Modified Strapdown Inertial Navigator Error Models

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ABSTRACT

This paper revisits the problem of error modeling for strapdown INS for the purpose of navigation sensor blending with a Kalman filter. This problem has been addressed repeatedly over the last 30 years, and different perspectives on INS error modeling have emerged. This paper reviews these, focussing in particular on the properties and relative advantages and disadvantages of the ϕ-angle and ψ-angle error models. This paper then addresses some new concepts for the design of a Kalman filter model for integrated navigation. The sometimes troublesome problem of the explicit occurrence of the specific force in both error models is averted with the modified ϕ-angle and ψ-angle error models proposed herein, in which the explicit representation of the specific force is canceled via a transformation of the velocity error states.

INTRODUCTION

Two dominant strapdown navigator error models have emerged in the literature: the ϕ-angle and ψ-angle error models. Both models describe the same strapdown navigator errors in different frames of reference, hence they are equivalent. Benson [1] proved their equivalence analytically and using simulations. Goshen-Meskin and Bar-Itzhack [2] extended Benson's analysis and presented a unified approach to inertial navigation system error modeling.

Both error models include in their respective descriptions of acceleration errors the misresolved specific force due to platform misalignments. This acceleration error term contains in both cases an explicit occurrence of the specific force vector. This can present some practical difficulties in implementing the transition matrix for a discrete-time Kalman filter. If the vehicle exhibits high dynamic bandwidth, then a direct implementation of either model requires a high-speed integration in the transition matrix computation to adequately capture the specific force bandwidth. This integration already occurs in the strapdown navigator.

This paper uses the following notation. x^a denotes the vector x resolved in the a-coordinate frame. C_b^a denotes the direction cosine matrix (DCM) from the b-frame to the a-frame. The following coordinate frames are defined: the b-frame is the inertial measurement unit (IMU) *body frame*, and is assumed here to define the gyro and accelerometer input axes. The a-frame is the *true (wander angle) navigation frame*, which is locally level at the true navigator position with a well-defined wander azimuth with respect to true North. The strapdown navigator maintains a DCM from the body frame to its estimate of the wander angle frame defined as follows: $\hat{C}_b^a = C_b^p$. The p-frame is called the *platform frame*, and is misaligned with respect to the true wander angle frame. The *computer frame* (c-frame) is the navigation frame maintained by the strapdown navigator, and is the locally level navigation frame at the computed position.

ERROR MODELS

Phi-Angle Error Model

The ϕ-angle error model, given by (1a)-(1c), is obtained from a linear perturbation analysis of the strapdown inertial navigation equations with respect to a true navigation frame. This representation has certain advantages for error control of a strapdown navigator. A Kalman filter implementation of the ϕ-angle error model will have error states comprising position, velocity and misalignment errors whose estimates can be used directly to control the strapdown navigator integration processes, since the estimated errors are with respect to a true navigation frame. On the other hand, these state variables are linear combinations of the dominant rate-stable and Schuler dynamic modes. From an analysis perspective, these modes are not separated by this representation. Furthermore, this model has several terms requiring explicit representation in a Kalman filter implementation (for example Coriolis acceleration error due to earth and transport rate errors)

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which are not required in the ψ-angle error model.

$$\Delta \dot{\boldsymbol{r}}_1^a = -\boldsymbol{\rho}_a^a \times \Delta \boldsymbol{r}_1^a + \boldsymbol{\delta\theta} \times \boldsymbol{v}^a + \Delta \boldsymbol{v}_1^a \tag{1a}$$

$$\Delta \dot{\boldsymbol{v}}_1^a = \boldsymbol{f}^a \times \boldsymbol{\phi} + \Delta \boldsymbol{g}_1^a - \left(2\boldsymbol{\Omega}^a + \boldsymbol{\rho}_a^a\right) \times \Delta \boldsymbol{v}_1^a - \left(2\Delta\boldsymbol{\Omega}^a + \Delta\boldsymbol{\rho}^a\right) \times \boldsymbol{v}_1^a + \Delta \boldsymbol{f}^a \tag{1b}$$

$$\dot{\boldsymbol{\phi}} = -\left(\boldsymbol{\Omega}^a + \boldsymbol{\rho}_a^a\right) \times \boldsymbol{\phi} + \left(\Delta\boldsymbol{\Omega}^a + \Delta\boldsymbol{\rho}^a\right) - \boldsymbol{\varepsilon}^a \tag{1c}$$

where:

