Lecture 6

02/10/2016
(Jackson, Chapter 9)

recap: the vector-potential $\vec{A}(\vec{r},t)$ in the radiation zone (harmonic sources of the EM field)

$$\begin{cases} \vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r})\, e^{-i\omega t} \\ \rho(\vec{r},t) = \rho_\omega(\vec{r})\, e^{-i\omega t} \end{cases} \quad r \gg \lambda, r'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int \vec{J}_\omega(\vec{r}')\, e^{-i\vec{k}\vec{r}'} d^3r'$$

Case: $kr' = 2\pi \frac{r'}{\lambda} \ll 1$ $(\ell = 0)$ $\Rightarrow$ $\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int \vec{J}_\omega(\vec{r}')\, d^3r'}$

Electric Dipole Radiation

We can evaluate the integral: $\boxed{\int \vec{J}_\omega(\vec{r}')\, d^3r'}$

Density of the electric charge for the system of point charges: $\boxed{\rho(\vec{r}) = \sum_i q_i\, \delta(\vec{r} - \vec{r}_i)}$ and $\boxed{\vec{J}(\vec{r}) = \sum_i q_i \vec{v}_i\, \delta(\vec{r} - \vec{r}_i)}$

$\boxed{\vec{r}_i = \vec{r}_i(t)}$ where $\boxed{\vec{v}_i = \frac{d\vec{r}_i}{dt}}$

$$\int \vec{J}(\vec{r}')\, d^3r' = \sum_i q_i \int \vec{v}_i\, \delta(\vec{r}' - \vec{r}_i(t))\, d^3r' = \sum_i q_i \vec{v}_i\,;$$

$$\int \vec{J}(\vec{r}')\, d^3r' = \sum_i q_i \frac{d}{dt}\vec{r}_i(t) = \frac{d}{dt}\underbrace{\left(\sum_i q_i \vec{r}_i(t)\right)}_{\vec{p}}$$

$$\boxed{\int \vec{J}(\vec{r},t)\, d^3\vec{r} = \frac{d}{dt}\vec{p}(t)}$$

The dipole moment: $\boxed{\vec{p} = \sum q_i \vec{r}_i}$

For the harmonic motion:

$$\int \vec{J}(\vec{r},t)\, d^3\vec{r} = e^{-i\omega t} \int \vec{J}_\omega(\vec{r})\, d^3r' = \frac{d}{dt}\left(\vec{p}_\omega e^{-i\omega t}\right)$$

(Continuous charge $\vec{p} = \int \rho(\vec{r})\, \vec{r}\, d^3r$)

$$\Rightarrow \boxed{\int \vec{J}_\omega(\vec{r}')\, d^3r' = -i\omega \vec{p}_\omega}$$

$$\boxed{\begin{aligned} &\frac{d}{dt}\vec{p}(t) = -i\omega \vec{p}_\omega \\ &\vec{p}(t) = \vec{p}_\omega e^{-i\omega t} \end{aligned}}$$

Expressions for the $\vec{A}(\vec{r},t)$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \dot{\vec{p}}(t)}$$

or

$$\boxed{\vec{A}(\vec{r},t) = -\frac{i\omega\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \vec{p}_\omega}$$

$$\boxed{\vec{A}(\vec{r},t) = e^{-i\omega t} \vec{A}_\omega(\vec{r})}$$

where

$$\boxed{\vec{A}_\omega(\vec{r}) = -\frac{i\omega\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{p}_\omega}$$

Magnetic field $\vec{B}$ in the radiation zone

$(r \gg d, \lambda)$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{i\omega\mu_0}{4\pi} e^{-i\omega t} \vec{\nabla}_r \times \left(\frac{e^{ikr}}{r} \vec{p}_\omega \right)$$

$\vec{p}_\omega = const.$

Math.: $\boxed{\vec{\nabla} \times (f(\vec{r}) \cdot \vec{b}(\vec{r})) = \vec{\nabla} f(r) \times \vec{b}(\vec{r}) + f(r) \vec{\nabla} \times \vec{b}(\vec{r})}$ $\to 0$

