微扰 GR

$$\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \Rightarrow \begin{cases} G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu} \\ T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \end{cases} \Rightarrow \begin{cases} G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \\ \bar{G}_{\mu\nu} = 8\pi G\, \bar{T}_{\mu\nu} \\ \delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu} \end{cases}$$

$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p})$

背景宇宙

$$\bar{g}_{\mu\nu} = \begin{bmatrix} -1 & \\ & a^2(t)\,\delta_{ij} \end{bmatrix} \xrightarrow{\bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}} \begin{cases} H^2 = \frac{8\pi G}{3}\bar{\rho} \\ \ddot{a}/a = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) \\ \dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G(\bar{\rho} + \bar{p}) \end{cases} \quad H \equiv \dot{a}/a$$

$$(t, x, y, z) \xrightarrow{d\eta = \frac{dt}{a(\eta)}} (\eta, x, y, z) \Rightarrow \bar{g}_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu}$$

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{da}{ad\eta}\frac{a}{dt} = \dot{a} \Rightarrow \begin{cases} \mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}\, a^2 \\ \mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p})\, a^2 \end{cases}$$

$$\mathcal{H}' = a\ddot{a} = a^2(\dot{H} + H^2).$$

$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}).$

$$w = \frac{\bar{p}}{\bar{\rho}}, \quad c_s^2 = \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} = \frac{\bar{p}'}{\bar{\rho}'}$$

背景关系: $\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2$

$w = -\frac{1}{3} \to \mathcal{H} = \text{const}$

$w > -\frac{1}{3} \to$ expansion

$w < -\frac{1}{3} \to$ inflation

$$\frac{w'}{1+w} = 3\mathcal{H}(w - c_s^2)$$

$$\bar{p}' = w\bar{\rho}' + w'\bar{\rho} = -3\mathcal{H}(1+w)c_s^2\bar{\rho}$$

$$1 + w = -\frac{2\dot{H}}{3H^2} = \frac{2}{3}\left[1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right] \qquad \text{仅考虑 } w > -1.$$

$$1 + c_s^2 = -\frac{\ddot{H}}{3H\dot{H}} = \frac{\mathcal{H}'' - 4\mathcal{H}\mathcal{H}' + 2\mathcal{H}^3}{3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}')}.$$

微扰宇宙

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}). \quad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$$

$$g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - h^{\mu\nu}).$$

$$h_{\mu\nu} = \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}, \quad \delta g_{\mu\nu} = a^2 h_{\mu\nu}$$

$D \equiv -\frac{1}{6}h_i{}^i$, E 无迹 $(E_i{}^i = 0)$

$$h^{\mu\nu} = \begin{bmatrix} -2A & +B_i \\ +B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}$$

$$g_{\mu\nu} = a^2(\eta)\begin{bmatrix} 1 - 2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix}$$

规范变换, 微扰量的规范变换 $\boxed{\xi}$

标量: $\widetilde{\delta s} = \delta s - \partial_\mu s\, \xi^\mu \to \widetilde{\delta s} = \delta s - \bar{s}'\xi^0$

矢量: $\widetilde{\delta w}^\mu = \delta w^\mu + \xi^\mu{}_{,\nu}\bar{w}^\nu - \bar{w}^\mu{}_{,\nu}\xi^\nu \to \begin{cases} \widetilde{\delta w}^0 = \delta w^0 + \xi^0{}_{,0}\bar{w}^0 - \bar{w}^0{}_{,0}\xi^0 \\ \widetilde{\delta w}^i = \delta w^i + \xi^i{}_{,0}\bar{w}^0 \end{cases}$

张量: $\widetilde{\delta A}^\mu{}_\nu = \delta A^\mu{}_\nu + \xi^\mu{}_{,\rho}\bar{A}^\rho{}_\nu - \xi^\sigma{}_{,\nu}\bar{A}^\mu{}_\sigma - \bar{A}^\mu{}_{\nu,\alpha}\xi^\alpha$

$$\Rightarrow \begin{cases} \widetilde{\delta A}^0{}_0 = \delta A^0{}_0 - \bar{A}^0{}_{0,0}\xi^0 \\ \widetilde{\delta A}^0{}_i = \delta A^0{}_i + \frac{1}{3}\xi^0{}_{,i}\bar{A}^k{}_k - \xi^0{}_{,i}\bar{A}^0{}_0 \\ \widetilde{\delta A}^i{}_0 = \delta A^i{}_0 + \xi^i{}_{,0}\bar{A}^0{}_0 - \frac{1}{3}\xi^i{}_{,0}\bar{A}^k{}_k \\ \widetilde{\delta A}^i{}_j = \delta A^i{}_j - \frac{1}{3}\delta^i_j\bar{A}^k{}_{k,0}\xi^0 \end{cases}$$

张量的无迹部分 规范不变.

由上推出度规(A B D E)的规范变换.

$$\bar{g}_{\mu\nu}=a^2(\eta)\eta_{\mu\nu},\quad \bar{g}_{\mu\nu,0}=2a'a\,\eta_{\mu\nu}$$

$$\widetilde{\delta g}_{\mu\nu}=\delta g_{\mu\nu}-\xi^{\rho}{}_{,\mu}\bar{g}_{\rho\nu}-\xi^{\sigma}{}_{,\nu}\bar{g}_{\mu\sigma}-\bar{g}_{\mu\nu,0}\xi^0$$

$$=\delta g_{\mu\nu}+a^2\left[-\xi^{\rho}{}_{,\mu}\eta_{\rho\nu}-\xi^{\sigma}{}_{,\nu}\eta_{\mu\sigma}-2\frac{a'}{a}\eta_{\mu\nu}\xi^0\right]$$

$$\Rightarrow\begin{cases}\tilde{A}=A-\xi^0{}_{,0}-\frac{a'}{a}\xi^0 & \tilde{D}=D+\frac{1}{3}\xi^k{}_{,k}+\frac{a'}{a}\xi^0\\ \tilde{B}_i=B_i+\xi^i{}_{,0}-\xi^0{}_{,i} & \tilde{E}_{ij}=E_{ij}-\frac{1}{2}(\xi^i{}_{,j}+\xi^j{}_{,i})+\frac{1}{3}\delta_{ij}\xi^k{}_{,k}\end{cases}$$

A、B、D、E，并不遵循标矢张的规范变换法则

但是，均匀+各向同性~~要求~~仅允许空间转动，不允许 boost

在此意义下 A D 是标量，B 是矢量，E_{ij} 是张量.

分解(A B D E)到标、矢、张

$$B_i=-B_{,i}+B_i^V,\quad B^V_{i,i}=0\ (无源).$$

$$E_{ij}=\left[\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2\right]E-\frac{1}{2}\left[E_{i,j}+E_{j,i}\right]+E_{ij}^T$$

$$\begin{cases}E_{i,i}=0 & 无源\\ E^T_{ij,i}=0,\ E^T_{ii}=0\end{cases}$$

对称无迹横向

$9-3-1-3=2$

4个标量：A B D E

2个矢量：B_i^V、E_i：各2自由度

1个张量：E_{ij}^T 2自由度，规范不变.

傅里叶空间中的微扰

$$E=\sum\frac{E_{\vec{k}}}{k^2}e^{i\vec{k}\cdot\vec{x}},\quad B=\sum\frac{B_{\vec{k}}}{k}e^{i\vec{k}\cdot\vec{x}},\quad E_i=\sum\frac{E_{i\vec{k}}}{k}e^{i\vec{k}\cdot\vec{x}},\quad f=\sum f_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}$$

$$B_i=-B_{,i}+B_i^V\Rightarrow B_i=-\frac{ik_i}{k}B+B_i^V$$

$$E_{ij}=\left[\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2\right]E-\frac{1}{2}\left[E_{i,j}+E_{j,i}\right]+E_{ij}^T$$

$$\Rightarrow E_{ij}=\left[-k_ik_j+\frac{1}{3}\delta_{ij}k^2\right]\frac{E}{k^2}-\frac{1}{2}\left[\frac{ik_j}{k}E_i+\frac{ik_i}{k}E_j\right]+E_{ij}^T$$

约束：$B^V_{i,i}=0\Rightarrow ik_iB_i^V=0$，$E_{i,i}=0\Rightarrow\frac{ik_i}{k}E_i=0$

$$E^T_{ij,i}=0\Rightarrow ik_iE^T_{ij}=0$$

$$E^T_{ii}=0$$

$$E_{ij}^T=\begin{bmatrix}E_{11}^T & E_{12}^T & 0\\ E_{12}^T & -E_{11}^T & 0\\ 0 & 0 & 0\end{bmatrix}$$

取 $k_i=(0,0,k)\Rightarrow -B_{,i}=(0,0,-iB)$，$B_i^V=(B_1,B_2,0)$，$E_i=(E_1,E_2,0)$

$$\delta g_{\mu\nu}=\delta g_{\mu\nu}{}_{标+矢+张}=a^2\begin{bmatrix}-2A & & & +iB\\ & 2(-D+\frac{1}{3}E) & & \\ & & 2(-D+\frac{1}{3}E) & \\ +iB & & & 2(-D-\frac{2}{3}E)\end{bmatrix}$$

$$+a^2\begin{bmatrix} & -B_1 & -B_2 & \\ -B_1 & & & -iE_1\\ -B_2 & & & -iE_2\\ & -iE_1 & -iE_2 & \end{bmatrix}+a^2\begin{bmatrix}0&0&0&0\\0&2E_{11}^T&2E_{12}^T&0\\0&2E_{12}^T&-2E_{11}^T&0\\ \end{bmatrix}$$

$h_+\equiv 2E_{11}^T$

$h_\times\equiv 2E_{12}^T$

傅里叶空间中的 ABDE 的规范变换.

$\tilde A = A - \xi^{0\prime} - \frac{a'}{a}\xi^0$ $\qquad$ $\tilde D = D + \frac{1}{3} i k_k \xi^k + \frac{a'}{a}\xi^0$

$\tilde B_i = B_i + \xi^{i\prime} - i k_i \xi^0$ $\qquad$ $\tilde E_{ij} = E_{ij} - \frac{1}{2}(i k_j \xi_i + i k_i \xi_j) + \frac{1}{3}\delta_{ij}(i k_k)\xi^k$

标量微扰：仅有四个标量 ABDE. $\quad g_{\mu\nu} = \bar g_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$.

$$h_{\mu\nu} = \begin{bmatrix} -2A & B_{,i} \\ B_{,i} & -2D\delta_{ij} + 2E_{,ij} - \frac{2}{3}\delta_{ij}E_{kk} \end{bmatrix} \quad -2\underbrace{\left[D + \frac{1}{3}E_{,kk}\right]}_{\equiv\psi}\delta_{ij} + 2E_{,ij}$$

$B_i = B_{,i} + B^V_i$

在'xPand'中，把我的 A 写作他的 ϕ，他的分解为. ~~E_{ij}~~

现试图分解 ξ^μ: $\xi^\mu = (\xi^0, \xi^i)$, $\xi^i = -\xi_{,i} + \xi^i_{tr}$ $\quad$ $\partial_i \xi^i_{tr} = 0$ 无旋 $\quad$ 舍去，因为它仅带来了矢量微扰

$\boxed{\tilde A = A - \xi^{0\prime} - \frac{a'}{a}\xi^0}$

$\tilde B_i = B_i - \xi_{,0i} - \xi^0_{,i} \Rightarrow -\tilde B_{,i} = -B_{,i} - \xi'_{,i} - \xi^0_{,i} \Rightarrow \boxed{\tilde B = B + \xi' + \xi^0}$ $\quad$ ~~$\tilde E = E + \xi$~~

$\boxed{\tilde D = D - \frac{1}{3}\xi_{,kk} + \frac{a'}{a}\xi^0}$

$\tilde E_{ij} = E_{ij} + \xi_{,ij} - \frac{1}{3}\delta_{ij}\xi_{,kk} \Rightarrow \tilde E_{,ij} - \frac{1}{3}\delta_{ij}\tilde E_{,kk} = E_{,ij} - \frac{1}{3}\delta_{ij}E_{,kk} + \xi_{,ij} - \frac{1}{3}\delta_{ij}\xi_{,kk}$

$\tilde\psi = \psi + \frac{a'}{a}\xi^0$

特殊的傅立叶变换: $\quad E, \Pi, \mu^v \Downarrow \frac{E}{k^2}, \frac{\Pi}{k^2}, \frac{\mu}{k^2} \qquad B, \xi, v \Downarrow \frac{B}{k}, \frac{\xi}{k}, \frac{v}{k}$

$\Phi = A + \mathcal{H}(B - E') + (B - E')'$

$\tilde\Phi = A - \cancel{\xi^{0\prime}} - \cancel{\frac{a'}{a}\xi^0} + \frac{a'}{a}(B + \cancel{\xi'} + \cancel{\xi^0} - E' - \cancel{\xi'}) + (B - E' + \cancel{\xi^0})' = \Phi$

$\Psi = \psi - \mathcal{H}(B - E')$

$\tilde\Psi = \psi + \cancel{\frac{a'}{a}\xi^0} - \mathcal{H}(B - E' + \cancel{\xi^0}) = \Psi$.

共形牛顿规范. $\xi = -E$, $\xi^0 = -B + E'$

$\Rightarrow \tilde E = 0$, $\tilde B = 0$. 记作: $B^N = E^N = 0$, 此时 $A^N = \Phi$, $D^N = \Psi$.

$$h_{\mu\nu} = \begin{bmatrix} -2\Phi & \\ & -2\Psi \end{bmatrix} \quad g_{\mu\nu} = a^2\begin{bmatrix} -1-2\Phi & \\ & \delta_{ij}(1-2\Psi) \end{bmatrix}$$

可以用 xPand 方便地算出此时的曲率张量

能量张量微扰 $\quad T^{\mu\nu} = (\bar\rho + \bar p)\bar u^\mu \bar u^\nu + \bar p \bar g^{\mu\nu}$,

均匀: $\bar\rho = \bar\rho(\eta)$, $\bar p = \bar p(\eta)$

各向同性: $\bar u^\mu = (\bar u^0, 0, 0, 0)$

$\bar u_\mu \bar u^\mu = 1 \Rightarrow \bar u^\mu = (\frac{1}{a}, 0, 0, 0)$

$\rho \equiv \bar\rho + \delta\rho \equiv \bar\rho(1 + \delta)$

$p \equiv \bar p + \delta p \qquad u^i = \bar u^i + \delta u^i = \delta u^i \equiv \frac{1}{a}v_i$, $\; v_i = \frac{dx^i}{d\eta}$

$u^\mu = \bar u^\mu + \delta u^\mu = (\frac{1}{a} + \delta u^0, \frac{1}{a}v_1, \frac{1}{a}v_2, \frac{1}{a}v_3)$

$g_{\mu\nu}u^\mu u^\nu = -1 \Rightarrow u^\mu = \frac{1}{a}(1-A, v_i)$; $u_\mu = a(-1-A, v_i - B_i)$

$$T^\mu{}_\nu = \bar T^\mu{}_\nu + \delta T^\mu{}_\nu = \begin{bmatrix} -\bar\rho & \\ & \bar p\,\delta^i{}_j \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar\rho+\bar p)(v_i - B_i) \\ -(\bar\rho+\bar p)v_i & \delta p\cdot\delta^i{}_j \end{bmatrix}$$

以上未考虑各向异性压强

$\delta T^i{}_j = \delta p\,\delta^i{}_j + \Sigma_{ij}$, $\delta p = \frac{1}{3}\delta T^k{}_k$, $\Sigma_{ij} = \bar p\,\Pi_{ij}$ 无迹对称

$\boxed{\text{能量张量微扰的分解}}$

$$v_i = -v_{,i} + v_i^V, \quad v^V_{i,i} = 0$$

$$\Pi_{ij} = [\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2]\Pi - \frac{1}{2}[\Pi_{i,j} - \Pi_{j,i}] + \Pi^T_{ij} \quad \Big\{ \; T^T_{ij} = 0 \text{ 横向}$$

$\boxed{\text{能量张量微扰的规范变换}}$

$\widetilde{\delta\rho} = \delta\rho - \bar\rho'\xi^0$ $\quad\tilde\Pi_{ij} = \Pi_{ij}$ 规范不变

$\widetilde{\delta p} = \delta p - \bar p'\xi^0$

$\tilde v_i = v_i - \xi^i{}_{,0}$ $\quad\tilde\delta = \delta - \frac{\bar\rho'}{\bar\rho}\xi^0 = \delta + 3\mathcal{H}(1+w)\xi^0$

仅有标量微扰时: $v_i = -v_{,i}$, $\xi^i = -\xi_{,i}$ $\Rightarrow \tilde v = v + \xi'$, $\tilde\Pi = \Pi$

再取共形牛顿规范: $\delta\rho^N = \delta\rho + \bar\rho'\xi^0 = \delta\rho + \bar\rho'(B-E') = \delta\rho - 3\mathcal{H}(1+w)\bar\rho(B-E')$

$B^N = E^N = 0$ $\quad\delta p^N = \delta p - 3\mathcal{H}(1+w)c_s^2\bar\rho(B-E')$

$v^N = v - E'$, $\quad\Pi^N = \Pi$

×Pand: $g_{\mu\nu} = a^2\begin{bmatrix} -1-2\phi & B_{,i}+B_i \\ B_{,i}+B_i & h_{\mu\nu} + 2E_{,ij} + E_{i,j} + E_{j,i} + 2E_{ij} - 2\psi \end{bmatrix}$

×Pand 好像不能表示各向异性压强, ~~计算 能量张量微扰的结果也怪怪的~~

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*共形牛顿规范, 标量微扰 $\quad\delta G^\mu{}_\nu = 8\pi G\delta T^\mu{}_\nu$

$\Rightarrow 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi = -4\pi G a^2\delta\rho^N$

$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi G a^2(\bar\rho+\bar p)v^N_{,i}$

$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = 4\pi G a^2\delta p^N = \frac{3}{2}\mathcal{H}^2\delta p^N/\bar\rho$ ④

$(\partial_i\partial_j - \frac{1}{3}\delta^i{}_j\nabla^2)(\Psi - \Phi) = 8\pi G a^2\bar p(\partial_i\partial_j - \frac{1}{3}\delta^i{}_j\nabla^2)\Pi$

微扰量, 积分常数为零 $\Rightarrow \begin{cases} \Psi - \Phi = 8\pi G a^2\bar p\Pi = 3\mathcal{H}^2 w\Pi & ① \\ \Psi' + \mathcal{H}\Phi = 4\pi G a^2(\bar\rho+\bar p)v^N & ③ \end{cases}$

代入前式: $\nabla^2\Psi = 4\pi G a^2\delta\rho^N + 3\mathcal{H}\cdot 4\pi G a^2(\bar\rho+\bar p)v^N$

$= 4\pi G a^2\bar\rho[\delta^N + 3\mathcal{H}(1+w)v^N]$

$\underline{3\mathcal{H}^2 = 8\pi G\bar\rho a^2}$ $\;\frac{3}{2}\mathcal{H}^2[\delta^N + 3\mathcal{H}(1+w)v^N]$ ②

能量动量连续性方程. $w' = \left(\frac{\bar p}{\bar\rho}\right)' = \frac{\bar p' - w\bar\rho'}{\bar\rho}$

$T^{\mu}{}_{\nu;\mu}=0 \Rightarrow \bar\rho' = -3\mathcal{H}(\bar\rho+\bar p)$. 连续性方程 欧拉方程.

$$\Rightarrow \begin{cases} (\delta\rho^N)' = -3\mathcal{H}(\delta\rho^N+\delta p^N) + (\bar\rho+\bar p)(\nabla^2 v^N + 3\Psi') \\ (\bar\rho+\bar p)(v^N)' = -(\bar\rho+\bar p)' v^N - 4\mathcal{H}(\bar\rho+\bar p)v^N + \delta p^N + \frac{2}{3}\bar p\nabla^2\Pi + (\bar\rho+\bar p)\Phi \end{cases}$$

$$\Rightarrow \begin{cases} (\delta^N)' = (1+w)(\nabla^2 v^N + 3\Psi') + 3\mathcal{H}\left(w\delta^N - \frac{\delta p^N}{\bar\rho}\right) \\ (v^N)' = -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + \frac{\delta p^N}{\bar\rho+\bar p} + \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi + \Phi \end{cases}$$

$$\cancel{\mathcal{H}(1-3w) - \frac{w'}{1+w} = \mathcal{H}\left(1-3\frac{\bar p}{\bar\rho}\right) - \frac{\bar p' + w\bar\rho' \times 3\mathcal{H}(\bar\rho+\bar p)}{\bar\rho+\bar p}}$$ MMA 瞬间可以验证 ...

$$\cancel{(\bar\rho+\bar p)\cdots = -\mathcal{H}(\bar\rho+\bar p) + 3\mathcal{H}\frac{\bar p}{\bar\rho}(\bar\rho+\bar p) - \bar p' - 3\mathcal{H}w(\bar\rho+\bar p)}$$

字
$$\Rightarrow \begin{cases} (\delta^N)' = -(1+w)(kv^N - 3\Psi') + 3\mathcal{H}(w\delta^N - \delta p^N/\bar\rho) \\ (v^N)' = -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + k\frac{\delta p^N}{\bar\rho+\bar p} - \frac{2}{3}k\frac{w}{1+w}\Pi + k\Phi. \end{cases}$$

流体演化方程并不独立于场方程，但与之地位相同.

理想流体、牛顿规范、标量微扰

↳ $\Pi=0$. ↳ $B^N=0$, $E^N=0$ ↳ $B_i^V=0$, $E_i=0$, $E_{ij}^T=0$

$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\left[\delta^N + 3\mathcal{H}(1+w)v^N\right]$$

由上页①: $\Psi = \Phi \Rightarrow$

② $\cancel{\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2\delta\rho^N}$

③ $\cancel{\Psi'}$ $\Phi' + \mathcal{H}\Phi = 4\pi G a^2(\bar\rho+\bar p)v^N = \frac{3}{2}\mathcal{H}^2\bar\rho(1+w)v^N$

④ ...

熵微扰: $\mathcal{S} = \mathcal{H}\left(\frac{\delta p}{\bar p'} - \frac{\delta\rho}{\bar\rho'}\right) = \frac{1}{3(1+w)}\left[\frac{\delta\rho}{\bar\rho} - \frac{1}{c_s^2}\frac{\delta p}{\bar\rho}\right]$ 规范不变.

$$\mathcal{H}^{-2}\Phi'' + 3(1+c_s^2)\mathcal{H}^{-1}\Phi' + 3(c_s^2-w)\Phi = c_s^2\mathcal{H}^{-2}\nabla^2\Phi - \frac{9}{2}c_s^2(1+w)\mathcal{S}$$

绝热微扰: $\mathcal{S}=0$, $\delta p = c_s^2\delta\rho$

$$\Rightarrow \mathcal{H}^{-2}\Phi_k'' + 3(1+c_s^2)\mathcal{H}^{-1}\Phi_k' + 3(c_s^2-w)\Phi_k = -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\Phi_k$$

连续性方程: $(\delta^N)' = (1+w)(\nabla^2 v^N + 3\Psi') + 3\mathcal{H}(w\delta^N - \delta p^N/\bar\rho)$.

$$(v^N)' = -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + \frac{\delta p^N}{\bar\rho+\bar p} + \Phi$$

$$H^{-2}\ddot\Phi_k + (4+3c_s^2)H^{-1}\dot\Phi_k + 3(c_s^2-w)\Phi_k = -\left(\frac{c_s k}{H}\right)^2\Phi_k$$

超视界: $k/\mathcal{H} \ll 1$. $\Rightarrow \ddot\Phi_k - \frac{1}{\dot H}(\ddot H - H\dot H)\dot\Phi_k - \frac{1}{\dot H}(H\ddot H - 2\dot H^2)\Phi_k = 0$.

