RW SVT2 v1

1 Background

1.1 Comoving RW

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2\right) && (1.1)\\
U^\mu &= \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}U^0, \quad \frac{dx^i}{dt} = 0, \quad g_{\mu\nu}U^\mu U^\nu = -1, \implies U^\mu = \delta^\mu_0 && (1.2)\\
T_{\mu\nu} &= (\rho(t)+p(t))U_\mu U_\nu + p(t)g_{\mu\nu} && (1.3)
\end{aligned}
$$

1.2 Conformal RW $\Omega(T,R)$

$$
\begin{aligned}
ds^2 &= \Omega^2(T,R)\left(-dT^2 + dR^2 + R^2 d\theta^2\right) = \Omega^2(T,R)\tilde{g}_{\mu\nu}dx^\mu dx^\nu && (1.4)\\
x^\mu &= (t,r,\theta), \qquad X^\mu = (T,R,\theta) && (1.5)\\
T &= f(t,r), \quad R = g(t,r)\\
&\quad (\text{Ex: } k = -\tfrac{1}{L^2},\ T = e^{\frac{\tau}{L}}\cosh\chi,\ R = e^{\frac{\tau}{L}}\sinh\chi,\ dt = a(t)d\tau,\ \sinh\chi = \tfrac{r}{L}) && (1.6)\\
T^{(T,R)}_{\mu\nu} &= (\rho+p)U_\mu U_\nu + p g_{\mu\nu} && (1.7)\\
\rho &= \rho(T,R), \qquad p = p(T,R) && (1.8)\\
U^\mu \equiv U^\mu_{(T,R)} &= \frac{\partial X^\mu}{\partial x^\nu}U^\nu_{(t,r)} = \frac{\partial X^\mu}{\partial t} = \left(\frac{\partial T}{\partial t}, \frac{\partial R}{\partial t}, 0\right), \quad U_\mu = g_{\mu\nu}U^\nu = \Omega^2\left(-\frac{\partial T}{\partial t}, \frac{\partial R}{\partial t}, 0\right)^T && (1.9)\\
U_\mu \equiv U_\mu^{(T,R)} &= \frac{\partial x^\nu}{\partial X^\mu}U_\nu^{(t,r)} = -\frac{\partial t}{\partial X^\mu} = -\left(\frac{\partial t}{\partial T}, \frac{\partial t}{\partial R}, 0\right)^T, \quad U^\mu = g^{\mu\nu}U_\nu = \Omega^{-2}\left(\frac{\partial t}{\partial T}, -\frac{\partial t}{\partial R}, 0\right) && (1.10)\\
&\implies \Omega^2\left(\frac{\partial T}{\partial t}\right) = \left(\frac{\partial t}{\partial T}\right), \qquad \Omega^2\left(\frac{\partial R}{\partial t}\right) = -\left(\frac{\partial t}{\partial R}\right)\\
U^\mu U_\mu = -1 &\implies \boxed{\Omega^2 = \left[\left(\frac{\partial T}{\partial t}\right)^2 - \left(\frac{\partial R}{\partial t}\right)^2\right]^{-1} = \left[\left(\frac{\partial t}{\partial T}\right)^2 - \left(\frac{\partial t}{\partial R}\right)^2\right]} && (1.11)
\end{aligned}
$$

(Holds for a comoving $g^{(t,r)}_{\mu\nu}$ (i.e. timelike $U^\mu_{(t,r)}$) transformed to conformal flat form)

$$\tilde{U}_\mu \equiv \Omega^{-1} U_\mu = \Omega \left(-\frac{\partial T}{\partial t}, \frac{\partial R}{\partial t}, 0 \right)^T = -\Omega^{-1} \left(\frac{\partial t}{\partial T}, \frac{\partial t}{\partial R}, 0 \right)^T, \qquad \tilde{g}^{\mu\nu} \tilde{U}_\mu \tilde{U}_\nu = -1 \tag{1.12}$$

$$\tilde{U}^\mu = \tilde{g}^{\mu\nu} \tilde{U}_\nu, \qquad \tilde{U}_0 = -\tilde{U}^0, \quad \tilde{U}_r = \tilde{U}^r \tag{1.13}$$

