temp

桜井 雪子

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Contents

1 Mechanics

1.1 action principle & EoM

$$S[q(t)] = \int_{t_1}^{t_2} \mathrm{d}t L(t, q, \dot{q})$$

$$\delta S \simeq -\int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{(q)}}\right) - \frac{\partial L}{\partial q} \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{(q)}}\right) - \frac{\partial L}{\partial q} = 0$$

1.2 Noether's theorem

$$\tilde{t} = t + \delta t, \tilde{q} = q + \delta q$$

$$\delta_s := \tilde{t} - t, \delta_s q(t) := \tilde{q}(t) - q(t) \Rightarrow \delta_s \dot{q} = (\delta_s q)^{\cdot}$$

$$\Delta q(t) := \tilde{q}(\tilde{t}) - q(t) \xlongequal{\delta_s := \tilde{t} - t} \tilde{q}(t + \delta_s t) - q(t) \xlongequal{Taylor} \tilde{q}(t) + \delta_s t \dot{\tilde{q}}(t) - q(t) \xlongequal{\delta_s q(t) := \tilde{q}(t) - q(t) \Rightarrow \dot{\tilde{q}} = \dot{q} + (\delta_s q)^{\cdot}} \delta_s q + \delta_s t \dot{q}$$

$$\Delta \dot{q}(t) := \frac{\mathrm{d}\tilde{q}(\tilde{t})}{\mathrm{d}\tilde{t}} - \frac{\mathrm{d}q(t)}{\mathrm{d}t} = (\delta_s q)^{\cdot} - \delta_s t \ddot{q} \neq (\Delta q(t))^{\cdot}$$

$$\Delta L = \cdots = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}} \delta_s q + L \delta_s t \right] \equiv \frac{\mathrm{d}F}{\mathrm{d}t}$$

$$Q \coloneqq \left[\frac{\partial L}{\partial \dot{q}}\delta_s q + L\delta_s t\right] - F = p\delta_s q + L\delta_s t - F \Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

Quantum Field Theory by Mark Srednicki Section 22 中对作用量变分的写法更简洁。

2 Quantum Field Theory

2.1 LSZ

2.1.1 Scalar Field

$$a_1^\dagger = \int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)a^\dagger(\vec{k})$$

$$\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{+\infty}\mathrm{d}t\partial_0 a_1^\dagger \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\partial_0\left[\mathrm{e}^{\mathrm{i}kx}\overleftrightarrow{\partial_0}\varphi_1(x)\right] \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big(\partial_0^2+\omega^2\big)\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\Big(\partial_0^2+\vec{k}^2+m^2\Big)\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big(\partial_0^2+m^2\big)\varphi_1(x)+\varphi_1(x)\Big(-\vec{\nabla}^2\Big)\mathrm{e}^{\mathrm{i}kx} \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big(\partial_0^2+m^2\big)\varphi_1(x)+\vec{\nabla}\mathrm{e}^{\mathrm{i}kx}\vec{\nabla}\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big(\partial_0^2+m^2\big)\varphi_1(x)-\mathrm{e}^{\mathrm{i}kx}\Big(\vec{\nabla}^2\Big)\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\Big(\partial_0^2-\vec{\nabla}^2+m^2\Big)\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^3k\delta(\vec{k}-\vec{k}_1)\int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big(-\partial^2+m^2\big)\varphi_1(x) \\
&= -\mathrm{i}\int \mathrm{d}^4x\mathrm{e}^{+\mathrm{i}k_1x}\big(-\partial_1^2+m^2\big)\varphi_1(x) \\
\Rightarrow a_{1'}(+\infty) - a_{1'}(-\infty) &= +\mathrm{i}\int \mathrm{d}^4x\mathrm{e}^{-\mathrm{i}k_{1'}x}\big(-\partial_{1'}^2+m^2\big)\varphi_{1'}(x)
\end{aligned}$$

$$\begin{aligned}
|i\rangle =& a_1^\dagger(-\infty)\cdots|0\rangle \\
\langle f| =& \langle 0|a_{1'}(+\infty)\cdots
\end{aligned}$$

