

Quantum Mechanics III

HW 10

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- 7.4 Consider the two-level atom embedded in the big environment of the quantized electromagnetic field along the lines of section 7.5 using the interaction Hamiltonian (7.59). Show that the relaxation terms of the form (5.6) ensue. There will also be a level shift called *Lamb Shift*, but at the present stage of the modeling it is infinite and you should ignore it. In a full analysis the Lamb shift is “renormalized” so that it becomes finite.

The atom-field interaction Hamiltonian is

$$H_{AR} = - \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \mathbf{D} \cdot \hat{\mathbf{e}}_\lambda(\mathbf{k}) (|2\rangle \langle 1| a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}^\dagger |1\rangle \langle 2|).$$

We will work with a combined index n and shorten the notation to

$$H_{AR} = \sum_n G_n (|2\rangle \langle 1| a_n + a_n^\dagger |1\rangle \langle 2|)$$

where

$$G_n(\mathbf{D}, \lambda, \mathbf{k}) = - \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \mathbf{D} \cdot \hat{\mathbf{e}}_\lambda(\mathbf{k}).$$

Converting to the interaction picture with respect to H_0

$$H_0 = H_A + H_E = \hbar\omega_1 |1\rangle \langle 1| + \hbar\omega_2 |2\rangle \langle 2| + \sum_n \hbar\omega_n a_n^\dagger a_n$$

we have for the interaction perturbation

$$H_{AR}^I = e^{iH_0 t/\hbar} H_{AR} e^{-iH_0 t/\hbar}.$$

To compute this, first take the result from the atomic hamiltonian

$$e^{iH_A t/\hbar} H_{AR} e^{-iH_A t/\hbar} = \sum_n G_n (e^{i(\omega_2 - \omega_1)t} |2\rangle \langle 1| a_n + e^{-i(\omega_2 - \omega_1)t} |1\rangle \langle 2| a_n^\dagger).$$

Now we need to take this result between the exponentials of the environment

$$e^{iH_E} \left[\sum_n G_n (e^{i(\omega_2 - \omega_1)t} |2\rangle \langle 1| a_n + e^{-i(\omega_2 - \omega_1)t} |1\rangle \langle 2| a_n^\dagger) \right] e^{-iH_E}.$$

Some steps necessary to compute this:

$$[a_i, n_j] = \delta_{ij} a_k, \quad [a_i^\dagger, n_j] = -\delta_{ij} a_i^\dagger$$

BCH formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]] + \dots$$

$$e^{iH_E t/\hbar} a_k e^{-iH_E t/\hbar} = e^{-i\omega_k t} a_k, \quad e^{iH_E t/\hbar} a_k^\dagger e^{-iH_E t/\hbar} = e^{i\omega_k t} a_k^\dagger.$$

All together then we have the interaction potential

$$H_{AR}^I = \sum_n G_n (e^{i(\omega_{21}-\omega_n)t} |2\rangle \langle 1| a_n + e^{-i(\omega_{21}-\omega_n)t} |1\rangle \langle 2| a_n^\dagger).$$

The time dependence of the density operator $\rho(t)$ corresponding to the combined system may be written out as an perturbative expansion of the Liouville-von Neumann equation

$$\rho(t + \delta t) = \rho(t) - \frac{i}{\hbar} \int_t^{t+\delta t} dt_1 [V(t_1), \rho(t)] + \left(\frac{-i}{\hbar}\right)^2 \int_t^{t+\delta t} dt_1 \int_t^{t_1} dt_2 [V(t_1), [V(t_2), \rho(t)]] + \dots \quad (1)$$

