

TABLE OF CONTENTS

1. Introduction	1
1.1 Overview	6
2. Formalism	9
2.1 Einstein Gravity	10
2.1.1 Fluctuations Around Flat in the Harmonic Gauge	14
2.2 Conformal Gravity	14
2.2.1 Conformal Invariance	17
2.2.2 Fluctuations Around Flat in the Transverse Gauge	23
2.2.3 On the Energy Momentum Tensor	25
2.3 Cosmological Geometries	30
3. Scalar, Vector, Tensor (SVT) Decomposition	33
3.1 SVT3	35
3.1.1 SVT3 in Terms of $h_{\mu\nu}$ in a Conformal Flat Background	36
3.1.2 SVT3 in Terms of the Traceless $k_{\mu\nu}$ in a Conformal Flat Background	38
3.1.3 Gauge Structure and Asymptotic Behavior	39
3.2 SVTD	48
3.2.1 Gauge Structure ($D = 4$)	53
3.3 Relating SVT3 to SVT4	55

Chapter 1

Introduction

In the theory of cosmological fluctuations, the perturbative equations of motion have historically been acknowledged as forming a quite difficult and complex set of coupled non-linear differential tensor equations.¹ As a result, extensive methods of simplifying the equations of motion and eliminating non-physical gauge modes have been continually developed in order to construct the perturbative solutions, with such methods broadly falling into one of two categories.

The first entails the construction and imposition of a suitable gauge condition that aims to reduce the equations of motion into a simplified form such that a solution is readily obtainable. Here, no further decomposition upon the fluctuations is performed, with solutions being expressed directly in terms of the metric tensor fluctuation and the energy momentum tensor components. While use of such gauges are quite effective at facilitating solutions within Minkowski background geometries, their efficacy begins to diminish when faced with backgrounds

¹ For instance, within Weinberg’s cosmology book [1], he notes that even after performing additional simplifications, solving the cosmological perturbative equations is still “fearsomely complicated”.

Chapter 2

Formalism

Before we can enter the discussion of the technical methods used to decompose and simplify the cosmological fluctuation equations, we must first introduce the necessary formalism describing the interaction of gravitation and matter. The general procedure, repeated for both standard and conformal gravity, consists of varying a classical gravitational action (a general coordinate scalar) with respect to the metric, with stationary solutions yielding the equations of motion. The metric is then decomposed into zeroth and first order contributions where we obtain the background and perturbed fluctuation equations, respectively. Serving as a prototypical example of what is to come, we illustrate the form of the fluctuation equations in their simplest configuration, namely within a source-less Minkowski background geometry. Following convention [30], we impose a standard gauge condition (e.g, the harmonic or transverse gauge), allowing us to solve the equations of motion exactly.

In the case of conformal gravity, there are particular properties not shared within Einstein gravity [11] that deserve special attention which are also explored

with $g_\lambda^\ell(r)$ necessarily obeying

$$\left[(1 - kr^2) \frac{\partial^2}{\partial r^2} + \frac{(2 - 3kr^2)}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell + 1)}{r^2} + \lambda^2 \right] g_\lambda^\ell(r) = 0. \quad (2.57)$$

From here, we proceed with an interesting step and perform an incoherent averaging over all allowed spatial modes associated with a given ω . Upon calculating the sum over all modes, for each ω we obtain [37]

$$T_S^{\mu\nu} = \frac{\omega^2 (g^{\mu\nu} + 4U^\mu U^\nu)}{6\pi^2 a^4(t)} = \frac{(\lambda^2 + k^2) (g^{\mu\nu} + 4U^\mu U^\nu)}{6\pi^2 a^4(t)}, \quad (2.58)$$

where U^μ is a unit timelike vector. With (2.58) being traceless, the incoherent averaging over the spatial modes has yielded an energy momentum tensor of the perfect fluid form, namely

$$T_S^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}, \quad \rho = 3p, \quad (2.59)$$

for appropriate values of ρ and p . Inspecting (2.58), we see that if $\omega^2 = 0$, $T_S^{\mu\nu} = 0$ and if $\omega^2 = \lambda^2 + k$, we can satisfy $T_S^{\mu\nu} = 0$ non-trivially if and only if k is negative. Thus, we proceed with k negative. In performing an incoherent averaging for T_S^{00} (recalling that we are taking $\omega = 0$ here), we obtain [37]

$$T_S^{00} = \frac{1}{6} \sum_{\ell, m} \left[\sum_{i=1}^3 \gamma^{ii} |\partial_i (g_{(-k)^{1/2}}^\ell Y_\ell^m(\theta, \phi))|^2 + k |g_{(-k)^{1/2}}^\ell Y_\ell^m(\theta, \phi)|^2 \right]. \quad (2.60)$$

It has been shown in [37] that the sum in (2.60) in fact vanishes identically. With scalar field modes providing a positive contribution to $T_S^{\mu\nu}$, the negative contributions of the gravitational field from its negative spatial curvature serve

Now, in standard Einstein gravity with a non-zero background $T^{\mu\nu}$, neither the fluctuation in the background Einstein tensor or the fluctuation in the background $T^{\mu\nu}$ will separately be gauge invariant. It is only the perturbation of the entire $R^{\mu\nu} - g^{\mu\nu} R^\alpha_\alpha/2 + 8\pi G T^{\mu\nu}$ that is gauge invariant. Namely, one must impose the background equations of motion to the fluctuation equations to ensure gauge invariance. Moreover, there are no nontrivial background solutions to $G_{(0)}^{\mu\nu} = 0$ - all solutions demand a vanishing curvature tensor. However, within conformal gravity, any background that is conformal to flat will cause the background fluctuations to vanish and we have identified two scenarios in which the $T_S^{\mu\nu}$ itself vanishes non-trivially. Consequently, the background equations of motion serve no role in enforcing gauge invariance within $4\alpha_g \delta W^{\mu\nu} = \delta T^{\mu\nu}$, and thus $\delta W^{\mu\nu}$ and $\delta T^{\mu\nu}$ are separately gauge invariant. Specifically, we shall find that in any background that is conformal to flat, $\delta W^{\mu\nu}$ can be expressed entirely in terms of the gauge invariant components of the metric. Through the following chapters, we will illustrate the role of gauge invariance explicitly in both standard and conformal gravity using a Scalar, Vector, Tensor formulation as well as through the imposition of gauge conditions.

2.3 Cosmological Geometries

The cosmological principle asserts that on a large enough scale, the structure of spacetime is homogeneous and isotropic. Allowing for expansion or contraction of the universe over time, the generic metric that satisfies these criteria is the Fried-

Chapter 3

Scalar, Vector, Tensor (SVT) Decomposition

In the field of perturbative cosmology, it is standard to first introduce a 3+1 decomposition of the metric perturbation followed by a decomposition into scalars, vectors, and tensors according to the underlying background 3-space. A la the SVT decomposition [3, 4, 5, 6], referred to as SVT3 with this work.

In this chapter, we first develop the SVT3 formalism by separately forming relations between SVT3 components and integrals over $h_{\mu\nu}$, as well as their inversions (i.e. higher derivative relations between $h_{\mu\nu}$ and SVT3 quantities.) In utilizing the SVT3 decomposition in conformal gravity, we also form the analogous relations between the traceless $K_{\mu\nu}$ and the requisite SVT3 components.

We then investigate the behavior of the SVT3 quantities under gauge transformations, and along with reference to the flat space gauge invariant Einstein tensor $\delta G_{\mu\nu}$, we construct the set of gauge invariant quantities. (In fact, such a gauge invariant construction lies behind the core utility of implementing the SVT3 decomposition in the first place). In forming the gauge invariants, we highlight their dependence upon underlying assumptions of being asymptotically

but rather must be determined on a case by case basis according the specific background geometry.

3.1 SVT3

The discussion of the three dimensional SVT expansion begins by taking a flat background geometry of the form $ds^2 = dt^2 - \delta_{ij}dx^i dx^j$ where δ_{ij} represents a generic flat 3-space metric (equating to the Kronecker delta for a Minkowski background). Upon introducing a metric fluctuation $h_{\mu\nu}$ and performing a 3+1 decomposition, the geometry may be written as ¹

$$\begin{aligned} ds^2 &= (-\eta_{\mu\nu} - h_{\mu\nu})dx^\mu dx^\nu \\ &= (1 + 2\phi)dt^2 - 2(\tilde{\nabla}_i B + B_i)dt dx^i - [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E \\ &\quad + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j, \end{aligned} \tag{3.2}$$

where $\tilde{\nabla}_i = \partial/\partial x^i$ and $\tilde{\nabla}^i = \delta^{ij}\tilde{\nabla}_j$ (with Latin indices) are defined with respect to the background three-space metric δ_{ij} . In addition, the SVT3 components within

¹ In application to cosmological backgrounds, we will find it convenient to decompose the fluctuation around a conformal to flat background by incorporating an explicit factor of $\Omega^2(x)$, with the perturbed geometry taking the form

$$\begin{aligned} ds^2 &= \Omega^2(x) \left[(1 + 2\phi)dt^2 - 2(\tilde{\nabla}_i B + B_i)dt dx^i - [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E \right. \\ &\quad \left. + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right]. \end{aligned} \tag{3.1}$$

Here $\Omega(x)$ is an arbitrary function of the coordinates, where $\tilde{\nabla}_i = \partial/\partial x^i$ (with Latin index) and $\tilde{\nabla}^i = \delta^{ij}\tilde{\nabla}_j$ (not $\Omega^{-2}\delta^{ij}\tilde{\nabla}_j$) are defined with respect to the background 3-space metric δ_{ij} . SVT3 elements obey the same relations as in (3.3), i.e. transverse and traceless with respect to the background 3-space metric.

3.1.2 SVT3 in Terms of the Traceless $k_{\mu\nu}$ in a Conformal Flat Background

We have shown in Sec. 2.2 that in conformal to flat backgrounds, the perturbed Bach tensor $\delta W_{\mu\nu}$ may be expressed entirely in terms of the traceless $K_{\mu\nu}$. As such, it will prove useful to be able to express the SVT components in terms of the traceless part of $f_{\mu\nu}$. Defining $K_{\mu\nu} = \Omega^2 k_{\mu\nu}$, we have

$$K_{\mu\nu} = h_{\mu\nu} - (1/4)\Omega^2\eta_{\mu\nu}\Omega^{-2}\eta^{\alpha\beta}h_{\alpha\beta} = h_{\mu\nu} - (1/4)\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}, \quad (3.6)$$

whereby we factor out the conformal factor to form the traceless $k_{\mu\nu}$ as

$$k_{\mu\nu} = f_{\mu\nu} - (1/4)\eta_{\mu\nu}[-f_{00} + \delta^{ij}f_{ij}]. \quad (3.7)$$

Substituting $f_{\mu\nu}$ in terms of this $k_{\mu\nu}$, we obtain from (3.5) the following integral relations for the SVT components:

$$\begin{aligned} k_{00} &= \frac{3}{4}f_{00} + \frac{1}{4}\delta^{k\ell}f_{k\ell}, & k_{0i} &= f_{0i}, & k_{ij} &= f_{ij} + \frac{1}{4}\delta_{ij}f_{00} - \frac{1}{4}\delta_{ij}\delta^{k\ell}f_{k\ell}, \\ \phi &= -\frac{1}{2}f_{00}, & B &= \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i k_{0i}, & B_i &= k_{0i} - \tilde{\nabla}_i B, \\ \psi &= \frac{1}{4} \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^k \tilde{\nabla}_y^\ell k_{k\ell} - \frac{3}{4}k_{00} + \frac{1}{2}f_{00}, \\ E &= \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \left[\frac{3}{4} \int d^3z D^{(3)}(\mathbf{y} - \mathbf{z}) \tilde{\nabla}_z^k \tilde{\nabla}_z^\ell k_{k\ell} - \frac{1}{4}k_{00} \right], \\ E_i &= \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \left[\tilde{\nabla}_y^j k_{ij} - \tilde{\nabla}_i^y \int d^3z D^{(3)}(\mathbf{y} - \mathbf{z}) \tilde{\nabla}_z^k \tilde{\nabla}_z^\ell k_{k\ell} \right], \\ 2E_{ij} &+ 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i = k_{ij} - \frac{1}{2}\delta_{ij}k_{00} \\ &+ \frac{1}{2}\delta_{ij} \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^k \tilde{\nabla}_y^\ell k_{k\ell}. \end{aligned} \quad (3.8)$$

Here can see that all SVT3 components can be expressed in terms of $k_{\mu\nu}$ along with a single component of $f_{\mu\nu} = \Omega^{-2}(x)h_{\mu\nu}$, namely f_{00} . Recalling that $\delta W_{\mu\nu}$ can

to requiring that only $\tilde{\nabla}_k \tilde{\nabla}^k E_{ij} - \delta_{ij} \tilde{\nabla}_k \tilde{\nabla}^k \psi - \tilde{\nabla}_i \tilde{\nabla}_j \psi$ be gauge invariant and that only $\tilde{\nabla}_k \tilde{\nabla}^k E_{ij}$ be transverse.

3

To see how method a), constraining the asymptotic behavior of $h_{\mu\nu}$, may resolve the issues of integration by parts, we shall take $h_{\mu\nu}$ to be localized in space and oscillating in time. Specifically, for each mode we will set $h_{ij} = \epsilon_{ij}(q)e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$ with $\omega(q)$ as yet undefined (and thus not necessarily equal to q), and where $\epsilon_{ij}(q)$ serves as the polarization tensor. As a localized packet, we constrain the form of the polarization tensor by excluding any functional dependence of the form $\delta(q)$ or $\delta(q)/q$. Thus, referring to (3.13) and (3.14), for spatially localized fluctuations comprising a single mode, the quantities ψ and E_{ij} given in (3.13) and (3.14) evaluate to

$$\psi = e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \frac{[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^2},$$

³ In a similar manner, we may also integrate the remaining SVT3 components, obtaining

$$\begin{aligned} 2\phi &= -h_{00}, \quad B = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}, \\ B_i &= h_{0i} - \tilde{\nabla}_i \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}, \\ E &= \frac{1}{4} \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \int d^3z D^{(3)}(\mathbf{y} - \mathbf{z}) \left[3 \tilde{\nabla}_z^k \tilde{\nabla}_z^\ell h_{k\ell} - \tilde{\nabla}_k^z \tilde{\nabla}_z^k (\delta^{k\ell} h_{k\ell}) \right], \\ E_i &= \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \int d^3z D^{(3)}(\mathbf{y} - \mathbf{z}) \left[\tilde{\nabla}_k^z \tilde{\nabla}_z^k \nabla_z^j h_{ij} - \nabla_i^z \tilde{\nabla}_z^k \tilde{\nabla}_z^\ell h_{k\ell} \right] \end{aligned} \quad (3.16)$$

As constructed, we see that $\tilde{\nabla}^i B_i = 0$. However to show $\nabla^i E_i = 0$, we need to be able to integrate by parts. Using (3.10) and (3.12) directly, we can then show that both $\tilde{\nabla}_k \tilde{\nabla}^k \tilde{\nabla}_\ell \tilde{\nabla}^\ell (\phi + \dot{B} - \ddot{E})$ and $\tilde{\nabla}_k \tilde{\nabla}^k \tilde{\nabla}_\ell \tilde{\nabla}^\ell (B_i - \dot{E}_i)$ are gauge invariant, with the gauge invariance of $\phi + \dot{B} - \ddot{E}$ and $B_i - \dot{E}_i$ themselves then following when defining B , B_i , E and E_i according to (3.16). Hence, granted the freedom to integrate by parts, we can show that for fluctuations around flat spacetime all of the six ψ , E_{ij} , $\phi + \dot{B} - \ddot{E}$ and $B_i - \dot{E}_i$ quantities that appear in $\delta G_{\mu\nu}$ as given in (3.9) are gauge invariant.

$$\begin{aligned}
E_{ij} &= e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \left[\frac{[q^2\epsilon_{ij}(q) - q_i q^k \epsilon_{kj}(q) - q_j q^k \epsilon_{ki}(q) + q_i q_j \delta^{kl} \epsilon_{kl}(q)]}{2q^2} \right. \\
&\quad \left. + \frac{(\delta_{ij}q^2 + q_i q_j)[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^4} \right]. \tag{3.17}
\end{aligned}$$

With application of $\tilde{\nabla}^j$, one may confirm the transverse relation $\tilde{\nabla}^j E_{ij} = 0$. To construct a wave packet, we sum over all modes viz. $h_{ij} = \sum_q a_q \epsilon_{ij}(q) e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$, to then obtain

$$\begin{aligned}
\psi &= \sum_q a_q e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \frac{[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^2}, \\
E_{ij} &= \sum_q a_q e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \left[\frac{[q^2\epsilon_{ij}(q) - q_i q^k \epsilon_{kj}(q) - q_j q^k \epsilon_{ki}(q) + q_i q_j \delta^{kl} \epsilon_{kl}(q)]}{2q^2} \right. \\
&\quad \left. + \frac{(\delta_{ij}q^2 + q_i q_j)[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^4} \right], \tag{3.18}
\end{aligned}$$

where again $\tilde{\nabla}^j E_{ij} = 0$. Since the set of all $e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$ plane waves is complete for fluctuations around flat, any mode can be expanded as a general sum $h_{ij} = \sum_q a_q \epsilon_{ij}(q) e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$, with it following that (3.18) then holds for the complete plane wave basis. Hence, by constructing the ψ and E_{ij} in a localized plane-wave basis, we confirm the transverse relation $\tilde{\nabla}^j E_{ij} = 0$ without encountering issues related to integration by parts.

