Technical Note: INS State Error Model

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I. Introduction

In this Technical Note we consider the estimation state error model for an inertial navigation system (INS). We explain why the dimension of $\boldsymbol{x}$ and the dimension of $\delta\boldsymbol{x}$ can be distinct (see [1]), and how to use $\delta\boldsymbol{x}$ to correct $\boldsymbol{x}$. First we define the state propagation model in Section II, then derive the error model in Section III, and finally use $\delta\boldsymbol{x}$ to correct $\boldsymbol{x}$ in Section IV.

II. State Propagation

Let $\boldsymbol{x} \in \mathbb{R}^{n_s}$ denote the rover state vector, where

$$\boldsymbol{x}(t) = [\mathbf{p}^\intercal(t), \mathbf{v}^\intercal(t), \mathbf{q}^\intercal(t), \mathbf{b}_a^\intercal(t), \mathbf{b}_g^\intercal(t)]^\intercal \in \mathbb{R}^{n_s},$$

where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{b}_a$, $\mathbf{b}_g$ each in $\mathbb{R}^3$ represent the position, velocity, accelerometer bias and gyro bias vectors, respectively, $\mathbf{q} \in \mathbb{R}^4$ represents the attitude quaternion ($n_s = 16$).

The kinematic equations for the rover state are

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where $\boldsymbol{f} : \mathbb{R}^{n_s} \times \mathbb{R}^6 \mapsto \mathbb{R}^{n_s}$ represents the kinematics, and $\boldsymbol{u} \in \mathbb{R}^6$ is the vector of specific forces and angular rates. The function $\boldsymbol{f}$ is accurately known (see eqn. 11.31-11.33 in [2], and [3]). Nature integrates eqn. (1) to produce $\boldsymbol{x}(t)$.

Let τ_i denote the time instants at which inertial measurement unit (IMU) measurements are valid. Assume there is a prior for the initial state: $\boldsymbol{x}(t_0) \sim \mathcal{N}(\boldsymbol{x}_0, \mathbf{P}_0)$. Given the initial condition $\boldsymbol{x}_0$ and the IMU measurements $\tilde{\mathbf{u}}(\tau_i) = \mathbf{u}(\tau_i) + \mathbf{b}(\tau_i) + \boldsymbol{\omega}_u(\tau_i)$ of $\mathbf{u}(\tau_i)$, with additive stochastic errors $\boldsymbol{\omega}_u(\tau_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{Qd})$ and $\mathbf{b} = [\mathbf{b}_a^\intercal, \mathbf{b}_g^\intercal]^\intercal$, a navigation system propagates an estimate of the vehicle state as the solution of

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{f}(\hat{\boldsymbol{x}}(t), \tilde{\mathbf{u}}(t)), \tag{2}$$

where $\hat{\boldsymbol{x}}(t)$ denotes the real-time estimate of $\boldsymbol{x}(t)$.

The solution of (2) over the interval $t \in [\tau_{i-1}, \tau_i]$ from the initial condition $\boldsymbol{x}_{i-1}$ is represented as the operator:

$$\phi(\boldsymbol{x}_{i-1}, \mathbf{u}_{i-1}) = \boldsymbol{x}_{i-1} + \int_{\tau_{i-1}}^{\tau_i} \boldsymbol{f}(\boldsymbol{x}(\tau), \mathbf{u}(\tau))\, d\tau \tag{3}$$

where $\hat{\boldsymbol{x}}_{i+1} = \phi(\hat{\boldsymbol{x}}_{i-1}, \hat{\mathbf{u}}_{i-1})$, with $\hat{\mathbf{u}}_{i-1} = \tilde{\mathbf{u}}_{i-1} - \hat{\mathbf{b}}_{i-1}$. Define $\mathbf{U}_{k-1} = \{\tilde{\mathbf{u}}(\tau_i) \text{ for } \tau_i \in [t_{k-1}, t_k]\}$. The integral operator in (3) can be iterated for all IMU measurements in $\mathbf{U}_k$ to propagate the state from t_{k-1} to t_k: $\hat{\boldsymbol{x}}_k = \boldsymbol{\Phi}(\hat{\boldsymbol{x}}_{k-1}, \mathbf{U}_{k-1})$. It is shown herein that $\hat{\boldsymbol{x}}_k - \boldsymbol{\Phi}(\hat{\boldsymbol{x}}_{k-1}, \mathbf{U}_{k-1}) = \mathbf{w}_k$ can be modeled with covariance $\mathbf{Q}_{\mathbf{D}k}$.

