Quantum Mechanics Preliminary Examination Problems

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2015 Winter

1. Consider a finite set of operators B_i. Let H be a Hamiltonian which *commutes* with each B_i; i.e., $[H, B_i] = 0$ for all i. Suppose the $|a_n\rangle$'s form a complete set of eigenstates of H satisfying $H|a_n\rangle = a_n|a_n\rangle$.

   (a) Let us choose one particular value of i and one particular value of n. Under what circumstances can it be deduced that $B_i|a_n\rangle$ is proportional to $|a_n\rangle$?

   If $[H, B_i] = 0$ then $|a_n\rangle$ is an eigenket of B_i. To show this

   $$[H, B_i]|a_n\rangle = 0$$

   $$H(B_i|a_n\rangle) = a_n(B_i|a_n\rangle).$$

   Thus $B_i|a_n\rangle$ is an eigenket of H with eigenvalue a_n. If the spectrum of H is non-degenerate, we deduce that $B_i|a_n\rangle$ must be proportional to $|a_n\rangle$ up to a constant, which we denote b_n^i

   $$B_i|a_n\rangle = b_n^i|a_n\rangle.$$

   (b) Show that if the above is true for all i and for all n, then $[B_i, B_j] = 0$ for all i, j.

   $$[B_i, B_j]|a_n\rangle \quad \to \quad (b_n^i b_n^j - b_n^j b_n^i)|a_n\rangle = 0|a_n\rangle$$

   thus $[B_i, B_j] = 0$.

   (c) How can you reconcile the rule stated in part (b) with the fact that for angular momentum operators L_i, we can have a situation where $[L_i, H] = 0$ but $[L_i, L_j] \neq 0$ when $i \neq j$?

   For $[L_i, H] = 0$, we also have $[\mathbf{L}^2, H] = 0$. Therefore, we know our Hamiltonian must be proportional to $\mathbf{L}^2$. As such, the system is degenerate in L_i. For example, if the z-component of angular momentum were up or down, the energy eigenvalue would remain the same. Since this system is degenerate in L_i, our rule from part (b) does not hold. There is really a lot more to this. The important idea revolves around a complete set of mutually compatible observables.

2. Consider a quantum particle in 1D with mass m and energy $E = -E' < 0$ bound in the double δ-function potential $V(x) = -c_0\delta(x-L) - c_0\delta(x+L)$ where $c_0 > 0$.

   (a) Derive the transcendental equation for the ground state energy E_0 and show (by plotting an appropriate freehand graph) that a solution of this equation exists, for all (positive) values of c_0.

   Confer problem 2.27 Griffiths. Since the potential is even, we will have even and odd solutions. The ground state energy always occurs as an even solution as it can be proven that the even solution contains no nodes (no nodes equates to lower energy). Therefore, we need to find the even wavefunction for the ground state. The energy is strictly negative, and so we denote

$$k \equiv \sqrt{\frac{2m|E|}{\hbar^2}}.$$

   Our wavefunction goes as

$$\psi(x) = \begin{cases} Ae^{kx} + Be^{-kx} & \text{for} \quad 0 < x < L \\ Ce^{-kx} & \text{for} \quad x > L \end{cases}$$

   and our boundary conditions are

   i. $\psi(L)_L = \psi(L)_R; \quad \psi$ is everywhere continuous

   ii. $\left.\dfrac{d\psi}{dx}\right|_0 = 0$

   iii. $\Delta \left.\dfrac{d\psi}{dx}\right|_L = -\dfrac{2m}{\hbar^2}c_0\psi(L)$

   where the last conditions follows from

$$\lim_{\epsilon\to 0}\left[-\frac{\hbar^2}{2m}\int_{L-\epsilon}^{L+\epsilon} dx\,\frac{d^2\psi}{dx^2} + \int_{L-\epsilon}^{L+\epsilon} dx\,V(x)\psi(x)\right] = \lim_{\epsilon\to 0}\left[\int_{L-\epsilon}^{L+\epsilon} dx\,E\psi(x)\right].$$

   Let's first impose the even boundary condition ii., in which we have

$$Ak - Bk = 0 \quad \to \quad A = B$$

   Now lets impose B.C. i.,

$$Ae^{kL} + Ae^{-kL} = Ce^{-kL}$$

$$A(e^{2kL} + 1) = C.$$

   Now we have a wavefunction given as

$$\psi(x) = \begin{cases} A(e^{kx} + e^{-kx}) & \text{for} \quad 0 < x < L \\ A(e^{2kL} + 1)e^{-kx} & \text{for} \quad x > L \end{cases}$$