1.3 Field Equations

$$G_{\mu\nu} = -\tilde{g}_{\mu\nu} \Omega^{-1} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \Omega + \tilde{g}_{\mu\nu} \Omega^{-2} \tilde{\nabla}_\alpha \Omega \tilde{\nabla}^\alpha \Omega - 2\Omega^{-2} \tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega + \Omega^{-1} \tilde{\nabla}_\nu \tilde{\nabla}_\mu \Omega \tag{1.14}$$

$$g^{\mu\nu} G_{\mu\nu} = -2\Omega^{-3} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \Omega + \Omega^{-4} \tilde{\nabla}_\alpha \Omega \tilde{\nabla}^\alpha \Omega \tag{1.15}$$

$$T_{\mu\nu} = \Omega^2 (\rho + p) \tilde{U}_\mu \tilde{U}_\nu + \Omega^2 p \tilde{g}_{\mu\nu} \tag{1.16}$$

$$g^{\mu\nu} T_{\mu\nu} = 2p - \rho \tag{1.17}$$

$$\Delta_{00} = -\dot{\Omega}^2 \Omega^{-2} + {U_0}^2 \rho \Omega^2 + p(-\Omega^2 + {U_0}^2 \Omega^2) + \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega - \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tag{1.18}$$

$$\Delta_{0i} = U_0 (p U_i \Omega^2 + U_i \rho \Omega^2) + \Omega^{-1} \tilde{\nabla}_i \dot{\Omega} - 2\dot{\Omega} \Omega^{-2} \tilde{\nabla}_i \Omega \tag{1.19}$$

$$\begin{aligned} \Delta_{ij} = {} & -\dot{\Omega}^2 \tilde{g}_{ij} \Omega^{-2} + \ddot{\Omega} \tilde{g}_{ij} \Omega^{-1} + U_i U_j \rho \Omega^2 + p(\tilde{g}_{ij} \Omega^2 + U_i U_j \Omega^2) - \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega \\ & + \tilde{g}_{ij} \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega - 2\Omega^{-2} \tilde{\nabla}_i \Omega \tilde{\nabla}_j \Omega + \Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i \Omega \end{aligned} \tag{1.20}$$

$$g^{\mu\nu} \Delta_{\mu\nu} = 2p - \rho - \dot{\Omega}^2 \Omega^{-4} + 2\ddot{\Omega} \Omega^{-3} - 2\Omega^{-3} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + \Omega^{-4} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tag{1.21}$$

1.3.1 Polar Components

$$g^{\mu\nu} \Delta_{\mu\nu} = 2p - \rho - \dot{\Omega}^2 \Omega^{-4} + (\tilde{\nabla}_1 \Omega)^2 \Omega^{-4} + 2\ddot{\Omega} \Omega^{-3} - 2r^{-1} \Omega^{-3} \tilde{\nabla}_1 \Omega - 2\Omega^{-3} \tilde{\nabla}_1 \tilde{\nabla}_1 \Omega \tag{1.22}$$

$$\Delta_{00} = -\dot{\Omega}^2 \Omega^{-2} - (\tilde{\nabla}_1 \Omega)^2 \Omega^{-2} - p\Omega^2 + p{U_0}^2 \Omega^2 + {U_0}^2 \rho \Omega^2 + r^{-1} \Omega^{-1} \tilde{\nabla}_1 \Omega + \Omega^{-1} \tilde{\nabla}_1 \tilde{\nabla}_1 \Omega \tag{1.23}$$

$$\Delta_{11} = -\dot{\Omega}^2 \Omega^{-2} - (\tilde{\nabla}_1 \Omega)^2 \Omega^{-2} + \ddot{\Omega} \Omega^{-1} + p\Omega^2 + p{U_r}^2 \Omega^2 + {U_r}^2 \rho \Omega^2 - r^{-1} \Omega^{-1} \tilde{\nabla}_1 \Omega \tag{1.24}$$