III. Error Model Derivation

Due to initial condition errors, system calibration errors, and measurement noise, an estimation state error $\delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ develops over time. The error state vector is

$$\delta\boldsymbol{x} = [\delta\mathbf{p}^\intercal,\ \delta\mathbf{v}^\intercal,\ \boldsymbol{\rho}^\intercal,\ \delta\mathbf{b}_a{}^\intercal,\ \delta\mathbf{b}_g{}^\intercal]^\intercal \in \mathbb{R}^{15},$$

where $\delta\mathbf{p}$, $\delta\mathbf{v}$, $\boldsymbol{\rho}$, $\delta\mathbf{b}_a$, $\delta\mathbf{b}_g$ each in $\mathbb{R}^3$ represent the position, velocity, attitude, accelerometer bias, and gyro bias errors, respectively. The dynamics and stochastic properties of the estimation error are well understood, and can be found in Section 11.4 of [2]. The definition of $\boldsymbol{\rho}$ is derived in Section III-B. However, first consider the state space linearization of $\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ defined in Section III-A.

A. State Space Linearization

Let the state space model for a system with input $\mathbf{u}$ be defined as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mathbf{u}).$$

Define the nominal input and state trajectory as $\mathbf{u}_o(t)$ and $\boldsymbol{x}_o(t)$, respectively, then

$$\dot{\boldsymbol{x}}_o = \boldsymbol{f}(\boldsymbol{x}_o, \mathbf{u}_o).$$

The error state vector is $\delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_o(t)$, thus

$$\begin{aligned}\delta\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \mathbf{u}) - \boldsymbol{f}(\boldsymbol{x}_o, \mathbf{u}_o)\\ &= \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_o\end{aligned}$$

Using a Taylor series expansion to approximate $\boldsymbol{f}(\boldsymbol{x}, \mathbf{u})$,

$$\begin{aligned}\delta\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}_o, \mathbf{u}_o) &+ \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \mathbf{u})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_o(t), \mathbf{u}_o(t)} \delta\boldsymbol{x}\\ &+ \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \mathbf{u})}{\partial \mathbf{u}}\right|_{\boldsymbol{x}_o(t), \mathbf{u}_o(t)} \delta\mathbf{u} + h.o.t.'s\\ &- \boldsymbol{f}(\boldsymbol{x}_o, \mathbf{u}_o)\end{aligned}$$

where $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_o$. Simplifying, and dropping the higher order terms ($h.o.t.'s$),

$$\delta\dot{\boldsymbol{x}} = \mathbf{F}(t)\delta\boldsymbol{x}(t) + \mathbf{G}(t)\delta\mathbf{u}(t)$$

where $\mathbf{F}(t) = \frac{\partial \mathbf{f}(\boldsymbol{x}, \mathbf{u})}{\partial \boldsymbol{x}}\big|_{\boldsymbol{x}_o(t), \mathbf{u}_o(t)}$, and $\mathbf{G}(t) = \frac{\partial \mathbf{f}(\boldsymbol{x}, \mathbf{u})}{\partial \mathbf{u}}\big|_{\boldsymbol{x}_o(t), \mathbf{u}_o(t)}$, which is accurate for small $\|\delta\boldsymbol{x}\|$ and $\|\delta\mathbf{u}\|$.

B. Attitude Error

In INS state propagation there exists a small error $\delta\boldsymbol{\Omega}$ between the true and estimated angular rotations. The true rotation is represented by the rotation matrix $\mathbf{R}_a^b$ from frame a to b, and the estimated rotation represented by $\hat{\mathbf{R}}_a^b$. The angular error is a multiplication defined as

$$\delta\boldsymbol{\Omega} = (\hat{\mathbf{R}}_a^b)^\intercal \mathbf{R}_a^b$$

such that

$$\mathbf{R}_a^b = \hat{\mathbf{R}}_a^b \delta\hat{\boldsymbol{\Omega}}$$

This section develops the attitude error model.

For small angular perturbations, a small angle approximation is assumed valid. Define the infinitesimal rotation vector as $\delta\boldsymbol{\theta} = [\delta\theta_1, \delta\theta_2, \delta\theta_3]^\intercal$. As shown in Section 2.5.5 in [2] the vector transformation from frame a to frame b is defined as

$$\begin{aligned}\mathbf{v}^b &= \begin{pmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{pmatrix} \mathbf{v}^a \\ &= (\mathbf{I}_3 - \delta\boldsymbol{\Theta})\mathbf{v}^a,\end{aligned}$$

the reverse transformation is

$$\mathbf{v}^a = (\mathbf{I}_3 + \delta\boldsymbol{\Theta})\mathbf{v}^b, \tag{4}$$

where $\delta\boldsymbol{\Theta} = [\delta\boldsymbol{\theta}\times]$ represents the skew-symmetric matrix of $\delta\boldsymbol{\theta}$.