可验证，解为: $\Phi_k(t) = A_k\left[1 - \frac{H(t)}{a(t)}\int_0^t a(t')dt'\right] + B_k\frac{H(t)}{a(t)}$.

气体型理想流体：$p=p(\rho)$.

$$\begin{cases}\Phi_{\vec k}''+3(1+c_s^2)\mathcal{H}\Phi_{\vec k}'+3(c_s^2-w)\Phi_{\vec k}+c_s^2k^2\Phi_{\vec k}=0 \Rightarrow \Phi_{\vec k}(\eta). & ① \\ v_{\vec k}^N(\eta)=\dfrac{2k}{3(1+w)}(\mathcal{H}^{-2}\Phi'+\mathcal{H}^{-1}\Phi) & ② \end{cases}$$

$$\delta_{\vec k}^N(\eta)=-\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi-3(1+w)\left(\frac{\mathcal{H}}{k}\right)v^N=-\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi-2(\mathcal{H}^{-1}\Phi'+\Phi).\ ③$$

物质主导宇宙中的标量微扰. $\bar p=0,\ w=0,\ c_s^2=0,\ \Pi=0.\ \delta p=0.$

$$\mathcal{H}^2=\frac{8\pi G}{3}\bar\rho a^2,\ \mathcal{H}'=-\frac{4\pi G}{3}\bar\rho a^2 \Rightarrow \mathcal{H}^2+2\mathcal{H}'=0 \Rightarrow \mathcal{H}=\frac{2}{\eta}$$

① $\Phi_{\vec k}''+3\mathcal{H}\Phi_{\vec k}'=0 \Rightarrow \Phi_{\vec k}''+\frac{6}{\eta}\Phi_{\vec k}'=0 \Rightarrow \Phi_{\vec k}(\eta,\vec x)=C_1(\vec x)+C_2(\vec x)\eta^{-5}$

η^{-5} 快速衰减，$\Phi_{\vec k}(\eta,\vec x)=\Phi_{\vec k}(\vec x)$，$\Phi(\eta,\vec x)=\Phi(\vec x)$.

② $v_{\vec k}^N(\eta)=\frac{2k}{3\mathcal{H}}\Phi_{\vec k}=\frac{1}{3}k\eta\Phi_{\vec k} \Rightarrow v^N(\eta)=\frac{1}{3}\eta\Phi$

③ $\delta_{\vec k}^N(\eta)=-\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi-2\Phi=-\left[2+\frac{1}{6}k^2\eta^2\right]\Phi$

$\begin{cases}\text{超视界},\ k/\mathcal{H}\ll 1. \Rightarrow \delta_{\vec k}^N(\eta)=-2\Phi \\ \text{次视界},\ k/\mathcal{H}\gg 1. \Rightarrow \delta_{\vec k}^N(\eta)=-\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi\propto\eta^2\end{cases}$ 进入视界后，相对密度微扰开始增长

辐射主导宇宙中的标量微扰 $p=\frac{1}{3}\rho \Rightarrow w=c_s^2=\frac{1}{3},\ \delta p=\frac{1}{3}\delta\rho.$

$$\mathcal{H}^2=\frac{8\pi G}{3}\bar\rho a^2,\ \mathcal{H}'=-\frac{8\pi G}{3}\bar\rho a^2 \Rightarrow \mathcal{H}'+\mathcal{H}^2=0,\ \mathcal{H}=\frac{1}{\eta}$$

① $\Phi_{\vec k}''+4\mathcal{H}\Phi_{\vec k}'+\frac{1}{3}k^2\Phi_{\vec k}=0$ 舍去衰减解：$\Phi_{\vec k}(\eta)=A_{\vec k}\dfrac{\sin(k\eta/\sqrt3)-k\eta/\sqrt3\cos(k\eta/\sqrt3)}{(k\eta/\sqrt3)^3}$.

$\Rightarrow \Phi_{\vec k}''+\frac{4}{\eta}\Phi_{\vec k}'+\frac{1}{3}k^2\Phi_{\vec k}=0$

超视界：~~$\Phi_{\vec k}(\eta)$~~ $k\eta\ll1. \Rightarrow \Phi_{\vec k}=\frac{1}{3}A_{\vec k}=\text{const}$

② $v_{\vec k}^N(\eta)=\frac{1}{2}k\eta\Phi_k$

③ $\delta_{\vec k}^N(\eta)=-2\Phi_k$

次视界：$k\eta\gg1.$ $\Phi_{\vec k}(\eta)\sim-3A_{\vec k}\dfrac{\cos(k\eta/\sqrt3)}{(k\eta)^2}$

② $v_{\vec k}^N(\eta)\propto k^{3/2}\cdot\sin(k\eta/\sqrt3)$

③ $\delta_{\vec k}^N(\eta)\propto k^{3/2}\cdot\cos(k\eta/\sqrt3)$.

其他规范. 共动曲率微扰：$\mathcal{R}=-\psi^c$.

选择不同的规范，就是为 A, B, D, E 或(和) $\delta\rho$, δp, v 中的某几个指定特殊值

1. 共动规范 comoving gauge.

$$\left.\begin{matrix}① v=B \\ ② v=0\end{matrix}\right\}\ v^c=B^c=0 \Rightarrow \begin{cases}\tilde v=v+\xi' \\ \tilde B=B+\xi'+\xi^0\end{cases} \Rightarrow \begin{cases}\xi'=-v \Rightarrow \xi\ \text{可差一个积分常数} \\ \xi^0=v-B\end{cases}$$

求出了 ξ 和 ξ^0 之后，代入其他的 $\tilde{\ }=\cdot+\xi\cdots$，可得到量在共动规范下的形式

例如，$A^C=\tilde{A}=A-(v-B)'-\mathcal{H}(v-B)$.

想从 N 规范到 C 规范，则有：$A^C=A^N-(v^N-B^N)'-\mathcal{H}(v^N-B^N)$.

$$\xlongequal{A^N=\Phi,\ B^N=0}\ \Phi-v^{N\prime}-\mathcal{H}v^N$$

$\boxed{\mathcal{R}}\equiv-\psi^c=-\psi-\mathcal{H}(v-B)=-\psi^N-\mathcal{H}(v^N-B^N)$ $\quad\psi^N=\Psi,\ B^N=0$

$$=-\Psi-\frac{2}{3(1+w)}(\mathcal{H}^{-1}\Psi'+\Phi).$$

$$\rightarrow\mathcal{R}'=-\frac{2\mathcal{H}^{-1}}{3(1+w)}\Psi''-\frac{4+6c_s^2}{3(1+w)}\Psi'-\frac{2}{3(1+w)}\Phi'+2\mathcal{H}\frac{w-c_s^2}{1+w}\Phi$$

$$\Rightarrow\mathcal{H}^{-1}\mathcal{R}'=\frac{2}{3(1+w)}\left(\frac{k}{\mathcal{H}}\right)^2\left[c_s^2\Psi+\frac{1}{3}(\Psi-\Phi)\right]+3c_s^2\,S$$

超视界尺度 + 绝热微扰 → $\mathcal{R}=\text{const}$

进一步说明：一般的微扰可以分为 绝热($S=0$) 和 零曲率($\mathcal{R}=0$) 部分，设外

(等价于) $\mathcal{H}^{-1}\mathcal{R}'=-c_s^2\left(\frac{\delta^c}{1+w}-3S\right)-\frac{2}{3}\frac{w}{1+w}\nabla^2\Pi$.

2. 混合规范：标记物理量的所用规范，即可写出组合规范的方程.

$$\begin{cases}\nabla^2\Psi=\frac{3}{2}\mathcal{H}^2\delta^c\\ \Psi-\Phi=3\mathcal{H}^2w\Pi\\ \Psi'+\mathcal{H}\Phi=\frac{3}{2}\mathcal{H}^2(1+w)v^N\\ \Psi''+(2+3c_s^2)\mathcal{H}\Psi'+\mathcal{H}\Phi'+3(c_s^2-w)\mathcal{H}^2\Phi+\frac{1}{3}\nabla^2(\Phi-\Psi)=\frac{3}{2}\mathcal{H}^2\frac{\delta p^c}{\bar{\rho}}\end{cases}$$

$\delta^c=\delta^N+3\mathcal{H}(1+w)k^{-1}v^N$

$$\begin{cases}\delta_c'-3\mathcal{H}w\delta_c=(1+w)\nabla^2v^N+2\mathcal{H}w\nabla^2\Pi\\ v_N'+\mathcal{H}v_N=\frac{\delta p^c}{\bar{\rho}+\bar{p}}+\frac{2}{3}\frac{w}{1+w}\nabla^2\Pi+\Phi.\end{cases}$$

理想流体($\Pi=0$, $\Psi=\Phi$) 可以再简化.

绝热理想流体：$\mathcal{R}=-\Phi-\frac{2}{3(1+w)}(\mathcal{H}^{-1}\Phi'+\Phi)$

$$\Rightarrow\frac{2}{3}\mathcal{H}^{-1}\Phi'+\frac{5+3w}{3}\Phi=-(1+w)\mathcal{R}$$

对 $w=\text{const}$，舍去衰减解，$\Phi=-\frac{3+3w}{5+3w}\mathcal{R}$

3. 一致能量规范

$$\delta\rho^u=0\Rightarrow\widetilde{\delta\rho}=\delta\rho-\bar{\rho}'\xi^0\rightarrow\xi^0=\delta\rho/\bar{\rho}'$$

$$\psi^u=\psi+\mathcal{H}\xi^0=\psi+\mathcal{H}\cdot\delta\rho/\bar{\rho}'$$

$$\zeta\equiv-\psi^u=-\psi+\frac{\delta}{3(1+w)}=-\psi^c+\frac{\delta^c}{3(1+w)}=\mathcal{R}+\frac{2}{9(1+w)}\mathcal{H}^{-2}\nabla^2\Phi$$

超视界尺度 $\zeta=\mathcal{R}$

4. 平坦空间规范 $\quad\psi^a=0\Rightarrow\xi^0=-\mathcal{H}^{-1}\psi$

$$\delta^a=\delta-3(1+w)\psi=\delta^c+3(1+w)\mathcal{R}=3(1+w)\zeta$$

5. 同步规范.

$A^z = B_i{}^z = 0 \Rightarrow A^z = B^z = 0. \Rightarrow \begin{cases} \xi^{0\prime\prime} + \mathcal{H}\xi^{0\prime} = A \\ \xi^{\prime} = -\xi^{0} - B \end{cases}$ 仅决定规微扰的空间部分

临时转换一下记号.

$h \equiv -6D^z,\ \eta \equiv \psi^z = D^z + \frac{1}{3}\nabla^2 E^z,\ \mu \equiv 2E^z.$

在傅立叶空间, $h + \mu + 6\eta = 0$

$$h_{ij} = -2D^z\delta_{ij} + 2\left(-\frac{k_ik_j}{k^2} + \frac{1}{3}\delta_{ij}\right)E^z = \hat{k}_i\hat{k}_j h + \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)6\eta$$

据定义, $\Phi = A^z + \mathcal{H}(B^z - E^{z\prime}) + (B^z - E^{z\prime})' = \frac{1}{2k^2}\left[h'' + 6\eta'' + \mathcal{H}(h' + 6\eta')\right]$

$$\Psi = \eta - \frac{1}{2k^2}\mathcal{H}(h' + 6\eta').$$

然后讲义上列了列方程: 求爱因斯坦张量的微扰, 列场方程, 列连续性方程

流体分量. $T^\mu{}_\nu = \sum_i T^\mu{}_{\nu(i)} \Rightarrow \bar{T}^\mu{}_\nu = \sum_i \bar{T}^\mu{}_{\nu(i)},\ \delta T^\mu{}_\nu = \sum_i \delta T^\mu{}_{\nu(i)}.$

分解是为了使用纯净物的状态方程: $p=0$, $p = \rho/3 \cdots$

$\bar{\rho} = \sum_i \bar{\rho}_i \qquad w = \bar{p}/\bar{\rho} = \sum w_i \frac{\bar{\rho}_i}{\bar{\rho}} \qquad \delta\rho = \sum \delta\rho_i = \sum \bar{\rho}_i \cdot \delta_i$

$\bar{p} = \sum_i \bar{p}_i = \sum_i w_i\bar{\rho}_i \qquad c_s^2 = \bar{p}'/\bar{\rho}' = \sum c_{si}^2 \frac{\bar{\rho}_i'}{\bar{\rho}'} \qquad \delta p = \sum \delta p_i$

$\delta = \delta\rho/\bar{\rho} = \sum \frac{\bar{\rho}_i}{\bar{\rho}}\delta_i$

$(\bar{\rho} + \bar{p})v_l = \sum(\bar{\rho}_i + \bar{p}_i)v_{l(i)}. \Rightarrow v_l = \sum \frac{\bar{\rho}_i + \bar{p}_i}{\bar{\rho} + \bar{p}} v_{l(i)}.$

$\bar{p}\Pi_{lm} = \Sigma_{lm} = \sum \Sigma_{lm(i)} = \sum \bar{p}_i\Pi_{lm(i)} \Rightarrow \Pi_{lm} = \sum \Pi_{lm(i)}\frac{\bar{p}_i}{\bar{p}} = \sum \Pi_{lm(i)}\frac{w_i\bar{\rho}_i}{w\bar{\rho}}$

流体分量微扰的规范变换

$\widetilde{\delta\rho}_i = \delta\rho_i - \bar{\rho}_i'\xi^0,\quad \tilde{\delta}_i = \delta_i - \frac{\bar{\rho}_i'}{\bar{\rho}_i}\xi^0$

$\widetilde{\delta p}_i = \delta p_i - \bar{p}_i'\xi^0$

$\tilde{v}_i = v_i + \xi'$ $\qquad v_i - v_j$ (两个流体元相对速度)规范不变

$\tilde{\Pi}_{ij} = \Pi_{ij}$ $\qquad \Pi_{ij}$ 规范不变

熵微扰: $S_{ij} = -3\mathcal{H}\left(\frac{\delta\rho_i}{\bar{\rho}_i'} - \frac{\delta\rho_j}{\bar{\rho}_j'}\right)$

原则上, 只能写出一套场方程, 因为我们只有一个时空/度规, 度规可不能分解成各流体成分

若流体成分间无能量交换: $\bar{\rho}_i' = -3\mathcal{H}(\bar{\rho}_i + \bar{p}_i)$

$\overset{S_i - S_j \equiv}{\Rightarrow} S_{ij} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j},\ \frac{w_i'}{1+w_j} = 3\mathcal{H}(w_i - c_{si}^2)$「若有能量交换, 则会出现相互作用项」

$(\delta_i^N)' = (1+w_i)(\nabla^2 v_i^N + 3\Psi') + 3\mathcal{H}(w_i\delta_i^N - \delta p_i^N/\bar{\rho}_i)$

$(v_i^N)' = -\mathcal{H}(1-3w_i)v_i^N - \frac{w_i'}{1+w_i}v_i^N + \frac{\delta p_i^N}{\bar{\rho}_i + \bar{p}_i} + \frac{2}{3}\frac{w_i}{1+w_i}\nabla^2\Pi_i + \Phi$

$S_i' = \delta_i'\frac{1}{1+w_i} - \frac{\delta_i w_i'}{(1+w_i)^2} = \nabla^2 v_i^N + 3\Psi' + \frac{3\mathcal{H}}{1+w_i}(w_i\cancel{\delta_i^N} - \delta p_i^N/\bar{\rho}_i) - \frac{3\mathcal{H}}{1+w_i}(w_i - c_{si}^2)\delta_i$

$S_{ij}' = \nabla^2(v_i - v_j) - 9\mathcal{H}(c_i^2 S_i - c_j^2 S_j),\quad S_i = \mathcal{H}(\delta p_i/\bar{\rho}_i' - \delta\rho_i/\bar{\rho}_i)$

进一步，气体型流体：$p_i = p_i(\rho_i)$. $\Rightarrow S'_{ij} = \nabla^2(v_i - v_j)$. 导出 $S'_{ij} = -\frac{k}{\mathcal{H}}(v_i - v_j)$

S_{ij} 超视界下为常数

简化的「物质+辐射」宇宙.

理想流体，物质 $p=0$，辐射 $p=\rho/3$，~~不~~无相互作用.

没有 DE，重子和光子相互作用，中微子和光子均已解耦.

$\boxed{\text{背景方程}}$ $\rho = \rho_r + \rho_m$, $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$

$p_r = \rho_r/3$, $p_m = 0$. $\Rightarrow w_r = c_{s\,r}^2 = 1/3$, $w_m = c_{s\,m}^2 = 0$

$y = a/a_{eq} = \rho_m/\rho_r$, 其 $a_{eq} = \Omega_r/\Omega_m$.

$$\frac{\rho_r}{\rho} = \frac{1}{1+y}, \quad \frac{\rho_m}{\rho} = \frac{y}{1+y}, \quad \frac{\rho_r + p_r}{\rho + p} = \frac{4}{4+3y}, \quad \frac{\rho_m + p_m}{\rho + p} = \frac{3y}{4+3y}$$

$$w = \frac{p}{\rho} = \frac{\rho_r/3}{\rho_r + \rho_m} = \frac{1}{3+3y}, \quad c_s^2 = \frac{p'}{\rho'} = \frac{\rho_r'/3}{\rho_r' + \rho_m'} = \frac{-4a^{-1}\rho_r/3}{-4a^{-1}\rho_r - 3a^{-1}\rho_m} = \frac{4/3}{4+3y}$$

场方程：$\mathcal{H}^2 = \frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3}\rho a^2 \Rightarrow y = 2\left(\frac{\eta}{\eta_3}\right) + \left(\frac{\eta}{\eta_3}\right)^2$

~~$\eta_3 = \frac{\eta_{eq}}{\sqrt{2}-1} = (\sqrt{2}+1)\eta_{eq} = \frac{2}{H_0}\frac{\sqrt{\Omega_r}}{\Omega_m}$~~

$$\mathcal{H} = a'/a = y'/y = \frac{\eta + \eta_3}{\eta_3\eta + \eta^2/2} \qquad \eta = \eta_3 \cdot (\sqrt{1+y} - 1)$$

$\eta \ll \eta_3$，辐射主导，$\eta \gg \eta_3$，物质主导 $\qquad \mathcal{H} = \frac{\sqrt{1+y}}{y} \cdot \frac{2}{\eta_3}$

$\boxed{\text{微扰}}$ $\delta = \frac{1}{1+y}\delta_r + \frac{y}{1+y}\delta_m$, $v = \frac{4}{4+3y}v_r + \frac{3y}{4+3y}v_m$

可解出 $\delta_r = \delta_r(\delta, S)$

$S_{mr} = \delta_m/(1+w_m) - \delta_r/(1+w_r) = \delta_m - 3\delta_r/4$

$$\Rightarrow \zeta = \frac{1}{3(1+w)}\left[\frac{\delta\rho}{\rho} - \frac{1}{c_s^2}\frac{\delta p}{\rho}\right] = \frac{y}{4+3y}S$$

$$\mathcal{H}^{-2}\delta_c'' + (1 - 6w + 3c_s^2)\mathcal{H}^{-1}\delta_c' - \frac{3}{2}(1 + 8w - 6c_s^2 - 3w^2)\delta_c = -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\left(\delta_c - \frac{y}{1+y}S\right)$$

$S' = -k(v_m - v_r)$, $S'' = -k(v_m' - v_r')$.

此时（$w_m = c_{s\,m}^2 = 0$, $w_r = c_{s\,r}^2 = 1/3$, $\Pi = 0$），$S'' = \mathcal{H}k(v_m - v_r) + \mathcal{H}k v_r^N + \frac{1}{4}k^2\delta_r^N$

而 $\delta_r^N = \delta_r^c - 3\mathcal{H}(1+w)k^{-1}v^N$

$$S'' = \frac{1}{4}k^2\delta_r^c - \cancel{\frac{3}{4}}\mathcal{H}\cancel{(1+w)}k\cdot\left[\frac{4}{4+3y}v_r + \frac{3y}{4+3y}v_m\right] + \mathcal{H}k(v_m - \cancel{v_r}) + \mathcal{H}k\cancel{v_r^N}$$

$$= \mathcal{H}k\cdot\frac{4}{4+3y}\underbrace{(v_m - v_r)}_{\to -k^{-1}S'} + \frac{1}{4}k^2\delta_r^c$$

$$\Rightarrow \mathcal{H}^{-2}S'' + 3c_s^2\mathcal{H}^{-1}S' = \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\left[\frac{1}{1+w}\delta^c - (1-3c_s^2)S\right]$$

之前一直用的时间 η，现在换到时间 $y = 2(\eta/\eta_3) + (\eta/\eta_1)^2$

$$\mathcal{H}^{-1} f' = \cancel{\frac{dy}{d\eta}} \cdot \frac{df}{dy} \cdot \frac{a}{da/dy} = y \cdot \frac{df}{dy}$$

$$\mathcal{H}^{-2} f'' = y^2 \frac{d^2 f}{dy^2} + \frac{1}{2}(1-3w) y \frac{df}{dy}$$

$$\begin{cases} y^2 \dfrac{d^2\delta^c}{dy^2} + \dfrac{3}{2}(1-5w+2c_s^2) y \dfrac{d\delta^c}{dy} - \dfrac{3}{2}(1+8w-6c_s^2-3w^2)\delta^c = -\left(\dfrac{k}{\mathcal{H}}\right)^2 c_s^2\left(\delta^c - \dfrac{y}{1+y} S\right) \\ y^2 \dfrac{d^2 S}{dy^2} + \dfrac{1}{2}(1-3w+6c_s^2) y \dfrac{dS}{dy} = (k/\mathcal{H})^2 \cdot \left[\dfrac{1+y}{4+3y}\delta^c - \dfrac{y}{4+3y} S\right] \end{cases}$$

$$\boxed{\delta^c, S} \longrightarrow \Phi = -\frac{3}{2}\left(\frac{\mathcal{H}}{k}\right)^2 \delta^c \Rightarrow v^N = \frac{2}{3(1+w)} \cdot \left(\frac{k}{\mathcal{H}}\right)(\mathcal{H}^{-1}\Phi' + \Phi)$$

$$\Rightarrow \delta^N = \delta^c - 3\left(\frac{\mathcal{H}}{k}\right)(1+w) v^N,\quad \mathcal{R} = -\Phi - \frac{2}{3(1+w)\mathcal{H}}(\Phi' + \mathcal{H}\Phi)$$

initial 时期：$y \ll 1$，$\mathcal{H} = \dfrac{\sqrt{1+y}}{y} \cdot \dfrac{\mathcal{H}_{eq}}{2} \Rightarrow \mathcal{H}^2 = \dfrac{\mathcal{H}_{eq}^2}{2y^2}$

$$\begin{cases} 1-5w+2c_s^2 = 1 - 5\cdot\dfrac{1}{3+3y} + 2\cdot\dfrac{4/3}{4+3y} \sim O(y). \\ 1+8w-6c_s^2-3w^2 = 1+8\cdot\dfrac{1}{3+3y} - 6\cdot\dfrac{4/3}{4+3y} - 3\cdot\left(\dfrac{1}{3+3y}\right)^2 \sim \dfrac{4}{3} + O(y). \end{cases}$$

$$\Rightarrow y^2 \frac{d^2\delta^c}{dy^2} - 2\delta^c = -\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\delta^c - yS) = -\frac{2}{3}\left(\frac{k}{\mathcal{H}_{eq}}\right)^2(y^2\delta^c - y^3 S)$$

$$1-3w+6c_s^2 = 1 - 3\cdot\frac{1}{3+3y} + 6\cdot\frac{4/3}{4+3y} = 2 + O(y).$$

$$\Rightarrow y^2 \frac{d^2 S}{dy^2} + y\frac{dS}{dy} = \frac{1}{2}\left(\frac{k}{\mathcal{H}_{eq}}\right)^2 \cdot (y^2\delta^c - y^3 S).$$

早期，超视界：$\begin{cases} y^2\, d^2\delta^c/dy^2 - 2\delta^c = 0 \\ y^2\, d^2 S/dy^2 + y\, dS/dy = 0 \end{cases} \Rightarrow \begin{cases} \delta^c_{\vec{k}} = A_{\vec{k}}\, y^2 + B_{\vec{k}}\, y^{-1} \sim A_{\vec{k}}\, y^2 \\ S_{\vec{k}} = C_{\vec{k}} + D_{\vec{k}} \ln y \xrightarrow{y<1} C_{\vec{k}} \end{cases}$

$y \to 0$，衰减误差散，说明取的近似太粗略了.