- $\boldsymbol{v}^a$ is the true velocity of the navigation frame,
- $\boldsymbol{\rho}_a^a$ is the transport rate vector (angular rate with respect to the earth) of the true navigation frame,
- $\boldsymbol{\Omega}$ is the earth rate vector,
- $\boldsymbol{f}$ is the specific force vector comprising gravity plus vehicle accelerations,
- $\Delta \boldsymbol{f}^a$ is the vector of accelerometer errors resolved in the true navigation frame,
- $\boldsymbol{\varepsilon}^a$ is the vector of gyro errors resolved in the true navigation frame,
- $\boldsymbol{\delta\theta}$ is the misalignment vector of the computer frame with respect to the true navigation frame as a consequence of position error, and the following dynamics [1]:

$$\dot{\boldsymbol{\delta\theta}} = -\boldsymbol{\rho}^c \times \boldsymbol{\delta\theta} + \boldsymbol{\delta\rho}^a \tag{2}$$

$\boldsymbol{\delta\theta}$ can be characterized as follows:

$$\delta\theta_x = -\frac{\Delta r_y^a}{r_e + h} \tag{3a}$$

$$\delta\theta_y = \frac{\Delta r_x^a}{r_e + h} \tag{3b}$$

$$\delta\theta_z = \Delta\lambda \sin L + \Delta\alpha_c \tag{3c}$$

where:

- $r_e + h$ is the IMU's distance from the earth's centre,
- $\Delta\lambda$ is the computed latitude error,
- L is the IMU longitude
- $\Delta\alpha$ is the error in computed wander angle, given by α_p-α_c where α_p is the actual platform wander angle and α_c is the computed wander angle.

Psi-Angle Error Model

The ψ-angle error model, given by (4a)-(4c), is obtained from a linear perturbation analysis of the strapdown inertial navigation errors in the computer frame, which is the navigation frame maintained by the inertial navigator. The computer frame is a "known" reference frame, hence perturbations of the computer frame angular position and angular rate are zero [2]. This leads to a simpler model than the ϕ-angle error model.

$$\Delta \dot{\boldsymbol{r}}_2^c = -\boldsymbol{\rho}_c^c \times \Delta \boldsymbol{r}_2^c + \Delta \boldsymbol{v}_2^c \tag{4a}$$

$$\Delta \dot{\boldsymbol{v}}_2^c = \boldsymbol{f}^c \times \boldsymbol{\psi} + \Delta \boldsymbol{g}_2^c - \left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}_c^c\right) \times \Delta \boldsymbol{v}_2^c + \Delta \boldsymbol{f}^c \tag{4b}$$

$$\dot{\boldsymbol{\psi}} = -\left(\boldsymbol{\Omega}^c + \boldsymbol{\rho}_c^c\right) \times \boldsymbol{\psi} - \boldsymbol{\varepsilon}^c \tag{4c}$$

where $\boldsymbol{\rho}_c^c$ is the transport rate of the computer frame.

Relationship Between Error Models

The ϕ and ψ misalignments are related as follows:

$$\boldsymbol{\phi} = \boldsymbol{\psi} + \boldsymbol{\delta\theta} \tag{5}$$

The velocity errors are related as follows:

$$\Delta \boldsymbol{v}_2^a = \Delta \boldsymbol{v}_1^c - \boldsymbol{\delta\theta} \times \boldsymbol{v}^c \tag{6}$$

$\dot{\boldsymbol{\psi}} = \boldsymbol{\omega}_{cp}^p$ describes the angular rate of the platform frame with respect to the computer frame, and comprises only the misresolved navigation frame angular rate plus the gyro errors. The ψ-angle dynamics are for this reason called the *rate-stable dynamics*. By contrast, $\dot{\boldsymbol{\phi}} = \boldsymbol{\omega}_{ap}^p$ contains both the rate-stable dynamics and the Schuler dynamics.