Define the derivative of the rotation matrix from frame a to b as

$$\dot{\mathbf{R}}_a^b(t) = \lim_{\delta t \mapsto 0} \frac{\mathbf{R}_a^b(t+\delta t) - \mathbf{R}_a^b(t)}{\delta t} \tag{5}$$

For small δt, the rotation $\mathbf{R}_a^b(t+\delta t)$ is the rotation from frame a to b at time t, followed by the rotation from frame a to b at time $t + \delta t$,

$$\mathbf{R}_a^b(t+\delta t) = \mathbf{R}_{b(t)}^{b(t+\delta t)} \mathbf{R}_a^b(t) \tag{6}$$

where $\mathbf{R}_{b(t)}^{b(t+\delta t)}$ represents the small angle rotation $\delta\boldsymbol{\theta} = \boldsymbol{\omega}_{ab}^b \delta t$, for finite instantaneous angular velocity $\boldsymbol{\omega}_{ab}^b$ of frame b with respect to frame a represented in frame b. Applying eqn. (4),

$$\mathbf{R}_{b(t)}^{b(t+\delta t)} = \mathbf{I} - \boldsymbol{\Omega}_{ab}^b \delta t \tag{7}$$

where $\boldsymbol{\Omega}_{ab}^b = [\boldsymbol{\omega}_{ab}^b\times]$. Substituting eqns. (6) and (7) into (5),

$$\begin{aligned}\dot{\mathbf{R}}_a^b(t) &= \lim_{\delta t \mapsto 0} \frac{(\mathbf{I} - \boldsymbol{\Omega}_{ab}^b \delta t)\mathbf{R}_a^b - \mathbf{R}_a^b(t)}{\delta t} \\ &= -\boldsymbol{\Omega}_{ab}^b \mathbf{R}_a^b(t).\end{aligned} \tag{8}$$

Transformation from $\boldsymbol{\Omega}_{ab}^a$ to $\boldsymbol{\Omega}_{ab}^b$ is defined as

$$\boldsymbol{\Omega}_{ab}^b = \mathbf{R}_a^b \boldsymbol{\Omega}_{ab}^a \mathbf{R}_b^a \tag{9}$$

and reversing frame b with respect to frame a is defined as

$$\boldsymbol{\Omega}_{ab}^a = -\boldsymbol{\Omega}_{ba}^a,$$

then eqn. (8) is modified to be

$$\dot{\mathbf{R}}_a^b = \mathbf{R}_a^b \boldsymbol{\Omega}_{ba}^a. \tag{10}$$

Using eqn. (10), define the time derivative of the small-angle attitude error of the ECEF-frame e with respect to body-frame b represented in frame b,

$$\dot{\mathbf{R}}_b^e = \mathbf{R}_b^e \boldsymbol{\Omega}_{eb}^b, \tag{11}$$

where $\boldsymbol{\Omega}_{eb}^b = [\boldsymbol{\omega}_{eb}^b\times]$, and the rotation from body to ECEF-frame is $\mathbf{R}_b^e$. Based on the kinematics of eqn. (11), the estimate $\dot{\hat{\mathbf{R}}}_b^e$ is

$$\dot{\hat{\mathbf{R}}}_b^e = \hat{\mathbf{R}}_b^e(\hat{\boldsymbol{\Omega}}_{ib}^b - \hat{\boldsymbol{\Omega}}_{eb}^b),$$

where $\hat{\boldsymbol{\Omega}}_{ib}^b = [\hat{\boldsymbol{\omega}}_{ib}^b\times]$ with $\hat{\boldsymbol{\omega}}_{ib}^b = \mathbf{u} - \hat{\boldsymbol{x}}$ is calculated based on gyro inputs for the inertial-frame i, $\hat{\boldsymbol{\Omega}}_{ie}^e = [\hat{\boldsymbol{\omega}}_{eb}^b\times]$ with the Earth-rotation rate vector defined as $\boldsymbol{\omega}_{eb}^b \approxeq \boldsymbol{\omega}_{ie}^e = [0, 0, \omega_{ie}]^\intercal$ for Earth-rotation rate ω_{ie}.

