases} defined & for \quad l \cdot T \le t \le [l+1] \cdot T + T_{cp} \\ 0 & otherwise \end{cases}$$

Generally signal $\underline{u}_l(t)$ overlaps the consecutive signal $\underline{u}_{l+1}(t)$ by a time span of T_{cp} . This overlap is due to the channel impulse response $\underline{g}(t)$. It leads to interference with the first part of the consecutive signal.

So the useful part of $\underline{u}_l(t)$ is restricted to time interval

$l \cdot T + T_{cp} \le t \le [l+1] \cdot T$

It is therefore the task of the receiver's signal processing to extract these signal intervals from the OFDM signal .

$$\underline{r}(t) = \sum_{l=-\infty}^{\infty} \underline{u}_l(t) + \underline{n}(t)$$

$$\underline{u}_l(t) = \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \int_0^{T_{cp}} \underline{g}(\tau) \cdot \phi_k(t-\tau-l \cdot T) \cdot d\tau$$

Inserting the definition equation for the waveform function we get

$$\underline{u}_l(t) = \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \int_0^{T_{cp}} \underline{g}(\tau) \cdot e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t-\tau-l \cdot T - T_{cp})} \cdot d\tau$$

and

$$\underline{u}_l(t) = \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t-l \cdot T - T_{cp})} \cdot \int_0^{T_{cp}} \underline{g}(\tau) \cdot e^{-j \cdot 2\pi \cdot \Delta f \cdot k \cdot \tau} \cdot d\tau$$

The integral is just the Fourier transform $\underline{G}(f)$ evaluated at frequencies $k \cdot \Delta f$.

$$\underline{G}(k \cdot \Delta f) = \int_0^{T_{cp}} \underline{g}(\tau) \cdot e^{-j \cdot 2\pi \cdot \Delta f \cdot k \cdot \tau} \cdot d\tau$$

Hence

$$\underline{u}_l(t) = \sum_{k=0}^{N-1} \underline{G}(k \cdot \Delta f) \cdot \underline{x}_{k,l} \cdot e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t-l \cdot T - T_{cp})}$$

Using the signal in the interval

$l \cdot T + T_{cp} \le t \le [l+1] \cdot T$

we want to reconstruct data symbols $\underline{x}_{k,l}$.

$$\int_{l \cdot T + T_{cp}}^{[l+1] \cdot T} \underline{u}_l(t) \cdot e^{-j \cdot 2\pi \cdot \Delta f \cdot k' \cdot (t-l \cdot T - T_{cp})} \cdot dt = \sum_{k=0}^{N-1} \underline{G}(k \cdot \Delta f) \cdot \underline{x}_{k,l} \cdot \int_{l \cdot T + T_{cp}}^{[l+1] \cdot T} e^{j \cdot 2\pi \cdot \Delta f \cdot k \cdot (t-l \cdot T - T_{cp})} \cdot e^{-j \cdot 2\pi \cdot \Delta f \cdot k' \cdot (t-l \cdot T - T_{cp})} \cdot dt$$

$$\int_{l\cdot T+T_{cp}}^{[l+1]\cdot T} \underline{u}_l(t)\cdot e^{-j\cdot 2\pi\cdot\Delta f\cdot k'\cdot(t-l\cdot T-T_{cp})}\cdot dt = \sum_{k=0}^{N-1}\underline{G}(k\cdot\Delta f)\cdot\underline{x}_{k,l}\cdot\int_{l\cdot T+T_{cp}}^{[l+1]\cdot T} e^{j\cdot 2\pi\cdot\Delta f\cdot[k-k']\cdot(t-l\cdot T-T_{cp})}\cdot dt$$

$$=\sum_{k=0}^{N-1}\underline{G}(k\cdot\Delta f)\cdot\underline{x}_{k,l}\cdot\int_{0}^{T_p} e^{j\cdot 2\pi\cdot\Delta f\cdot[k-k']\cdot t}\cdot dt$$

$$\int_{l\cdot T+T_{cp}}^{[l+1]\cdot T} \underline{u}_l(t)\cdot e^{-j\cdot 2\pi\cdot\Delta f\cdot k'\cdot(t-l\cdot T-T_{cp})}\cdot dt = \begin{cases}\underline{G}(k\cdot\Delta f)\cdot\underline{x}_{k,l}\cdot T_p & \text{for } k'=k\\ 0 & k'\neq k\end{cases}$$