$$\begin{aligned}
\langle f|i\rangle =& \langle 0|a_{1'}(+\infty)\cdots a_1(-\infty)\cdots|0\rangle \\
=& \Big\langle 0\Big|\mathrm{T}a_{1'}(+\infty)\cdots a_1^\dagger(-\infty)\cdots\Big|0\Big\rangle \\
=& \Big\langle 0\Big|\mathrm{T}[a_{1'}(+\infty)-a_{1'}(-\infty)]\cdots\Big[a_1^\dagger(-\infty)-a_1^\dagger(+\infty)\Big]\cdots\Big|0\Big\rangle \\
=& \left\langle 0\middle|\mathrm{T}\left[\mathrm{i}\int \mathrm{d}^4x_{1'}\mathrm{e}^{-\mathrm{i}k_{1'}x_{1'}}(-\partial_{1'}^2+m^2)\varphi(x_{1'})\right]\cdots\left[\mathrm{i}\int \mathrm{d}^4x_1\mathrm{e}^{-\mathrm{i}k_1x_1}(-\partial_1^2+m^2)\varphi(x_1)\right]\cdots\middle|0\right\rangle \\
=& \left[\mathrm{i}\int \mathrm{d}^4x_{1'}\mathrm{e}^{-\mathrm{i}k_{1'}x_{1'}}(-\partial_{1'}^2+m^2)\right]\cdots\left[\mathrm{i}\int \mathrm{d}^4x_1\mathrm{e}^{-\mathrm{i}k_1x_1}(-\partial_1^2+m^2)\right]\cdots\langle 0|\mathrm{T}\varphi(x_{1'})\cdots\varphi(x_1)\cdots|0\rangle
\end{aligned}$$

2.2 Symmetry

2.2.1 Discrete Symmetry

Notes on Quantum Field Theory by Yuchen Wang 的处理更加严谨。

Scalar Field

$$\mathcal{P}^{\mu}{}_{\nu} = \left(\mathcal{P}^{-1}\right)^{\mu}{}_{\nu} = \mathrm{diag}(+1,-1,-1,-1)$$

$$U(\mathcal{P})^{-1}\varphi(x)U(\mathcal{P}) \xlongequal{P\equiv U(\mathcal{P})} P^{-1}\varphi(x)P \xlongequal{U(\Lambda)^{-1}\varphi(x)U(\Lambda)=\varphi(\Lambda^{-1}x)} \pm\varphi(\mathcal{P}^{-1}x) \xlongequal{\mathcal{P}=\mathcal{P}^{-1}} \pm\varphi(\mathcal{P}x)$$

$$\mathcal{T}^{\mu}{}_{\nu} = \left(\mathcal{T}^{-1}\right)^{\mu}{}_{\nu} = \mathrm{diag}(-1,+1,+1,+1)$$

$$U(\mathcal{T})^{-1}\varphi(x)U(\mathcal{T}) \xlongequal{T\equiv U(\mathcal{T})} T^{-1}\varphi(x)T \xlongequal{U(\Lambda)^{-1}\varphi(x)U(\Lambda)=\varphi(\Lambda^{-1}x)} \pm\varphi(\mathcal{T}^{-1}x) \xlongequal{\mathcal{T}=\mathcal{T}^{-1}} \pm\varphi(\mathcal{T}x)$$

$$P^{-2}\varphi(x)P^2 = T^{-2}\varphi(x)T^2 = \varphi(x)$$

$$P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x), T^{-1}\mathcal{L}(x)T = +\mathcal{L}(\mathcal{T}x)$$

$$P^{-1}P^{\mu}P = \mathcal{P}^{\mu}{}_{\nu}P^{\nu}, T^{-1}P^{\mu}T = -\mathcal{T}^{\mu}{}_{\nu}P^{\nu}, T^{-1}\mathrm{i}T = -\mathrm{i}$$

$$Z^{-1}\varphi_a(x)Z = \eta_a\varphi_a(x), \eta_a = \pm 1 \Rightarrow Z^2 = 1$$

C is a Z_2 operator.

$$C^{-1}\varphi(x)C = \varphi^{\dagger}(x) \xrightarrow{\varphi=(\varphi_1+\mathrm{i}\varphi_x)/\sqrt{2}} \begin{cases} C^{-1}\varphi_1(x)C = +\varphi_1(x) \\ C^{-1}\varphi_2(x)C = -\varphi_2(x) \end{cases} \Rightarrow C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$$

Spinor Field

Dirac / Majorana Field $\Psi(x)$

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x), D(\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}) = 1_{4\times 4} + \frac{\mathrm{i}}{2}\delta\omega_{\mu\nu}S^{\mu\nu}, S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^{\mu},\gamma^{\nu}]$$

$$P^{-2}\Psi(x)P^2 = D(\mathcal{P})^2\Psi(\mathcal{P}^2x) = \pm\Psi(x)$$

$$P^{-1}\vec{p}P = -\vec{p}, P^{-1}\vec{J}P = +\vec{J} \Rightarrow P^{-1}b_s^{\dagger}(\vec{p})P = \eta b_s^{\dagger}(-\vec{p}), P^{-1}d_s^{\dagger}(\vec{p})P = \eta d_s^{\dagger}(-\vec{p}), \eta^2 = 1$$

