Comments in *Supersymmetric Quantum Mechanics* by David Tong
David Tong 《超对称量子力学》中的评论

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目录

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0 Introduction

It will come as no surprise to hear that there is a close relationship between mathematics and physics. Yet, for many centuries, the relationship was more than a little one sided. There was, in the language of marriage counsellors, a lack of equitable reciprocity. Physicists took, but gave little in return. Admittedly there were exceptions, some of them rather important like Newton's development of calculus. Nonetheless, it remains true that mathematics is a tool that us physicists cannot live without, while many mathematicians have no more use of physics than they do of chemistry or botany.

In the last few decades, this narrative has started to change. Physicists have been giving back. As our understanding of quantum field theories has grown, we have un- covered increasingly sophisticated mathematical structures lurking within. These are largely, but not exclusively, the structures that arise in geometry and topology. Using physicist's methods and techniques to solve quantum fields theories has revealed connections to these mathematical ideas. Initially this gave new ways of deriving results well known to mathematicians. But, as the quantum field theories became more in- volved, so too did the mathematics until physicists were able to discover new results that came as a complete surprise to mathematicians. Prominent among these is an idea called mirror symmetry, a novel relationship between different manifolds.

You might reasonably wonder what advantage physicists have over mathematicians in this game. After all, we're certainly not smarter. (At least, not most of us.) And yet, there are times when we are able to leapfrog mathematicians and then turn around and present them with new results that sit firmly within their area of expertise. This seems unfair, like physicists have some kind of secret weapon that mathematicians are unable to wield. And we

听到数学和物理学之间存在密切的关系也就不足为奇了。然而，几个世纪以来，这种关系不仅仅是一点点片面的。用婚姻顾问的话说，缺乏公平的互惠。物理学家索取了，但几乎没有给予任何回报。诚然，也有例外，其中一些相当重要，比如牛顿对微积分的发展。尽管如此，数学仍然是我们物理学家离不开的工具，而许多数学家对物理学的使用并不比对化学或植物学的使用更多。

在过去的几十年里，这种说法开始发生变化。物理学家一直在回馈。随着我们对量子场理论理解的加深，我们发现了潜伏在其中的日益复杂的数学结构。这些主要但不完全是几何和拓扑中出现的结构。使用物理学家的方法和技术来解决量子场理论已经揭示了与这些数学思想的联系。最初，这为数学家提供了推导结果的新方法。但是，随着量子场理论变得越来越复杂，数学也变得越来越复杂，直到物理学家能够发现令数学家完全惊讶的新结果。其中最突出的是一种称为镜像对称的想法，这是不同流形之间的一种新颖关系。

您可能有理由想知道在这个游戏中物理学家比数学家有什么优势。毕竟，我们当然并不聪明。(至少，我们大多数人不是。) 然而，有时我们能够超越数学家，然后转身向他们展示完全属于他们专业领域的新结果。这似乎不公平，就像物理学家拥有某种数学家无法使用的秘密武器一样。我们确实这么做了。

do. In fact, we have two. The first is the path integral. The second, a wilful disregard for rigour.

These two weapons are not unrelated. The path integral approach to quantum field theory has so far evaded attempts to be placed on a rigorous footing, at least beyond quantum mechanics. This means that most often the physicist's approach to these questions does not meet the mathematician's bar for proof. Physics is perhaps better thought of as an idea generating machine, giving new insights into areas of mathematics that can subsequently be proven using more traditional methods. Happily, in most cases, these subsequent proofs have turned out to be much more than an exercise in dotting i's and crossing $\hbar$'s. Mathematicians take their own path to a problem, developing new ideas along the way, and these then feed back into our understanding of quantum field theory. Over the past few decades this process has resulted in a harmonious and extraordinarily fruitful relationship between communities of physicists and mathematicians.

This interaction has revolutionised certain areas of mathematics. For example, it's difficult to envisage a thriving field of symplectic geometry without mirror symmetry. But it has also changed what we mean by "mathematical physics". Towards the end of the 20th century, this was viewed as a rather a dry subject and mostly involved bringing a mathematician's level of pedantry to bear on problems that physicists care about, but with little insight flowing back into the underlying physics. Now, this situation has been reversed, with interesting and exciting ideas flowing in both directions. To emphasise the shift of focus, this new activity is sometimes rebranded "physical mathematics".

Much of this interplay between physics and mathematics takes place in the arena of supersymmetric field theories. (There are important exceptions, Witten's Fields medal winning work on knot polynomials in Chern Simons theory among them.) Supersymmetric theories are a class of quantum field theories that have a symmetry relating bosons and fermions. There is, so far, no experimental evidence that supersymmetry is a symmetry of our world. But supersymmetric theories have a number of special prop- erties that allow us to make much more progress in solving them than would otherwise be possible. It is often in these solutions to supersymmetric field theories that we find results of interest to mathematicians.

The purpose of these lectures is to take the first first few steps along this journey. Sadly we will not reach the heights of the subject like mirror symmetry or knot invariants, both of which require quantum field theories in higher dimensions ($d = 1 + 1$

事实上, 我们有两个。第一个是路径积分。第二，故意无视严格性。

这两种武器并非毫无关联。迄今为止, 量子场论的路径积分方法还没有被建立在严格的基础上, 至少超出了量子力学的范围。这意味着物理学家解决这些问题的方法通常不符合数学家的证明标准。物理学也许更适合被认为是一种想法产生机器，它为数学领域提供了新的见解，随后可以使用更传统的方法来证明。令人高兴的是, 在大多数情况下, 这些后续的证明不仅仅是给 i 加点和给 $\hbar$ 画叉的练习。数学家们走自己的路来解决问题, 一路上提出新的想法，然后这些想法反馈到我们对量子场论的理解中。在过去的几十年里，这一过程在物理学家和数学家群体之间建立了和谐且卓有成效的关系。

这种相互作用彻底改变了数学的某些领域。例如, 如果没有镜像对称性, 就很难想象辛几何领域会蓬勃发展。但它也改变了我们所说的“数学物理”的含义。到了 20 世纪末, 这被认为是一门相当枯燥的学科，主要涉及用数学家的迂腐水平来解决物理学家关心的问题，但很少有洞察力回到底层物理学。现在，这种情况已经发生了逆转，有趣且令人兴奋的想法在两个方向流动。为了强调焦点的转移，这项新活动有时被重新命名为“物理数学”。

物理学和数学之间的这种相互作用大部分发生在超对称场论领域。（有一些重要的例外，其中包括陈西蒙斯理论中关于结多项式的威滕菲尔兹奖获奖作品。超对称理论是一类量子场论，具有与玻色子和费米子相关的对称性。到目前为止，还没有实验证据表明超对称性是我们世界的对称性。但是超对称理论有许多特殊的属性，使我们能够在解决它们方面取得比其他方式更多的进展。通常，在这些超对称场论的解中，我们发现了数学家感兴趣的结果。

这些讲座的目的是在这一旅程中迈出最初的几步。遗憾的是，我们无法达到镜像对称或扭结不变量那样的高度，这两者都需要更高维度的量子场论（分别为 $d = 1 + 1$ 和

and $d = 2 + 1$ respectively). Instead, we will restrict ourselves to $d = 0 + 1$ dimensional quantum field theories, also known as quantum mechanics. We will study a number of examples of supersymmetric quantum mechanics and, in solving them, recover some of the highlights of 20th century geometry, including ideas of de Rham, Hodge, Morse, Atiyah and Singer.

I should warn you that the level of rigour when addressing the more mathematical aspect of these lectures will be mediocre at best. Anyone with a real interest in these ideas is encouraged to learn both the underlying mathematics and physics to truly appreciate how the two connect. But that is not the path we will take here. Instead, these lectures will assume only a basic knowledge in differential geometry (at the level, say, of my lectures on General Relativity.) We will then use supersymmetric quantum mechanics as a vehicle to take us deeper into the mathematician's territory, allowing us to take a peek at some of the beautiful vistas that await.

$d = 2 + 1$）。相反，我们将把自己限制在 $d = 0 + 1$ 维量子场理论，也称为量子力学。我们将研究一些超对称量子力学的例子，并在解决它们的过程中恢复 20 世纪几何学的一些亮点，包括德拉姆、霍奇、莫尔斯、阿蒂亚和辛格的思想。

我应该警告你，这些讲座中涉及数学方面的严谨程度充其量也只是平庸。我们鼓励任何对这些想法真正感兴趣的人学习基础数学和物理，以真正理解两者之间的联系。但这不是我们在这里要走的道路。相反，这些讲座将仅假设微分几何的基础知识（例如，在我关于广义相对论的讲座的水平上）。然后，我们将使用超对称量子力学作为工具，带我们更深入地进入数学家的领域，使我们能够欣赏一些等待着的美丽景色。

1 Introducing Supersymmetric Quantum Mechanics

Introducing 超对称量子力学

1.1 Supersymmetry Algebra

超对称代数

1.1.1 A First Look at the Energy Spectrum

能谱初探

In this section, we discuss some basic facts about supersymmetric quantum mechanics. Our focus will be on a simple class of quantum mechanical systems that, while they have a certain elegance, won't exhibit any deep mathematics. Instead, we will treat them as a proving ground, allowing us to build some intuition for supersymmetry while developing a number of useful calculational techniques. We'll then bring these to bear on problems with a deeper mathematical pedigree in Section 3.