Imposing the last boundary condition now,

$$(-kAe^{kL} - kAe^{-kL}) - (kAe^{kL} - kAe^{-kL}) = -\frac{2m}{\hbar^2}c_0 A(e^{kL} + e^{-kL})$$

$$-2kAe^{kL} = -\frac{2m}{\hbar^2}c_0 A(e^{kL} + e^{-kL})$$

$$k = \frac{m}{\hbar^2}c_0(1 + e^{-2kL})$$

$$\left(\frac{\hbar^2}{m}\right)\frac{k}{c_0} - 1 = e^{-2kL}.$$

This is the transcendental equation we were looking for. In dimensionless variables we have

$$z \equiv 2kL; \quad a \equiv \frac{\hbar^2}{2mc_0 L}$$

$$az - 1 = e^{-z}$$

which is simply the graph of a decaying exponential and a line with positive slope. We see that there is exactly one solution.

(b) Derive the transcendental equation for the energy E_1 of the first (and only) excited state, show (by plotting an appropriate freehand graph) that it has a solution only if the constant c_0 is above a certain threshold c_{min}, i.e. $c_0 > c_{min}$, and determine the value of c_{min}.

The next excited state will be the odd solution. Following the same routine as earlier we start with

$$\psi(x) = \begin{cases} Ae^{kx} + Be^{-kx} & \text{for} \quad 0 < x < L \\ Ce^{-kx} & \text{for} \quad x > L \end{cases}$$

except boundary condition ii. this time is changed to

i. $\psi(L)_L = \psi(L)_R$; ψ is everywhere continuous

ii. $\psi(0) = 0$

iii. $\Delta \left.\frac{d\psi}{dx}\right|_L = -\frac{2m}{\hbar^2}c_0\psi(L)$.

Starting from boundary condition ii. we have

$$A = -B.$$

Now applying this to B.C. i., we have

$$Ae^{kL} - Ae^{-kL} = Ce^{-kL}$$

$$A(e^{2kL} - 1) = C$$

and so our wavefunction goes as

$$\psi(x) = \begin{cases} A(e^{kx} - e^{-kx}) & \text{for} \quad 0 < x < L \\ A(e^{2kL} - 1)e^{-kx} & \text{for} \quad x > L \end{cases}$$

Moving on to B.C. iii,

$$-kAe^{kL} + kAe^{-kL} - (kAe^{kL} + kAe^{-kL}) = -\frac{2m}{\hbar^2} c_0 A(e^{kL} - e^{-kL})$$

$$\begin{aligned} -2kAe^{kL} &= -\frac{2m}{\hbar^2} c_0 A(e^{kL} - e^{-kL}) \\ k &= \frac{m}{\hbar^2} c_0 (1 - e^{-2kL}) \\ -\frac{\hbar^2}{mc_0} k + 1 &= e^{-2kL}. \end{aligned}$$

In the same dimensionless units,

$$z \equiv 2kL; \quad a \equiv \frac{\hbar^2}{2mc_0 L}$$

we have

$$-az + 1 = e^{-z}.$$

Note that a is positive. If we are careful in analyzing the graph, we can see that there can only be one intersection if the slope of of the line is greater than the slope of the exponential at the y-intercept $z = 0$. The slope of the exponential is easily found to be

$$(e^{-z})'|_0 = -1$$

and thus we find our condition is

$$a < 1.$$

Imposing this in terms of our previous variables we have

$$\frac{\hbar^2}{2mc_0 L} < 1 \quad \rightarrow \quad c_0 > \frac{\hbar^2}{2mL}.$$

Therefore we can have up to exactly two bound states if $c_0 > c_{min} = \frac{\hbar^2}{2mL}$.

2014 Summer

1. Let us define $D = \frac{1}{2}(xp + px)$, where x is the position operator and p the momentum operator in one dimension.

   (a) Calculate $[D, x^m]$ and $[D, p^n]$ where m and n are integers.

