Inertial Navigator Error Models For Large Heading Uncertainty

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Abstract - This paper is concerned with the development of a modified ψ-angle error model for a strapdown inertial navigator whose platform azimuth is completely unknown, as would be the case during initial coarse heading alignment. The paper is a review of [5], and provides a framework for developing large azimuth uncertainty error models. It then presents the development of a ψ-angle error model as a generalization of the ψ-angle small error model described by [1], and a modified ψ-angle error model as a generalization of the modified ψ-angle small error model presented in [2], [3].

INTRODUCTION

Heading initialization in an INS or aided-INS is typically considered a transient function to be performed once during an initial ground alignment or in-air alignment. Its purpose is to bring the initial heading error down to a magnitude that is adequate for free-inertial navigation or is consistent with the small error assumption used by a fine alignment Kalman filter. The problem of aided INS alignment in-air or otherwise in-motion using a Kalman filter has been addressed in numerous papers over the last 20 years. Most of these assume some form of coarse heading initialization to within a few degrees accuracy that admits the use of a fine alignment Kalman filter. [4] presents a complete approach to INS airstart without heading initialization, and can be considered an alternative approach to that presented here, differing in the way heading uncertainty is modeled.

This paper is concerned with the development of a modified ψ-angle error model for a strapdown navigator whose platform horizontal misalignments are small and whose heading error is arbitrary on $[-\pi,\pi]$. The analysis given in this paper seeks to avoid any assumptions about the vehicle dynamics or the quality of the inertial sensors to the extent possible. The development given here is a generalization of the ψ-angle error model [1] and the modified ψ-angle error model [2], [3]. The term "modified" describes a velocity error transformation which effects the cancellation of the explicit occurrence of specific force in the ψ-angle error model. The intention of the large heading error model formulation given here is to transition to the small-error modified ψ-angle error model [2], [3] with as little disruption as possible when the heading uncertainty becomes sufficiently small. A single Kalman filter implementation can then be designed to handle both large and small heading errors.

Two error models result from this analysis, depending on the type of constructed position and/or velocity measurement that the error model must describe. One model is called a *platform-consistent* (PC) error model, which describes velocity errors in terms of a platform-referenced velocity as obtained by transforming a body-referenced velocity to the platform frame. The body-to-platform direction cosine matrix (DCM) generated by the strapdown navigator is known without error, hence body and platform frame referenced velocities are equivalent within the known DCM transformation. The other model is called a *geographic-consistent* (GC) error model, which describes velocity errors in terms of velocity resolved in the computed navigation frame, or computer frame, as obtained by transforming geographic velocity to the computer frame. The geographic to computer frame DCM generated by the strapdown navigator is known without significant error if the strapdown navigator position errors are assumed to be sufficiently small. The geographic and computer frame referenced velocities are then equivalent within the known DCM transformation. Both the PC and GC error models converge to the same small error modified ψ-angle error model as the azimuth error diminishes.

This paper begins with a presentation of background material comprising a mixture of information from past publications and development of some useful identities. Some definitions are well-known to practitioners of aided INS design, and are included here for completeness. The PC and GC large heading error ψ-angle error models are then developed in a manner that parallels the small error ψ-angle error model development given in [1]. The modified large heading error ψ-angle error models are then derived via velocity transformations that parallel [2], [3]. Finally, some concluding remarks and useful observations are presented.

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BACKGROUND

Coordinate Systems

The following coordinate frame definitions are required in the subsequent development. These are standard definitions in the literature and are included here for completeness.

The *body frame* (*b-frame*) is the frame in which the accelerations and angular rates generated by the strapdown accelerometers and gyros are resolved. To maintain simplicity, no distinction is made here between inertial sensor axes and inertial measurement unit (IMU) body axes.

The *true wander angle navigation frame* (*a-frame*) is the true local level navigation frame at the true position. The *earth frame* (*e-frame*) is fixed to the rotating earth with the x-axis coincident with the polar axis and the z-axis intersecting the equator at the Greenwich meridian.