在本节中，我们讨论有关超对称量子力学的一些基本事实。我们的重点将是一类简单的量子力学系统，虽然它们具有一定的优雅性，但不会表现出任何深奥的数学。相反，我们会将它们视为试验场，使我们能够对超对称性建立一些直觉，同时开发一些有用的计算技术。然后，我们将在第 3 节中将这些内容应用于具有更深入数学谱系的问题。

As an aside: there's only one other place in physics where we care about the over- all value of the ground state energy, and that's the cosmological constant in general relativity. So far, sadly, no plausible link has been found between the value of the cosmological constant and the supersymmetry algebra.

顺便说一句：物理学中只有另一个地方我们关心基态能量的整体价值，那就是广义相对论中的宇宙常数。遗憾的是，到目前为止，我们还没有发现宇宙常数的值和超对称代数之间存在合理的联系。

Finally, one last piece of terminology. If a ground state with energy $E = 0$ exists, then we say that supersymmetry is unbroken. If the ground state has energy $E > 0$ then we say that supersymmetry is broken. This language is really adopted from higher dimensions where symmetries that do not leave the vacuum invariant are said to be "spontaneously broken". In the present

最后，最后一个术语。如果存在能量 $E = 0$ 的基态，则我们说超对称性未破缺。如果基态的能量 $E > 0$，则我们说超对称性被破坏。这种语言实际上是从更高维度采用的，在更高维度中，不保持真空不变量的对称性被称为“自发破缺”。在目前的情况下，我们说如

context we say that supersymmetry is broken if the vacuum is not annihilated by the supercharges: the connection to symmetries will become clearer as we proceed.

果真空没有被超电荷湮灭，超对称性就会被打破：随着我们的继续，与对称性的联系将变得更加清晰。

1.2 A Particle in a Potential

势中的单粒子

1.2.1 Ground States

基态

The magic of supersymmetry means that, at least for the ground state, the Schrödinger equation has morphed from a challenging second order differential equation into a pair of decoupled, first order differential equations. Note that this same trick doesn't work to figure out the excited states of the theory. We can't solve for the whole spectrum. But we can solve for the ground state.

超对称的魔力意味着,至少对于基态,Schrödinger 方程已经从一个具有挑战性的二阶微分方程变成了一对解耦的一阶微分方程。请注意,同样的技巧无法弄清楚理论的激发状态。我们无法解决整个频谱。但是我们可以解决基态。

Usually in a double well potential, the particle can lower its energy by tunnelling through the barrier and sitting in a superposition of both states. But that's not the case here because the two wavefunctions live in different components of spin space. This kills the possibility for tunnelling. Instead, the supersymmetric set-up is closer to our naive, classical guess of the ground states, with a Gaussian around each minima giving a good approximation to the ground state. Our arguments above tell us that the energy of this two-fold degenerate ground state is necessarily $E > 0$ We will say more about tunnelling in this system and how to compute the actual energy in Section 2.2.

通常在双势阱中，粒子可以通过隧道穿过势垒并处于两种状态的叠加来降低其能量。但这里的情况并非如此，因为这两个波函数位于自旋空间的不同组成部分。这消除了隧道效应的可能性。相反,超对称设置更接近于我们对基态的朴素经典猜测，每个最小值周围都有一个高斯分布, 可以很好地近似基态。我们上面的论点告诉我们，这个两倍简并基态的能量必然是 $E > 0$ 我们将在第 2.2 节中详细介绍该系统中的隧道效应以及如何计算实际能量。

We started with three states that we thought had the smallest energy - one for each minima - but only one survives as the true $E = 0$ ground state. The other two states must have some small, but non-zero energy. These states are the Gaussian localised in the middle vacuum, and the combination of states localised on the outside minima that is orthogonal to the ground state. Although it is far from obvious from staring at the potential, supersymmetry tells us that the energies of these states must be degenerate.

我们从我们认为能量最小的三种状态开始——每个极小值对应一种状态——但只有一种状态能够作为真正的 $E = 0$ 基态存在。其他两个状态必须具有一些小但非零的能量。这些状态是位于中真空的高斯状态，以及位于与基态正交的外部最小值上的状态组合。尽管从盯着势来看还远不明显，但超对称性告诉我们这些状态的能量一定是简并的。

1.2.2 The Witten Index

Witten 指标

Before we proceed, a few comments. Since $\mathcal{I}$ doesn't depend on β, you might wonder why we don't just set $\beta = 0$ and consider $\mathrm{Tr}(-1)^F$. Indeed, often the Witten index is written in this way as shorthand, but it's a dangerous thing to do. The quantity $\mathrm{Tr}(-1)^F$ is an infinite series of $+1$ and -1 and by pairing terms together in various ways you can get any answer that you like. Including $\mathrm{e}^{-\beta H}$ in the definition acts as a regulator for this sum, rendering it finite. Of course, it's a familiar regulator because it also appears in the partition function in statistical mechanics.

在我们继续之前，先发表一些评论。由于 $\mathcal{I}$ 不依赖于 β，你可能想知道为什么我们不直接设置 $\beta = 0$ 并考虑 $\mathrm{Tr}(-1)^F$。事实上，Witten 指数经常以这种方式写成速记，但这是一件危险的事情。数量 $\mathrm{Tr}(-1)^F$ 是 $+1$ 和 -1 的无限级数，通过以各种方式将术语配对在一起，您可以获得您喜欢的任何答案。定义中包含 $\mathrm{e}^{-\beta H}$ 作为该总和的调节器，使其成为有限的。当然，它是一个熟悉

The same arguments that show $\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{I}=0$ also show that $\mathcal{I}$ is independent of the parameters of the Hamiltonian H. This was demonstrated in the examples above although, as we also saw, it comes with a caveat: if you change the Hamiltonian too dramatically then you can lose states in your Hilbert space and this will change $\mathcal{I}$. This happens for the particle on a line whenever we change the power of the leading term in $h(x)$.

The Witten index counts the difference between the bosonic and fermionic $E=0$ states. However, in the simple examples considered above, it actually counts the number of $E=0$ states, positive if they're bosonic, negative if they're fermionic. One might wonder if, in practice, it always does this. Indeed, there's is some intuition that suggests this is the case. If there's no good reason for pairs of states to be stuck at $E=0$ then, as you vary parameters in the potential, it's tempting to think that they will be lifted to $E>0$.

However, it's not difficult to exhibit examples where, for example, $\mathcal{I}=0$ but there are a pair of bosonic and fermionic $E=0$ states. A particularly simple example arises from particle moving on a circle S^1 of radius R. The supercharge (1.5) and Hamiltonian (1.7) take the same form as before and are characterised by a periodic function $h(x)=h(x+2\pi R)$. We can follow our earlier footsteps to find a two parameter family of ground states labelled by $\alpha,\beta\in\mathbb{C}$

$$\Psi(x)=\alpha\begin{pmatrix}\mathrm{e}^{-h}\\0\end{pmatrix}+\beta\begin{pmatrix}0\\\mathrm{e}^{+h}\end{pmatrix}$$

This time, because the particle lives on a circle, there is no issue with the normalisability of the wavefunction. We see that the system has two linearly independent $E=0$ ground states for any choice of h. Yet, because one ground states lives in $\mathcal{H}_B$ and the other in $\mathcal{H}_F$, the Witten index of this system is $\mathcal{I}=0$. The potential (in blue) and wavefunctions (in orange and green) for $h(x)=\sin(x/R)$ are shown in Figure 4.

For this particle on the circle, the pair of states sticks at $E=0$ as we change the parameters of h, even though these ground states are not protected by the Witten index. One might wonder if there's a deeper reason for this. There is and it's related to the deeper mathematical concept of cohomology. We'll look at this further in Section 3.

的调节器，因为它也出现在统计力学的配分函数中。

显示 $\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{I}=0$ 的相同论证也表明 $\mathcal{I}$ 独立于哈密顿量 H 的参数。这在上面的示例中得到了证明，但正如我们也看到的那样，它带有一个警告：如果您将哈密顿量更改得太大，那么您可能会丢失希尔伯特空间中的状态，这将改变 $\mathcal{I}$。每当我们改变 $h(x)$ 中首项的幂时，这条线上的粒子就会发生这种情况。

Witten 指数计算了玻色子和费米子 $E=0$ 状态之间的差异。然而，在上面考虑的简单示例中，它实际上计算 $E=0$ 状态的数量，如果它们是玻色子，则为正，如果它们是费米子，则为负。人们可能想知道，在实践中，它是否总是这样做。事实上，有一些直觉表明情况确实如此。如果没有充分的理由让状态对停留在 $E=0$，那么当你改变势能参数时，很容易认为它们会被提升到 $E>0$。

然而，展示例子并不困难，例如，$\mathcal{I}=0$，但存在一对玻色子和费米子 $E=0$ 状态。一个特别简单的例子是粒子在半径为 R 的圆 S^1 上移动。增压 (1.5) 和哈密顿量 (1.7) 采用与之前相同的形式，并以周期函数 $h(x)=h(x+2\pi R)$ 为特征。我们可以按照之前的脚步找到一个由 $\alpha,\beta\in\mathbb{C}$ 标记的二参数基态族

这次，因为粒子生活在一个圆上，所以波函数的归一化性不存在问题。我们看到，对于 h 的任何选择，系统都有两个线性独立的 $E=0$ 基态。然而，由于一个基态位于 $\mathcal{H}_B$ 中，另一个基态位于 $\mathcal{H}_F$ 中，因此该系统的 Witten 指数为 $\mathcal{I}=0$。$h(x)=\sin(x/R)$ 的势（蓝色）和波函数（橙色和绿色）如图 4 所示。

对于圆上的这个粒子，当我们改变 h 的参数时，这对状态保持在 $E=0$，即使这些基态不受 Witten 指数的保护。人们可能想知道这是否有更深层次的原因。确实存在，并且它与更深层的上同调数学概念有关。我们将在第 3 节中进一步讨论这一点。

Finally, one last comment before we move on. The manipulations of the Witten index rely on the discreteness of the energy spectrum. There are more subtle situations, where a particle moves on a non-compact space without a potential, where the energy spectrum is continuous and, despite the bose-fermi degeneracy in the spectrum, strange things can happen that mean that $\mathcal{I}$ does, in fact, depend on β. We will not encounter situations of this kind in these lectures.

