SVT Decomposition

The most general form we may take for our perturbation to the FRW metric is

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \gamma_{ij}dx^i dx^j + h_{\mu\nu}dx^\mu dx^\nu\right].$$

Seperating out $h_{\mu\nu}$ into space and time components,

$$ds^2 = a^2(\tau)\left[-(1+\psi)d\tau^2 + w_i dx^i d\tau + (\phi\gamma_{ij} + S_{ij})\right].$$

Here we have taken the trace of h_{ij} and placed it in ϕ so that S_{ij} is a symmetric traceless tensor. Given the 3 vector w_i, we may decompose it into its scalar (curl-free) and vector (divergence-free) components

$$w_i = w_i^{(S)} + w_i^{(V)}$$

where

$$w_i^{(S)} = \nabla_i w$$
$$\nabla^i w_i^{(V)} = 0.$$

Our symmetric traceless tensor S_{ij} may also be broken up accordingly

$$S_{ij} = S_{ij}^{(S)} + S_{ij}^{(V)} + S_{ij}^{(T)}$$

where

$$S_{ij}^{(S)} = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S$$
$$S_{ij}^{(V)} = \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right), \qquad \nabla^i S_i = 0$$
$$\nabla^i S_{ij}^{(T)} = 0.$$

Note that the vector S_i is divergence-less, which is as we expect for a true vector, and that S_{ij}^T is transverse. Additionally, the actual scalar, vector, and tensor functions are not unique. If we put all these together, our metric takes the form

$$ds^2 = a(\tau^2)\left\{-(1+\psi)d\tau^2 + (\nabla_i w + w_i)dx^i d\tau + \left[\phi\gamma_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S + \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right) + S_{ij}^T\right]\right\}.$$

Mode superscripts have been dropped, so we make take a vector like w_i to be a true vector $\nabla^i w_i = 0$. Counting the degrees of freedom, we have 4 scalar fields (ψ, w, S, ϕ), 2 two-component vectors (w_i, S_i), and one traceless transverse symmetric tensor S_{ij}^T, $4 + 4 + 2 = 10$.

$$h_{00} = -\psi$$
$$h_{0i} = \nabla_i w + w_i$$
$$h_{ij} = \phi\gamma_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S + \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right) + S_{ij}^T$$

In flat space $h = \psi + 3\phi$ and

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\nu\lambda} - \partial_\nu\partial^\lambda h_{\mu\lambda} + \partial_\mu\partial_\nu h^\lambda{}_\lambda\right)$$

$$\begin{aligned}\delta R_{00} &= \frac{1}{2}\left(\partial^\lambda\partial_\lambda h_{00} - 2\partial_0\partial^\lambda h_{0\lambda} + \ddot{h}\right) \\ &= \frac{1}{2}\left[-\partial_i\partial^i\psi - 2\partial_0(\nabla^2 w + \partial^i w_i) + 2\ddot{\phi}\right]\end{aligned}$$

Under the transverse or synchronous gauge, we see that δR_{00} relates scalars.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$\begin{aligned}\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\left(h_{\alpha\beta}R^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right) \\ \delta G^\mu_\nu &= \delta(g^{\mu\lambda}G_{\lambda\nu}) = h^{\mu\lambda}G_{\lambda\nu} + g^{\mu\lambda}\delta G_{\lambda\nu}\end{aligned}$$

Synchronous:

$$\begin{aligned}\delta G^0{}_0 &= g^{00}\delta G_{00} = -a^{-2}(\tau)\delta G_{00} \\ \delta G_{00} &= \delta R_{00} - \frac{1}{2}h_{00}R + \frac{1}{2}a^2(\tau)\left(h_{\alpha\beta}R^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)\end{aligned}$$

$h_{\mu\nu}$ is not trace-free. In conformal gravity, we may do the standard perscription with perturbations involving $h_{\mu\nu}$ and we will find that the fluctation equations do not depend on the trace, h. Equivalently, we may instead work with traceless "gauge" $K_{\mu\nu}$. What we cannot do within the code is remove the trace terms h, and then use the full gauge $h_{\mu\nu}$. However, we can do the substitution $h_{\mu\nu} \to K_{\mu\nu}$ without problems.