$\delta W_{\mu\nu} = \delta T_{\mu\nu}$ (SVT) Matthew v2

According to (E6), via orthogonal projection to the four velocity U^μ, we may decompose a rank 2 $T_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} + U_\mu q_\nu + U_\nu q_\mu + \pi_{\mu\nu} \tag{1}$$

where

$$U^\mu q_\mu = 0, \qquad U^\nu \pi_{\mu\nu} = 0, \qquad \pi_{\mu\nu} = \pi_{\nu\mu}, \qquad g^{\mu\nu}\pi_{\mu\nu} = U^\mu U^\nu \pi_{\mu\nu} = 0. \tag{2}$$

We will expand the above $T_{\mu\nu}$ up to first order as

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}. \tag{3}$$

For a flat background viz. $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, it follows that $W^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu} = 0$. Hence the full $T_{\mu\nu}$ of (3) will be entirely first order. The first order quantities will be defined according to

$$\rho^{(1)} = \delta\rho, \qquad p^{(1)} = \delta p, \qquad U^{(1)} = \delta U, \qquad q^{(1)}_\mu = q_\mu, \qquad \pi^{(1)}_{\mu\nu} = \pi_{\mu\nu}. \tag{4}$$

where the scalars, vectors, and tensors are defined in terms of flat projectors as

$$\begin{aligned} \delta\rho =& U^\sigma_{(0)} U^\tau_{(0)} \delta T_{\sigma\tau}, \qquad \delta p = \frac{1}{3} P^{\sigma\tau}_{(0)} \delta T_{\sigma\tau}, \qquad q_\mu = -P_\mu{}^\sigma U^\tau_{(0)} \delta T_{\sigma\tau} \\ \pi_{\mu\nu} =& \left[\frac{1}{2} P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2} P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3} P^{(0)}_{\mu\nu} P^{\sigma\tau}_{(0)}\right] \delta T_{\sigma\tau}. \end{aligned} \tag{5}$$

Now the fluctuation goes as

$$\delta T_{\mu\nu} = (\delta\rho + \delta p) U^{(0)}_\mu U^{(0)}_\nu + g^{(0)}_{\mu\nu} \delta p + U^{(0)}_\mu q_\nu + U^{(0)}_\nu q_\mu + \pi_{\mu\nu}. \tag{6}$$

Since we will shortly be conformally transforming to the Roberston Walker background, the coordinates are taken as comoving, i.e. $\frac{dx^i}{dt} = 0$, and thus the four velocity is

$$U^\mu_{(0)} = \delta^\mu_0, \qquad U^{(0)}_\mu = -\delta^0_\mu \tag{7}$$

in which $\delta T_{\mu\nu}$ becomes

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)\delta^0_\mu \delta^0_\nu + \eta_{\mu\nu}\delta p - \Omega\delta^0_\mu q_\nu - \Omega\delta^0_\nu q_\mu + \pi_{\mu\nu}, \tag{8}$$

$$\delta T_{00} = \delta\rho \tag{9}$$

$$\delta T_{0i} = -q_i \tag{10}$$

$$\delta T_{ij} = \delta_{ij}\delta p + \pi_{ij}. \tag{11}$$

To bring $\delta T_{\mu\nu}$ closer to form of $\delta W_{\mu\nu}$ in the SVT basis, we follow appendix E and introduce

$$Q = \int d^3y D^3(x - y)\tilde{\nabla}^i_y q_i \tag{12}$$

such that

$$q_i = Q_i + \tilde{\nabla}_i Q, \qquad \tilde{\nabla}^i Q_i = 0. \tag{13}$$

For $\pi_{\mu\nu}$, we recall that (evalauted in the geoemetry of (20)) it obeys

$$g^{\mu\nu}\pi_{\mu\nu} = U^{\mu}U^{\nu}\pi_{\mu\nu} = 0. \tag{14}$$

Via (E21), we may decompose the five component $\pi_{\mu\nu}$ into a transverse traceless π_{ij}, a divergenceless π_i, and a scalar π as