最后，在我们继续之前，还有最后一条评论。Witten 指数的操纵依赖于能谱的离散性。还有更微妙的情况，其中粒子在没有势能的非紧空间上移动，能量谱是连续的，尽管谱中存在玻色费米简并性，但可能会发生奇怪的事情，这意味着 $\mathcal{I}$ 事实上，依赖于 β。在这些讲座中我们不会遇到这种情况。

1.3 The Supersymmetric Action

超对称作用量

There is one fairly large omission in our discussion so far. As presented above, super- symmetric Hamiltonians have a nice algebraic structure. But we have no inkling of why supersymmetry has anything to do with symmetry!

到目前为止，我们的讨论中有一个相当大的遗漏。如上所述，超对称哈密顿量具有良好的代数结构。但我们不知道为什么超对称性与对称性有任何关系！

Usually in quantum mechanics, Hermitian operators that commute with the Hamiltonian correspond to conserved quantities and conserved quantities come, via Noether's theorem, from symmetries. This suggests that perhaps $Q + Q^\dagger$ is somehow the conserved charge associated to a symmetry. But what symmetry?

通常在量子力学中，与哈密顿量交换的厄米算子对应于守恒量,而守恒量通过 Noether 定理来自对称性。这表明 $Q+Q^\dagger$ 可能是与对称性相关的守恒电荷。但什么是对称性呢?

Often the Lagrangian framework is a better starting point when looking for symmetries. To this end, we would like to introduce a Lagrangian for our supersymmetric theory of a particle on a line. We know well how to think of position and momentum in the Lagrangian setting. But how do we incorporate the discrete $\mathbb{C}^2$ factor in the Hilbert space that gave us the all-important $\mathbb{Z}^2$ grading?

在寻找对称性时，Lagrangian 框架通常是一个更好的起点。为此，我们想为直线上粒子的超对称理论引入拉格朗日。我们很清楚如何考虑 Lagrangian 设置中的位置和动量。但是我们如何将离散的 $\mathbb{C}^2$ 因子合并到希尔伯特空间中，从而为我们提供最重要的 $\mathbb{Z}^2$ 分类呢?

The answer is that we should turn to fermions. In higher dimensions, adding a fermion to a Lagrangian gives another field. But in quantum mechanics, fermions simply offer a different way of describing some discrete aspect of the physics.

答案是我们应该转向费米子。在更高维度中，将费米子添加到拉格朗日量中会产生另一个场。但在量子力学中,费米子只是提供了一种不同的方式来描述物理学的某些离散方面。

Note that their kinetic terms are first order, like the Dirac action that we met in Quantum Field Theory, albeit without the intricacies of gamma matrices. We will first show that this action is equivalent to the supersymmetric Hamiltonian (1.7) describing a particle with an internal degree of freedom moving on a line. We'll then understand how to think of the supercharges Q in the Lagrangian formulation.

请注意,它们的动力学项是一阶的,就像我们在量子场论中遇到的 Dirac 作用一样,尽管没有 gamma 矩阵的复杂性。我们将首先证明这个作用相当于描述具有内部自由度的沿直线运动的粒子的超对称哈密顿量(1.7)。然后我们将了解如何考虑 Lagrangian 公式中的超荷 Q。

There is, however, a small subtlety awaiting us. We think of the Lagrangian as a classical object in which x and $\dot{x} = p$ be placed in any order. Relatedly, ψ and $\psi^\dagger$ are viewed as "classical Grassmann variables" in the action, which means that if one moves past the other then we just pick up a minus sign. But in

然而,有一个小微妙之处等待着我们。我们将拉格朗日函数视为一个经典对象，其中 x 和 $\dot{x} = p$ 可以按任意顺序放置。相关地，ψ 和 $\psi^\dagger$ 在动作中被视为“经典格拉斯曼变量”，这意味着如果一个移动超过另一个，那

the Hamiltionian, these are all to be thought of as quantum operators and, because of the commutation relations (1.15), ordering matters. Which ordering should we take?

In most other contexts, there is no way to fix this ambiguity and it reflects the fact that there are different ways to quantise a classical theory. However, for us, we do have a way to fix the ambiguity since the resulting Hamiltonian should be supersymmetric.

1.3.1 Supersymmetry as a Fermionic Symmetry

Note that these swap bosonic fields x for fermionic fields ψ. This is the characteristic feature of supersymmetry that distinguishes it from other symmetries. For this to make sense, the infinitesimal transformation parameter ϵ must be a Grassmann valued object.

The Supercharge is a Noether Charge

Finally, we can make good on our promise and see that the supercharges Q and $Q^\dagger$ are indeed Noether charges for supersymmetry. Usually when the action has a symmetry, we can construct the Noether charge by allowing the transformation parameter to depend on time. Things are no different here. We vary the action with $\epsilon = \epsilon(t)$. There are two steps where things differ from our previous calculation: first when we vary the kinetic terms, and again at the last where we see that the variation of the Lagrangian is a total derivative which requires an integration by parts.

It's slightly odd that the variation of the action involves $\dot{\epsilon}^\dagger$ but not $\dot{\epsilon}$. We can trace this to our choice of fermion kinetic term $\psi^\dagger\dot{\psi}$, which is asymmetric between ψ and $\psi^\dagger$. We could instead start with the more symmetric choice

We can now go full circle. In the operator framework of quantum mechanics, the Noether charges generate the symmetry. Again, supersymmetry is no different.

1.4 A Particle Moving in Higher Dimensions

1.4.1 A First Look at Morse Theory

This means that our supersymmetric quantum mechanics will describe a particle moving in R^n with 2^n internal states.

There's a useful geometrical way to think about these states. At the top of the pyramid depicted above we have wavefunctions that look like $\phi(x)|0\rangle$: these are just functions over R^n.

么我们只会拾取一个负号。但在哈密尔顿量纲中，这些都被视为量子算子，并且由于交换关系 (1.15)，排序很重要。我们应该采取哪种顺序?

在大多数其他情况下，没有办法解决这种歧义，它反映了一个事实，即有不同的方法来量化经典理论。然而，对我们来说，我们确实有办法解决歧义，因为所得的哈密顿量应该是超对称的。

超对称性作为费米子对称性

请注意，这些将玻色子场 x 交换为费米子场 ψ。这是超对称性区别于其他对称性的特征。为了使这一点有意义，无穷小变换参数 ϵ 必须是 Grassmann 值对象。

超荷是 Noether 荷

最后,我们可以兑现我们的承诺,看到超级电荷 Q 和 $Q^\dagger$ 确实是超对称性的 Noether 荷。通常当作用具有对称性时,我们可以通过允许变换参数依赖于时间来构造诺特电荷。这里的情况并没有什么不同。我们用 $\epsilon = \epsilon(t)$ 来改变操作。有两个步骤与我们之前的计算有所不同：首先，我们改变了动力学项；最后，我们看到 Lagrangian 的变化是全导数,需要分部分积分。

有点奇怪的是,动作的变化涉及 $\dot{\epsilon}^\dagger$ 而不是 $\dot{\epsilon}$。我们可以将其追溯到我们对费米子动力学项 $\psi^\dagger\dot{\psi}$ 的选择，它在 ψ 和 $\psi^\dagger$ 之间是不对称的。我们可以从更对称的选择开始

现在我们可以回到原点了。在量子力学算子框架中, 诺特电荷产生对称性。同样, 超对称性也不例外。

在更高维度中运动的单粒子

Morse 理论初探

这意味着我们的超对称量子力学将描述一个在 R^n 中运动且内部状态为 2^n 的粒子。

有一种有用的几何方法来思考这些状态。在上面描述的金字塔顶部，我们有看起来像 $\phi(x)|0\rangle$ 的波函数：这些只是 R^n 上的函数。

At the next level, the wavefunctions look like $\phi(x)\psi(x)^{\dagger i}|0\rangle$ and come with an internal index $i = 1, \ldots, n$. We usually think of objects on R^n that carry such an index as vectors. However, as we now explain, the anti-symmetric nature of the Grassmann variable means that it's much more natural to think about these states as one-forms on R^n.