The *computer frame* (*c-frame*) is the frame which the INS computer assumes to be the true navigation frame. It is locally level at the computed position, and hence differs in orientation from the true wander angle frame by misalignment angle $\delta\theta$ comprising the position and wander angle errors. The computed wa-to-earth frame DCM is in fact the computer-to-earth frame DCM, i.e.

$$\hat{C}_a^e = C_c^e \tag{1}$$

The *platform frame* (*p-frame*) is the frame in which the transformed accelerations from the accelerometers and angular rates from the gyros are resolved. It differs in orientation from the true wander angle frame by misalignment angle ϕ. The computed body-to-wa frame DCM is in fact the body-to-platform DCM, i.e.

$$\hat{C}_b^a = C_b^p \tag{2}$$

The *computed platform frame* (*c'-frame*) is a coordinate frame whose x-y plane is coplanar with the p-frame and whose azimuth coincides with the c-frame azimuth. This implies that:

$$C_c^p = C_{c'}^p C_c^{c'} = \left(I - \left(\bar{\psi}_{xy} \times\right)\right) C_c^{c'} \tag{3}$$

where $\bar{\psi}_{xy} = \begin{bmatrix} \psi_x & \psi_y & 0 \end{bmatrix}^T$

and

$$C_c^{c'} = \begin{bmatrix} \cos\psi_z & \sin\psi_z & 0 \\ -\sin\psi_z & \cos\delta\psi_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The sequence of rotations in (3) implies the angle $\bar{\psi}$ describes the misalignment of the platform frame with respect to the computer frame. Also $\bar{\phi} = \bar{\psi} + \delta\bar{\theta}$.

Small Error ψ-Angle Error Model [1]

Benson [1] gives a first order (linear) perturbation analysis of the strapdown inertial navigation errors in the computer frame which leads to the small-error ψ-angle error model. The true strapdown velocity equation in the computer frame is given by:

$$\dot{\bar{v}}_t^c = \bar{f}_t^c + \bar{g}_t^c - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \bar{v}_t^c \tag{4}$$

where $\bar{\Omega}^c$ is the earth rate resolved in the computer frame,

$\bar{\rho}_c^c$ is the transport rate of the computer frame resolved in the computer frame,

$\bar{f}_t^c$ is the true specific force resolved in the computer frame,

$\bar{g}_t^c$ is the true gravity vector resolved in the computer frame.

The strapdown navigator computes the following velocity equation:

$$\dot{\hat{\bar{v}}} = \bar{f}_t^p + \delta\bar{f}^p + \hat{\bar{g}} - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \hat{\bar{v}} \tag{5}$$

where $\bar{f}_t^p$ is the true specific force resolved in the platform frame,

$\delta\bar{f}^p$ is the specific force error due to accelerometer errors,

$\hat{\bar{g}}$ is the computed gravity vector.

The computer frame is a "known" reference frame, hence perturbations of the computer frame angular position and angular rate are zero [1]. The computer frame's angular rate $\bar{\Omega}^c + \bar{\rho}_c^c$ is by definition of the computer frame known without error. The standard small-error ψ-angle error model derived in [1] is given by the following equations:

$$\Delta\dot{\vec{r}}^c = -\vec{\rho}_c^c \times \Delta\vec{r}^c + \Delta\vec{v}^c \tag{6}$$

$$\Delta\dot{\vec{v}}^c = \vec{f}_t^p \times \vec{\psi} + \delta\vec{f}^p + \Delta\vec{g}^c - \left(2\vec{\Omega}^c + \vec{\rho}_c^c\right) \times \Delta\vec{v}^c \tag{7}$$

$$\dot{\vec{\psi}} = -\left(\vec{\Omega}^c + \vec{\rho}_c^c\right) \times \vec{\psi} - \vec{\varepsilon}^p \tag{8}$$

where $\Delta\vec{r}^c$ is the position error resolved in the computer frame,

$\Delta\vec{v}^c$ is the velocity error resolved in the computer frame,

$\vec{\psi}$ is the platform misalignment vector resolved in the platform frame,

$\vec{\varepsilon}^p$ is the vector of gyro errors resolved in the platform frame,

The gravity error is due to the misresolution of gravity by the tilted computer frame, and is given by:

$$\Delta\vec{g}^c = \begin{bmatrix} -\frac{\Delta r_x^c}{r_e} & -\frac{\Delta r_y^c}{r_e} & \frac{2\Delta r_z^c}{r_e} \end{bmatrix}^T g$$

$$= \begin{bmatrix} -\omega_s^2 \Delta r_x^c & -\omega_s^2 \Delta r_y^c & 2\omega_s^2 \Delta r_z^c \end{bmatrix} \tag{9}$$

where ω_s is the Schuler frequency. Note that this term describes the Schuler oscillations of both the platform frame and the computer frame. The misalignment vector $\vec{\psi}$ describing the misalignment of the platform frame with respect to the computer frame is driven only by the misresolved earth and transport rates and the gyro errors. The $\vec{\psi}$ misalignment vector is thus called *rate-stable* in that it does not exhibit the marginally stable dynamics of the Schuler loop.