We assume the density operator starts in the vacuum state of the environment

$$\rho(t) = \rho_A(t) \otimes |0\rangle \langle 0|_E$$

and we choose to trace away the environment to get the time evolution of the system. With this we compute the trace of each term in (1) (denoting $\rho_A(t) \equiv \rho(t)$). The first order term vanishes

$$\text{Tr}_E([V(t_1), \rho(t)]) = 0$$

The second order term can be split into four parts. One part is

$$\begin{aligned} \text{Tr}_E(V(t_1)V(t_2)\rho(t)) &= \sum_{nn'} G_n G_{n'} \left(\langle 0| \left[e^{i(\omega_{21}-\omega_n)t_1} |2\rangle \langle 1| a_n + e^{-i(\omega_{21}-\omega_n)t_1} |1\rangle \langle 2| a_n^\dagger \right] \right. \\ &\quad \times \left. \left[e^{i(\omega_{21}-\omega_{n'})t_2} |2\rangle \langle 1| a_{n'}' + e^{-i(\omega_{21}-\omega_{n'})t_2} |1\rangle \langle 2| a_{n'}'^\dagger \right] |0\rangle \right) \rho(t) \end{aligned}$$

The annihilation operators annihilate the vacuum and only $n = n'$ terms survive. We must also note the orthogonality of the atomic states. We are left with a trace of

$$\text{Tr}_E(V(t_1)V(t_2)\rho(t)) = \sum_n G_n^2 e^{i(\omega_{21}-\omega_n)(t_1-t_2)} |2\rangle \langle 2| \rho(t)$$

The remaining three terms of the commutator from the perturbation are evaluated similarly. At this point, we note that our results take a similar form as that in the script. The result of the trace of the integration of each term is

$$\begin{aligned} \left(\frac{-i}{\hbar}\right)^2 \text{Tr}_E \left[\int_t^{t+\delta t} dt_1 \int_t^{t_1} dt_2 V(t_1)V(t_2)\rho(t) \right] &\approx \delta t [-(\gamma - i\delta) |2\rangle \langle 2| \rho(t)] \\ \left(\frac{-i}{\hbar}\right)^2 \text{Tr}_E \left[\int_t^{t+\delta t} dt_1 \int_t^{t_1} dt_2 \rho(t)V(t_2)V(t_1) \right] &\approx \delta t [-(\gamma + i\delta)\rho(t) |2\rangle \langle 2|] \\ \left(\frac{-i}{\hbar}\right)^2 \text{Tr}_E \left[\int_t^{t+\delta t} dt_1 \int_t^{t_1} dt_2 V(t_1)\rho(t)V(t_2) \right] &\approx \delta t [(\gamma - i\delta) |1\rangle \langle 2| \rho(t) |2\rangle \langle 1|] \\ \left(\frac{-i}{\hbar}\right)^2 \text{Tr}_E \left[\int_t^{t+\delta t} dt_1 \int_t^{t_1} dt_2 V(t_2)\rho(t)V(t_1) \right] &\approx \delta t [(\gamma + i\delta) |1\rangle \langle 2| \rho(t) |2\rangle \langle 1|] \end{aligned}$$

where γ and δ follow the definitions in the text.

Summing these results

$$\rho(t + \delta t) = \rho(t) + \delta t \left(-(\gamma - i\delta) |2\rangle \langle 2| \rho(t) - (\gamma + i\delta)\rho(t) |2\rangle \langle 2| + 2\gamma |1\rangle \langle 2| \rho(t) |2\rangle \langle 1| \right)$$

which may be brought to the form

$$\dot{\rho} = i\delta[|2\rangle\langle 2|, \rho] + \gamma(2|1\rangle\langle 2|\rho|2\rangle\langle 1| - |2\rangle\langle 2|\rho - \rho|2\rangle\langle 2|).$$

We have the Liouville-von Neumann term for a modified Hamiltonian and the relaxation terms in the Lindblad form. Taking expectation values between the two atomic states, we may easily confirm eq. (5.9) for $\Gamma = 2\gamma$.

$$\dot{\rho}_{22} = -\Gamma\rho_{22}, \quad \dot{\rho}_{11} = \Gamma\rho_{22}, \quad \dot{\rho}_{21} = -\frac{1}{2}\Gamma\rho_{21}, \quad \dot{\rho}_{12} = -\frac{1}{2}\Gamma\rho_{12}.$$

For some reason I find that only the state $|2\rangle\langle 1|\rho(t)|1\rangle\langle 2|a_na_n^\dagger$ is nonvanishing in the interaction term. But this doesn't follow the proper Lindblad form of $|1\rangle\langle 2|\rho(t)|2\rangle\langle 1|$ for a two level system..