The gravity errors are related as follows:

$$\Delta \boldsymbol{g}_2^c = \Delta \boldsymbol{g}_1^a + \boldsymbol{\delta\theta} \times \boldsymbol{g}^c \tag{7a}$$

$\Delta \boldsymbol{g}_2$, thus describes the errors in computed gravity $\Delta \boldsymbol{g}_1$ due to altitude error Δr_z given by:

$$\Delta \boldsymbol{g}_1^a \cong \begin{bmatrix} 0 & 0 & \dfrac{2g\Delta r_z}{r_e + h} \end{bmatrix}^T = diag\begin{bmatrix} 0 & 0 & 2\omega_s^2 \end{bmatrix} \Delta \boldsymbol{r}^c \tag{7b}$$

plus the Schuler dynamics $\boldsymbol{\delta\theta} \times \boldsymbol{g}$ given by:

$$\boldsymbol{\delta\theta} \times \boldsymbol{g}^c \cong \begin{bmatrix} \frac{g\Delta r_y}{r_e + h} & \frac{g\Delta r_x}{r_e + h} & 0 \end{bmatrix}^T = diag\begin{bmatrix}\omega_s^2 & \omega_s^2 & 0\end{bmatrix} \boldsymbol{\Delta r}^c \tag{7c}$$

where ω_s is the Schuler frequency given by:

$$\omega_s = \sqrt{\frac{g}{r_e + h}}$$

Modified Error Models

Both error models (1) and (4) contain explicit occurrences of the specific force $\boldsymbol{f}$ that the vehicle experiences in order to describe the misresolved specific force. In order to implement either model in a Kalman filter, some means of estimating the specific force must be implemented. If the raw IMU data are available, then the specific force vector can be computed directly from the incremental velocities generated by the IMU. If low-grade inertial sensors are used, then the directly measured specific force vector from the accelerometer triad may be noisy and prone to large errors. If these are not available, as is likely the case for most INS's and AHRS's, then the specific force must be computed from the estimated gravity and the computed velocity.

In either case, the implementation of either error model in a transition matrix in a discrete-time Kalman filter will require a high-speed integration to adequately capture the bandwidth of the specific force. This high-speed integration is already being performed by the strapdown navigator. It therefore makes sense to use this fact in designing a practical error model for use in a Kalman filter.

Modified Psi-Angle Error Model

The following velocity error transformation is defined for the purpose of canceling the specific force term:

$$\boldsymbol{\delta v} = \Delta \boldsymbol{v}_I^c + \boldsymbol{\psi} \times \hat{\boldsymbol{v}} = \Delta \boldsymbol{v}_I^c - \hat{\boldsymbol{v}} \times \boldsymbol{\psi} \tag{8}$$

where $\hat{\boldsymbol{v}}$ is the computed velocity. This can be described as the difference between the computed velocity resolved in the platform frame and the true velocity resolved in the true navigation frame, i.e. $\boldsymbol{\delta v} = \hat{\boldsymbol{v}}^p - \boldsymbol{v}^a$. Strictly speaking, this is not a true velocity error, since it cannot be described as a computed minus true velocity in a common reference frame (except of course when the vehicle velocity is zero). In a broader sense, (8) can be considered a velocity error.

The time derivative is given by:

$$\Delta \dot{\boldsymbol{v}}^c = \boldsymbol{\delta \dot{v}} + \dot{\hat{\boldsymbol{v}}} \times \boldsymbol{\psi} + \hat{\boldsymbol{v}} \times \dot{\boldsymbol{\psi}} \tag{9}$$