While we have demonstrated the role asymptotic behavior plays within tradeoff of transverse behavior vs. gauge invariance, it is also of importance to consider under conditions the SVT3 decomposition of $h_{\mu\nu}$ may be afforded in the first place. We revisit the SVT3 derivation constructed in [42] with an eye towards boundary conditions and asymptotic behavior.

Let us suppose that we are given a general vector A_i and we desire to extract

SVT3 components takes the form

$$h_{0\mu} = \begin{pmatrix} -2\phi \\ B_1 + \tilde{\nabla}_1 B \\ B_2 + \tilde{\nabla}_2 B \\ B_3 + \tilde{\nabla}_3 B \end{pmatrix}. \quad (3.28)$$

Now, with the full four-dimensional coordinate transformation mixing each of the four components of $h_{0\mu}$, we see that the transformation of ϕ may induce a contribution due to a vector B_i .

Thus to decompose the $h_{\mu\nu}$ into a set of scalars, vectors, and tensors that remain closed under the Poincare group, we must develop a formalism that matches the underlying space-time dimensionality; namely an SVT4 formalism. We proceed to do so here in a flat spacetime, following the series of steps given within [41]. It is no additional overhead to generalize this to D dimensions here, with even further generalization to arbitrary curved spacetimes found in detail within Appendix A.

We defined Greek indices to range over the full D-dimensional space and begin with the construction of a symmetric rank two tensor $F_{\mu\nu}$, taken to be transverse and traceless in the full D-dimensional space.⁴ Accounting for the dimensionality and the transverse traceless constraints, $F_{\mu\nu}$ tensor will have $D(D+1)/2 - D - 1 = (D+1)(D-2)/2$ components. To facilitate the decomposition, we introduce a D-dimensional vector W_μ . In terms of this W_μ and h , and motivated

⁴ The previously introduced E_{ij} was only transverse and traceless in a 3-dimensional subspace.

Chapter 4

Construction and Solution of SVT Fluctuation Equations

We now apply the SVT3 and SVT4 decompositions of $h_{\mu\nu}$ developed in detail within Ch. 3 to the fluctuation equations themselves and proceed to find exact solutions within backgrounds relevant to cosmology. Of particular importance, for each background geometry we analyze the conditions necessary for the decomposition theorem (cf. Sec. 3.4) to hold. Repeatedly, we find that a separation of the scalar, vector, and tensor sectors of the fluctuations may be generically achieved (i.e. without any assumptions) by application of higher derivatives which effectively project out the transverse and longitudinal components. Hence, for each geometry we may achieve an SVT separation at the equation level in terms of fourth or higher order fluctuation equations. Once this achieved, we may then analyze precisely which conditions may be required in order to recover the naive decomposition one obtains when immediately separating scalars, vectors, and tensors at the outset. Since one cannot validate the decomposition theorem in general for arbitrary backgrounds, we evaluate the SVT3 and SVT4 fluctuations for a number of specific geometries.

fluctuation is given not by E_{ij} but by $2E_{ij}/H^2\tau^2$, then with $\tau = e^{-Ht}/H$, through the $\cos(k\tau)/k^2$ term we find that at large comoving time E_{ij}/τ^2 behaves as e^{2Ht} , viz. the standard de Sitter fluctuation exponential growth.

4.1.2 Robertson Walker $k = 0$ Radiation Era

In comoving coordinates a spatially flat Robertson-Walker background metric takes the form $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$. In the radiation era where a perfect fluid pressure p and energy density ρ are related by $\rho = 3p$, the background energy-momentum tensor is given by the traceless

$$T_{\mu\nu} = p(4U_\mu U_\nu + g_{\mu\nu}), \quad (4.23)$$

where $g^{\mu\nu}U_\mu U_\nu = -1$, $U^0 = 1$, $U_0 = -1$, $U^i = 0$, $U_i = 0$. With this source the background Einstein equations $G_{\mu\nu} = -T_{\mu\nu}$ with $8\pi G = 1$ fix $a(t)$ to be $a(t) = t^{1/2}$. In conformal to flat coordinates we set $\tau = \int dt/t^{1/2} = 2t^{1/2}$, with the conformal factor being given by $\Omega(\tau) = \tau/2$. In conformal to flat coordinates the background pressure is of the form $p = 4/\tau^4$ while $U^0 = 2/\tau$, $U_0 = -\tau/2$. In this coordinate system the SVT3 fluctuation metric as written with an explicit conformal factor is of the form

$$ds^2 = \frac{\tau^2}{4} \left[(1 + 2\phi)d\tau^2 - 2(\tilde{\nabla}_i B + B_i)d\tau dx^i - [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right], \quad (4.24)$$

and the fluctuation energy-momentum tensor is of the form

$$\delta T_{\mu\nu} = \delta p(4U_\mu U_\nu + g_{\mu\nu}) + p(4\delta U_\mu U_\nu + 4U_\mu \delta U_\nu + h_{\mu\nu}). \quad (4.25)$$

where the delta function terms are solutions to $\tilde{\nabla}^4 \alpha = 0$. Finally, we recall that α and γ are related through $\tilde{\nabla}^4(\tau\alpha + 2\gamma) = 0$, with the coefficients thus obeying

$$m_1(\mathbf{k}) + 2a_1(\mathbf{k}) = 0, \quad n_1(\mathbf{k}) + 2b_1(\mathbf{k}) = 0. \quad (4.51)$$

Having determined α and γ , we can now determine $\delta p - 16\psi/\tau^4$ from the $\Delta_{00} = 0$ equation, and obtain

$$\begin{aligned} \delta p - \frac{16}{\tau^4}\psi &= -\frac{8}{\tau^4}(\alpha - \dot{\gamma}) - \frac{8}{3\tau^3}\tilde{\nabla}^2\gamma \\ &= \sum_{\mathbf{k}} \left[\frac{8}{\tau^4} \left(\frac{2}{\tau} + \frac{\partial}{\partial \tau} \right) + \frac{8k^2}{3\tau^3} \right] [a_1(\mathbf{k})j_1(k\tau/\sqrt{3}) + b_1(\mathbf{k})y_1(k\tau/\sqrt{3})] e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\quad + \text{delta function terms} \\ &= \sum_{\mathbf{k}} a_1(\mathbf{k}) \left[\frac{8k}{\tau^4\sqrt{3}}j_0(k\tau/\sqrt{3}) + \frac{8k^2}{3\tau^3}j_1(k\tau/\sqrt{3}) \right] \\ &\quad + \sum_{\mathbf{k}} b_1(\mathbf{k}) \left[\frac{8k}{\tau^4\sqrt{3}}y_0(k\tau/\sqrt{3}) + \frac{8k^2}{3\tau^3}y_1(k\tau/\sqrt{3}) \right] \\ &\quad + \text{delta function terms}. \end{aligned} \quad (4.52)$$

To determine $B_i - \dot{E}_i$ we apply $\tilde{\nabla}^j$ to $\Delta_{ij} = 0$, to obtain

$$\tilde{\nabla}^2 \left[\frac{1}{\tau}(B_i - \dot{E}_i) + \frac{1}{2}(\dot{B}_i - \ddot{E}_i) \right] = \tilde{\nabla}_i \left[\frac{2}{\tau}(\dot{\alpha} - \ddot{\gamma}) + \frac{2}{3\tau}\tilde{\nabla}^2\gamma \right], \quad (4.53)$$

from which it follows that

$$\tilde{\nabla}^i \tilde{\nabla}^2 \left[\frac{1}{\tau}(B_i - \dot{E}_i) + \frac{1}{2}(\dot{B}_i - \ddot{E}_i) \right] = \tilde{\nabla}^2 \left[\frac{2}{\tau}(\dot{\alpha} - \ddot{\gamma}) + \frac{2}{3\tau}\tilde{\nabla}^2\gamma \right] = 0. \quad (4.54)$$

On now applying $\tilde{\nabla}^2$ to (4.53) we obtain

$$\tilde{\nabla}^4 \left[\frac{1}{\tau}(B_i - \dot{E}_i) + \frac{1}{2}(\dot{B}_i - \ddot{E}_i) \right] = 0, \quad (4.55)$$

Combining these equations then gives

$$3(\dot{\alpha} - \ddot{\gamma}) + \tilde{\nabla}^2 \gamma = 0. \quad (4.72)$$

We recognize (4.40) as the $\tilde{\nabla}^2$ derivative of (4.72) and recognize (4.42) as the $\tilde{\nabla}^4$ derivative of (4.71).

Similarly, in the V, V_i sector we recognize the two equations that appear in (4.38) as the ∇^2 derivative of (4.63) and the $\tilde{\nabla}^i$ derivative of (4.64), with (4.39) being the curl of (4.63). In the $B_i - \dot{E}_i$ sector we recognize (4.55) as the $\tilde{\nabla}^j \tilde{\nabla}^2$ derivative of (4.66), and in the E_{ij} sector we recognize (4.60) as the $\tilde{\nabla}^4$ derivative of (4.67). Consequently we see that if spatially asymptotic boundary conditions are such that the only solutions to $\Delta_{\mu\nu} = 0$ are also solutions to (4.62) to (4.68) (i.e. vanishing of all delta function terms and integration constants that would lead to non-vanishing asymptotics), then the decomposition theorem follows. Otherwise it does not. Finally, we should note that, as constructed, in the matter sector we have found solutions for the gauge-invariant quantities $\delta p - 16\psi/\tau^4$, and $V - \tau^2\psi/2$. However since ψ is not gauge invariant, by choosing a gauge in which $\psi = 0$, we would then have solutions for δp and V alone.

4.1.3 General Robertson Walker Constructing the Fluctuations

Having seen how things work in a particular background Robertson-Walker case (radiation era with $k = 0$, Sec. 4.1.2), we now present a general analysis that can be applied to any background Robertson-Walker geometry with any background

will be able to test for the validity of the decomposition theorem without actually needing to know the specific form of such a relation at all, or even needing to specify any particular form for the background $\Omega(\tau)$ either for that matter.

4.1.4 Robertson Walker $k = -1$

The Scalar Sector

We have seen that the scalar sector evolution equations (4.94), (4.98), (4.99) and (4.100) involve derivatives of the form $\tilde{\nabla}^2$, $\tilde{\nabla}^2 + 3k$ where the coefficient of k is either zero or positive, while the vector and tensor sectors equations (4.95), (4.103) and (4.112) also involve derivatives such as $\tilde{\nabla}^2 - 2k$, $\tilde{\nabla}^2 - 3k$, and $\tilde{\nabla}^2 - 6k$ in which the coefficient of k is negative. As the implications of boundary conditions are very sensitive to the sign of the coefficient of k , and we will need to monitor both positive and negative coefficient cases below. In implementing evolution equations that involve products of derivative operators such as the generic $(\tilde{\nabla}^2 + \alpha)(\tilde{\nabla}^2 + \beta)F = 0$ (F denotes scalar, vector or tensor), we can satisfy these relations by $(\tilde{\nabla}^2 + \alpha)F = 0$, by $(\tilde{\nabla}^2 + \beta)F = 0$, or by $F = 0$. The decomposition theorem will only follow if boundary conditions prevent us from satisfying $(\tilde{\nabla}^2 + \alpha)F = 0$ or $(\tilde{\nabla}^2 + \beta)F = 0$ with non-zero F , leaving $F = 0$ as the only remaining possibility. It is the purpose of this section to explore whether or not boundary conditions do force us to $F = 0$ in any of the scalar, vector or tensor sectors. While a decomposition theorem would immediately hold if they do, as we will show in Sec. 4.1.4 the interplay of the vector and tensor sectors in the $\Delta_{ij} = 0$ relation

which $\chi \rightarrow 0$ behavior (the insertion of either χ^ℓ or $\chi^{-\ell-1}$ into (4.135) generates the other, with both behaviors thus being required in any $\hat{S}_\ell, \hat{f}_\ell \hat{S}_\ell$ pair), and to determine which is which we thus need to construct the asymptotic solutions directly.

For $A_S = 0, \nu = i$, the relevant $f(\nu^2)$ given in (4.132) are $\cosh \chi$ and $\sinh \chi$. Consequently, we find the first few $S_\ell^{(i)}, i = 1, 2$ solutions to $\tilde{\nabla}_a \tilde{\nabla}^a S = 0$ to be of the form

$$\begin{aligned}
\hat{S}_0^{(1)}(A_S = 0) &= \frac{\cosh \chi}{\sinh \chi}, & \hat{S}_0^{(2)}(A_S = 0) &= 1, \\
\hat{S}_1^{(1)}(A_S = 0) &= \frac{1}{\sinh^2 \chi}, & \hat{S}_1^{(2)}(A_S = 0) &= \frac{\cosh \chi}{\sinh \chi} - \frac{\chi}{\sinh^2 \chi}, \\
\hat{S}_2^{(1)}(A_S = 0) &= \frac{\cosh \chi}{\sinh^3 \chi}, & \hat{S}_2^{(2)}(A_S = 0) &= 1 + \frac{3}{\sinh^2 \chi} - \frac{3\chi \cosh \chi}{\sinh^3 \chi}, \\
\hat{S}_3^{(1)}(A_S = 0) &= \frac{4}{\sinh^2 \chi} + \frac{5}{\sinh^4 \chi}, \\
\hat{S}_3^{(2)}(A_S = 0) &= \frac{2 \cosh \chi}{\sinh \chi} + \frac{15 \cosh \chi}{\sinh^3 \chi} - \frac{12\chi}{\sinh^2 \chi} - \frac{15\chi}{\sinh^4 \chi}.
\end{aligned} \tag{4.136}$$

From this pattern we see that the solutions that are bounded at $\chi = \infty$ are badly-behaved at $\chi = 0$, while the solutions that are well-behaved at $\chi = 0$ are unbounded at $\chi = \infty$. Thus all of these $A_S = 0$ solutions are excluded by a requirement that solutions be bounded at $\chi = \infty$ and be well-behaved at $\chi = 0$.