All of this suggests that we should make the identification between Grassmann variables and forms

$$\psi^{\dagger i} \longleftrightarrow \mathrm{d}x^i \wedge$$

在下一个级别,波函数看起来像 $\phi(x)\psi(x)^{\dagger i}|0\rangle$ 并带有内部索引 $i = 1, \ldots, n$。我们通常将 R^n 上带有此类索引的对象视为向量。然而，正如我们现在所解释的，格拉斯曼变量的反对称性质意味着将这些状态视为 R^n 上的一种形式更为自然。

所有这些都表明我们应该对格拉斯曼变量和形式进行认同

The Supersymmetric Hamiltonian

The generalisation of the story above is now the following: for each negative eigenvalue $\lambda_k < 0$, we should excite the corresponding collection of fermions $e_k^j \psi^{\dagger j}$. Meanwhile, for each positive eigenvalue $\lambda_k > 0$, we should just leave well alone: we're better off in the unexcited state. At a given critical point $x = X$, the semi-classical ground state then sits in the part of the Hilbert space given by

In the geometrical language, this means that the ground state wavefunction is a p-form, where $p = \mu(X)$ is the Morse index.

Note that we're not assuming that all critical points of h correspond to true $E = 0$ ground states of the theory. It may well be that some get lifted to non-zero energy and, later in these lectures, we'll put in some effort to understand when this happens. But that's not relevant for computing the Witten index since any such states must get lifted in pairs and so cancel out.

The same formula (1.29) also holds for our earlier model with a single x and ψ. There a maximum of h was necessarily followed by a minimum, so the sum over critical points could never exceed $+1$ or drop below -1. Now, however, we could have multiple ground states. For example, we could have a situation where all the critical points X have $\mu(X)$ even. In this case, they all contribute $+1$ to the Witten index and each of them must correspond to a true, $E = 0$ ground state of the system.

超对称 Hamiltonian

上述故事的概括如下：对于每个负特征值 $\lambda_k < 0$，我们应该激发相应的费米子集合 $e_k^j \psi^{\dagger j}$。同时，对于每个正特征值 $\lambda_k > 0$，我们应该不管它：我们在非激励状态下会更好。在给定的临界点 $x = X$ 处，半经典基态位于由下式给出的希尔伯特空间部分

在几何语言中，这意味着基态波函数是 p 形式，其中 $p = \mu(X)$ 是 Morse 指数。

请注意，我们并不假设 h 的所有临界点都对应于理论的真实 $E = 0$ 基态。很可能有些能量会被提升到非零能量，在这些讲座的后面，我们将努力理解这种情况何时发生。但这与计算 Witten 指数无关，因为任何此类状态都必须成对提升，从而抵消。

相同的公式 (1.29) 也适用于我们之前具有单个 x 和 ψ 的模型。h 的最大值后面必然有最小值，因此临界点的总和永远不会超过 $+1$ 或低于 -1。然而，现在我们可以有多个基态。例如，我们可能会遇到这样一种情况，即所有临界点 X 都有 $\mu(X)$ 为偶数。在这种情况下，它们都为 Witten 指数贡献了 $+1$，并且它们中的每一个都必须对应于系统的真实 $E = 0$ 基态。

1.4.2 More Supersymmetry and Holomorphy

Hamiltonian that can be written in this form is said to have $N = 2q$ supersymmetries, with the 2 because each Q is complex. In this convention, the kind of quantum mechanics that we considered up until now is said to have $N = 2$ supersymmetry. (I should warn you that the nomenclature for counting supersymme-

更多超对称和全纯

可以写成这种形式的哈密顿量据说具有 $N = 2q$ 超对称性，其中 2 是因为每个 Q 都是复数。在这个惯例中，我们到目前为止所考虑的量子力学被认为具有 $N = 2$ 超对称性。(我应该警告你，量子力学中计算超对称

try generators in quantum mechanics is not completely standard: things settle down once we go to higher dimensional quantum field theories.)

发生器的术语并不完全标准：一旦我们进入更高维的量子场论，事情就会稳定下来。）

At first glance, it looks like these are simply different Hamiltonians. However, all is not lost: these two Hamiltonians coincide if the function $h(x, y)$ obeys

乍一看，这些似乎只是不同的哈密顿量。然而，一切并没有丢失：如果函数 $h(x, y)$ 服从（以下条件），这两个 Hamiltonian 重合

$$\frac{\partial^2 h}{\partial x^i\,\partial x^j} = -\frac{\partial^2 h}{\partial y^i\,\partial y^j} \text{ and } \frac{\partial^2 h}{\partial x^i\,\partial y^j} = \frac{\partial^2 h}{\partial y^i\,\partial x^j}$$

There's a much nicer way of writing these conditions: as we will now see, they are telling us that $h(x, y)$ is related to a holomorphic function.

有一种更好的方式来编写这些条件：正如我们现在所看到的，它们告诉我们 $h(x, y)$ 与全纯函数相关。

Complex Variables

复数变量

Here the word "complex" is in inverted com- mas because our original Grassmann variables were already complex; we just introduce different linear combinations

这里“复数”这个词用引号引起来，因为我们最初的格拉斯曼变量已经是复数了；我们只是引入不同的线性组合

Supersymmetric Lagrangians of this kind, involving complex scalar fields and fermions, are usually referred to as Landau-Ginzburg theories. This is a nod to the Landau-Ginzburg theories that we met when discussing phase transitions in Statistical Physics. But it's not a very good nod. In particular, the theory (1.25) with just a single supersymmetry is just as much related to the kinds of models that Landau and Ginzburg considered but is never given this name in the context of supersymmetry. It's best to think of the name "Landau-Ginzburg" for the Lagrangian (1.33) as merely a quirk of history and forget that the term is also used elsewhere in physics.

这种涉及复标量场和费米子的超对称拉格朗日量通常被称为 Landau-Ginzburg 理论。这是对我们在讨论统计物理学中的相变时遇到的 Landau-Ginzburg 理论的认可。但这并不是一个很好的点头。特别是，只有一个超对称性的理论(1.25)与 Landau 和 Ginzburg 考虑的模型类型同样相关，但在超对称性的背景下从未被赋予这个名称。最好将拉格朗日量 (1.33) 的名称“Landau-Ginzburg”视为历史的一个巧合，而忘记该术语在物理学的其他领域也有使用。

The Landau-Ginzburg Lagrangian depends on a single holomorphic function $W(z)$. This is known as the superpotential. The fact that extended supersymmetry comes hand in hand with holomorphy and associated ideas in complex analysis is extremely important. We will not discuss quantum mechanics with $N = 4$ supersymmetry in these lectures, but it's not for want of interesting content. In particular, there is a beautiful relationship to a form of complex geometry known as "Kähler geometry" that underlies many of the most interesting results in this subject.

Landau-Ginzburg Lagrangian 依赖于单个全纯函数 $W(z)$。这被称为超潜力。扩展的超对称性与复分析中的全纯性和相关思想齐头并进，这一事实极其重要。在这些讲座中，我们不会讨论具有 $N = 4$ 超对称性的量子力学，但这并不是因为缺乏有趣的内容。特别是，它与一种被称为“Kähler 几何”的复杂几何形式有着美妙的关系，它是该学科中许多最有趣的结果的基础。

Furthermore, when we go to higher dimensional field theories, supersymmetry generators are associated to spinors and these necessarily have more than one component. This means that in, for example, $d = 3 + 1$ dimensions, the simplest supersymmetric theories have the form (1.33) and are based on complex, rather than real variables. In that context, the holomorphy of the superpotential goes a long way towards allowing us to solve some

此外，当我们研究高维场论时，超对称发生器与旋量相关联，并且它们必然具有多个分量。这意味着，例如，在 $d = 3 + 1$ 维度中，最简单的超对称理论具有 (1.33) 的形式，并且基于复变量而不是实变量。在这种情况下，超势的全纯性对于让我们解决超对称量子场理论的一些复杂特征大有帮助。超

complicated features of supersymmetric quantum field theories. This is covered in some detail in the lectures on Supersymmetric Field Theory.

对称场论讲座对此进行了详细介绍。

The Ground States

We know from our discussion in Section 1.4.1 what we should do next: we compute the Morse index for each critical point, meaning the number of positive eigenvalues of the Hessian of h. But this is trivial for a holomorphic function $W(z)$.

We learn that in theories with $N = 4$ supersymmetry, every critical point of W is a true $E = 0$ ground state of the quantum theory.

基态

从第 1.4.1 节的讨论中我们知道下一步应该做什么：我们计算每个临界点的 Morse 指数，即 h 的 Hessian 矩阵的正特征值的数量。但这对于全纯函数 $W(z)$ 来说是微不足道的。

我们了解到，在具有 $N = 4$ 超对称性的理论中，W 的每个临界点都是量子理论的真正 $E = 0$ 基态。

1.4.3 Less Supersymmetry and Spinors

更少超对称和旋量

It's also possible to consider theories with less supersymmetry than our starting point. In fact, this is easy to achieve. We return to our theory with $N = 2$ supersymmetry and impose a reality condition on the Grassmann variables

也可以考虑比我们的起点具有更少超对称性的理论。事实上，这很容易实现。我们回到 $N = 2$ 超对称性的理论，并对 Grassmann 变量施加现实条件

$$\psi^{\dagger i} = \psi^i$$

Real quantum mechanical Grassmann variables like this are called Majorana modes or Majorana fermions.

