SVT Boundary Conditions

Taking h_{0i} as an example, if we decompose h_{0i} according to the identity

$$h_{0i} = \underbrace{h_{0i} - \nabla_i \int D\nabla^j h_{0j}}_{B_i} + \nabla_i \underbrace{\int D\nabla^j h_{0j}}_{B}, \tag{0.1}$$

then naturally it follows that

$$\nabla^k B_k = 0, \qquad \nabla^k h_{0k} = \nabla^k \nabla_k B. \tag{0.2}$$

Thus a decomposition into transverse and longitudinal components appears to have been satisfied, and proper behavior of B only requires that $\int D\nabla^j h_{ij}$ does not diverge.

By directly taking $B = \int D\nabla^j h_{0j}$, it would seem we do not have to specify any specific behavior of B on the boundary to achieve decomposition.

However, in using Green's identity

$$(\nabla_i \nabla^i D)B = D(\nabla_i \nabla^i B) + \nabla^i[(\nabla_i D)B - D(\nabla_i B)] \tag{0.3}$$

$$B = \int D(\nabla_i \nabla^i B) + \oint dS^i[(\nabla_i D)B - D(\nabla_i B)], \tag{0.4}$$

we see that since $\nabla_i \nabla^i B = \nabla^i h_{0i}$ and since we have defined $B = \int D\nabla^j h_{0j}$ it must follow that

$$B = B + \oint dS^i[(\nabla_i D)B - D(\nabla_i B)] \tag{0.5}$$

$$\implies 0 = \oint dS^i[(\nabla_i D)B - D(\nabla_i B)]. \tag{0.6}$$

Put in different terms, Green's identity allows us to decompose any scalar into harmonic and non-harmonic parts (harmonic according to math convention, meaning $\nabla^2 f(x) = 0$)

$$B = \underbrace{\int D(\nabla_i \nabla^i B)}_{B^{NH}} + \underbrace{\oint dS^i[(\nabla_i D)B - D(\nabla_i B)]}_{B^H}. \tag{0.7}$$

We see that our initial definition of B via $B = \int D\nabla^j h_{0j}$ is in fact the non-harmonic projection B^{NH} by virtue of $\nabla^2 B = \nabla^j h_{0j}$. Thus such a definition automatically enforces that B vanish on the boundary.

Given the integral relation for ψ (according to a definition that never requires an integration by parts) and its derivative relation,

$$\psi = \frac{1}{4}\left[\int D\nabla^l h_{kl} - g^{ab}h_{ab}\right] \tag{0.8}$$

$$\nabla^2 \psi = \frac{1}{4}\nabla^2\left[\nabla^l h_{kl} - g^{ab}h_{ab}\right], \tag{0.9}$$

we see that $\psi \neq \int D\nabla^2\psi$, and thus is not required to vanish on the boundary. However, if we define the integral relation for ψ as

$$\psi = \frac{1}{4}\left[\int D(\nabla^l h_{kl} - g^{ab}h_{ab})\right], \tag{0.10}$$

then it does follow that ψ is nonharmonic, namely $\psi = \int D\nabla^2\psi$ and thus must vanish on the boundary.

The tradeoff between having an E_{ij} that is automatically transverse + traceless but not itself gauge invariant, vs an E_{ij} that is automatically gauge invariant but requires integration by parts to be transverse and traceless is discussed below (2.9) in the decomposition paper.