Hence we get $\underline{x}_{k,l}$ by

$$\underline{x}_{k,l} = \frac{1}{T_p}\cdot\underline{G}(k\cdot\Delta f)^{-1}\cdot\int_{l\cdot T+T_{cp}}^{[l+1]\cdot T}\underline{u}_l(t)\cdot e^{-j\cdot 2\pi\cdot\Delta f\cdot k\cdot(t-l\cdot T-T_{cp})}\cdot dt$$

2.3 digital signal processing

2.3.1 digital transmitter

Each OFDM symbol is constructed from a set of complex data symbols denoted by $\underline{x}_{k,l}$. In this notation we have indices k, l to indicate the

- sub-carrier index → $k \in [0, N-1]$
- index of the OFDM symbol → l

For each index l → (OFDM symbol) the N data symbols $\underline{x}_{k,l}$ are transformed into a new sequence $\underline{y}_{m,l}$ using equation

$$y_{m,l} = \begin{cases} A\cdot\sum_{k=0}^{N-1}\underline{x}_{k,l}\cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[m-l\cdot N]} & \text{for}\quad l\cdot N\leq m\leq (l+1)\cdot N-1\\ 0 & \text{otherwise}\end{cases}$$

Sequence $\underline{y}_l$

$\underline{y}_l = [\underline{y}_{(l\cdot N+0),l}, \underline{y}_{(l\cdot N+1),l}, \dots, \underline{y}_{([l+1]\cdot N-2),l}, \underline{y}_{([l+1]\cdot N-1),l}]$

cannot be used directly for transmission. Instead $\underline{y}_l$ is extended to a longer sequence $\underline{z}_l$ by adding a *finite length prefix*. For reasons explained later in this document the prefix is constructed from the last L samples of sequence $\underline{y}_l$.

Note :

L is generally chosen larger than the maximum number of *significant* samples of the channel impulse response. The new sequence (also known as OFDM symbol) is thus the concatenation of two sequences

$\underline{z}_l = [\underline{y}_{([l+1]\cdot N-L),l}, \underline{y}_{([l+1]\cdot N-L+1),l}, \dots, \underline{y}_{([l+1]\cdot N-2),l}, \underline{y}_{([l+1]\cdot N-1),l}] , [\underline{y}_{(l\cdot N+0),l}, \underline{y}_{(l\cdot N+1),l}, \dots, \underline{y}_{([l+1]\cdot N-2),l}, \underline{y}_{([l+1]\cdot N-1),l}]$

Since we use a repetition of samples of $\underline{y}_l$for the prefix the part of sequence $\underline{z}_l$ where

$0 \leq n \leq L-1$

equals

$\underline{z}_l[n+N]$

This property is similar to periodicity and the prefix is commonly denoted as "*cyclic prefix"* for this reason.

The total length of sequence $\underline{z}_l$ is $N_{tot} = N + L$

The transmitted sequence is then just the juxtaposition of sub-sequences $\underline{z}_{m,l}$.

$$\underline{z}_{m,l} = \begin{cases} A \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot [m-L-l\cdot N]} & for \quad l \cdot N \le m \le (l+1) \cdot N + L - 1 \\ 0 & otherwise \end{cases}$$

For indices m in the range $[l\cdot N, \ldots, l\cdot N-1+L]$ samples $\underline{z}_{m,l}$ are just the cyclic prefix.

Defining the pulse function $p[m]$

$$p[m] = \begin{cases} 1 & for \quad 0 \le m \le N + L - 1 \\ 0 & otherwise \end{cases}$$

we may rewrite sub-sequences $\underline{z}_{m,l}$ by

$$\underline{z}_{m,l} = A \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot [m-L-l\cdot N]} \cdot p[m - l \cdot (N+L)]$$

Using the last equation the transmitted sequence $\underline{z}[m]$ is expressed as

$$\underline{z}[m] = \sum_{l=-\infty}^{\infty} \underline{z}_{m,l} = A \cdot \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot [m-L-l\cdot N]} \cdot p[m - l \cdot (N+L)]$$

Many publications on OFDM signal processing define a time discrete waveform $\phi_k[m]$ by

$$\phi_k[m] = \begin{cases} e^{j \cdot \frac{2\pi}{N} \cdot k \cdot [m-L]} \cdot p[m] & 0 \le m \le N + L - 1 \\ 0 & otherwise \end{cases}$$

which gives

$$\underline{z}[m] = \sum_{l=-\infty}^{\infty} \underline{z}_{m,l} = A \cdot \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \phi_k[m - l \cdot (N+L)]$$

2.3.2 time discrete channel

The discrete channel impulse response is denoted by $g[n]$. We will always assume a finite number $L+1$ of samples.