像这样的真正的量子力学格拉斯曼变量被称为 Majorana 模式或 Majorana 费米子。

This is usually referred to as $N = 1$ supersymmetry. (You will sometimes see the terminology $N = 1/2$ supersymmetry in the literature, counting complex supercharges rather than real.)

这通常称为 $N = 1$ 超对称性。（有时您会在文献中看到术语 $N = 1/2$ 超对称性，计算的是复杂的增压而不是真实的。）

Here our interest lies in a very specific property of these theories: how should we think of the internal degrees of freedom generated by the real fermions ψ^i?

这里我们的兴趣在于这些理论的一个非常具体的属性：我们应该如何思考由实费米子 ψ^i 生成的内部自由度?

This means that the fermions in this theory should be viewed as gamma matrices! The Clifford algebra has a unique irreducible representation of dimension $2^{n/2}$ if n is even and $2^{(n-1)/2}$ if n is odd. This strongly suggests that the internal degrees of freedom of the particle described by the action (1.34) have something to do with spinors on $\mathbb{R}^n$.

这意味着该理论中的费米子应该被视为 gamma 矩阵！如果 n 为偶数，则 Clifford 代数具有维度 $2^{n/2}$ 的唯一不可约表示；如果 n 为奇数，则具有 $2^{(n-1)/2}$。这强烈表明由作用 (1.34) 描述的粒子的内部自由度与 $\mathbb{R}^n$ 上的旋量有关。

This is precisely the dimension of a Dirac spinor on R^n.

这正是 R^n 上狄拉克旋量的维数。

There is more to say about these spinors. Under a rotation in R^n, the Dirac spinor transforms in the representation generated by $\Sigma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$. (See the lectures on Quantum Field Theory for more details of this.) However, in even dimension, as we have here, this is not an irreducible representation. It is composed of two smaller representations known as chiral spinors or Weyl spinors.

关于这些旋量还有更多要说的。在 R^n 的旋转下,Dirac 旋量变换为 $\Sigma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$。(有关更多详细信息，请参阅量子场论讲座。) 然而，在偶数维中，正如我们在这里所看到的，这不是一个不可约的表示。它由两个较小的表示形式组成，称为手性旋量或 Weyl 旋量。

These arise because we can always construct an operator $\hat{\gamma}$ that is analogous to γ^5 in four dimensions.

In the context of our supersymmetric quantum mechanics, this $\hat{\gamma}$ operator has a very natural meaning. The eigenvalues are simply states with an even or odd number of $c^\dagger$ operators excited. In other words, this plays the role of our fermion number.

This means that $\hat{\gamma}$ determines whether states live in $\mathcal{H}_B$ or $\mathcal{H}_F$.

The punchline of this argument is that quantising real fermions, appropriate for $N = 1$ supersymmetry, gives Dirac spinors on $\mathbb{R}^n$, at least for n even. These have dimension $2^{n/2}$. Meanwhile, while quantising complex fermions, appropriate for $N = 2$ supersymmetry, gives forms on $\mathbb{R}^n$. These have dimension 2^n. We'll have use for quantum mechanics with $N = 1$ supersymmetry in Section 3.3 where we discuss the Atiyah-Singer index theorem.

As an aside, clearly the construction of spinors and forms on $\mathbb{R}^n$ from Grassmann degrees of freedom is closely related. This also suggests that you can take $2^{n/2}$ different Dirac spinors and bundle them together to look like forms. Such a construction is called Kähler-Dirac fermions. It won't play a role in these lectures, but arises in a number of other areas of physics including topological twisting of field theories and lattice gauge theory where it goes by the name of staggered fermions.

The Case of n Odd: A Subtle Anomaly

We still have to understand the case of n odd. Here there is a surprise. Quantum mechanical theories with an odd number of Majorana modes don't make any sense! They are an example of what is sometimes called an *anomalous* quantum theory: a seemingly sensible classical theory that cannot be quantised.

For us, this means that theories with $N = 1$ supersymmetry are restricted to describe a particle moving in an even dimensional space, like $\mathbb{R}^n$ with n even.

出现这些问题是因为我们总是可以构造一个类似于四个维度中的 γ^5 的运算符 $\hat{\gamma}$。

在超对称量子力学的背景下，这个 $\hat{\gamma}$ 运算符具有非常自然的含义。特征值只是由偶数或奇数个 $c^\dagger$ 运算符激发的状态。换句话说，这扮演了我们的费米子数的角色。

这意味着 $\hat{\gamma}$ 决定状态是否位于 $\mathcal{H}_B$ 或 $\mathcal{H}_F$ 中。

这个论点的要点是，量子化实费米子，适用于 $N = 1$ 超对称性，给出 $\mathbb{R}^n$ 上的 Dirac 旋量，至 少对于 n 偶数。它们的维度为 $2^{n/2}$。同时，在量子化复费米子时，适用于 $N = 2$ 超对称性，给出了 $\mathbb{R}^n$ 上的形式。它们的维度为 2^n。我们将在第 3.3 节中讨论 Atiyah-Singer 指数定理，使用具有 $N = 1$ 超对称性的量子力学。

顺便说一句，显然，来自 Grassmann 自由度的 $\mathbb{R}^n$ 上的形式和旋量的构造是密切相关的。这也表明您可以采用 $2^{n/2}$ 不同的 Dirac 旋量并将它们捆绑在一起以看起来像形式。这种结构称为 Kähler-Dirac 费米子。它不会在这些讲座中发挥作用，但会出现在物理学的许多其他领域，包括场论的拓扑扭曲和晶格规范理论，其名称为交错费米子。

n 为奇数的情况：微妙的异常

我们仍然需要理解 n 奇数的情况。这里有一个惊喜。具有奇数个 Majorana 模式的量子力学理论没有任何意义！它们是有时被称为异常量子理论的一个例子：一种看似合理但无法量子化的经典理论。

对我们来说，这意味着具有 $N = 1$ 超对称性的理论仅限于描述在偶数维空间中移动的粒子，例如 $\mathbb{R}^n$ 和 n 偶数。

2 Supersymmetry and the Path Integral

超对称和路径积分

In addition to making the symmetry aspect of supersymmetry manifest, the Lagrangian description of the quantum mechanics has one additional advantage: it allows us to bring the path integral to bear on the problem.

We'll make plenty of use of the path integral in later stud-

除了使超对称性的对称性变得明显之外，量子力学的 Lagrangian 描述还有一个额外的优点：它允许我们用路径积分来解决问题。

我们将在以后的超对称系统研究中大量

ies of supersymmetric systems. The purpose of this section is to understand some of the basic properties of the quantum mechanical path integral and how we can use it to compute quantities of interest in supersymmetric theories.

使用路径积分。本节的目的是了解量子力学路径积分的一些基本属性，以及如何使用它来计算超对称理论中感兴趣的量。

2.1 The Partition Function and the Index

配分函数和指标

Our goal now is to manipulate (2.2) so that the left-hand-side looks like the partition function Z. There are a number of differences that we need to fix.

where now the boundary conditions just tell us that we should integrate over all possible closed paths. Equivalently, we can implement this condition by insisting that we work in periodic Euclidean time, so that τ is a coordinate on a circle S^1, with

$$\tau \equiv \tau + \beta$$

Although we've derived this punchline in the context of quantum mechanics, it also works in quantum field theory. If you want to compute the thermal partition function of any quantum field theory, you simply need to work in periodic, Euclidean time. This will tell you information about the equilibrium properties of the system at temperature $T = 1/\beta$.

我们现在的目标是操纵 (2.2), 使左侧看起来像配分函数 Z。我们需要修复许多差异。

现在边界条件只是告诉我们应该整合所有可能的闭合路径。等价地，我们可以通过坚持在 Euclidean 周期时间内工作来实现这个条件，因此 τ 是圆 S^1 上的坐标，其中

尽管我们是在量子力学的背景下得出这个妙语的,但它也适用于量子场理论。如果你想计算任何量子场理论的热配分函数，你只需要在周期性的 Euclidean 时间中工作。这将告诉您有关系统在温度 $T = 1/\beta$ 时的平衡特性的信息。

2.1.1 An Example: The Harmonic Oscillator

例子：谐振子

That leaves us with the first infinite product in (2.5) to deal with. And that's more tricky because it diverges. To better understand such terms, we should really go back and dissect the path integral to figure out where it came from. (For example, the partition function should be dimensionless but this term has dimension of $[\text{Energy}]^{2\infty}$ which is a hint that we didn't define our measure very well.) However, in the spirit of this course we're going to treat this term as blithely as possible. And, for those physicists of a blithe disposition, there are few tools more useful than zeta function regularisation.