$\boxed{\text{initial 时期の绝热模式}}$ $C_{\vec{k}} = D_{\vec{k}} = 0$，$S \ll \delta^c$

$$\begin{cases} y^2 \cdot d^2\delta^c/dy^2 - 2\delta^c = 0 \\ y^2 \cdot d^2 S/dy^2 + y\, dS/dy = \dfrac{1}{2}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 y^2 \delta^c \end{cases} \Rightarrow \begin{cases} \delta^c_{\vec{k}} = A_{\vec{k}}\, y^2 + B_{\vec{k}}\, y^{-1}. \\ S_{\vec{k}} = \dfrac{1}{32}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 A_{\vec{k}}\, y^4 + \dfrac{1}{2}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 B_{\vec{k}}\, y \end{cases}$$

叫绝热模式但已不再绝热

$$\begin{cases} B_{\vec{k}} = 0，\text{增长模式}，S_{\vec{k}} = \dfrac{1}{64}\left(\dfrac{k}{\mathcal{H}}\right)^2 \delta^c_{\vec{k}} \\ A_{\vec{k}} = 0，\text{衰减模式}，S_{\vec{k}} = \dfrac{1}{4}\left(\dfrac{k}{\mathcal{H}}\right)^2 \delta^c_{\vec{k}} \end{cases}$$

$\boxed{\text{initial 时期的等曲率模式}}$ $A_{\vec{k}} = B_{\vec{k}} = 0$，$\delta^c \ll S$

isocurvature

$$\begin{cases} y^2 d^2\delta^c/dy^2 - 2\delta^c = \dfrac{2}{3}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 y^3 S \\ y^2 d^2 S/dy^2 + y\, dS/dy = 0 \end{cases} \Rightarrow \begin{cases} \delta^c = \dfrac{1}{6}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 C_{\vec{k}}\, y^3 + \dfrac{1}{6}\left(\dfrac{k}{\mathcal{H}_{eq}}\right)^2 D_{\vec{k}} \cdot \left[y^3 \ln y - \dfrac{5}{6} y^3\right] \\ S_{\vec{k}} = C_{\vec{k}} + D_{\vec{k}} \ln y \end{cases}$$

initial 时期, $w=1/3$.(仅有辐射).

$$\Phi=-\frac{3}{2}\left(\frac{\mathcal{H}}{k}\right)^2\delta^c,\quad v^\nu=\frac{1}{2}\left(\frac{k}{\mathcal{H}}\right)(\mathcal{H}\Phi'+\Phi).$$

→ 仅傅立叶空间中!

$$\delta^\nu=\delta^c-4\left(\frac{\mathcal{H}}{k}\right)v^\nu,\quad \mathcal{R}=\Phi-\frac{1}{2}\mathcal{H}^{-1}\Phi'-\frac{3}{2}\Phi.$$

绝热+增长(仅有 A_k). $\Phi=-\frac{3}{4}\left(\frac{\mathcal{H}_{eq}}{k}\right)^2A$

$$\mathcal{R}=\frac{9}{8}\left(\frac{\mathcal{H}_{eq}}{k}\right)^2A$$

不含时

$$v^\nu=\frac{1}{2}\Phi\cdot\frac{k}{\mathcal{H}}\propto\frac{k}{\mathcal{H}_{eq}}y$$

$$\delta^\nu\propto\Phi$$ 不含时.

等曲率+增长(仅有 C_k). $\Phi\propto y$

$S=C_k=\text{const}$

$\delta^c\propto y^3$

$$\mathcal{R}\propto y=\frac{1}{4}y\cdot C_k$$

$$v^\nu\propto y^2$$

$$\delta^\nu\propto\cdots y^3+\cdots y\propto y$$

$y\ll1$.

衰减模式衰减掉后,系统可由 $(S,\mathcal{R})$ 表示

$$\mathcal{R}_k=\frac{9}{8}\left(\frac{\mathcal{H}_{eq}}{k}\right)^2A_k+\frac{1}{4}yC_k=\mathcal{R}_k(\text{rad})+\frac{1}{4}yS_k(\text{rad})$$

$$S_k=\frac{1}{32}\left(\frac{k}{\mathcal{H}_{eq}}\right)^2A_ky^4+C_k=\frac{1}{9}\left(\frac{k}{\mathcal{H}_{eq}}\right)^4\mathcal{R}_k(\text{rad})y^4+S_k(\text{rad}).$$

大尺度($k\ll k_{eq}=\mathcal{H}_{eq}$)全程演化.

$$\begin{cases}\mathcal{H}^{-1}\mathcal{R}_k'=c_s^2\left[\frac{1}{1+w}\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi_k+(1-3c_s^2)S_k\right]\\ \mathcal{H}^{-2}S_k''+3c_s^2\mathcal{H}^{-1}S_k'=\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\left[\frac{1}{1+w}\delta_k^c-(1-3c_s^2)S_k\right]\end{cases}$$

超视界 $k\ll\mathcal{H}\Rightarrow S_k=\text{const}=S_k(\text{rad})$.

$\mathcal{R}_k=\text{const}=\mathcal{R}_k(\text{rad})$. ⇔ 绝热过程, $k\ll k_{eq}$.

超视界 $k\ll\mathcal{H}\Rightarrow\mathcal{H}^{-1}\mathcal{R}_k'=c_s^2(1-3c_s^2)S_k\Rightarrow\mathcal{R}_k'=\frac{4}{(4+3y)^2}S_k$

$$\mathcal{R}_k=\mathcal{R}_k(\text{rad})+\frac{y}{4+3y}S_k(\text{rad}).$$

「没看懂这一段在干嘛」

code "CAMB" 的初条件,我们先写出共动规范下的,再变换到同步规范下的.

用刚刚讨论的"δ^c, S"方程组,但系数和 w 和 c_s^2 用 y 写出来:

$$\begin{cases}\eta_3^2\delta_c''+\frac{2}{\sqrt{1+y}}\frac{5+3y}{4+3y}\eta_3\delta_c'-\frac{16+38y+30y^2+9y^3}{(1+y)(4+3y)}\cdot\frac{2}{y^2}\delta_c=-\frac{(k\eta_3)^2}{3(1+\frac{3}{4}y)}\left[\delta_c-\frac{y}{1+y}S\right]\\ \eta_3^2S''+\frac{\sqrt{1+y}}{1+y\cdot3/4}\cdot\frac{2}{y}\cdot\eta_3S'=\frac{(k\eta_3)^2}{4(1+y\cdot3/4)}\left[(1+y)\delta_c-yS\right]\end{cases}$$

$x=\eta/\eta_3\Rightarrow y=2x+x^2$, 系数由 y 再换到 x, 考虑早期, $\eta\ll\eta_3\Rightarrow x\ll1$.

构造一个到一定次数的多项式解,代进去求解系数. 发现有 2 个自由系数.

从解中可以识别出一个绝热模式和一个常曲率模式.

$$\mathcal{H}=\frac{\eta+\eta_3}{\eta_3\eta+\eta^2/2}=\frac{1}{\eta_3}\cdot\frac{1+x}{x+x^2/2}$$

$$w=\frac{1}{3}\cdot\frac{1}{1+2x+x^2}$$

$\omega\equiv 2/\eta_3$

常曲率模式: $\delta^c=\frac{1}{12}\omega k^2\eta^3$

$S=1-\frac{1}{36}\omega k^2\eta^3+\frac{1}{192}\omega^2k^2\eta^4$

$\Rightarrow \begin{cases}\Phi=-\omega\eta/8\\ v^N=-\omega k\eta^2/8\\ \delta^N=\omega\eta/2\\ \mathcal{R}=\omega\eta/4\end{cases}$

可计算、δ^c_m, δ^c_r, v_m-v_r, $v^N_m-v^N_r$ $\rightarrow =-S'/k$.

最后做规范变换 ⇒ 同步规范 (A^Z 和 $B^Z=0$) + $v^Z_m=0$.

绝热模式: $\delta^c=\frac{4}{9}k^2\eta^2-\frac{1}{4}\omega k^2\eta^3$

$S=\frac{1}{144}k^4\eta^4-\frac{1}{1200}\omega k^4\eta^5$

$\Rightarrow \Phi, v^N, \delta^N, \mathcal{R}\rightarrow C\rightarrow Z.$

最后验证了CAMB给的初值

宇宙学常数之影响

宇宙学常数Λ → 暗能量. DE $p_\Lambda=-\rho_\Lambda$, $w_\Lambda=-1$.

宇宙学常数与暗能量不可辨.

→ 不可被微扰: $\delta_\Lambda=\delta p_\Lambda=0$, $\Pi_\Lambda=0$.

$\boxed{\Lambda+\text{可微扰成分}}$ $\rho=\rho_p+\rho_\Lambda$, $p=p_p+p_\Lambda$. $\Omega\equiv\rho_p/\rho$

$w=\Omega(1+w_p)-1$

$\delta=\Omega\delta_p$. $\delta p=\Omega\,\delta p_p$. $v=v_p$.

$\frac{w}{1+w}$ 和 $\Pi=\frac{w_p}{1+w_p}\Pi_p$.

$c_s^2=p'/\rho'=p_p'/\rho_p'=c_p^2$

可写下四个场方程和两个连续性方程.

导出: $\Omega'=-3\mathcal{H}\Omega(1-\Omega)(1+w_p)$

$\Omega''=-\mathcal{H}^2\Omega(1-\Omega)(1+w_p)\left[15/2+27/2\,w-9c_s^2\right]$

← 暂时不知如何导出,应该用到了背景关系.

$\boxed{\Lambda+\text{理想流体}}$ → $\Pi_p=0$, $\Psi=\Phi$. 消失了一个场方程,两个连续性方程可化简.

~~$(\delta_p^c)'-3\mathcal{H}w_p\cdot\delta_p^c=(1+w_p)\nabla^2v_p^N$ & $\nabla^2\Gamma$~~

$[\delta_p^c-3(1+w_p)S]$

$\nabla^2\Phi=\frac{3}{2}\mathcal{H}^2\Omega\delta_p^c$ & $\Phi''+3(1+c_s^2)\mathcal{H}\Phi'+3(c_s^2-w)\mathcal{H}^2\Phi=\frac{3}{2}\mathcal{H}^2\Omega c_s^2$

$\nabla^2\Phi'=(\frac{3}{2}\mathcal{H}^2\Omega\delta_p^c)'=\frac{3}{2}\mathcal{H}^2\Omega\delta_p^{c\prime}+\frac{3}{2}\mathcal{H}^2\underline{\Omega'}\delta_p^c+\frac{3}{2}\mathcal{H}\times(2\underline{\mathcal{H}'})\Omega\delta_p^c$

背景关系: $\mathcal{H}'=-\frac{1}{2}(1+3w)\mathcal{H}^2$

$\nabla^2\Phi''=[(\nabla^2\Phi)']'=\cdots$ 「之前导出过一个类似的」

$\boxed{\Lambda+\text{物质}}$ $\delta p_p=w_p=c_p^2=0$, ⇒ $c_s^2=0$. $w=\Omega-1$

$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega\,\delta_m^c \qquad \delta_m^{N\prime} = \nabla^2 v_N + 3\Phi'$$

$$\Phi' + \mathcal{H}\Phi = \frac{3}{2}\mathcal{H}^2\Omega\, v^N \qquad \delta_m^{c\prime} = \nabla^2 v_N$$

$$\Phi'' + 3\mathcal{H}\Phi' + 3(1-\Omega)\mathcal{H}^2\Phi = 0 \qquad v_N' + \mathcal{H}v_N = \Phi.$$

$$\Rightarrow \mathcal{H}^{-2}\delta_m^{c\prime\prime} + \mathcal{H}^{-1}\delta_m^{c\prime} = \frac{3}{2}\Omega\,\delta_m^c$$

$$= H^{-2}\ddot{\delta}_m^c + 2H^{-1}\dot{\delta}_m^c$$

$$\Rightarrow \frac{d^2\delta_m^c}{d\ln a^2} + \left[2 + \frac{d\ln H}{d\ln a}\right]\frac{d\delta_m^c}{d\ln a} = \frac{3}{2}\Omega\,\delta_m^c$$

真实宇宙：「从此，密度微扰、速度微扰都默认牛顿规范」

$$\rho = \rho_b + \rho_c + \rho_\gamma + \rho_\nu + \underbrace{\rho_{DE}}_{\text{无微扰}}$$

$$\delta p_b = \delta p_c = 0$$

$p_b = p_c = 0$, $\quad p_\gamma/\rho_\gamma = p_\nu/\rho_\nu = 3 \Rightarrow \delta p_\gamma/\delta\rho_\gamma = \delta p_\nu/\delta\rho_\nu = 1/3$

$w_b = w_c = c_b^2 = c_c^2 = 0$, $\quad w_\gamma = w_\nu = c_\gamma^2 = c_\nu^2 = 1/3$

$p_{DE} = -\rho_{DE}$.

CDM不参加相互作用，中微子认为已解耦. b和γ存在散射.

(共形牛顿规范 + 标量微扰)

b, c: $\delta_{b/c}' = \nabla^2 v_{bc} + 3\Phi'$, $v_{b/c}' = -\mathcal{H}v_{b/c} + \Phi$ } 对于b，有额外碰撞项.

γ, ν: $\delta_{\gamma/\nu}' = \frac{4}{3}\nabla^2 v_{\gamma\nu} + 4\Phi'$, $v_{\gamma/\nu}' = \frac{1}{4}\delta_\nu + \frac{1}{6}\nabla^2\Pi_\nu + \Phi$ } 对于γ，有额外碰撞项.

忽略δ'中的碰撞项，即仅v_b'和v_γ'中有.

精确的碰撞项来自于CMB物理，玻尔兹曼分布.

取CDM的同步规范，$v_c^Z = 0$.

傅立叶空间中，有

$$\delta_c' = -\frac{1}{2}h' \qquad \delta_\gamma' = -\frac{4}{3}k v_\gamma - \frac{2}{3}h'$$

$$v_c' = 0 \qquad v_\gamma' = \frac{1}{4}k\delta_\gamma - \frac{1}{6}k\Pi_\gamma + \text{碰撞}$$

$$\delta_b' = -k v_b - \frac{1}{2}h' \qquad \delta_\nu' = -\frac{4}{3}k v_\nu - \frac{2}{3}h'$$

$$v_b' = -\mathcal{H}v_b + \text{碰撞} \qquad v_\nu' = \frac{1}{4}k\delta_\nu - \frac{1}{6}k\Pi_\nu$$

原初时期：电子正电子湮灭及BBN之后，辐射主导

认为衰减模式已消失，认为 $k \ll \mathcal{H}$

对于绝热扰动，$v_i = v$, $\frac{\delta_i}{1+w_i} = \frac{\delta}{1+w}$

相对熵扰动：$S_{ij} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j}$, $\quad S_{i\gamma} = \frac{\delta_i}{1+w_i} - \frac{3}{4}\delta_\gamma$, $i \neq \gamma$

四个场方程，以及共动曲率微扰 $\mathcal{R}$

原初时期/早期辐射主导时期：

1. 超视界 $k \ll \mathcal{H}$.

2. 辐射主导：$\rho_\gamma, \rho_\nu \gg \rho_b, \rho_c, \rho_{DE} \Rightarrow w = c_s^2 = \frac{1}{3}, \Rightarrow \mathcal{H} = \frac{1}{\eta}$

可忽略 v_γ' 中的碰撞项（但不可忽略 v_b' 中的）

3. 早于重合成(recombination)未去耦解耦，$v_b = v_\gamma$, $\Pi_\gamma = 0$.

4. $m_\nu = 0$

$$\Rightarrow \delta = \delta_r = f_\gamma \delta_\gamma + f_\nu \delta_\nu, \quad f_\gamma = \frac{\rho_\gamma}{\rho_\gamma + \rho_\nu}, \quad f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}$$

$\Rightarrow$ γ 和 ν 的连续性方程：

$$\mathcal{H}^{-1}\delta_{\gamma/\nu}' = -\frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\gamma + 4\mathcal{H}^{-1}\Psi'$$

$$\mathcal{H}^{-1}v_\gamma' = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\gamma + \left(\frac{k}{\mathcal{H}}\right)\Phi$$

$$\mathcal{H}^{-1}v_\nu' = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\nu + \left(\frac{k}{\mathcal{H}}\right)\Phi - \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)\Pi_\nu$$

场方程：$\mathcal{H}^{-1}\Psi' + \Phi = -\frac{1}{2}\delta$

$$\mathcal{H}^{-1}\Psi' + \Phi = 2\frac{\mathcal{H}}{k}v$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') - \Phi = \frac{1}{2}\delta \qquad \mathcal{H} = \frac{1}{\eta}$$

$$(k/\mathcal{H})^2(\Psi - \Phi) = f_\nu \Pi_\nu$$

共动曲率微扰 $\frac{2}{3}\mathcal{H}^{-1}\Psi' + 2\Psi = -\frac{4}{3}\mathcal{R} + \frac{2}{3}(\Psi - \Phi)$.

$\boxed{\text{绝热}}$ $S_{\nu\gamma} = S_{b\gamma} = S_{c\gamma} = 0$, $\delta_\nu = \delta_\gamma = \delta$, $v_\nu = v_\gamma = v$

考虑平凡解，$\Phi = const$, $\Psi = const$

$$\Rightarrow \delta_\nu = \delta_\gamma = \delta = -2\Phi$$

$$v_\nu = v_\gamma = v = \frac{1}{2}\left(\frac{k}{\mathcal{H}}\right)\Phi = \frac{1}{2}k\eta\Phi \qquad \mathcal{R} = -\Psi - \frac{1}{2}\Phi$$

$$\Pi_\nu = \frac{1}{f_\nu}\left(\frac{k}{\mathcal{H}}\right)^2(\Psi - \Phi) = \frac{\Psi - \Phi}{f_\nu}(k\eta)^2$$

根CMB物理，$\mathcal{H}^{-1}\Pi_\nu' = \frac{8}{5}\frac{k}{\mathcal{H}}v_\nu$

$$\Rightarrow 2\cdot\frac{\Psi - \Phi}{f_\nu}\cancel{k^2}\cancel{\eta} = \frac{8}{5}\cancel{k}\cdot\frac{1}{2}k\eta\Phi, \quad \Rightarrow \Psi = \left(1 + \frac{2}{5}f_\nu\right)\Phi$$

$\boxed{\text{物质}}$ $\mathcal{H}^{-1}\delta_{b/c}' + \left(\frac{k}{\mathcal{H}}\right)v_{b/c} - 3\mathcal{H}^{-1}\Psi' = 0$

$$\mathcal{H}^{-1}v_c' + v_c - \left(\frac{k}{\mathcal{H}}\right)\Phi = 0$$

$$\mathcal{H}^{-1}v_b' + v_b - \left(\frac{k}{\mathcal{H}}\right)\Phi = a \cdot n_e \cdot \sigma_T \cdot \frac{4\rho_\gamma}{3\rho_b}(v_\gamma - v_b).$$

来自于 CMB 物理

由上，有：$\mathcal{H} = \frac{1}{\eta}$, $\Phi = const$, $\Psi = const$; 可化简

完全绝热解，$\delta_c = \delta_b = \frac{3}{4}\delta = -\frac{3}{2}\Phi$, $v_c = v_b = v = \frac{1}{2}k\eta\Phi$

$\boxed{\text{b和c的熵微扰}}$ ~~$S_{\nu\gamma}$~~ $S_{i\gamma} = \frac{\delta_i}{1+w_i} - \frac{3}{4}\delta_\gamma$, $S_{i\gamma}' = -k(v_i - v_\gamma)$

考虑b和c仍有熵微扰：$\delta_b = \frac{3}{4}\delta + S_{b\gamma}$, $\delta_c = \frac{3}{4}\delta + S_{c\gamma}$.

b: $v_b : v_\gamma = v \Rightarrow S_{b\gamma} = \text{const}.\ \ \delta_b = \text{const}$

c: $v_{rel} \equiv v_c - v \Rightarrow \eta v_c' + v_c - k\eta\Phi = 0 \Rightarrow v_{rel} \propto \eta^{-1} \to 0$

$\Rightarrow S_{c\gamma} \to \text{const},\ \ \delta_c \to \text{const}$

原初时期三个模式：绝热 $\Phi \xrightarrow{\mathcal{R}}$，c带曲率 $S_{c\gamma}$，b带曲率 $S_{b\gamma}$

ADI　　　CDI　　　BDI.

速度

中微子速度带曲率模式 NDI.　　~~中微子~~「共有5种模式，最后一种 NVI 难于以激发」

中微子解耦前，ν，γ 分别与物质相互作用 $\Rightarrow v_\nu = v_\gamma$

…解了一堆很简单的微分方程，不知道在干什么.

紧束缚近似

$$\begin{cases} v_b' = -\mathcal{H} v_b + k\Phi + \dfrac{1}{R\,\eta_{roll}}(v_\gamma - v_b) \\ v_\gamma' = \dfrac{1}{4} k\delta_\gamma - \dfrac{1}{6} k\Pi_\gamma + k\Phi + \dfrac{1}{\eta_{roll}}(v_b - v_\gamma) \end{cases}$$

ν解耦前：$v_b \approx v_\gamma$

$$\Rightarrow v_b = v_\gamma - R\cdot\eta_{roll}(v_b' + \mathcal{H} v_b - k\Phi)$$

$$\approx v_\gamma - R\cdot\eta_{roll}(v_\gamma' + \mathcal{H} v_\gamma - k\Phi)$$

$$\Rightarrow (1+R)\,v_\gamma' \approx \frac{1}{4}k\delta_\gamma - \mathcal{H} R\, v_\gamma + (1+R)k\Phi.$$

紧束缚：$v_b = v_\gamma,\ \Pi_\gamma = 0$

下面一节做了这样一件事情，$y = 2x + x^2$，$\mathcal{H} = \dfrac{1}{x}\cdot\dfrac{1+x}{1+\frac{1}{2}x}$，$w = 1/3(1+y)$.

把所有东西按 x 展开

然后用 MMA 求解，得到了 ADI. matter DI，NDI

超视界微扰，及其与原初微扰的关系.

现在认为，微扰产生于中微子解耦.

之前一直讨论的是原始时期，我们想知道原初时期与 rad 微扰产生时期 * 的关系

超视界：$\mathcal{R} = \text{const} \Rightarrow \mathcal{R}(rad) = \mathcal{R}(*)$ ADI

$$\begin{bmatrix} \mathcal{R}_k(rad) \\ S_{c\gamma,k}(rad) \\ S_{b\gamma,k}(rad) \\ S_{\nu\gamma,k}(rad) \end{bmatrix} = \begin{bmatrix} 1 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \begin{bmatrix} \mathcal{R}_k(*) \\ S_{c\gamma,k}(*) \\ S_{b\gamma,k}(*) \\ S_{\nu\gamma,k}(*) \end{bmatrix}$$

统计均匀各向同性初始条件　　(使用高斯型微扰，与中心极限定理也有一定关系)

没有办法得到所有的初始值，也不可能得到现在的宏观观测. 只好从统计的角度理解

$\mathcal{R}_{-k} = \mathcal{R}_k^* \Rightarrow \langle \mathcal{R}_k \rangle = 0$

$\langle |\mathcal{R}_k|^2 \rangle \equiv P_\mathcal{R}(k)$. 功率谱　$P_\mathcal{R}(\vec{k}) = P_\mathcal{R}(k)$

25. 大尺度, $k \ll k_{eq} = \mathcal{H}_{eq}$. 超视界.

ignore DE

ignore m_ν.