$g[n] = [\, g[0], g[1], \ldots, g[L] \,]$

It is expected however that only samples up to some index $M < L$ are significant .

The discrete channel performs a convolution of the input sequences $\underline{z}[m]$.

Let r[m] be the channel's output signal.

$$r[m] = \sum_{n=-\infty}^{\infty} z[n] \cdot g[m-n]$$

$$r[m] = A \cdot \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \phi_k[n - l \cdot (N+L)] \cdot g[m-n]$$

The last expression seems daunting. However some simplifications are possible by taking into account that waveform functions ϕ_k[.] and impulse response g[.] both are defined over a finite range of indices.

For a given time index m index we have

$0 \le m\text{-}n \le L$

which restricts n to the range

$-L+m \le n \le m$

So we can introduce finite summation limits for n .

$$r[m] = A \cdot \sum_{l=-\infty}^{\infty} \sum_{n=-L+m}^{m} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \phi_k[n - l \cdot (N+L)] \cdot g[m-n]$$

For a more convenient notation we express index m by

$$m = l_m \cdot (N+L) + r$$

l_m and r are integers. r is defined on the interval $r \in [0, N+L-1]$. Hence any value of m is uniquely defined by the index tuple (l_m, r) .

Wavefunctions $\phi_k[n - l \cdot (N+L)]$ are defined for

$0 \le n - l \cdot (N+L) \le N+L-1$

$-(N+L) + 1 + n \le l \cdot (N+L) \le n$

Inserting the minimum possible values of n

$n_{min} = -L + m$

$n_{max} = m$

we get:

$-(N+L) + 1 - L + m \le l \cdot (N+L) \le m$

and

$-(N+L) + 1 - L + l_m \cdot (N+L) + r \le l \cdot (N+L) \le l_m \cdot (N+L) + r$

$(l_m - 1) \cdot (N+L) + r + 1 - L \le l \cdot (N+L) \le l_m \cdot (N+L) + r$

$(l_m - 1) + (r + 1 - L)/(N+L) \le l \le l_m + r/(N+L)$

$(l_m - 1) + (r + 1 - L)/(N+L) \leq l \leq \lfloor l_m + r/(N+L) \rfloor = l_m$

and finally

$0 \leq r \leq (L-1) \rightarrow (l_m - 1) \leq l \leq l_m$

$$r\ m\ = r[l_m \cdot (N+L) + r] =$$

$$A \cdot \left(\sum_{n=-L+m}^{m} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \ _k\left[n - l_m \cdot (N+L)\right] \cdot g[m-n] + \sum_{n=-L+m}^{m} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot \ _k\left[n - (l_m - 1) \cdot (N+L)\right] \cdot g[m-n] \right)$$

$$\phi_k[m] = \begin{cases} e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[m-L]} \cdot p[m] & 0 \le m \le N+L-1 \\ 0 & otherwise \end{cases}$$

$$r[m] = r[l_m \cdot (N+L) + r] =$$
$$A \cdot \sum_{n=-L+m}^{m} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[n-l_m\cdot(N+L)-L]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot m} \cdot p[n-l_m\cdot(N+L)] \cdot g[m-n]$$

The value of $p[n-l_m\cdot(N+L)]$ is always 1 for n ∈ [-L+m, m] . So we have

$$r[m] = r[l_m \cdot (N+L) + r] =$$
$$A \cdot \sum_{n=-L+m}^{m} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[n-l_m\cdot(N+L)-L]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot m} \cdot g[m-n]$$

Looking at g[m-n] for index n running from (-L+m) to m we observe that [m-n] runs from L to 0 regardless of the specific choice for index m.
r runs from L to (N+L-1) . Hence we define r' .