However, $\zeta(s)$ is defined for all values of s. The idea is that we use this to give meaning to divergent sums. For example, we could think of the sum of all positive integers as $\zeta(-1) = -1/12$. Although these zeta function games seem rather inane when you first meet them, the magic is that they tend to give the right answers when used to regulate divergences in quantum field theory. (For example, in the lectures on String Theory we first invoked the unconvincing $\zeta(-1) = -1/12$ argument to compute the critical dimension of the string, and then spent a significant amount of

这让我们需要处理 (2.5) 中的第一个无限乘积。这更加棘手, 因为它存在发散。为了更好地理解这些项，我们真的应该回顾并剖析路径积分以找出它的来源。(例如, 配分函数应该是无量纲的,但这项的维数为 $[\text{Energy}]^{2\infty}$, 这暗示我们没有很好地定义我们的度量。)但是，本着本课程的精神，我们将尽可能轻松地对待这个术语。而且，对于那些性格开朗的物理学家来说, 没有什么工具比 zeta 函数正则化更有用了。

然而，$\zeta(s)$ 是为 s 的所有值定义的。我们的想法是，我们用它来赋予发散的总和意义。例如，我们可以将所有正整数的总和视为 $\zeta(-1) = -1/12$。尽管这些 zeta 函数游戏在你第一次见到时看起来相当愚蠢，但神奇的是，当它们用于调节量子场论中的发散时，它们往往会给出正确的答案。(例如，在弦理论讲座中，我们首先调用了令人难以置信的 $\zeta(-1) = -1/12$ 参数来计算弦的临界

time rederiving this using conformal field theory techniques where the divergences were absent.)

尺寸，然后花费大量时间使用不存在发散的共形场论技术重新推导它。）

2.1.2 Fermions: Periodic or Anti-Periodic?

费米子：周期还是反周期?

Relatedly, there are two natural partition functions that we could construct for fermions. In addition to the thermal partition function $\mathrm{Tr}\,\mathrm{e}^{-\beta H}$, we could also consider the quantity $\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H}$. In supersymmetric quantum mechanics, $\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta}$ is the Witten index and is necessarily an integer. But, for a general fermionic system it is just a different way to sum the states, weighted by an extra minus sign. I'll refer to the quantity $\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H}$ as an "index" in both supersymmetric and non-supersymmetric theories, although strictly this terminology should be reserved for the former case.

It seems plausible that inserting a factor of $(-1)^F$ in the trace would flip the sign of the fermion as we go around the Euclidean temporal circle. But which boundary condition corresponds to the thermal partition function, and which to the index?

相关地，我们可以为费米子构造两个自然配分函数。除了热分配函数 $\mathrm{Tr}\,\mathrm{e}^{-\beta H}$ 之外，我们还可以考虑数量 $\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H}$。在超对称量子力学中，$\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H}$ 是 Witten 指标，并且必然是整数。但是，对于一般的费米子系统来说，这只是一种不同的状态求和方式，由一个额外的负号加权。我将把数量 $\mathrm{Tr}(-1)^F\mathrm{e}^{-\beta H}$ 称为超对称和非超对称理论中的“指标”，尽管严格来说这个术语应该保留给前者的情况。

当我们绕 Euclidean 时间圆运行时，在迹中插入一个因子 $(-1)^F$ 会翻转费米子的符号，这似乎是合理的。但是哪个边界条件对应于热分配函数，哪个边界条件对应于指标呢?

As always, the right answer can be found by going back to first principles and looking at how one constructs the path integral from small, but finite, time steps.

与往常一样，通过回到第一原理并研究如何从小而有限的时间步构建路径积分，可以找到正确的答案。

The Fermionic Oscillator

费米子谐振子

Clearly the index isn't independent of β for this simple model: that is only true for supersymmetric systems.

显然，对于这个简单模型，该指数并不独立于 β：这只适用于超对称系统。

It will not have escaped your attention that the path integral calculation was a lot of work to get the partition function for a two state system. However, as we come to consider more complicated quantum mechanical models, including higher dimensional field theories, the path integral starts to come into its own and, ultimately, is much more convenient than canonical quantisation.

您一定会注意到，为了获得二态系统的配分函数，路径积分计算需要大量工作。然而，当我们开始考虑更复杂的量子力学模型，包括高维场论时，路径积分开始发挥作用，并且最终比规范量子化方便得多。

2.1.3 The Witten Index Revisited

重新审视 Witten 指标

It's useful to understand why, from the path integral perspective, the Witten index is always an integer for supersymmetric theories. After all, something magical must happen where we do an infinite dimensional integral but, regardless of the parameters in the integrand, we always get an integer. How does this come about? The answer is a rather special property of supersymmetric path integrals known as localisation.

从路径积分的角度来看，理解为什么 Witten 指数对于超对称理论总是整数是有用的。毕竟，当我们进行无限维积分时，一定会发生一些神奇的事情，但无论被积函数中的参数如何，我们总是得到一个整数。这是怎么发生的? 答案是超对称路径积分的一个相当特殊的属性，称为局域化。

The bosonic field $x(\tau)$ is always periodic: $x(\tau) = x(\tau + \beta)$. But that means that the supersymmetry transformations (2.14)

玻色子场 $x(\tau)$ 始终是周期性的：$x(\tau) = x(\tau+\beta)$。但这意味着超对称变换（2.14）仅在

only hold if ψ is also periodic: $\psi(\tau) = \psi(\tau + \beta)$.

ψ 也是周期性的情况下成立：$\psi(\tau) = \psi(\tau + \beta)$。

As we've seen, if we wish to compute the thermal partition function $Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H}$ using the path integral then we must give the fermions anti-periodic boundary conditions. But, in doing so, we break supersymmetry. In contrast, if we wish to compute the Witten index $\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H}$ then the path integral enjoys supersymmetry. This makes intuitive sense. In general, the full partition function Z is no easier to compute for a supersymmetric theory than a non-supersymmetric theory. But the Witten index is much easier. And, from the path integral perspective, this manifests itself because of the transformations (2.14).

正如我们所看到的，如果我们希望使用路径积分计算热配分函数 $Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H}$，那么我们必须给出费米子反周期边界条件。但是，这样做就打破了超对称性。相反，如果我们希望计算 Witten 指标 $\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H}$ 则路径积分具有超对称性。这很直观。一般来说，超对称理论的完整配分函数 Z 并不比非超对称理论更容易计算。但 Witten 指标要容易得多。并且，从路径积分的角度来看，这由于变换而显现出来（2.14）。

The extra term in the integrand has a special form because it is itself a supersymmetry variation. To see this, it's useful to use the supersymmetry generators that we introduced in (1.21). With a rescaled potential λh and Euclidean time, these become

被积函数中的额外项具有特殊形式，因为它本身就是超对称变体。为了看到这一点，使用我们在 (1.21) 中引入的超对称生成器很有用。通过重新调整势 λh 和欧几里得时间，这些变为

(Note that there's no danger of a boundary term here because τ parameterises a circle and all fields are periodic.)

（请注意，这里不存在边界项的危险，因为 τ 参数化了一个圆，并且所有域都是周期性的。）

But we also know that the action is invariant under supersymmetry and, as we showed in (1.22), this can be written as $\mathcal{Q}_\lambda^\dagger S_E = 0$. This means that our final expression is a path integral of a total supersymmetry variation,

但我们也知道，作用在超对称性下是不变的，正如我们在 (1.22) 中所示，这可以写成 $\mathcal{Q}_\lambda^\dagger S_E = 0$。这意味着我们的最终表达式是总超对称性变化的路径积分，

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger \mathcal{Q}_\lambda^\dagger \left(-\mathrm{e}^{-S_E} \oint \mathrm{d}\tau h'\psi \right)$$

The integrand is said to be Q-exact. The all-important point is that the integral of any Q-exact quantity always vanishes.

据说该被积函数是 Q 精确的。最重要的一点是任何 Q 精确数量的积分总是消失。

We're left with the bosonic functional derivatives $\delta/\delta x(t)$. Here we have a total derivative, albeit of a functional kind and we would expect such an integral to be given by the boundary term. The question is: what should we consider to be the boundary of this functional space? Large $x(t)$? Wildly varying $x(t)$? Either way, the boundary term vanishes. This is because there is an exponential suppression from the action e^{-S_E} that asymptotes quickly to zero for anything that you might reasonably consider to be the boundary. The upshot of these arguments is that

我们剩下玻色子泛函导数 $\delta/\delta x(t)$。这里我们有一个全导数，尽管是函数类型，并且我们期望这样的积分由边界项给出。问题是：我们应该将什么视为这个功能空间的边界？$x(t)$ 很大？$x(t)$ 变化很大？无论哪种方式，边界项都会消失。这是因为 e^{-S_E} 操作存在指数抑制，对于您可能合理地认为是边界的任何内容，它都会快速渐近到零。这些争论的结果是

Now we're in business. Because the Witten index is independent of λ, we can calculate it in the limit $\lambda \to \infty$. Here the potential term in the action suppresses all contribution except for the a finite number of constant maps,

现在我们开始做生意了。因为 Witten 指标与 λ 无关，所以我们可以在 $\lambda \to \infty$ 的极限下计算它。这里，动作中的势项抑制了除了有限数量的常量映射之外的所有贡献，

$$x(\tau) = X \mathrm{s.t.} h'(X) = 0$$

There are the critical points of h. The phenomenon of an integral - in this case an infinite dimensional functional integral - receiving contributions from just a handful of points is known as *localisation*. It is a property of supersymmetric path integrals that is not shared by most other quantum systems.