$$\Delta_{22} = -\dot{\Omega}^2 r^2 \Omega^{-2} + r^2 (\tilde{\nabla}_1 \Omega)^2 \Omega^{-2} + \ddot{\Omega} r^2 \Omega^{-1} + p r^2 \Omega^2 - r^2 \Omega^{-1} \tilde{\nabla}_1 \tilde{\nabla}_1 \Omega \tag{1.25}$$

$$\Delta_{01} = p U_0 U_r \Omega^2 + U_0 U_r \rho \Omega^2 + \Omega^{-1} \tilde{\nabla}_1 \dot{\Omega} - 2\dot{\Omega} \Omega^{-2} \tilde{\nabla}_1 \Omega \tag{1.26}$$

$$\Delta_{12} = \Delta_{02} = 0 \tag{1.27}$$

$$\Delta_{22} = 0 \implies \boxed{p = \Omega^{-4} \big(\dot{\Omega}^2 - (\tilde{\nabla}_1 \Omega)^2 - \ddot{\Omega} \Omega + \Omega \tilde{\nabla}_1 \tilde{\nabla}_1 \Omega \big)} \tag{1.28}$$

$$g^{\mu\nu} \Delta_{\mu\nu} = 0 \implies \boxed{\rho = r^{-1} \Omega^{-4} \big(\dot{\Omega}^2 r - r(\tilde{\nabla}_1 \Omega)^2 - 2\Omega \tilde{\nabla}_1 \Omega \big)} \tag{1.29}$$

$$\Delta_{00} = 0 \implies \boxed{U_0^2 = -(-2\dot{\Omega}^2 r + \ddot{\Omega} r \Omega + \Omega \tilde{\nabla}_1 \Omega) \big(2\dot{\Omega}^2 r - 2r(\tilde{\nabla}_1 \Omega)^2 - 2\Omega \tilde{\nabla}_1 \Omega + r\Omega(-\ddot{\Omega} + \tilde{\nabla}_1 \tilde{\nabla}_1 \Omega) \big)^{-1}} \tag{1.30}$$

$$\Delta_{11} = 0 \Longrightarrow \boxed{U_r^2 = \big(2r(\tilde\nabla_1\Omega)^2 + \Omega\tilde\nabla_1\Omega - r\Omega\tilde\nabla_1\tilde\nabla_1\Omega\big)\big(2\dot\Omega^2 r - 2r(\tilde\nabla_1\Omega)^2 - \ddot\Omega r\Omega - 2\Omega\tilde\nabla_1\Omega + r\Omega\tilde\nabla_1\tilde\nabla_1\Omega\big)^{-1}} \tag{1.31}$$

$$\Delta_{01} = 0 \Longrightarrow \boxed{U_0U_r = r(\Omega\tilde\nabla_1\dot\Omega - 2\dot\Omega\tilde\nabla_1\Omega)\big(-2\dot\Omega^2 r + 2r(\tilde\nabla_1\Omega)^2 + \ddot\Omega r\Omega + 2\Omega\tilde\nabla_1\Omega - r\Omega\tilde\nabla_1\tilde\nabla_1\Omega\big)^{-1}} \tag{1.32}$$