$$\pi_{ij} = -\frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}, \tag{15}$$

where we have restricted to $D = 3$ according to $U^{\mu}U^{\nu}\pi_{\mu\nu} = 0$. Now $\delta T_{\mu\nu}$ can be expressed in the SVT form as

$$\begin{aligned}
&\delta T_{00} = \Omega^{-2}\delta\rho,\\
&\delta T_{0i} = -\Omega^{-2}(Q_i + \tilde{\nabla}_i Q),\\
&\delta T_{ij} = \Omega^{-2}\left[\delta_{ij}\delta p - \frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}\right]
\end{aligned} \tag{16}$$

Such a $\delta T_{\mu\nu}$ must be gauge invariant since $T^{(0)}_{\mu\nu} = 0$. In addition, it must be covariantly conserved and traceless, conditions which when imposed yield the following constraints:

$$\delta\rho = 3\delta p \tag{17}$$

$$-\partial_t\rho = \tilde{\nabla}_i\tilde{\nabla}^i Q \tag{18}$$

$$0 = \partial_t(Q^i + \tilde{\nabla}^i Q) + \tilde{\nabla}^i\delta p + \frac{4}{3}\tilde{\nabla}^i\tilde{\nabla}^k\tilde{\nabla}_k\pi + \tilde{\nabla}_k\tilde{\nabla}^k\pi^i. \tag{19}$$

To bring $\delta T_{\mu\nu}$ closer to form of $\delta W_{\mu\nu}$ in the SVT basis, we follow appendix E and introduce

$$Q = \int d^3y D^3(x-y)\tilde{\nabla}^i_y q_i \tag{20}$$

such that

$$q_i = Q_i + \tilde{\nabla}_i Q, \qquad \tilde{\nabla}^i Q_i = 0. \tag{21}$$

For $\pi_{\mu\nu}$, we recall that (evalauted in the geoemetry of (20)) it obeys

$$g^{\mu\nu}\pi_{\mu\nu} = U^{\mu}U^{\nu}\pi_{\mu\nu} = 0. \tag{22}$$

Via (E21), we may decompose the five component $\pi_{\mu\nu}$ into a transverse traceless π_{ij}, a divergenceless π_i, and a scalar π as

$$\pi_{ij} = -\frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}, \tag{23}$$

where we have restricted to $D = 3$ according to $U^{\mu}U^{\nu}\pi_{\mu\nu} = 0$. Now (24-26) can be expressed in the SVT form as

$$\begin{aligned}
&\delta T_{00} = \delta\rho,\\
&\delta T_{0i} = -(Q_i + \tilde{\nabla}_i Q),\\
&\delta T_{ij} = \delta_{ij}\delta p - \frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}
\end{aligned} \tag{24}$$

From (20), it follows

$$Q = -\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t\delta\rho. \tag{25}$$

Applying $\tilde{\nabla}_i$ to (19) and inserting (17-18) yields

$$0 = -\partial_t^2\delta\rho + \frac{1}{3}\tilde{\nabla}_k\tilde{\nabla}^k\delta\rho + \frac{4}{3}\tilde{\nabla}^l\tilde{\nabla}_l\tilde{\nabla}^k\tilde{\nabla}_k\pi \tag{26}$$

in which we may solve for π as

$$\pi = \frac{3}{4}\int d^3y D^3(\mathbf{x}-\mathbf{y})\left[\int d^3z D^3(\mathbf{y}-\mathbf{z})\partial_t^2\delta\rho - \frac{1}{3}\delta\rho\right]. \tag{27}$$

Now we insert Q and π back into (19) and solve for Q_i and π_i

$$Q_i = -\tilde{\nabla}_k\tilde{\nabla}^k\int dt\ \pi_i \tag{28}$$

$$\pi_i = -\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t Q_i. \tag{29}$$