For $A_S = -3, \nu = 2i$, the relevant $f(\nu^2)$ given in (4.132) are $\cosh 2\chi$ and $\sinh 2\chi$. Consequently, the first few solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a - 3)S = 0$ are of the form

$$\hat{S}_0^{(1)}(A_S = -3) = \cosh \chi, \quad \hat{S}_0^{(2)}(A_S = -3) = 2 \sinh \chi + \frac{1}{\sinh \chi},$$

$$\begin{aligned}
\hat{S}_1^{(1)}(A_S = -3) &= \sinh \chi, \quad \hat{S}_1^{(2)}(A_S = -3) = 2 \cosh \chi - \frac{\cosh \chi}{\sinh^2 \chi}, \\
\hat{S}_2^{(1)}(A_S = -3) &= 2 \cosh \chi - \frac{3 \cosh \chi}{\sinh^2 \chi} + \frac{3\chi}{\sinh^3 \chi}, \quad \hat{S}_2^{(2)}(A_S = -3) = \frac{1}{\sinh^3 \chi}, \\
\hat{S}_3^{(1)}(A_S = -3) &= 2 \sinh \chi - \frac{5}{\sinh \chi} - \frac{15}{\sinh^3 \chi} + \frac{15\chi \cosh \chi}{\sinh^4 \chi}, \\
\hat{S}_3^{(2)}(A_S = -3) &= \frac{\cosh \chi}{\sinh^4 \chi}.
\end{aligned} \tag{4.137}$$

From this pattern we again see that the solutions that are bounded at $\chi = \infty$ are badly-behaved at $\chi = 0$, while the solutions that are well-behaved at $\chi = 0$ are unbounded at $\chi = \infty$. Thus all of these $A_S = -3$ solutions are also excluded by a requirement that solutions be bounded at $\chi = \infty$ and be well-behaved at $\chi = 0$.

With all of these $A_S = 0$, $A_S = -3$ solutions being excluded we must realize (4.94), (4.98), (4.99) and (4.100) by

$$-2\dot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + 2k\gamma + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\hat{V} = 0, \tag{4.138}$$

$$\begin{aligned}
&2\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha} - \ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + \Omega^2\delta\hat{p} + 2k(\alpha + 2\dot{\Omega}\Omega^{-1}\gamma)] \\
&= 0,
\end{aligned} \tag{4.139}$$

$$\alpha + 2\dot{\Omega}\Omega^{-1}\gamma = 0, \tag{4.140}$$

$$2\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) - 2\dot{\Omega}\Omega^{-1}(\dot{\alpha} - \ddot{\gamma}) - 4\ddot{\Omega}\Omega^{-1}(\alpha - \dot{\gamma}) + \Omega^2\delta\hat{p} = 0. \tag{4.141}$$

These equations are augmented by (4.85), (4.88) and (4.89)

$$6\dot{\Omega}^2\Omega^{-2}(\alpha - \dot{\gamma}) + \delta\hat{\rho}\Omega^2 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma = 0, \tag{4.142}$$

$$\begin{aligned}
& + \frac{2 \cos \theta \partial_1 V_1}{\sin \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 V_2}{\sinh^2 \chi} + \frac{2 \cosh \chi \partial_2 V_1}{\sinh^3 \chi} + \frac{3 \cos \theta \partial_2 V_2}{\sin \theta \sinh^4 \chi} \\
& + \frac{\partial_2 \partial_2 V_2}{\sinh^4 \chi} + \frac{\partial_3 \partial_3 V_2}{\sin^2 \theta \sinh^4 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a V^3 = & - \frac{2V_3}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 V_3}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 V_3}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 V_3}{\sin^2 \theta \sinh^4 \chi} \\
& + \frac{2 \cosh \chi \partial_3 V_1}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 V_2}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 V_3}{\sin^4 \theta \sinh^4 \chi}. \tag{4.146}
\end{aligned}$$

To explore the structure of the $k = -1$ vector sector we seek solutions to

$$(\tilde{\nabla}_a \tilde{\nabla}^a + A_V)V_i = 0. \tag{4.147}$$

(Here V_i is to denote the full combinations of vector components that appear in (4.95) and (4.103).) In (4.147) we have introduced a generic vector sector constant A_V , whose values in (4.95) and (4.103) are $(2, -1, -2)$.

Conveniently, we find that the equation for V_1 involves no mixing with V_2 or V_3 , and can thus be solved directly. On setting $V_1(\chi, \theta, \phi) = g_{1,\ell}(\chi)Y_\ell^m(\theta, \phi)$, the equation for V_1 reduces to

$$\left[\frac{d^2}{d\chi^2} + 4 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + 2 + A_V + \frac{2}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} \right] g_{1,\ell} = 0. \tag{4.148}$$

To check the $\chi \rightarrow \infty$ and $\chi \rightarrow 0$ limits, we take the solutions to behave as $e^{\lambda\chi}$ (times an irrelevant polynomial in χ) and χ^n in these two limits. For (4.148) the limits give

$$\begin{aligned}
\lambda^2 + 4\lambda + 2 + A_V &= 0, \quad \lambda = -2 \pm (2 - A_V)^{1/2}, \quad \lambda(A_V = 2) = (-2, -2), \\
\lambda(A_V = -1) &= -2 \pm \sqrt{3}, \quad \lambda(A_V = -2) = (0, -4), \\
n(n-1) + 4n + 2 - \ell(\ell+1) &= 0, \quad n = \ell - 1, -\ell - 2. \tag{4.149}
\end{aligned}$$

Thus for $A_V = 2$ and $A_V = -1$ both solutions are bounded at infinity, while for $A_V = -2$ one solution is bounded at infinity. Moreover, for each value of A_V one of the solutions will be well-behaved as $\chi \rightarrow 0$ for any $\ell \geq 1$ while the other solution will not be. Thus for $A_V = 2$ there will always be one $\ell \geq 1$ solution that is bounded at $\chi = \infty$ and well-behaved at $\chi = 0$. To determine whether we can obtain a solution that is bounded in both limits for $A_V = -1$, $A_V = -2$ we need to explicitly find the solutions in closed form.

To this end we need to put (4.148) into the form of a differential equation whose solutions are known. We thus set $g_{1,\ell} = \alpha_\ell / \sinh \chi$, to find that (4.148) takes the form

$$\left[\frac{d^2}{d\chi^2} + 2 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} + A_V - 1 \right] \alpha_\ell = 0. \quad (4.150)$$

We recognize (4.150) as being in the form given in (4.128), which we discussed above, with $\nu^2 = A_V - 2$.

Thus for $A_V = 2$, viz. $\nu = 0$ in (4.132) and $f(\nu^2 = 0) = \chi$, χ^2 , we find $V_\ell^{(i)}$, $i = 1, 2$ solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)V_1 = 0$ of the form

$$\begin{aligned} \hat{V}_0^{(1)}(A_V = 2) &= \frac{1}{\sinh^2 \chi}, & \hat{V}_0^{(2)}(A_V = 2) &= \frac{\chi}{\sinh^2 \chi}, \\ \hat{V}_1^{(1)}(A_V = 2) &= \frac{\cosh \chi}{\sinh^3 \chi}, & \hat{V}_1^{(2)}(A_V = 2) &= \frac{1}{\sinh^2 \chi} - \frac{\chi \cosh \chi}{\sinh^3 \chi}, \\ \hat{V}_2^{(1)}(A_V = 2) &= \frac{2}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi}, \\ \hat{V}_2^{(2)}(A_V = 2) &= \frac{3 \cosh \chi}{\sinh^3 \chi} - \frac{2\chi}{\sinh^2 \chi} - \frac{3\chi}{\sinh^4 \chi}, \\ \hat{V}_3^{(1)}(A_V = 2) &= \frac{2 \cosh \chi}{\sinh^3 \chi} + \frac{5 \cosh \chi}{\sinh^5 \chi}, \end{aligned}$$

$$\hat{V}_3^{(2)}(A_V = 2) = \frac{11}{\sinh^2 \chi} + \frac{15}{\sinh^4 \chi} - \frac{6\chi \cosh \chi}{\sinh^3 \chi} - \frac{15\chi \cosh \chi}{\sinh^5 \chi}. \quad (4.151)$$

The just as required by (4.149), the $\hat{V}_\ell^{(2)}(A_V = 2)$ solutions with $\ell \geq 1$ are bounded at $\chi = \infty$ and well-behaved at $\chi = 0$. Since they can thus not be excluded by boundary conditions at $\chi = \infty$ and $\chi = 0$ (though boundary conditions do exclude modes with $\ell = 0$), solutions to (4.95) and (4.103) do not become the vector sector solutions associated with (4.113). Thus if we implement (4.103) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)V_i = 0$, the decomposition theorem will fail in the vector sector for modes with $\ell \geq 1$. Thus an equation such as (4.103) will be solved by

$$(\tilde{\nabla}_a \tilde{\nabla}^a - 1)(\tilde{\nabla}_b \tilde{\nabla}^b - 2) \left[\frac{1}{2}(\dot{B}_i - \ddot{E}_i) + \dot{\Omega} \Omega^{-1}(B_i - \dot{E}_i) \right] = V_i, \quad (4.152)$$

and not by

$$\frac{1}{2}(\dot{B}_i - \ddot{E}_i) + \dot{\Omega} \Omega^{-1}(B_i - \dot{E}_i) = 0. \quad (4.153)$$

Thus (4.103) is solved by the χ dependence of $B_i - \dot{E}_i$ and not by its τ dependence, i.e., not by the $B_i - \dot{E}_i = 1/\Omega^2$ dependence on τ that one would have obtained from the decomposition-theorem-required (4.153). This then raises the question of what does fix the τ dependence in the vector sector. We will address this issue below.

For $A_V = -2$ we see that $\nu^2 = -4$ and that $f(\nu^2) = \cosh 2\chi, \sinh 2\chi$. However in the scalar case discussed above where $\nu^2 = A_S - 1$, ν^2 would also obey $\nu^2 = -4$ if $A_S = -3$. Thus for $A_V = -2$ we can obtain the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a - 2)V_1 = 0$ directly from (4.137), and after implementing $g_{1,\ell} = \alpha_\ell / \sinh \chi$

we obtain

$$\begin{aligned}
\hat{V}_0^{(1)}(A_V = -2) &= \frac{\cosh \chi}{\sinh \chi}, & \hat{V}_0^{(2)}(A_V = -2) &= 2 + \frac{1}{\sinh^2 \chi}, \\
\hat{V}_1^{(1)}(A_V = -2) &= 1, & \hat{V}_1^{(2)}(A_V = -2) &= 2 \frac{\cosh \chi}{\sinh \chi} - \frac{\cosh \chi}{\sinh^3 \chi}, \\
\hat{V}_2^{(1)}(A_V = -2) &= 2 \frac{\cosh \chi}{\sinh \chi} - \frac{3 \cosh \chi}{\sinh^3 \chi} + \frac{3\chi}{\sinh^4 \chi}, & \hat{V}_2^{(2)}(A_V = -2) &= \frac{1}{\sinh^4 \chi}, \\
\hat{V}_3^{(1)}(A_V = -2) &= 2 - \frac{5}{\sinh^2 \chi} - \frac{15}{\sinh^4 \chi} + \frac{15\chi \cosh \chi}{\sinh^5 \chi}, \\
\hat{V}_3^{(2)}(A_V = -2) &= \frac{\cosh \chi}{\sinh^5 \chi}.
\end{aligned} \tag{4.154}$$

As required by (4.149), the $\hat{V}_2^{(2)}(A_V = -2)$ and $\hat{V}_3^{(2)}(A_V = -2)$ solutions are bounded at $\chi = \infty$. However, they are not well-behaved at $\chi = 0$. Since they thus can be excluded by boundary conditions at $\chi = \infty$ and $\chi = 0$, if we implement (4.103) by $(\tilde{\nabla}_a \tilde{\nabla}^a - 2)V_i = 0$, the only allowed solution will be $V_i = 0$, and the decomposition theorem will then follow.

Finally, for $A_V = -1$, viz. $\nu = i\sqrt{3}$, $f(\nu^2) = e^{\chi\sqrt{3}}, e^{-\chi\sqrt{3}}$, the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a - 1)V_1 = 0$ are of the form

$$\begin{aligned}
\hat{V}_0^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^2 \chi}, & \hat{V}_0^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^2 \chi}, \\
\hat{V}_1^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^3 \chi} [\sqrt{3} \sinh \chi - \cosh \chi], \\
\hat{V}_1^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^3 \chi} [-\sqrt{3} \sinh \chi - \cosh \chi], \\
\hat{V}_2^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^4 \chi} [3 - 3\sqrt{3} \cosh \chi \sinh \chi + 5 \sinh^2 \chi], \\
\hat{V}_2^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^4 \chi} [3 + 3\sqrt{3} \cosh \chi \sinh \chi + 5 \sinh^2 \chi], \\
\hat{V}_3^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^5 \chi} \left[15\sqrt{3} \sinh \chi + 14\sqrt{3} \sinh^3 \chi - 15 \cosh \chi \right.
\end{aligned}$$

$$\begin{aligned}
& -24 \cosh \chi \sinh^2 \chi \Big], \\
\hat{V}_3^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^5 \chi} \Big[-15\sqrt{3} \sinh \chi - 14\sqrt{3} \sinh^3 \chi - 15 \cosh \chi \\
& -24 \cosh \chi \sinh^2 \chi \Big]. \tag{4.155}
\end{aligned}$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{V}_\ell^{(1)}(A_V = -1) - \hat{V}_\ell^{(2)}(A_V = -1)$ with $\ell \geq 1$ are well-behaved at $\chi = 0$. Thus if we implement (4.103) by $(\tilde{\nabla}_a \tilde{\nabla}^a - 1)V_i = 0$, we are not forced to $V_i = 0$, with the decomposition theorem not then following in this sector.

