Lecture 7

02/15/2016

recap: electric dipole radiation $(r \gg \lambda, d)$

$$\begin{cases} \vec{B} = i\vec{k}\times\vec{A} = \dfrac{\mu_0 c k^2}{4\pi}(\hat{e}_k \times \vec{p})\dfrac{e^{ikr}}{r} \\ \vec{E} = c(\vec{B}\times\hat{e}_k) \end{cases}$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{e^{i(kr-\omega t)}}{r}\cdot\int_{} \vec{J}_\omega(\vec{r}\,')\,d^3r'}$$

$$\int \vec{J}_\omega\, d^3r' = \frac{d}{dt}\vec{p}(t) = \dot{\vec{p}}(t)$$

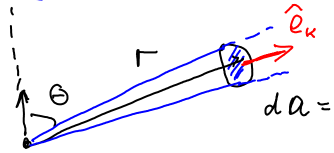
$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{c}{\mu_0}\left(\overset{a}{\vec{B}}\times\overset{b}{\hat{e}_k}\times\overset{c}{\vec{B}}\right) =$$

$$= c\left[\vec{e}_k\frac{B^2}{\mu_0} + \vec{B}\cancelto{0}{(\hat{e}_k\vec{B})}\right]$$

$$\boxed{\vec{S} = \hat{e}_k c\left[\frac{B^2}{2\mu_0} + \frac{B^2\varepsilon_0}{2\mu_0\varepsilon_0}\right] = \hat{e}_k c\,[w_B + w_E]}$$

$$\begin{cases} w_B = \dfrac{B^2}{2\mu_0} \\ w_E = \dfrac{\varepsilon_0 E^2}{2} \end{cases}$$

$$\vec{S} = \vec{e}_k\,\frac{\mu_0 c^3 k^4}{16\pi^2 r^2}\,p_\omega^2 \sin^2\theta\cdot\cos^2(kr-\omega t)$$



$da = r^2\,d\Omega$

$$dP = \vec{S}\cdot d\vec{a} = S\,r^2\,d\Omega$$

Intensity: $\boxed{I(\theta) = \dfrac{dP}{d\Omega} = S\cdot r^2}$

$$\boxed{I(\theta) = \frac{\mu_0\omega^4}{16\pi^2 c}\,p_\omega^2 \sin^2\theta\,\cos^2(\omega t - kr)}$$

Poynting Vector and Maxwell Equations

radiation zone: $r \gg d \implies \vec{j} = 0$ and $\rho = 0$

The Maxwell equation:

$$\begin{aligned} \vec{\nabla}\times\vec{B} &= \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \Big\| \cdot\vec{E} \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \quad \Big\| \cdot\vec{B} \end{aligned}$$

$$\frac{1}{\mu_0}\vec{E}(\vec{\nabla}\times\vec{B}) = \frac{\varepsilon_0}{2}\frac{\partial\vec{E}^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon_0 E^2}{2}\right) = \frac{\partial w_E}{\partial t}$$

$$\frac{1}{\mu_0}\vec{B}(\vec{\nabla}\times\vec{E}) = -\frac{1}{2\mu_0}\frac{\partial\vec{B}^2}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\vec{B}^2}{2\mu_0}\right) = -\frac{\partial w_B}{\partial t}$$

where $w_E = \dfrac{\varepsilon_0 E^2}{2}$, $w_B = \dfrac{B^2}{2\mu_0}$ densities of the electric and magnetic energies

$$\frac{\partial}{\partial t}(w_E + w_B) = \frac{1}{\mu_0}\left[\vec{E}(\vec{\nabla}\times\vec{B}) - \vec{B}(\vec{\nabla}\times\vec{E})\right] =$$

$$= -\vec{\nabla}\left[\frac{\vec{E}\times\vec{B}}{\mu_0}\right] = -\vec{\nabla}\cdot\vec{S}; \quad \text{where } \boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$$

The density of the EM energy $w = w_E + w_B$

Continuity Equation:

$$\boxed{\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$$

Math: $\vec{\nabla}(\vec{E}\times\vec{B}) = \vec{B}(\vec{\nabla}\times\vec{E}) - \vec{E}(\vec{\nabla}\times\vec{B})$

Magnetic dipole emission

$$\vec{A}_w(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_w(\vec{r}\,')e^{-i\vec{k}\vec{r}\,'}d^3r' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\int d^3r'\,\vec{J}_w(\vec{r}\,')\left(1 - i\vec{k}\vec{r}\,' + \dots\right)$$

electric dipole

magnetic dipole or electric quadrupole

$$\vec{A}_w(\vec{r}) = \vec{A}_w^{(p)} + \vec{A}_w^{(m)} + \vec{A}_w^{(Q)}$$

$$\vec{A}_w^{(p)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_w(\vec{r}\,')d^3r' \qquad \vec{A}_w^{(m)} + \vec{A}_w^{(Q)} = -i\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_w(\vec{r}\,')\vec{k}\vec{r}\,'d^3r$$

$$\int \vec{J}_w(\vec{r}\,')\vec{k}\vec{r}\,'d^3r' = k\int \vec{J}_w(\vec{r}\,')\,\hat{e}_k\vec{r}\,'d^3r'$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$$

$$\boxed{(\hat{e}_k\vec{r}\,')\vec{J}_w \equiv \frac{1}{2}\left[(\hat{e}_k\vec{r}\,')\vec{J}_w + (\hat{e}_k\vec{J}_w)\vec{r}\,'\right] + \frac{1}{2}(\vec{r}\,'\times\vec{J}_w)\times\hat{e}_k}$$

$$\hat{e}_k\times(\vec{r}\,'\times\vec{J}_w) = \vec{r}\,'(\hat{e}_k\vec{J}_w) - \vec{J}_w(\vec{r}\,'\hat{e}_k)$$

Electric Quadrupole:

$$\vec{A}_w^{Q} = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{1}{2}\int\left[(\hat{e}_k\vec{r}\,')\vec{J}_w + (\hat{e}_k\vec{J}_w)\vec{r}\,'\right]d^3r'$$

Magnetic dipole field:

$$\vec{A}_w(\vec{r}\,') = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\left(\frac{\vec{r}\,'\times\vec{J}_w(\vec{r}\,')}{2}\times\hat{e}_k\right)d^3r' = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{e}_k\times\vec{m}$$

where

$$\boxed{\vec{m} = \frac{1}{2}\int(\vec{r}\,'\times\vec{J}_w)\,d^3r}$$

is the magnetic dipole moment.