Define the small-angle rotations $\boldsymbol{\rho} = [\epsilon_x, \epsilon_y, \epsilon_z]^\intercal$ which align the true frame with computed frame. The values ϵ_x and ϵ_y are commonly referred to as tilt-errors, and ϵ_z is referred to as yaw error. To determine a dynamic model for $\boldsymbol{\rho}$, first define

$$\mathbf{R}_b^e = (\mathbf{I} + \mathbf{P})\hat{\mathbf{R}}_e^b \tag{12}$$

where $\mathbf{P} = [\boldsymbol{\rho}\times]$. Then set the derivative of eqn. (12) equal to the left side of eqn. (11)

$$\dot{\mathbf{P}}\hat{\mathbf{R}}_b^e + (\mathbf{I} + \mathbf{P})\dot{\hat{\mathbf{R}}}_b^e = (\mathbf{I} + \mathbf{P})\hat{\mathbf{R}}_b^e(\boldsymbol{\Omega}_{ib}^b - \boldsymbol{\Omega}_{eb}^b). \tag{13}$$

Define

$$\begin{aligned}\delta\boldsymbol{\Omega}_{ib}^b &= \boldsymbol{\Omega}_{ib}^b - \hat{\boldsymbol{\Omega}}_{ib}^b \\ \delta\boldsymbol{\Omega}_{eb}^b &= \boldsymbol{\Omega}_{eb}^b - \hat{\boldsymbol{\Omega}}_{eb}^b\end{aligned}$$

eqn. (13) simplifies to

$$\begin{aligned}\dot{\mathbf{P}}\hat{\mathbf{R}}_b^e + (\mathbf{I} + \mathbf{P})\dot{\hat{\mathbf{R}}}_b^e &= (\mathbf{I} + \mathbf{P})\hat{\mathbf{R}}_b^e(\boldsymbol{\Omega}_{ib}^b - \boldsymbol{\Omega}_{eb}^b + \delta\boldsymbol{\Omega}_{ib}^b - \delta\boldsymbol{\Omega}_{eb}^b) \\ &= (\mathbf{I} + \mathbf{P})\dot{\hat{\mathbf{R}}}_b^e \\ &\quad + (\mathbf{I} + \mathbf{P})\hat{\mathbf{R}}_b^e(\delta\boldsymbol{\Omega}_{ib}^b - \delta\boldsymbol{\Omega}_{eb}^b) \\ \dot{\mathbf{P}} &= \hat{\mathbf{R}}_b^e(\delta\boldsymbol{\Omega}_{ib}^b - \delta\boldsymbol{\Omega}_{eb}^b)\hat{\mathbf{R}}_e^b\end{aligned} \tag{14}$$

where $\mathbf{P}\hat{\mathbf{R}}_b^e(\delta\boldsymbol{\Omega}_{ib}^b - \delta\boldsymbol{\Omega}_{eb}^b)\hat{\mathbf{R}}_e^b$ was dropped due to second-order error terms. Since $\dot{\mathbf{P}} = [\dot{\boldsymbol{\rho}}\times]$, using eqn. (9), eqn. (14) can be defined as

$$\dot{\boldsymbol{\rho}} = \hat{\mathbf{R}}_b^e(\delta\boldsymbol{\omega}_{ib}^b - \delta\boldsymbol{\omega}_{eb}^b) \tag{15}$$

where $\delta\boldsymbol{\omega}_{ib}^b \triangleq \delta\mathbf{b}_g$ is the error in the gyro measurement of the body-frame inertial-relative angular rate, and $\delta\boldsymbol{\omega}_{eb}^b$ is the error in the Earth-rate estimate.

The inertial rotation rate of the Earth-frame represented in the body-frame is

$$\boldsymbol{\omega}_{ie}^b = \mathbf{R}_e^b \boldsymbol{\omega}_{ie}^e$$

which can be manipulated to derive an expression for $\delta\boldsymbol{\omega}_{ie}^b$

$$\begin{aligned}\hat{\boldsymbol{\omega}}_{ie}^b + \delta\boldsymbol{\omega}_{ie}^b &= \hat{\mathbf{R}}_e^b(\mathbf{I} - \mathbf{P})(\hat{\boldsymbol{\omega}}_{ie}^e + \delta\boldsymbol{\omega}_{ie}^e) \\ &= \hat{\mathbf{R}}_e^b(\delta\boldsymbol{\omega}_{ie}^e - \mathbf{P}\hat{\boldsymbol{\omega}}_{ie}^e),\end{aligned} \tag{16}$$

where higher order terms have been removed. Applying the relationships defined for $\hat{\mathbf{R}}_e^b$, $\mathbf{R}_e^b$, and the definition $\mathbf{P} = [\boldsymbol{\rho}\times]$, substitute eqn. (16) into eqn. (15)

$$\dot{\boldsymbol{\rho}} + \hat{\boldsymbol{\Omega}}_{ie}^e \boldsymbol{\rho} = -\delta\boldsymbol{\omega}_{ie}^e + \hat{\mathbf{R}}_b^e \delta\boldsymbol{\omega}_{ib}^b \tag{17}$$