2 Fluctuations

$$ds^2 = \Omega^2(T,R)\,[g_{\mu\nu} + f_{\mu\nu}]\,dx^\mu dx^\nu \tag{2.1}$$

$$f_{00} = -2\phi, \qquad f_{0i} = B_i + \tilde\nabla_i B, \qquad f_{ij} = -2\tilde g_{ij}\psi + 2\tilde\nabla_i\tilde\nabla_j E + \tilde\nabla_i E_j + \tilde\nabla_j E_i \tag{2.2}$$

$$\tilde g^{\mu\nu} f_{\mu\nu} = 2\phi - 4\psi + 2\tilde\nabla_a\tilde\nabla^a E, \qquad \tilde g^{ij} f_{ij} = -4\psi + 2\tilde\nabla_a\tilde\nabla^a E \tag{2.3}$$

$$\tilde U^\mu\delta\tilde U_\mu = \tfrac{1}{2}\tilde U^\mu\tilde U^\nu f_{\mu\nu} \tag{2.4}$$

$$-\tilde U_0\delta\tilde U_0 + \tilde U_r\delta\tilde U_r = -\tilde U_0^2\phi - \tilde U_0\tilde U_r(B_1 + \tilde\nabla_1 B) + \tilde U_r^2(-2\psi + 2\tilde\nabla_r\tilde\nabla_r E + 2\tilde\nabla_r E_r) \tag{2.5}$$

$$\delta\tilde U_i = \tilde\nabla_i V + V_i \tag{2.6}$$

$$\delta\tilde U_0 = B_1\tilde U_r + \tilde U_0\phi + \tilde U_0{}^{-1}\tilde U_r\big(V_1 + \tilde U_r\psi + \tilde U_0\tilde\nabla_1 B + \tilde\nabla_1 V - \tilde U_r(\tilde\nabla_1 E_1 + \tilde\nabla_1\tilde\nabla_1 E)\big) \tag{2.7}$$

$$\delta T_{\mu\nu} = \Omega^2\left[(\rho+p)(\delta\tilde U_\mu\tilde U_\nu + \delta\tilde U_\nu\tilde U_\mu) + (\delta\rho + \delta p)\tilde U_\mu\tilde U_\nu + \delta p\tilde g_{\mu\nu} + pf_{\mu\nu}\right] \tag{2.8}$$

$$g^{\mu\nu}\delta T_{\mu\nu} = 2(\rho+p)\tilde U^\mu\delta\tilde U_\mu - \delta\rho + 2\delta p + p(\tilde g^{\mu\nu} f_{\mu\nu}) \tag{2.9}$$

$$\begin{aligned}\delta G_{\mu\nu} = {} & \tfrac12\tilde\nabla_\alpha\tilde\nabla^\alpha f_{\mu\nu} - \tfrac12\tilde g_{\mu\nu}\tilde\nabla_\alpha\tilde\nabla^\alpha f - f_{\mu\nu}\Omega^{-1}\tilde\nabla_\alpha\tilde\nabla^\alpha\Omega - \tfrac12\tilde g_{\mu\nu}\Omega^{-1}\tilde\nabla_\alpha\Omega\tilde\nabla^\alpha f + \tfrac12\Omega^{-1}\tilde\nabla_\alpha f_{\mu\nu}\tilde\nabla^\alpha\Omega\\ & + f_{\mu\nu}\Omega^{-2}\tilde\nabla_\alpha\Omega\tilde\nabla^\alpha\Omega + \tilde g_{\mu\nu}\Omega^{-1}\tilde\nabla^\alpha\Omega\tilde\nabla_\beta f_\alpha{}^\beta + \tfrac12\tilde g_{\mu\nu}\tilde\nabla_\beta\tilde\nabla_\alpha f^{\alpha\beta} + \tilde g_{\mu\nu} f^{\alpha\beta}\Omega^{-1}\tilde\nabla_\beta\tilde\nabla_\alpha\Omega\\ & - \tilde g_{\mu\nu} f_{\alpha\beta}\Omega^{-2}\tilde\nabla^\alpha\Omega\tilde\nabla^\beta\Omega - \tfrac12\Omega^{-1}\tilde\nabla^\alpha\Omega\tilde\nabla_\mu f_{\nu\alpha} - \tfrac12\tilde\nabla_\mu\tilde\nabla_\alpha f_\nu{}^\alpha - \tfrac12\Omega^{-1}\tilde\nabla^\alpha\Omega\tilde\nabla_\nu f_{\mu\alpha} - \tfrac12\tilde\nabla_\nu\tilde\nabla_\alpha f_\mu{}^\alpha\\ & + \tfrac12\tilde\nabla_\nu\tilde\nabla_\mu f\end{aligned} \tag{2.10}$$