The Tensor Sector

For $k = -1$ the transverse-traceless tensor sector modes need to satisfy

$$\begin{aligned}
\tilde{\gamma}^{ab} T_{ab} &= T_{11} + \frac{T_{22}}{\sinh^2 \chi} + \frac{T_{33}}{\sin^2 \theta \sinh^2 \chi} = 0, \\
\tilde{\nabla}_a T^{a1} &= -\frac{\cosh \chi T_{22}}{\sinh^3 \chi} - \frac{\cosh \chi T_{33}}{\sin^2 \theta \sinh^3 \chi} + \frac{\cos \theta T_{12}}{\sin \theta \sinh^2 \chi} + \frac{2 \cosh \chi T_{11}}{\sinh \chi} + \partial_1 T_{11} \\
&\quad + \frac{\partial_2 T_{12}}{\sinh^2 \chi} + \frac{\partial_3 T_{13}}{\sin^2 \theta \sinh^2 \chi} = 0, \\
\tilde{\nabla}_a T^{a2} &= -\frac{\cos \theta T_{33}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta T_{22}}{\sin \theta \sinh^4 \chi} + \frac{2 \cosh \chi T_{12}}{\sinh^3 \chi} + \frac{\partial_1 T_{12}}{\sinh^2 \chi} + \frac{\partial_2 T_{22}}{\sinh^4 \chi} \\
&\quad + \frac{\partial_3 T_{23}}{\sin^2 \theta \sinh^4 \chi} = 0, \\
\tilde{\nabla}_a T^{a3} &= \frac{\cos \theta T_{23}}{\sin^3 \theta \sinh^4 \chi} + \frac{2 \cosh \chi T_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 T_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_2 T_{23}}{\sin^2 \theta \sinh^4 \chi} \\
&\quad + \frac{\partial_3 T_{33}}{\sin^4 \theta \sinh^4 \chi} = 0. \tag{4.156}
\end{aligned}$$

Under these conditions the components of $\tilde{\nabla}_a \tilde{\nabla}^a T^{ij}$ evaluate to

$$\begin{aligned}
\tilde{\nabla}_a \tilde{\nabla}^a T^{11} &= T_{11} \left(6 + \frac{6}{\sinh^2 \chi} \right) + \frac{6 \cosh \chi \partial_1 T_{11}}{\sinh \chi} + \partial_1 \partial_1 T_{11} + \frac{\cos \theta \partial_2 T_{11}}{\sin \theta \sinh^2 \chi} \\
&\quad + \frac{\partial_2 \partial_2 T_{11}}{\sinh^2 \chi} + \frac{\partial_3 \partial_3 T_{11}}{\sin^2 \theta \sinh^2 \chi},
\end{aligned}$$

$$\begin{aligned}
\tilde{\nabla}_a \tilde{\nabla}^a T^{22} &= \frac{4T_{22}}{\sinh^6 \chi} - \frac{4T_{22}}{\sin^2 \theta \sinh^6 \chi} + \frac{4T_{11}}{\sinh^4 \chi} - \frac{2T_{22}}{\sinh^4 \chi} - \frac{2T_{11}}{\sin^2 \theta \sinh^4 \chi} \\
&+ \frac{2T_{11}}{\sinh^2 \chi} - \frac{2 \cosh \chi \partial_1 T_{22}}{\sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{22}}{\sinh^4 \chi} + \frac{4 \cosh \chi \partial_2 T_{12}}{\sinh^5 \chi} \\
&+ \frac{\cos \theta \partial_2 T_{22}}{\sin \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 T_{22}}{\sinh^6 \chi} - \frac{4 \cos \theta \partial_3 T_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 T_{22}}{\sin^2 \theta \sinh^6 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{33} &= \frac{2T_{33}}{\sin^4 \theta \sinh^6 \chi} (1 - \sinh^2 \chi) + T_{11} \left(\frac{2}{\sin^4 \theta \sinh^4 \chi} + \frac{2}{\sin^2 \theta \sinh^2 \chi} \right) \\
&- \frac{4 \cos \theta \cosh \chi T_{12}}{\sin^3 \theta \sinh^5 \chi} - \frac{4 \cos \theta \partial_1 T_{12}}{\sin^3 \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_1 T_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{33}}{\sin^4 \theta \sinh^4 \chi} \\
&+ \frac{4 \cos \theta \partial_2 T_{11}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta \partial_2 T_{33}}{\sin^5 \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 T_{33}}{\sin^4 \theta \sinh^6 \chi} + \frac{4 \cosh \chi \partial_3 T_{13}}{\sin^4 \theta \sinh^5 \chi} \\
&+ \frac{\partial_3 \partial_3 T_{33}}{\sin^6 \theta \sinh^6 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{12} &= T_{12} \left(-\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{2 \cosh \chi \partial_1 T_{12}}{\sinh^3 \chi} + \frac{\partial_1 \partial_1 T_{12}}{\sinh^2 \chi} \\
&+ \frac{2 \cosh \chi \partial_2 T_{11}}{\sinh^3 \chi} + \frac{\cos \theta \partial_2 T_{12}}{\sin \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 T_{12}}{\sinh^4 \chi} - \frac{2 \cos \theta \partial_3 T_{13}}{\sin^3 \theta \sinh^4 \chi} \\
&+ \frac{\partial_3 \partial_3 T_{12}}{\sin^2 \theta \sinh^4 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{13} &= -\frac{2T_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{2 \cosh \chi \partial_1 T_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 \partial_1 T_{13}}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 T_{13}}{\sin^3 \theta \sinh^4 \chi} \\
&+ \frac{\partial_2 \partial_2 T_{13}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_3 T_{11}}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 T_{12}}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 T_{13}}{\sin^4 \theta \sinh^4 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{23} &= T_{23} \left(\frac{2(1 - \sinh^2 \chi)}{\sin^2 \theta \sinh^6 \chi} - \frac{1}{\sin^4 \theta \sinh^6 \chi} \right) + \frac{2 \cos \theta \partial_1 T_{13}}{\sin^3 \theta \sinh^4 \chi} \\
&- \frac{2 \cosh \chi \partial_1 T_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{23}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_2 T_{13}}{\sin^2 \theta \sinh^5 \chi} + \frac{\cos \theta \partial_2 T_{23}}{\sin^3 \theta \sinh^6 \chi} \\
&+ \frac{\partial_2 \partial_2 T_{23}}{\sin^2 \theta \sinh^6 \chi} + \frac{2 \cosh \chi \partial_3 T_{12}}{\sin^2 \theta \sinh^5 \chi} + \frac{2 \cos \theta \partial_3 T_{22}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 T_{23}}{\sin^4 \theta \sinh^6 \chi}.
\end{aligned} \tag{4.157}$$

Following our analysis of the vector sector, in the $k = -1$ tensor sector we seek solutions to

$$(\tilde{\nabla}_a \tilde{\nabla}^a + A_T)T_{ij} = 0. \tag{4.158}$$

To solve (4.159) we set $h_{11,\ell} = \gamma_\ell / \sinh^2 \chi$ to obtain:

$$\left[\frac{d^2}{d\chi^2} + 2 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} - 2 + A_T \right] \gamma_\ell = 0. \quad (4.161)$$

We recognize (4.161) as being (4.128), and can set $\nu^2 = A_T - 3$ in (4.132), viz. $\nu^2 = (-1, 0, 3)$ for $A_T = 2, 3, 6$. For $A_T = 2$ we see that $\nu^2 = -1$. However in the scalar case discussed above where $\nu^2 = A_S - 1$, ν^2 would also obey $\nu^2 = -1$ if $A_S = 0$. Thus for $A_T = 2$ we can obtain the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)T_{11} = 0$ directly from (4.136), and after implementing $h_{11,\ell} = \gamma_\ell / \sinh^2 \chi$ we obtain $T_\ell^{(1)}$, $T_\ell^{(2)}$ solutions to (4.159) of the form

$$\begin{aligned} \hat{T}_0^{(1)}(A_T = 2) &= \frac{\cosh \chi}{\sinh^3 \chi}, & \hat{T}_0^{(2)}(A_T = 2) &= \frac{1}{\sinh^2 \chi}, \\ \hat{T}_1^{(1)}(A_T = 2) &= \frac{1}{\sinh^4 \chi}, & \hat{T}_1^{(2)}(A_T = 2) &= \frac{\cosh \chi}{\sinh^3 \chi} - \frac{\chi}{\sinh^4 \chi}, \\ \hat{T}_2^{(1)}(A_T = 2) &= \frac{\cosh \chi}{\sinh^5 \chi}, \\ \hat{T}_2^{(2)}(A_T = 2) &= \frac{1}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi} - \frac{3\chi \cosh \chi}{\sinh^5 \chi}, \\ \hat{T}_3^{(1)}(A_T = 2) &= \frac{4}{\sinh^4 \chi} + \frac{5}{\sinh^6 \chi}, \\ \hat{T}_3^{(2)}(A_T = 2) &= \frac{2 \cosh \chi}{\sinh^3 \chi} + \frac{15 \cosh \chi}{\sinh^5 \chi} - \frac{12\chi}{\sinh^4 \chi} - \frac{15\chi}{\sinh^6 \chi}. \end{aligned} \quad (4.162)$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{T}_\ell^{(2)}(A_T = 2)$ with $\ell \geq 2$ are well-behaved at $\chi = 0$. Thus if we implement (4.112) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)T_{ij} = 0$, we are not forced to $T_{ij} = 0$, with the decomposition theorem not then following in the tensor sector.

For $A_T = 3$ we see that $\nu^2 = 0$. However in the vector case discussed above

where $\nu^2 = A_V - 2$, ν^2 would also obey $\nu^2 = 0$ if $A_V = 2$. Thus for $A_T = 3$ we can obtain the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 3)T_{11} = 0$ directly from (4.151), and after implementing $h_\ell^{11} = \alpha_\ell / \sinh \chi$ we obtain

$$\begin{aligned}
\hat{T}_0^{(1)}(A_T = 3) &= \frac{1}{\sinh^3 \chi}, & \hat{T}_0^{(2)}(A_T = 3) &= \frac{\chi}{\sinh^3 \chi}, \\
\hat{T}_1^{(1)}(A_T = 3) &= \frac{\cosh \chi}{\sinh^4 \chi}, & \hat{T}_1^{(2)}(A_T = 3) &= \frac{1}{\sinh^3 \chi} - \frac{\chi \cosh \chi}{\sinh^4 \chi}, \\
\hat{T}_2^{(1)}(A_T = 3) &= \frac{2}{\sinh^3 \chi} + \frac{3}{\sinh^5 \chi}, \\
\hat{T}_2^{(2)}(A_T = 3) &= \frac{3 \cosh \chi}{\sinh^4 \chi} - \frac{2\chi}{\sinh^3 \chi} - \frac{3\chi}{\sinh^5 \chi}, \\
\hat{T}_3^{(1)}(A_T = 3) &= \frac{2 \cosh \chi}{\sinh^4 \chi} + \frac{5 \cosh \chi}{\sinh^6 \chi}, \\
\hat{T}_3^{(2)}(A_T = 3) &= \frac{11}{\sinh^3 \chi} + \frac{15}{\sinh^5 \chi} - \frac{6\chi \cosh \chi}{\sinh^4 \chi} - \frac{15\chi \cosh \chi}{\sinh^6 \chi}. \quad (4.163)
\end{aligned}$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{T}_\ell^{(2)}(A_T = 3)$ with $\ell \geq 2$ are well-behaved at $\chi = 0$. Thus if implement (4.112) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 3)T_{ij} = 0$, we are not forced to $T_{ij} = 0$, with the decomposition theorem not then following.

A similar outcome occurs for $A_T = 6$, and even though we do not evaluate the $A_T = 6$ solutions explicitly, according to (4.160) all solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 6)T_{11} = 0$ with $A_T = 6$ are bounded at $\chi = \infty$ (behaving as $e^{-3\chi} \cos(\sqrt{3}\chi)$ and $e^{-3\chi} \sin(\sqrt{3}\chi)$), with one set of these solutions being well-behaved at $\chi = 0$ for all $\ell \geq 2$. Thus if we implement (4.112) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 6)T_{ij} = 0$, we are not forced to $T_{ij} = 0$, with the decomposition theorem again not following in the tensor sector.

so that these particular scalar and vector modes can interface. As a check, with the vector sector needing $\ell \geq 1$ we differentiate $\hat{S}_1^{(2)}(A_S = 0)$ to obtain

$$\begin{aligned} \frac{d}{d\chi} \hat{S}_1^{(2)}(A_S = 0) &= \frac{d}{d\chi} \left[\frac{\cosh \chi}{\sinh \chi} - \frac{\chi}{\sinh^2 \chi} \right] \\ &= -\frac{2}{\sinh^2 \chi} + \frac{2\chi \cosh \chi}{\sinh^3 \chi} = -2\hat{V}_1^{(2)}(A_V = 2). \end{aligned} \quad (4.165)$$

Similarly, if we differentiate the vector field (4.148) with respect to χ we obtain

$$\begin{aligned} &\left[\frac{d^2}{d\chi^2} + 6\frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + 10 + A_V + \frac{6}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} \right] \frac{dg_{1,\ell}}{d\chi} \\ &+ 2(2 + A_V) \frac{\cosh \chi}{\sinh \chi} g_{1,\ell} = 0. \end{aligned} \quad (4.166)$$

Comparing with the tensor (4.159) we see that up to an overall normalization we can identify $dg_{1,\ell}/d\chi$ with the tensor $h_{11,\ell}$ for modes that obey $A_V = -2$ and $A_T = 2$, so that these particular vector and tensor modes can interface. As a check, with the tensor sector needing $\ell \geq 2$ we differentiate $\hat{V}_2^{(1)}(A_V = -2)$ to obtain

$$\begin{aligned} \frac{d}{d\chi} \hat{V}_2^{(1)}(A_V = -2) &= \frac{d}{d\chi} \left[\frac{2 \cosh \chi}{\sinh \chi} - \frac{3 \cosh \chi}{\sinh^3 \chi} + \frac{3\chi}{\sinh^4 \chi} \right] \\ &= \frac{4}{\sinh^2 \chi} + \frac{12}{\sinh^4 \chi} - \frac{12\chi \cosh \chi}{\sinh^5 \chi} \\ &= 4\hat{T}_2^{(2)}(A_T = 2). \end{aligned} \quad (4.167)$$

Thus while we can interface $A_S = 0$ and $A_V = 2$, we cannot interface $A_V = 2$ with any of the tensor modes. Rather, we must interface the $A_V = -2$ vector modes with the $A_T = 2$ tensor modes. With none of the scalar sector

modes meeting the boundary conditions at both $\chi = \infty$ and $\chi = 0$ anyway, the scalar sector must satisfy $\Delta_{\mu\nu} = 0$ by itself, with the scalar term contribution to $\Delta_{\mu\nu} = 0$ then having to vanish, just as required of the decomposition theorem. However, in the vector and tensor sectors we can achieve a common χ behavior if we set $B_1 - \dot{E}_1 = p_1(\tau)\hat{V}_2^{(1)}(A_V = -2)$, $E_{11} = q_{11}(\tau)\hat{T}_2^{(2)}(A_T = 2)$, since then the $\Delta_{11} = 0$ equation reduces to

$$\Delta_{11} = \left[\frac{1}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi} - \frac{3\chi \cosh \chi}{\sinh^5 \chi} \right] \times \left[8\dot{\Omega}\Omega^{-1}p_1(\tau) + 4\dot{p}_1(\tau) - \ddot{q}_{11}(\tau) + 2q_{11}(\tau) - 2\dot{\Omega}\Omega^{-1}\dot{q}_{11}(\tau) - 2q_{11}(\tau) \right] = 0. \quad (4.168)$$