背景方程: $y = (\eta/\eta_1)^2 + 2(\eta/\eta_1)$, $\mathcal{H} = \frac{\eta+\eta_1}{\eta_1\eta + \eta^2/2}$, $w = \frac{1}{3(1+y)}$, $c_s^2 = \frac{4}{3(4+3y)}$. $S = \frac{y}{4+3y} S_{mr}$

$\begin{cases} m \to b+c \\ \gamma \to \gamma+\nu \end{cases}$ 且 有各向异性压强.

$f_b \equiv \rho_b/\rho_m$, $f_c = \rho_c/\rho_m$

超视界: 放弃了辐射主导 → 研究从辐射主导到物质主导.

放弃了 γ 和 b 的紧束缚

$$a\frac{d\mathcal{R}}{da} = \mathcal{H}^{-1}\mathcal{R}' = y\frac{d\mathcal{R}}{dy} = 3c_s^2 S + \cdots \cancel{\left(\frac{k}{\mathcal{H}}\right)^2} \cdots = 3c_s^2 S = \frac{4}{4+3y}\cdot\frac{y}{4+3y} S_{mr}$$

$y \equiv a/a_{eq}$ ($k \ll \mathcal{H}$) const

绝热模式: $S_{mr} = 0 \Rightarrow \mathcal{R} = \text{const}$

$$\Rightarrow \frac{2}{3}\mathcal{H}'\Psi' + \frac{5+3w}{3}\Psi = -(1+w)\mathcal{R} + \frac{2}{3}(\Psi - \Phi).$$

物质主导 ($\Pi = 0$, $\Psi = \Phi$), $w = 0$, $\mathcal{H} = 2/\eta$.

$$\Rightarrow \frac{1}{3}\eta\Psi' + \frac{5}{3}\Psi = -\mathcal{R} \Rightarrow \Psi = -\frac{3}{5}\mathcal{R} + \cancel{C\eta^{-5}}$$ decay

$$\Rightarrow \Psi = \Phi = -\frac{3}{5}\mathcal{R}(\text{rad}).$$

$$\delta = \delta_b = \delta_c = \delta_m = -2\Phi \propto \mathcal{R}(\text{rad}).$$

常曲率模式: $\mathcal{R}(y) = \int_{y_0}^{y} d\mathcal{R} = \frac{4}{9} S_{mr}\int_0^y \frac{dy}{(y+4/3)^2} = \frac{y}{3y+4} S_{mr}$

$y \to +\infty$, $\mathcal{R}(+\infty) \sim \frac{1}{3} S_{mr}(\text{rad})$.

物质主导. $\Phi(+\infty) = \Psi(+\infty) = -\frac{3}{5}\mathcal{R}(+\infty)$.

通过视界: 已是物质主导 $\Rightarrow c_s^2 = 0$, $\Psi = \Phi$, $\mathcal{R} = \text{const}$

「后面没看懂」

26. Sachs-Wolfe 效应.

光子测地线: $\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0$.

$p^\mu = dx^\mu/du$ ···

$$\frac{dp^0}{d\eta} + (\mathcal{H} + \Psi')p^0 + 2\Psi_{,k}p^k + [\mathcal{H} - 2\mathcal{H}(\Phi+\Psi) - \Phi']\frac{\delta_{ij}p^ip^j}{p^0} = 0.$$

p^μ 在坐标系中, 而 $p^{\hat{\mu}} \equiv \sqrt{|g_{\mu\mu}|}\, p^\mu$ 在局域正交系中.

$q^0 \equiv a p^{\hat{0}}$, $q^i = a p^{\hat{i}}$ $\Rightarrow p^0 = a^{-2}(1-\Psi)q^0$, $p^i = a^{-2}(1+\Phi)q^i$

~~$= a^{-2}(1-\Phi)q^0$~~ ~~$= a^{-2}(1+\Psi)q^i$~~

$$\Rightarrow (1-\Phi)\frac{dq}{d\eta} = q\cdot\frac{d\Phi}{d\eta} - \eta\Phi' - 2q^k\Phi_{,k} + \eta\Phi'$$

$$\frac{1}{q}\frac{dq}{d\eta} = \frac{d\Phi}{d\eta} - \Phi' + \Phi' - 2\frac{\vec{q}\cdot\nabla\Phi}{q}$$

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \frac{dx^k}{d\eta}\frac{\partial}{\partial x^k}, \quad \frac{\vec{q}}{q} \approx \frac{p^k}{p^0} = \frac{dx^k}{d\eta} \Rightarrow \frac{\vec{q}}{q}\cdot\nabla\Phi = -2\left(\frac{d\Phi}{d\eta} - \frac{\partial\Phi}{\partial\eta}\right)$$

$$\frac{1}{q}\frac{dq}{d\eta} = -\frac{d\Phi}{d\eta} + \Phi' + \Phi' \approx \frac{1}{T}\frac{dT}{d\eta}$$

$$\left(\frac{\delta T}{T}\right)_{\text{from}} \Rightarrow \delta(E/E) = \frac{\delta q}{q} = \int\frac{dq}{q} = \Phi(x)\Big|_{x_{obs}}^{x_*} + \int_{\eta_*}^{\eta_{obs}}\left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Phi}{\partial\eta}\right)d\eta.$$

「看不懂…」

27. 大尺度的近似.

近似: 1. ignore ν和γ的各向异性, $\Pi=0$, $\Phi=\Phi$. $\delta_\gamma=\delta_\nu=\delta_r=\delta$

ν绝热, 没有NDI模式 $\qquad v_\gamma=v_\nu=v_r=v$

2. ignore b, 仅有 ~~CMD~~ CDM.

即又回到了只分 r 和 $m(c)$ 的情况, 且使用牛顿(N)规范

场方程:

$$\begin{cases} \mathcal{H}^{-1}\Phi' + \Phi + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi = \frac{1}{2}\delta. \Rightarrow k^2\Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 (\rho_m\delta_m + \rho_r\delta_r). & ① \\ \mathcal{H}^{-1}\Phi' + \Phi = \frac{3}{2}(1+w)\frac{\mathcal{H}}{k}v & ② \end{cases}$$

连续性方程:

$$\begin{cases} \delta_m' + k v_m = 3\Phi' & \delta_r' + \frac{4}{3}k v_r = 4\Phi' \\ v_m' + \mathcal{H}v_m = k\Phi & v_r' - \frac{1}{4}k\delta_r = k\Phi \end{cases}$$

未知量: $\delta_m, v_m, \delta_r, v_r, \Phi$. — ①-②得一个约束条件.

$\boxed{\text{辐射主导时期}}$ $k \gg k_{eq}$.

$$y = \frac{a}{a_{eq}} = 2\frac{\eta}{\eta_3} \ll 1, \quad \mathcal{H} = \frac{1}{\eta}, \quad w = c_s^2 = 1/3. \quad \delta p = \frac{1}{3}\cdot\delta\rho$$

辐射流体:

$$\begin{cases} \eta\delta' + \frac{4}{3}k\eta v = 4\eta\Phi' \\ \eta v' - \frac{1}{4}k\eta\delta = k\eta\Phi \end{cases}$$

场:

$$\begin{cases} \eta\Phi' + \Phi + \frac{1}{3}(k\eta)^2\Phi = \frac{1}{2}\delta \\ \eta\Phi' + \Phi = \frac{2}{k\eta}v \end{cases}$$

$$\Rightarrow \Phi'' + \frac{4}{\eta}\Phi' + \frac{1}{3}k^2\Phi = 0.$$

with $k \gg \mathcal{H} \sim 1/\eta \Rightarrow k\eta \gg 1. \Rightarrow \Phi(\eta) = -9\Phi(rad)\frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2}$ 振荡衰减

v和δ振荡, 振幅不变.

物质：

流体：$\begin{cases} \eta\,\delta_m' + k\eta\, v_m = 3\eta\,\Phi' \\ \eta\, v_m' + v_m = k\eta\,\Phi \end{cases} \Rightarrow \delta_m'' + \frac{1}{\eta}\delta_m' = \underbrace{3\Phi'' + \frac{3}{\eta}\Phi' - k^2\Phi}_{\text{直接代入RD得到的解}} \equiv F(k,\eta)$.

$$\Rightarrow \delta_m(\eta) = C_1 + C_2\ln(k\eta) - \int_0^\eta d\eta'\, F(k,\eta')\,\eta'\left[\ln(k\eta') - \ln(k\eta)\right].$$

$k\eta \ll 1$ $\int\cdots \to 0$ & $C_2 \equiv 0$, $\delta_m(0) \sim \text{const}$

$k\eta \gg 1$. $\delta_m \propto \ln(0.62\,k\eta)$.

$\boxed{\text{物质主导时期}}$

辐射主导时，δ_r 幅值不变，δ_c 对数增加。

可能存在这样一种情况：对于背景，是辐射主导；对于微扰项，物质主导。

考虑此情况，仅有物质微扰项（影响 Φ），但背景既有 m 又有 r.

$\longrightarrow \delta_m \gg \delta_r$.

$$\begin{cases} \delta_m' + k\, v_m = 3\Phi' \\ v_m' + \mathcal{H}\, v_m = k\,\Phi \\ k^2\Phi = -4\pi G a^2\, \underbrace{\rho_m\,\delta_m}_{\rho\cdot\frac{y}{y+1}\cdot\delta_m} = \frac{3}{2}\mathcal{H}^2\frac{y}{y+1}\delta_m. \end{cases}$$

$$\Rightarrow y^2\cdot\frac{d^2\delta_m}{dy^2} + \frac{3y+2}{2(y+1)}\frac{d\delta_m}{dy} = \frac{3}{2}\frac{y}{y+1}\delta_m$$

$y \gg y_{eq} \Rightarrow \delta_m^{(1)} \propto y$, $\delta_m^{(2)} \propto y^{-3/2}$

定入两个通解常数，与 $\delta_m \propto \ln(0.62\,k\eta)$ 对比，要求 $\delta_m(y_m)$ 与 $\frac{d\delta_m(y_m)}{dy}\Big|_{y=y_m}$ 相等。

最后舍去衰减模式 $\delta_m \propto y$.

28. b & γ 流体声学振荡

上一节忽略了 b，即 m 只有 c；现考虑 b.

$$\delta_b' = -k\,v_b + 3\Phi'$$

$$v_b' = -\mathcal{H}\,v_b + k\Phi + \frac{1}{R\,\eta_{coll}}(v_\gamma - v_b).$$

$$\delta_\gamma' = -\frac{4}{3}k\,v_\gamma + 4\Phi'$$

$$v_\gamma' = \frac{1}{4}k\,\delta_\gamma - \frac{1}{6}k\,\Pi_\gamma + k\Phi + \frac{1}{R\,\eta_{coll}}(v_b - v_\gamma).$$

$R = 3\rho_b/4\rho_\gamma$，$R' = \mathcal{H}R$.

$\eta_{coll} = 1/a\, n_e\sigma_T$

紧束缚极限：$\eta_{coll} \ll \mathcal{H}^{-1}$. $v_\gamma - v_b$ 和 Π 很小，舍去 Π.

$$\Rightarrow \Theta'' + \frac{R'}{1+R}\Theta' + C_s^2k^2\Theta = F_\Sigma(\eta) = \frac{1}{3}k^2\Phi + \Phi'' + \frac{R'}{1+R}\Phi',\quad \Theta = \delta_\gamma/4.$$

受迫阻尼振动

$\boxed{\text{视界内}}$ $k \gg \mathcal{H}$

先忽略强迫项，设 $\Theta = A(\eta)\, e^{iB(\eta)}$

解得 $\Theta_{\vec{k}} = A_{\vec{k}} (1+R)^{-1/4} e^{i(kr_s+\phi)}$

$v_r = -i\sqrt{3} A_{\vec{k}} (1+R)^{-3/4} e^{i(kr_s+\phi)}$

考虑强迫项．．．．．．．

$\Theta_{\vec{k}}(\eta) = \cdots + -(1+R)\Psi_{\vec{k}}$

光子解耦时 kr 振荡结束，光子均匀但各向异性

30. 张量微扰

仅有张量微扰：$A = D = B_i = 0$，$E_{ij} = E^T_{ij} \Rightarrow ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j + 2E^T_{ij}dx^i dx^j]$

$\delta\rho = \delta p = v_i = 0$，$\Pi_{ij} = \Pi^T_{ij}$ $\quad E_i^{T\,i} = 0$ & $E^T_{ij,i} = 0$

$\Rightarrow$ 场方程：$E^{T\,\prime\prime}_{ij} + 2\mathcal{H}E^{T\,\prime}_{ij} - \nabla^2 E^T_{ij} = 8\pi G a^2 \bar{p}\, \Pi^T_{ij}$

是一波动方程．

略去 Π^T_{ij}，考虑 $\hat{z} \parallel \vec{k}$，$E^T_{11} = E^T_{22} = h_+$，$E^T_{12} = E^T_{21} = h_\times$．$\delta g_{\mu\nu} = a^2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow h'' + 2\mathcal{H}h' + k^2 h = 0$

$\boxed{\text{物质主导宇宙}}$ $a(\eta) \propto \eta^2$，$\mathcal{H} = 2/\eta$．

$\Rightarrow h'' + \frac{4}{\eta}h' + k^2 h = 0$．

解，略去「$k\eta \to 0$ 时发散」项，得：$h = c \times \left[\frac{\sin k\eta}{(k\eta)^3} - \frac{\cos k\eta}{(k\eta)^2}\right]$

「早期，$\eta \to 0$，$k\eta \to 0$，$k \ll \eta^{-1} \sim \mathcal{H}$．$k^{-1} \gg \mathcal{H}^{-1}$ 超视界」

$k\eta \ll 1$，$h \to \text{const}$

$k\eta \gg 1$ $\quad h \propto \frac{\cos(k\eta)}{(k\eta)^2}$

$\boxed{\text{辐射主导宇宙}}$ $a(\eta) \propto \eta$，$\mathcal{H} = 1/\eta$．$\Rightarrow h'' + \frac{2}{\eta}h' + k^2 h = 0$．

解 $\Rightarrow h \propto \frac{\sin k\eta}{k\eta}$．

$k\eta \ll 1$，$h \to \text{const}$

$\boxed{\text{辐射+物质}}$ $y(\eta) = \frac{a}{a_{eq}} = \frac{\rho_m}{\rho_r} = 2\cdot\frac{\eta}{\eta_3} + \left(\frac{\eta}{\eta_3}\right)^2$，$\mathcal{H}(\eta) = \frac{\eta_3 + \eta}{\eta_3\cdot\eta + \eta^2/2}$

视界外：$k \ll \mathcal{H}$，$h(0) = \text{const}$

视界内：$k \gg \mathcal{H}$；$h(+\infty)$ 振荡，amplitude $\propto a^{-1}$

大尺度和小尺度下，h 振荡的振幅都有 $\propto a^{-1}$

真空能/暗能量/宇宙学常数 $w=-1$

* $k \ll \mathcal{H}$, $h(\eta) = const$

$k \gg \mathcal{H}$, $h(\eta)$ 振荡. amplitude $\propto a^{-1}$

宇宙加速膨胀, 共动视界收缩 ($\mathcal{H}^{-1}$), $h(\vec{k},\eta)$ 冻结且与历史有关.

Cosmotur bacory 2

4. 背景宇宙.

$$ds^2 = a^2(\eta)\left[-dt^2 + dx^2 + dy^2 + dz^2\right] \qquad homo \to \bar{\varphi}_I = \bar{\varphi}_I(\eta).$$

$\Rightarrow$ eom $\bar{\varphi}_I'' + 2\mathcal{H}\bar{\varphi}_I' = -a^2 \frac{\partial V}{\partial \varphi_I}$

eng tensor. $\bar{T}^0_0 = -\frac{1}{2} a^{-2} \bar{\varphi}_I' \bar{\varphi}_I' - V \equiv -\bar{\rho}$ $\qquad \bar{T}^0_i = \bar{T}^i_0 = 0$

$$\bar{T}^i_j = \delta^i_j\left[\frac{1}{2} a^2 \bar{\varphi}_I' \bar{\varphi}_I' + V\right] = \delta^i_j \bar{p}$$

场方程: $\mathcal{H}^2 = 8\pi G \bar{\rho} a^2/3$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}).$$

5. 微扰宇宙.

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix}$$

A, B_i, D, E_{ij} 是微扰

E_{ij} 无迹

$$g = a^2(-1-2A) \times a^2[(1-2D) + 2E_{11}] \times \cdots \approx -a^8(1+2A-6D).$$

$$\sqrt{-g} \approx a^4(1+A-3D).$$

标量场: $\varphi_I'' + 2\mathcal{H}\varphi_I' - \nabla^2\varphi_I - [A' + 3D' - B_{i,i}]\varphi_I' = -a^2(1+2A)\frac{\partial V}{\partial \varphi_I}$

$$\varphi_I = \bar{\varphi}_I(\eta) + \delta\varphi_I(\eta, \vec{x}), \quad V(\varphi_I) = V(\bar{\varphi}_I) + \frac{\partial V}{\partial \varphi_I}\delta\varphi_I$$

$$V_I \equiv \frac{\partial V}{\partial \varphi_I}, \quad V_I(\varphi_I) = V_I(\bar{\varphi}_I) + V_{IJ}\delta\varphi_J$$

标量场微扰项: $\delta\varphi_I'' + 2\mathcal{H}\delta\varphi_I' - \nabla^2\delta\varphi_I + a^2 V_{IJ}\delta\varphi_J = -2a^2 V_I A + \varphi_I'[A' + 3D' - B_{i,i}]$

* 只出现 $B_{i,i} = -B^S_{,ii}$, 故标量场微扰项的运动方程只有标量微扰, 故只讨论标量微扰

$$T^0_0 = -\frac{1}{2}a^{-2}(\bar{\varphi}_I' + \delta\varphi_I')(\bar{\varphi}_I' + \delta\varphi_I')^{(1-2A)} - V - V_I\,\delta\varphi_I$$

$$\Rightarrow \quad \delta T^0_0 = -a^{-2}(\varphi_I'\delta\varphi_I' - \varphi_I'\varphi_I' A) - V_I\delta\varphi_I \equiv -\delta\rho$$

同理. $\delta T^0_i = -a^{-2}\varphi_I'\partial_i(\delta\varphi_I) = -a^{2}\varphi_I'(\delta\varphi_I)_{,i}$

$\delta T^i_0 = a^{-2}[\varphi_I'(\delta\varphi_I)_{,i} - \varphi_I'\varphi_I' B_i]$

$\delta T^i_j / \delta^i_j = a^{-2}[\varphi_I'\delta\varphi_I' - \varphi_I'\varphi_I' A] - V_I\delta\varphi_I \equiv \delta p$. 没有各向异性压强, $\Phi = \Psi$.

引力场方程: $3\mathcal{H}(\mathcal{H}A + D') - \nabla^2(\underline{\psi} - \mathcal{H}B) = -4\pi G[\varphi_I'\delta\varphi_I' - \varphi_I'\varphi_I' A + a^2 V_I\delta\varphi_I]$

$$\psi' + \mathcal{H}A = 4\pi G\,\varphi_I'\delta\varphi_I$$

$$\begin{cases} (2\mathcal{H}' + \mathcal{H}')A + \mathcal{H}A' + \psi'' + 2\mathcal{H}\psi' = 4\pi G[\varphi_I'\delta\varphi_I' - \varphi_I'\varphi_I' A - a^2 V_I\delta\varphi_I] \\ \psi \equiv D + \frac{1}{3}\nabla^2 E, \qquad A - \psi + 2\mathcal{H}(B - E') + (B - E')' = 0 \end{cases}$$

规范变换. $\widetilde{\delta\varphi_I} = \delta\varphi_I - \bar{\varphi}_I'\xi^0$

6. 平坦空间度规

$\psi_Q = 0$. $\qquad 0 = \psi_Q = \psi + \mathcal{H}\xi^0 \Rightarrow \xi^0 = -\mathcal{H}^{-1}\psi$

记: $Q_I \equiv \delta\varphi_I{}^Q = \delta\varphi_I + \frac{\bar{\varphi}_I'}{\mathcal{H}}\psi$.

标量场方程: $Q_I'' + 2\mathcal{H}Q_I' - \nabla^2 Q_I + a^2 V_{IJ}Q_J = -2a^2 V_I \cancel{A_Q} + \varphi_I'\cancel{A_Q'} - \varphi_I'\nabla^2(E_Q' - B_Q)$

引力场方程 ②: $\mathcal{H}A_Q = 4\pi G\,\varphi_{QI}'\,Q_I = 0 \Rightarrow A_Q = 0$

$\Psi = \Phi = \psi - \mathcal{H}(B - E') = \psi_Q - \mathcal{H}(B_Q - E_Q') = \mathcal{H}(E_Q' - B_Q)$.

$-\varphi_I'\nabla^2(E_Q' - B_Q) = -\mathcal{H}^{-1}\varphi_I'\nabla^2\Phi = -\mathcal{H}^{-1}\varphi_I' \times (4\pi G a^2 \underline{\delta\rho^c})$.

comoving 共动规范: $v^c = B^c = 0 \Rightarrow \delta T^0_i = 0$, $B = 0$.

$\varphi_I'(\delta\varphi_I^c)_{,i} = 0 \Rightarrow \delta\varphi^c = 0$

$$\begin{cases} (\delta T^0_i)^c = \delta T^0_i + (\rho + p)\xi^0{}_{,i} = 0 \\ B^c = B + \xi' + \xi^0 = 0 \end{cases} \Rightarrow \begin{cases} \xi^0{}_{,i} = -\frac{\delta T^0_i}{\rho + p} \Rightarrow \xi^0 = \frac{\varphi_I'\delta\varphi_I}{\varphi_I'\varphi_I'} \\ \xi' = -B - \xi^0 \end{cases}$$

$$\Rightarrow \underline{\delta\rho^c} = \delta\rho - \rho'\xi^0 = a^{-2}[\varphi_I'(\delta\varphi_I' - \varphi_I'A) - (\varphi_I'' - \mathcal{H}\varphi_I')\delta\varphi_I]$$

$$= a^{-2}[\qquad\qquad\qquad]^Q$$

$$= a^{-2}[\varphi_I'(Q_I' - \varphi_I'\cancel{A_Q}) - (\varphi_I'' - \mathcal{H}\varphi_I')Q_I]$$

$$\Rightarrow \nabla^2\Phi = 4\pi G a^2\delta\rho^c = 4\pi G\,\varphi_I'\left[Q_I' + \frac{\mathcal{H}'}{\mathcal{H}}Q_I - \frac{\varphi_I''}{\varphi_I}Q_I\right]$$

$$\varphi_I'\nabla^2(E_Q' - B_Q) = \mathcal{H}^{-1}\varphi_I'\nabla^2\Phi = \frac{4\pi G}{\mathcal{H}}\varphi_I'\varphi_J'\left[Q_J' + \frac{\mathcal{H}'}{\mathcal{H}}Q_J - \frac{\varphi_J''}{\varphi_J}Q_J\right].$$

$$\Rightarrow Q_I'' + 2\mathcal{H}Q_I' - \nabla^2 Q_I + \left[a^2 V_{IJ} - \frac{8\pi G}{a^2}\left(\frac{a^2}{\mathcal{H}}\varphi_I'\varphi_J'\right)'\right]Q_J = 0.$$

7. 单场

$\boxed{\text{背景}}$. $\rho = \frac{1}{2} a^{-2}(\varphi')^2 + V$, $p = \frac{1}{2} a^{-2}(\varphi')^2 - V$, $w = p/\rho$.