   Typically, we strive to use the equation

$$[f(A), B] = \frac{\partial f}{\partial A}[A, B]$$

which is valid only if the commutators commute, i.e. $[A,[A,B]] = [B,[A,B]] = 0$. Let's see if this last condition holds for the operators given:

$$\begin{aligned}
[x, D] &= \frac{1}{2}[x, xp] + \frac{1}{2}[x, px] \\
&= \frac{1}{2}([x, x]p + x[x, p]) + \frac{1}{2}([x, p]x + p[x, x]) \\
&= \frac{1}{2}(i\hbar x) + \frac{1}{2}(i\hbar x) \\
&= i\hbar x
\end{aligned}$$

$$\begin{aligned}
[p, D] &= \frac{1}{2}[p, xp] + \frac{1}{2}[p, px] \\
&= \frac{1}{2}([p, x]p + x[p, p]) + \frac{1}{2}([p, p]x + p[p, x]) \\
&= \frac{1}{2}(-i\hbar p) + \frac{1}{2}(-i\hbar p) \\
&= -i\hbar p.
\end{aligned}$$

Now we can observe that

$$\begin{aligned}
[x,[x,D]] = [x,(i\hbar x)] = 0; &\quad [D,[x,D]] = [D,(i\hbar x)] = -i\hbar[x,D] = \hbar^2 x \\
[p,[p,D]] = [p,(-i\hbar p)] = 0; &\quad [D,[p,D]] = [D,(-i\hbar p)] = i\hbar[p,D] = \hbar^2 p.
\end{aligned}$$

From the right hand equations, apparently we cannot use our simple derivative rule for commutators because the commutators don't commute. Not to worry, we can still easily compute the desired commutators by simply invoking the identity that we have already used multiple times, namely

$$[A, BC] = [A, B]C + B[A, C].$$

Applying this to our quantities of interest we have

$$\begin{aligned}
[x^m, D] &= \frac{1}{2}[x^m, xp] + \frac{1}{2}[x^m, px] = \frac{1}{2}([x^m, x]p + x[x^m, p]) + \frac{1}{2}([x^m, p]x + p[x^m, x]) \\
&= xmx^{m-1}i\hbar = mi\hbar x^m
\end{aligned}$$

$$\begin{aligned}
[p^m, D] &= \frac{1}{2}[p^m, xp] + \frac{1}{2}[p^m, px] = \frac{1}{2}([p^m, x]p + x[p^m, p]) + \frac{1}{2}([p^m, p]x + p[p^m, x]) \\
&= pmp^{m-1}(-i\hbar) = -mi\hbar p^m
\end{aligned}$$

where this time we have in fact used our derivative rule since it is valid here. Therefore we alas have

$$\begin{aligned}
[D, x^m] &= -im\hbar x^m \\
[D, p^m] &= im\hbar p^m.
\end{aligned}$$

(b) Consider the Hamiltonian operator $H = \dfrac{p^2}{2m} + V(x)$ with the potential $V(x) = \alpha x^\beta$ where α and β are real non-zero constants. Calculate $U(\lambda)HU^\dagger(\lambda)$ with $U(\lambda) = \exp(i\lambda D/\hbar)$.

First note that D is hermitian. Therfore, assuming λ is real, we have

$$U^\dagger(\lambda) = \exp(-i\lambda D^\dagger/\hbar) = \exp(-i\lambda D/\hbar).$$

This unitary operator is some mixture between both translation in position space and momentum space. Interesting. And we are essentially computing the expectation value of the Hamiltonian (expectation energy). Anyway, we can now use the hint to decompose our problem as

$$\begin{aligned}
U(\lambda)HU^\dagger(\lambda) &= \exp(i\lambda D/\hbar)H\exp(-i\lambda D/\hbar) \\
&= H + \left(\frac{i\lambda}{1!\hbar}\right)[D,H] + \left(\frac{i\lambda}{2!\hbar}\right)^2[D,[D,H]] + \left(\frac{i\lambda}{3!\hbar}\right)^3[D,[D,[D,H]]].
\end{aligned}$$

To solve this we need to calculate

$$\begin{aligned}
[D,H] &= [D, p^2/2m + \alpha x^\beta] \\
&= \frac{1}{2m}[D,p^2] + \alpha[D,x^\beta] \\
&= \frac{p^2}{2m}(2i\hbar) + (-\beta i\hbar)\alpha x^\beta.
\end{aligned}$$

We can start to see the pattern emerging. To allow easier computation, note that

$$[D,p^2] = 2i\hbar p^2; \quad [D,x^\beta] = -\beta i\hbar x^\beta.$$

Now observe

$$\begin{aligned}
[D,[D,H]] &= \frac{1}{2m}[D,[D,p^2]] + \alpha[D,[D,x^\beta]] \\
&= \frac{1}{2m}(2i\hbar)[D,p^2] + \alpha(-\beta i\hbar)[D,x^\beta] \\
&= \frac{p^2}{2m}(2i\hbar)^2 + (-\beta i\hbar)^2\alpha x^\beta \\
[D,[D,[D,H]]] &= \frac{1}{2m}(2i\hbar)^2[D,p^2] + \alpha(-\beta i\hbar)^2[D,x^\beta] \\
&= \frac{p^2}{2m}(2i\hbar)^3 + (-\beta i\hbar)^3\alpha x^\beta.
\end{aligned}$$