$$P^{-1}\Psi(x)P = \cdots \Rightarrow \begin{cases} \eta = -\mathrm{i} \Rightarrow D(\mathcal{P}) = +\mathrm{i}\beta \\ \eta = +\mathrm{i} \Rightarrow D(\mathcal{P}) = -\mathrm{i}\beta \end{cases}, \beta = \begin{bmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{bmatrix}$$

$$T^{-2}\Psi(x)T^2 = D(\mathcal{T})^2\Psi(\mathcal{P}^2x) = \pm\Psi(x)$$

$$T^{-1}\vec{p}T = -\vec{p}, T^{-1}\vec{J}T = -\vec{J} \Rightarrow T^{-1}b_s^{\dagger}(\vec{p})T = \zeta_s b_{-s}^{\dagger}(-\vec{p}), T^{-1}d_s^{\dagger}(\vec{p})T = \zeta_s d_{-s}^{\dagger}(-\vec{p})$$

$$T^{-1}\Psi(x)T = \cdots \Rightarrow \begin{cases} \zeta_s = +s \Rightarrow D(\mathcal{T}) = +\mathcal{C}\gamma_5 \\ \zeta_s = -s \Rightarrow D(\mathcal{T}) = -\mathcal{C}\gamma_5 \end{cases}, \mathcal{C} = \begin{bmatrix} -\varepsilon^{ab} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{b}} \end{bmatrix}, \gamma_5 = \begin{bmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{bmatrix}$$

Weyl Field

$$\Psi = \begin{bmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{bmatrix} \Rightarrow \begin{cases} P^{-1}\chi_a(x)\ P = \mathrm{i}\xi^{\dagger\dot{a}}(\mathcal{P}x) \\ P^{-1}\xi^{\dagger\dot{a}}(x)\,P = \mathrm{i}\chi_a(x)(\mathcal{P}x) \\ P^{-1}\chi^{\dagger\dot{a}}(x)P = \mathrm{i}\xi_a(x)(\mathcal{P}x) \\ P^{-1}\xi_a(x)\ P = \mathrm{i}\chi^{\dagger\dot{a}}(\mathcal{P}x) \end{cases} \begin{cases} T^{-1}\chi_a(x)\ T = +\chi^a(\mathcal{T}x) \\ T^{-1}\xi^{\dagger\dot{a}}(x)\,T = -\xi^{\dagger}_{\dot{a}}(\mathcal{T}x) \\ T^{-1}\chi^{\dagger\dot{a}}(x)T = -\chi^{\dagger}_{\dot{a}}(\mathcal{T}x) \\ T^{-1}\xi_a(x)\ T = +\xi^a(\mathcal{T}x) \end{cases}$$

Fermion Bilinear

$$P^{-1}\Psi(x)P = \mathrm{i}\beta\Psi(\mathcal{P}x) \Rightarrow P^{-1}\bar{\Psi}(x)P = -\mathrm{i}\bar{\Psi}(\mathcal{P}x)\beta \Rightarrow P^{-1}\big[\bar{\Psi}A\Psi\big]P = \bar{\Psi}[\beta A\beta]\Psi$$

$$\begin{aligned}\beta 1\beta &= +\,1\\ \beta\mathrm{i}\gamma_5\beta &= -\,\mathrm{i}\gamma_5\\ \beta\gamma_0\beta &= +\,\gamma^0\\ \beta\gamma^i\beta &= -\,\gamma^i\\ \beta\gamma^0\gamma_5\beta &= -\,\gamma^0\gamma_5\\ \beta\gamma^i\gamma_5\beta &= +\,\gamma^0\gamma_5\end{aligned} \Rightarrow \begin{aligned}P^{-1}\big[\bar{\Psi}\Psi\big]P &= +\,\big[\bar{\Psi}\Psi\big]\\ P^{-1}\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]P &= -\,\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]\\ P^{-1}\big[\bar{\Psi}\gamma^\mu\Psi\big]P &= +\,\mathcal{P}^\mu{}_\nu\big[\bar{\Psi}\gamma^\nu\Psi\big]\\ P^{-1}\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big]P &= -\,\mathcal{P}^\mu{}_\nu\big[\bar{\Psi}\gamma^\nu\gamma_5\Psi\big]\end{aligned}$$

$$T^{-1}\Psi(x)T = \mathcal{P}\gamma_5\Psi(\mathcal{T}x) \Rightarrow T^{-1}\bar{\Psi}(x)T = \bar{\Psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1} \xRightarrow{T^{-1}AT=A^*} T^{-1}\big[\bar{\Psi}A\Psi\big]T = \bar{\Psi}\big[\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\big]\Psi$$