引力场 $\begin{cases} \mathcal{H}^2 = \frac{8\pi G}{3} \rho a^2 = \frac{4\pi G}{3}[\varphi'^2 + 2a^2 V] \\ \mathcal{H}' = -\frac{4\pi G}{3}(\rho + 3p) a^2 = -\frac{8\pi G}{3}[\varphi'^2 - a^2 V] \end{cases}$

标量场: $\varphi'' + 2\mathcal{H}\varphi' + a^2 V_\varphi = 0$. $V_\varphi \equiv \frac{dV}{d\varphi}$

$\Rightarrow \rho' = a^{-2}\varphi'\varphi'' - a^{-3}(\varphi')^2 a' + V_\varphi \cdot \varphi' = -3\mathcal{H} a^{-2} \varphi'^2$

$p' = \cdots\cdots = -3\mathcal{H} a^{-2}\varphi'^2 - 2V_\varphi \varphi'$

$c_s^2 = p'/\rho' = 1 + \frac{2a^2 V_\varphi}{3\mathcal{H}\varphi'}$

$\boxed{\text{场微扰方程}}$

$$Q_k'' + 2\mathcal{H} Q_k' + k^2 Q_k + a^2 V_{\varphi\varphi} Q_k = \frac{8\pi G}{a^2}\left[\frac{a^2}{\mathcal{H}}\varphi'^2\right]' Q_k, \quad Q \equiv \delta\varphi^Q$$

$\boxed{\text{共动曲率微扰}}$

$\mathcal{R} \equiv -\psi^c = -\psi - \mathcal{H}\xi^0$, $\xi^0 = \frac{\varphi_I' \delta\varphi_I^0}{\varphi_I' \varphi_I'} = \frac{\delta\varphi}{\varphi'}$

$= -\psi^Q - \mathcal{H} \cdot \frac{\delta\varphi^Q}{\varphi^{Q\prime}} = -\frac{\mathcal{H}}{\varphi^{Q\prime}} Q$

$\mathcal{R}' = -\frac{\mathcal{H}}{\varphi^{Q\prime}}\left[Q' + \frac{\mathcal{H}''}{\mathcal{H}} Q - \frac{\varphi''}{\varphi'} Q\right]$

$= -\frac{\mathcal{H} a^2}{\varphi'^2}\delta\rho^c = -\frac{2}{3\mathcal{H}}\frac{\rho}{\rho + p}\nabla^2\Phi$ $\leftarrow \begin{cases} \nabla^2\Phi = 4\pi G a^2 \rho^c \\ \rho + p = a^{-2}\varphi'^2 \end{cases}$

$\Rightarrow \mathcal{H}^{-1}\mathcal{R}' = \frac{2}{3}\frac{\rho}{\rho + p}\left(\frac{k}{\mathcal{H}}\right)^2 \Phi$. $\rightarrow$ 超视界下, $\frac{k}{\mathcal{H}} \to 0$. $\mathcal{R} = const$

总熵微扰: $\mathcal{S} \equiv \mathcal{H}[\delta p/p' - \delta\rho/\rho']$

$$= \frac{a^2 V_\varphi}{6\pi G(\varphi')^2(3\mathcal{H}\varphi' + 2a^2 V_\varphi)}\nabla^2\Phi.$$

$$\mathcal{S}_k = -\frac{2}{9}\frac{\rho}{\rho + p}\cdot\frac{2a^2 V_\varphi}{3\mathcal{H}\varphi' + 2a^2 V_\varphi}\left(\frac{k}{\mathcal{H}}\right)^2 \Phi_k$$

$\boxed{\text{慢滚暴胀}}$ $\rightarrow$ 暂不讨论微扰, 使用宇宙时 t

场方程 $\begin{cases} \ddot{\varphi} + 3H\dot{\varphi} + V' = 0, \quad V' \equiv dV/d\varphi. \\ H^2 = \frac{8\pi G}{3}(\dot{\varphi}^2/2 + V) \equiv \frac{1}{3M_{Pl}^2}(\dot{\varphi}^2/2 + V). \end{cases}$

慢滚近似: $\dot{\varphi}^2 \ll V$, $|\ddot{\varphi}| \ll |3H\dot{\varphi}|$.

$\Rightarrow \begin{cases} 3H\dot{\varphi} + V' = 0 \\ H^2 = \frac{8\pi G}{3} V \end{cases} \Rightarrow \begin{cases} V' = -3H\dot{\varphi} \\ V = 3M^2 H^2 \end{cases}$ $\quad H^{-1}\dot{\varphi} = -M^2\frac{V'}{V}.$

$\Rightarrow \begin{cases} V''\dot{\varphi} = -3H\ddot{\varphi} - 3\dot{H}\dot{\varphi} \\ V'\dot{\varphi} = 3M^2 \cdot 2H\dot{H} \end{cases}$ $\Rightarrow \dot{H} = -\frac{V'^2}{6V}$, $\ddot{\varphi} = \frac{M^2}{3}\cdot\frac{V''V'}{V} - \frac{M^2}{6}\cdot\frac{V'^3}{V^2}$

慢滚参数：$\varepsilon_V = \frac{M^2}{2}\left(\frac{V'}{V}\right)^2$，$\eta_V = M^2 \frac{V''}{V}$，$\xi_V = M^4 \frac{V'''V'}{V^2}$

得到一系列方程：
$$\cancel{\quad} H^{-2}\dot{H} = -\varepsilon \qquad \rho = (1+\tfrac{1}{3}\varepsilon)V$$
$$(H^{-1}\dot{\varphi})^2 = 2M^2\varepsilon \qquad p = (-1+\tfrac{1}{3}\varepsilon)V$$
$$H^{-2}\ddot{\varphi} = H^{-1}\dot{\varphi}(\varepsilon-\eta) \qquad w = p/\rho = -1+\tfrac{2}{3}\varepsilon$$
$$H^{-1}\dot{\varepsilon} = -2\varepsilon\eta + 4\varepsilon^2 \qquad c_s^2 = \dot{p}/\dot{\rho} \approx -1-\tfrac{2}{3}\varepsilon+\tfrac{2}{3}\eta.$$
$$H^{-1}\dot{\eta} = 2\varepsilon\eta - \xi$$

$\boxed{\text{演化到视界退出}}$ → 考虑~~对~~微扰在视界内演化到微扰到视界外。

$$H^{-2}\ddot{Q}_{\vec{k}} + 3H^{-1}\dot{Q}_{\vec{k}} + \left(\frac{k}{aH}\right)^2 Q_{\vec{k}} = \left[\frac{8\pi G}{a^3H^2}\frac{d}{dt}\left(\frac{a^3}{H}\dot{\varphi}^2\right) - H^{-2}V''\right]Q_{\vec{k}}$$

$$\Rightarrow \frac{8\pi G}{\cancel{a^3}H^2}\left[\frac{3\cancel{a^3}\cancel{\dot{a}}}{\cancel{H}}\dot{\varphi}^2 + \frac{\cancel{a^3}}{H^2}\dot{H}\dot{\varphi}^2 + \frac{2\cancel{a^3}}{H}\dot{\varphi}\ddot{\varphi}\right] - H^{-2}V'' \qquad V'', V', V$$

$$= \frac{\cancel{M_{pl}^2}}{\cancel{8\pi G}}\left[3\left(-M^2\frac{V'}{V}\right)^2 + \left(-\frac{V'^2}{6V}\right)\cdot\left(-M^2\frac{V'}{V}\right)^2\cdot\frac{3M^2}{V} + \frac{3M^2}{V}\cdot 2\cdot\left(-M^2\frac{V'}{V}\right)\ddot{\varphi}\right] - \frac{3M}{V}V''$$

$$= 6\varepsilon - 3\eta + 6\varepsilon^2 - 4\varepsilon\eta. \approx 6\varepsilon - 3\eta.$$

$$\Rightarrow H^{-2}\ddot{Q}_{\vec{k}} + 3H^{-1}\dot{Q}_{\vec{k}} + \left(\frac{k}{aH}\right)^2 Q_{\vec{k}} = (6\varepsilon-3\eta)Q_{\vec{k}}$$

$$\mathcal{H}^{-2}Q''_{\vec{k}} + \cancel{3}2\mathcal{H}Q'_{\vec{k}} + k^2Q_{\vec{k}} = \mathcal{H}^2(6\varepsilon-3\eta_V)Q_{\vec{k}}$$

$$\xrightarrow{u\equiv aQ} u'' + \left(k^2 - \frac{a''}{a}\right)u = \mathcal{H}^2(6\varepsilon - 3\eta_V)u. \qquad = \mathcal{H}^2(2-\varepsilon). \qquad = 1-\varepsilon$$

背景. $\mathcal{H} = a'/a = aH = a$，$a''/a = \mathcal{H}^2(1+\mathcal{H}'/\mathcal{H}^2)$，$\mathcal{H}'/\mathcal{H}^2 = 1 + \dot{H}/H^2$

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1-\varepsilon \Rightarrow \mathcal{H} = \frac{da}{a\,d\eta} = \frac{-1}{(1-\varepsilon)\eta} \Rightarrow a \propto (-\eta)^\wedge\left(-\frac{1}{1-\varepsilon}\right), \eta<0$$

$$\mathcal{H}^2 \approx \frac{1+2\varepsilon}{\eta^2}$$

微扰. $u'' + [k^2 - \mathcal{H}^2(2+5\varepsilon-3\eta_V)]u = 0.$

$u'' + \left[k^2 - \frac{1}{\eta^2}(2+9\varepsilon-3\eta_V)\right]u = 0.$ 贝塞尔方程.

早期. $-k\eta \to \infty$，$Q_{\vec{k}} \propto \frac{1}{a\sqrt{k}} e^{-ik\eta}$.

晚期. $-k\eta \to 0$，$Q_{\vec{k}} \propto \frac{\sqrt{-\eta}}{a}(-k\eta)^{-\nu}$，$\nu = \frac{3}{2} + 3\varepsilon - \eta_V$

$$\Rightarrow Q_{\vec{k}} \propto (-\eta)^{\eta_V - 2\varepsilon} \sim \text{const}.$$

$\boxed{\text{标量微扰的产生}}$

暴胀时期视界尺度极小，次视界（sub horizon）演化中量子效应明显.

闵氏时空中的真空~~涨落~~涨落

$$\ddot{\varphi}_{\vec{k}} + (k^2+m^2)\varphi_{\vec{k}} = 0. \qquad w_{\vec{k}}(t) = V^{-1/2}\frac{1}{\sqrt{2E_{\vec{k}}}}e^{-iE_{\vec{k}}t}$$

正则量子化. $\hat{\varphi}(t,x) = \sum \hat{\varphi}_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}$. $\hat{\varphi}_{\vec{k}}(t) = w_{\vec{k}}(t)\hat{a}_{\vec{k}} + w^*_{\vec{k}}(t)\hat{a}^\dagger_{-\vec{k}}$

功率谱: $P_\varphi(\vec{k}) = V\frac{k^3}{2\pi^2}\langle|\varphi_{\vec{k}}|^2\rangle$ 　　方差 $\langle|\delta\varphi(\vec{x})|^2\rangle = \int_0^\infty \frac{dk}{k} P_\varphi(\vec{k})$.

真空功率谱: $P_\varphi(\vec{k}) = V\cdot\frac{k^3}{2\pi^2}\langle 0|\hat{\varphi}_{\vec{k}}^\dagger\hat{\varphi}_{\vec{k}}|0\rangle = V\frac{k^3}{2\pi^2}|w_k|^2$

<u>暴胀中的真空涨落</u> 背景 φ 演化缓慢,被认为是"真空",微扰项 Q 演化迅速,被认为是"粒子"

$Q_{\vec{k}}(\eta) = w_{\vec{k}}(\eta) = C_{\vec{k}}\, a^{-1}\sqrt{-\eta}\; H_\nu^{(1)}(-k\eta)$, $a(\eta) \propto (-\eta)^\wedge(-\frac{1}{1-\varepsilon_V})$.

次视界: $k \gg \mathcal{H} \sim \frac{1}{-\eta}$, $w_{\vec{k}}(\eta) \to C_{\vec{k}}\sqrt{\frac{2}{\pi}}\frac{1}{a\sqrt{k}}e^{-ik\eta} \to (aL)^{-3/2}\frac{1}{\sqrt{2E_{\vec{k}}}}e^{-iE_{\vec{k}}t}$

$k^{-1} \ll \mathcal{H}^{-1} \to$ 哈勃视界,认为背景不演化. $\eta = t/a$ 　$E_{\vec{k}} = k/a$

$\hat{Q}_{\vec{k}}(\eta) = w_k(\eta)\,\hat{a}_{\vec{k}} + w_k^*(\eta)\,\hat{a}^\dagger_{-\vec{k}}$ 　　$\to$ 与闵氏时空无异

$P_Q(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2$

超视界: $k \ll \mathcal{H}$, $(k^{-1} \gg \mathcal{H}^{-1})$, $w_k(\eta) \propto (-\eta)^{\eta_V - 2\varepsilon}$, 涨落"冻结"

大尺度,认为是经典情况,

$P_Q(k) \propto (-a\eta)^{-2}(-k\eta)^{2\eta_V - 6\varepsilon} \propto k^{2\eta_V - 6\varepsilon}$,

$\propto(-\eta)^{2\eta_V - 4\varepsilon}$.

<u>原初功率谱</u>

$\mathcal{R} = -\frac{H}{\dot{\varphi}}Q \Rightarrow P_{\mathcal{R}}(\eta, k) = \left(\frac{H}{\dot{\varphi}}\right)^2 P_Q(\eta, k)$.

$k \ll \mathcal{H}$: $P_{\mathcal{R}} \propto \left(\frac{H}{-a\eta\dot{\varphi}}\right)^2(-k\eta)^{3-2\nu} \propto k^{3-2\nu}\Big/\left(\frac{H}{a\dot{\varphi}}\right)^2(-\eta)^{1-2\nu}$
　　　　尺度 / 时间

$\left(a^{-1}\frac{1}{H^{-1}\dot{\varphi}}\right)^2(-\eta)^{1-2\nu} = \left(a^{-1}\frac{1}{-M^2\cdot V'/V}\right)^2(-\eta)^{1-2\nu}$ 　　$1-2\nu = -2-6\varepsilon+2\eta_V$

$= \frac{a^{-2}}{2\cdot M^2\cdot\frac{M^2}{2}(V'/V)^2}(-\eta)^{1-2\nu} = \frac{a^{-2}}{2M^2\varepsilon_V}(-\eta)^{1-2\nu}$ $\propto(-\eta)^0$

与时间无关. 　　　　$a^{-2} \propto (-\eta)^{2+2\varepsilon}$

故可找一个特殊时间计算谱

$$P_{\mathcal{R}}(k) = \left[2^{2\nu-3}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2(1-\varepsilon)^{2\nu-1}\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2\right]_{k=aH=\mathcal{H}}$$

<u>分阶段计算谱指数</u> 谱指数 $n_s = d\ln P_{\mathcal{R}}/d\ln k$

▽ 一阶. $\nu = 3/2 \Rightarrow P_{\mathcal{R}}(k) = \left[\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2\right]_{k=aH=\mathcal{H}} = \frac{1}{24\pi^2}\cdot\frac{1}{M^2}\cdot\frac{V}{\varepsilon}\Big|_{k=\mathcal{H}}$

$\frac{d\ln k}{dt} = \frac{d\ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = H + \frac{-\epsilon H^2}{H} = (1-\varepsilon)H$. $\approx \frac{1}{H}\frac{d}{dt}$

$\frac{d}{d\ln k} = \frac{dt}{d\ln k}\frac{d}{dt} = \frac{dt}{d\ln k}\cdot\frac{d\varphi}{dt}\cdot\frac{d}{d\varphi} \approx -\frac{M^2}{1-\varepsilon}\frac{V'}{V}\frac{d}{d\varphi} \approx -M^2\frac{V'}{V}\frac{d}{d\varphi}$.

$\Rightarrow n_s = \frac{d\ln P_{\mathcal{R}}}{d\ln k} = \frac{d}{d\ln k}\ln\left(\frac{V}{\varepsilon}\right) \Rightarrow n_s = -6\varepsilon + 2\eta$.

▽ 二阶. 不取 $\nu = 3/2$, 直接算 $\frac{d\ln P_{\mathcal{R}}}{d\ln k}$.

回到暴场方程：$\ddot{\varphi}+3H\dot{\varphi}+V'=0$，$H^2=\frac{1}{3M}(\dot{\varphi}^2/2+V)$.

考虑到慢滚近似下有关系，$\ddot{\varphi}\approx H\dot{\varphi}(\varepsilon-\eta)$，$\dot{\varphi}^2/2=M^2H^2\varepsilon$

代回：$\begin{cases} H\dot{\varphi}(\varepsilon-\eta)+3H\dot{\varphi}+V'=0 \\ 3MH^2=\cancel{\frac{1}{3}}\cdot MH^2\varepsilon+V \end{cases} \Rightarrow \begin{cases} H\dot{\varphi}=-\dfrac{V'}{3+\varepsilon-\eta} \\ H^2=\dfrac{V}{(3-\varepsilon)M^2} \end{cases}$

$\Rightarrow \varepsilon_H=-\dot{H}/H^2\approx \varepsilon-\frac{4}{3}\varepsilon^2+\frac{2}{3}\varepsilon\eta$.

$(H^{-1}\dot{\varphi})^2\approx 2M^2\varepsilon_H$

$H^{-1}\dot{\varepsilon}\approx 4\varepsilon^2-2\varepsilon\eta-\frac{8}{3}\varepsilon^3+\frac{8}{3}\varepsilon^3\eta-\frac{2}{3}\varepsilon\eta^2$

$H^{-1}\dot{\varepsilon}_H\approx \varepsilon\times[4\varepsilon-2\eta-\frac{40}{3}\varepsilon^2+\frac{36}{3}\varepsilon\eta-2\eta^2-\frac{2}{3}\xi]$

可算 $n_s=\cdots\cdots\cdots$ (很长).

n_s是一重要观测参量，$n_s=-0.035\pm0.004$ Planck 2018.

8. 曲率微扰. 之前讨论的都是单场，但从全文来看，一般仍研究N个标量场的情况

$\mathcal{R}=-\psi^c=-\psi-\mathcal{H}\xi^o$，$\xi^o=\frac{\varphi_I'\delta\varphi_I}{\varphi_I'\varphi_I'}$

$=-\psi^Q-\mathcal{H}\cdot\frac{\varphi_I^{Q\prime}\delta\varphi_I^Q}{\varphi_I^{Q\prime}\varphi_I^{Q\prime}}$

$=0-\mathcal{H}\cdot\frac{1}{(\rho+p)a^2}\varphi_I'Q_I=4\pi G\cdot\frac{\mathcal{H}}{\mathcal{H}'-\mathcal{H}^2}\varphi_I'Q_I$

9. 慢滚暴胀 (N个标量场)

场 $\begin{cases}\ddot{\varphi}_I+3H\dot{\varphi}_I=-V_I \\ H^2=\frac{8\pi G}{3}\bar{\rho}=\frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}_I\dot{\varphi}_I+V\right]\end{cases}$

慢滚近似：$|\ddot{\varphi}_I|\ll|3H\dot{\varphi}_I|$，$\dot{\varphi}_I\dot{\varphi}_I\ll V$

慢滚方程：$\begin{cases}3H\dot{\varphi}_I=-V_I \\ 3M^2H^2=V\end{cases} \Rightarrow H\dot{\varphi}_I=\mathcal{H}\varphi_I'=-M^2\frac{V_I}{V}$

背景场沿势的梯度下降.

求导：$\begin{cases}3\dot{H}\dot{\varphi}_I+3H\ddot{\varphi}_I=-V_{IJ}\dot{\varphi}_J \\ \cancel{3}\,6M^2H\dot{H}=V_I\dot{\varphi}_I\end{cases} \Rightarrow \begin{cases}\dot{H}=-\dfrac{V_IV_I}{6V} \\ \ddot{\varphi}_I=\dfrac{M^2}{3}\cdot\dfrac{\cancel{M}V_{IJ}V_J}{V}-\dfrac{M^2}{6}\dfrac{V_IV_JV_J}{V^2}\end{cases}$

$\boxed{\text{慢滚参数}}$

$\varepsilon_{IJ}=\frac{M^2}{2}\frac{V_IV_J}{V^2}$，$\eta_{IJ}=M^2\frac{V_{IJ}}{V}$，$\varepsilon=\varepsilon_{II}$

$\Rightarrow H^{-2}\dot{H}=-\varepsilon$，$(H^{-1}\dot{\varphi}_I)^2=2M^2\varepsilon_{II}$ (不求和)，$H^{-2}\ddot{\varphi}_I=\varepsilon H^{-1}\dot{\varphi}^{-1}-\eta_{IJ}H^{-1}\dot{\varphi}_J$

$H^{-1}\dot{\varepsilon}_{IJ}=4\varepsilon\varepsilon_{IJ}-\varepsilon_{IK}\eta_{JK}-\eta_{IK}\varepsilon_{JK}$，$H^{-1}\dot{\varepsilon}=4\varepsilon^2-\varepsilon_{IK}\eta_{IK}$.

$\boxed{\text{背景演化}}$

「认为 ε 是常数」

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1+\frac{\dot H}{H^2} = 1-\varepsilon \Rightarrow \mathcal{H} = \frac{-1}{(1-\varepsilon)\eta} = \frac{a'}{a} \Rightarrow a \propto (-\eta)^{\wedge}\left(\frac{-1}{1-\varepsilon}\right).$$

$$a''/a = \mathcal{H}^2(2-\varepsilon).$$

$\boxed{\text{微扰项的演化}}$

$$H^{-2}\ddot Q_I + 3H^{-1}\dot Q_I + \left(\frac{k}{aH}\right)^2 Q_I = H^{-2}\left[\frac{8\pi G}{a^3}\frac{d}{dt}\left(\frac{a^3}{H}\dot\varphi_I\dot\varphi_J\right) - V_{IJ}\right]Q_J$$

$$\text{RHS} = [6\varepsilon_{IJ} - 3\eta_{IJ} + 6\varepsilon\varepsilon_{IJ} - 2\varepsilon_{IK}\eta_{JK} - 2\eta_{IK}\varepsilon_{JK}]Q_J \approx [6\varepsilon_{IJ} - 3\eta_{IJ}]Q_J.$$

方程的左边与单场的形式相同，右边出现了 $M_{IJ}\cdot Q_J$ 的形式，可以通过重新线性组合出新的 $\bar Q_I$，使得 $\bar Q_J$ 是 M_{IJ} 的本征矢。——我们假设已经这样做，则可得解，包含汉克尔函数什么的。 → 依赖于 M_{IJ} 实对称。

$$\mathcal{R} = 4\pi G \cdot \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}\varphi_I' Q_I \approx \frac{1}{2M^2\varepsilon}\mathcal{H}^{-1}\varphi_I' Q_I \approx \frac{1}{2\varepsilon V} V_I\cdot Q_I$$

再做一次线性组合（转动），使得 Q 的第一分量为 $Q_\sigma \equiv \frac{V_I\cdot Q_I}{\Sigma V_I}$，对 φ_I' 也做此转动

10. 双场

$$\vec\varphi = (\varphi_1, \varphi_2), \quad \tan\theta \equiv \dot\varphi_2/\dot\varphi_1$$

$\boxed{\text{绝热分量和熵分量}}$

绝热场分量：σ，沿从某个绝热点出发的迹线

熵　场分量：s，与该迹线正交　　背景中：$s=\dot s=\ddot s=0$.