$$r' = r - L$$

$$r[r'] =$$
$$A \cdot \sum_{u=0}^{L} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[-L+u+m-l_m\cdot(N+L)-L]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot[l_m\cdot(N+L)+r'+L]} \cdot g[L-u]$$

$$r[r'] =$$
$$A \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot[l_m\cdot(N+L)+L]} \cdot \sum_{u=0}^{L} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[u+r'-L]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot r'} \cdot g[L-u]$$

The receiver's task is to recover transmitted data symbol $\underline{x}_{k,l}$ of the l^{th} OFDM symbol. To this end we evaluate

$$o[k'] = \sum_{r'=0}^{N-1} r[r'] \cdot e^{-j\cdot\frac{2\pi}{N}\cdot k'\cdot r'}$$
$$= A \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot[l_m\cdot(N+L)+L]} \cdot \sum_{r'=0}^{N-1} \sum_{u=0}^{L} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[u+r'-L]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot r'} \cdot e^{-j\cdot\frac{2\pi}{N}\cdot k'\cdot r'} \cdot g[L-u]$$

Using w = L – u we obtain

$$o[k'] = A \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot[l_m\cdot(N+L)+L]} \cdot \sum_{r'=0}^{N-1} \sum_{w=0}^{L} \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j\cdot\frac{2\pi}{N}\cdot k\cdot[r'-w]} \cdot e^{j\cdot 2\pi\cdot\frac{\Delta f}{f_s}\cdot r'} \cdot e^{-j\cdot\frac{2\pi}{N}\cdot k'\cdot r'} \cdot g[w]$$

Re-ordering this equation yields

$$o[k'] = A \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{w=0}^{L} g[w] \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot w} \cdot \sum_{r'=0}^{N-1} e^{j \cdot \frac{2\pi}{N} \cdot [k-k'] \cdot r'} \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot r'}$$

A closed form solution does not seem possible. But for a small frequency offset Δf a reasonable good approximation of $e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot r'}$ is

$$e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot r'} \cong 1 + j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot r' \quad \{ for \quad 0 \le r' \le N-1$$

Then samples o[k'] are expressed as

$$o[k'] = A \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{w=0}^{L} g[w] \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot w} \cdot \sum_{r'=0}^{N-1} e^{j \cdot \frac{2\pi}{N} \cdot [k-k'] \cdot r'}$$
$$+ j \cdot A \cdot 2\pi \cdot \left(\frac{\Delta f}{f_s} \right) \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{w=0}^{L} g[w] \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot w} \cdot \sum_{r'=0}^{N-1} r' \cdot e^{j \cdot \frac{2\pi}{N} \cdot [k-k'] \cdot r'}$$

Using property

$$\sum_{r'=0}^{N-1} e^{j \cdot \frac{2\pi}{N} \cdot [k-k'] \cdot r'} = \begin{cases} N & k = k' \\ 0 & otherwise \end{cases}$$

and defining

$$M[k-k'] = \sum_{r'=0}^{N-1} r' \cdot e^{j \cdot \frac{2\pi}{N} \cdot [k-k'] \cdot r'}$$

we get

$$o[k'] = A \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot N \cdot \underline{x}_{k',l} \cdot \sum_{w=0}^{L} g[w] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k' \cdot w}$$
$$+ j \cdot A \cdot 2\pi \cdot \left(\frac{\Delta f}{f_s} \right) \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot M[k-k'] \cdot \sum_{w=0}^{L} g[w] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot w}$$

And having seen that

$\sum_{w=0}^{L} g[w] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k' \cdot w}$ is just the DFT of the channel impulse evaluated at frequency instant k' .

$$\underline{G}[k'] = \sum_{w=0}^{L} g[w] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k' \cdot w}$$

$$o[k'] = A \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot N \cdot \underline{x}_{k',l} \cdot \underline{G}[k']$$
$$+ j \cdot A \cdot \frac{N \cdot (N-1)}{2} \cdot 2\pi \cdot \left(\frac{\Delta f}{f_s} \right) \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot M[k-k'] \cdot \underline{G}[k]$$