$$\begin{aligned}\gamma_5\mathcal{C}^{-1}[1]^* &= +\,1\\ \gamma_5\mathcal{C}^{-1}[\mathrm{i}\gamma_5]^*\mathcal{C}\gamma_5 &= -\,\mathrm{i}\gamma_5\\ \gamma_5\mathcal{C}^{-1}[\gamma_0]^*\mathcal{C}\gamma_5 &= +\,\gamma^0\\ \gamma_5\mathcal{C}^{-1}\big[\gamma^i\big]^*\mathcal{C}\gamma_5 &= -\,\gamma^i\\ \gamma_5\mathcal{C}^{-1}\big[\gamma^0\gamma_5\big]^*\mathcal{C}\gamma_5 &= +\,\gamma^0\gamma_5\\ \gamma_5\mathcal{C}^{-1}\big[\gamma^i\gamma_5\big]^*\mathcal{C}\gamma_5 &= -\,\gamma^0\gamma_5\end{aligned} \Rightarrow \begin{aligned}T^{-1}\big[\bar{\Psi}\Psi\big]T &= +\,\big[\bar{\Psi}\Psi\big]\\ T^{-1}\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]T &= -\,\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]\\ T^{-1}\big[\bar{\Psi}\gamma^\mu\Psi\big]T &= -\,\mathcal{T}^\mu{}_\nu\big[\bar{\Psi}\gamma^\nu\Psi\big]\\ T^{-1}\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big]T &= -\,\mathcal{T}^\mu{}_\nu\big[\bar{\Psi}\gamma^\nu\gamma_5\Psi\big]\end{aligned}$$

$$C^{-1}\Psi(x)C = \mathcal{C}\bar{\Psi}^{\mathrm{T}}(x) \Rightarrow C^{-1}\bar{\Psi}(x)C = \Psi^{\mathrm{T}}(x)\mathcal{C} \Rightarrow C^{-1}\big[\bar{\Psi}A\Psi\big]C = \Psi^{\mathrm{T}}\mathcal{C}A\mathcal{C}\bar{\Psi}^{\mathrm{T}} = -\bar{\Psi}\big[\mathcal{C}^{\mathrm{T}}A^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\big]\Psi = -\bar{\Psi}\big[\mathcal{C}^{-1}A^{\mathrm{T}}\mathcal{C}\big]\Psi$$

$$\begin{aligned}\mathcal{C}^{-1}[1]^{\mathrm{T}}\mathcal{C} &= +\,1\\ \mathcal{C}^{-1}[\mathrm{i}\gamma_5]^{\mathrm{T}}\mathcal{C} &= +\,\mathrm{i}\gamma_5\\ \mathcal{C}^{-1}[\gamma^\mu]^{\mathrm{T}}\mathcal{C} &= -\,\gamma^\mu\\ \mathcal{C}^{-1}[\gamma^\mu\gamma_5]^{\mathrm{T}}\mathcal{C} &= +\,\gamma^\mu\gamma_5\end{aligned} \Rightarrow \begin{aligned}C^{-1}\big[\bar{\Psi}\Psi\big]C &= +\,\bar{\Psi}\Psi\\ C^{-1}\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]C &= +\,\bar{\Psi}\mathrm{i}\gamma_5\Psi\\ C^{-1}\big[\bar{\Psi}\gamma^\mu\Psi\big]C &= -\,\bar{\Psi}\gamma^\mu\Psi\\ C^{-1}\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big]C &= +\,\bar{\Psi}\gamma^\mu\gamma_5\Psi\end{aligned}$$

$$(\mathcal{P}\mathcal{T})^\mu{}_\nu = -1_{4\times 4} \Rightarrow \begin{aligned}(CPT)^{-1}\big[\bar{\Psi}\Psi\big](CPT) &= +\,\big[\bar{\Psi}\Psi\big]\\ (CPT)^{-1}\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big](CPT) &= +\,\big[\bar{\Psi}\mathrm{i}\gamma_5\Psi\big]\\ (CPT)^{-1}\big[\bar{\Psi}\gamma^\mu\Psi\big](CPT) &= -\,\big[\bar{\Psi}\gamma^\mu\Psi\big]\\ (CPT)^{-1}\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big](CPT) &= -\,\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big]\end{aligned}$$

Vector Field

$$P^{-1}\vec{p}P = -\vec{p}, P^{-1}\vec{J}P = +\vec{J} \Rightarrow P^{-1}a^\dagger_\lambda(\vec{k})P = \eta_\lambda a^\dagger_\lambda(-\vec{k})$$

$$P^{-1}A^\mu(x)P = \cdots \Rightarrow P^{-1}A^\mu(x)P = -\eta\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x)$$

$$T^{-1}\vec{p}T = -\vec{p}, T^{-1}\vec{J}T = -\vec{J} \Rightarrow T^{-1}a^\dagger_\lambda(\vec{k})T = \zeta_\lambda a^\dagger_{-\lambda}(-\vec{k})$$

$$T^{-1}A^\mu(x)T = \cdots \Rightarrow T^{-1}A^\mu(x)T = \zeta\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x)$$