$$\begin{aligned}g^{\mu\nu}\delta G_{\mu\nu} = {} & -\Omega^{-2}\tilde\nabla_\alpha\tilde\nabla^\alpha f - \tfrac32\Omega^{-3}\tilde\nabla_\alpha\Omega\tilde\nabla^\alpha f + \tfrac12\Omega^{-3}\tilde\nabla_\alpha f^\beta{}_\beta\tilde\nabla^\alpha\Omega + f^\beta{}_\beta\Omega^{-4}\tilde\nabla_\alpha\Omega\tilde\nabla^\alpha\Omega\\ & + 2\Omega^{-3}\tilde\nabla^\alpha\Omega\tilde\nabla_\beta f_\alpha{}^\beta + \tfrac12\Omega^{-2}\tilde\nabla_\beta\tilde\nabla_\alpha f^{\alpha\beta} + 3f^{\alpha\beta}\Omega^{-3}\tilde\nabla_\beta\tilde\nabla_\alpha\Omega + \tfrac12\Omega^{-2}\tilde\nabla_\beta\tilde\nabla^\beta f^\alpha{}_\alpha\\ & - f^\alpha{}_\alpha\Omega^{-3}\tilde\nabla_\beta\tilde\nabla^\beta\Omega - 3f_{\alpha\beta}\Omega^{-4}\tilde\nabla^\alpha\Omega\tilde\nabla^\beta\Omega\end{aligned} \tag{2.11}$$