Define

$$\delta\boldsymbol{\omega}_{ie}^e = \boldsymbol{\omega}_{ie}^e - \hat{\boldsymbol{\omega}}_{ie}^e.$$

The Taylor series expansion of $\boldsymbol{\omega}_{ie}^e$ is

$$\boldsymbol{\omega}_{ie}^e = \hat{\boldsymbol{\omega}}_{ie}^e(\hat{\mathbf{p}}, \hat{\mathbf{v}}) + \left[\frac{\partial\hat{\boldsymbol{\omega}}_{ie}^e}{\partial\hat{\mathbf{p}}}, \frac{\partial\hat{\boldsymbol{\omega}}_{ie}^e}{\partial\hat{\mathbf{v}}}\right]\left[\begin{array}{c}\mathbf{p} - \hat{\mathbf{p}} \\ \mathbf{v} - \hat{\mathbf{v}}\end{array}\right] + \ldots$$

Rearranging the first-order approximation

$$\begin{aligned}\delta\boldsymbol{\omega}_{ie}^e &= \left[\frac{\partial\hat{\boldsymbol{\omega}}_{ie}^e}{\partial\hat{\mathbf{p}}}, \frac{\partial\hat{\boldsymbol{\omega}}_{ie}^e}{\partial\hat{\mathbf{v}}}\right]\left[\begin{array}{c}\delta\mathbf{p} \\ \delta\mathbf{v}\end{array}\right] \\ &= \mathbf{0}\end{aligned} \tag{18}$$

Combining the result in eqn. (18) with eqn. (17), the time derivative of the attitude error is

$$\begin{aligned}\dot{\boldsymbol{\rho}} &= \hat{\mathbf{R}}_b^e \delta\boldsymbol{\omega}_{ib}^b - \hat{\boldsymbol{\Omega}}_{ie}^e \boldsymbol{\rho} \\ &= \hat{\mathbf{R}}_b^e \delta\mathbf{b}_g - \hat{\boldsymbol{\Omega}}_{ie}^e \boldsymbol{\rho}\end{aligned} \tag{19}$$

C. Velocity Error

The rate of change of the Earth-referenced velocity is

$$\dot{\mathbf{v}}_{eb}^e = \mathbf{f}_{ib}^e + \mathbf{g}_b^e(\mathbf{r}_{eb}^e) - 2\boldsymbol{\Omega}_{ie}^e \mathbf{v}_{eb}^e,$$

where $\mathbf{f}_{ib}^b$ is the specific force of the inertial-frame i, with respect to the body-frame b represented in the body-frame. The local gravity vector in the ECEF-frame is $\mathbf{g}_b^e$, which may be computed by a precise gravity model [4] for the position in ECEF-frame $\mathbf{r}_{eb}^e$. The velocity in ECEF-frame is $\mathbf{v}_{eb}^e$.

The time derivative of the velocity error is

$$\begin{aligned}\delta\dot{\mathbf{v}}_{eb}^e = \hat{\mathbf{f}}_{ib}^e - \mathbf{f}_{ib}^e + \mathbf{g}_b^e(\hat{\mathbf{r}}_{eb}^e) \\ - \mathbf{g}_b^e(\mathbf{r}_{eb}^e) - 2\boldsymbol{\Omega}_{ie}^e(\hat{\mathbf{v}}_{eb}^e - \mathbf{v}_{eb}^e).\end{aligned} \tag{20}$$

The specific force term in eqn. (20) is defined as

$$\hat{\mathbf{f}}_{ib}^e - \mathbf{f}_{ib}^e \approxeq -[(\hat{\mathbf{R}}_b^e \hat{\mathbf{f}}_{ib}^b)\times]\delta\boldsymbol{\rho}_{eb}^e + \hat{\mathbf{R}}_b^e \delta\mathbf{b}_a, \tag{21}$$

where $\delta\mathbf{b}_a$ is the error in accelerometer measurement estimate.

