

# Vector RW PDE's

## 1 Polar RW

$$\tilde{g}_{ij} = \text{Diag} \left( \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right) \quad (1.1)$$

$$A^i = \tilde{\nabla}_a \tilde{\nabla}^a B^i \quad (1.2)$$

$$\tilde{\nabla}_a B^a = B_1(2r^{-1} - 3kr) + B_2 \cos \theta r^{-2} (\sin \theta)^{-1} + (1 - kr^2) \partial_1 B_1 + r^{-2} \partial_2 B_2 + r^{-2} (\sin \theta)^{-2} \partial_3 B_3 \quad (1.3)$$

$$\begin{aligned} A^1 = & B_1(k - 2r^{-2} + k^2 r^2) + B_2 \cos \theta (-2r^{-3} + 2kr^{-1}) (\sin \theta)^{-1} + (2r^{-1} - 7kr + 5k^2 r^3) \partial_1 B_1 \\ & + (1 - 2kr^2 + k^2 r^4) \partial_1 \partial_1 B_1 + \cos \theta (-k + r^{-2}) (\sin \theta)^{-1} \partial_2 B_1 + (-2r^{-3} + 2kr^{-1}) \partial_2 B_2 \\ & + (-k + r^{-2}) \partial_2 \partial_2 B_1 + (-2r^{-3} + 2kr^{-1}) (\sin \theta)^{-2} \partial_3 B_3 + (-k + r^{-2}) (\sin \theta)^{-2} \partial_3 \partial_3 B_1 \end{aligned} \quad (1.4)$$

$$\text{Substituting } \partial_2 B_2 \quad (1.5)$$

$$\begin{aligned} A^1 = & B_1(-9k + 2r^{-2} + 7k^2 r^2) + (4r^{-1} - 11kr + 7k^2 r^3) \partial_1 B_1 + (1 - 2kr^2 + k^2 r^4) \partial_1 \partial_1 B_1 \\ & + \cos \theta (-k + r^{-2}) (\sin \theta)^{-1} \partial_2 B_1 + (-k + r^{-2}) \partial_2 \partial_2 B_1 + (-k + r^{-2}) (\sin \theta)^{-2} \partial_3 \partial_3 B_1 \end{aligned} \quad (1.6)$$

$$(1.7)$$

$$\begin{aligned} A^2 = & B_2(2kr^{-2} - r^{-4} (\sin \theta)^{-2}) - kr^{-1} \partial_1 B_2 + (-k + r^{-2}) \partial_1 \partial_1 B_2 + (2r^{-3} - 2kr^{-1}) \partial_2 B_1 \\ & + \cos \theta r^{-4} (\sin \theta)^{-1} \partial_2 B_2 + r^{-4} \partial_2 \partial_2 B_2 - 2 \cos \theta r^{-4} (\sin \theta)^{-3} \partial_3 B_3 + r^{-4} (\sin \theta)^{-2} \partial_3 \partial_3 B_2 \end{aligned} \quad (1.8)$$

$$\text{Substituting } \partial_3 B_3 \quad (1.9)$$

$$\begin{aligned} A^2 = & B_2(-2r^{-4} + 2kr^{-2} + r^{-4} (\sin \theta)^{-2}) + B_1 \cos \theta (4r^{-3} - 6kr^{-1}) (\sin \theta)^{-1} \\ & + \cos \theta (-2k + 2r^{-2}) (\sin \theta)^{-1} \partial_1 B_1 - kr^{-1} \partial_1 B_2 + (-k + r^{-2}) \partial_1 \partial_1 B_2 + (2r^{-3} - 2kr^{-1}) \partial_2 B_1 \\ & + 3 \cos \theta r^{-4} (\sin \theta)^{-1} \partial_2 B_2 + r^{-4} \partial_2 \partial_2 B_2 + r^{-4} (\sin \theta)^{-2} \partial_3 \partial_3 B_2 \end{aligned} \quad (1.10)$$

$$(1.11)$$

$$\begin{aligned} A^3 = & 2kB_3 r^{-2} (\sin \theta)^{-2} - kr^{-1} (\sin \theta)^{-2} \partial_1 B_3 + (-k + r^{-2}) (\sin \theta)^{-2} \partial_1 \partial_1 B_3 \\ & - \cos \theta r^{-4} (\sin \theta)^{-3} \partial_2 B_3 + r^{-4} (\sin \theta)^{-2} \partial_2 \partial_2 B_3 + (2r^{-3} - 2kr^{-1}) (\sin \theta)^{-2} \partial_3 B_1 \\ & + 2 \cos \theta r^{-4} (\sin \theta)^{-3} \partial_3 B_2 + r^{-4} (\sin \theta)^{-4} \partial_3 \partial_3 B_3 \end{aligned} \quad (1.12)$$