对于某特定的背景场，σ 和 s 构成了一个「在场迹线领域」上有效的坐标系。

考虑这样构造出 (σ, s)：

$$\begin{pmatrix}\dot\sigma\\ \dot s\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\dot\varphi_1\\ \dot\varphi_2\end{pmatrix} = S^T\begin{pmatrix}\dot\varphi_1\\ \dot\varphi_2\end{pmatrix}, \quad \begin{pmatrix}\dot\varphi_1\\ \dot\varphi_2\end{pmatrix} = S\begin{pmatrix}\dot\sigma\\ \dot s\end{pmatrix}$$

$$\dot s = 0 \Rightarrow \dot\sigma^2 = \dot\varphi_1^2 + \dot\varphi_2^2, \quad \dot\varphi_1\cdot\sin\theta = \dot\varphi_2\cdot\cos\theta$$

$$\dot\varphi_1 = \dot\sigma\cos\theta, \quad \dot\varphi_2 = \dot\sigma\sin\theta$$

$$\Rightarrow \ddot\varphi_1 = \cdots\cdots, \quad \ddot\varphi_2 = \cdots$$

一阶导：$\begin{pmatrix}\overset{\partial V/\partial\sigma}{V_\sigma}\\ V_s\end{pmatrix} = S^T\begin{pmatrix}\overset{\partial V/\partial\varphi_1}{V_1}\\ V_2\end{pmatrix}$

此空间中的函数是势场 $V(\varphi_1, \varphi_2)$，$V(\sigma, s)$.

二阶导：$V_{I'J'} = S^K{}_{I'}S^L{}_{J'}V_{KL}$.

三阶导：$V_{I'J'K'} = S^L{}_{I'}S^M{}_{J'}S^N{}_{K'}V_{LMN}$.

二阶导：$\begin{pmatrix}V_{\sigma\sigma} & V_{\sigma s}\\ V_{\sigma s} & V_{ss}\end{pmatrix} = S^T\begin{pmatrix}V_{11} & V_{12}\\ V_{12} & V_{22}\end{pmatrix}S.$

$I'J'$ 可取 σ, s；KL 可取 $1,2\ (\varphi_1, \varphi_2)$.

$\boxed{\text{背景}}$

$$\begin{cases}\ddot\varphi_1 + 3H\dot\varphi_1 + V_1 = 0\\ \ddot\varphi_2 + 3H\dot\varphi_2 + V_2 = 0\end{cases} \Rightarrow \begin{cases}\ddot\sigma\cos\theta - \dot\sigma\dot\theta\sin\theta + 3H\dot\sigma\cos\theta + \overset{V_\sigma\cos\theta - V_s\sin\theta}{V_1} = 0\\ \ddot\sigma\sin\theta + \dot\sigma\dot\theta\cos\theta + 3H\dot\sigma\sin\theta + \underset{V_\sigma\sin\theta + V_s\cos\theta}{V_2} = 0\end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\sigma} + 3H\dot{\sigma} + V_\sigma = 0 \\ V_s + \dot{\sigma}\dot{\theta} = 0 \end{cases} \Rightarrow \cancel{-V_1\sin\theta + V_2\cos\theta + \dot{\sigma}\dot{\theta} = 0}$$

$$\cancel{+\dot{\sigma}\ddot{\theta} + \ddot{\sigma}\dot{\theta} = 0}$$

$$\cancel{\Rightarrow -V_{11}\dot{\varphi}_1\sin\theta - V_{12}\dot{\varphi}_2\sin\theta + V_{21}\dot{\varphi}_1\cos\theta + V_{22}\dot{\varphi}_2\cos\theta}$$

$$\cancel{-V_1\cos\theta\cdot\dot{\theta} - V_2\sin\theta\cdot\dot{\theta}}$$

$$V_s = -V_1\sin\theta + V_2\cos\theta \Rightarrow \dot{V}_s = -V_{11}\dot{\varphi}_1\sin\theta - V_{12}\dot{\varphi}_2\sin\theta - (V_1\cos\theta\,\dot{\theta} \to -\dot{\theta}V_\sigma$$

$$+V_{21}\dot{\varphi}_1\cos\theta + V_{22}\dot{\varphi}_2\cos\theta - V_2\sin\theta)\dot{\theta}$$

$$\Rightarrow \dot{V}_s = V_{\sigma s}\dot{\sigma} - \dot{\theta}V_\sigma$$

$$\Rightarrow \ddot{\theta} - 3H\dot{\theta} + V_{\sigma s} - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\theta} = 0$$

$\boxed{\text{微扰项}}$ $\begin{pmatrix}\delta\sigma\\ \delta s\end{pmatrix} = S^T\begin{pmatrix}\delta\varphi_1\\ \delta\varphi_2\end{pmatrix}$, $\begin{pmatrix}\delta\varphi_1\\ \delta\varphi_2\end{pmatrix} = S\begin{pmatrix}\delta\sigma\\ \delta s\end{pmatrix}$. $\delta V = V_1\delta\varphi_1 + V_2\delta\varphi_2 = V_\sigma\delta\sigma + V_s\delta s.$

$$\Rightarrow \delta\sigma = \cos\theta\,\delta\varphi_1 + \sin\theta\,\delta\varphi_2$$

$$\delta\dot{\sigma} = \cancel{\cos} - \sin\theta\,\dot{\theta}\,\delta\varphi_1 + \cos\theta\,\delta\dot{\varphi}_1 + \cos\theta\,\dot{\theta}\,\delta\varphi_2 + \sin\theta\,\delta\dot{\varphi}_2$$

$$\delta\ddot{\sigma} = \cdots\cdots$$

$$\Rightarrow \delta\varphi = \cos\theta\,\delta\sigma - \sin\theta\,\delta s$$

$$\delta\dot{\varphi} = -\sin\theta\,\dot{\theta}\,\delta\sigma + \cos\theta\,\delta\dot{\sigma} - \cos\theta\,\dot{\theta}\,\delta s - \sin\theta\,\delta\dot{s}$$

$$\delta\ddot{\varphi} = \cdots\cdots$$

$$\Rightarrow \dot{\varphi}_1\delta\dot{\varphi}_1 + \dot{\varphi}_2\delta\dot{\varphi}_2 = \dot{\sigma}(\delta\dot{\sigma} - \dot{\theta}\,\delta s) = \dot{\sigma}\,\delta\dot{\sigma} + V_s\,\delta s.$$

运动方程:

$$\Rightarrow \ddot{\vec{\delta\varphi}} + 3H\dot{\vec{\delta\varphi}} - \frac{1}{a^2}\nabla_x^2\vec{\delta\varphi} + \nabla_\varphi\nabla_\varphi V\cdot\vec{\delta\varphi} = -2\nabla_\varphi VA + \dot{\vec{\varphi}}\left[\dot{A} + 3\dot{D} + \frac{1}{a}\nabla_x^2 B\right]$$

$$\Rightarrow \begin{cases} \delta\ddot{\sigma} + 3H\delta\dot{\sigma} + \left(-\frac{1}{a^2}\nabla^2 + V_{\sigma\sigma} - \dot{\theta}^2\right)\delta\sigma = -2V_\sigma A + \dot{\sigma}(\dot{A} + 3\dot{D} + \frac{1}{a}\nabla^2 B) + 2\frac{d}{dt}(\dot{\theta}\,\delta s) - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\theta}\delta s \\ \delta\ddot{s} + 3H\delta\dot{s} + \left(-\frac{1}{a^2}\nabla^2 + V_{ss} - \dot{\theta}^2\right)\delta s = -2\frac{\dot{\theta}}{\dot{\sigma}}\left[\dot{\sigma}(\delta\dot{\sigma} - \dot{\sigma}A) - \ddot{\sigma}\,\delta\sigma\right]. \end{cases}$$

若迹线是直的($\dot{\theta}=0$),则 $\delta\sigma$ 与 δs 解耦,$\delta\sigma$ 的方程就是单场的方程,与 $\delta\rho, \delta p$ 挂钩,故与度规微扰挂钩,δs 不与它们相关.

若迹线是弯的,则 $\delta\sigma$ 与 δs 存在交互. RHS of $\delta s \propto$ 单场多熵微扰

$\boxed{\text{在平坦空间规范下(的场方程)}}$ $\to \psi_Q = 0$

流体
$$\begin{cases} \delta\rho = \delta\rho(\varphi_1, \varphi_2) = \delta\rho(\sigma, s) = \dot{\sigma}(\delta\dot{\sigma} - \dot{\theta}\,\delta s) - \dot{\sigma}^2 A + \delta V \\ \delta p = \cdots\cdots = \dot{\sigma}(\delta\dot{\sigma} - \dot{\theta}\,\delta s) - \dot{\sigma}^2 A - \delta V \\ \delta q = \delta T^0_i = \cdots\cdots = \dot{\sigma}\,\delta\sigma \end{cases}$$

引力场：$\nabla^2\Phi = 4\pi G a^2 \delta\rho^c$，$\Phi = \Phi$.

$$\delta\rho^c = \delta\rho - \dot\rho\,\xi^0,\quad \xi^0 = \frac{\dot\varphi_1\delta\varphi_1 + \dot\varphi_2\delta\varphi_2}{\dot\varphi_1^2 + \dot\varphi_2^2} = \frac{\dot\sigma\,\delta\sigma}{\dot\sigma^2}$$

$$\dot\rho = \frac{d}{dt}\left(\dot\varphi_1^2/2 + \dot\varphi_2^2/2 + V\right) = \cdots = \dot\sigma\ddot\sigma + V_\sigma\dot\sigma$$

$$\Rightarrow \delta\rho^c = \dot\sigma\,\delta\dot\sigma - \ddot\sigma\,\delta\sigma - \dot\sigma^2 A + 2V_s\,\delta s.$$

场微扰的规范变换：
$$\begin{cases} \widetilde{\delta\sigma} = \delta\sigma - \dot\sigma\,\xi^0 \\ \widetilde{\delta s} = \delta s - \dot s\,\xi^0 = \delta s. \ \text{规范不变.} \end{cases}$$

在空间平坦规范下：

$$\delta\ddot\sigma + 3H\,\delta\dot\sigma + \left[-\frac{1}{a^2}\nabla^2 + V_{\sigma\sigma} - \dot\theta^2 - \frac{8\pi G}{a^3}\frac{d}{dt}\left(\frac{a^3}{H}\dot\sigma^2\right)\right]\delta\sigma = 2\frac{d}{dt}(\dot\theta\,\delta s) - 2\left(\frac{V_\sigma}{\dot\sigma} + \frac{\dot H}{H}\right)\dot\theta\,\delta s$$

$$\delta\ddot s + 3H\,\delta\dot s + \left[-\frac{1}{a^2}\nabla^2 + V_{ss} + 3\dot\theta^2\right]\delta s = -\frac{\dot\theta}{\dot\sigma}\cdot\frac{1}{2\pi G}\cdot\frac{1}{a^2}\nabla^2\Phi$$

$\boxed{\text{曲率微扰和熵微扰}}$

$$\mathcal{R} = -H\frac{\dot\varphi_I^{a'}\cdot\delta\varphi_I^{a}}{\dot\varphi_I^{a'}\cdot\dot\varphi_I^{a'}} = -H\frac{\dot\varphi_1\delta\varphi_1 + \dot\varphi_2\delta\varphi_2}{\dot\varphi_1^2 + \dot\varphi_2^2} = -H\frac{\delta\sigma}{\dot\sigma}$$

$$\dot{\mathcal{R}} = -\dot H\frac{\delta\sigma}{\dot\sigma} - H\frac{\delta\dot\sigma}{\dot\sigma} + H\frac{\delta\sigma}{\dot\sigma^2}\ddot\sigma$$

$$= \frac{H}{\dot H}\frac{1}{a^2}\nabla^2\Phi - 2H\frac{\dot\theta}{\dot\sigma}\delta s.$$

类比熵微扰：$S = H\frac{\delta s}{\dot\sigma} \Rightarrow \dot{\mathcal{R}} \xlongequal{k \ll H} -2\dot\theta S$

$\boxed{\text{慢滚近似}}$ 慢滚：
$$\begin{cases} H^2 = \frac{V}{3M^2} \\ H^{-1}\dot\varphi_I = -M^2\frac{V_I}{V} \end{cases} \Rightarrow \begin{cases} H^{-1}\dot\sigma = -M^2\frac{V_\sigma}{V} \\ H^{-1}\overset{=0}{\dot s} = -M^2\frac{V_s}{V} \Rightarrow V_s = 0. \end{cases}$$

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} = \frac{M^2}{2}\cdot\frac{(\nabla V)^2}{V^2} = \frac{M^2}{2}\cdot\frac{V_\sigma^2}{V^2}.$$

$$\varepsilon_{11} = \varepsilon\cos^2\theta,\quad \varepsilon_{22} = \varepsilon\sin^2\theta,\quad \varepsilon_{12} = \varepsilon\sin\theta\cdot\cos\theta.$$

$$\eta_{\sigma\sigma} = M^2\frac{V_{\sigma\sigma}}{V},\quad \eta_{ss} = M^2\frac{V_{ss}}{V},\quad \eta_{\sigma s} = M^2\frac{V_{\sigma s}}{V}.$$

由 $\begin{pmatrix} V_{\sigma\sigma} & V_{\sigma s} \\ V_{\sigma s} & V_{ss}\end{pmatrix} = S^T\begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22}\end{pmatrix}S$ 得到：

$$\eta_{\sigma\sigma} = \cos^2\theta\,\eta_{11} + 2\cos\theta\sin\theta\,\eta_{12} + \sin^2\theta\,\eta_{22}$$
$$\eta_{ss} = \sin^2\theta\,\eta_{11} - 2\cos\theta\sin\theta\,\eta_{12} + \cos^2\theta\,\eta_{22}$$
$$\eta_{\sigma s} = -\sin\theta\cos\theta\,\eta_{11} + (\cos^2\theta - \sin^2\theta)\,\eta_{12} + \cos\theta\sin\theta\,\eta_{22}.$$

$$\eta_{11} + \eta_{22} = \eta_{\sigma\sigma} + \eta_{ss}.$$

$$\xi_{I'J'K'} \equiv M^4\cdot\frac{V_\sigma V_{I'J'K'}}{V^2}$$

$$\begin{cases} H^{-1}\dot\varepsilon = 4\varepsilon^2 - \varepsilon_{IK}\eta_{IK} = 4\varepsilon^2 - 2\varepsilon\,\eta_{\sigma\sigma} \\ H^{-2}\ddot\varphi_I = \varepsilon H^{-1}\dot\varphi^{-1} - \eta_{IJ}H^{-1}\dot\varphi_J \Rightarrow H^{-2}\ddot\sigma = H^{-1}\dot\sigma(\varepsilon - \eta_{\sigma\sigma}). \end{cases}$$

$$H^{-1}\dot\theta = -\eta_{\sigma s} \propto -V_{\sigma s}$$

$$\begin{cases} H^{-1}\eta_{\sigma\sigma}' = 2\varepsilon\eta_{\sigma\sigma} - 2\eta_{\sigma s}^2 - \xi_{\sigma\sigma\sigma} \\ H^{-1}\eta_{\sigma s}' = 2\varepsilon\eta_{\sigma s} + \eta_{\sigma s}(\eta_{\sigma\sigma} - \eta_{ss}) - \xi_{\sigma\sigma s} \\ H^{-1}\eta_{ss}' = 2\varepsilon\eta_{ss} + 2\eta_{\sigma s}^2 - \xi_{\sigma ss}. \end{cases}$$

$\boxed{\text{演化到视界退出}}$ 先做 $\underline{\eta}$ 最后面的那个转动

$$U^T M U = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}^T \begin{bmatrix} \varepsilon + 2\varepsilon_{11} - \eta_{11} & 2\varepsilon_{12} - \eta_{12} \\ 2\varepsilon_{12} - \eta_{12} & \varepsilon + 2\varepsilon_{22} - \eta_{22} \end{bmatrix} \begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\Rightarrow \tan[2\Theta] = 2\times\frac{2\varepsilon_{12} - \eta_{12}}{2(\varepsilon_{11} - \varepsilon_{22}) - (\eta_{11} - \eta_{22})}$$

$$2\lambda = 4\varepsilon - (\eta_{11} + \eta_{22}) \pm \sqrt{[2(\varepsilon_{11} - \varepsilon_{22}) - (\eta_{11} - \eta_{22})]^2 + 4(2\varepsilon_{12} - \eta_{12})^2}$$

$$\begin{pmatrix} a\,\delta\sigma \\ a\,\delta s \end{pmatrix} = S^T U \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow,\ v_I = a W_I = L^{-3/2}\, 2^{\lambda_I}\frac{\Gamma(\frac{3}{2}+\lambda_I)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\eta)^{-1-\lambda_I}$$

$$\Rightarrow \langle |v_I|^2\rangle = \cdots,\ \langle v_1 v_2^*\rangle = 0$$

$\downarrow$

$$a^2\langle|\delta\sigma|^2\rangle = \cdots,\ a^2\langle\delta\sigma\,\delta s^*\rangle = \cdots,\ a^2\langle|\delta s|^2\rangle = \cdots$$

$\downarrow$

$$P_{\sigma *}(k),\ C_{\sigma s *}(k),\ P_{s*}(k).$$

$\boxed{\text{视界外的演化}}$ $\downarrow$

$$P_{\mathcal{R}*}(k) = \left(\frac{H_*}{\dot\sigma_*}\right)^2 P_{\sigma*}(k),\ C_{\mathcal{R}s*}(k) = -\left(\frac{H_*}{\dot\sigma_*}\right)^2 P_{\sigma*}(k),\ P_{\mathcal{S}*}(k) = \left(\frac{H_*}{\dot\sigma_*}\right)^2 P_{s*}(k)$$

captial.

传播函数 $\begin{pmatrix} \mathcal{R}_k(t) \\ S_k(t) \end{pmatrix} = T_k(t, t_*)\begin{pmatrix} \mathcal{R}_{k*} \\ S_{k*} \end{pmatrix}$

① 绝热微扰保持绝热

② 绝热微扰 $\mathcal{R}$ 不变

$$\Rightarrow T_k(t, t_*) = \begin{pmatrix} 1 & T_{\mathcal{R}S}(t, t_*) \\ 0 & T_{SS}(t, t_*). \end{pmatrix}$$

$$P_{\mathcal{R}}(k) = L^3\frac{k^3}{2\pi^2}\langle|\mathcal{R}_k|^2\rangle.$$

$\boxed{\text{一阶原初~~微~~扰谱指数.}}$ 从零阶的谱开始计算 $\begin{cases} P_{\mathcal{R}*}(k) = \left(\frac{H_*^2}{2\pi\,\dot\sigma_*}\right)^2 = P_*^{(0)}(k) \\ C_{\mathcal{R}S*}(k) = 0 \\ P_{S*}(k) = \end{cases}$

$$n_{\mathcal{R}} = \frac{d\ln P_{\mathcal{R}}(k)}{d\ln k}$$

$\boxed{\text{可选的重参数化}}$ 具体的暴胀模型~~送~~会自带一些参数，可~~把~~观测量按它们~~作~~展开以和观测值相对~~应~~比.

$\boxed{\text{与观测的对比}}$ 说实话没有看…

11. 原初张量微扰

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = a^2\begin{bmatrix} -1 & & & \\ & 1+h_+ & h_\times & \\ & h_\times & 1-h_+ & \\ & & & 1 \end{bmatrix}$$ ~~$\psi_{+,\times} = \frac{M}{\sqrt{2}} h_{+,\times}$~~

$\psi_{+,\times} \equiv \frac{M}{\sqrt{2}} h_{+,\times}$ → 行为与无质量实标量场相同，故其谱函数也类似

$$P_\psi(k) = L^3 \cdot \frac{k^3}{2\pi^2}\langle|\psi(\vec{k})|^2\rangle = \left[H/2\pi\right]^2_{k=aH}$$

$$\Rightarrow P_T(k) = L^3 \cdot \frac{k^3}{2\pi^2}\langle h_{\mu\nu}(\vec{k}) h^{\mu\nu}(\vec{k})^*\rangle = \frac{8}{M^2}\cdot\left(\frac{H}{2\pi}\right)^2\Big|_{k=aH}.$$ 张量微扰在视界外不演化·

同时，$\langle h_a(\vec{k}) h_b(\vec{k}')^*\rangle = \frac{2\pi^2}{L^3 k^3}\delta_{ab}\delta_{\vec{k}\vec{k}'}\frac{1}{4}P_T(k)$.

利用慢滚条件：$P_T = \frac{2}{3\pi^2 M^4}V$，$n_T = -2\varepsilon$，$q_T \equiv \frac{d n_T}{d\ln k} = -8\varepsilon^2 + 4\varepsilon\eta_{\sigma\sigma}$

张标转化率 $r \equiv \frac{P_T}{P_R}$，$r_* \equiv \frac{P_T}{P_{R*}}$

单场时，P_R在视界外不演化，$P_R = P_{R*} \Rightarrow r = r_*$

多场时，P_R在视界外演化.

$$P_{R*} = P_*^{(0)} = \frac{H^2}{4\pi^2\dot{\sigma}^2} = \frac{1}{24\pi^2 M^4}\frac{V}{\varepsilon}, \quad r_* = 16\varepsilon$$

由上节(我没写…) $\Rightarrow P_R = (1+T_{RS}^2)P_*^{(0)} = P_*^{(0)}/\sin^2\Delta = P_*^{(0)}/1-|r|$.

$$r = \cancel{r_* \sin\theta}\ r_* \sin^2\Delta = 16\varepsilon\cdot\sin^2\Delta$$

⇒ 一致性关系(不依赖于暴胀模型)：$n_T = -\frac{1}{8}r_* = -\frac{r}{8(1-|r|)}$ & $q_T = n_T(n_T - n_{\sigma\sigma})$.

「原初张量微扰仍未被观测到」(2015) $r < 0.07$.

单场 $\varepsilon < 0.0044$.

一致性关系. $n_T > -0.009$.

12. 无相互作用场

考虑N个单场，除引力外无其他相互作用.

$$V(\varphi_I) = \sum_I V_I(\varphi_I), \Rightarrow \frac{\partial V}{\partial\varphi_I\partial\varphi_J} = 0,\quad I\neq J.$$

$$V_I' = \frac{\partial V}{\partial\varphi_I} = \frac{\partial V_I}{\partial\varphi_I},\quad V_I'' = \frac{\partial^2 V_I}{\partial\varphi_I^2}$$

$$\begin{cases} \rho = \sum\rho_I & \rho_I = \dot{\varphi}_I^2/2 + V_I \\ p = \sum p_I & p_I = \cdots\cdots \end{cases} \Rightarrow \boxed{背景} \begin{cases} H^2 = \frac{8\pi G}{3}\rho = \frac{1}{3M^2}\sum\left(\frac{\dot{\varphi}_I^2}{2} + V_I\right) \\ \ddot{\varphi}_I + 3H\dot{\varphi}_I + V_I' = 0. \end{cases}$$

慢滚近似：$H^2 = \frac{V}{3M^2}$，$H^{-1}\dot{\varphi}_I = -M^2\frac{V_I'}{V}$，$\frac{\dot{H}}{H^2} = -\frac{M^2}{2V^2}\sum_I(V_I')^2 = -\varepsilon$

$$\varepsilon_I \equiv \frac{M^2}{2}\left(\frac{V_I'}{V}\right)^2,\quad \eta_I \equiv M^2\frac{V_I''}{V}. \Rightarrow \varepsilon_{IJ} = \sqrt{\varepsilon_I\varepsilon_J},\quad \eta_{IJ} = \eta_I\delta_{IJ}$$

对于不同的场，慢滚近似在不同的时间失效.