The gravity term in eqn. (20) is defined as

$$\begin{aligned}\mathbf{g}_b^e(\hat{\mathbf{r}}_{eb}^e) - \mathbf{g}_b^e(\mathbf{r}_{eb}^e) &\approxeq (\hat{\gamma}_{ib}^e - \gamma_{ib}^e) - \boldsymbol{\Omega}_{ie}^e \boldsymbol{\Omega}_{ie}^e \delta\mathbf{r}_{eb}^e \\ &\approxeq -\frac{2\hat{\gamma}_{ib}^e(\hat{\mathbf{r}}_{eb}^e)}{r_{eS}^e(\hat{L}_b)||\hat{\mathbf{r}}_{eb}^e||_2}\delta\mathbf{r}_{eb}^e,\end{aligned} \tag{22}$$

where $\hat{\gamma}_{ib}^e(\hat{\mathbf{r}}_{eb}^e)$ is the gravitational acceleration evaluated at the estimated position $\hat{\mathbf{r}}_{eb}^e$, and $r_{eS}^e(\hat{L}_b)$ is the geocentric radius at the Earth's surface evaluated at the estimated geodetic latitude $\hat{L}_b$ (see Section 2.4.7 and 14.2.3 of [5])

Substituting eqns. (21) and (22) into (20), the time derivative of the velocity error is

$$\begin{aligned}\delta\dot{\mathbf{v}}_{eb}^e = -[(\hat{\mathbf{R}}_b^e \hat{\mathbf{f}}_{ib}^b)\times]\delta\boldsymbol{\rho}_{eb}^e - 2\boldsymbol{\Omega}_{ie}^e \delta\mathbf{v}_{eb}^e \\ - \frac{2\hat{\gamma}_{ib}^e(\hat{\mathbf{r}}_{eb}^e)}{r_{eS}^e(\hat{L}_b)||\hat{\mathbf{r}}_{eb}^e||_2}\delta\mathbf{r}_{eb}^e + \hat{\mathbf{R}}_b^e \delta\mathbf{b}_a.\end{aligned} \tag{23}$$

D. Position Error

The time derivative of the ECEF-frame position is simply the velocity, defined as

$$\dot{\mathbf{r}}_{eb}^e = \mathbf{v}_{eb}^e.$$

The time derivative of the position error is simply the velocity error,

$$\delta\dot{\mathbf{r}}_{eb}^e = \delta\mathbf{v}_{eb}^e. \tag{24}$$

E. Sensor Error

A complete accelerometer measurement model $\tilde{\mathbf{f}}_{is}^s$ with measurements in the inertial frame i with respect to the sensor-frame s represented in the sensor frame, can be defined as

$$\tilde{\mathbf{f}}_{is}^s = (\mathbf{I} - \delta\mathbf{SF}_a)(\mathbf{f}_{is}^s - \delta\mathbf{b}_a - \delta\mathbf{nl}_a - \boldsymbol{\nu}_a), \tag{25}$$

where $\delta\mathbf{SF}_a$ is a diagonal matrix representing the uncompensated accelerometer scale factor errors, $\delta\mathbf{b}_a$ represents the uncompensated accelerometer bias, $\delta\mathbf{nl}_a$ represents the uncompensated accelerometer nonlinearity, and $\boldsymbol{\nu}_a$ represents the white Gaussian accelerometer measurement noise.

Similarly, a complete gyro measurement model $\tilde{\boldsymbol{\omega}}_{is}^s$ can be defined as

$$\tilde{\boldsymbol{\omega}}_{is}^s = (\mathbf{I} - \delta\mathbf{SF}_g)(\boldsymbol{\omega}_{is}^s - \delta\mathbf{b}_g - \delta\mathbf{k}_g - \boldsymbol{\nu}_g), \tag{26}$$

where $\delta\mathbf{SF}_g$ is a diagonal matrix representing the uncompensated gyro scale factor errors, $\delta\mathbf{b}_g$ represents the uncompensated gyro bias, $\delta\mathbf{k}_g$ represents the uncompensated gyro g-sensitivity, and $\boldsymbol{\nu}_g$ represents the white Gaussian gyro measurement noise.