2.3 Mark Srednicki's Spinor Notation

$$[\varepsilon_{ab}] = \big[\varepsilon_{\dot{a}\dot{b}}\big] = \begin{bmatrix}0 & -1\\ +1 & 0\end{bmatrix}, \big[\varepsilon^{ab}\big] = \big[\varepsilon^{\dot{a}\dot{b}}\big] = \begin{bmatrix}0 & +1\\ -1 & 0\end{bmatrix}, \sigma^\mu_{a\dot{a}} = (I,\vec{\sigma}), \bar{\sigma}^{\mu a\dot{a}} = (I,-\vec{\sigma})$$

```
In := PauliMatrix@Range[0,3]
Out = {{{1,0},{0,1}},{{0,1},{1,0}},{{0,-I},{I,0}},{{1,0},{0,-1}}}
```

$$g_{\mu\nu}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}, \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = -2g^{\mu\nu}$$

2.4 Scattering Amplitude in Spinor Field

1. Feynman diagram
2. Spinor technology
3. Gamma matrix technology
4. Average over initial spins and Sum over final spins

2.4.1 Spinor Technology

Dirac Equation

$$\begin{aligned}\big(+\not{p}+m\big)u_s(\vec{p})&=0\\ \big(-\not{p}+m\big)v_s(\vec{p})&=0\end{aligned}\Rightarrow\begin{aligned}\bar{u}_s(\vec{p})\big(+\not{p}+m\big)&=0\\ \bar{v}_s(\vec{p})\big(-\not{p}+m\big)&=0\end{aligned}$$

$$\begin{aligned}\bar{u}_{s'}(\vec{p})u_s(\vec{p})&=\bar{u}_{s'}(\vec{0})u_s(\vec{0})=+2m\delta_{s's}\\ \bar{v}_{s'}(\vec{p})v_s(\vec{p})&=\bar{v}_{s'}(\vec{0})v_s(\vec{0})=-2m\delta_{s's}\\ \bar{u}_{s'}(\vec{p})v_s(\vec{p})&=\bar{u}_{s'}(\vec{0})v_s(\vec{0})=\quad 0\\ \bar{v}_{s'}(\vec{p})u_s(\vec{p})&=\bar{v}_{s'}(\vec{0})u_s(\vec{0})=\quad 0\end{aligned}$$

Gordon Identity

$$\begin{aligned}+2m\bar{u}_{s'}(\vec{p}')\gamma^\mu u_s(\vec{p})&=\bar{u}_{s'}(\vec{p}')[(p'+p)^\mu-2\mathrm{i}S^{\mu\nu}(p'-p)_\nu]u_s(\vec{p})\\ -2m\bar{v}_{s'}(\vec{p}')\gamma^\mu v_s(\vec{p})&=\bar{v}_{s'}(\vec{p}')[(p'+p)^\mu-2\mathrm{i}S^{\mu\nu}(p'-p)_\nu]v_s(\vec{p})\\ +2m\bar{u}_{s'}(\vec{p}')\gamma^\mu v_s(\vec{p})&=\bar{u}_{s'}(\vec{p}')[(p'-p)^\mu+2\mathrm{i}S^{\mu\nu}(p'+p)_\nu]v_s(\vec{p})\\ -2m\bar{v}_{s'}(\vec{p}')\gamma^\mu u_s(\vec{p})&=\bar{v}_{s'}(\vec{p}')[(p'-p)^\mu+2\mathrm{i}S^{\mu\nu}(p'+p)_\nu]u_s(\vec{p})\\ 0&=\bar{u}_{s'}(\vec{p}')[(p'+p)^\mu-2\mathrm{i}S^{\mu\nu}(p'-p)_\nu]\gamma_5u_s(\vec{p})\\ 0&=\bar{v}_{s'}(\vec{p}')[(p'+p)^\mu-2\mathrm{i}S^{\mu\nu}(p'-p)_\nu]\gamma_5v_s(\vec{p})\end{aligned}\Rightarrow\begin{aligned}\bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p})=\ \bar{v}_{s'}(\vec{p})\gamma^\mu v_s(\vec{p})&=2p^\mu\delta_{s's}\\ \bar{u}_{s'}(\vec{p})\gamma^0 v_s(-\vec{p})=\bar{v}_{s'}(\vec{p})\gamma^0 u_s(-\vec{p})&=\quad 0\end{aligned}$$

$$\begin{aligned}\gamma^\mu\not{p}&=\frac{1}{2}\big\{\gamma^\mu,\not{p}\big\}+\frac{1}{2}\big[\gamma^\mu,\not{p}\big]=-p^\mu-2\mathrm{i}S^{\mu\nu}p_\nu\\ \not{p}'\gamma^\mu&=\frac{1}{2}\big\{\gamma^\mu,\not{p}'\big\}-\frac{1}{2}\big[\gamma^\mu,\not{p}'\big]=-p'^\mu-2\mathrm{i}S^{\mu\nu}p'_\nu\end{aligned}$$

plus or minus, times γ_5 or not, sandwich them with $\bar{u}\&u$ or $\bar{u}\&v$ or $\bar{v}\&u$ or $\bar{v}\&v$, 16 scenarios in total