- 7.6 By attempting to construct an eigenstate of the boson creation operator a^\dagger in the Fock state basis, show that it does not have any eigenstates at all.

Eigenvalue equation for a^\dagger :

$$a^\dagger |\psi\rangle = \lambda |\psi\rangle.$$

Expand $|\psi\rangle$ in a basis of single mode Fock states $|n\rangle$ (we may expand over all modes and all particle numbers, but $a_{\mathbf{k}}^\dagger$ acting on any other momentum of $\mathbf{k} \neq \mathbf{k}'$ vanishes, so we restrict our space to a single mode)

$$\begin{aligned} a^\dagger |\psi\rangle &= \lambda |\psi\rangle \\ a^\dagger \sum_{n=0}^{\infty} c_n |n\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\ \sum_{n=0}^{\infty} \sqrt{n+1} c_n |n+1\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\ \sum_{n=1}^{\infty} \sqrt{n} c_{n-1} |n\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\ \sum_{n=1}^{\infty} \sqrt{n} c_{n-1} |n\rangle &= \lambda c_0 |0\rangle + \lambda \sum_{n=1}^{\infty} \sqrt{n} c_n |n\rangle. \end{aligned}$$

This leads to

$$\lambda c_0 |0\rangle + \sum_{n=1}^{\infty} (\lambda c_n - \sqrt{n} c_{n-1}) |n\rangle = 0$$

As the states are orthogonal, each coefficient must vanish independently

$$\begin{aligned} \lambda c_0 &= 0 \\ \lambda c_1 &= \sqrt{1} c_0 \\ \lambda c_2 &= \sqrt{2} c_1 \\ &\dots \\ \lambda c_n &= \sqrt{n} c_{n-1} \end{aligned}$$

For a nonzero eigenvalue, $\lambda \neq 0$, we must have $c_0 = 0$. However, through the recursion it then follows that all $c_n = 0$. Thus a non-zero eigenstate of the creation operator cannot exist.

7.7 Consider a beam splitter that combines two incoming photon modes a_1 and a_2 into two new photon modes A_1 and A_2 .

- (a) Assume that the transformation between the modes is linear, so that $A_1 = u_{11}a_1 + u_{12}a_2$ and $A_2 = u_{21}a_1 + u_{22}a_2$. Show that the matrix with the element u_{ij} must be unitary.

Now consider an ideal 50/50 beam splitter, for which the (unitary) transformation is $A_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2$, $A_2 = -\frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2$.

- (b) Suppose that we put one photon in each incoming mode a_1 and a_2 . Show that at the output both photons will be observed to come out from the same port, either in the mode A_1 or A_2 .

Hint: A state with one photon in the modes a_1 and a_2 may be written $a_1^\dagger a_2^\dagger |0\rangle$.

- (a) Recall the commutation relations for boson modes

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.$$

This also applies to the A modes. Using the linear composition of A modes in terms of a modes, we equate them to the commutator

$$\begin{aligned} [A_1, A_2^\dagger] &= [u_{11}a_1 + u_{12}a_2, u_{21}^*a_1^\dagger + u_{22}^*a_2^\dagger] \\ &= u_{11}u_{21}^*[a_1, a_1^\dagger] + u_{12}u_{22}^*[a_2, a_2^\dagger] \\ &= u_{11}u_{21}^* + u_{12}u_{22}^* \\ &= 0. \end{aligned}$$

Thus

$$u_{11}u_{21}^* = -u_{12}u_{22}^* \quad (2)$$

and it follows

$$u_{11}^*u_{21} = -u_{12}^*u_{22}. \quad (3)$$

Repeat the same procedure for A_1 and A_2 , yielding

$$[A_1, A_1^\dagger] \Rightarrow |u_{11}|^2 + |u_{12}|^2 = 1 \quad (4)$$

$$[A_2, A_2^\dagger] \Rightarrow |u_{21}|^2 + |u_{22}|^2 = 1 \quad (5)$$

Now, a 2x2 unitary matrix must satisfy

$$U^\dagger = U^{-1}$$

$$\begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix} = \frac{1}{u_{11}u_{22} - u_{12}u_{21}} \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}$$