## 2 Polar Conformal RW

$$\tilde{g}_{ij} = (1 + \rho^2 k/4)^{-2} \text{Diag}(1, \rho^2, \rho^2 \sin^2 \theta) \quad (2.1)$$

$$A^i = \tilde{\nabla}_a \tilde{\nabla}^a B^i \quad (2.2)$$

$$\begin{aligned} \tilde{\nabla}_a B^a &= B_1(2\rho^{-1} + \frac{1}{2}k\rho) + B_2 \cos \theta (\frac{1}{2}k + \rho^{-2} + \frac{1}{16}k^2\rho^2)(\sin \theta)^{-1} + (1 + \frac{1}{2}k\rho^2 + \frac{1}{16}k^2\rho^4)\partial_1 B_1 \\ &\quad + (\frac{1}{2}k + \rho^{-2} + \frac{1}{16}k^2\rho^2)\partial_2 B_2 + (\frac{1}{2}k + \rho^{-2} + \frac{1}{16}k^2\rho^2)(\sin \theta)^{-2}\partial_3 B_3 \end{aligned} \quad (2.3)$$

$$\begin{aligned} A^1 &= B_1(\frac{3}{2}k - 2\rho^{-2} + \frac{9}{8}k^2\rho^2 + \frac{5}{32}k^3\rho^4) + B_2 \cos \theta (-2\rho^{-3} - k\rho^{-1} + \frac{1}{16}k^3\rho^3 + \frac{1}{128}k^4\rho^5)(\sin \theta)^{-1} \\ &\quad + (2\rho^{-1} + \frac{5}{2}k\rho + \frac{9}{8}k^2\rho^3 + \frac{7}{32}k^3\rho^5 + \frac{1}{64}k^4\rho^7)\partial_1 B_1 \\ &\quad + (1 + k\rho^2 + \frac{3}{8}k^2\rho^4 + \frac{1}{16}k^3\rho^6 + \frac{1}{256}k^4\rho^8)\partial_1 \partial_1 B_1 \\ &\quad + \cos \theta (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)(\sin \theta)^{-1}\partial_2 B_1 \\ &\quad + (-2\rho^{-3} - k\rho^{-1} + \frac{1}{16}k^3\rho^3 + \frac{1}{128}k^4\rho^5)\partial_2 B_2 \\ &\quad + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)\partial_2 \partial_2 B_1 \\ &\quad + (-2\rho^{-3} - k\rho^{-1} + \frac{1}{16}k^3\rho^3 + \frac{1}{128}k^4\rho^5)(\sin \theta)^{-2}\partial_3 B_3 \\ &\quad + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)(\sin \theta)^{-2}\partial_3 \partial_3 B_1 \end{aligned} \quad (2.4)$$

$$\text{Substituting } \partial_2 B_2 \quad (2.5)$$

$$\begin{aligned} A^1 &= B_1(\frac{5}{2}k + 2\rho^{-2} + \frac{7}{8}k^2\rho^2 + \frac{3}{32}k^3\rho^4) + (4\rho^{-1} + \frac{7}{2}k\rho + \frac{9}{8}k^2\rho^3 + \frac{5}{32}k^3\rho^5 + \frac{1}{128}k^4\rho^7)\partial_1 B_1 \\ &\quad + (1 + k\rho^2 + \frac{3}{8}k^2\rho^4 + \frac{1}{16}k^3\rho^6 + \frac{1}{256}k^4\rho^8)\partial_1 \partial_1 B_1 \\ &\quad + \cos \theta (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)(\sin \theta)^{-1}\partial_2 B_1 \\ &\quad + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)\partial_2 \partial_2 B_1 \\ &\quad + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)(\sin \theta)^{-2}\partial_3 \partial_3 B_1 \end{aligned} \quad (2.6)$$

