Phys. 6430 - THEORY OF RELATIVITY - HOMEWORK 3

Due in class on Thursday, October 13.

1. **Conserved quantities.**

   a). By examining the geodesic equations prove that if the metric $g_{\mu\nu}$ does not depend on one of the coordinates x_β, then $\frac{dx_\beta}{d\tau}$ is constant along the particles trajectory.

   b). Show that if a vector field ξ_μ satisfies Killing's equations

   $$\xi_{\beta;\alpha} + \xi_{\alpha;\beta} = 0$$

   then along a geodesic $\xi_\alpha \frac{dx^\alpha}{d\tau} = const.$

   c). Find ten Killing fields of Minkowski spacetime.

2. **Linearized Einstein equations.**

   Derive the linearized Einstein equations in the limit of weak gravity.

   a). Take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and treat $h_{\mu\nu}$ as small. Show that under the linearized general coordinate transformation h transforms as

   $$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}.$$

   where $\xi_\mu(x)$ are arbitrary functions.

   b). Now define a "tensor"

   $$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\lambda_\lambda$$

   Show that the following is an acceptable gauge condition ("Lorentz gauge")

   $$\bar{h}^{\mu\nu}_{,\nu} = 0$$

   c). Show that in this gauge the linearized Einstein equations read

   $$\Delta\bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}$$

   where Δ is the D'Alembert operator.

3. **Postnewtonian corrections due to rotation.**

Here you will calculate the first corrections to Newtonian solution cused by a source that rotates.

a). Suppose a spherical body of uniform density ρ and radius R rotates rigidly abut the x^3 axis with constant angular velocity ω. Write down the components of $T^{0\nu}$ in a Lorentz frame at rest with respect to the center of mass of the body, assuming ρ, R and ω are independent of time. For each component work to the lowest order in ωR.

b). The general solution to the equation $\nabla^2 f = g$ which vanishes at infinity is given by

$$f(x) = -\frac{1}{4\pi} \int \frac{g(y)}{|x-y|} d^3y.$$

Use this to solve the linearized Einstein equation ofr $\bar{h}^{00}$ and $\bar{h}^{0i}$. Obtain the solutions only outside the body and only to the lowest nonvanishing order in $1/r$, where r is the distance to the body's center of mass. Express the result for $\bar{h}^{0i}$ in terms of the body's angular momentum. Find the metric tensor within this approximation, and transform it to spherical coordinates.

c). Because the metric is independent of time and the azimuthal angle ϕ, particles orbiting this body will have $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$ constant along their trajectories. Consider a particle of nonzero rest mass in a circular orbit of radius r in the equatorial plane. To lowest order in ω calculate the difference between its orbital period in the positive sense (i.e. rotating in the sense of the central body's rotation) and in the negative sense.

d). Take the central body to be Sun ($M = 2 \times 10^{30}$ kg, $R = 7 \times 10^8$ m, $\omega = 3 \times 10^{-6}\ s^{-1}$.) and the orbiting particle Earth ($r = 1.5 \times 10^{11}\ m$). What would be the difference in the year between positive and negative orbits?