Technical Note: INS Noise Propagation

Paul F. Roysdon   Jay A. Farrell

***Abstract*— Due to space limitations in [1] and [2], this Technical Note is supplied to explain the state and noise propagation for an inertial navigation system (INS), between two aiding measurement times. The temporal propagation of the state error and noise is required for optimal state estimation.**

I. Introduction

Let $\boldsymbol{x} \in \mathbb{R}^{n_s}$ denote the rover state vector, where

$$\boldsymbol{x}(t) = [\mathbf{p}^\intercal(t), \mathbf{v}^\intercal(t), \mathbf{q}^\intercal(t), \mathbf{b}_a^\intercal(t), \mathbf{b}_g^\intercal(t)]^\intercal \in \mathbb{R}^{n_s},$$

where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{b}_a$, $\mathbf{b}_g$ each in $\mathbb{R}^3$ represent the position, velocity, accelerometer bias and gyro bias vectors, respectively, $\mathbf{q} \in \mathbb{R}^4$ represents the attitude quaternion ($n_s = 16$).

Let $\boldsymbol{x}_v(t) = [\mathbf{p}^\intercal(t), \mathbf{v}^\intercal(t), \mathbf{q}^\intercal(t)]^\intercal \in \mathbb{R}^{10}$ represent the vehicle state position, velocity and attitude. Let $\boldsymbol{x}_c(t) = [\mathbf{b}_a^\intercal(t), \mathbf{b}_g^\intercal(t)]^\intercal \in \mathbb{R}^6$ represent the IMU calibration terms: accelerometer bias and gyro bias. Then $\boldsymbol{x}(t)$ can be represented as $\boldsymbol{x}(t) = [\boldsymbol{x}_v^\intercal(t), \boldsymbol{x}_c^\intercal(t)]^\intercal$.

Let τ_i denote the time instants of the i^{th} IMU measurements of $\mathbf{u}$, as defined in Section II.A in [1] and [2]. Let $\boldsymbol{x}_i = \boldsymbol{x}(\tau_i)$ and $\mathbf{u}_i = \mathbf{u}(\tau_i)$.

Let the state estimate time propagation be represented as

$$\hat{\boldsymbol{x}}_{i+1} \doteq \boldsymbol{\phi}(\hat{\boldsymbol{x}}_i, \tilde{\mathbf{u}}_i),$$

where the vehicle state estimate is $\hat{\boldsymbol{x}}_{v,i+1} \doteq \boldsymbol{\phi}_v(\hat{\boldsymbol{x}}_{v,i}, \tilde{\mathbf{u}}_i)$.

Let the true state time propagation be represented as

$$\boldsymbol{x}_{i+1} = \boldsymbol{\phi}(\boldsymbol{x}_i, \mathbf{u}_i),$$

and the true vehicle state as $\boldsymbol{x}_{v,i+1} = \boldsymbol{\phi}_v(\boldsymbol{x}_{v,i}, \mathbf{u}_i)$.

Define the state error as

$$\delta\boldsymbol{x}_i = \boldsymbol{x}_i \ominus \hat{\boldsymbol{x}}_i \in \mathbb{R}^{n_e},$$

where the symbol '$\ominus$', which is discussed in [3], represents the subtraction operation for position, velocity and bias states, and the multiplication operation of the attitude states. The fact that $n_s = 16$ and $n_e = 15$ is discussed in [3]. Let $\delta\boldsymbol{x}_{v,i} \in \mathbb{R}^9$ represent the vehicle state error for position, velocity and attitude. Let $\delta\boldsymbol{x}_{c,i} \in \mathbb{R}^6$ represent the error in the IMU calibration terms: accelerometer bias and gyro bias. Then $\delta\boldsymbol{x}_i$ can be represented as $\delta\boldsymbol{x}_i = [\delta\boldsymbol{x}_{v,i}^\intercal, \delta\boldsymbol{x}_{c,i}^\intercal]^\intercal$.