$$(2.7)$$

$$\begin{aligned} A^2 &= B_2(k^2 + 2k\rho^{-2} + \frac{1}{8}k^3\rho^2 + (-\frac{3}{8}k^2 - \rho^{-4} - k\rho^{-2} - \frac{1}{16}k^3\rho^2 - \frac{1}{256}k^4\rho^4)(\sin \theta)^{-2}) \\ &\quad + (\frac{1}{2}k\rho^{-1} + \frac{3}{8}k^2\rho + \frac{3}{32}k^3\rho^3 + \frac{1}{128}k^4\rho^5)\partial_1 B_2 \\ &\quad + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)\partial_1 \partial_1 B_2 + (2\rho^{-3} + k\rho^{-1} - \frac{1}{16}k^3\rho^3 - \frac{1}{128}k^4\rho^5)\partial_2 B_1 \\ &\quad + \cos \theta (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin \theta)^{-1}\partial_2 B_2 \\ &\quad + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)\partial_2 \partial_2 B_2 \\ &\quad + \cos \theta (-\frac{3}{4}k^2 - 2\rho^{-4} - 2k\rho^{-2} - \frac{1}{8}k^3\rho^2 - \frac{1}{128}k^4\rho^4)(\sin \theta)^{-3}\partial_3 B_3 \\ &\quad + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin \theta)^{-2}\partial_3 \partial_3 B_2 \end{aligned} \quad (2.8)$$

$$\text{Substituting } \partial_3 B_3 \quad (2.9)$$

$$\begin{aligned} A^2 &= B_2(\frac{1}{4}k^2 - 2\rho^{-4} - \frac{1}{128}k^4\rho^4 + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin \theta)^{-2}) \\ &\quad + B_1 \cos \theta (4\rho^{-3} + 3k\rho^{-1} + \frac{3}{4}k^2\rho + \frac{1}{16}k^3\rho^3)(\sin \theta)^{-1} \\ &\quad + \cos \theta (2k + 2\rho^{-2} + \frac{3}{4}k^2\rho^2 + \frac{1}{8}k^3\rho^4 + \frac{1}{128}k^4\rho^6)(\sin \theta)^{-1}\partial_1 B_1 \\ &\quad + (\frac{1}{2}k\rho^{-1} + \frac{3}{8}k^2\rho + \frac{3}{32}k^3\rho^3 + \frac{1}{128}k^4\rho^5)\partial_1 B_2 \end{aligned}$$

$$\begin{aligned}
& + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)\partial_1\partial_1B_2 + (2\rho^{-3} + k\rho^{-1} - \frac{1}{16}k^3\rho^3 - \frac{1}{128}k^4\rho^5)\partial_2B_1 \\
& + \cos\theta(\frac{9}{8}k^2 + 3\rho^{-4} + 3k\rho^{-2} + \frac{3}{16}k^3\rho^2 + \frac{3}{256}k^4\rho^4)(\sin\theta)^{-1}\partial_2B_2 \\
& + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)\partial_2\partial_2B_2 \\
& + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\partial_3\partial_3B_2
\end{aligned} \tag{2.10}$$

$$(2.11)$$

$$\begin{aligned}
A^3 &= B_3(k^2 + 2k\rho^{-2} + \frac{1}{8}k^3\rho^2)(\sin\theta)^{-2} + (\frac{1}{2}k\rho^{-1} + \frac{3}{8}k^2\rho + \frac{3}{32}k^3\rho^3 + \frac{1}{128}k^4\rho^5)(\sin\theta)^{-2}\partial_1B_3 \\
& + (k + \rho^{-2} + \frac{3}{8}k^2\rho^2 + \frac{1}{16}k^3\rho^4 + \frac{1}{256}k^4\rho^6)(\sin\theta)^{-2}\partial_1\partial_1B_3 \\
& + \cos\theta(-\frac{3}{8}k^2 - \rho^{-4} - k\rho^{-2} - \frac{1}{16}k^3\rho^2 - \frac{1}{256}k^4\rho^4)(\sin\theta)^{-3}\partial_2B_3 \\
& + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\partial_2\partial_2B_3 \\
& + (2\rho^{-3} + k\rho^{-1} - \frac{1}{16}k^3\rho^3 - \frac{1}{128}k^4\rho^5)(\sin\theta)^{-2}\partial_3B_1 \\
& + \cos\theta(\frac{3}{4}k^2 + 2\rho^{-4} + 2k\rho^{-2} + \frac{1}{8}k^3\rho^2 + \frac{1}{128}k^4\rho^4)(\sin\theta)^{-3}\partial_3B_2 \\
& + (\frac{3}{8}k^2 + \rho^{-4} + k\rho^{-2} + \frac{1}{16}k^3\rho^2 + \frac{1}{256}k^4\rho^4)(\sin\theta)^{-4}\partial_3\partial_3B_3
\end{aligned} \tag{2.12}$$