Now we relate all the elements using (2), (3), (4), and (5).

$$\begin{aligned} u_{22}|u_{11}|^2 + u_{22}|u_{12}|^2 &= u_{22} \\ u_{22}|u_{11}|^2 - u_{21}u_{11}^*u_{12} &= u_{22} \\ \Rightarrow u_{11}^* &= \frac{u_{22}}{u_{22}u_{11} - u_{21}u_{12}} \\ \Rightarrow (U^\dagger)_{11} &= (U^{-1})_{11} \end{aligned}$$

$$\begin{aligned}
& -u_{12}|u_{22}|^2 - u_{12}|u_{21}|^2 = -u_{12} \\
& -u_{12}|u_{21}|^2 + u_{11}u_{21}^*u_{22} = -u_{12} \\
& \Rightarrow u_{21}^* = -\frac{u_{12}}{u_{22}u_{11} - u_{21}u_{12}} \\
& \Rightarrow (U^\dagger)_{21} = (U^{-1})_{21} \\
& u_{12}^*u_{11}u_{22} - |u_{11}|^2u_{21} = -u_{21} \\
& \Rightarrow u_{12}^* = -\frac{u_{21}}{u_{22}u_{11} - u_{21}u_{12}} \\
& \Rightarrow (U^\dagger)_{12} = (U^{-1})_{12} \\
& -u_{22}^*u_{12}u_{21} + |u_{21}|^2u_{11} = -u_{21} \\
& \Rightarrow u_{22}^* = \frac{u_{11}}{u_{22}u_{11} - u_{21}u_{12}} \\
& \Rightarrow (U^\dagger)_{22} = (U^{-1})_{22}.
\end{aligned}$$

Thus we establish

$$U^\dagger = U^{-1}.$$

(b) we have for the state $a_1^\dagger a_2^\dagger |0\rangle = |\psi\rangle$

$$\begin{aligned}
A_1 &= \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 = \alpha_1 a_1 + \beta a_2 \\
A_2 &= -\frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 = \alpha_2 a_1 + \beta a_2
\end{aligned}$$

The measurement probability for sequence $A_i A_j$ in the state $|11\rangle = a_1^\dagger a_2^\dagger |0\rangle$ is

$$\begin{aligned}
P(A_i A_j) &= \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger A_i A_j a_1^\dagger a_2^\dagger | 0 \rangle \\
&= \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger [A_i A_j, a_1^\dagger a_2^\dagger] | 0 \rangle \\
&= \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger A_i [A_j, a_1^\dagger a_2^\dagger] | 0 \rangle \\
&= \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger A_i (a_1^\dagger [A_j, a_2^\dagger] + [A_j, a_1^\dagger] a_2^\dagger) | 0 \rangle
\end{aligned}$$

The results above follow from annihilation of the state from annihilation operators. Looking at the last two commutators

$$\begin{aligned}
[A_i, a_1^\dagger] &= [\alpha_i a_1, a_1^\dagger] + [\beta a_2, a_1^\dagger] \\
&= \alpha_i
\end{aligned}$$

and likewise

$$[A_i, a_2^\dagger] = \beta.$$

Thus

$$P(A_i A_j) = \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger \beta (\alpha_i a_1 a_1^\dagger + \alpha_j a_2 a_2^\dagger) | 0 \rangle$$

$$= \frac{1}{2} \langle 0 | a_2 a_1 A_j^\dagger A_i^\dagger \beta (\alpha_i + \alpha_j) | 0 \rangle$$

Now noting from the earlier assignment that

$$\alpha_1 + \alpha_2 = 0, \quad 2\alpha_1 = 2\beta, \quad 2\alpha_2 = -2\beta$$

we see that unless $i = j$, the probability $P(A_i A_j) = 0$. Therefore, at the output both photons will be observed to come from the same port, either in the mode A_1 or A_2 .