Special Gauge Matthew v8

Setup

Metric decomposed to first order:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}). \tag{1}$$

We then split $h_{\mu\nu}$ into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \tag{2}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. We impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega. \tag{3}$$

With $J = -3$, the trace for $\Omega(\tau) = \frac{1}{H\tau}$ becomes

$$\eta^{\mu\nu}\delta G_{\mu\nu} = (-4R\tau^{-2} - \tfrac{3}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}\tau^{-1}\partial_0 + 3P\tau^{-1}\partial_0 + R\tau^{-1}\partial_0)h \tag{4}$$

With $J = -4$, the trace for $\Omega(\tau) = \frac{1}{H\tau}$ becomes

$$\eta^{\mu\nu}\delta G_{\mu\nu} = (-3R\tau^{-2} - \tfrac{3}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}\tau^{-1}\partial_0 + 2P\tau^{-1}\partial_0 + R\tau^{-1}\partial_0)h \tag{5}$$

We seek to see if it is possible to find coefficients of P and R such that this reduces to the box operator onto a factor of $\Omega(\tau)$. There are two possible forms

$$C\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h) = C(\eta^{\mu\nu}\partial_\mu\partial_\nu - 6\tau^{-2} + 4\partial_0\tau^{-1})h \tag{6}$$

and

$$C\Omega^{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^{-2} h) = C(\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\tau^{-2} - 4\partial_0\tau^{-1})h \tag{7}$$

where C is just an overall coefficient.

$J = -3$, $\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h)$

Matching coefficients from (4) with (6) we find

$$C = -\frac{3}{4} + P \tag{8}$$

$$4C = -\frac{3}{2} + 3P + R \tag{9}$$

$$-6C = -4R. \tag{10}$$

These three linearly independent equations will uniquely specify C, P and R. Their solution is

$$C = -\frac{3}{2}, \qquad J = -3, \qquad P = -\frac{3}{4}, \qquad R = -\frac{9}{4}. \tag{11}$$

With these constants, the fluctuation equations become:

$$\eta^{\mu\nu}\delta G_{\mu\nu} = -\frac{3}{2}\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h) \tag{12}$$

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (\tfrac{5}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \partial_0\partial_0)h. \tag{13}$$

$$\delta G_{01} = \tfrac{1}{2}\tau^{-1}\partial_1 K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{01} + (-\tfrac{7}{8}\tau^{-1}\partial_1 + \partial_1\partial_0)h. \tag{14}$$

$$\begin{aligned}\delta G_{11} = {}& \tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{9}{4}\tau^{-2} - \tfrac{5}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{7}{4}\tau^{-1}\partial_0 \\ & + \partial_1\partial_1)h.\end{aligned} \tag{15}$$

$$\delta G_{12} = \tfrac{1}{2}\tau^{-1}\partial_2 K_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} + \partial_2\partial_1 h. \tag{16}$$

$J = -3$, $\Omega^2\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^{-2}h)$

Matching coefficients from (4) with (7) we find

$$C = -\frac{3}{4} + P \tag{17}$$

$$-4C = -\frac{3}{2} + 3P + R \tag{18}$$

$$-2C = -4R. \tag{19}$$

Their solution is

$$C = -\frac{1}{10}, \qquad J = -3, \qquad P = \frac{13}{20}, \qquad R = -\frac{1}{20}. \tag{20}$$

$$\eta^{\mu\nu}\delta G_{\mu\nu} = -\frac{1}{10}\Omega^2\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^{-2}h) \tag{21}$$

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (-\tfrac{3}{40}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{2}{5}\tau^{-1}\partial_0 - \tfrac{2}{5}\partial_0\partial_0)h. \tag{22}$$

$$\delta G_{01} = \tfrac{1}{2}\tau^{-1}\partial_1 K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{01} + (\tfrac{9}{40}\tau^{-1}\partial_1 - \tfrac{2}{5}\partial_1\partial_0)h. \tag{23}$$

$$\begin{aligned}\delta G_{11} = {}& \tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{20}\tau^{-2} + \tfrac{3}{40}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{20}\tau^{-1}\partial_0 \\ & - \tfrac{2}{5}\partial_1\partial_1)h.\end{aligned} \tag{24}$$

$$\delta G_{12} = \tfrac{1}{2}\tau^{-1}\partial_2 K_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} - \tfrac{2}{5}\partial_2\partial_1 h. \tag{25}$$