Spin Sum

$$\begin{aligned}u_s(\vec{p})\bar{u}_s(\vec{p})&=\frac{1}{2}(1-s\gamma_5\not{z})\big(-\not{p}+m\big)\Rightarrow\sum_{s=\pm}u_s(\vec{p})\bar{u}_s(\vec{p})=-\not{p}+m\\ v_s(\vec{p})\bar{v}_s(\vec{p})&=\frac{1}{2}(1-s\gamma_5\not{z})\big(-\not{p}-m\big)\Rightarrow\sum_{s=\pm}v_s(\vec{p})\bar{v}_s(\vec{p})=-\not{p}-m\end{aligned}$$

CPT

C

$$\mathcal{C}=\begin{bmatrix}0&-1&0&0\\+1&0&0&0\\0&0&0&-1\\0&0&+1&0\end{bmatrix}\Rightarrow\begin{cases}\mathcal{C}^T=\mathcal{C}^\dagger=\mathcal{C}^{-1}=-\mathcal{C}\\ \beta\mathcal{C}+\mathcal{C}\beta=0\\ \mathcal{C}^{-1}\gamma^\mu\mathcal{C}=-(\gamma^\mu)^{\mathrm{T}}\Rightarrow\mathcal{C}^{-1}K^j\mathcal{C}=-\big(K^j\big)^{\mathrm{T}}\end{cases}$$

$$
\begin{aligned}
\mathcal{C}\bar{u}_s(0)^{\mathrm{T}} &= v_s(0), \quad \mathcal{C}\bar{v}_s(0)^{\mathrm{T}} = \quad u_s(0) \\
\Rightarrow \mathcal{C}\bar{u}_s(\vec{p})^{\mathrm{T}} &= v_s(\vec{p}), \quad \mathcal{C}\bar{v}_s(\vec{p})^{\mathrm{T}} = \quad u_s(\vec{p}) \\
\Rightarrow u_s^*(\vec{p}) \quad &= \mathcal{C}\beta v_s(\vec{p}), \quad v_s^*(\vec{p}) = \mathcal{C}\beta u_s(\vec{p})
\end{aligned}
$$

P

$$
\begin{aligned}
\beta u_s(0) &= +\, u_s(0), \quad \beta v_s(0) = \; -v_s(0), \beta K^i + K^i \beta = 0 \\
\Rightarrow u_s(-\vec{p}) &= +\, \beta u_s(\vec{p}), v_s(-\vec{p}) = -\beta v_s(\vec{p})
\end{aligned}
$$

T

$$
\begin{aligned}
\gamma_5 u_s(0) \; &= +\, s v_{-s}(0), \quad \gamma_5 v_s(0) = -s u_{-s}(0), \;\; \gamma_5 K^i = K^i \gamma_5 \\
\Rightarrow \gamma_5 u_s(\vec{p}) \; &= +\, s v_{-s}(\vec{p}), \quad \gamma_5 v_s(\vec{p}) = -s u_{-s}(\vec{p}) \\
\Rightarrow u_{-s}^*(-\vec{p}) &= -\, s\mathcal{C}\gamma_5 u_s(\vec{p}), v_{-s}^*(-\vec{p}) = -s\mathcal{C}\gamma_5 v_s(\vec{p})
\end{aligned}
$$

2.4.2 Gamma Matrix Technology

Introduction

$$
\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} 1_{4\times 4}, \gamma_5^2 = 1_{4\times 4}, \{\gamma^\mu, \gamma_5\} = 0, \mathrm{Tr} 1_{4\times 4} = 4
$$

$$
\mathrm{Tr}[\text{odd \# of } \gamma^\mu] = 0, \mathrm{Tr}[\gamma_5 \text{ odd \# of } \gamma^\mu] = 0
$$