We can now rewrite our desired product as

$$\begin{aligned}
\exp(i\lambda D/\hbar)H\exp(-i\lambda D/\hbar) &= \frac{p^2}{2m}\sum_{n=0}^{\infty}\frac{(i\lambda)^n}{\hbar^n n!}(2i\hbar)^n + \alpha x^\beta\sum_{n=0}^{\infty}\frac{(i\lambda)^n}{\hbar^n n!}(-\beta i\hbar)^n \\
U(\lambda)HU^\dagger(\lambda) &= \frac{p^2}{2m}\exp(-2\lambda) + \alpha x^\beta\exp(\beta\lambda).
\end{aligned}$$

(c) There is a value for β in the potential $V(x) = \alpha x^\beta$ for which the Hamiltonian in part (b) transforms as $U(\lambda)HU^\dagger(\lambda) = f(\lambda)H$. What is the function $f(\lambda)$?

We can easily see that for $\beta = -2$ we could express

$$U(\lambda)HU^\dagger(\lambda) = \exp(-2\lambda)H$$

and thus

$$f(\lambda) = \exp(-2\lambda).$$

Hints: Recall the identity for two non-commuting linear operators A and B:

$$\exp(\lambda A)B\exp(-\lambda A) = B + \frac{\lambda^1}{1!}[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + ...$$

You may do the mathematics formally, ignoring issues such as the precise definitions and domains of various operators.

2. Problem

   (a)

   (b)

   (c)

3. A particle of mass m and electric charge q is constrained to move in a tightly confining ring of radius R; call the remaining coordinate along the ring x. The motion along x is free, i.e., there are no forces acting on the particle in the direction x. Determine:

   (a) Eigenvalues and eigenfunctions of energy.

   (b) The maximum value of the electric current I in the first excited state.

   Hint: The current density of a quantum particle is $\mathbf{j} = \frac{i\hbar q}{2m}(\psi\nabla\psi^* - \psi\nabla\psi)$.

4. Problem

   (a)

   (b)

   (c)

5. Consider the Hamiltonian

$$H = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + V|2\rangle\langle 1| + V^*|1\rangle\langle 2|$$

with $|V| \ll |E_2 - E_1|$.

   (a) Find the eigenvalues of energy and the corresponding normalized eigenstates up to the lowest nontrivial order in the strength of the perturbation V. Denote these by E_1', $|1'\rangle$ and E_2', $|2'\rangle$, with $E_1' \to E_1$ as $V \to 0$ and so on.

(b) Suppose we are studying transitions from yet another state $|g\rangle$ to the states $|1\rangle$ and $|2\rangle$ governed by the operator D, and have the known transition matrix elements $\langle 1|D|g\rangle = d$, $\langle 2|D|g\rangle = 0$. At this level the transition $g \to 2$ is evidently forbidden. However, the perturbation V leads to a small admixture of the original state $|1\rangle$ in the state $|2'\rangle$. Thus a transition that to an observer unaware of the existence of the perturbation V might seem to be $g \to 2$ is possible after all. Find the corresponding matrix element $\langle 2'|D|g\rangle$.

2014 Winter

1. Let $H = H_{kin} + V(\mathbf{x})$ be a single-particle Hamiltonian operator with $H_{kin} = \frac{\mathbf{p}^2}{2m}$ and m the mass of the particle. Consider the operator $D = \frac{1}{2}(\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x})$.

   (a) Calculate the commutators $[D, \mathbf{x}]$ and $[D, \mathbf{p}]$.

   (b) Calculate $[D, F(\mathbf{x})]$ and $[D, G(\mathbf{p})]$ where F and G are differentiable functions. You may want to work out the commutators in position or momentum space.

   (c) Let $H\,|E_i\rangle = E_i\,|E_i\rangle$. Calculate $[D, H]$ and prove that $2\,\langle E_i|H_{kin}|E_i\rangle = \langle E_i|\mathbf{x}\cdot\nabla V(\mathbf{x})|E_i\rangle$, which is the quantum mechanical virial theorem.