Definition of Heading Error

Figure 1 shows the relationships among the following computed and true azimuth components: platform heading ψ_p, wander angle α and true heading ψ_t. The platform azimuth and wander angle in a strapdown navigator can be arbitrarily defined within the constraint that their difference equals the true heading. The heading error can then be characterized as follows:

$$\delta\psi = \hat{\psi} - \psi_t = \left(\hat{\psi}_p - \hat{\alpha}\right) - \left(\psi_p - \alpha\right)$$

$$= \delta\psi_p - \delta\alpha$$

where $\hat{\psi}$ is the computed heading. The degree of freedom in defining heading results in a degree of freedom in defining heading error. This is used here to simplify the subsequent analysis by defining the platform and true wander angle frames to have the same azimuth. This implies that the platform azimuth is known without error, i.e. $\delta\psi_p = 0$, and that the platform azimuth error with respect to the true navigation frame is zero, i.e. $\phi_z = 0$. By making this definition, only one azimuth error needs to be considered. The heading error is then given by $\delta\psi = -\delta\alpha$ where $\delta\alpha$ is the error in the computed wander angle. This implies

$$\psi_p = \psi_t + \alpha = \psi_t + \hat{\alpha} + \psi_z - \delta\theta_D$$

$$\delta\psi = -\delta\alpha = \psi_z - \delta\theta_D$$

If the strapdown navigator position error is controlled by an accurate aiding position, then $\delta\theta_D$ is negligibly small and the heading error is approximated by the platform azimuth misalignment with respect to the computer frame ψ_z, hereafter referred to simply as the *azimuth misalignment*. Because of the above construction, azimuth misalignment and heading error are assumed to be approximately equivalent.

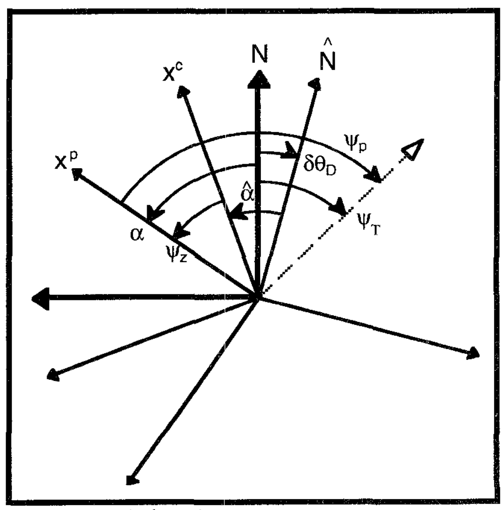

Figure 1: Azimuth errors

Lemmas and Identities

The following identities involving (3) are useful in the subsequent development, and can be demonstrated by simple algebraic expansion:

Lemma 1

Define the "extended" misalignment vector $\bar{\psi}_e$ as follows:

$$\bar{\psi}_e \equiv \begin{bmatrix} \psi_x \\ \psi_y \\ \sin\psi_z \\ \cos\psi_z - 1 \end{bmatrix} \tag{10}$$

Define the matrix representation of the "extended cross-product type (+)" operator as follows:

$$(\bar{a}\times)_{e+} \equiv \left[\begin{array}{c|c} (\bar{a}\times) & \begin{matrix} a_x \\ a_y \\ 0 \end{matrix} \end{array}\right] \tag{11}$$

Then for any 3×1 vector $\bar{a} = [a_x, a_y, a_z]^T$

$$\left(C_c^p - I\right)\bar{a} = (\bar{a}\times)_{e+}\bar{\psi}_e + \bar{f}_z(\bar{a},\bar{\psi}) \tag{12}$$

where

$$\bar{f}_z(\bar{a},\bar{\psi}) \equiv \begin{bmatrix} 0 \\ 0 \\ f_z(\bar{a},\bar{\psi}) \end{bmatrix} \tag{13}$$

and

$$\begin{aligned} f_z(\bar{a},\bar{\psi}) \equiv\ & a_x\left(\psi_x \sin\psi_z + \psi_y(\cos\psi_z - 1)\right) \\ & + a_y\left(\psi_y \sin\psi_z - \psi_x(\cos\psi_z - 1)\right) \end{aligned} \tag{14}$$

A similar result applies to $I - C_p^c$ given by:

$$\left(I - C_p^c\right) = -\left(C_c^p - I\right)^T$$

Lemma 2

Define the matrix representation of the "extended cross-product type (-)" operator as follows:

$$(\bar{a}\times)_{e-} \equiv \left[\begin{array}{c|c} (\bar{a}\times) & \begin{matrix} -a_x \\ -a_y \\ 0 \end{matrix} \end{array}\right] \tag{15}$$