Tr

$$
\mathrm{Tr}[\gamma^\mu \gamma^\nu] \xlongequal{\mathrm{Tr}[AB]=\mathrm{Tr}[BA]} \mathrm{Tr}[\gamma^\nu \gamma^\mu] \xlongequal{\mathrm{Tr}\,A+\mathrm{Tr}\,B=\mathrm{Tr}[A+B]} \frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \xlongequal{\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}1_{4\times4}} -g^{\mu\nu}\,\mathrm{Tr}\,1_{4\times 4} = -4g^{\mu\nu}
$$

$$
\begin{gathered}
\Rightarrow \mathrm{Tr}\big[\not{a}\not{b}\big] = -4(ab) \\
\not{a}\not{b} + \not{b}\not{a} = a_\mu b_\nu \{\gamma^\mu, \gamma^\nu\} = -2(ab) \\
\Rightarrow \mathrm{Tr}\big[\not{a}\not{b}\not{c}\not{d}\big] = 4[(ad)(bc) - (ac)(bd) + (ab)(cd)]
\end{gathered}
$$

γ_5

$$
\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{\mathrm{i}}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma
$$

$$
\begin{aligned}
&\mathrm{Tr}[\gamma_5] = 0 \\
&\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0 \\
&\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}
\end{aligned}
$$

Sandwich

$$
\begin{aligned}
&\gamma^\mu\gamma_\mu = -d \\
&\gamma^\mu\not{a}\gamma_\mu = (d-2)\not{a} \\
&\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(ab) - (d-4)\not{a}\not{b} \\
&\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = 2\not{c}\not{b}\not{a} - (d-4)\not{a}\not{b}\not{c}
\end{aligned}
$$

2.5 Spinor Helicity

massless = all Mandelstam variables $\gg m^2$

$$
\begin{aligned}
u_s(\vec{p})\bar{u}_s(\vec{p}) =& \frac{1}{2}(1+s\gamma_5)\left(-\not{p}\right) \\
v_s(\vec{p})\bar{v}_s(\vec{p}) =& \frac{1}{2}(1-s\gamma_5)\left(-\not{p}\right)
\end{aligned}
$$

$$v_s(\vec{p}) = u_{-s}(\vec{p})$$

$$u_-\bar{u}_-(\vec{p}) = \frac{1}{2}(1-\gamma_5)\left(-\not{p}\right) \xlongequal[p_{a\dot{a}}\equiv p_\mu\sigma^\mu_{a\dot{a}},p^{a\dot{a}}=\varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}}p_{c\dot{c}}=p_\mu\bar{\sigma}^{\mu\dot{a}a}]{\frac{1}{2}(1-\gamma_5)=\begin{bmatrix}1&0\\0&0\end{bmatrix},\gamma^\mu=\begin{bmatrix}0&\sigma^\mu\\\bar{\sigma}^\mu&0\end{bmatrix}} \begin{bmatrix}0&-p_{a\dot{a}}\\0&0\end{bmatrix}$$

twistor = commuting spinor ϕ_a

$$u_-(\vec{p}) = \begin{bmatrix}\phi_a\\0\end{bmatrix} \Rightarrow \bar{u}_-(\vec{p}) \xlongequal{\varphi^*_{\dot{a}}\equiv(\phi_{\dot{a}})^*} \begin{bmatrix}0&\phi^*_{\dot{a}}\end{bmatrix}, u_-(\vec{p})\bar{u}_-(\vec{p}) = \begin{bmatrix}0&\phi_a\phi^*_{\dot{a}}\\0&0\end{bmatrix} \Rightarrow p_{a\dot{a}} = -\phi_a\phi^*_{\dot{a}}$$

$$\xrightarrow{\phi^{*\dot{a}}=\varepsilon^{\dot{a}\dot{c}}\phi^*_{\dot{c}}} u_+(\vec{p}) = \begin{bmatrix}0\\\phi^{*\dot{a}}\end{bmatrix}, \bar{u}_+(\vec{p}) = \begin{bmatrix}\phi^a&0\end{bmatrix}$$

$$
\begin{aligned}
|p] =& u_-(\vec{p}) = v_+(\vec{p}) = \phi_a \\
|p\rangle =& u_+(\vec{p}) = v_-(\vec{p}) = \phi^{*\dot{a}} \\
[p| =& \bar{u}_+(\vec{p}) = \bar{v}_-(\vec{p}) = \phi^a \\
\langle p| =& \bar{u}_-(\vec{p}) = \bar{v}_+(\vec{p}) = \phi^*_{\dot{a}}
\end{aligned}
\Rightarrow
\begin{aligned}
\langle k||p\rangle =& \langle kp\rangle \\
[k|\,|p] =& [kp] \\
\langle k|\,|p] =& 0 \\
[k|\,|p\rangle =& 0
\end{aligned}
\Rightarrow
\begin{aligned}
[pk] =& \phi^a\kappa_a, [kp]+[pk]=0 \\
\langle pk\rangle =& \phi^*_{\dot{a}}\kappa^{*\dot{a}}, \langle kp\rangle+\langle pk\rangle=0
\end{aligned}, \langle pk\rangle = [pk]^*
$$