This relation has a non-trivial solution of the form

$$4p_1(\tau) - \dot{q}_{11}(\tau) = \frac{1}{\Omega^2(\tau)}, \quad (4.169)$$

to thereby relate the τ dependencies of the vector and tensor sectors. With the other components of V_i and T_{ij} being constructed in a similar manner, as such we have provided an exact interface solution in the vector and tensor sectors. However, it only falls short in one regard. Both of $\hat{V}_2^{(1)}(A_V = -2)$ and $\hat{T}_2^{(2)}(A_T = 2)$ are well-behaved at $\chi = 0$ and $\hat{T}_2^{(2)}(A_T = 2)$ vanishes at $\chi = \infty$. However, $\hat{V}_2^{(1)}(A_V = -2)$ does not vanish at $\chi = \infty$, as it limits to a constant value. Imposing a boundary condition that the vector and tensor modes have to vanish at $\chi = \infty$ then excludes this solution, with the decomposition theorem then being

4.2.3 General Robertson Walker

The Background

Let us take the background metric and the 3-space Ricci tensor to be of the form

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = \Omega^2(\tau) (d\tau^2 - \tilde{\gamma}_{ij}dx^i dx^j), \quad \tilde{R}_{ij} = -2k\tilde{\gamma}_{ij}. \quad (4.272)$$

Given the symmetry of the 4-geometry, the 4-space Ricci tensor and the 4-space Einstein tensor can be written as

$$\begin{aligned} R_{\mu\nu} &= (A + B)U_\mu U_\nu + g_{\mu\nu}B, \\ G_{\mu\nu} &= \frac{1}{2}Ag_{\mu\nu} - \frac{1}{2}Bg_{\mu\nu} + AU_\mu U_\nu + BU_\mu U_\nu, \end{aligned} \quad (4.273)$$

where A and B are functions of τ alone and U^μ is a unit 4-vector that obeys $g_{\mu\nu}U^\mu U^\nu = -1$. With a background perfect fluid radiation era or matter era source of the form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \quad (4.274)$$

where ρ and p are functions of τ , the background Einstein equations are of the form

$$\begin{aligned} \Delta_{\mu\nu}^{(0)} &= \frac{1}{2}Ag_{\mu\nu} - \frac{1}{2}Bg_{\mu\nu} + g_{\mu\nu}p + AU_\mu U_\nu + BU_\mu U_\nu + pU_\mu U_\nu \\ &\quad + U_\mu U_\nu \rho = 0, \end{aligned} \quad (4.275)$$

with solution

$$A = -\frac{1}{2}(3p + \rho) = -3\dot{\Omega}^2\Omega^{-4} + 3\ddot{\Omega}\Omega^{-3},$$

$$\begin{aligned}
& +2z^{-1}\tilde{\nabla}_1 E_{23} + 2z^{-1}\tilde{\nabla}_2 E_{13} - 2z^{-1}\tilde{\nabla}_3 E_{12}, \\
\Delta_{13} &= -z^{-1}\tilde{\nabla}_1 \alpha + z^{-1}\tilde{\nabla}_1 \delta + \tilde{\nabla}_3 \tilde{\nabla}_1 \delta + \frac{1}{2}\tilde{\nabla}_1 (\dot{B}_3 - \ddot{E}_3) + \frac{1}{2}\tilde{\nabla}_3 (\dot{B}_1 - \ddot{E}_1) \\
& \quad - \ddot{E}_{13} + \tilde{\nabla}^2 E_{13} + 2z^{-1}\tilde{\nabla}_1 E_{33}, \\
\Delta_{23} &= -z^{-1}\tilde{\nabla}_2 \alpha + z^{-1}\tilde{\nabla}_2 \delta + \tilde{\nabla}_3 \tilde{\nabla}_2 \delta + \frac{1}{2}\tilde{\nabla}_2 (\dot{B}_3 - \ddot{E}_3) + \frac{1}{2}\tilde{\nabla}_3 (\dot{B}_2 - \ddot{E}_2) \\
& \quad - \ddot{E}_{23} + \tilde{\nabla}^2 E_{23} + 2z^{-1}\tilde{\nabla}_2 E_{33}, \tag{4.289}
\end{aligned}$$

where $\tilde{\nabla}^2 = \delta^{ab}\tilde{\nabla}_a \tilde{\nabla}_b$. With $\Delta_{\mu\nu}$ being gauge invariant we recognize α , δ , $B_i - \dot{E}_i$ and E_{ij} as being gauge invariant. We thus see that one of the gauge-invariant combinations, viz. δ , depends on both scalars and vectors. Since our only purpose here is in establishing that one of the gauge-invariant SVT3 combinations does depend on both scalars and vectors, we shall not seek to solve $\Delta_{\mu\nu} = 0$ in this particular case. Though if we were to we would only find expressions for α , δ , $B_i - \dot{E}_i$ and E_{ij} , and not for the separate scalar and vector components of δ .

4.3.2 SVT3 Fluctuations Around a General Conformal to Flat Background

In [42] it was shown that for the arbitrary conformal to flat SVT3 fluctuations of the form

$$\begin{aligned}
ds^2 &= \Omega^2(\mathbf{x}, t) \left[(1 + 2\phi)dt^2 - 2(\partial_i B + B_i)dt dx^i - [(1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E \right. \\
& \quad \left. + \partial_i E_j + \partial_j E_i + 2E_{ij}]dx^i dx^j \right] \tag{4.290}
\end{aligned}$$

with general $\Omega(\mathbf{x}, t)$, the metric sector gauge-invariant combinations are

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad \eta = \psi - \Omega^{-1}\dot{\Omega}(B - \dot{E}) + \Omega^{-1}\tilde{\nabla}^i \Omega(E_i + \tilde{\nabla}_i E),$$

Chapter 5

Construction and Imposition of Gauge Conditions

In the context of conformal gravity, we continue work done in obtaining solutions to the cosmological fluctuation equations [11] by constructing a gauge condition that is invariant under conformal transformations. Referred to as the conformal gauge, we find that in backgrounds that are conformal to flat, the conformal gravity cosmological fluctuation equations can be brought to an exceedingly simple form, comprised only of a single term. This calculation requires a series of many steps which are given in detail within the chapter, consisting of first motivating and constructing the conformal gauge, composing the fluctuation equations and using curvature identities and covariant derivatives to reduce its form, and finally imposing the conformal gauge itself, with expansion of covariant derivatives into flat Minkowski partial derivatives to yield (5.32), one of the seminal results of this thesis. Using the conformal properties detailed in Sec. 2.2.1, we find the extra degree of symmetry afforded by conformal invariance provides significant simplifications regarding the trace and moreover allows a very straightforward treatment of the entire cosmological fluctuation equations.

5.1 The Conformal Gauge and General Solutions in Conformal Gravity

5.1.1 The Conformal Gauge

As discussed in Sec. 2.1.1, we recall that under an infinitesimal transformation of coordinates of the form

$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x), \quad (5.1)$$

the metric perturbation $h_{\mu\nu}$ will transform as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \nabla_\nu \epsilon_\mu - \nabla_\mu \epsilon_\nu, \quad (5.2)$$

with contravariant components transforming similarly as

$$h^{\mu\nu} \rightarrow h^{\mu\nu} + \nabla^\nu \epsilon^\mu + \nabla^\mu \epsilon^\nu. \quad (5.3)$$

Here all covariant derivatives are taken with respect to the background metric $g_{\mu\nu}^{(0)}$. For every solution $h_{\mu\nu}$ that satisfies the fluctuation equations (i.e. $\delta G_{\mu\nu} = \delta T_{\mu\nu}$ or $\delta W_{\mu\nu} = \delta T_{\mu\nu}$), there exists a transformed $h'_{\mu\nu}$ that will also serve as a solution. Thus with the freedom of fixing the four possible arbitrary space-time functions $\epsilon^\mu(x)$, one may eliminate the four gauge degrees of freedom within $h_{\mu\nu}$ by imposing a gauge condition satisfying four equations. Within Ch. 2 we have already explored the form of the Einstein fluctuation equations in the harmonic gauge as well as the conformal gravity fluctuation equations in the transverse gauge.

5.1.2 $\delta W_{\mu\nu}$ in an Arbitrary Background

Composing the Fluctuation Equations

With the conformal gauge in hand, we proceed in a sequence of steps in order to implement it with the fluctuation equations. Prior to perturbing $W_{\mu\nu}$ we present a useful identity

$$\nabla_\beta \nabla_\nu T_{\lambda\mu} = \nabla_\nu \nabla_\beta T_{\lambda\mu} + R_{\lambda\sigma\nu\beta} T_{\mu}^{\sigma} - R_{\sigma\mu\nu\beta} T_{\lambda}^{\sigma}, \quad (5.11)$$

which holds for any rank two tensor. We then express $W^{\mu\nu}$ as

$$\begin{aligned} W_{\mu\nu} = & -\frac{1}{6} g_{\mu\nu} \nabla_\beta \nabla^\beta R_{\alpha}^{\alpha} + \nabla_\beta \nabla^\beta R_{\mu\nu} - \frac{1}{3} \nabla_\mu \nabla_\nu R_{\alpha}^{\alpha} - R^{\beta\sigma} R_{\sigma\mu\beta\nu} \\ & - R^{\beta\sigma} R_{\sigma\nu\beta\mu} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{3} R_{\alpha}^{\alpha} R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} (R_{\alpha}^{\alpha})^2. \end{aligned} \quad (5.12)$$

We set the metric as the most general $g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ denotes an arbitrary background metric and $\delta g_{\mu\nu} = h_{\mu\nu}$ an arbitrary and general fluctuation. Upon perturbing $W^{\mu\nu}$ we then obtain

$$\begin{aligned} \delta W_{\mu\nu}(h_{\mu\nu}) = & \frac{1}{2} h_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - g_{\mu\nu} h^{\alpha\beta} R_{\alpha}^{\gamma} R_{\beta\gamma} - \frac{2}{3} h^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + \frac{1}{3} g_{\mu\nu} h^{\alpha\beta} R_{\alpha\beta} R \\ & - \frac{1}{6} h_{\mu\nu} R^2 + h^{\alpha\beta} R_{\alpha}^{\gamma} R_{\mu\beta\nu\gamma} + h^{\alpha\beta} R_{\alpha}^{\gamma} R_{\mu\gamma\nu\beta} - \frac{1}{6} h_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha} R - h^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} R_{\mu\nu} \\ & + \frac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} R + \frac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_{\gamma} \nabla^{\gamma} R_{\alpha\beta} + \frac{1}{3} h^{\alpha\beta} \nabla_{\mu} \nabla_{\nu} R_{\alpha\beta} + \frac{1}{3} R \nabla_{\alpha} \nabla^{\alpha} h_{\mu\nu} \\ & + R_{\mu\beta\nu\gamma} \nabla_{\alpha} \nabla^{\gamma} h^{\alpha\beta} + R_{\mu\gamma\nu\beta} \nabla_{\alpha} \nabla^{\gamma} h^{\alpha\beta} - \frac{1}{3} R \nabla_{\alpha} \nabla_{\mu} h_{\nu}^{\alpha} - \frac{1}{3} R \nabla_{\alpha} \nabla_{\nu} h_{\mu}^{\alpha} \\ & - \frac{1}{6} \nabla_{\alpha} h_{\mu\nu} \nabla^{\alpha} R + \frac{1}{6} g_{\mu\nu} \nabla^{\alpha} R \nabla_{\beta} h_{\alpha}^{\beta} - \nabla_{\alpha} h^{\alpha\beta} \nabla_{\beta} R_{\mu\nu} - \frac{2}{3} R_{\mu\nu} \nabla_{\beta} \nabla_{\alpha} h^{\alpha\beta} \\ & + \frac{1}{3} g_{\mu\nu} R \nabla_{\beta} \nabla_{\alpha} h^{\alpha\beta} + \frac{1}{2} R_{\nu}^{\alpha} \nabla_{\beta} \nabla_{\alpha} h_{\mu}^{\beta} - R^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} h_{\mu\nu} + \frac{1}{2} R_{\mu}^{\alpha} \nabla_{\beta} \nabla_{\alpha} h_{\nu}^{\beta} \\ & - \frac{1}{2} R_{\nu}^{\alpha} \nabla_{\beta} \nabla^{\beta} h_{\mu\alpha} - \frac{1}{2} R_{\mu}^{\alpha} \nabla_{\beta} \nabla^{\beta} h_{\nu\alpha} + \frac{1}{2} \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \nabla^{\alpha} h_{\mu\nu} - \frac{1}{2} \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \nabla_{\mu} h_{\nu}^{\alpha} \end{aligned}$$

and

$$\begin{aligned}
\delta C_{\lambda\mu\nu\kappa}(h) &= \left[\frac{1}{8}g_{\mu\nu}R_{\kappa\lambda} - \frac{1}{8}g_{\lambda\nu}R_{\kappa\mu} - \frac{1}{8}g_{\kappa\mu}R_{\lambda\nu} + \frac{1}{8}g_{\kappa\lambda}R_{\mu\nu} \right. \\
&\quad \left. + \frac{1}{24}g_{\kappa\mu}g_{\lambda\nu}R - \frac{1}{24}g_{\kappa\lambda}g_{\mu\nu}R - \frac{1}{4}R_{\kappa\nu\lambda\mu} \right] h \\
&= \frac{1}{4}hC_{\lambda\mu\nu\kappa}.
\end{aligned} \tag{5.22}$$

Inspection of (5.22) reveals that if the background Weyl tensor $C_{\lambda\mu\nu\kappa}$ is zero then it follows that $\delta C_{\lambda\mu\nu\kappa}$ has no dependence upon h . Given a vanishing background Weyl tensor, we may also observe that from (2.20) $\delta W^{\mu\nu}$ can therefore be expressed as

$$\delta W^{\mu\nu} = 2\nabla_\kappa \nabla_\lambda \delta C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} \delta C^{\mu\lambda\nu\kappa}, \tag{5.23}$$

and thereby also be independent of h .

Differential Commutations

Before we can express (5.14) as a form ready for application of the conformal gauge condition, we must first commute the differential operators as per (5.11) and (5.16). On performing the commutations for $\delta W_{\mu\nu}(K_{\mu\nu})$ we obtain

$$\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) &= \frac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}K_\nu^\alpha R_{\alpha\beta}R_\mu^\beta - \frac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} \\
&\quad - \frac{1}{2}K_\mu^\alpha R_{\alpha\beta}R_\nu^\beta + \frac{1}{3}g_{\mu\nu}K^{\alpha\beta}R_{\alpha\beta}R + \frac{1}{3}K_\nu^\alpha R_{\mu\alpha}R + \frac{1}{3}K_\mu^\alpha R_{\nu\alpha}R - \frac{1}{6}K_{\mu\nu}R^2 \\
&\quad - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\kappa}R_{\alpha\gamma\beta\kappa} - \frac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} - K_\nu^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_\alpha^\gamma R_{\mu\gamma\nu\beta} \\
&\quad + 2K^{\alpha\beta}R_{\alpha\gamma\beta\kappa}R_\mu^{\gamma\kappa} - K_\mu^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \frac{1}{3}R\nabla_\alpha \nabla^\alpha K_{\mu\nu} - \frac{1}{6}K_{\mu\nu}\nabla_\alpha \nabla^\alpha R \\
&\quad + \frac{1}{2}R_\nu^\alpha \nabla_\alpha \nabla_\beta K_\mu^\beta + \frac{1}{2}R_\mu^\alpha \nabla_\alpha \nabla_\beta K_\nu^\beta - \frac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \frac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha^\beta
\end{aligned}$$

5.1.3 $\delta W_{\mu\nu}$ in a Conformal to Flat Minkowski Background Implementing the Conformal Gauge Condition

With (5.24) now being ready for insertion of the conformal gauge, we evaluate (5.24) in the conformal to flat background given in (2.23) recalling that we take $\Omega(x)$ to be a completely general and arbitrary spacetime function. In the $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ background the gauge condition $\nabla_\nu K^{\mu\nu} = \frac{1}{2}K^{\mu\nu}g^{\alpha\beta}\partial_\nu g_{\alpha\beta}$ takes the form

$$\begin{aligned}
\nabla_\nu K^{\mu\nu} - \frac{1}{2}K^{\mu\nu}\Omega^{-2}\eta^{\alpha\beta}\eta_{\alpha\beta}\partial_\nu\Omega^2 &= \nabla_\nu K^{\mu\nu} - 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \\
&= \partial_\nu K^{\mu\nu} + 6\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega - 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \\
&= \partial_\nu K^{\mu\nu} + 2\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega = \Omega^{-2}\partial_\nu(\Omega^2 K^{\mu\nu}) \\
&= 0.
\end{aligned} \tag{5.26}$$

We can factor out a contribution of $\Omega^2(x)$ from the fluctuation by setting $K^{\mu\nu} = \Omega^{-2}(x)k^{\mu\nu}$ and $K_{\mu\nu} = \Omega^2(x)k_{\mu\nu}$. For clarification, here indices on $k^{\mu\nu}$ and $k_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}k^{\alpha\beta}$ are raised and lowered with $\eta_{\mu\nu}$. As a result, (5.26) can thus be expressed as the familiar transverse form $\partial_\nu k^{\mu\nu} = 0$, noting that the conformal gauge condition is such that there is no longer a dependence upon the conformal factor.