In our case $\vec{b}(\vec{r}) = \vec{p}_\omega = const \to 0$

$$\boxed{\vec{B} = -\frac{i\omega\mu_0}{4\pi} e^{-i\omega t} \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \times \vec{p}_\omega};$$

We may calculate this gradient:

$$\vec{\nabla}\left(\frac{e^{ikr}}{r}\right) = \frac{\vec{\nabla} e^{ikr}}{r} + e^{ikr} \vec{\nabla}\left(\frac{1}{r}\right) = ik \frac{\vec{e}_r e^{ikr}}{r} - \frac{e^{ikr} \vec{e}_r}{r^2};$$

$$\left(\vec{\nabla} r = \frac{\vec{r}}{r} = \vec{e}_r \right)$$

wave vector $\vec{k} = k\vec{e}_r$

$$\vec{\nabla}\left(\frac{e^{ikr}}{r}\right) = i\vec{k} \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \cong i\vec{k} \frac{e^{ikr}}{r}$$

$kr \gg 1$, $\frac{1}{kr} \ll 1$

$$\vec{B} = -\frac{i\omega\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ikr}}{r} \cdot i\vec{k} \times \vec{p}_\omega = \frac{\mu_0}{4\pi} \frac{ck^2 e^{ikr-i\omega t}}{r} (\vec{e}_r \times \vec{p}_\omega) \quad (*)$$

(we use $\omega = ck$)

$$\boxed{\vec{B} = \frac{\mu_0 ck^2}{4\pi} (\vec{e}_k \times \vec{p}_\omega) \frac{e^{ikr-i\omega t}}{r}}$$

$$\vec{B} = \vec{B}_\omega \cdot e^{-i\omega t}$$

(Definition $\vec{e}_k = \frac{\vec{k}}{k} \equiv \vec{e}_r = \frac{\vec{r}}{r}$)

$$\boxed{\vec{B}_\omega = \frac{\mu_0 ck^2}{4\pi} (\vec{e}_k \times \vec{p}_\omega) \frac{e^{ikr}}{r}}$$

Electric field: $\vec{E}(\vec{r}, t)$

From the Maxwell equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}; \quad \frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$$

$$(\vec{E} = \vec{E}_\omega \cdot e^{-i\omega t})$$

$$\boxed{\vec{E} = \frac{i}{\omega} \frac{1}{\mu_0 \varepsilon_0} \vec{\nabla} \times \vec{B} = \frac{i c^2}{\omega} e^{-i\omega t} \vec{\nabla} \times \vec{B}_\omega}$$

$\left(\frac{1}{i} = -i\right)$ $\quad 1/\mu_0\varepsilon_0 = c^2$

$$\vec{\nabla} \times \vec{B}_\omega = \frac{\mu_0 c k^2}{4\pi} \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \vec{p}_\omega \right) = i\vec{k} \times \vec{B}_\omega \quad \text{(It was calculated above)}$$

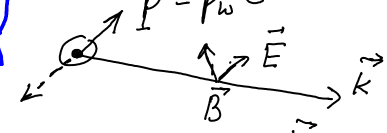
$$\boxed{\vec{E} = -\frac{c^2}{\omega} (\vec{k} \times \vec{B}_\omega) e^{-i\omega t} = \frac{c\cancel{k}}{\cancel{\omega}} (\vec{B} \times \vec{e}_k)}$$

$$\boxed{\vec{E} = c(\vec{B} \times \vec{e}_k) = \frac{\mu_0 \omega^2}{4\pi} (\vec{e}_k \times \vec{p}_\omega \times \vec{e}_k) \frac{e^{ikr-i\omega t}}{r}}$$

Radiation Energy Flux

$$\boxed{\begin{aligned}\vec{B} &= \frac{\mu_0 c k^2}{4\pi}(\vec{e}_k \times \vec{p})\frac{e^{i(kr-\omega t)}}{r}\\ \vec{E} &= c\,(\vec{B}\times\vec{e}_k)\end{aligned}}$$