3 Field Equations

$$\begin{aligned}\Delta_{00} = {} & 2\dot\psi\dot\Omega\Omega^{-1} + \delta\rho U_0{}^2\Omega^2 + \delta p(-\Omega^2 + U_0{}^2\Omega^2) + \delta U_0(2pU_0\Omega^2 + 2U_0\rho\Omega^2) + \dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a B\\ & - \dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\dot E - \tilde\nabla_a\tilde\nabla^a\psi + (2\Omega^{-1}\tilde\nabla_a\dot\Omega - 2\dot\Omega\Omega^{-2}\tilde\nabla_a\Omega)\tilde\nabla^a B\\ & + \psi(2\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\Omega - 2\Omega^{-2}\tilde\nabla_a\Omega\tilde\nabla^a\Omega) + \phi(-2p\Omega^2 + 2\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\Omega - 2\Omega^{-2}\tilde\nabla_a\Omega\tilde\nabla^a\Omega)\\ & - \Omega^{-1}\tilde\nabla^a\Omega\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a E + 2\Omega^{-2}\tilde\nabla^a\Omega\tilde\nabla_b\tilde\nabla_a E\tilde\nabla^b\Omega\\ & - 2\Omega^{-1}\tilde\nabla_b\tilde\nabla_a\Omega\tilde\nabla^b\tilde\nabla^a E + B^a(2\Omega^{-1}\tilde\nabla_a\dot\Omega - 2\dot\Omega\Omega^{-2}\tilde\nabla_a\Omega) - \Omega^{-1}\tilde\nabla^a\Omega\tilde\nabla_b\tilde\nabla^b E_a\\ & + (2\Omega^{-2}\tilde\nabla_a\Omega\tilde\nabla_b\Omega - 2\Omega^{-1}\tilde\nabla_b\tilde\nabla_a\Omega)\tilde\nabla^b E^a\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
\Delta_{0i} \quad = \quad & \delta p U_i U_0 \Omega^2 + \delta\rho U_i U_0 \Omega^2 + \delta U_0 (p U_i \Omega^2 + U_i \rho \Omega^2) \\
& + (-\dot{\Omega}^2 \Omega^{-2} + \ddot{\Omega}\Omega^{-1} + p\Omega^2 - \Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \Omega + \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \tilde{\nabla}_i B - \tilde{\nabla}_i \dot{\psi} - \dot{\Omega}\Omega^{-1} \tilde{\nabla}_i \phi \\
& + \dot{\psi} \Omega^{-1} \tilde{\nabla}_i \Omega - \Omega^{-1} \tilde{\nabla}^a \Omega \tilde{\nabla}_i \tilde{\nabla}_a \dot{E} + \delta U_i (p U_0 \Omega^2 + U_0 \rho \Omega^2) + \tfrac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^a B_i - \tfrac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E}_i \\
& + \tfrac{1}{2} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}^a B_i - \tfrac{1}{2} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \dot{E}_i \\
& + B_i (-\dot{\Omega}^2 \Omega^{-2} + \ddot{\Omega}\Omega^{-1} + p\Omega^2 - \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) - \tfrac{1}{2} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}_i B^a \\
& - \tfrac{1}{2} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}_i \dot{E}^a
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
\Delta_{ij} \quad = \quad & -\ddot{\psi} \tilde{g}_{ij} + \tilde{g}_{ij} \phi (2\dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega}\Omega^{-1}) - \dot{\phi}\dot{\Omega} \tilde{g}_{ij} \Omega^{-1} - \dot{\psi}\dot{\Omega} \tilde{g}_{ij} \Omega^{-1} + \delta\rho U_i U_j \Omega^2 \\
& + \tilde{g}_{ij} \psi (2\dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega}\Omega^{-1} - 2p\Omega^2) + \delta p (\tilde{g}_{ij} \Omega^2 + U_i U_j \Omega^2) - \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a B - \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \dot{B} \\
& + \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{E} + \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} - \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \phi + \tilde{g}_{ij} (-2\Omega^{-1} \tilde{\nabla}_a \dot{\Omega} + 2\dot{\Omega}\Omega^{-2} \tilde{\nabla}_a \Omega) \tilde{\nabla}^a B \\
& - \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \dot{B} - \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \phi - \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \psi + \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a E \\
& - 2\tilde{g}_{ij} \Omega^{-2} \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}_a E \tilde{\nabla}^b \Omega + 2\tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}_a \Omega \tilde{\nabla}^b \tilde{\nabla}^a E + 2p\Omega^2 \tilde{\nabla}_i \tilde{\nabla}_j E + \Omega^{-1} \tilde{\nabla}_i \Omega \tilde{\nabla}_j \psi \\
& + \Omega^{-1} \tilde{\nabla}_i \psi \tilde{\nabla}_j \Omega + \dot{\Omega}\Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i B + \tilde{\nabla}_j \tilde{\nabla}_i \dot{B} - \tilde{\nabla}_j \tilde{\nabla}_i \ddot{E} - \dot{\Omega}\Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i \dot{E} \\