¹ Before we also evaluate the (5.24) within a conformal Minkowski background in the conformal gauge given in (5.26), we observe that the gauge condition

$$\nabla_\nu K^{\mu\nu} = 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \tag{5.27}$$

¹ Within (5.26) we note that we have taken $\Omega(x)$ to be a general function of the coordinates so that we additionally encompass the special case of Robertson-Walker geometries with arbitrary spatial curvature k . As given in Appendix B, $\Omega(x)$ will only depend on the time coordinate t for $k = 0$, while for $k \neq 0$, $\Omega(x)$ will depend on both t and the radial coordinate r .

$$\begin{aligned}
& [\cos \theta \sin \theta \cos^2 \phi K_{x'x'} + \cos \theta \sin \theta \sin^2 \phi K_{y'y'} - \sin \theta \cos \theta K_{z'z'}], \\
K_{p'\theta} &= r' \cos \theta \cos \phi K_{p'x'} + r' \cos \theta \sin \phi K_{p'y'} - r' \sin \theta K_{p'z'} \\
&= \frac{\sinh \chi}{(\cosh p + \cosh \chi)} [\cos \theta \cos \phi K_{p'x'} + \cos \theta \sin \phi K_{p'y'} - \sin \theta K_{p'z'}].
\end{aligned} \tag{5.49}$$

We may continue to compute the analogous expressions for $K_{\theta\phi}$, $K_{\phi\phi}$, $K_{r'\phi}$ and $K_{p'\phi}$. Within (5.49), we see that the $r' = \sinh \chi / (\cosh p + \cosh \chi)$ prefactor has leading comoving time behavior as t^0 if $p = \chi$, (i.e. $t = r$ with both t and r large) or as t^{-1} if $p \gg \chi$ (i.e. $t \gg r$). These correspond to the lightlike and timelike modes respectively.

In transforming from (B.14) to (B.4), we note that the angular sector is unaffected by the transformation. Thus the angular sector fluctuations $K_{\theta\theta}$, $K_{\theta\phi}$, $K_{\phi\phi}$ associated with the comoving Robertson-Walker geometry given in (B.4) will have leading order growth as t^4 when including the contribution of the prefactor in (5.49) for lightlike coordinates and as t^2 for the timelike case. Moreover, since the lightlike $ds^2 = 0$ is both general coordinate invariant and conformal invariant, lightlike modes associated with the (B.15) metric will transform into lightlike modes associated with the metric (B.4). Finally, we note that a t^4 growth for the lightlike modes is a rather significant growth rate, one that cannot be obtained from standard Einstein gravity when using the same radiation matter source.

To further address the non-angular modes, we recall that the full coordinate

Now investigating the leading order behavior for large $p \gg \chi$, the transformations take the form

$$\begin{aligned} \frac{\partial p'}{\partial p} &= \frac{\partial r'}{\partial \chi} = \frac{d \cosh \chi}{t}, & \frac{\partial p'}{\partial \chi} &= \frac{\partial r'}{\partial p} = -\frac{d \sinh \chi}{t}, \\ \frac{\partial p}{\partial t} &= \frac{1}{t}, & \frac{\partial \chi}{\partial r} &= \frac{1}{L \cosh \chi}. \end{aligned} \quad (5.53)$$

As we go from the (p', r') coordinates to (p, χ) coordinates, incorporating the t^4 dependence of $\Omega^2(p, \chi)p'$ and including the prefactor in (5.49), we proceed analogous to before and determine growth of $K_{tt} \sim t^0$, $K_{tr} \sim t^1$, $K_{t\theta} \sim t^1$, $K_{t\phi} \sim t^1$, while $K_{rr} \sim t^2$, $K_{r\theta} \sim t^2$, $K_{r\phi} \sim t^2$, $K_{\theta\theta} \sim t^2$, $K_{\theta\phi} \sim t^2$ and $K_{\phi\phi} \sim t^2$.² Hence, for large $p \gg \chi$, K_{tr} , $K_{t\theta}$ and $K_{t\phi}$ all have t^1 leading order time behavior, and K_{rr} , $K_{r\theta}$, $K_{r\phi}$, $K_{\theta\theta}$, $K_{\theta\phi}$ and $K_{\phi\phi}$ all have t^2 leading order behavior, with t^2 being the overall leading growth.

² Such suppression in this case can be viewed as the following: if χ is negligible then so is r , and thus the spatial part of the metric in (B.4) effectively becomes flat.

Chapter 6

Conclusions

arise in regards to spatially asymptotic

at hand that one may use to solve the fluctuation equations, within this work we instill both.

the perturbative equations of motion have historically been acknowledged as forming a rather complex set of coupled non-linear differential tensor equations. For instance, within Weinberg's cosmology book [1], he notes that even after performing additional simplifications that these equations are still “fearsomely complicated”.

extensive methods of simplifying the equations of motion and eliminating non-physical gauge modes are required in order to construct the perturbative solutions

*touch on decomposition theorem. maybe a little mock example *conformal gravity - while we do not dig into details on qm here, has been proposed as a ... citations. dope as properties *overview section *imposing gauges or for intrinsic gauge invariants. difficult problem to solve, complicated fluctuations. as we move

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Appendix A

SVT by Projection

A.1 The 3 + 1 Decomposition

For geometries with constant spatial curvature, (e.g. Roberston Walker, Minkowski) it is especially convenient to utilize projections that decouple the time and spatial components of symmetric rank two tensors. Thus, we introduce the standard co-variant 3+1 decomposition of a symmetric rank two tensor $T_{\mu\nu}$ in a 4-dimensional geometry with metric $g_{\mu\nu}$. To facilitate decomposition, we need only make use of a 4-vector U^μ obeying $g_{\mu\nu}U^\mu U^\nu = -1$ and a projector

$$P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \quad (\text{A.1})$$

that obeys

$$U_\mu P^{\mu\nu} = 0, \quad P_{\mu\nu} P^{\mu\nu} = g_{\mu\nu} P^{\mu\nu} = 3, \quad P_{\mu\sigma} P^\sigma_\nu = P_{\mu\nu}. \quad (\text{A.2})$$

In terms of the projector we can write

$$\begin{aligned} T_{\mu\nu} &= g_\mu^\sigma g_\nu^\tau T_{\sigma\tau} = P_\mu^\sigma P_\nu^\tau T_{\sigma\tau} - U_\mu U^\sigma P_\nu^\tau T_{\sigma\tau} \\ &\quad - P_\mu^\sigma U_\nu U^\tau T_{\sigma\tau} + U_\mu U_\nu U^\sigma U^\tau T_{\sigma\tau}. \end{aligned} \quad (\text{A.3})$$

On introducing

$$\begin{aligned} \rho &= U^\sigma U^\tau T_{\sigma\tau}, \quad p = \frac{1}{3} P^{\sigma\tau} T_{\sigma\tau}, \quad q_\mu = -P_\mu^\sigma U^\tau T_{\sigma\tau}, \\ \pi_{\mu\nu} &= \left[\frac{1}{2} P_\mu^\sigma P_\nu^\tau + \frac{1}{2} P_\nu^\sigma P_\mu^\tau - \frac{1}{3} P_{\mu\nu} P^{\sigma\tau} \right] T_{\sigma\tau}, \end{aligned} \quad (\text{A.4})$$

which obey

$$U^\mu q_\mu = 0, \quad U^\nu \pi_{\mu\nu} = 0, \quad \pi_{\mu\nu} = \pi_{\nu\mu}, \quad g^{\mu\nu} \pi_{\mu\nu} = P^{\mu\nu} \pi_{\mu\nu} = 0, \quad (\text{A.5})$$

we can rewrite $T_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} + U_\mu q_\nu + U_\nu q_\mu + \pi_{\mu\nu}, \quad (\text{A.6})$$

a familiar form that may for instance be found in [5]. As constructed, the ten-component $T_{\mu\nu}$ has been covariantly decomposed into two one-component 4-scalars, one three-component 4-vector that is orthogonal to U_μ and one five-component traceless, rank two tensor that is also orthogonal to U_μ .

A.2 Vector Fields

We recall from Sec. 2.1 that the general $h_{\mu\nu}$ has ten components and with the freedom to impose four coordinate transformations, these may be reduced to six physical components. In terms of the SVT decompositions, the SVT3 and SVT4 formalisms yield fluctuation equations that only depend on six SVT combinations in the SVT3 or SVT4 expansions of $h_{\mu\nu}$. We recall that the SVT3 and SVT4 components are related to the components of $h_{\mu\nu}$ via integral relations such as that given in (3.16), for example

$$B = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}, \quad B_i = h_{0i} - \tilde{\nabla}_i \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}. \quad (\text{A.7})$$

From (A.7) one can observe that components are intrinsically non-local, with their very existence requiring that the associated integrals exist. Consequently, we see that asymptotic boundary conditions are a necessary and irreducible component in their construction. As discussed in [43] and [42], we introduce another method to implement gauge invariance using non-local operators, namely the projection operator approach, with such an approach being equivalent to the SVT formalism used in Ch. 3.

To discuss the application of the projection operator approach to rank two tensors such as $h_{\mu\nu}$ we first apply it to a four-dimensional gauge field A_μ . Thus in analog to (A.7) we set

$$\begin{aligned} A_\mu &= A_\mu^T + \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha = A_\mu^T + A_\mu^L, \\ A_\mu^T &= A_\mu - \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha, \quad A_\mu^L = \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha, \end{aligned} \quad (\text{A.8})$$

where $\partial_\mu \partial^\mu D(x - x') = \delta^4(x - x')$, where A_μ^T obeys the transverse condition $\partial^\mu A_\mu^T = 0$, and where A_μ^L is longitudinal. The utility of this expansion is that under $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ the transverse A_μ^T transforms as

$$\begin{aligned} A_\mu^T &\rightarrow A_\mu + \partial_\mu \chi - \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha - \partial_\mu \int d^4x' D(x - x') \partial^\alpha \partial_\alpha \chi \\ &= A_\mu^T, \end{aligned} \quad (\text{A.9})$$

where we perform an integration by parts. Thus with integration by parts the transverse A_μ^T is automatically gauge invariant. In addition we note that A_μ^T obeys

$$\partial_\nu \partial^\nu A_\mu^T = \partial_\nu \partial^\nu A_\mu - \partial_\mu \partial^\nu A_\nu = \partial^\nu F_{\nu\mu}. \quad (\text{A.10})$$

Thus just as with the use of the non-local SVT formalism for gravity, the use of the non-local A_μ^T enables us to write the Maxwell equations entirely in terms of

gauge-invariant quantities. With A_μ^L being the derivative of a scalar function it is pure gauge, and thus cannot appear in the gauge-invariant Maxwell equations. Moreover, while there may be an integration by parts issue for A_μ^T , there is none for $\partial_\nu \partial^\nu A_\mu^T$ as it is equal to the gauge-invariant quantity $\partial_\nu \partial^\nu A_\mu - \partial_\mu \partial^\alpha A_\alpha$, just as it must be since the Maxwell equations are gauge invariant. In the SVT language, with (A.7) and (A.9) only involving scalars and vectors, we can think of (A.7) as an SV3 decomposition of the 3-component h_{0i} , and (A.9) as an SV4 decomposition of the 4-component A_μ .

An alternate way of understanding these results is to introduce a projection operator

$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial}{\partial x^\mu} \int d^4x' D(x-x') \frac{\partial}{\partial x'^\nu}, \quad (\text{A.11})$$

as we can then rewrite A_μ^T as

$$A_\mu^T = \Pi_{\mu\nu} A^\nu. \quad (\text{A.12})$$

As introduced, $\Pi_{\mu\nu}$ obeys the projector algebra relations

$$\begin{aligned} \Pi_{\mu\nu} \Pi^\nu{}_\sigma &= \Pi_{\mu\sigma}, \\ \Pi_{\mu\nu} A^{T\nu} &= A_\mu^T - \partial_\mu \int d^4x' D(x-x') \partial_\nu A^{T\nu}(x') = A_\mu^T, \\ \Pi_{\mu\nu} A^{L\nu} &= \partial_\mu \int d^4x' D(x-x') \partial_\nu A^\nu(x') - \partial_\mu \int d^4x' D(x-x') \times \\ &\quad \partial_\nu \partial^\nu \int d^4x'' D(x'-x'') \partial_\sigma A^\sigma(x'') = 0. \end{aligned} \quad (\text{A.13})$$

In the SVT4 language we set $A_\mu = A_\mu^T + \partial_\mu A$, and can thus identify

$$A_\mu^T = \Pi_{\mu\nu} A^\nu, \quad A_\mu^L = \partial_\mu A = (\eta_{\mu\nu} - \Pi_{\mu\nu}) A^\nu. \quad (\text{A.14})$$

For vector fields the SVT formalism is thus equivalent to the projector formalism. Having now established this equivalence for vector fields, we turn now to tensor fields.

