Cosmological Fluctuations in Standard and Conformal Gravity

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Doctoral Degree Final Examination

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- Introduction and Formalism
- Conformal Gravity

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  - Cosmological Geometries
  - Einstein Gravity
  - Perturbation Theory
  - Gauge Transformations

Cosmological Geometries

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker (flat, spherical, hyperbolic), de Sitter ($dS_4 \subset \mathrm{RW}$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

$$ds^2 = \Omega(x)^2 \big(-dt^2 + dx^2 + dy^2 + dz^2 \big)$$

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -3k\tilde{g}_{ij}, \qquad R = -6k$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$$

Cosmological Geometries R.W.

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with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

set $k = 0$ (flat), simple conformal to flat form

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$$

$k = 1$ (spherical)

$$ds^2 = a(\tau)^2\Big[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\Big]$$

Set $\sin\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\Big[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\Big]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p+\chi)/2], \quad p' - r' = \tan[(p-\chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos\chi}, \quad r' = \frac{\sin\chi}{\cos p + \cos\chi} \end{aligned}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1+(p'+r')^2][1+(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]$$

$k = -1$ (hyperbolic)

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

Set $\sin\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\left[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\right]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tanh[(p+\chi)/2], \quad p' - r' = \tanh[(p-\chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh\chi}, \quad r' = \frac{\sinh\chi}{\cosh p + \cosh\chi} \end{aligned}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1-(p'+r')^2][1-(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]$$

Einstein Hilbert action

$$I_{\mathsf{EH}} = -\frac{1}{16\pi G}\int d^4x(-g)^{1/2}g^{\mu\nu}R_{\mu\nu}.$$

Functional variation w.r.t $g_{\mu\nu}$ yields Einstein tensor,

$$\frac{16\pi G}{(-g)^{1/2}}\frac{\delta I_{\mathsf{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha},$$

likewise, variation of matter action I_{M} w.r.t $g_{\mu\nu}$ yields Energy Momentum tensor

$$\frac{2}{(-g)^{1/2}}\frac{\delta I_{\mathsf{M}}}{\delta g_{\mu\nu}} = T_{\mu\nu}.$$

Requiring sum of actions to be stationary gives us Einstein field equations

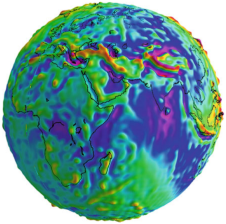
$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha} = -8\pi G T^{\mu\nu},$$

subject to Bianchi identity

$$\nabla_{\mu}R^{\mu\nu} = \frac{1}{2}\nabla^{\nu}R^{\mu}{}_{\mu} \implies \nabla_{\mu}G^{\mu\nu} = 0.$$

Decompose metric into background and fluctuation, truncating at linear order

$$
\begin{aligned}
g_{\mu\nu}(x) &= g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad g^{\mu\nu}_{(0)} h_{\mu\nu} \equiv h \\
G_{\mu\nu} &= G_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) \\
G^{(0)}_{\mu\nu} &= R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(0)\alpha}_{\alpha} \\
\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R^{(0)\alpha}_{\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}{}_{\alpha}.
\end{aligned}
$$



[1]

Likewise perturb $T_{\mu\nu}$ around background

$$
T_{\mu\nu} = T_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu})
$$

Form background and first order equations of motion (upon setting $8\pi G = 1$)

$$
\begin{aligned}
\Delta^{0}_{\mu\nu} &= G^{(0)}_{\mu\nu} + T^{(0)}_{\mu\nu} = 0 \\
\Delta_{\mu\nu} &= \delta G^{(0)}_{\mu\nu} + \delta T^{(0)}_{\mu\nu} = 0
\end{aligned}
$$

[1] Walter, U. (2019). Correction to: Astronautics. In Astronautics (pp. C1–C1). Springer International Publishing.

- Under coordinate transformation $x^\mu \to x^\mu - \epsilon^\mu(x)$, with $\epsilon^\mu \sim \mathcal{O}(h)$, the peturbed metric transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$$

- For every solution $h_{\mu\nu}$ to $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$, a transformed $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ will also serve as a solution
- Set of four $\epsilon^\mu(x)$ define gauge freedom under coordinate transformation
- 10 components in $h_{\mu\nu}$, 4 coordinate transformations, leads to 6 independent degrees of freedom
- Under $x^\mu \to x^\mu - \epsilon^\mu(x)$, the perturbed tensors transform as

$$\delta G_{\mu\nu} \to \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda G^{(0)}_{\mu\nu} \epsilon^\lambda$$
$$\delta T_{\mu\nu} \to \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda T^{(0)}_{\mu\nu} \epsilon^\lambda.$$

- If background $G^{(0)}_{\mu\nu} = 0$, then $\delta G_{\mu\nu}$ separately gauge invariant; likewise for vanishing background energy momentum tensor
- If $G^{(0)}_{\mu\nu} \neq 0$, then only the entire $\Delta_{\mu\nu} = \delta G_{\mu\nu} + T_{\mu\nu}$ gauge invariant

Perturbed equations of motion $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ form a rather complex and extensive set of coupled non-linear tensor PDE's

Much effort involved in simplifying, decoupling, and solving them

Two main approaches

- Fix the gauge by constraining $h_{\mu\nu}$, e.g. transverse gauge $\nabla^\mu h_{\mu\nu} = 0$, then solve fluctuation equations directly in terms of $h_{\mu\nu}$
    - Simplification usually not effective in more general curved backgrounds
- Decompose $h_{\mu\nu}$ into a basis of scalars, vectors, and tensors, express in terms of gauge invariant combinations, and solve fluctuation equations with possible decoupling between modes
    - SVT Decomposition, de facto approach in modern cosmology

$$
\begin{aligned}
\delta G_{ij} \quad = \quad & -\tfrac{1}{2}\ddot{f}_{ij} + \tfrac{1}{2}\ddot{f}_{00}\tilde{g}_{ij} + \tfrac{1}{2}\ddot{f}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}f_{ab} + 3kf_{ij} - \dot{\Omega}^2 f_{ij}\Omega^{-2} - \dot{\Omega}^2\tilde{g}_{ij}f_{00}\Omega^{-2} \\
& -\dot{f}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{f}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{f}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}f_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}f_{00}\Omega^{-1} \\
& +2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}f_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{f}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{f}_{0a} \\
& -4\tilde{g}^{ba}\tilde{g}_{ij}f_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b f_{ij} \\
& -2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b f_{0a} - \tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a f\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}f_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
& +\tilde{g}^{ba}f_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a f_{ij} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a f - 2\tilde{g}^{ba}f_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& -\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i f_{ja} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j f_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d f_{cb} \\
& +\tfrac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c f_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}f_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \tfrac{1}{2}\tilde{\nabla}_i\dot{f}_{0j} \\
& -\tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i f_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i f_{0j} + \tfrac{1}{2}\tilde{\nabla}_j\dot{f}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j f_{ib} \\
& +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j f_{0i} + \tfrac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i f,
\end{aligned}
\tag{1}
$$