Based on eqns. (25) and (26), a complete IMU sensor error model would require 60 additional states to the INS error state vector $\delta\boldsymbol{x} = [\delta\mathbf{p}^\intercal,\ \delta\mathbf{v}^\intercal,\ \delta\boldsymbol{\theta}^\intercal]^\intercal$ (see Sections 11.6.2 and 11.6.3 of [2]). Therefore we define a simplified model of the accelerometer and gyro measurements with respect to the body-frame, $\tilde{\mathbf{f}}_{ib}^b$ and $\tilde{\boldsymbol{\omega}}_{ib}^b$ respectively, defined as

$$\begin{aligned}\tilde{\mathbf{f}}_{ib}^b &= \mathbf{f}_{ib}^b + \Delta\mathbf{f}_{ib}^b \\ \tilde{\boldsymbol{\omega}}_{ib}^b &= \boldsymbol{\omega}_{ib}^b + \Delta\boldsymbol{\omega}_{ib}^b\end{aligned}$$

where $\Delta\mathbf{f}_{ib}^b$ and $\Delta\boldsymbol{\omega}_{ib}^b$ represent the specific force and angular rate measurement errors. These measurement errors contain both random noise and instrument calibration factors. The instrument calibration factors can be compensated through state estimation of $\Delta\hat{\mathbf{f}}_{ib}^b$ and $\Delta\hat{\boldsymbol{\omega}}_{ib}^b$, such that the estimated specific force and angular rate measurements are

$$\begin{aligned}\hat{\mathbf{f}}_{ib}^b &= \tilde{\mathbf{f}}_{ib}^b + \Delta\hat{\mathbf{f}}_{ib}^b \\ \hat{\boldsymbol{\omega}}_{ib}^b &= \tilde{\boldsymbol{\omega}}_{ib}^b + \Delta\hat{\boldsymbol{\omega}}_{ib}^b.\end{aligned}$$

Eqn. (23) models the effect of the uncalibrated portion of the accelerometer error on the velocity error,

$$\delta\mathbf{f}_{ib}^b = \Delta\mathbf{f}_{ib}^b - \Delta\hat{\mathbf{f}}_{ib}^b,$$

while eqn. (19) models the uncalibrated portion of the gyro error on the attitude error,

$$\delta\boldsymbol{\omega}_{ib}^b = \Delta\boldsymbol{\omega}_{ib}^b - \Delta\hat{\boldsymbol{\omega}}_{ib}^b.$$

The accelerometer and gyro instrument errors can be modeled as

$$\Delta \mathbf{f}_{ib}^{b} = \mathbf{F}_{va}\boldsymbol{x}_a + \boldsymbol{\nu}_a \tag{27}$$

$$\Delta \boldsymbol{\omega}_{ib}^{b} = \mathbf{F}_{\rho g}\boldsymbol{x}_g + \boldsymbol{\nu}_g. \tag{28}$$

A complete discussion of $\mathbf{F}_{va}$, $\boldsymbol{x}_a$, $\mathbf{F}_{\rho g}$ and $\boldsymbol{x}_g$ is provided in Sections 11.6.2 and 11.6.3 of [2], for now we will assume that $\mathbf{F}_{va} = \mathbf{I}$ and $\mathbf{F}_{\rho g} = \mathbf{I}$.

Given eqns. (27) and (28), the estimates are

$$\Delta \hat{\mathbf{f}}_{ib}^{b} = \mathbf{F}_{va}\hat{\boldsymbol{x}}_a$$
$$\Delta \hat{\boldsymbol{\omega}}_{ib}^{b} = \mathbf{F}_{\rho g}\hat{\boldsymbol{x}}_g,$$

where $\hat{\boldsymbol{x}}_a$ and $\hat{\boldsymbol{x}}_g$ are initialized off-line.

Define

$$\delta\boldsymbol{x}_a = \boldsymbol{x}_a - \hat{\boldsymbol{x}}_a$$
$$\delta\boldsymbol{x}_g = \boldsymbol{x}_g - \hat{\boldsymbol{x}}_g,$$

the instrument calibration errors are

$$\delta \mathbf{f}_{ib}^{b} = \mathbf{F}_{va}\delta\boldsymbol{x}_a + \boldsymbol{\nu}_a$$
$$\delta \boldsymbol{\omega}_{ib}^{b} = \mathbf{F}_{\rho g}\delta\boldsymbol{x}_g + \boldsymbol{\nu}_g.$$

The time derivative of the instrumentation errors is

$$\delta\dot{\boldsymbol{x}}_a = \mathbf{F}_{aa}\delta\boldsymbol{x}_a + \boldsymbol{\omega}_a \tag{29}$$

$$\delta\dot{\boldsymbol{x}}_g = \mathbf{F}_{gg}\delta\boldsymbol{x}_g + \boldsymbol{\omega}_g, \tag{30}$$

where the spectral densities $\boldsymbol{\omega}_a$ and $\boldsymbol{\omega}_g$ are typically specified by the IMU manufacturer and can be found on the Allen Variance plots provided in an IMU data-sheet (for example see Section 2.4 & 4 of [6]). The values $\mathbf{F}_{aa}$ and $\mathbf{F}_{gg}$ are often selected as the inverse of the time-correlation of the sensor errors, e.g. $\mathbf{F}_{aa} = -\frac{1}{\tau_a}\mathbf{I}_3$ and $\mathbf{F}_{gg} = -\frac{1}{\tau_g}\mathbf{I}_3$, where τ_a and τ_a are the correlation times of the accelerometer and gyro errors.