### 3 $\sinh\chi$ RW

$$\tilde{g}_{ij} = \text{Diag}(1, \sinh^2\chi, \sinh^2\chi \sin^2\theta) \tag{3.1}$$

$$A^i = \tilde{\nabla}_a \tilde{\nabla}^a B^i \tag{3.2}$$

#### 3.1 $B_i = g_i(\chi)$

$$\tilde{\nabla}_a B^a = \dot{g}_1 + g_2 \cos\theta (\sin\theta)^{-1} (\sinh\chi)^{-2} + 2g_1 \cosh\chi (\sinh\chi)^{-1} = 0 \tag{3.3}$$

$$A^1 = -2g_1 + \ddot{g}_1 - 2g_2 \cos\theta \cosh\chi (\sin\theta)^{-1} (\sinh\chi)^{-3} - 2g_1 (\sinh\chi)^{-2} + 2 \cosh\chi \dot{g}_1 (\sinh\chi)^{-1} \tag{3.4}$$

$$\text{Substituting } g_2 \tag{3.5}$$

$$A^1 = \ddot{g}_1 + g_1(2 + 2(\sinh\chi)^{-2}) + 4 \cosh\chi \dot{g}_1 (\sinh\chi)^{-1} \tag{3.6}$$

$$A^2 = -g_2 (\sin\theta)^{-2} (\sinh\chi)^{-4} - 2g_2 (\sinh\chi)^{-2} + \ddot{g}_2 (\sinh\chi)^{-2} \tag{3.7}$$

$$A^3 = -2g_3 (\sin\theta)^{-2} (\sinh\chi)^{-2} + \ddot{g}_3 (\sin\theta)^{-2} (\sinh\chi)^{-2} \tag{3.8}$$

#### 3.2 $B_i = h_i(\chi) \cos\theta$

$$\tilde{\nabla}_a B^a = \cos\theta \dot{h}_1 + h_2 (\sin\theta)^{-1} (\sinh\chi)^{-2} - 2h_2 \sin\theta (\sinh\chi)^{-2} + 2h_1 \cos\theta \cosh\chi (\sinh\chi)^{-1} = 0 \tag{3.9}$$

$$\begin{aligned}
A^1 &= \cos\theta \ddot{h}_1 + h_2(-2 \cosh\chi (\sin\theta)^{-1} (\sinh\chi)^{-3} + 4 \cosh\chi \sin\theta (\sinh\chi)^{-3}) \\
&+ h_1(-2 \cos\theta - 4 \cos\theta (\sinh\chi)^{-2}) + 2 \cos\theta \cosh\chi \dot{h}_1 (\sinh\chi)^{-1}
\end{aligned} \tag{3.10}$$

$$\text{Substituting } h_2 \tag{3.11}$$

$$A^1 = 2h_1 \cos \theta + \cos \theta \ddot{h}_1 + 4 \cos \theta \cosh \chi \dot{h}_1 (\sinh \chi)^{-1} \quad (3.12)$$

$$\begin{aligned} A^2 = & h_2 \left( -2 \cos \theta (\sinh \chi)^{-4} - \cos \theta (\sin \theta)^{-2} (\sinh \chi)^{-4} - 2 \cos \theta (\sinh \chi)^{-2} \right) \\ & - 2h_1 \cosh \chi \sin \theta (\sinh \chi)^{-3} + \cos \theta \ddot{h}_2 (\sinh \chi)^{-2} \end{aligned} \quad (3.13)$$

$$A^3 = -2h_3 \cos \theta (\sin \theta)^{-2} (\sinh \chi)^{-2} + \cos \theta \ddot{h}_3 (\sin \theta)^{-2} (\sinh \chi)^{-2} \quad (3.14)$$