Indeed, taking the $\lambda \to \infty$ limit should be viewed as suppressing the non-linear inter- actions in the potential. The statement that the Witten index is independent of λ is equivalently to saying that the one-loop approximation is, in fact, exact.

h 存在临界点。积分（在本例中为无限维泛函积分）仅从少数点接收贡献的现象称为局部化。这是大多数其他量子系统所不具备的超对称路径积分的性质。

事实上，采用 $\lambda \to \infty$ 限制应该被视为抑制势能中的非线性相互作用。Witten 指标独立于 λ 的说法相当于说单圈近似实际上是精确的。

2.2 Instantons

瞬子

Much of our story so far has revolved around understanding the structure of ground states in supersymmetric systems. A common theme - one familiar from other quantum mechanical models - is that the existence of multiple classical ground states does not necessarily mean that there are multiple quantum ground states.

In this section, we develop a more hands-on understanding of how ground states are lifted. Once again, our tool of choice will be the path integral and, as we will see, this provides a particularly direct way to think about quantum tunnelling and related phenomena. We will explore how this works in some detail, first in ordinary quantum mechanical systems and then in those that exhibit supersymmetry.

Tunnelling is particularly easy to understand from the path integral perspective. It arises from paths that start at one minima and end up at another. If the parameters in the potential are such that we can do a semi-classical analysis, then the amplitude for tunnelling is dominated by the classical paths that minimise S_E. There is a rather cute way of finding these paths.

We write the action (2.18) by completing the square

If we fix the end points x_i and x_f to be two distinct minima, then the action is minimised when this inequality is saturated with the most stringent $\pm$ sign. This means that if $h(x_f) > h(x_i)$, we should solve the equation

到目前为止，我们的大部分故事都是围绕着理解超对称系统中的基态结构展开的。一个常见的主题（在其他量子力学模型中很常见）是多个经典基态的存在并不一定意味着存在多个量子基态。

在本节中，我们将更深入地了解基态是如何提升的。我们选择的工具将再次是路径积分，正如我们将看到的，这提供了一种特别直接的方式来思考量子隧道效应和相关现象。我们将首先在普通量子力学系统中，然后在表现出超对称性的系统中，详细探讨它是如何工作的。

从路径积分的角度来看，隧道特别容易理解。它产生于从一个最小值开始并以另一个最小值结束的路径。如果势能中的参数使得我们可以进行半经典分析，那么隧道效应的幅度将由最小化 S_E 的经典路径主导。有一种相当可爱的方法可以找到这些路径。

我们通过配平方来写出作用量（2.18）

如果我们将端点 x_i 和 x_f 固定为两个不同的最小值，那么当这个不等式用最严格的 $\pm$ 符号饱和时，动作就会最小化。这意味着如果 $h(x_f) > h(x_i)$，我们应该求解方程

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}h}{\mathrm{d}x}$$

Solutions to this equation are known as *instantons*. The name is chosen (by 't Hooft) to mimic the names give to particles but, as will see, these solutions are not localised in space but in (Euclidean) time and so occur just for an instant. If $h(x_f) < h(x_i)$, we should solve the other equation

该方程的解称为 瞬子。这个名称（由't Hooft 命名）选择是为了模仿粒子的名称，但是，正如我们将看到的，这些解决方案并不局限于空间，而是局限于（Euclidean）时间，因此只发生一瞬间。如果 $h(x_f) < h(x_i)$，我们应该解另一个方程

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\frac{\mathrm{d}h}{\mathrm{d}x}$$

Solutions to this equations are called *anti-instantons*. They interpolate between the two vacua in the opposite direction to instantons.

For more general $h(x)$ the exact solution of the instanton may be harder to come by but its simple to get an intuitive feel for its properties. Viewed from the usual perspective of Lagrangian dynamics, the Euclidean action (2.18) describes a particle moving in a potential $-V(x)$. This is shown on the right-hand side of Figure 6 for the double well potential. The instanton (or anti-instanton) describes a particle that starts at one maximum of $-V(x)$ at $\tau \to -\infty$ and then rolls down and up to another maximum, reaching the peak only at $\tau \to +\infty$.

If $V(x)$ has multiple minima, then we can only find solutions to the instanton equations (2.20) and (2.21) that interpolate between *neighbouring* minima. This is because these are first order equations of motion, and once you sit at a critical point of h you have necessarily stopped. That doesn't mean that there is no tunnelling between multiple vacua: indeed, as we'll see shortly, in non-supersymmetric quantum mechanics it is usually approximate solutions to the classical equations of motion that dominate proceedings.

该方程的解称为反瞬子。它们以与瞬子相反的方向在两个真空之间进行插值。

对于更一般的 $h(x)$ 来说，瞬子的精确解可能更难获得，但对其属性有一个直观的感受很简单。从拉格朗日动力学的通常角度来看，欧几里得作用 (2.18) 描述了一个粒子在势 $-V(x)$ 中运动。这显示在图 6 右侧的双阱电势中。瞬子（或反瞬子）描述了一个粒子，它从 $-V(x)$ 的一个最大值开始，在 $\tau \to -\infty$ 处，然后向下和向上滚动到另一个最大值，仅在 $\tau \to -\infty$ 处达到峰值 $\tau \to +\infty$。

如果 $V(x)$ 有多个最小值，那么我们只能找到在 邻近最小值之间插值的瞬子方程 (2.20) 和 (2.21) 的解。这是因为这些是一阶运动方程，一旦到达 h 的临界点，您就必然会停止。这并不意味着多个真空之间不存在隧道效应：事实上，正如我们很快就会看到的，在非超对称量子力学中，通常是主导过程的经典运动方程的近似解。

2.2.1 Tunnelling

Let's first remind ourselves what we qualitatively expect from the ground states. Around each minima, the potential looks like a harmonic oscillator (2.23) and we can then construct approximations to the ground states as Gaussian wavefunctions, localised around each of the minima

For any even potential $V(x) = V(-x)$, the energy eigenstates are also eigenstates of the parity operator, meaning that they are either even or odd functions. A better approximation to the low-lying energy eigenstates must therefore be

But the true ground state of any quantum system has no node, meaning that $\psi(x) \neq 0$ for any finite x. (Given a wavefunction $\psi(x)$ with a node, we can consider $|\psi(x)|$ and then smooth out the cusp to lower the expected energy.) So it must be that $\psi_+(x)$ is an approximation to the ground state, while $\psi_-(x)$ is an approximation to the first excited state.

So to determine the ground state energy, we just need to compute the path integral and extract the large T behaviour. We can then find the ground state energy in the exponent.

It should be thought of as a measure of the difficulty in getting up and over (or, more precisely, through) the barrier between

隧穿

让我们首先提醒自己，我们对基态的定性期望是什么。在每个最小值周围，势看起来像一个谐振子 (2.23)，然后我们可以将基态近似构造为高斯波函数，位于每个最小值周围

对于任何偶势 $V(x) = V(-x)$，能量本征态也是奇偶算子的本征态，这意味着它们要么是偶函数，要么是奇函数。因此，对低位能量本征态更好的近似是

但任何量子系统的真实基态都没有节点，这意味着对于任何有限的 x，$\psi(x) \neq 0$。（给定一个带有节点的波函数 $\psi(x)$，我们可以考虑 $|\psi(x)|$，然后平滑尖点以降低预期能量。）所以它一定是 $\psi_+(x)$ 是基态的近似值，而 $\psi_-(x)$ 是第一激发态的近似值。

因此，为了确定基态能量，我们只需要计算路径积分并提取大 T 行为。然后我们可以找到指数中的基态能量。

它应该被视为衡量站起来并越过（或更准确地说，穿过）两个最小值之间障碍的难

the two minima.

The semi-classical approximation is valid whenever we can ignore the $\mathcal{O}(\delta x^3)$ contributions relative to the δx^2 contributions in the path integral. To understand the circumstances under which this holds, we should look more closely at the action and identify a dimensionless coupling constant g which multiplies all higher order terms. Perturbation theory is then valid when $g \ll 1$. A simpler way to view things is to rescale the potential $h(x) \to \lambda h(x)$. Then the semi-classical approximation is valid in the limit $\lambda \gg 1$ where we have a steep potential, with deep minima. Under this rescaling, the action of the instanton (2.26) becomes

This is the requirement that we will use for the semi-classical approximation to be valid. The results that we will get below will receive corrections of order $1/S_{\rm inst}$.

In the language of quantum field theory, neglecting the higher order δx^3 terms is tantamount to computing one-loop diagrams but not two-loop or higher. In normal circumstances, we would be doing perturbation theory around the classical vacuum $x(\tau) = \pm a$, in which case we would have $V'' = \omega^2$, a constant. The difference here is that we're now doing perturbation theory around the background of the instanton profile.

The kind of instanton calculations that we're performing here are often referred to as *non-perturbative*. This refers to the fact that tunnelling phenomena of this kind can't be captured by perturbation theory around any single vacuum. However, the phrase "non-perturbative" is also a little misleading: we're still doing perturbation theory, just around a non-trivial solution.