The strapdown navigator computes a velocity from the transformed specific force vector resolved in the platform frame given by:

$$\dot{\hat{\boldsymbol{v}}} = \boldsymbol{f}^p + \hat{\boldsymbol{g}} - \left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \hat{\boldsymbol{v}} \tag{10}$$

which when substituted with (4c) into (9) yields:

$$\begin{aligned} \Delta \dot{\boldsymbol{v}}^c = {} & \boldsymbol{\delta \dot{v}} + \boldsymbol{f}^p \times \boldsymbol{\psi} + \hat{\boldsymbol{g}} \times \boldsymbol{\psi} - \left(\left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \hat{\boldsymbol{v}}\right) \times \boldsymbol{\psi} \\ & - \hat{\boldsymbol{v}} \times \left(\left(\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \boldsymbol{\psi}\right) - \hat{\boldsymbol{v}} \times \hat{C}_b^a \boldsymbol{\varepsilon}^b \end{aligned} \tag{11}$$

Substituting (8) and (4b) into (11) and canceling the $\boldsymbol{f}^p \times \boldsymbol{\psi}$ terms on both sides (which is the whole purpose of this transformation) results in the following expression:

$$\begin{aligned} & \boldsymbol{\delta \dot{v}} + \hat{\boldsymbol{g}} \times \boldsymbol{\psi} - \left(\left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \hat{\boldsymbol{v}}\right) \times \boldsymbol{\psi} \\ & - \hat{\boldsymbol{v}} \times \left(\left(\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \boldsymbol{\psi}\right) - \hat{\boldsymbol{v}} \times \hat{C}_b^b \boldsymbol{\varepsilon}^b \\ & = \hat{C}_b^a \boldsymbol{\Delta a}^b + \boldsymbol{\Delta g}_2^c - \left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \left(\boldsymbol{\delta v} + \hat{\boldsymbol{v}} \times \boldsymbol{\psi}\right) \end{aligned} \tag{12}$$

where $\boldsymbol{\Delta g}^c$ is given by (7). (12) contains the following expression assembled from terms on both sides of the equation:

$$\begin{aligned} & \left(\left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \hat{\boldsymbol{v}}\right) \times \boldsymbol{\psi} + \hat{\boldsymbol{v}} \times \left(\left(\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \boldsymbol{\psi}\right) \\ & - \left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \left(\hat{\boldsymbol{v}} \times \boldsymbol{\psi}\right) \\ & = \left(\boldsymbol{\omega} \times \hat{\boldsymbol{v}}\right) \times \boldsymbol{\psi} + \hat{\boldsymbol{v}} \times \left(\boldsymbol{\omega} \times \boldsymbol{\psi}\right) - \boldsymbol{\omega} \times \left(\hat{\boldsymbol{v}} \times \boldsymbol{\psi}\right) \\ & - \hat{\boldsymbol{v}} \times \left(\boldsymbol{\Omega}^c \times \boldsymbol{\psi}\right) \\ & = - \hat{\boldsymbol{v}} \times \left(\boldsymbol{\Omega}^c \times \boldsymbol{\psi}\right) \end{aligned} \tag{13}$$

where $\boldsymbol{\omega} = 2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c$.

The transformed velocity error model becomes:

$$\begin{aligned} \boldsymbol{\delta \dot{v}} = {} & \hat{C}_b^a \boldsymbol{\Delta a}^b + \boldsymbol{\Delta g}_2^c - \hat{\boldsymbol{g}} \times \boldsymbol{\psi} - \left(2\boldsymbol{\Omega}^c + \boldsymbol{\rho}^c\right) \times \boldsymbol{\delta v} \\ & - \hat{\boldsymbol{v}} \times \left(\boldsymbol{\Omega}^c \times \boldsymbol{\psi}\right) + \hat{\boldsymbol{v}} \times \hat{C}_b^a \boldsymbol{\varepsilon}^b \end{aligned} \tag{14a}$$

The position error model is obtained from substitution of (8) into (4a) to obtain:

$$\Delta\dot{r}^c = -\hat{\rho}\times\Delta r^c + \delta v + \hat{v}\times\psi \tag{14b}$$

The ψ-misalignment dynamics are those given by (4c) and repeated here for completeness:

$$\dot{\psi} = -\left(\Omega^c + \rho^c_c\right)\times\psi - \varepsilon^c \tag{14c}$$

To recover the true velocity error with respect to the true navigation frame for the purpose of controlling the strapdown navigator errors, the transformations (6) and (8) are inverted as follows:

$$\Delta v^a_I = \delta v - \psi\times\hat{v} - \delta\theta\times\hat{v} = \delta v - \phi\times\hat{v} \tag{15}$$

(14a,b,c) is called the *modified ψ-angle error model*. This model can be constructed from data readily available from the strapdown navigator or INS output.