Let the IMU measurement be defined as

$$\tilde{\mathbf{u}}(\tau_i) \triangleq \mathbf{u}(\tau_i) - \mathbf{b}(\tau_i) - \boldsymbol{\omega}_u(\tau_i) \in \mathbb{R}^6,$$

with additive stochastic errors $\boldsymbol{\omega}_u(\tau_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q_d})$ and $\mathbf{b} = [\mathbf{b}_a^\intercal, \mathbf{b}_g^\intercal]^\intercal$. The sensor bias $\mathbf{b}$ represents time correlated measurement errors, and $\boldsymbol{\omega}_u(\tau_i)$ represents the white measurement errors. Let the estimate $\hat{\mathbf{u}}_i \triangleq \tilde{\mathbf{u}}_i + \hat{\mathbf{b}}_i$, where $\hat{\mathbf{b}}_i$ is the estimate of $\mathbf{b}_i$ (i.e. $\hat{\boldsymbol{x}}_{c,i}$). Let the measurements $\tilde{\mathbf{u}}(\tau_i)$ be defined for IMU measurement times i, between aiding measurement times k, such that $\tau_i \in [t_{k-1}, t_k]$.

Define

$$\begin{aligned}
\delta\mathbf{u}_i &\triangleq \mathbf{u}_i - \hat{\mathbf{u}}_i \\
&= \mathbf{u}_i - \tilde{\mathbf{u}}_i - \hat{\mathbf{b}}_i \\
&= \mathbf{u}_i - (\mathbf{u}_i - \mathbf{b}_i - \boldsymbol{\omega}_{u,i}) - \hat{\mathbf{b}}_i \\
&= \delta\mathbf{b}_i + \boldsymbol{\omega}_{u,i},
\end{aligned}$$

where $\delta\mathbf{b}_i \triangleq \mathbf{b}_i - \hat{\mathbf{b}}_i$, and $\delta\mathbf{b}_i$ (i.e. $\delta\mathbf{b}_i = \delta\boldsymbol{x}_{c,i}$) is a state calibration term.

Linearization of the state error, using Taylor series to first order, yields

$$\begin{aligned}
\delta\boldsymbol{x}_{v,i+1} &= \boldsymbol{\phi}_v(\boldsymbol{x}_{v,i}, \mathbf{u}_i) - \boldsymbol{\phi}_v(\hat{\boldsymbol{x}}_{v,i}, \hat{\mathbf{u}}_i) \\
&= \boldsymbol{\phi}_v(\hat{\boldsymbol{x}}_{v,i}, \hat{\mathbf{u}}_i) + \left.\frac{\partial\boldsymbol{\phi}_v}{\partial\boldsymbol{x}_{v,i}}\right|_{\hat{\boldsymbol{x}}_{v,i}} \delta\boldsymbol{x}_{v,i} \\
&\quad + \left.\frac{\partial\boldsymbol{\phi}_v}{\partial\mathbf{u}_i}\right|_{\hat{\mathbf{u}}_i} \delta\mathbf{u}_i - \boldsymbol{\phi}_v(\hat{\boldsymbol{x}}_{v,i}, \hat{\mathbf{u}}_i) \\
&= \left.\frac{\partial\boldsymbol{\phi}_v}{\partial\boldsymbol{x}_{v,i}}\right|_{\hat{\boldsymbol{x}}_{v,i}} \delta\boldsymbol{x}_{v,i} + \left.\frac{\partial\boldsymbol{\phi}_v}{\partial\mathbf{u}_i}\right|_{\hat{\mathbf{u}}_i} \delta\mathbf{u}_i \\
&= \mathbf{A}_i\delta\boldsymbol{x}_{v,i} + \mathbf{B}_i\delta\mathbf{u}_i \\
&= \mathbf{A}_i\delta\boldsymbol{x}_{v,i} + \mathbf{B}_i\delta\mathbf{b}_i + \mathbf{B}_i\boldsymbol{\omega}_{u,i},
\end{aligned} \tag{1}$$