Applying scaling factors (1/A) , (1/N) and correcting for frequency dependent gain (1/$\underline{G}$[k']) yields

$$\frac{1}{A} \cdot \frac{1}{N} \cdot \frac{1}{\underline{G}[k']} \cdot o[k'] = e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \underline{x}_{k',l}$$
$$+ j \cdot \frac{1}{N} \cdot 2\pi \cdot \left(\frac{\Delta f}{f_s} \right) \cdot e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]} \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot M[k-k'] \cdot \frac{\underline{G}[k]}{\underline{G}[k']}$$

Ignoring the common phase term $e^{j \cdot 2\pi \cdot \frac{\Delta f}{f_s} \cdot [l_m \cdot (N+L)+L]}$ we observe that only for $\Delta f = 0$ the data symbol $\underline{x}_{k',l}$ is recovered exactly. With frequency offset Δf there will be always some cross talk and all data symbols contribute to this. Moreover cross talk increases with the number of sub-carriers.

4 A simple simulator

4.1 scope

Demonstrate basic principle of a time discrete OFDM transmission system.
As a first step a simulation program will be written using Python + some extension modules. Once this approach works similar programs will be written in MATLAB and SCILAB .

4.2 simulation parameters

`N`................................ number of subcarriers ; for numeric efficiency N should be a power of 2
`L`................................ length of cyclic prefix
`N_OFDM`..................... number of OFDM symbols used in simulation
`channel_imp`........... vector of complex samples of the channel impulse response
`f_offset_n`............. normalised frequency offset $\frac{\Delta f}{f_s}$

4.3 QAM source

A Python module `QAMSources.py` provides several QAM data generators. Currently supported modulation methods are :

1. 4-QAM → `Source4QAM(block_len)`
2. 16-QAM → `Source16QAM(block_len)`
3. 32-QAM → `Source32QAM(block_len)`
4. 64-QAM → `Source64QAM(block_len)`
5. 128-QAM → `Source128QAM(block_len)`
6. 256-QAM → `Source256QAM(block_len)`

example:

`qam_vec = Source4QAM(block_len)`
`block_len` .... number of 4-QAM symbols
`qam_vec`......................... complex numpy array of randomly generated 4-QAM symbols ;
`len(qam_vec) = block_len`

4.4 OFDM modulator

4.4.1 IFFT block

Let the l'th QAM data vector be represented by a complex vector $\underline{x}_{k,l}$ with $k \in [0, N-1]$. The IFFT block of the OFDM modulator processes the elements of vector into a new complex vector $\underline{y}_{k,l}$ using equation.

$$y_{m,l} = A \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot m}$$

This is just the inverse discrete Fourier transform scaled by a factor A . In Python (module numpy) as in MATLAB the inverse FFT functions ifft() provide the operation

$$\frac{1}{N} \cdot \sum_{k=0}^{N-1} \underline{x}_{k,l} \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot m}$$

To correct for the scaling factor (1/N) introduced by the ifft() function we multiply by N .

example

```
y_vec = N*numpy.fft.ifft(qam_vec)  # Python + numpy module
```

4.4.2 obtaining and adding the cyclic prefix

`y_vec` is a complex vector with `N` elements

```
y_vec[0], y_vec[1], …, y_vec[N-2], y_vec[N-1]
```

The last `L` elements of `y_vec` make up the samples of the cyclic prefix .

```
cyclic_prefix = [y_vec[N-L], …, y_vec[N-2], y_vec[N-1]]
```

In Python `cyclic_prefix` is obtained through a slicing operation of the vector `y_vec` (MATLAB and SCILAB have similar indexing operations) .

```
cyclic_prefix = y_vec[N-L:]
```

The cyclic prefix is *prepended* to vector `y_vec`. The extended vector is denoted `ofdm_vec` . In Python this is achieved using

```
ofdm_vec = numpy.concatenate( (cyclic_prefix, y_vec) )
```

or directly without explicitly creating vector `cyclic_prefix`

```
ofdm_vec = numpy.concatenate( (y_vec[N-L:], y_vec) )
```

4.5 time discrete channel

Complex samples of the channel impulse response are stored in a vector `g_channel_imp` . We assume a length of `M` samples. To make the organisation of program flow simpler the restriction

```
M < N
```

is imposed. But this is really not much of a restriction since for successful operation (no intersymbol interference of adjacent OFDM symbols → no ISI) we require

```
M < L < N
```

Stated in a different way: The length `M` of the channel impulse response shall be less than the length `L` of the cyclic prefix.