有一个场φ_J的慢滚条件失效之后，φ_J快速到达$V_J=0$处，ρ_J很快减小，之后φ_J对度规和其他标量场的影响几乎没有，剩余的场继续一起慢滚

$\boxed{\text{微扰}}$ 「牛顿规范」：$B=E=0$，$A=D=\Phi$

$$\begin{cases} \dot{\Phi}+H\Phi=4\pi G\cdot\dot{\varphi}_I\,\delta\varphi_I^N \\ \delta\ddot{\varphi}_I^N+3H\,\delta\dot{\varphi}_I^N+\frac{k^2}{a^2}\delta\varphi_I^N+V_I''\,\delta\varphi_I^N=-2V_I'\Phi+4\dot{\varphi}_I\dot{\Phi} \end{cases}$$

共动规范下：$\delta\rho^c=\sum\delta\rho_I^c$，$\delta\rho_I^c=\dot{\varphi}_I\,\delta\dot{\varphi}_I^N+V_I'\,\delta\varphi_I^N+3H\,\dot{\varphi}_I\,\delta\varphi_I^N-\dot{\varphi}_I^2\Phi$.

$\delta^c\equiv\delta\rho^c/\rho$，$\delta_I^c\equiv\delta\rho_I^c/\rho_I$.

暴胀期间：$\rho_I=\dot{\varphi}_I^2/2+V_I\sim V_I$．$\rho_I+p_I\ll\rho_I$.

$\Delta_I\equiv\frac{\delta\rho_I^c}{\rho_I+p_I}=\frac{\delta\rho_I^c}{\dot{\varphi}_I^2}=\frac{d}{dt}\left(\frac{\delta\varphi_I^N}{\dot{\varphi}_I}\right)-\Phi$. 相对熵微扰 $S_{IJ}=\Delta_I-\Delta_J$

$\boxed{\text{绝热超视界解}}$ 课程I中，理想流体，一般绝热超视界微扰解：

$$\begin{cases} \Phi_k(t)=\left[A_k\int_0^t a\,dt-B_k\right]\left(-\frac{H}{a}\right)+A_k \\ \dot{\Phi}_k(t)=\cdots\cdots\left[\frac{-\dot{H}+H^2}{a}\right]-A_kH. \end{cases}$$

背景方程：$\ddot{\varphi}_I+3H\dot{\varphi}_I+V_I'=0$.

微扰方程：$\delta\ddot{\varphi}_I^N+3H\,\delta\dot{\varphi}_I^N+V_I''\delta\varphi_I^N=-2V_I'\Phi+4\dot{\varphi}_I\dot{\Phi}=\cdots$

解得：$\delta\varphi_I^N=\frac{1}{a}\left[A_k\int_0^t a\,dt-B_k\right]\dot{\varphi}_I\Rightarrow\frac{d}{dt}\left(\frac{\delta\varphi_I^N}{\dot{\varphi}_I}\right)=\Phi\Rightarrow\Delta_I=0\Rightarrow\delta\rho_I^c=0$

$\mathcal{R}_k=-\Phi_k+\frac{H}{\dot{H}}(\dot{\Phi}_k+H\Phi_k)=\cdots=\text{const}$

$\boxed{\text{超视界慢滚近似}}$

超视界近似下，空间每部分独立演化，

「微扰项」场方程：$H^{-2}\,\delta\ddot{\varphi}_I^N+3H^{-1}\,\delta\dot{\varphi}_I^N+\cancel{\frac{1}{a^2}\left(\frac{k}{H}\right)^{+2}\delta\varphi_I^N}+H^{-2}V_I''\,\delta\varphi_I^N=-2H^{-2}V_I'\Phi+4H^{-2}\dot{\varphi}_I\dot{\Phi}$

背景场：$H^{-2}\ddot{\bar{\varphi}}_I+3H^{-1}\dot{\bar{\varphi}}_I+H^{-2}V_I'(\bar{\varphi}_I)=0$

$\varphi_I=\bar{\varphi}_I+\delta\varphi_I$.

全场：$H^{-2}\ddot{\varphi}_I+3H^{-1}\dot{\varphi}_I+H^{-2}V_I'(\varphi_I)=-2H^{-2}V_I'(\bar{\varphi}_I)\Phi+4H^{-1}\dot{\bar{\varphi}}_I H^{-1}\dot{\Phi}$

$\text{RHS}=-2H^{-2}V_I'(\bar{\varphi}_I)\Phi+4H^{-1}\dot{\bar{\varphi}}_I H^{-1}\dot{\Phi}\sim-2H^{-2}V_I'(\bar{\varphi}_I)\Phi$

微扰项慢滚方程：$3H\,\delta\dot{\varphi}_I^N+V_I''\,\delta\varphi_I^N=-2V_I'\Phi$.

引力场：$\Phi=\frac{1}{2M^2}\sum H^{-1}\dot{\varphi}_I\,\delta\varphi_I^N=-\frac{1}{2V}\sum V_I'\,\delta\varphi_I^N$

$M=1/\sqrt{8\pi G}$

双场 $N=2$.

$$\varphi_1\delta\varphi_1^N+\varphi_2\delta\varphi_2^N$$

$$\begin{cases}\Phi=\dfrac{1}{2M^2H}(\cancel{\varphi_I\delta\varphi_I^N})=-\dfrac{1}{2V}(V_1'\delta\varphi_1^N+V_2'\delta\varphi_2^N).\\ 3H\delta\varphi_1^N+V_1''\delta\varphi_1^N=-2V_1'\Phi\\ 3H\delta\varphi_2^N+V_2''\delta\varphi_2^N=-2V_2'\Phi.\end{cases}$$

仅保留增长模式,舍去衰减模式: $\Phi=-C_1\dfrac{\dot H}{H^2}+\dfrac{1}{3}C_3\dfrac{V_1(V_2')^2-V_2(V_1')^2}{V^2}$

$$\begin{cases}\delta\varphi_1^N/\dot\varphi_1=C_1\cdot\dfrac{1}{H}-2C_3\dfrac{HV_2}{V}\\ \delta\varphi_2^N/\dot\varphi_2=C_1\cdot\dfrac{1}{H}+2C_3\dfrac{HV_1}{V}\end{cases}$$

~~$\delta\varphi_I^N/\dot\varphi_I=C_1\frac{1}{H}\pm C_3\times 2\frac{HV}{}$~~

C_1部分代表绝热模式: $\delta\varphi^N=\dfrac{C_1}{H}\dot\varphi$

$$\begin{cases}H^2=V/3M^2\\ H^{-1}\dot\varphi_2=-M^2V_2'/V\end{cases}$$

C_3部分代表等曲率模式:

$$\Delta_1=\frac{d}{dt}(\delta\varphi_1^N/\dot\varphi_1)-\Phi=\cancel{\ }-2C_3\frac{\dot HV_2}{V}-2C_3\frac{HV_2'\dot\varphi_2}{V}+2C_3\frac{HV_2(V_1'\dot\varphi_1+V_2'\dot\varphi_2)}{V^2}$$

$$=\frac{1}{3}C_3\frac{(V_2')^2}{V}.$$

$$\Delta_2=-\frac{1}{3}C_3(V_1')^2/V\qquad S_{12}=\Delta_1-\Delta_2=\frac{C_3}{3}\cdot\frac{V_1'^2+V_2'^2}{V}=\frac{2}{3}C_3\cdot\frac{V}{M^2}\varepsilon$$

$$\delta\rho_1^c=\Delta_1\dot\varphi_1^2=\frac{C_3M^2}{9V^2}(V_1'V_2')^2,\quad\delta\rho_2^c=-\delta\rho_1^c,\quad\underline{\delta\rho^c=\delta\rho_1^c+\delta\rho_2^c=0.}$$

$\boxed{\text{(微扰的)产生}}$ $\{\delta\varphi_I^a=\delta\varphi_I^N+H^{-1}\dot\varphi_I\Phi\Rightarrow\begin{cases}\delta\sigma^a=\delta\sigma^N+H^{-1}\dot\sigma\Phi\\ \delta s\text{ 规范不变}\end{cases}$

上面可以用反解出 C_1,C_3.(用 φ_1,φ_2 表示),作做变换,可用 (σ,s)表示.

$\boxed{\text{穿过重加热的等曲率模式}}$

熵微扰在重加热的过程中被抹去,想存活下来并产生原初等曲率模式,需要受守恒量保护,或场和辐射解耦.

设 φ_1 衰减到辐射中去,φ_2成为CDM且暴胀在慢滚条件被破坏时停止.但 φ_1和 φ_2未必同时停止,需分类讨论.

<u>设φ_2首先慢滚失效,φ_1主导能量密度</u>

$\ddot\varphi_2+3H\dot\varphi_2+V_2'=0$ 中 $\ddot\varphi_2$不可舍去. $V_2<\rho_2\ll V_1\sim V\sim\rho_1\sim\rho$

φ_2到达 V_2最小值.取0级近似. $V_2=\frac{1}{2}m^2\varphi_2^2$.

$\Rightarrow\varepsilon_2\approx M^2\cdot m^2V_2/V^2,\ \eta_2\approx M^2\cdot m^2/V,\quad\varepsilon_2=\dfrac{V_2}{V}\eta_2\ll\eta_2,\quad\eta_2$最大.

φ_2的场方程: $\begin{cases}\ddot\varphi_2+3H\dot\varphi_2+m^2\varphi_2=0\\ \delta\ddot\varphi_2+3H\delta\dot\varphi_2+m^2\delta\varphi_2=-2m^2\cancel{\ }\varphi_2\Phi+4\cancel{\dot\varphi_2}\dot\Phi\end{cases}$

舍去衰减模式，$\frac{\delta\varphi_2^N}{\varphi_2}=2C_3H\frac{V_1}{V}\approx 2C_3H_*\Rightarrow \underline{\delta\varphi_2^N}=2C_3H\varphi_2\underline{=-\frac{2}{3}C_3m^2\varphi_2}$

$\eta_2=M^2\cdot m^2/V\Rightarrow m^2=3\eta_2H^2$，$\eta_2$逐渐 >1，H会 $\ll m$.

$\Rightarrow \ddot{\varphi}_2+m^2\varphi_2=0.\Rightarrow \varphi_2=\varepsilon\sin[m(t-t_1)]$ 即在势底振荡

$\delta\varphi_2^N=\frac{2}{3}C_3m^2\varphi_2$，$\rho_2=\frac{1}{2}(\dot{\varphi}_2^2+m^2\varphi_2^2)\approx\frac{1}{2}m\varepsilon^2$

$\rho_2=\frac{1}{2}(\dot{\varphi}_2^2+m^2\varphi_2^2)$

$\dot{\rho}_2=\dot{\varphi}_2\ddot{\varphi}_2+m^2\dot{\varphi}_2\varphi_2$

$\Rightarrow \dot{\rho}_2+3H\rho_2=\dot{\varphi}_2(\ddot{\varphi}_2+m^2\varphi_2)+\frac{3}{2}H(\dot{\varphi}_2^2+m^2\varphi_2^2)$

$=\dot{\varphi}_2(-3H\dot{\varphi}_2)+\frac{3}{2}H(\dot{\varphi}_2^2+m^2\varphi_2^2)=\frac{3}{2}H(m^2\varphi_2^2-\dot{\varphi}_2^2)$

$\Rightarrow \dot{\rho}_2+3H\rho_2$振荡，长期的行为为：$\dot{\rho}_2+3H\rho_2=0\Rightarrow\rho_2\propto a^{-3}$.

$S=\delta m-\frac{3}{4}\delta r=\delta_m^c-\frac{3}{4}\delta_r^c\approx\delta_m^c$　　$\delta\rho^c=0=\delta\rho_m^c+\delta\rho_r^c$

$|\delta_m^c|=|\delta\rho_m^c/\rho_m|\gg|\delta\rho_r^c/\rho_r|=|\delta_r^c|$

$\delta\rho_2^c=\dot{\varphi}_2\delta\dot{\varphi}_2^N+V_2'\delta\varphi_2^N+3H\dot{\varphi}_2\delta\varphi_2^N-\cancel{\dot{\varphi}_2^2\Phi}$

§

$3H\dot{\varphi}_2\ll V_2'=m^2\varphi_2$

$S=\delta_m^c\approx\frac{\delta\rho_2^c}{\rho_2}\approx 2\frac{\dot{\varphi}_2\delta\dot{\varphi}_2^N+m^2\varphi_2\delta\varphi_2^N}{\dot{\varphi}_2^2+m^2\varphi_2^2}=-\frac{4}{3}C_3m^2\frac{\cancel{\dot{\varphi}_2^2+m^2\varphi_2^2}}{\cancel{\dot{\varphi}_2^2+m^2\varphi_2^2}}$

$\Rightarrow S_{\vec{k}}(rad)=-\frac{4}{3}m^2C_3(\vec{k})$.

13. 双暴胀

「是12节的一个特例」.

$V(\varphi,\chi)=\frac{1}{2}m_\varphi^2\varphi^2+\frac{1}{2}m_\chi^2\chi^2$，$m_\varphi<m_\chi$

$m_\varphi\ll m_\chi$时，相当于有两个暴胀时期　　φ的暴胀持续时间更长

$(\varphi,\chi)\to(r=\sqrt{\varphi^2+\chi^2},\ \alpha=\arctan\frac{\chi}{\varphi})$.　　$\alpha\neq\theta\equiv\dot{\chi}/\dot{\varphi}$

场方程：
$$\begin{cases}H^2=\frac{8\pi G}{3}\rho=\frac{4\pi G}{3}[\dot{\varphi}^2+\dot{\chi}^2+m_\varphi^2\varphi^2+m_\chi^2\chi^2]\\ \ddot{\varphi}+3H\dot{\varphi}+m_\varphi^2\varphi=0,\quad \ddot{\chi}+3H\dot{\chi}+m_\chi^2\chi=0.\end{cases}$$

$\boxed{\text{慢滚近似}}$
$$\begin{cases}H^2=\frac{1}{3M^2}V=\frac{1}{6M^2}(m_\varphi^2\varphi^2+m_\chi^2\chi^2).\Rightarrow V=3M^2H^2\\ 3H\dot{\varphi}+m_\varphi^2\varphi=0.\Rightarrow H^{-1}\dot{\varphi}=-M^2\frac{m_\varphi^2}{V}\varphi,\quad H^{-1}\dot{\chi}=-M^2\frac{m_\chi^2}{V}\chi\end{cases}$$

e-folding

单场暴胀时，我们~~讨论了~~可以解出 $N\equiv-\ln\frac{a}{a_{end}}=\frac{1}{6}m(t-t_{end})^2$，$\varphi=2M\sqrt{N}$.

类似地，用N作为时间，可解出双暴胀：

$$\begin{cases}r=2M\sqrt{N}\\ N=N_0\cdot\frac{(\sin\alpha)^\wedge(\frac{2}{R^2-1})}{(\cos\alpha)^\wedge(\frac{2R^2}{R^2-1})}\end{cases}\qquad \tan\theta=R^2\tan\alpha,\quad R\equiv m_\chi/m_\varphi>1.$$

早期，χ占主导．$V_\chi>V_\varphi$

晚期，φ占主导．$V_\chi<V_\varphi$

$V_\chi=V_\varphi\Rightarrow\tan\theta=R\equiv\frac{m_\chi}{m_\varphi}$.

$\Rightarrow R\gg1$时 $N\approx N_0$

此模型有3个参数：m_φ, m_χ, N_0

设：① $V_\varphi > V_\chi$.

手放. ② 一段时间后 χ 停止暴胀，但 φ 仍继续.

$$\varepsilon_\varphi \equiv \frac{M^2}{2}\left(\frac{V_\varphi'}{V}\right)^2 = 2\left(\frac{M}{\varphi}\right)^2 \frac{1}{1+2R^2(\chi/\varphi)^2+R^4(\chi/\varphi)^4}$$

$$\varepsilon_\chi \equiv \frac{M^2}{2}\left(\frac{V_\chi'}{V}\right)^2 = R^4(\chi/\varphi)^2 \varepsilon_\varphi$$

$$\eta_\varphi \equiv M^2 V_\varphi''/V = 2\left(\frac{M}{\varphi}\right)^2 \frac{1}{1+R^2(\chi/\varphi)^2}$$

$$\eta_\chi \equiv M^2 V_\chi''/V = R^2 \times \eta_\varphi$$

$V_\varphi = V_\chi \Rightarrow \chi/\varphi = 1/R \Rightarrow \varepsilon_\varphi = \frac{1}{2}\left(\frac{M}{\varphi}\right)^2$, $\qquad \varepsilon_\chi = \frac{1}{2}\left(\frac{RM}{\varphi}\right)^2$

$\qquad \eta_\varphi = \left(\frac{M}{\varphi}\right)^2$, $\qquad \eta_\chi = \left(\frac{RM}{\varphi}\right)^2$ 最大.

慢滚成立时，$\eta_\chi < 1 \Rightarrow \varphi > RM$.

~~当 $R \gg 1$ 时,~~ 当 $R \gg 1$ 时，$r = \sqrt{\varphi^2+\chi^2} \approx \varphi$, $N \approx N_0$.

$\varphi \approx r \approx 2M\sqrt{N_0} > RM$, $\underline{N_0 > R^2/4}$.

$V_\varphi = V_\chi \Rightarrow \dot\chi/\chi = R^2 \dot\varphi/\varphi$

$\Rightarrow V_\chi/V_\varphi = V_\chi'\dot\chi / V_\varphi'\dot\varphi = R^2 \gg 1$.

慢滚失效后，$V_\varphi \gg V_\chi$, $\chi/\varphi \ll 1/R$.

$\varepsilon_\varphi \approx \eta_\varphi \approx 2(M/\varphi)^2$, $\varepsilon_\chi \approx 2\left(\frac{M}{\varphi}\right)^2 R^4\left(\frac{\chi}{\varphi}\right)^2 \ll 2\left(\frac{M}{\varphi}\right)^2 R^2 = \eta_\chi$

~~$\eta_\varphi \approx 2$~~

$\boxed{\text{微扰}}$. $\Phi = -C_1 \cdot \frac{\dot H}{H^2} + \frac{2}{3} C_3 \frac{(m_\chi^2 - m_\varphi^2) m_\chi^2 m_\varphi^2 \chi^2 \varphi^2}{(m_\chi^2\chi^2 + m_\varphi^2\varphi^2)^2}$ $\qquad -\dot H/H^2 = \varepsilon = \cdots$

$$\delta\varphi^N/\dot\varphi = C_1\frac{1}{H} - 2C_3 H \frac{m_\chi^2\chi^2}{m_\chi^2\chi^2 + m_\varphi^2\varphi^2}$$

$$\delta\chi^N/\dot\chi = C_1\frac{1}{H} + 2C_3 H \frac{m_\varphi^2\varphi^2}{\cdots}$$

晚期，C_3 衰减.「等曲率部分」 所谓等曲率部分仅仅在原初期间（辐射主导早期）对 $\mathcal{R}$ 无贡献而已

设 χ 成为 CDM, φ 成为 SM 粒子

14. Curvaton 曲率子

仍是双场模型，一个场 φ（暴胀子）代表暴胀，另一个场 χ（曲率子）代表原初扰动

假设：① inflation 的微扰很小；curvaton 的微扰很大.

② inflation 的~~微扰~~能量密度~~很大~~ > curvaton 的.

认为暴胀期间 inf 时空无微扰

认为暴胀期间 curvaton 较为自由. $m^2 \ll H^2 \Rightarrow |V_{\chi\chi}| \ll H^2 \Rightarrow |\eta_{\chi\chi}| \ll 1$.

→ 暴胀期间: $\begin{cases} \ddot{\bar{\chi}} + 3H\dot{\bar{\chi}} + V_\chi = 0 \\ \ddot{\delta\chi} + 3H\dot{\delta\chi} + V_{\chi\chi}\delta\chi = 0 \end{cases}$ 若 V 是二次的, 或 $V_\chi \approx 0$, $V_{\chi\chi} \approx 0$, 则 $\delta\chi/\chi = \text{const}$

↓

curvaton 慢滚; $H < m_\chi$ 时, curvaton 开始振荡, 但假设这发生在暴胀和重加热之后 ←

此时 ~~infl~~ inflaton 的能量~~组件~~已被辐射.

$\chi_{amp} = \bar{\chi}_{amp} + \delta\chi_{amp}$. $\rho_\chi = \frac{1}{2} m^2 \chi_{amp}^2$

$\bar{\rho}_\chi = \frac{1}{2} m^2 \bar{\chi}_{amp}^2$

$\delta\rho_\chi = \frac{1}{2} m^2 (2\bar{\chi}_{amp} \cdot \delta\chi_{amp} + \delta\chi_{amp}^2)$.

$\delta_\chi = \delta\rho_\chi / \rho_\chi = 2\frac{\delta\chi_{amp}}{\bar{\chi}_{amp}} + (\frac{\delta}{\quad})^2 \approx 2\frac{\delta\chi_{amp}}{\bar{\chi}_{amp}} \sim \text{const}$

→ 假设 curvaton 振荡时间很长. 起初 $\rho_\chi \ll \rho_r$. 但 $\rho_\chi \propto a^{-3}$ 而 $\rho_r \propto a^{-4}$

不论管 curvaton 的背景能量可能大可能小, 但 curvaton 的微扰变得重要.

$$\zeta_i \equiv -\psi - H\frac{\delta\rho_i}{\dot{\rho}_i} = -\psi + \frac{\delta_i}{3(1+w_i)}$$

若内部流体之间没有能量交换, 则 ζ_i 守恒, $\frac{\delta p_i}{\delta\rho_i} = \frac{\bar{p}_i'}{\bar{\rho}_i'}$

curvaton 振荡期间: $\rho = \rho_r + \rho_\chi$, $w_r = 1/3$, $w_\chi = 0$.

$\zeta_r = -\psi + \frac{\delta_r}{4}$, $\zeta_\chi = -\psi + \frac{\delta_\chi}{3}$ 两者守恒.

$\zeta = (1-f_\chi)\zeta_r + f_\chi\zeta_\chi$, $f_\chi = \frac{\rho_\chi + p_\chi}{\rho + p}$, 总的 ζ 演化.

设 $\zeta_r = 0 \Rightarrow f_\chi \approx 0$. 则开始振荡时 $\zeta = 0$. 随后 $\zeta = f_\chi\zeta_\chi$

→ curvaton 停止振荡时 $(f_\chi \approx 1)$, 立即衰减. 假定 ~~assume~~ 产物为热化了的. 会再次重加热 ~~inflation~~

设衰减是迅速的. $\zeta = f_{dec.}\zeta_\chi = \frac{2}{3} f_{dec} \frac{\delta\chi}{\chi}$

curvaton 再加热之后. $\zeta \equiv r\zeta_\chi = \frac{2}{3} r \frac{\delta\chi}{\chi}$ ζ 即是原初曲率微扰

curvaton 可能产生非高斯微扰 $\Phi(\vec{x}) = \Phi_G(\vec{x}) + f_{NL} \cdot \Phi_G(\vec{x})^2$

15. 共形度规

$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$. → Weyl 变换. (还存在其他共形变换).

$\Rightarrow \tilde{g}^{\mu\nu} = \omega^{-2} g^{\mu\nu}$, $g_{\mu\nu} = \omega^{-2}\tilde{g}_{\mu\nu}$, $g^{\mu\nu} = \omega^2 \tilde{g}^{\mu\nu}$ ↳ 要与坐标变换中的 Weyl 变换相区分.

行列式: $\tilde{g} = \omega^8 g$, $g = \omega^{-8}\tilde{g}$

$$\begin{cases} \Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}] \\ \tilde{\Gamma}^\sigma{}_{\mu\nu} = \cdots = \Gamma^\sigma{}_{\mu\nu} + \frac{1}{\omega}[\delta^\sigma{}_\nu \partial_\mu\omega + \delta^\sigma{}_\mu \partial_\nu\omega - g^{\sigma\rho} g_{\mu\nu} \partial_\rho\omega] \end{cases}$$

$\tilde{\nabla}_\alpha \tilde{g}_{\mu\nu} = \nabla_\alpha g_{\mu\nu} = 0$ 与度规相适配的导数算符.