$J=-4$, $\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h)$

Matching coefficients from (5) with (6) we find

$$C = -\frac{3}{4} + P \tag{26}$$
$$4C = -\frac{3}{2} + 2P + R \tag{27}$$
$$-6C = -3R. \tag{28}$$

Since two equations are linearly dependent, their solution is

$$R = 2C, \qquad P = \frac{3}{4} + C, \qquad \to \qquad R = 2P - \frac{3}{2}. \tag{29}$$

Hence we may vary P such that the equations simplify the most (note that we could also have $C = 0$ which is explored below). For $R = 2P - \frac{3}{2}$ the fluctuation equations take the form

$$\eta^{\mu\nu}\delta G_{\mu\nu} = (P - \tfrac{3}{4})\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h) \tag{30}$$

$$\begin{aligned}\delta G_{00} = {} & (-2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-1}\partial_0)K_{00} + (\tfrac{3}{4}\tau^{-2} - P\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & + P\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_0\partial_0 - P\partial_0\partial_0)h.\end{aligned} \tag{31}$$

$$\begin{aligned}\delta G_{01} = {} & \tau^{-1}\partial_1 K_{00} + (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{01} + (-\tfrac{1}{2}\tau^{-1}\partial_1 + P\tau^{-1}\partial_1 + \tfrac{1}{4}\partial_1\partial_0 \\ & - P\partial_1\partial_0)h.\end{aligned} \tag{32}$$

$$\begin{aligned}\delta G_{11} = {} & \tau^{-2}K_{00} + 2\tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{3}{4}\tau^{-2} - P\tau^{-2} \\ & - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tau^{-1}\partial_0 + P\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_1\partial_1 - P\partial_1\partial_1)h.\end{aligned} \tag{33}$$

$$\delta G_{12} = \tau^{-1}\partial_2 K_{01} + \tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} + (\tfrac{1}{4}\partial_2\partial_1 - P\partial_2\partial_1)h. \tag{34}$$

Possible choices that allow simplification are $P \in (0, 1, \frac{1}{4}, \frac{1}{2}, \frac{3}{4})$. For $P = \frac{1}{4}$ the fluctuation equations reduce to

$$\eta^{\mu\nu}\delta G_{\mu\nu} = -\frac{1}{2}\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 h) \tag{35}$$

$$\delta G_{00} = (-2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-1}\partial_0)K_{00} + (\tfrac{1}{2}\tau^{-2} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{4}\tau^{-1}\partial_0)h. \tag{36}$$

$$\delta G_{01} = \tau^{-1}\partial_1 K_{00} + (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{01} - \tfrac{1}{4}\tau^{-1}\partial_1 h. \tag{37}$$

$$\begin{aligned}\delta G_{11} = {} & \tau^{-2}K_{00} + 2\tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{2}\tau^{-2} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{3}{4}\tau^{-1}\partial_0)h.\end{aligned} \tag{38}$$

$$\delta G_{12} = \tau^{-1}\partial_2 K_{01} + \tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12}. \tag{39}$$

$$J = -4, \quad \Omega^2 \eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^{-2}h)$$

Matching coefficients from (5) with (7) we find

$$C = -\frac{3}{4} + P \tag{40}$$

$$-4C = -\frac{3}{2} + 2P + R \tag{41}$$

$$-2C = -3R. \tag{42}$$

Here there is no solution for $C \neq 0$. However, if we do take

$$C = 0, \qquad J = -3, \qquad P = \frac{3}{4}, \qquad R = 0, \tag{43}$$

then we have a very simple equation for the trace, i.e.

$$\eta^{\mu\nu}\delta G_{\mu\nu} = \frac{3}{4\tau}\partial_0 h \tag{44}$$

and thus

$$h = \frac{4}{3}\int d\tau \ \tau(\eta^{\mu\nu}\delta G_{\mu\nu}). \tag{45}$$

The rest of the fluctuation equations take the form

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (-\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{8}\tau^{-1}\partial_0 - \tfrac{1}{2}\partial_0\partial_0)h. \tag{46}$$

$$\delta G_{01} = \tfrac{1}{2}\tau^{-1}\partial_1 K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{01} + (\tfrac{1}{4}\tau^{-1}\partial_1 - \tfrac{1}{2}\partial_1\partial_0)h. \tag{47}$$

$$\delta G_{11} = \tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{8}\tau^{-1}\partial_0 - \tfrac{1}{2}\partial_1\partial_1)h. \tag{48}$$

$$\delta G_{12} = \tfrac{1}{2}\tau^{-1}\partial_2 K_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} - \tfrac{1}{2}\partial_2\partial_1 h. \tag{49}$$