& + (-2\dot{\Omega}^2 \Omega^{-2} + 2\ddot{\Omega}\Omega^{-1} - 2\Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + 2\Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \tilde{\nabla}_j \tilde{\nabla}_i E + \tilde{\nabla}_j \tilde{\nabla}_i \phi \\
& - \Omega^{-1} \tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i \tilde{\nabla}_a E + \delta U_j (p U_i \Omega^2 + U_i \rho \Omega^2) + \delta U_i (p U_j \Omega^2 + U_j \rho \Omega^2) - \dot{B}^a \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \Omega \\
& + B^a \tilde{g}_{ij} (-2\Omega^{-1} \tilde{\nabla}_a \dot{\Omega} + 2\dot{\Omega}\Omega^{-2} \tilde{\nabla}_a \Omega) + \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b E_a \\
& + \tilde{g}_{ij} (-2\Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + 2\Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}_a \Omega) \tilde{\nabla}^b E^a + \tfrac{1}{2} \dot{\Omega}\Omega^{-1} \tilde{\nabla}_i B_j + \tfrac{1}{2} \tilde{\nabla}_i \dot{B}_j - \tfrac{1}{2} \tilde{\nabla}_i \ddot{E}_j \\
& - \tfrac{1}{2} \dot{\Omega}\Omega^{-1} \tilde{\nabla}_i \dot{E}_j + (-\dot{\Omega}^2 \Omega^{-2} + \ddot{\Omega}\Omega^{-1} + p\Omega^2 - \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \tilde{\nabla}_i E_j \\
& + \tfrac{1}{2} \dot{\Omega}\Omega^{-1} \tilde{\nabla}_j B_i + \tfrac{1}{2} \tilde{\nabla}_j \dot{B}_i - \tfrac{1}{2} \tilde{\nabla}_j \ddot{E}_i - \tfrac{1}{2} \dot{\Omega}\Omega^{-1} \tilde{\nabla}_j \dot{E}_i \\
& + (-\dot{\Omega}^2 \Omega^{-2} + \ddot{\Omega}\Omega^{-1} + p\Omega^2 - \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + \Omega^{-2} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \tilde{\nabla}_j E_i - \Omega^{-1} \tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i E_a
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
g^{\mu\nu} \Delta_{\mu\nu} \quad = \quad & 2\delta p - \delta\rho - 2\dot{\phi}\dot{\Omega}\Omega^{-3} - 4\dot{\psi}\dot{\Omega}\Omega^{-3} - 2\ddot{\psi}\Omega^{-2} + (-2pU^a U_0 - 2U^a U_0 \rho) \tilde{\nabla}_a B - 2\dot{\Omega}\Omega^{-3} \tilde{\nabla}_a \tilde{\nabla}^a B \\
& - \Omega^{-2} \tilde{\nabla}_a \tilde{\nabla}^a \dot{B} + \Omega^{-2} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{E} + 2\dot{\Omega}\Omega^{-3} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} - \Omega^{-2} \tilde{\nabla}_a \tilde{\nabla}^a \phi + \Omega^{-2} \tilde{\nabla}_a \tilde{\nabla}^a \psi \\
& + (-6\Omega^{-3} \tilde{\nabla}_a \dot{\Omega} + 6\dot{\Omega}\Omega^{-4} \tilde{\nabla}_a \Omega) \tilde{\nabla}^a B - 2\Omega^{-3} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \dot{B} - 2\Omega^{-3} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \phi \\
& + \psi (-4p - 2pU_a U^a - 2U_a U^a \rho + 4\dot{\Omega}^2 \Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} - 2\Omega^{-3} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + 2\Omega^{-4} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \\
& + \phi (2p - 2p{U_0}^2 - 2{U_0}^2 \rho + 4\dot{\Omega}^2 \Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} - 2\Omega^{-3} \tilde{\nabla}_a \tilde{\nabla}^a \Omega + 2\Omega^{-4} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega) \\
& + 2\Omega^{-4} \tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b E + \tilde{\nabla}_a \tilde{\nabla}^a E (2p - 2\dot{\Omega}^2 \Omega^{-4} + 2\ddot{\Omega}\Omega^{-3} - 2\Omega^{-3} \tilde{\nabla}_b \tilde{\nabla}^b \Omega) \\
& + 2\Omega^{-3} \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a E + \tilde{\nabla}_b \tilde{\nabla}_a E (2pU^a U^b + 2U^a U^b \rho - 6\Omega^{-4} \tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega) \\
& + 6\Omega^{-3} \tilde{\nabla}_b \tilde{\nabla}_a \Omega \tilde{\nabla}^b \tilde{\nabla}^a E - 2\dot{B}^a \Omega^{-3} \tilde{\nabla}_a \Omega \\
& + B^a (-2pU_a U_0 - 2U_a U_0 \rho - 6\Omega^{-3} \tilde{\nabla}_a \dot{\Omega} + 6\dot{\Omega}\Omega^{-4} \tilde{\nabla}_a \Omega) + (2pU^a U^b + 2U^a U^b \rho) \tilde{\nabla}_b E_a \\
& + 2\Omega^{-3} \tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b E_a + (-6\Omega^{-4} \tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + 6\Omega^{-3} \tilde{\nabla}_b \tilde{\nabla}_a \Omega) \tilde{\nabla}^b E^a
\end{aligned} \tag{3.4}
$$