A.3 Transverse and Longitudinal Projection Operators for Flat Spacetime Tensor Fields

For tensor fields we introduce 4-dimensional flat spacetime transverse and longitudinal projection operators [42, 43]:

$$T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma} \eta_{\nu\tau} - \partial_\mu \int d^4x' D(x-x') \eta_{\nu\tau} \partial_\sigma - \partial_\nu \int d^4x' D(x-x') \eta_{\mu\sigma} \partial_\tau$$

$$\begin{aligned}
& + \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\tau, \\
L_{\mu\nu\sigma\tau} & = \partial_\mu \int d^4 x' D(x-x') \eta_{\nu\tau} \partial_\sigma + \partial_\nu \int d^4 x' D(x-x') \eta_{\mu\sigma} \partial_\tau \\
& - \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\tau.
\end{aligned} \tag{A.15}$$

As constructed, these projectors obey a standard projector algebra

$$\begin{aligned}
T_{\mu\nu\sigma\tau} T_{\alpha\beta}^{\sigma\tau} & = T_{\mu\nu\alpha\beta}, \quad L_{\mu\nu\sigma\tau} L_{\alpha\beta}^{\sigma\tau} = L_{\mu\nu\alpha\beta}, \\
T_{\mu\nu\sigma\tau} L_{\alpha\beta}^{\sigma\tau} & = 0, \quad L_{\mu\nu\sigma\tau} T_{\alpha\beta}^{\sigma\tau} = 0, \quad L_{\mu\nu\sigma\tau} + T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma} \eta_{\nu\tau}.
\end{aligned} \tag{A.16}$$

In terms of these projectors we define transverse and longitudinal components $h_{\mu\nu}^T$ and $h_{\mu\nu}^L$ of $h_{\mu\nu}$ according to

$$\begin{aligned}
T_{\mu\nu\sigma\tau} h^{\sigma\tau} & = h_{\mu\nu}^T = h_{\mu\nu} - \partial_\mu \int d^4 x' D(x-x') \partial_\sigma h^\sigma{}_\nu(x') \\
& \quad - \partial_\nu \int d^4 x' D(x-x') \partial_\kappa h^\kappa{}_\mu(x') \\
& \quad + \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\kappa h^{\sigma\kappa}(x''), \\
L_{\mu\nu\sigma\tau} h^{\sigma\tau} & = h_{\mu\nu}^L = \partial_\mu \int d^4 x' D(x-x') \partial_\sigma h^\sigma{}_\nu(x') + \partial_\nu \int d^4 x' D(x-x') \partial_\kappa h^\kappa{}_\mu(x') \\
& \quad - \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\kappa h^{\sigma\kappa}(x'').
\end{aligned} \tag{A.17}$$

Assuming integration by parts these components obey

$$\partial_\nu h^{T\mu\nu} = 0, \quad \partial_\nu h^{L\mu\nu} = \partial_\nu h^{\mu\nu}. \tag{A.18}$$

With $h_{\mu\nu}^T$ transforming as $h_{\mu\nu}^T \rightarrow h_{\mu\nu}^T$ under $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ as long as we can integrate by parts, we see that, as introduced, $h_{\mu\nu}^T$ is both transverse and gauge invariant.

On evaluation we obtain

$$\begin{aligned}
\frac{1}{2} [\partial_\mu \partial_\nu h^T + \partial_\alpha \partial^\alpha h_{\mu\nu}^T] - \frac{1}{2} \eta_{\mu\nu} \partial_\sigma \partial^\sigma h^T & = \frac{1}{2} [\partial_\mu \partial_\nu h - \partial_\mu \partial_\lambda h^\lambda{}_\nu - \partial_\nu \partial_\lambda h^\lambda{}_\mu \\
& \quad + \partial_\alpha \partial^\alpha h_{\mu\nu}] - \frac{1}{2} \eta_{\mu\nu} [\partial_\alpha \partial^\alpha h - \partial_\sigma \partial_\lambda h^{\sigma\lambda}],
\end{aligned} \tag{A.19}$$

where h^T is given by

$$h^T = \eta^{\alpha\beta} h_{\alpha\beta}^T = h - \partial_\nu \int d^4 x' D(x-x') \partial_\sigma h^{\sigma\nu}(x'), \tag{A.20}$$

with $h = \eta^{\alpha\beta} h_{\alpha\beta}$. On recognizing the right-hand side of (A.19) as $\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R = \delta G_{\mu\nu}$, we obtain

$$\delta G_{\mu\nu} = \frac{1}{2}[\partial_\mu\partial_\nu h^T + \partial_\alpha\partial^\alpha h_{\mu\nu}^T] - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\partial^\sigma h^T. \quad (\text{A.21})$$

We thus write the perturbed Einstein tensor entirely in terms of the non-local, gauge invariant, six degree of freedom $h_{\mu\nu}^T$.

To make contact with the SVT4 expansion we insert

$$h_{\mu\nu} = -2\eta_{\mu\nu}\chi + 2\partial_\mu\partial_\nu F + \partial_\mu F_\nu + \partial_\nu F_\mu + 2F_{\mu\nu} \quad (\text{A.22})$$

into $h_{\mu\nu}^T$, to obtain

$$h_{\mu\nu}^T = -2\eta_{\mu\nu}\chi + 2F_{\mu\nu} + 2\partial_\mu\partial_\nu \int d^4D(x-x')\chi(x'), \quad h^T = -6\chi. \quad (\text{A.23})$$

With $\delta G_{\mu\nu}$ being written in terms of the projected $h_{\mu\nu}^T$, we see that it is written in terms of the SVT4 $F_{\mu\nu}$ and χ . However as written, $h_{\mu\nu}^T$ contains an integral term in (A.23). To eliminate it we extend transverse projection to transverse-traceless projection.

A.4 Transverse-Traceless Projection Operators for Flat Spacetime Tensor Fields

In [43] and [42] two further projectors were introduced

$$\begin{aligned} Q_{\mu\nu\sigma\tau} &= \frac{1}{3} \left[\eta_{\mu\nu} - \partial_\mu\partial_\nu \int d^4x' D(x-x') \right] \left[\eta_{\sigma\tau} - \partial'_\sigma \int d^4x'' D(x'-x'') \partial''_\tau \right], \\ P_{\mu\nu\sigma\tau} &= T_{\mu\nu\sigma\tau} - Q_{\mu\nu\sigma\tau}. \end{aligned} \quad (\text{A.24})$$

They obey the projector algebra

$$\begin{aligned} T_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} &= Q_{\mu\nu\alpha\beta}, \quad Q_{\mu\nu\sigma\tau} T^{\sigma\tau}_{\alpha\beta} = Q_{\mu\nu\alpha\beta}, \quad Q_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} = Q_{\mu\nu\alpha\beta}, \\ P_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} &= 0, \quad Q_{\mu\nu\sigma\tau} P^{\sigma\tau}_{\alpha\beta} = 0, \quad P_{\mu\nu\sigma\tau} P^{\sigma\tau}_{\alpha\beta} = P_{\mu\nu\alpha\beta}. \end{aligned} \quad (\text{A.25})$$

The projector $P_{\mu\nu\sigma\tau}$ projects out the traceless piece of $h_{\mu\nu}^T$, while $Q_{\mu\nu\sigma\tau}$ projects out its complement, and they implement

$$P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^T = h_{\mu\nu}^{T\theta}, \quad Q_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^T = h_{\mu\nu}^T - h_{\mu\nu}^{T\theta}, \quad (\text{A.26})$$

with $h_{\mu\nu}^{T\theta}$ being both traceless and transverse. With $Q_{\mu\nu}{}^{\sigma\tau}$ implementing $Q_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^L = 0$, $P_{\mu\nu}{}^{\sigma\tau}$ implements $P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^L = 0$ as well, to thus implement

$$P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau} = h_{\mu\nu}^{T\theta}. \quad (\text{A.27})$$

$P_{\mu\nu\sigma\tau}$ is thus a traceless projector not just for the transverse $h_{\mu\nu}^T$ but for the full $h_{\mu\nu}$ as well. We can thus introduce its complementary projection operator $U_{\mu\nu\sigma\tau} = \eta_{\mu\sigma}\eta_{\nu\tau} - P_{\mu\nu\sigma\tau}$, as it obeys

$$\begin{aligned} P_{\mu\nu\sigma\tau} U^{\sigma\tau\alpha\beta} &= 0, \quad U_{\mu\nu\sigma\tau} P^{\sigma\tau\alpha\beta} = 0, \quad U_{\mu\nu\sigma\tau} U^{\sigma\tau}_{\alpha\beta} = U_{\mu\nu\alpha\beta}, \\ U_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau} &= h_{\mu\nu} - h_{\mu\nu}^{T\theta} = h_{\mu\nu}^{L\theta} + \frac{1}{3} \eta_{\mu\nu} \eta^{\sigma\tau} h_{\sigma\tau} \\ &\quad - \frac{1}{3} \partial_\mu \partial_\nu \int d^4 y D(x-y) \eta^{\sigma\tau} h_{\sigma\tau}, \end{aligned} \quad (\text{A.28})$$

Given (A.26) and (A.24) we obtain

$$h_{\mu\nu}^{T\theta} = h_{\mu\nu}^T - \frac{1}{3} \eta_{\mu\nu} \eta^{\sigma\kappa} h_{\sigma\kappa}^T + \frac{1}{3} \partial_\mu \partial_\nu \int d^4 y D(x-y) \eta^{\sigma\kappa} h_{\sigma\kappa}^T, \quad (\text{A.29})$$

Inserting (A.23) into (A.29) yields

$$h_{\mu\nu}^{T\theta} = 2F_{\mu\nu}, \quad (\text{A.30})$$

with χ dropping out. Finally, in terms of $h_{\mu\nu}^{T\theta}$ we can rewrite (A.21) as

$$\delta G_{\mu\nu} = \frac{1}{2} \partial_\alpha \partial^\alpha h_{\mu\nu}^{T\theta} - \frac{1}{3} \eta_{\mu\nu} \partial_\sigma \partial^\sigma h^T + \frac{1}{3} \partial_\mu \partial_\nu h^T. \quad (\text{A.31})$$

Then with

$$F_{\mu\nu} = \frac{1}{2} h_{\mu\nu}^{T\theta}, \quad \chi = -\frac{1}{6} h^T, \quad (\text{A.32})$$

we can rewrite (A.31) as

$$\delta G_{\mu\nu} = \partial_\alpha \partial^\alpha F_{\mu\nu} + 2\eta_{\mu\nu} \partial_\alpha \partial^\alpha \chi - 2\partial_\mu \partial_\nu \chi. \quad (\text{A.33})$$

We recognize (A.33) as the expression for $\delta G_{\mu\nu}$ as given in (3.40) when $D = 4$, and with $h_{\mu\nu}^T$ and thus $h_{\mu\nu}^{T\theta}$ and h^T being gauge invariant, we confirm that given integration by parts $F_{\mu\nu}$ and χ are gauge invariant, just as noted in Sec. 3.2. Thus with (A.32) we establish the equivalence of the SVT4 decomposition and the projection operator technique.

As a further example of this equivalence we note that for conformal gravity fluctuations around a flat spacetime background (4.282) takes the form

$$\begin{aligned} \delta W_{\mu\nu} &= \frac{1}{2} \left(\partial_\sigma \partial^\sigma \partial_\tau \partial^\tau K_{\mu\nu} - \partial_\sigma \partial^\sigma \partial_\mu \partial^\alpha K_{\alpha\nu} - \partial_\sigma \partial^\sigma \partial_\nu \partial^\alpha K_{\alpha\mu} \right. \\ &\quad \left. + \frac{2}{3} \partial_\mu \partial_\nu \partial^\alpha \partial^\beta K_{\alpha\beta} + \frac{1}{3} \eta_{\mu\nu} \partial_\sigma \partial^\sigma \partial^\alpha \partial^\beta K_{\alpha\beta} \right), \end{aligned} \quad (\text{A.34})$$

where all derivatives are four-dimensional derivatives with respect to a flat Minkowski metric, and where $K_{\mu\nu}$ is given by $K_{\mu\nu} = h_{\mu\nu} - (1/4)\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}$. Inserting (A.17) and (A.29) into (A.34) yields

$$\delta W_{\mu\nu} = \frac{1}{2}\partial_\sigma\partial^\sigma\partial_\tau\partial^\tau h_{\mu\nu}^{T\theta}. \quad (\text{A.35})$$

With the insertion of (A.22) into (A.34) yielding

$$\delta W_{\mu\nu} = \partial_\sigma\partial^\sigma\partial_\tau\partial^\tau F_{\mu\nu}, \quad (\text{A.36})$$

(cf. (4.284) with $\Omega = 1$), we recover (A.30), and again confirm the equivalence of the SVT4 decomposition and the projection operator technique.

A.5 Transverse and Longitudinal Projection Operators for Curved Spacetime Tensor Fields

For curved spacetime with background metric $g_{\mu\nu}$ it is convenient to define a 2-index propagator

$$[g^\nu_\beta\nabla_\tau\nabla^\tau + \nabla_\beta\nabla^\nu]D^\beta_\sigma(x, x') = g^\nu_\sigma(-g)^{-1/2}\delta^4(x - x'). \quad (\text{A.37})$$

In terms of it we introduce [43]

$$\begin{aligned} T_{\mu\nu\sigma\tau} &= g_{\mu\sigma}g_{\nu\tau} - \nabla_\mu \int d^4x' (-g)^{1/2} D_{\nu\sigma}(x, x') \nabla_\tau \\ &\quad - \nabla_\nu \int d^4x' (-g)^{1/2} D_{\mu\sigma}(x, x') \nabla_\tau, \\ L_{\mu\nu\sigma\tau} &= \nabla_\mu \int d^4x' (-g)^{1/2} D_{\nu\sigma}(x, x') \nabla_\tau + \nabla_\nu \int d^4x' (-g)^{1/2} D_{\mu\sigma}(x, x') \nabla_\tau. \end{aligned} \quad (\text{A.38})$$

These projection operators close on the projector algebra given in (A.16). As such, they effect $T_{\mu\nu\sigma\tau}h^{\sigma\tau} = h_{\mu\nu}^T$, and $L_{\mu\nu\sigma\tau}h^{\sigma\tau} = h_{\mu\nu}^L$, where

$$h_{\mu\nu}^T = h_{\mu\nu} - \nabla_\mu \int d^4x' (-g)^{1/2} D^\nu_\sigma(x, x') \nabla_\tau h^{\sigma\tau} - \nabla_\nu \int d^4x' (-g)^{1/2} D^\mu_\sigma(x, x') \nabla_\tau h^{\sigma\tau}, \quad (\text{A.39})$$

$$h_{\mu\nu}^L = \nabla_\mu \int d^4x' (-g)^{1/2} D^\nu_\sigma(x, x') \nabla_\tau h^{\sigma\tau} + \nabla_\nu \int d^4x' (-g)^{1/2} D^\mu_\sigma(x, x') \nabla_\tau h^{\sigma\tau}. \quad (\text{A.40})$$

The utility of constructing these projected states is that under a gauge transformation $h_{\mu\nu}$ transforms into $h_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu$. However, we see that this is precisely the structure of $h_{\mu\nu}^L$. The longitudinal component of $h_{\mu\nu}$ can thus be removed by a gauge transformation, and the fluctuation Einstein equations can only depend on the 6-component $h_{\mu\nu}^T$. However, unlike the flat background case where one can write $\delta G_{\mu\nu}$ itself entirely in terms of $h_{\mu\nu}^T$, in the curved background case there must be a background $T_{\mu\nu}$, and thus it is only in the full $\delta G_{\mu\nu} + 8\pi G \delta T_{\mu\nu}$ that the metric fluctuations can be described entirely by $h_{\mu\nu}^T$. If we introduce a quantity $\delta T_{\mu\nu}^T$ in which the dependence on ϵ_μ has been excluded (i.e. under a gauge transformation $\delta T_{\mu\nu} \rightarrow \delta T_{\mu\nu}^T$ plus a function of ϵ_μ , and this function of ϵ_μ cancels against an identical function of ϵ_μ in $\delta G_{\mu\nu}$), then following the commuting of some derivatives, the fluctuation equations take the form [43]

$$\begin{aligned} \delta G_{\mu\nu} + 8\pi G \delta T_{\mu\nu} &= \frac{1}{2} [\nabla_\mu \nabla_\nu h^T + R^\sigma{}_\mu h^T_{\sigma\nu} + R^\sigma{}_\nu h^T_{\sigma\mu} - 2R_{\mu\lambda\nu\sigma} h^{T\lambda\sigma} + \nabla_\alpha \nabla^\alpha h^T_{\mu\nu}] \\ &- \frac{1}{2} R^\sigma{}_\sigma h^T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} h^{T\alpha\beta} - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla^\alpha h^T + 8\pi G \delta T_{\mu\nu}^T = 0. \end{aligned} \quad (\text{A.41})$$

The SVT4 fluctuations around a de Sitter background as given in (4.210) to (4.213) and around a general Robertson-Walker background as given in (4.280) are special cases of (A.41), with the only metric fluctuations that appear in (4.213) and (4.280) being $F_{\mu\nu}$ and χ , viz. just the six degrees of freedom associated with $h_{\mu\nu}^T$.