The time discrete channel performs aperiodic convolution as defined by equation

$$r[k] = \sum_{p=0}^{P} i[p] \cdot h[k-p]$$

i[] .... samples of the input sequence

h[] .... samples of the channel impulse response

r[k] .... samples of the output sequence → channel output

In the specific case of this simulation program sequence i[] are the samples of `ofdm_vec` . Hence index p is in [0, N+L-1] .

Impulse response h[] are the samples of `g_channel_imp` .

The resulting sequence r[] has a length of (N+L+M-1)

r[0], r[1], … , r[N+L-1], r[N+L], …, r[N+L+M-2]

The last (M-1) samples of r[] overlap with first (M-1) samples of the consecutive filtered OFDM symbol. Filtering an OFDM symbols therefore requires several signal processing steps:

```
# the channel ...
r_vec = num.convolve(ofdm_vec, g_channel_impulse)

# split into vectors r1_vec, r2_vec
r1_vec = r_vec[0:N+L]
r2_vec = r_vec[N+L:]

# correct for channel output of last OFDM-symbol
r1_vec[0:M-1] = r1_vec[0:M-1] + overlap_old

# save overlap part of current OFDM symbol for next symbol
overlap_old = r2_vec
```

4.6 frequency offset

To apply a frequency offset to array `r1_vec` we first create a vector `fo_vec` of (N+L) samples

$$e^{j \cdot 2\pi \cdot \left(\frac{\Delta f}{f_s}\right) \cdot m} \qquad 0 \le m \le (N+L)-1$$

```
# frequency offset of receiver
fo_vec = num.exp(2.j *num.pi * f_offset_n * num.arange(N+L))
```

multiplying array `r1_vec` by `fo_vec` yields the OFDM symbol `r1_vec_fo` with frequency offset.

```
# apply frequency offset
r1_vec_fo = r1_vec * fo_vec
```

4.7 discard cyclic prefix

The first L samples are the cyclic prefic possibly garbled by ISI. So the receiver discards this part of the array r1_vec_fo. The remaining part is stored in an array ofdm_rec of length N .

```
# discard cyclic prefix part
ofdm_rec = r1_vec_fo[L:]
```

ofdm_rec is equivalent to sequence r[r'] in section 2.3.3 .

4.8 retrieving the data vector

Getting back to the transmitted data symbols $\underline{x}_{k,l}$ is a two step procedure.

1. application of FFT (In case of a transparent channel we are done. The FFT yields the sequence of samples $\underline{x}_{k,l}$.)
2. frequency dependent gain correction due to the channel's impulse response

We already derived an equation for the recovered data :

$$\underline{x}_{k',l} = \frac{1}{A} \cdot \frac{1}{\underline{G}[k']} \cdot \frac{1}{N} \cdot o[k'] = \frac{1}{A} \cdot \frac{1}{\underline{G}[k']} \cdot \frac{1}{N} \cdot \sum_{r'=0}^{N-1} r[r'] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k' \cdot r'}$$

The part

$$\sum_{r'=0}^{N-1} r[r'] \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k' \cdot r'}$$

is accomplished applying the fft() function available in Pyton/Numpy or MATLAB . Additionally a correction factor (1/N) must be applied. And in case there was a scaling factor of $A \neq 1$ there must be a gain correction (1/A) for this.

Samples of frequency dependent channel gain $\underline{G}[k']$ are obtained from the FFT of the discrete channel impulse response. The values of the impulse response are stored in array h . Using

```
G = num.fft.fft(h, N)
```

just provides the samples $\underline{G}[k']$ as array G . In our simplistic simulator the impulse response is fixed. Hence we only need to compute it once. The same channel applies to **all** transmitted OFDM symbols.

```
data_rec = (1/N)*num.fft.fft(ofdm_rec) / G
```

If all went well data_rec is the sequence of transmitted data symbols $\underline{x}_{k,l}$. These data blocks can be further processed to obtain information such as

- bit error rate
- signal constellation diagrams of individual sub-carriers
- etc.

Some examples are provided in the next sub-chapters.

4.9 post processing

Currently the simulation tool only supports display of the constellation diagram (I-/Q plot).

4.9.1 plot constellation diagram of subcarriers