h_{ij} Decomposition v2

1 $E_{ij}(h)$

$$
\begin{aligned}
R_{ijkl} &= k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ij}\tilde{g}_{jl}) && (1.1)\\
R_{ij} &= -2k\tilde{g}_{ij} && (1.2)\\
R &= -6k && (1.3)
\end{aligned}
$$

$$
h_{ij} = -2\psi\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \quad (1.4)
$$

$$
\begin{aligned}
(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)(2E_{ij}) &= (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)h_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\tilde{\nabla}_j\big[\tilde{\nabla}^a\tilde{\nabla}^b h_{ab} + (\tilde{\nabla}_a\tilde{\nabla}^a + 4k)(\tilde{g}^{bc}h_{bc})\big]\\
&\quad + \tfrac{1}{2}\tilde{g}_{ij}\big[(\tilde{\nabla}_a\tilde{\nabla}^a - 4k)\tilde{\nabla}^b\tilde{\nabla}^c h_{bc} - (\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b - 2k\tilde{\nabla}_a\tilde{\nabla}^a + 4k^2)(\tilde{g}^{bc}h_{bc})\big]\\
&\quad - (\tilde{\nabla}_a\tilde{\nabla}^a - 3k)(\tilde{\nabla}_i\tilde{\nabla}^b h_{jb} + \tilde{\nabla}_j\tilde{\nabla}^b h_{ib}) \qquad (1.5)
\end{aligned}
$$

1.1 Gauge Invariance

To test gauge invariance, we take

$$
h_{ij} = \tilde{\nabla}_i\epsilon_j + \tilde{\nabla}_j\epsilon_i, \qquad (\tilde{g}^{ab}h_{ab}) = 2\tilde{\nabla}_a\epsilon^a, \quad (1.6)
$$

$$
\epsilon_i = \tilde{\nabla}_i L + L_i, \qquad \tilde{\nabla}^i L_i = 0, \quad (1.7)
$$

and substitute into the RHS of (1.5). The result vanishes, confirming gauge invariance.

1.2 Asanka's 8th Order Result

Applying $(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)(\tilde{\nabla}_b\tilde{\nabla}^b - 6k)$ to (1.5), we have

$$
\begin{aligned}
&(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)(\tilde{\nabla}_b\tilde{\nabla}^b - 6k)(\tilde{\nabla}_c\tilde{\nabla}^c - 2k)(\tilde{\nabla}_d\tilde{\nabla}^d - 3k)(2E_{ij}) =\\
&45k^4h_{ij} - 15k^4\tilde{g}_{ij}h - \tfrac{9}{2}k^3\tilde{\nabla}_a\tilde{\nabla}^a h_{ij} - \tfrac{9}{2}k^3\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a h - 9k^2\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i h - 3k^3\tilde{\nabla}_a\tilde{\nabla}_i h_j{}^a\\
&+15k^3\tilde{\nabla}_a\tilde{\nabla}_j h_i{}^a + 18k^3\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h^{ab} - 9k^2\tilde{\nabla}_b\tilde{\nabla}_a\tilde{\nabla}_j\tilde{\nabla}_i h^{ab} - \tfrac{51}{2}k^2\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h_{ij}\\
&+\tfrac{21}{2}k^2\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h - \tfrac{3}{2}k\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i h + \tfrac{33}{2}k^2\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_i h_j{}^a\\
&+\tfrac{39}{2}k^2\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_j h_i{}^a - 6k^2\tilde{g}_{ij}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}_a h^{ab} - \tfrac{3}{2}k\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}_a\tilde{\nabla}_j\tilde{\nabla}_i h^{ab}\\
&-2k\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h_{ij} + \tfrac{3}{2}k\tilde{g}_{ij}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h + \tfrac{1}{2}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i h\\
&+\tfrac{7}{2}k\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_i h_j{}^a + \tfrac{5}{2}k\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_j h_i{}^a - \tfrac{5}{2}k\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}_a h^{ab}\\
&+\tfrac{1}{2}\tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}_a\tilde{\nabla}_j\tilde{\nabla}_i h^{ab} + \tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h_{ij} - \tfrac{1}{2}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a h\\
&-\tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_i h_j{}^a - \tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}_j h_i{}^a\\
&+\tfrac{1}{2}\tilde{g}_{ij}\tilde{\nabla}_e\tilde{\nabla}^e\tilde{\nabla}_d\tilde{\nabla}^d\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}_a h^{ab} - 6k^3\tilde{\nabla}_i\tilde{\nabla}_a h_j{}^a - 6k^3\tilde{\nabla}_j\tilde{\nabla}_a h_i{}^a. \qquad (1.8)
\end{aligned}
$$

Expanding out Asanka's (137) in *SVT_in_RW_fullasanka.pdf*, equates to (1.8).