Then for any 3×1 vector $\bar{a} = [a_x, a_y, a_z]^T$

$$\left(I - C_p^c\right)\bar{a} = (\bar{a}\times)_{e-}\bar{\psi}_e - \bar{f}_{xy}(\bar{a},\bar{\psi}) \tag{16}$$

where

$$\begin{aligned} \bar{f}_{xy}(\bar{a},\bar{\psi}) &= \begin{bmatrix} f_x(\bar{a},\bar{\psi}) \\ f_y(\bar{a},\bar{\psi}) \\ 0 \end{bmatrix} \\ &= a_z \begin{bmatrix} \psi_x \sin\psi_z + \psi_y(\cos\psi_z - 1) \\ \psi_y \sin\psi_z - \psi_x(\cos\psi_z - 1) \\ 0 \end{bmatrix} \end{aligned} \tag{17}$$

Lemma 3

$$(\bar{a}\times)_{e+/-}\begin{bmatrix} \bar{b} \\ \hline 0 \end{bmatrix} = \bar{a}\times\bar{b} \tag{18}$$

Lemma 4

Define $\bar{\psi} = \begin{bmatrix} \psi_x & \psi_y & \sin\psi_z \end{bmatrix}^T$ and

$$\bar{\psi}_e \equiv \begin{bmatrix} \bar{\psi} \\ \cos\psi_z - 1 \end{bmatrix} \tag{19}$$

Then the following identities are apparent by inspection:

$$(\bar{a}\times)_{e+}\bar{\psi}_e = \bar{a}\times\bar{\psi} + \begin{bmatrix} a_x \\ a_y \\ 0 \end{bmatrix}(\cos\psi_z - 1) \tag{20}$$

$$(\bar{a}\times)_{e-}\bar{\psi}_e = \bar{a}\times\bar{\psi} - \begin{bmatrix} a_x \\ a_y \\ 0 \end{bmatrix}(\cos\psi_z - 1) \tag{21}$$

$$\left(\hat{\bar{g}}\times\right)_{e-}\bar{\psi}_e = \left(\begin{bmatrix} 0 \\ 0 \\ \hat{g} \end{bmatrix}\times\right)_{e-}\bar{\psi}_e = \hat{\bar{g}}\times\bar{\psi} \tag{22}$$

ψ-Angle Error Model Development

Velocity Errors

Following the perturbation analysis of Benson [1], subtracting (5) from (4) results in the following velocity error equation that is valid for any heading error:

$$\Delta\dot{\bar{v}}^c = \bar{f}_t^p - \bar{f}_t^c + \delta\bar{f}^p + \Delta\bar{g}^c - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right)\times\Delta\bar{v}^c \tag{23}$$

where $\Delta\bar{g}^c$ is given by (9). The misresolved specific force can be expressed in two different ways as follows:

$$\bar{f}_t^p - \bar{f}_t^c = \left(I - C_p^c\right)\bar{f}_t^p \tag{24}$$

$$= \left(C_c^p - I\right)\bar{f}_t^c \tag{25}$$

If the first representation (24) is used, then Lemma 2 leads to the following expression:

$$\begin{aligned}\bar{f}_t^p - \bar{f}_t^c &\cong \left(I - C_p^c\right)\hat{\bar{f}}^p \\ &= \left(\hat{\bar{f}}^p \times\right)_{e-}\bar{\psi}_e - \bar{f}_{xy}\left(\hat{\bar{f}}^p, \bar{\psi}\right)\end{aligned} \tag{26}$$

(26) describes the misresolved specific force in terms of the measured specific force $\hat{\bar{f}}^p$ resolved in the platform frame, which is directly accessible from the transformed strapdown accelerometers. Substituting (26) into (23) results in the following *PC velocity error model*:

$$\begin{aligned}\Delta\dot{\bar{v}}^c &\cong \left(\hat{\bar{f}}^p \times\right)_{e-}\bar{\psi}_e - \bar{f}_{xy}\left(\hat{\bar{f}}^p, \bar{\psi}\right) \\ &+\delta\bar{f}^p + \Delta\bar{g}^c - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right)\times\Delta\bar{v}^c\end{aligned} \tag{27}$$

This model contains a 'nonlinear term' $\bar{f}_{xy}\left(\hat{\bar{f}}^p, \bar{\psi}\right)$ containing products of desired error state variables as described by (17). This term is not negligible with respect to the other acceleration errors in (27) when the azimuth uncertainty is large. It can be neglected only when the azimuth uncertainty becomes sufficiently small to make the nonlinear term acceptably small against the remaining 'linear' acceleration error terms.