A.6 D-dimensional SVTD Transverse-Traceless Projection Operators for Curved Spacetime Tensor Fields

Rather than generalize the general curved spacetime transverse and longitudinal projection technique to the general transverse-traceless case, we have instead found it more convenient to generalize the SVTD discussion given in Secs. 3.2 and 4.2.2 to general curved spacetime background fluctuations. To this end we take $h_{\mu\nu}$ to be of the form:

$$h_{\mu\nu} = 2F_{\mu\nu} + W_{\mu\nu} + S_{\mu\nu}, \quad (\text{A.42})$$

where

$$\begin{aligned} W_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu} \nabla^\alpha W_\alpha, \\ S_{\mu\nu} &= \frac{1}{D-1} (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.43})$$

with $D(x, x')$ obeying

$$\nabla_\alpha \nabla^\alpha D^{(D)}(x, x') = [-g(x)]^{-1/2} \delta^{(D)}(x - x'). \quad (\text{A.44})$$

From (A.43) we obtain

$$g^{\mu\nu} W_{\mu\nu} = 0, \quad g^{\mu\nu} S_{\mu\nu} = h, \quad (\text{A.45})$$

$$\nabla^\nu h_{\mu\nu} = \nabla^\nu W_{\mu\nu} + \nabla^\nu S_{\mu\nu} \quad (\text{A.46})$$

as the conditions that $F_{\mu\nu}$ be transverse and traceless. From (A.46) we obtain

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \nabla_\alpha \nabla_\nu - \frac{2}{D} \nabla_\nu \nabla_\alpha \right] W^\alpha &= \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1} (\nabla_\nu \nabla_\alpha \nabla^\alpha \\ &\quad - \nabla_\alpha \nabla^\alpha \nabla_\nu) \times \\ &\quad \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.47})$$

and by commuting derivatives can rewrite (A.47) as

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \left(\frac{D-2}{D} \right) \nabla_\nu \nabla_\alpha - R_{\nu\alpha} \right] W^\alpha &= \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1} R_{\nu\alpha} \nabla^\alpha \times \\ &\quad \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'). \end{aligned} \quad (\text{A.48})$$

To solve for W_μ it is convenient to use the bitensor formalism in which we define $G_\alpha^{(D)\beta}(x, x') = e_\alpha^a(x) e_a^\beta(x')$ where the D-dimensional $e_\alpha^a(x)$ vierbeins obey $g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x)$, with a and b referring to a fixed D-dimensional basis. With this bitensor definition $e_\alpha^a(x)$ and $e_a^\beta(x')$ are acting in separate spaces, but at $x = x'$ we obtain $G_\alpha^{(D)\beta}(x, x) = g_\alpha^\beta(x)$. On the introducing the propagator that satisfies

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \left(\frac{D-2}{D} \right) \nabla_\nu \nabla_\alpha - R_{\nu\alpha} \right] D_{(D)}^{\alpha\gamma}(x, x') &= G_\nu^{(D)\gamma}(x, x') [-g(x')]^{-1/2} \times \\ &\quad \delta^{(D)}(x - x'), \end{aligned} \quad (\text{A.49})$$

we can solve for W_μ as

$$W_\mu(x) = \int d^D x' [-g(x')]^{1/2} D_\mu^{(D)\sigma}(x, x') \left[\nabla_{x'}^\rho h_{\sigma\rho}(x') - \frac{1}{D-1} R_{\sigma\rho}(x') \nabla_{x'}^\rho \times \right.$$

$$\int d^D x'' [-g(x'')]^{1/2} D^{(D)}(x', x'') h(x'') \Big]. \quad (\text{A.50})$$

Next we decompose W_μ into transverse and longitudinal components viz.

$$\begin{aligned} W_\mu &= W_\mu^T + W_\mu^L = F_\mu + \nabla_\mu H, \quad \nabla^\mu F_\mu = 0, \\ H &= \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \nabla^\sigma W_\sigma(x'), \end{aligned} \quad (\text{A.51})$$

with $h_{\mu\nu}$ then taking the form

$$\begin{aligned} h_{\mu\nu} &= 2F_{\mu\nu} + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2\nabla_\mu \nabla_\nu H - \frac{2}{D} g_{\mu\nu} \nabla_\alpha \nabla^\alpha H \\ &+ \frac{1}{D-1} (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'). \end{aligned} \quad (\text{A.52})$$

Upon further defining

$$\begin{aligned} F &= H - \frac{1}{2(D-1)} \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \\ \chi &= \frac{1}{D} \nabla_\alpha \nabla^\alpha H - \frac{1}{2(D-1)} \nabla_\alpha \nabla^\alpha \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.53})$$

we may express $h_{\mu\nu}$ in the SVTD form:

$$h_{\mu\nu} = -2g_{\mu\nu} \chi + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}, \quad (\text{A.54})$$

where

$$\begin{aligned} \chi &= \frac{1}{D} \nabla^\sigma W_\sigma - \frac{1}{2(D-1)} h, \\ F_\mu &= W_\mu^T = W_\mu - \nabla_\mu \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \nabla^\sigma W_\sigma(x'), \\ F &= \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \left(\nabla^\sigma W_\sigma(x') - \frac{1}{2(D-1)} h(x') \right), \\ 2F_{\mu\nu} &= h_{\mu\nu} + 2g_{\mu\nu} \chi - 2\nabla_\mu \nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu. \end{aligned} \quad (\text{A.55})$$

We thus generalize the SVTD approach to the arbitrary D-dimensional curved spacetime background.

Appendix B

Conformal to Flat Cosmological Geometries

B.1 Robertson-Walker $k = 0$

In order to apply (5.32) to cosmology we need to write the Robertson-Walker and de Sitter background geometries in a conformal to flat Minkowski form. For a $k = 0$ Robertson-Walker background the comoving coordinate system metric takes the form

$$ds^2(\text{comoving}) = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \quad (\text{B.1})$$

The straightforward introduction of the conformal time

$$d\tau = \int \frac{dt}{a(t)} \quad (\text{B.2})$$

then allows us to write the conformal time metric as

$$ds^2(\text{conformal time}) = a^2(\tau)[d\tau^2 - dx^2 - dy^2 - dz^2]. \quad (\text{B.3})$$

B.2 Robertson-Walker $k > 0$

For a $k > 0$ or a $k < 0$ Robertson-Walker background the comoving and conformal time coordinate system metrics take the form

$$\begin{aligned} ds^2(\text{comoving}) &= dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \\ ds^2(\text{conformal time}) &= a^2(\tau) \left[d\tau^2 - \frac{dr^2}{1 - kr^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \right]. \end{aligned} \quad (\text{B.4})$$

To bring the RW geometries with non-zero k to a conformal to flat form requires coordinate transformations that involve both τ and r . For the $k > 0$ case first, it is convenient to set $k = 1/L^2$, and introduce $\sin \chi = r/L$, with the conformal time metric given in (B.4) then taking the form

$$ds^2 = L^2 a^2(p) [dp^2 - d\chi^2 - \sin^2 \chi d\theta^2 - \sin^2 \chi \sin^2 \theta d\phi^2], \quad (\text{B.5})$$

where $p = \tau/L$. Following e.g. [11] we introduce

$$\begin{aligned} p' + r' &= \tan[(p + \chi)/2], & p' - r' &= \tan[(p - \chi)/2], \\ p' &= \frac{\sin p}{\cos p + \cos \chi}, & r' &= \frac{\sin \chi}{\cos p + \cos \chi}, \end{aligned} \quad (\text{B.6})$$

so that

$$dp'^2 - dr'^2 = \frac{1}{4}[dp^2 - d\chi^2] \sec^2[(p + \chi)/2] \sec^2[(p - \chi)/2], \quad (\text{B.7})$$

$$\begin{aligned} \frac{1}{4}(\cos p + \cos \chi)^2 &= \cos^2[(p + \chi)/2] \cos^2[(p - \chi)/2] \\ &= \frac{1}{[1 + (p' + r')^2][1 + (p' - r')^2]}. \end{aligned} \quad (\text{B.8})$$

With these transformations the $k > 0$ line element then takes the conformal to flat form

$$ds^2 = \frac{4L^2 a^2(p)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [dp'^2 - dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2 \theta d\phi^2]. \quad (\text{B.9})$$

To bring the spatial sector of (B.9) to Cartesian coordinates we set $x' = r' \sin \theta \cos \phi$, $y' = r' \sin \theta \sin \phi$, $z' = r' \cos \theta$ and thus bring the line element to the form

$$ds^2 = L^2 a^2(p) (\cos p + \cos \chi)^2 [dp'^2 - dx'^2 - dy'^2 - dz'^2], \quad (\text{B.10})$$

where now $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. With these transformations (B.10) is now in the form given in (2.23).

B.3 Robertson-Walker $k < 0$

For the $k < 0$ case, it is convenient to set $k = -1/L^2$, and introduce $\sinh \chi = r/L$, with the conformal time metric given in (B.4) then taking the form

$$ds^2 = L^2 a^2(p) [dp^2 - d\chi^2 - \sinh^2 \chi d\theta^2 - \sinh^2 \chi \sin^2 \theta d\phi^2], \quad (\text{B.11})$$

where $p = \tau/L$. Next we introduce

$$\begin{aligned} p' + r' &= \tanh[(p + \chi)/2], & p' - r' &= \tanh[(p - \chi)/2], \\ p' &= \frac{\sinh p}{\cosh p + \cosh \chi}, & r' &= \frac{\sinh \chi}{\cosh p + \cosh \chi}, \end{aligned} \quad (\text{B.12})$$

so that

$$dp'^2 - dr'^2 = \frac{1}{4}[dp^2 - d\chi^2] \text{sech}^2[(p + \chi)/2] \text{sech}^2[(p - \chi)/2],$$

$$\begin{aligned}
\frac{1}{4}(\cosh p + \cosh \chi)^2 &= \cosh^2[(p + \chi)/2] \cosh^2[(p - \chi)/2] \\
&= \frac{1}{[1 - (p' + r')^2][1 - (p' - r')^2]}.
\end{aligned} \tag{B.13}$$

With these transformations the line element takes the conformal to flat form

$$ds^2 = \frac{4L^2 a^2(p)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [dp'^2 - dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2 \theta d\phi^2]. \tag{B.14}$$

The spatial sector can then be written in Cartesian form

$$ds^2 = L^2 a^2(p) (\cosh p + \cosh \chi)^2 [dp'^2 - dx'^2 - dy'^2 - dz'^2], \tag{B.15}$$

where again $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. We note that in transforming from (B.4) to (B.10) or to (B.15) we have only made coordinate transformations and not made any conformal transformation.

B.4 dS_4 and AdS_4 Background Solutions

While the conformal to flat Minkowski structures given in (B.3), (B.10) and (B.15) are purely kinematical, the explicit form of $a(t)$ can be determined once a dynamics has been specified. Thus in regard to a de Sitter or anti-de Sitter cosmology, a de Sitter or an anti-de Sitter geometry is just a particular case of a Robertson-Walker geometry in which $a(t)$ has a specific assigned value for each possible choice of spatial 3-curvature k . On writing the maximally 4-symmetric geometry condition $R_{\mu\nu} = -3\alpha g_{\mu\nu}$ in Robertson-Walker form one obtains

$$\dot{a}^2(t) + k = \alpha a^2(t). \tag{B.16}$$

(In terms of the scalar field model described in (2.48) – (2.52) we have $K = \alpha = -2\lambda_S S_0^2$.) Here α is positive for de Sitter and negative for anti-de Sitter. Allowable solutions to (B.16) depend on the values of α and k , and are of the form (see e.g. [10])

$$\begin{aligned}
a(t, \alpha > 0, k < 0) &= \left(-\frac{k}{\alpha}\right)^{1/2} \sinh(\alpha^{1/2}t), \\
a(t, \alpha > 0, k = 0) &= a(t=0) \exp(\alpha^{1/2}t), \\
a(t, \alpha > 0, k > 0) &= \left(\frac{k}{\alpha}\right)^{1/2} \cosh(\alpha^{1/2}t), \\
a(t, \alpha = 0, k < 0) &= (-k)^{1/2}t, \\
a(t, \alpha < 0, k < 0) &= \left(\frac{k}{\alpha}\right)^{1/2} \sin((-\alpha)^{1/2}t).
\end{aligned} \tag{B.17}$$

In these solutions (B.3), (B.10), and (B.15) all apply to a de Sitter or an anti-de Sitter cosmology.

B.5 dS_4 and AdS_4 Background Solutions - Radiation Era

For Robertson-Walker cosmologies we note that with slight modification we can extend the scalar field model given above to include a perfect fluid, with the energy-momentum tensor then being given by [35]

$$T_S^{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} - \frac{1}{6}S_0^2 \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha \right) - g^{\mu\nu}\lambda_S S_0^4, \quad (\text{B.18})$$

with the background conformal cosmology still obeying $T_S^{\mu\nu} = 0$ since the background Robertson-Walker geometry continues to obey $W_{\mu\nu} = 0$. On taking the perfect fluid energy-momentum tensor to be traceless radiation (viz. $\rho = 3p$, $\rho = A/a^4(t)$, $A > 0$) as needed in the early universe, and with $\alpha = -2\lambda_S S_0^2$ as before, the evolution equation takes the form

$$\dot{a}^2 + k = \alpha a^2 - \frac{2A}{S_0^2 a^2}, \quad (\text{B.19})$$

with allowed solutions to the cosmology being given by [35]

$$\begin{aligned} a(t, \alpha > 0, k < 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} - \frac{k\beta}{\alpha} \sinh^2(\alpha^{1/2}t) \right)^{1/2}, \\ a(t, \alpha > 0, k = 0, A > 0) &= \left(-\frac{A}{\lambda_S S_0^4} \right)^{1/4} \cosh^{1/2}(2\alpha^{1/2}t), \\ a(t, \alpha > 0, k > 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} + \frac{k\beta}{\alpha} \cosh^2(\alpha^{1/2}t) \right)^{1/2}, \\ a(t, \alpha = 0, k < 0, A > 0) &= \left(-\frac{2A}{kS_0^2} - kt^2 \right)^{1/2}, \\ a(t, \alpha < 0, k < 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} + \frac{k\beta}{\alpha} \sin^2((-\alpha)^{1/2}t) \right)^{1/2}, \end{aligned} \quad (\text{B.20})$$

where $\beta = (1 + 8A\alpha/k^2 S_0^2)^{1/2}$.