On the left-hand side, we've taken the tunnelling to happen over a time T; ultimately we will be interested in taking $T \to \infty$. We have also stressed that we're computing the contribution to the tunnelling from a single instanton and we'll subsequently see that this is just part of the story.

There is, however, a catch. In the background of the instanton, there is always one eigenvalue that is zero. Viewed naively, this would seem to tell us that the determinant vanishes, giving an infinite amplitude for tunnelling. This, it turns out, is not an infinity that we should try to regulate away, but instead an infinity that's means we should think more carefully about what we're calculating. Our first task, therefore, is to understand the physics behind this zero eigenvalue.

Understanding zero modes is an important part of any instanton computation. They typically arise, as in the present case, because the instanton solution is not unique, but labelled by a number of parameters known as *collective coordinates*. For us,

度的指标。

只要我们可以忽略路径积分中相对于 δx^2 贡献的 $\mathcal{O}(\delta x^3)$ 贡献, 半经典近似就有效。为了理解这种情况成立的情况，我们应该更仔细地观察该作用，并确定一个无量纲耦合常数 g，它乘以所有高阶项。当 $g \ll 1$ 时，微扰理论有效。一种更简单的看待事物的方法是将势 $h(x)$ $\lambda h(x)$。那么半经典近似在极限 $\lambda \gg 1$ 中是有效的, 其中我们有一个陡峭的潜力，具有深的最小值。在这种重新调整下，瞬子 (2.26) 的作用变为

这是我们要使半经典近似有效的要求。我们将在下面得到的结果将得到阶为 $1/S_{\rm inst}$ 的修正。

在量子场论的语言中，忽略高阶 δx^3 项相当于计算单环图，但不能计算双环或更高阶。在正常情况下,我们将围绕经典真空 $x(\tau) = \pm a$ 进行微扰理论, 在这种情况下, 我们将有 $V'' = \omega^2$,一个常数。这里的区别在于,我们现在正在围绕瞬子轮廓的背景进行微扰理论。

我们在这里执行的瞬子计算通常称为 非微扰。这是指这样的事实:这种隧道现象无法通过任何单一真空周围的微扰理论来捕获。然而，“非微扰”这个词也有点误导：我们仍在研究微扰理论，只是围绕着一个不平凡的解。

在左侧, 我们假设隧道发生的时间为 T; 最终我们会对将 $T \to \infty$ 感兴趣。我们还强调，我们正在计算单个瞬子对隧道效应的贡献，随后我们将看到这只是故事的一部分。

然而, 有一个问题。在瞬子的背景中, 总是有一个特征值为零。单纯地看,这似乎告诉我们行列式消失了，为隧道效应提供了无限的振幅。事实证明,这并不是一个我们应该试图调节掉的无穷大，而是一个意味着我们应该更仔细地思考我们正在计算的东西的无穷大。因此，我们的首要任务是理解零特征值背后的物理原理。

了解零模式是任何瞬子计算的重要组成部分。它们通常会出现，就像在本例中一样，因为瞬子解不是唯一的, 而是由许多称为 集体坐标的参数标记。对于我们来说，瞬子轮

the instanton profile has a single collective coordinate, τ_1. Any fluctuation, like (2.29), that can be thought of as a variation of a collective coordinate necessarily has zero eigenvalue. These fluctuations are called *zero modes*.

In the present case, the existence of the zero mode can be traced to the fact that the underling quantum mechanics enjoys time translation symmetry, while any particular instanton profile does not. In quantum field theory (or statistical field theory), we would refer to the zero mode as a "Goldstone boson" for time translational symmetry.

There are three things to take away from this. First, there are some slightly messy pre-factors that we've absorbed into K, which now include a ratio of the harmonic oscillator and instanton determinants. The exact expression for this ratio will not be particularly important in what follows and we won't make any attempt to compute it. However, the advantage of writing this as a ratio of determinants is that it makes it clear that it differs from 1 only due to physics in a region of width $1/\omega$ where the instanton profile is non-trivial, and $V''(x_{\rm inst})$ differs from ω^2. We'll see the utility of this shortly.

Second, the amplitude is suppressed by a factor of $\mathrm{e}^{-S_{\rm inst}}$. This is a characteristic feature of tunnelling in quantum mechanics. Finally, we see that the tunnelling amplitude from a single instanton has the slightly odd $T\mathrm{e}^{-T}$ behaviour. It turns out that the correct interpretation of this comes by considering not a lone instanton, but a whole slew of them.

2.2.2 The Dilute Gas Approximation

In the calculation above, we restricted to a single instanton solution that interpolates from one classical ground state to the other. However, we know that the interesting part of this instanton profile takes place over a region that is exponentially localised within a width $\sim 1/\omega$. That means that if we take an instanton, followed a long time later, by an anti-instanton, followed some time later still by another instanton, then this *almost* solves the classical equation of motion. It's not an exact solution because there are no exact classical solution with these properties. But, if the instantons and anti-instantons are separated by a distance $L \gg 1/\omega$, then the action of a string of n such objects is roughly

In computing the amplitude $\langle -a|\mathrm{e}^{-HT}|a\rangle$, we should sum over all possible numbers of instantons and anti-instantons. We just need one more instanton than anti-instanton to ensure that we end up in the opposite vacuum from where we started. In

廓有一个集体坐标, τ_1。任何波动, 如 (2.29), 可以被认为是集体坐标的变化，必然具有零特征值。这些波动称为 零模式。

在目前的情况下，零模式的存在可以追溯到以下事实：基础量子力学具有时间平移对称性，而任何特定的瞬子解则不然。在量子场论（或统计场论）中，我们将零模式称为时间平移对称性的 ""Goldstone 玻色子"。

从中可以得出三件事。首先,我们将一些稍微混乱的前置因素吸收到 K 中,其中现在包括谐振子和瞬子决定因素的比率。这个比率的确切表达式在接下来的内容中不会特别重要，我们不会尝试计算它。然而，将其写为行列式之比的优点在于，它清楚地表明它与 1 的不同仅是由于宽度为 $1/omega$ 的区域中的物理现象, 其中瞬时子解是非平凡的, 并且 $V''(x_{\rm inst})$ 与 ω^2 不同。我们很快就会看到它的用处。

其次,振幅被 $\mathrm{e}^{-S_{\rm inst}}$ 因子抑制。这是量子力学中隧道效应的一个特征。最后,我们看到单个瞬子的隧道振幅具有稍微奇怪的 $T\mathrm{e}^{-T}$ 行为。事实证明，对此的正确解释不是考虑单个瞬子，而是考虑整个瞬子。

稀薄气体近似

在上面的计算中，我们限制为从一个经典基态插值到另一个经典基态的单个瞬子解。然而，我们知道这个瞬子分布的有趣部分发生在宽度 $\sim 1/\omega$ 内呈指数局部化的区域。这意味着，如果我们采用一个瞬子，很长一段时间后是一个反瞬子，再过一段时间又是另一个瞬子, 那么这个 几乎就是经典运动方程的解。这不是一个精确解，因为没有具有这些性质的精确的经典解。但是，如果瞬子和反瞬子之间的距离为 $L \gg 1/\omega$，那么一串 n 这样的物体的作用大致为

在计算幅度 $\langle -a|\mathrm{e}^{-HT}|a\rangle$ 时, 我们应该对所有可能的瞬子和反瞬子数量进行求和。我们只需要比反瞬子多一个瞬子，以确保我们最终处于与开始位置相反的真空中。换句话

other words, n should be odd in (2.34).

We see the effect of summing over the dilute gas is to exponentiate the one-instanton contribution $KT\mathrm{e}^{-S_{\rm inst}}$.

We can also do a similar calculation to evaluate the amplitude $\langle +a|\mathrm{e}^{-HT}|+a\rangle = \langle -a|\mathrm{e}^{-HT}|-a\rangle$ for returning to our original vacuum. Everything is the same, except that we should now take the number n of instantons and anti-instantons to be even. Of course, $n = 0$ is allowed.

We see the promised energy splitting, proportional to the characteristic tunnelling amplitude $\mathrm{e}^{-S_{\rm inst}}$.

Of course, if we really want to do a good job then we should roll up our sleeves and compute the ratio of determinants that sits in K. But we can see the key piece of physics without doing this: the splitting of energy levels scales as $\mathrm{e}^{-S_{\rm inst}}$.

说，(2.34) 中的 n 应该是奇数。

我们看到对稀气体求和的效果是对单瞬时贡献 $KT\mathrm{e}^{-S_{\rm inst}}$ 求幂。

我们也可以进行类似的计算来评估幅度 $\langle +a|\mathrm{e}^{-HT}|+a\rangle = \langle -a|\mathrm{e}^{-HT}|-a\rangle$ 返回我们原来的真空。一切都是一样的，只是我们现在应该将瞬子和反瞬子的数量 n 取为偶数。当然，$n = 0$ 是允许的。

我们看到所承诺的能量分裂，与特征隧道振幅 $\mathrm{e}^{-S_{\rm inst}}$ 成正比。

当然，如果我们真的想做好工作，那么我们应该卷起袖子计算 K 中的决定因素的比率。但我们无需这样做就可以看到物理学的关键部分：能级分裂的尺度为 $\mathrm{e}^{-S_{\rm inst}}$。