3.0.1 Polar Components

We substitute the background quantities U_0, U_r, ρ, and p and the kinematic equation for δU_0 into $\Delta_{\mu\nu}$.

$$
\begin{aligned}
\Delta_{00} \quad = \quad & 2\dot{\psi}\dot{\Omega}\Omega^{-1} - \dot{\Omega} r^{-1} \Omega^{-1} \tilde{\nabla}_1 \dot{E} - r^{-1} \tilde{\nabla}_1 \psi - r^{-2} \Omega^{-1} \tilde{\nabla}_1 E \tilde{\nabla}_1 \Omega \\
& + \delta\rho \Big(2\dot{\Omega}^2 r \Omega^2 \big(2\dot{\Omega}^2 r - 2r(\tilde{\nabla}_1 \Omega)^2 - \ddot{\Omega} r \Omega - 2(\tilde{\nabla}_1 \Omega)\Omega + r(\tilde{\nabla}_1 \tilde{\nabla}_1 \Omega)\Omega \big)^{-1} - \ddot{\Omega} r \Omega^3 \big(2\dot{\Omega}^2 r \\
& - 2r(\tilde{\nabla}_1 \Omega)^2 - \ddot{\Omega} r \Omega - 2(\tilde{\nabla}_1 \Omega)\Omega + r(\tilde{\nabla}_1 \tilde{\nabla}_1 \Omega)\Omega \big)^{-1} - \Omega^3 \big(2\dot{\Omega}^2 r - 2r(\tilde{\nabla}_1 \Omega)^2 - \ddot{\Omega} r \Omega - 2(\tilde{\nabla}_1 \Omega)\Omega + r(\tilde{\nabla}_1 \tilde{\nabla}_1 \Omega)\Omega \big)^{-1} \tilde{\nabla}_1 \Omega \Big) \\
& + \delta p \Big(-\Omega^2 + 2\dot{\Omega}^2 r \Omega^2 \big(2\dot{\Omega}^2 r - 2r(\tilde{\nabla}_1 \Omega)^2 - \ddot{\Omega} r \Omega - 2(\tilde{\nabla}_1 \Omega)\Omega + r(\tilde{\nabla}_1 \tilde{\nabla}_1 \Omega)\Omega \big)^{-1} - \ddot{\Omega} r \Omega^3 \big(2\dot{\Omega}^2 r
\end{aligned}
$$

$$-2r(\tilde{\nabla}_1\Omega)^2 - \ddot{\Omega} r\Omega - 2(\tilde{\nabla}_1\Omega)\Omega + r(\tilde{\nabla}_1\tilde{\nabla}_1\Omega)\Omega\big)^{-1} - \Omega^3\big(2\dot{\Omega}^2 r - 2r(\tilde{\nabla}_1\Omega)^2 - \ddot{\Omega} r\Omega - 2(\tilde{\nabla}_1\Omega)\Omega$$
$$+r(\tilde{\nabla}_1\tilde{\nabla}_1\Omega)\Omega\big)^{-1}\tilde{\nabla}_1\Omega\Big) - E_1 r^{-2}\Omega^{-1}\tilde{\nabla}_1\Omega + ... \tag{3.5}$$

4 Field Equations (G.I. Form)

$$
\begin{aligned}
\alpha &= \phi + \psi + \dot{B} - \ddot{E} \\
\gamma &= \phi - \psi + \dot{B} - \ddot{E} + 2\Omega^{-1}\left[(B - \dot{E})\dot{\Omega} - (E_a + \tilde{\nabla}_a E)\tilde{\nabla}^a \Omega\right] \\
Q_i &= B_i - \dot{E}_i
\end{aligned} \tag{4.1}
$$