$\vec{p} = \vec{p}_\omega e^{-i\omega t}$



The Poynting vector (the density of EM energy flux):

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{c}{\mu_0}(\vec{B}\times\vec{e}_k)\times\vec{B} =$$

$$= \frac{c}{\mu_0}\{\vec{e}_k(\vec{B}^2) - \vec{B}(\cancel{\vec{e}_k\vec{B}})\} =$$

$$= c\frac{B^2}{\mu_0}\vec{e}_k \equiv c\left(\frac{B^2}{2\mu_0} + \frac{E^2}{2\mu_0 c^2}\right)\vec{e}_k$$

$$\Rightarrow \boxed{\vec{S} = c\,(w_B + w_E)\,\vec{e}_k}$$

energy density of magnetic field $w_B = \frac{B^2}{2\mu_0}$; energy density of electric field $w_E = \frac{E^2}{2\mu_0 c^2} = \frac{\varepsilon_0 E^2}{2}$

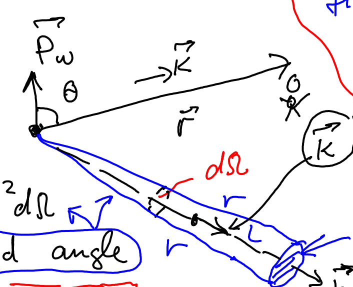
$$\boxed{\vec{S} = \frac{cB^2\vec{e}_k}{\mu_0} = \frac{\mu_0^{\cancel{2}}c^3 k^4 \vec{e}_k}{\cancel{\mu_0}16\pi^2 r^2}p_\omega^2 \sin^2\theta\cdot\cos^2(kr-\omega t)}$$

$$\boxed{B = |\vec{B}| = \frac{\mu_0 c k^2}{4\pi r}\cdot p_\omega \sin\theta\cdot\cos(kr-\omega t)}$$

$$|\vec{B}| = |\mathrm{Re}\{\vec{B}_\omega e^{-i\omega t}\}| = \frac{\mu_0 c k^2}{4\pi r}p_\omega \sin\theta\cdot\cos(kr-\omega t)$$

θ is the angle between $\vec{p}_\omega$ and $\vec{k}$

The power of EM waves:



$$dP = \vec{S}\cdot d\vec{a} = S\cdot r^2 d\Omega$$

solid angle

elemental area $da = r^2 d\Omega$

$\vec{n} \equiv \vec{e}_r \equiv \vec{e}_k$

Intensity of the EM Radiation

The energy radiated per sec into a unit solid angle

$$\boxed{I = \frac{dP}{d\Omega} = \frac{\mu_0 c^3 k^4}{16\pi^2}p_\omega^2 \sin^2\theta\cos^2(kr-\omega t)} \quad (*)$$

$$\omega = ck \Rightarrow \boxed{I \equiv I(\theta) = \frac{\mu_0\omega^4\cdot p_\omega^2}{16\pi^2 c}\sin^2\theta\cos^2(kr-\omega t)}$$

The factors $\omega^4 p_\omega^2\cdot\cos^2(kr-\omega t)$ are equal to $\ddot{\vec{p}}^2 \Rightarrow$

$$\boxed{I(\theta) = \frac{\mu_0}{16\pi^2 c}\ddot{\vec{p}}^2\sin^2\theta}$$

$\ddot{\vec{p}} = -\omega^2\vec{p}$ $\quad \mu_0\varepsilon_0 = \frac{1}{c^2}$

$$\boxed{I(\theta) = \frac{1}{4\pi\cdot 4\pi\varepsilon_0}\cdot\frac{\ddot{\vec{p}}^2}{c^3}\sin^2\theta}$$

"$\frac{1}{4\pi\varepsilon_0}$" $= 1$ (Gaussian units)

Gaussian System of units:

"$\frac{1}{4\pi\varepsilon_0}$" $= 1 \Rightarrow$

$$\boxed{I(\theta) = \frac{\ddot{\vec{p}}^2}{4\pi c^3}\sin^2\theta}$$

This formula is written for Gaussian system of units.