标量场 f: $\tilde{\nabla}_\mu f = \nabla_\mu f = \partial_\mu f$, $\tilde{\nabla}_\alpha \omega = \nabla_\alpha \omega = \partial_\alpha \omega$.

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \partial_\mu \partial_\nu f - \tilde{\Gamma}^\beta{}_{\mu\nu} \partial_\beta f$$

$$= \nabla_\mu \nabla_\nu f - \frac{1}{\omega}\left[\delta^\alpha{}_\mu \delta^\beta{}_\nu + \delta^\beta{}_\mu \delta^\alpha{}_\nu - g^{\alpha\beta} g_{\mu\nu}\right] \nabla_\alpha \omega \nabla_\beta f$$

$$\nabla_\mu \nabla_\nu f = \tilde{\nabla}_\mu \tilde{\nabla}_\nu f - \frac{1}{\omega}\left[\delta^\alpha{}_\mu \delta^\beta{}_\nu + \delta^\beta{}_\mu \delta^\alpha{}_\nu - \tilde{g}^{\alpha\beta} \tilde{g}_{\mu\nu}\right] \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta f$$

$$\Box f = \nabla^\mu \nabla_\mu f = g^{\mu\nu} \nabla_\mu \nabla_\nu f = \omega^2 \tilde{\Box} f - 2\omega \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu f$$

$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\mu\sigma}$, $\tilde{R}^\rho{}_{\sigma\mu\nu} - R^\rho{}_{\sigma\mu\nu} = \cdots$

$R_{\sigma\nu} \equiv R^\mu{}_{\sigma\mu\nu} = \cdots$, $\tilde{R}_{\sigma\nu} - R_{\sigma\nu} = \cdots$

$$\tilde{R} \equiv \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} = \omega^{-2} R - \frac{6}{\omega^3} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \omega.$$

由不带~的量计算带~的量如上列出.

由带~的量计算不带~的量有二法: ① ω 换作 $\frac{1}{\omega}$. 这样要计算 $\nabla_\alpha \nabla_\beta \frac{1}{\omega}$ 等

② 实际只需代入:

$$\nabla_\mu \nabla_\nu \omega = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega - \frac{1}{\omega}\left[\delta^\alpha{}_\mu \delta^\beta{}_\nu + \delta^\beta{}_\mu \delta^\alpha{}_\nu - \tilde{g}^{\alpha\beta} \tilde{g}_{\mu\nu}\right] \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega.$$

16. 标量-张量理论

φ $\quad g_{\mu\nu}$

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2} f(\varphi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) + \mathcal{L}_{mat}\right]$$

$f(\varphi) = 1/8\pi G = M_{pl}^2 = \text{const}$ 回到 GR.

共形变换 $\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$, $\omega^2 = \frac{f(\varphi)}{M_{pl}^2}$

$$\frac{1}{2}\sqrt{-g} f(\varphi) R = \sqrt{-\tilde{g}}\left\{\frac{1}{2} M_{pl}^2 \tilde{R} + \frac{1}{2} M_{pl}^2 \left[3\frac{f''}{f} - \frac{9}{2}\left(\frac{f'}{f}\right)^2\right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi + \frac{3}{2} M_{pl}^2 \frac{f'}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi\right\}$$

$$-\frac{\sqrt{-g}}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi = \sqrt{-\tilde{g}}\left[-\frac{1}{2} M_{pl}^2 \frac{1}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi\right]$$

$$-\sqrt{-g}\, V(\varphi) = -\sqrt{-\tilde{g}} \frac{M_{pl}^4}{f^2} V(\varphi).$$

$$S = \int d^4x \sqrt{-\tilde{g}}\left\{\frac{1}{2} M_{pl}^2 \tilde{R} + \frac{1}{2} M_{pl}^2 \left[3\frac{f''}{f} - \frac{9}{2}\left(\frac{f'}{f}\right)^2 - \frac{1}{f}\right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi + \frac{3}{2} M_{pl}^2 \frac{f'}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \frac{M_{pl}^4}{f^2} V(\varphi) + \tilde{\mathcal{L}}_{mat}\right\}$$

可分部积分,与化简

$$= \int d^4x \sqrt{-\tilde{g}}\left\{\frac{1}{2} M_{pl}^2 \tilde{R} - \frac{1}{2} M_{pl}^2 \left[\frac{3}{2}\left(\frac{f'}{f}\right)^2 + \frac{1}{f}\right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \frac{M_{pl}^4}{f^2} V(\varphi) + \tilde{\mathcal{L}}_{mat}\right\}$$

再给标量场做一个变换: $d\tilde{\varphi} \equiv M_{pl} \sqrt{\frac{3}{2}\left(\frac{f'}{f}\right)^2 + \frac{1}{f}}\, d\varphi$, $\tilde{V}(\varphi) \equiv \frac{M_{pl}^4}{f^2} V(\varphi)$.

$$S = \int d^4x \sqrt{-\tilde{g}}\left[\frac{1}{2} M_{pl}^2 \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\varphi} \tilde{\nabla}_\nu \tilde{\varphi} - \tilde{V}(\varphi) + \tilde{\mathcal{L}}_{mat}\right]$$

17. Higgs 暴胀

Higgs 是 SM 中的唯一基本标量场, 但在 GR 和最小耦合中却不是暴胀(势不足够平坦)

Higgs 中是 SU(2) 表示下的双态 $\phi = \frac{1}{\sqrt{2}}\begin{bmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{bmatrix}$

$V(\phi)=\lambda(\phi^\dagger\phi)^2-\mu^2(\phi^\dagger\phi)+\frac{1}{4}\frac{\mu^4}{\lambda}$，$\phi^\dagger\phi=\frac{1}{2}(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2)$.

$(\phi^\dagger\phi)=\frac{\mu^2}{2\lambda}\equiv\frac{\sigma^2}{2}$ 时 $V=0$.

不考虑ϕ的相位，仅考虑ϕ的幅值φ，$V(\varphi)=\frac{1}{4}\lambda\varphi^4-\frac{1}{2}\mu^2\varphi^2+\frac{1}{4}\frac{\mu^4}{\lambda}=\frac{1}{4}\lambda(\varphi^2-\sigma^2)^2$

$m_H^2=V''(\varphi)|_{\varphi=\sigma}=2\mu^2$ 可据观测得到 λ,μ,σ 的值.

Higgs暴胀的作用量：$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}M^2R+\underbrace{\frac{1}{2}\xi\varphi^2R}_{\text{非最小耦合}}-\frac{1}{2}g^{\mu\nu}D_\mu\varphi D_\nu\varphi-V(\varphi)+\mathcal{L}_{SM}\right]$

首先假设 $1\ll\xi\ll(M_{Pl}/\sigma)^2\sim10^{32}$. 之后再考虑 $1\ll\xi\ll10^{16}$

由上节标量张量理论. $f(\varphi)=M^2+\xi\varphi^2$，$d\tilde\varphi=M_{Pl}\frac{\sqrt{M^2+\xi\varphi^2+6\xi^2\varphi^2}}{M^2+\xi\varphi^2}d\varphi$.

①小场近似：$\varphi\ll M/\xi$. $d\tilde\varphi=d\varphi\cdot M_{Pl}/M=d\varphi$.

$\tilde\varphi=\varphi$

$\tilde V(\tilde\varphi)=\cdots=\frac{1}{4}\lambda(\tilde\varphi^2-\sigma^2)^2$

Einstein和Jordan框架此时无区别.

②大场近似：大场下，暴胀会发生，且Einstein和Jordan框架有区别.

但当暴胀结束时，φ变成小场，两种框架的最终结果应无区别.

$\varphi\gg M/\sqrt\xi\gg\sigma$，$d\tilde\varphi=\sqrt6M_{Pl}\frac{d\varphi}{\varphi}$

$\varphi=\varphi_0\cdot e^{\tilde\varphi/\sqrt6M_{Pl}}\xlongequal{\varphi_0\equiv M/\sqrt\xi}\frac{M}{\sqrt\xi}e^{\tilde\varphi/\sqrt6M_{Pl}}$

$\tilde V(\tilde\varphi)=\frac{\lambda M_{Pl}^4}{4\xi^2}\left[1+e^{-2\tilde\varphi/\sqrt6M_{Pl}}\right]^{-2}$

$\tilde\varepsilon\approx\frac{4}{3\xi^2}\left(\frac{\varphi}{M}\right)^{-4}$，$\tilde\eta\approx-\frac{4}{3\xi}\left(\frac{\varphi}{M}\right)^{-2}$，$\tilde\xi\approx\frac{16}{9\xi^2}\left(\frac{\varphi}{M}\right)^{-4}$.

$\tilde N(\tilde\varphi)=\frac{1}{M_{Pl}^2}\int_{\tilde\varphi_{end}}^{\tilde\varphi}\frac{\tilde V}{\tilde V'}d\tilde\varphi\approx\cdots\approx\frac{3}{4}\xi\left(\frac{\varphi}{M}\right)^2$

Einstein & Jordan：$\tilde a=\omega a$. $\tilde N=N+\ln[\bar\omega_{end}/\bar\omega]$

$P_R(k)=\frac{1}{24\pi^2}\cdot\frac{1}{M_{Pl}^4}\cdot\frac{\tilde V}{\varepsilon}=\frac{\lambda\tilde N^2}{72\pi^2\xi^2}$，$P_T(k)=\frac{2}{3\pi^2M_{Pl}^4}\tilde V$

Einstein框架下暴胀结束时，两框架的内容消失，功率谱只剩下原初标量和张量谱

18. Palatini 变分

在Palatini变换下，度规$g_{\mu\nu}$和联络$\Gamma^\alpha{}_{\beta\gamma}$独立.

度规定义了和局域几何，定义了两点间的测地线

联络定义了协变导数算符：$D_\beta v^\alpha=\partial_\beta v^\alpha+\Gamma^\alpha{}_{\beta\gamma}v^\gamma$

矢量平移：$\frac{D}{d\lambda}v^\alpha=\frac{dx^\beta}{d\lambda}D_\beta v^\alpha=\frac{dv^\alpha}{d\lambda}+\Gamma^\alpha{}_{\beta\lambda}\frac{dx^\beta}{d\lambda}v^\gamma=0$

曲率张量，$\frac{D}{d\lambda}\frac{dx^\alpha}{d\lambda}=\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0$

测地线

若联络是克氏联络，度规和联络定义的测地线一致；否则不一致，物理上要问哪一条是自由下落线

在GR中，总一致；在MG中，未必.

因此拿到一个MG作用量，有两种理论供选择：$\begin{cases}\text{度规的} \\ \text{Palatini 的}\end{cases}$

(度规适配和斯托克斯定理仅对克氏联络成立)

$\boxed{\text{Palatini GR}}$

Palatini-Hilbert 作用量：$S=\int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}(\Gamma)$. 由联络得到的曲率. 设联络无挠，$\Gamma^\lambda{}_{\mu\nu}=\Gamma^\lambda{}_{\nu\mu}$

对度规变分，运动方程：$R_{\mu\nu}(\Gamma)-\frac{1}{2}g_{\mu\nu}R(\Gamma)=0$

对联络变分：$\Gamma^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}+C^\lambda{}_{\mu\nu}$ 克氏符

$$\delta S=\int d^4x\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}=\int d^4x\sqrt{-g}\,g^{\mu\nu}\left[\nabla_\sigma(\delta\Gamma^\sigma{}_{\nu\mu})-\nabla_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})\right]$$

$$\begin{cases}\nabla_\sigma(\Gamma^\sigma{}_{\nu\mu})=\tilde{\nabla}_\sigma(\delta\Gamma^\sigma{}_{\nu\mu})+C^\sigma{}_{\sigma\lambda}\delta\Gamma^\lambda{}_{\nu\mu}-C^\lambda{}_{\sigma\nu}\delta\Gamma^\sigma{}_{\lambda\mu}-C^\lambda{}_{\sigma\mu}\delta\Gamma^\sigma{}_{\nu\lambda}\\ -\nabla_\nu(\Gamma^\lambda{}_{\lambda\mu})=-\tilde{\nabla}_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})-C^\lambda{}_{\nu\sigma}\delta\Gamma^\sigma{}_{\lambda\mu}+C^\sigma{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\sigma\mu}+C^\sigma{}_{\nu\mu}\delta\Gamma^\lambda{}_{\lambda\sigma}\end{cases}$$

$\tilde{\nabla}$：$\tilde{\Gamma}^\lambda{}_{\mu\nu}$ 克氏符所定义的协变导数.

$$\Rightarrow 2g^{\mu\nu}C^\sigma{}_{\sigma\lambda}+\delta^\mu{}_\lambda C^{\nu\sigma}{}_\sigma+\delta^\nu{}_\lambda C^{\mu\sigma}{}_\sigma-2C^{\nu\mu}{}_\lambda-2C^{\mu\nu}{}_\lambda=0$$

$$\Rightarrow C^{\mu\sigma}{}_\sigma=0 \quad \& \quad C^\sigma{}_{\sigma\lambda}=0 \quad \& \quad C_{\mu\nu\lambda}+C_{\nu\mu\lambda}=0$$

对前两个指标对称，对后两个指标反称 $C_{\mu\nu\lambda}=0$

$$\Rightarrow \Gamma^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}=\frac{1}{2}g^{\lambda\rho}\left[\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}\right]$$

GR下两种联络无差.

$\boxed{\text{Palatini } f(R)}$

$$S=\int d^4x\sqrt{-g}\,f(\hat{R}),\quad \hat{R}=\hat{R}(\Gamma)=g^{\mu\nu}\hat{R}_{\mu\nu}(\Gamma).$$

对度规变分：$F(\hat{R})\hat{R}_{\mu\nu}-\frac{1}{2}f(\hat{R})g_{\mu\nu}=0$，$F(\hat{R})=\dfrac{df(\hat{R})}{d\hat{R}}$

对联络变分：$\delta S=\int d^4x\sqrt{-g}\,F(\hat{R})g^{\mu\nu}\left[\hat{\nabla}_\sigma(\delta\Gamma^\sigma{}_{\nu\mu})-\hat{\nabla}_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})\right]=0$

做共形变换：$\tilde{g}_{\mu\nu}=F(\hat{R})g_{\mu\nu}\Rightarrow g^{\mu\nu}=F(\hat{R})\tilde{g}^{\mu\nu}$，$\sqrt{-g}=F(\hat{R})^{-2}\sqrt{-\tilde{g}}$

$$\delta S=\int d^4x\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\left[\hat{\nabla}_\sigma(\delta\Gamma^\sigma{}_{\nu\mu})-\hat{\nabla}_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})\right]$$

$\hat{\Gamma}=\tilde{\Gamma}+C$，仍有：

$$C=0,\quad \hat{\Gamma}=\tilde{\Gamma},\quad \hat{R}_{\mu\nu\sigma\rho}=\tilde{R}_{\mu\nu\sigma\rho},\quad \hat{R}_{\mu\nu}=\tilde{R}_{\mu\nu},\quad \hat{R}=F\cdot\tilde{R}$$

考虑物质：$F(\hat{R})\tilde{R}_{\mu\nu}-\frac{1}{2}f(\hat{R})g_{\mu\nu}=8\pi G\,T_{\mu\nu}$

$$g^{\mu\nu}\otimes\ \Rightarrow F(\hat{R})\hat{R}-2f(\hat{R})=8\pi G\,T.$$

$\boxed{\text{Palatini 标量-张量理论}}$

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}f(\varphi)g^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma})-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-V(\varphi)+\mathcal{L}_{mat}\right]$$

度规变分：$f(\varphi)\hat{R}_{\mu\nu}-\frac{1}{2}f(\varphi)\hat{R}g_{\mu\nu}=\partial_\mu\varphi\partial_\nu\varphi-g_{\mu\nu}\left[\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi+V(\varphi)\right]+T_{\mu\nu}$

联络变分：$\delta S=\int d^4x\sqrt{-g}\,\frac{1}{2}f(\varphi)g^{\mu\nu}\delta\hat{R}_{\mu\nu}=\int d^4x\sqrt{-g}\,\frac{1}{2}f(\varphi)g^{\mu\nu}\left[\hat{\nabla}_\alpha(\delta\Gamma^\alpha{}_{\nu\mu})-\hat{\nabla}_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})\right]=0$

共形变换：$\tilde{g}_{\mu\nu}=\omega^2g_{\mu\nu}$，$\omega^2=f(\varphi)/M_{Pl}^2\Rightarrow C_{\mu\nu\lambda}=0$

$\Rightarrow(\hat{R}-\tilde{R})_{\mu\nu\sigma\rho}=0$，$(\hat{R}-\tilde{R})_{\mu\nu}=0$，$\hat{R}=\omega^2\tilde{R}$

一个结论：$\mathcal{L}_{mat}$ 依赖于哪个联络，自由粒子走谁的测地线

换到 Einstein 框架．$S=\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}M_{Pl}^2\tilde{g}^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma})-\frac{1}{2}M_{Pl}^2\frac{1}{f}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi\right.$

$\left.-\frac{M_{Pl}^4}{f^2}V(\varphi)+\tilde{\mathcal{L}}_{mat}\right]$

$$\xlongequal[\tilde{V}(\tilde{\varphi})=\frac{M_{Pl}^4}{f^2}V(\varphi)]{d\tilde{\varphi}=\frac{M_{Pl}}{\sqrt{f}}d\varphi}\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}M_{Pl}^2\tilde{g}^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma})-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\varphi}\tilde{\nabla}_\nu\tilde{\varphi}-\tilde{V}(\tilde{\varphi})+\tilde{\mathcal{L}}_{mat}\right]$$

$\boxed{\text{Palatini Higgs 暴胀}}$ $\rightarrow f(\varphi)=M^2+\xi\varphi^2$，$V(\varphi)=\frac{1}{4}\lambda(\varphi^2-\sigma^2)^2$

$$d\tilde{\varphi}=\frac{M_{Pl}}{M}\cdot\frac{d\varphi}{\sqrt{1+\xi(\varphi/M)^2}}\Rightarrow\varphi=\frac{M}{\sqrt{\xi}}\sinh\left[\sqrt{\xi}\frac{\tilde{\varphi}}{M_{Pl}}\right]$$

$$\tilde{V}(\tilde{\varphi})=\frac{1}{4}\cdot\frac{\lambda M_{Pl}^4}{(M^2+\xi\varphi^2)^2}\left[\frac{M^2}{\xi}\sinh^2\left(\sqrt{\xi}\frac{\tilde{\varphi}}{M_{Pl}}\right)-\sigma^2\right]^2$$

小场近似：$\varphi\ll M/\sqrt{\xi}\Rightarrow\tilde{\varphi}=\varphi$，$\tilde{V}\approx V$．

大场近似：$\varphi\gg M/\sqrt{\xi}\Rightarrow\tilde{V}(\tilde{\varphi})\approx\frac{\lambda M_{Pl}^4}{4\xi^2}\left[1-8e^{-2\sqrt{\xi}\tilde{\varphi}/M_{Pl}}\right]$

$\Rightarrow\tilde{N}(\tilde{\varphi})\approx\frac{\varphi^2}{8M^2}$，$\tilde{\varepsilon}\approx\frac{1}{8\xi\tilde{N}^2}$，$\tilde{\eta}\approx-\frac{1}{\tilde{N}}$，$\tilde{\xi}=\frac{1}{\tilde{N}^2}$

$$P_{\mathcal{R}}=\frac{1}{24\pi^2}\cdot\frac{1}{M_{Pl}^4}\cdot\frac{\tilde{V}}{\tilde{\varepsilon}}\approx\frac{\lambda\tilde{N}^2}{12\pi^2\xi}$$

Palatini 下，原初引力波/张量微扰更小．

19. Einstein 框架下的 f(R) 引力

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2}\left[f(\chi)+f'(\chi)(R-\chi)\right]+\mathcal{L}_m\right\}$$

变量：$g_{\mu\nu}$ 度规，标量场 χ．　　要求 $f''(\chi)\neq0$，否则回到 GR．

变分 χ：$\delta S=\int d^4x\sqrt{-g}\times\frac{1}{2}f''(\chi)(R-\chi)\times\delta\chi\Rightarrow\chi=R$．

变分 $g^{\mu\nu}$：$\delta S=\delta S_1+\delta S_2+\delta S_3+\delta S_m$．

$$\delta S_1=\int d^4x\sqrt{-g}\times\frac{1}{2}\left[g_{\mu\nu}\nabla_\lambda\nabla^\lambda f'(\chi)-\nabla_\nu\nabla_\mu f'(\chi)\right]\delta g^{\mu\nu}$$

$$\delta S_2=\int d^4x\sqrt{-g}\times\frac{1}{2}f'(\chi)R_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta S_3=-\int d^4x\sqrt{-g}\times\frac{1}{4}\left[f(\chi)+f'(\chi)(R-\chi)\right]g_{\mu\nu}\delta g^{\mu\nu}$$

$\Rightarrow$ 运动方程：$f'(\chi)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\left[f(\chi)+f'(\chi)(R-\chi)\right]-\nabla_\mu\nabla_\nu f'(\chi)+g_{\mu\nu}\Box f'(\chi)=\underbrace{-\frac{\delta S_m}{\frac{1}{2}\sqrt{-g}\,\delta g^{\mu\nu}}}_{\equiv T_{\mu\nu}}$

代入 $x=R$，发现运动方程与 $f(R)$ 引力的相同

注：一个度规理论如何与一个标量-张量理论对应？$f(R)$ 中有更高的导数，多余的自由度被变换成标量场

定义：$\varphi \equiv f'(x)$，$W(\varphi) \equiv x(\varphi)\varphi - f(x(\varphi))$

↘假设单调

$$\Rightarrow S = \int d^4x \sqrt{-g}\left[\frac{1}{2}\left(\varphi R - W(\varphi)\right) + \mathcal{L}_m\right]$$

*是无动能项的标量-张量理论.

$$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu} \Rightarrow S = \int d^4x \sqrt{-\tilde g}\left\{\frac{1}{2}M_{Pl}^2 \tilde{R} - \frac{3}{4}M_{Pl}^2 \underbrace{\frac{1}{\varphi^2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi}_{\text{动能项}} - \frac{1}{2}\frac{M_{Pl}^2}{\varphi^2}W(\varphi) + \tilde{\mathcal{L}}_m\right\}$$

$$d\tilde{\varphi} \equiv M_{Pl}\cdot\sqrt{3/2}\cdot\frac{d\varphi}{\varphi} \Rightarrow \varphi = \exp\left[\sqrt{2/3}\cdot\tilde{\varphi}/M_{Pl}\right],\quad \tilde{V}(\tilde{\varphi}) \equiv \frac{1}{2}\frac{M_{Pl}^2}{\varphi^2}W(\varphi).$$

$$\Rightarrow S = \int d^4x\sqrt{-\tilde g}\left[\frac{1}{2}M_{Pl}^2\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\varphi}\tilde{\nabla}_\nu\tilde{\varphi} - \tilde{V}(\tilde{\varphi}) + \tilde{\mathcal{L}}_m\right]$$

$\boxed{\text{Starobinsky 暴胀}}$

$$f(R) = R + R^2/6M^2$$

$$\Rightarrow \varphi \equiv f'(x) = 1 + x/3M^2 \Rightarrow x = 3M^2(\varphi - 1).$$

$$W(\varphi) = \frac{3}{2}M^2(\varphi-1)^2$$

$$\tilde{V}(\tilde{\varphi}) = \frac{3}{2}M^2 M_{Pl}^2\left[1 - e^{-\sqrt{2/3}(\tilde{\varphi}/M_{Pl})}\right]^2$$

作替换：$\frac{\lambda}{\xi^2} \to 3\left(\frac{M}{M_{Pl}}\right)^2$，则是 Higgs 暴胀

从 Plank 的数据无法区分 Starobinsky 和 Higgs.

从未来 CMB 实验可区分.