If the second representation (25) is used, then Lemma 1 leads to the following expression:

$$\bar{f}_t^p - \bar{f}_t^c = \left(\bar{f}_t^c \times\right)_{e+}\bar{\psi}_e + \bar{f}_z\left(\bar{f}_t^c, \bar{\psi}\right) \tag{28}$$

$\bar{f}_t^c$ is the specific force resolved in the computer frame, and can theoretically be derived by reversing the strapdown velocity equation (4) given an accurate measurement of velocity. Practically this is not feasible because of the large noise that differentiation of measured velocity would generate.

Substituting (28) into (23) leads to the following *GC velocity error model*:

$$\begin{aligned}\Delta\dot{\bar{v}}^c &\cong \left(\bar{f}_t^c \times\right)_{e+}\bar{\psi}_e + \bar{f}_z\left(\bar{f}_t^c, \bar{\psi}\right) \\ &+\delta\bar{f}^p + \Delta\bar{g}^c - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right)\times\Delta\bar{v}^c\end{aligned} \tag{29}$$

(29) contains a 'nonlinear term' $\bar{f}_z\left(\bar{f}_t^c, \bar{\psi}\right)$ in the vertical specific force error component as described by (13). The nonlinear term can be considered either negligible or approximately random with respect to the larger long-term correlated vertical acceleration errors such as gravity error and accelerometer bias if the platform horizontal tilts are small and the vehicle experiences only short-term horizontal accelerations. A linear error model can then be obtained for implementation in a Kalman filter by simply dropping the nonlinear term or replacing it with a random noise model.

Platform Misalignments

The following development characterizes the platform misalignment angular rate vector $\dot{\bar{\psi}}_e$. From the definition (3):

$$\begin{aligned}\dot{C}_p^c &= \dot{C}_{c'}^c\left(I + \left(\bar{\psi}_{xy}\times\right)\right) + C_{c'}^c\left(\dot{\bar{\psi}}_{xy}\times\right) \\ &= C_{c'}^c\left(I + \left(\bar{\psi}_{xy}\times\right)\right)\left(\bar{\omega}_{cp}^p\times\right)\end{aligned}$$

Hence

$$\begin{aligned}&\dot{C}_{c'}^c + C_{c'}^c\left(\dot{\bar{\psi}}_{xy}\times\right)\left(I - \left(\bar{\psi}_{xy}\times\right)\right) \\ &= C_{c'}^c\left(I + \left(\bar{\psi}_{xy}\times\right)\right)\left(\bar{\omega}_{cp}^p\times\right)\left(I - \left(\bar{\psi}_{xy}\times\right)\right)\end{aligned} \tag{30}$$

We assume that the ψ_x and ψ_y misalignment components and their angular rates are small so that products of these terms are negligible consistent with a linear perturbation analysis. This assumption is valid if the strapdown navigator is aligned to within a few arc-minutes of tilt error. (30) then simplifies to the following:

$$\frac{d}{dt}\left(C_{c'}^c - I\right) + C_{c'}^c\left(\dot{\bar{\psi}}_{xy}\times\right) \cong C_{c'}^c\left(\bar{\omega}_{cp}^p\times\right) \tag{31}$$

which implies

$$\dot{\psi}_x = \beta_x \qquad \dot{\psi}_y = \beta_y$$

$$\frac{d}{dt} \sin\psi_z = \beta_z \cos\psi_z = \beta_z + \beta_z(\cos\psi_z - 1) \tag{32}$$

$$\frac{d}{dt}(\cos\psi_z - 1) = -\beta_z \sin\psi_z$$

where $\vec{\omega}_{cp}^{p} = \begin{bmatrix} \beta_x & \beta_y & \beta_z \end{bmatrix}^T$ is the angular rate of the platform with respect to the computer frame, resolved in the platform frame.

Representations of the strapdown navigator's computations of the body-to-navigation frame DCM and navigation-to-earth DCM are given by:

$$\dot{\hat{C}}_b^a = \hat{C}_b^a\left(\hat{\vec{\omega}}_{ib}^b \times\right) - \left(\hat{\vec{\omega}}_{ia}^a \times\right)\hat{C}_b^a \tag{33}$$

$$\dot{\hat{C}}_a^e = \left(\hat{\vec{\omega}}_{ea}^a \times\right)\hat{C}_a^e = \left(\vec{\rho}_c^c \times\right)\hat{C}_a^e \tag{34}$$