$$\langle pk\rangle[kp] = \left(\phi^*_{\dot{a}}\kappa^{*\dot{a}}\right)\left(\kappa^a\phi_a\right) = \left(\phi^*_{\dot{a}}\phi_a\right)\left(\kappa^a\kappa^{*\dot{a}}\right) = p_{\dot{a}a}k^{a\dot{a}} \xlongequal{\bar{\sigma}^{\mu\dot{a}a}\sigma^\nu_{a\dot{a}}=-2g^{\mu\nu}} -2p^\mu k_\mu = -2p\cdot k = -(p+k)^2$$

$$-\not{p} = \sum_{s=\pm} u_s(\vec{p})\bar{u}_s(\vec{p}) = u_+(\vec{p})\bar{u}_+(\vec{p}) + u_-(\vec{p})\bar{u}_-(\vec{p}) = |p\rangle\,[p| + |p]\,\langle p| = \begin{bmatrix}0&\phi_a\phi^*_{\dot{a}}\\\phi^{*\dot{a}}\phi^a&0\end{bmatrix}$$

Schouten Identity

$$\langle pq\rangle\langle rs\rangle + \langle pr\rangle\langle sq\rangle + \langle ps\rangle\langle qr\rangle = 0$$

$$\langle pq\rangle[qr]\langle rs\rangle[sp] = \mathrm{Tr}\left[\frac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s}\right] = 2[(pq)(rs)-(pr)(qs)+(ps)(qr)] + 2\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}p_\mu q_\nu r_\rho s_\sigma$$

$$\langle p|\gamma^\mu\,|k] = [k|\,\gamma^\mu|p\rangle, \langle p|\gamma^\mu\,|k]^* = \langle k|\gamma^\mu\,|p], \langle p|\gamma^\mu\,|p] = 2p^\mu, \langle p|\gamma^\mu|k\rangle = [p|\gamma^\mu|k] = 0$$

Fierz Identity

$$
\begin{aligned}
&-\frac{1}{2}\langle p|\gamma_\mu\,|q]\,\gamma^\mu = |q]\,\langle p| + |p\rangle\,[q| \\
&-\frac{1}{2}\,[p|\,\gamma_\mu|q\rangle\gamma^\mu = |q\rangle\,[p| + |p]\,\langle q| \Rightarrow [p|\,\gamma^\mu|q\rangle\langle r|\gamma_\mu\,|s] = 2[ps]\langle qr\rangle
\end{aligned}
$$

in Spinor Electrodynamics

2.6 Scattering Amplitude in QED

$$\sum_{\lambda=\pm}\varepsilon^\mu_\lambda\vec{k}\varepsilon^{\nu*}_\lambda(\vec{k}) \to g^{\mu\nu}$$

2.7 Anomaly

2.7.1 Chiral Gauge Theories and Anomaly

a single left-handed Weyl field ψ in a complex representation R:

$$\mathcal{L} = \mathrm{i}\psi^\dagger \bar{\sigma}^\mu D_\mu \psi - \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}, D_\mu = \partial_\mu - \mathrm{i}g A^a_\mu T^a_{\mathrm{R}}$$

- ψ is massless: $\mathrm{R} \otimes \mathrm{R}$ does not contain a singlet
- Lorentz invariance, gauge invariance
- charge and spin are correlated
- regularization
  - dimensional regularization: gamma matrices aren't well-define
  - Pauli–Villars regularization: is equivalent to adding an extra fermion field with mass: not be gauge invariance
- *Quantum Field Theory by Mark Srednicki*: 计算三光子单圈图时，无法使振幅 U(1) gauge invariance，存在多个左手 Weyl 场时可能恰好抵消，存在 nonabelian gauge 时，要看表示的 anomaly coefficient

2.8 Supersymmetry

$$\begin{aligned}
[Q_{aA}, P^\mu] = 0, \left[Q^\dagger_{\dot{a}A}, P^\mu\right] = 0 \\
[Q_{aA}, M^{\mu\nu}] = (S_L^{\mu\nu})_a{}^c Q_{cA}, \left[Q^\dagger_{\dot{a}A}, M^{\mu\nu}\right] = (S_R^{\mu\nu})_{\dot{a}}{}^{\dot{c}} Q_{cA} \\
\{Q_{aA}, Q_{bB}\} = Z_{AB}\varepsilon_{ab}, \left\{Q_{aA}, Q^\dagger_{\dot{a}B}\right\} = -2\delta_{AB}\sigma^\mu_{a\dot{a}} P_\mu, Z_{AB} + Z_{BA} = 0; \mathcal{N} = 1 \Rightarrow Z_{AB} = 0
\end{aligned}$$