where $\mathbf{A}_i = \frac{\partial\boldsymbol{\phi}_v}{\partial\boldsymbol{x}_{v,i}}\big|_{\hat{\boldsymbol{x}}_{v,i},\hat{\mathbf{u}}_i} \in \mathbb{R}^{9\times 9}$, and $\mathbf{B}_i = \frac{\partial\boldsymbol{\phi}_v}{\partial\mathbf{u}_i}\big|_{\hat{\boldsymbol{x}}_{v,i},\hat{\mathbf{u}}_i} \in \mathbb{R}^{9\times 6}$.

Let the model of the sensor bias be defined as a first-order Gauss-Markov process

$$\delta\mathbf{b}_{i+1} = \mathbf{F}_b\delta\mathbf{b}_i + \boldsymbol{\nu}, \tag{2}$$

where $\mathbf{F_b} \in \mathbb{R}^{6\times 6}$ is selected such that the bias errors are modeled as either random constants or random walk plus constants (see eqns. 11.106 and 11.107 of [4]), and $\boldsymbol{\nu} \sim \mathcal{N}(\mathbf{0}, \sigma_\nu\mathbf{I})$.

Rewriting eqns. (1) and (2) in matrix form:

$$\begin{bmatrix} \delta\boldsymbol{x}_{v,i+1} \\ \delta\boldsymbol{x}_{c,i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{F_b} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{x}_{v,i} \\ \delta\mathbf{b}_i \end{bmatrix} + \begin{bmatrix} \mathbf{B}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_{u,i} \\ \boldsymbol{\nu} \end{bmatrix}. \tag{3}$$

For analysis, in the following section let $\mathbf{F_b} = \mathbf{I}$.

II. Propagation of State Error

This section analyzes the error accumulation over the time interval $t \in [k-1, k]$ using superposition.

A. Propagation of Initial State Error

Consider eqn. (3) over the interval $t \in [k-1, k]$, where $\boldsymbol{\omega}_{u,i-1} = \mathbf{0}$ and $\boldsymbol{\nu} = \mathbf{0}$. Without loss of generality let $k = 1$, such that $t \in [0, 1]$. For each time instant, eqn. (3) can be represented in terms of $\mathbf{A}_i$, and $\mathbf{B}_i$, with initial condition errors $\delta \boldsymbol{x}_{v,0}$, and $\delta \mathbf{b}_0$:

$$
\begin{aligned}
\delta \boldsymbol{x}_1 &= \begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix} \\
\delta \boldsymbol{x}_2 &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,1} \\ \delta \mathbf{b}_1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}_1\mathbf{A}_0 & \mathbf{A}_1\mathbf{B}_0 + \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix} \\
\delta \boldsymbol{x}_3 &= \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,2} \\ \delta \mathbf{b}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1\mathbf{A}_0 & \mathbf{A}_1\mathbf{B}_0 + \mathbf{B}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}_2\mathbf{A}_1\mathbf{A}_0 & \mathbf{A}_2\mathbf{A}_1\mathbf{B}_0 + \mathbf{A}_2\mathbf{B}_1 + \mathbf{B}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix}. \quad (4)
\end{aligned}
$$