where $\hat{\vec{\omega}}_{ib}^b$ is the angular rate vector generated by the strapdown gyros and $\hat{\vec{\omega}}_{ia}^a = \vec{\omega}_{ic}^c = \vec{\Omega}^c + \vec{\rho}_c^c$. Most strapdown navigator implementations propagate the equivalent quaternion and then extract the DCM from the quaternion. The rightmost term in (33) describes the strapdown navigator's rotation of the computed platform by the computed earth rate and computer frame transport rate to compensate the platform for these angular rates with respect to the inertial reference. (34) describes the strapdown navigator's rotation of the computer frame with respect to the earth frame. The computer frame therefore has an angular rate $\vec{\omega}_{ic}^c$ with respect to the inertial reference, which in the platform frame is given by $C_c^p\vec{\omega}_{ic}^c$. In addition the platform is rotated by the gyro errors. The resulting angular rate of the platform frame with respect to the computer frame is given by:

$$\begin{aligned}\vec{\omega}_{cp}^p &= \left(I - C_c^p\right)\vec{\omega}_{ic}^c - \varepsilon^p \\ &= -\left(\vec{\omega}_{ic}^c \times\right)_{e+}\vec{\psi}_e - \vec{f}_z\left(\vec{\omega}_{ic}^c, \vec{\psi}\right) - \varepsilon^p\end{aligned} \tag{35}$$

From (35) and the definition of $f_z(.,.)$ in (13) it is clear than the derivatives of *sin* ψ_z and *cos* ψ_z - *1* as given by (32) will have non-negligible products of desired error states that are comparable in magnitude to the states themselves. Hence linearization is not straight-forward. Rather, we note that although *sin* ψ_z and *cos* ψ_z - *1* are expected to be large initially, their rates of change will be small, admitting the following approximations:

$$\frac{d}{dt}\sin\psi_z \cong 0, \qquad \frac{d}{dt}(\cos\psi_z - 1) \cong 0 \tag{36}$$

When the azimuth misalignment ψ_z diminishes to below a selected threshold, these approximations are revised to their small error approximations given by:

$$\frac{d}{dt}\sin\psi_z \cong \beta_z, \qquad \frac{d}{dt}(\cos\psi_z - 1) \cong 0 \tag{37}$$

which become increasingly valid with decreasing ψ_z. From (35) and (36) the misalignment error model becomes the following for large azimuth misalignments:

$$\dot{\vec{\psi}}_e = -\begin{bmatrix} \left(\vec{\omega}_{ic}^c \times\right)_{e\pm} \\ \hline 0_{1\times 4} \end{bmatrix}\vec{\psi}_e + \begin{bmatrix} 0_{2\times 1} \\ \omega_x\psi_y - \omega_y\psi_y \\ \hline 0 \end{bmatrix} - \begin{bmatrix} \vec{\varepsilon}^p \\ \hline 0 \end{bmatrix} \tag{38}$$

and the following for smaller azimuth misalignments that fall below the selected threshold:

$$\dot{\vec{\psi}}_e = -\begin{bmatrix} \left(\vec{\omega}_{ic}^c \times\right)_{e\pm} \\ \hline 0_{1\times 4} \end{bmatrix}\vec{\psi}_e - \begin{bmatrix} \varepsilon^p \\ \hline 0 \end{bmatrix} \tag{39}$$

Position Errors

The development from [1] which describes the position error dynamics does not rely on small angle approximations. The position error dynamics model (6) therefore describes the position error dynamics for large azimuth uncertainty.

MODIFIED ψ-ANGLE ERROR MODEL TRANSFORMATION

Modified GC Velocity Error Equation

The following GC velocity error transformation is designed to cause the cancellation of the specific force term in the velocity error model (29):

$$\delta\vec{v} = \Delta\vec{v}^c - \left(\vec{v}^c \times\right)_{e+}\vec{\psi}_e \tag{40}$$

where $\vec{v}^c$ is the a best estimate of true velocity resolved in the computer frame. If geographically referenced velocity such as from GPS is available, then $\vec{v}^c$ can be obtained as

$\bar{v}^c = \hat{C}_g^a \bar{v}^g \cong C_g^c \bar{v}^g$. The dynamics of $\bar{v}^c$ as a function of specific force resolved in the computer frame are described by (4). Differentiating (40) on both sides results in:

$$\delta\dot{\bar{v}} = \Delta\dot{\bar{v}}^c - \left(\dot{\bar{v}}^c \times\right)_{e+} \bar{\psi}_e - \left(\bar{v}^c \times\right)_{e+} \dot{\bar{\psi}}_e \tag{41}$$