Modified Phi-Angle Error Model

(1a,b,c) is the result of a well-defined linear perturbation analysis about the true navigation solution, hence there exist the terms Ω^a, ρ^a_a and v^a in (1a,b,c) which cannot be known exactly. These are replaced with Ω^c, ρ^c_c and $\hat{v}$ in the subsequent analysis, since these are what the strapdown navigator generates. The errors introduced with these replacements comprise products of perturbation quantities, which can be neglected consistent with the ground rules of linear perturbation analysis.

The previous approach to canceling the explicit specific force term can be applied to the ϕ-angle error model. The velocity transformation is given by:

$$\delta v = \Delta v^a_I + \phi\times\hat{v} \tag{16}$$

The development (8)-(13) is repeated for (1b) with (16) defining the velocity transformation. The resulting velocity error is given by:

$$\begin{aligned}\delta\dot{v} = \hat{C}^a_b\Delta a^b + \Delta g^a_I - \hat{g}\times\phi - \left(2\Omega^c + \rho^c_c\right)\times\delta v \\ -\hat{v}\times\left(\Omega^c\times\phi\right) + \hat{v}\times\Delta\Omega^a + \hat{v}\times\hat{C}^a_b\varepsilon^b\end{aligned} \tag{17a}$$

The position error is given by:

$$\Delta\dot{r}^a_I = -\rho^c_c\times\Delta r^a_I + \delta v + \left(\delta\theta - \phi\right)\times\hat{v} \tag{17b}$$

where

$$\dot{\phi} = -\left(\Omega^c + \rho^c_c\right)\times\phi + \left(\Delta\Omega^a + \Delta\rho^a\right) - \hat{C}^a_b\varepsilon^b \tag{17c}$$

and $\delta\theta$ is given by (3). (17a,b,c) is called the *modified ϕ-angle error model.*

The velocity transformation (16) is in fact the reverse of (15). This suggests that the modified ϕ-angle error model must be equivalent to the previous modified ψ-angle error model. The following is a demonstration of this equivalence. Equating the right-hand sides of (14a) and (17a) and canceling terms, there results:

$$\begin{aligned}\Delta g^a_I - \hat{g}\times\phi - \hat{v}\times\left(\Omega^c\times\phi\right) + \hat{v}\times\Delta\Omega^a \\ = \Delta g^c_2 - \hat{g}\times\psi - \hat{v}\times\left(\Omega^c\times\psi\right)\end{aligned} \tag{18}$$

which with (5) and (7a) reduces to a known relationship:

$$-\delta\theta\times\Omega^c = \Delta\Omega^a \tag{19}$$

(17b) is the same equation as (14b) when (5) is used in (17b). The equivalence of (17c) with (14c) and (2), (5) has been demonstrated in [1] and is not repeated here. This demonstrates the equivalence of the modified ϕ and ψ-angle error models.

CONCLUSIONS

Modified versions of the ϕ and ψ-angle error models have been developed, in which the explicit occurrence of the specific force in the acceleration error equations have been canceled by a velocity error transformation. This removes the need for a high-speed integration of the specific force in a Kalman filter transition matrix computation. This high-speed integration is performed by the strapdown navigator, and the velocity transformation (8) makes use of this fact.

The modified error models were shown to be equivalent. The choice of which modified error model should be used is dependent on practical considerations of implementation. The modified ψ-angle error model is the simpler of the two, and therefore appears to be the better choice. This model has been used in several successful Kalman filter implementations [3]-[6], including a real-time Kalman filter for the Helicopter Integrated Navigation System (HINS) [3].

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