Define F_s as the sample frequency of the sensor (e.g. IMU). As defined in Section II.A of both [1] and [2], let $\mathbf{U}_k = \{\tilde{\mathbf{u}}(\tau_i) \text{ for } \tau_i \in [t_{k-1}, t_k]\}$. As defined in Section III.A of both [1] and [2], let $\mathbf{X}_k = [\boldsymbol{x}(t_{k-L})^\intercal, \dots, \boldsymbol{x}(t_k)^\intercal]^\intercal \in \mathbb{R}^{n_s(L+1)}$ denote the vehicle trajectory over a sliding time window that contains L one second GPS measurement epochs: $[\mathbf{y}_{k-L+1}, \dots, \mathbf{y}_k]$. After F_s IMU time steps (i.e. $F_{s,k=1}$)

$$
\begin{aligned}
\delta \boldsymbol{x}_{F_s} &= \left\{ \prod_{i=1}^{F_s} \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right\} \begin{bmatrix} \delta \boldsymbol{x}_{v,0} \\ \delta \mathbf{b}_0 \end{bmatrix} && (5) \\
&= \boldsymbol{\Upsilon}(\hat{\mathbf{X}}_k, \mathbf{U}_k)[\delta \boldsymbol{x}_{v,0}, \delta \mathbf{b}_0]^\intercal, && (6)
\end{aligned}
$$

where the operator $\boldsymbol{\Upsilon}(\hat{\mathbf{X}}_k, \mathbf{U}_k)$ in eqn. (6) represents the product operation in eqn. (5), and $\hat{\mathbf{X}}_k$ is the estimate of $\mathbf{X}_k$. The product operation in eqn. (5) must follow the order of multiplications shown in eqn. (4).

B. Noise Propagation

Again consider eqn. (3) over the interval $t \in [k-1, k]$. Here we will analyze the effect of the noise terms $\boldsymbol{\omega}_u$ and $\boldsymbol{\nu}$, with $\delta \boldsymbol{x}_{v,0}$ and $\delta \mathbf{b}_0$ both zero.

To simplify notation, let

$$
\mathbf{C}_i \triangleq \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{D}_i \triangleq \begin{bmatrix} \mathbf{B}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},
$$

and

$$
\delta \boldsymbol{x}_i \triangleq \begin{bmatrix} \delta \boldsymbol{x}_{v,i} \\ \delta \boldsymbol{x}_{c,i} \end{bmatrix}, \quad \mathbf{n}_i \triangleq \begin{bmatrix} \boldsymbol{\omega}_{u,i} \\ \boldsymbol{\nu} \end{bmatrix}.
$$

Defining eqn. (3) using the terms above, we have

$$
\delta \boldsymbol{x}_{i+1} = \mathbf{C}_i \delta \boldsymbol{x}_i + \mathbf{D}_i \mathbf{n}_i. \quad (7)
$$

Performing operations on eqn. (7) (similar to the operations leading up to eqn. (4)),

$$
\begin{aligned}
\delta \boldsymbol{x}_1 &= \mathbf{C}_0 \delta \boldsymbol{x}_0 + \mathbf{D}_0 \mathbf{n}_0 \\
\delta \boldsymbol{x}_2 &= \mathbf{C}_1 \delta \boldsymbol{x}_1 + \mathbf{D}_1 \mathbf{n}_1 \\
&= \mathbf{C}_1(\mathbf{C}_0 \delta \boldsymbol{x}_0 + \mathbf{D}_0 \mathbf{n}_0) + \mathbf{D}_1 \mathbf{n}_1 \\
&= \mathbf{C}_1\mathbf{C}_0 \delta \boldsymbol{x}_0 + \mathbf{C}_1\mathbf{D}_0 \mathbf{n}_0 + \mathbf{D}_1 \mathbf{n}_1 \\
\delta \boldsymbol{x}_3 &= \mathbf{C}_2 \delta \boldsymbol{x}_2 + \mathbf{D}_2 \mathbf{n}_2 \\
&= \mathbf{C}_2(\mathbf{C}_1\mathbf{C}_0 \delta \boldsymbol{x}_0 + \mathbf{C}_1\mathbf{D}_0 \mathbf{n}_0 + \mathbf{D}_1 \mathbf{n}_1) + \mathbf{D}_2 \mathbf{n}_2 \\
&= \mathbf{C}_2\mathbf{C}_1\mathbf{C}_0 \delta \boldsymbol{x}_0 + \mathbf{C}_2\mathbf{C}_1\mathbf{D}_0 \mathbf{n}_0 + \mathbf{C}_2\mathbf{D}_1 \mathbf{n}_1 + \mathbf{D}_2 \mathbf{n}_2. \quad (8)
\end{aligned}
$$