Substitute (40) and (4) into (41) and apply Lemmas 3 and 4 to obtain the GC velocity error transformation:

$$\begin{aligned} \Delta\dot{\bar{v}}^c &= \delta\dot{\bar{v}} + \left(\bar{f}^c \times\right)_{e+} \bar{\psi}_e + \bar{g}^c \times \bar{\psi} \\ &-\left(\left(\left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \bar{v}^c\right) \times\right)_{e+} \bar{\psi}_e \\ &-\bar{v}^c \times \left(\left(\left(\bar{\Omega}^c + \bar{\rho}_c^c\right) \times\right)_{e+} \bar{\psi}_e\right) - \bar{v}^c \times \bar{\varepsilon}^p \end{aligned} \tag{42}$$

where $\bar{\Psi}$ is defined by (19), i.e. the z-component of $\bar{\Psi}$ is $\sin \psi_z$. (41) and (42) are substituted into (29), and the $\left(\bar{f}^c \times\right)_{e+} \bar{\psi}_e$ terms on both sides are canceled to obtain the modified GC velocity error model given by:

$$\begin{aligned} \delta\dot{\bar{v}} &= \delta\bar{f}^p + \Delta\bar{g}^c - \hat{\bar{g}} \times \bar{\psi} - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \delta\bar{v} \\ &-\bar{v}^c \times \left(\bar{\Omega}^c \times \bar{\psi}\right) + \bar{v}^c \times \bar{\varepsilon}^p + \bar{f}_z\left(\bar{f}^c, \bar{\psi}\right) + \bar{f}_c\left(\bar{\psi}, \bar{v}^c\right) \\ &+ \begin{bmatrix} \Omega_y^c v_z \\ -\Omega_x^c v_z \\ -\left(3\Omega_x^c + 2\rho_x^c\right)v_y + \left(3\Omega_y^c + 2\rho_y^c\right)v_x \end{bmatrix} \left(\cos \psi_z - 1\right) \end{aligned} \tag{43}$$

where $\bar{v}^c = [v_x, v_y, v_z]^T$. (43) has the form of the small error modified ψ-angle error model described in [2], [3] with the addition of extra terms. The nonlinear term $\bar{f}_z\left(\bar{f}^c, \bar{\psi}\right)$ is negligibly small under most operational scenarios of non-accelerating motion, and appears only in the z-component of velocity error. $\bar{f}_c\left(\bar{\psi}, \bar{v}^c\right)$ is a correction function to account for the choice of misalignment error model (38) or (39). When (39) describes the misalignment dynamics, then $\bar{f}_c\left(\bar{\psi}, \bar{v}^c\right) = \bar{0}$. When the approximated misalignment dynamics (38) are used, then $\bar{f}_c\left(\bar{\psi}, \bar{v}^c\right)$ is given by:

$$\bar{f}_c\left(\bar{\psi}, \bar{v}^c\right) = \begin{bmatrix} \left(\Omega_y^c + \rho_y^c\right)v_y & -\left(\Omega_x^c + \rho_x^c\right)v_y & 0 \\ -\left(\Omega_y^c + \rho_y^c\right)v_x & \left(\Omega_x^c + \rho_x^c\right)v_x & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{\psi} \tag{44}$$

(43) converges to the small error model described in [2], [3] as the azimuth uncertainty diminishes.

Modified PC Velocity Error Equation

The PC velocity error transformation is given by:

$$\delta\bar{v} = \Delta\bar{v}^c - \left(\hat{\bar{v}} \times\right)_{e-} \bar{\psi}_e \tag{45}$$

where $\hat{\bar{v}}$ is the computed velocity given by the strapdown velocity equation (5) based on the measured specific force. A development similar to (40)-(43) involving (45) yields:

$$\begin{aligned} \Delta\dot{\bar{v}}^c &= \delta\dot{\bar{v}} + \left(\hat{\bar{f}}^p \times\right)_{e-} \bar{\psi}_e + \bar{g}^c \times \bar{\psi} \\ &-\left(\left(\left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \hat{\bar{v}}\right) \times\right)_{e-} \bar{\psi}_e \\ &-\hat{\bar{v}} \times \left(\left(\left(\bar{\Omega}^c + \bar{\rho}_c^c\right) \times\right)_{e+} \bar{\psi}_e\right) - \hat{\bar{v}} \times \bar{\varepsilon}^p \end{aligned} \tag{46}$$

where $\bar{\Psi}$ is defined by (19). Substitute (45) and (46) into (27) and then use Lemma 4 to obtain the modified velocity error model given by:

$$\begin{aligned} \delta\dot{\bar{v}} &= \delta\bar{f}^p + \Delta\bar{g}^c - \hat{\bar{g}} \times \bar{\psi} - \left(2\bar{\Omega}^c + \bar{\rho}_c^c\right) \times \delta\bar{v} \\ &-\hat{\bar{v}} \times \left(\bar{\Omega}^c \times \bar{\psi}\right) + \hat{\bar{v}} \times \bar{\varepsilon}^p - \bar{f}_{xy}\left(\hat{\bar{f}}^p, \bar{\psi}\right) + \bar{f}_c\left(\bar{\psi}, \hat{\bar{v}}\right) \\ &+ \begin{bmatrix} -\left(3\Omega_y^c + 2\rho_y^c\right)\hat{v}_z \\ \left(3\Omega_x^c + 2\rho_x^c\right)\hat{v}_z \\ \Omega_x^c \hat{v}_y - \Omega_y^c \hat{v}_x \end{bmatrix} \left(\cos \psi_z - 1\right) \end{aligned} \tag{47}$$