F. Continuous-Time Error Model

The linearized dynamic model (see Section 11.5 of [2]) is defined as

$$\delta\dot{\boldsymbol{x}}(t) = \mathbf{F}(t)\delta\boldsymbol{x}(t) + \boldsymbol{\Gamma}\mathbf{q}. \tag{31}$$

where

$$\mathbf{q} = [\boldsymbol{\nu}_a^{\intercal}, \boldsymbol{\nu}_g^{\intercal}, \boldsymbol{\omega}_a^{\intercal}, \boldsymbol{\omega}_g^{\intercal}]^{\intercal}.$$

Substituting eqns. (19), (23), (24), (29), and (30) into (31),

$$\mathbf{F} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{F}_{21} & -2\boldsymbol{\Omega}_{ie}^{e} & \mathbf{F}_{23} & \hat{\mathbf{R}}_b^e & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -\boldsymbol{\Omega}_{ie}^{e} & \mathbf{0}_3 & \hat{\mathbf{R}}_b^e \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{F}_{44} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{F}_{55} \end{bmatrix}, \tag{32}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\hat{\mathbf{R}}_b^e & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \hat{\mathbf{R}}_b^e & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \tag{33}$$

where

$$\mathbf{F}_{21} = \frac{2\hat{\gamma}_{ib}^{e}(\hat{\mathbf{r}}_{eb}^{e})}{r_{eS}^{e}(\hat{L}_b)||\hat{\mathbf{r}}_{eb}^{e}||_2},$$
$$\mathbf{F}_{23} = -[(\hat{\mathbf{R}}_b^e\mathbf{f}_{ib}^{b})\times],$$
$$\mathbf{F}_{44} = -\frac{1}{\tau_a}\mathbf{I}_3,$$
$$\mathbf{F}_{55} = -\frac{1}{\tau_g}\mathbf{I}_3.$$

IV. State Correction

In general, the state correction $\delta\hat{\boldsymbol{x}}$ of the state vector $\hat{\boldsymbol{x}}$ for the time interval $[k-1, k]$, can be performed by

$$\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_{k-1} + \delta\boldsymbol{x}_k.$$

The subtleties of the state update are discussed in this section.

A. Position, Velocity, and Bias Updates

Position, velocity, accelerometer bias and gyro bias, each have corrections which are additive. The state correction step is

$$\hat{\mathbf{p}}_k = \hat{\mathbf{p}}_{k-1} + \delta\mathbf{p}_k$$
$$\hat{\mathbf{v}}_k = \hat{\mathbf{v}}_{k-1} + \delta\mathbf{v}_k$$
$$\hat{\mathbf{b}}_{a,k} = \hat{\mathbf{b}}_{a,k-1} + \delta\mathbf{b}_{a,k}$$
$$\hat{\mathbf{b}}_{g,k} = \hat{\mathbf{b}}_{g,k-1} + \delta\mathbf{b}_{g,k}.$$

B. Attitude Update

Attitude corrections are multiplicative (see Section 10.5 of [2]), thus the quaternion state correction setp is

$$\hat{\mathbf{q}}_k = \hat{\mathbf{q}}_{k-1}\bar{\mathbf{q}}_k$$

where the quaternion correction $\bar{\mathbf{q}} \in \mathbb{R}^{4\times 1}$ is converted from the small-angle error state $\delta\boldsymbol{\rho} \in \mathbb{R}^{3\times 1}$.

Define

$$\bar{\mathbf{q}}_s = \sqrt{1 - \left\|\frac{1}{2}\delta\boldsymbol{\rho}\right\|_2^2}$$
$$\bar{\mathbf{q}}_v = \frac{1}{2}\delta\boldsymbol{\rho}$$

where the scalar part of the quaternion is $\bar{\mathbf{q}}_s$, and the vector part is $\bar{\mathbf{q}}_v$. The quaternion correction $\bar{\mathbf{q}}$ is defined as

$$\bar{\mathbf{q}} = \begin{bmatrix} \bar{\mathbf{q}}_s \\ \bar{\mathbf{q}}_v \end{bmatrix}.$$

Quaternion mathematics are defined in Section D of [2].

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