For $i = F_s$, and $\delta \boldsymbol{x}_0 = \mathbf{0}$, the terms in eqn. (8) can be defined as

$$
\begin{aligned}
\mathbf{w}_{k-1} &= \left\{ \sum_{i=0}^{F_s-2} \left(\prod_{j=i+1}^{F_s-1} \mathbf{C}_j \right) \mathbf{D}_i \mathbf{n}_i \right\} + \mathbf{D}_{F_s-1} \mathbf{n}_{F_s-1} \quad (9) \\
&= \boldsymbol{\Gamma} \boldsymbol{\eta}.
\end{aligned}
$$

Let the product of $\mathbf{C}_j$ in eqn. (9) be defined as

$$
\mathbf{C}_j^p \triangleq \begin{cases} \prod_{j=q}^{p} \mathbf{C}_j = \mathbf{C}_p \cdots \mathbf{C}_{q-1}\mathbf{C}_q & \text{for } q \neq p \\ \mathbf{C}_q & \text{for } q = p \end{cases} \quad (10)
$$

where product operation in eqn. (10) must follow the order of operations shown in eqn. (8). Let $\boldsymbol{\Gamma}$ and $\boldsymbol{\eta}$ be defined as

$$
\begin{aligned}
\boldsymbol{\Gamma} &\triangleq \left[\mathbf{C}_1^{F_s-1}\mathbf{D}_0, \mathbf{C}_2^{F_s-1}\mathbf{D}_1, \dots, \mathbf{C}_{F_s-1}^{F_s-1}\mathbf{D}_{F_s-2}, \mathbf{D}_{F_s-1} \right] \\
\boldsymbol{\eta} &\triangleq [\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_{F_s-1}].
\end{aligned}
$$

C. Summary

Combining the results from Sections II-A and II-B, the linear state transition error model over $t \in [t_{k-1}, t_k]$ is

$$
\delta \boldsymbol{x}_k = \boldsymbol{\Upsilon}_{k-1} \delta \boldsymbol{x}_{k-1} + \mathbf{w}_{k-1}. \quad (11)
$$

with

$$
\begin{aligned}
\mathbf{Q}_\mathbf{D} &= \mathrm{Cov}(\mathbf{w}_{k-1}) \in \mathbb{R}^{n_e \times n_e} \\
&= E \left\langle \boldsymbol{\Gamma} \boldsymbol{\eta} \boldsymbol{\eta}^\intercal \boldsymbol{\Gamma}^\intercal \right\rangle \\
&= E \left\langle \boldsymbol{\Gamma} \begin{bmatrix} \boldsymbol{\eta}_0 \\ \boldsymbol{\eta}_1 \\ \vdots \\ \boldsymbol{\eta}_{F_s-1} \end{bmatrix} [\boldsymbol{\eta}_0 \ \boldsymbol{\eta}_1 \ \cdots \ \boldsymbol{\eta}_{F_s-1}] \boldsymbol{\Gamma}^\intercal \right\rangle \\
&= \boldsymbol{\Gamma} \begin{bmatrix} \mathbf{Q}_{\mathbf{d},0} & & \\ & \ddots & \\ & & \mathbf{Q}_{\mathbf{d},F_s-1} \end{bmatrix} \boldsymbol{\Gamma}^\intercal, \quad (12)
\end{aligned}
$$

where $\mathbf{Q}_{\mathbf{d},i} = \boldsymbol{\eta}_i \boldsymbol{\eta}_i^\intercal$. The stochastic properties of eqn. (12) are well understood, and can be found in Sections 4.7 and 7.2.5.2 of [4].

References

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