where $\hat{\bar{v}} = \begin{bmatrix} \hat{v}_x & \hat{v}_y & \hat{v}_z \end{bmatrix}^T$. (47) again has the form of the small error modified ψ-angle error model described in [2], [3] with the addition of extra terms. The nonlinear term $\bar{f}_{xy}\left(\hat{\bar{f}}^p, \bar{\psi}\right)$ is in general not negligible, and appears in the x and y components of the velocity error dynamics. This error model can as a result be troublesome to implement in a Kalman filter under conditions of large heading error.

$\bar{f}_c(\bar{\psi}, \hat{\bar{v}})$ is the correction function given by (44) that accounts for the choice of misalignment model (38) or (39). (47) also converges to the small error model described in [2], [3] as the azimuth misalignment diminishes.

Position Error Models

The GC and PC position error models are obtained from respective substitutions of (40) and (45) into (6) as follows:

GC Position Error Equation

$$\Delta\dot{\bar{r}}^c = -\bar{\rho}_c^c \times \Delta\bar{r}^c + \delta\bar{v} + \left(\bar{v}^c \times\right)_{e+} \bar{\psi}_e \tag{48}$$

PC Position Error Equation

$$\Delta\dot{\bar{r}}^c = -\bar{\rho}_c^c \times \Delta\bar{r}^c + \delta\bar{v} + \left(\hat{\bar{v}} \times\right)_{e-} \bar{\psi}_e \tag{49}$$

Concluding Remarks

The modified GC and PC strapdown navigator error equations arise from the two representations (24) and (25) of the misresolved specific force. The GC and PC error models converge to the same small error modified ψ-angle error model given in [2], [3] as the heading error diminishes. This provides for a continuous transition from a coarse heading alignment phase to "steady-state" integrated navigation without switching Kalman filter models. Which model to implement will depend on what aiding data are available for construction of the error model. For describing position and geographically referenced velocity measurements, such as from a GPS receiver, the modified GC error model must be used. For describing a body-referenced aiding velocity measurement, such as from a Doppler velocity sensor, the modified PC error model must be used. The nonlinear function in (47) must be deleted to obtain a linear error model in spite of the fact that it is not initially negligibly small.

An analysis of short term velocity errors highlights some interesting properties of the modified velocity error formulation. (43) and (47) are approximated in the short term by:

$$\delta\dot{\bar{v}} \cong \delta\bar{f}^p - \hat{\bar{g}} \times \bar{\psi}$$

This suggests that $\delta\bar{v}$ is driven in the short term by gravity misresolved by the misaligned platform. A large platform azimuth misalignment ψ_z will cause the misresolved earth rate to generate ramping horizontal tilt errors as described by (38), which in turn causes the horizontal components of $\delta\bar{v}$ to grow as a quadratic function of time. $\delta\bar{v}$ thus describes the quadratic velocity error that a gyrocompassing mechanism typically uses to seek North, regardless of vehicle dynamics. The effect of misresolved vehicle accelerations on velocity error $\Delta\bar{v}$ is then described by the velocity error transformations (40) and (45). The constructed Kalman filter measurement comprises the difference between computed velocity and a body-referenced aiding velocity $\bar{v}^b$, given by:

$$\begin{aligned}\hat{C}_a^b\hat{\bar{v}} - \bar{v}^b &= C_c^b C_p^c\left(\bar{v}_t^c + \delta\bar{v} + \left(\hat{\bar{v}} \times\right)_{e-} \bar{\psi}_e\right) - \bar{v}^b \\ &= C_c^b\left(C_p^c - I\right)\bar{v}_t^c + \hat{C}_a^b\left(\delta\bar{v} + \left(\hat{\bar{v}} \times\right)_{e-} \bar{\psi}_e\right) \\ &\cong \hat{C}_a^b\delta\bar{v}\end{aligned}$$

which is a direct observation on $\delta\bar{v}$ and through it on the heading error. A Kalman filter which processes this measurement will effectively implement a gyrocompassing mechanism for seeking North, in that the filter estimates the heading error via the observed quadratically growing $\delta\bar{v}$. This particular advantage of a Doppler velocity aiding measurement was noted in [4].

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