Lastly, we peform a conformal transformation $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$ such that we are working within the background geometry $ds^2 = -\Omega^2\eta_{\mu\nu}dx^\mu dx^\nu$. Under such a conformal transformation, $\delta T_{\mu\nu}$ transform as $\delta T_{\mu\nu} \to \Omega^{-2}\delta T_{\mu\nu}$. Finally, we can express $\delta T_{\mu\nu}$ in terms of 5 components consisting of $\delta\rho$, π_i and $\pi_{ij}^{T\theta}$ as

$$\begin{aligned}
\delta T_{00} &= \Omega^{-2}\delta\rho,\\
\delta T_{0i} &= \Omega^{-2}\left[\tilde{\nabla}_k\tilde{\nabla}^k\int dt\ \pi_i + \tilde{\nabla}_i\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t\delta\rho\right],\\
\delta T_{ij} &= \Omega^{-2}\bigg[\frac{1}{2}\delta_{ij}\delta\rho - \frac{1}{2}\delta_{ij}\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t^2\delta\rho\\
&\quad + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j\int d^3y D^3(\mathbf{x}-\mathbf{y})\left(\int d^3z D^3(\mathbf{y}-\mathbf{z})\partial_t^2\delta\rho - \frac{1}{3}\delta\rho\right) + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi_{ij}^{T\theta}\bigg].
\end{aligned} \tag{30}$$

This is to be contrasted with the S.V.T. decomposition of $\delta W_{\mu\nu}$:

$$\begin{aligned}
\delta W_{00} &= -\frac{2}{3\Omega^2}\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell\Psi,\\
\delta W_{0i} &= -\frac{2}{3\Omega^2}\tilde{\nabla}_i\tilde{\nabla}_\ell\tilde{\nabla}^\ell\partial_t\Psi + \frac{1}{2\Omega^2}\left[\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell\mathcal{Q}_i - \tilde{\nabla}_\ell\tilde{\nabla}^\ell\partial_t^2\mathcal{Q}_i\right],\\
\delta W_{ij} &= \frac{1}{3\Omega^2}\left[\delta_{ij}\tilde{\nabla}_\ell\tilde{\nabla}^\ell\partial_t^2\Psi + \tilde{\nabla}_\ell\tilde{\nabla}^\ell\tilde{\nabla}_i\tilde{\nabla}_j\Psi - \delta_{ij}\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell\Psi - 3\tilde{\nabla}_i\tilde{\nabla}_j\partial_t^2\Psi\right]\\
&\quad + \frac{1}{2\Omega^2}\left[\tilde{\nabla}_\ell\tilde{\nabla}^\ell\tilde{\nabla}_i\partial_t\mathcal{Q}_j + \tilde{\nabla}_\ell\tilde{\nabla}^\ell\tilde{\nabla}_j\partial_t\mathcal{Q}_i - \tilde{\nabla}_i\partial_t^3\mathcal{Q}_j - \tilde{\nabla}_j\partial_t^3\mathcal{Q}_i\right]\\
&\quad + \frac{1}{\Omega^2}\left[\tilde{\nabla}_\ell\tilde{\nabla}^\ell - \partial_t^2\right]^2 E_{ij}.
\end{aligned} \tag{31}$$

where $\delta W_{\mu\nu}$ has been carried out in the perturbed geometry

$$\begin{aligned}
ds^2 &= -g_{\mu\nu}dx^\mu dx^\nu = -\Omega^2(\eta_{\mu\nu}+f_{\mu\nu})dx^\mu dx^\nu\\
&= \Omega^2(x)\left[(1+2\phi)dt^2 - 2(\tilde{\nabla}_iB + B_i)dtdx^i - [(1-2\psi)\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_jE + \tilde{\nabla}_iE_j + \tilde{\nabla}_jE_i + 2E_{ij}]dx^idx^j\right].
\end{aligned} \tag{32}$$

and where we have defined

$$\Psi = \phi + \psi + \dot{B} - \ddot{E}, \qquad \mathcal{Q}_i = B_i - \dot{E}_i. \tag{33}$$

If we further define

$$\begin{aligned}
\delta\bar{\rho} &= -\frac{2}{3}\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell\Psi\\
\bar{\pi}_i &= \frac{1}{2}\big(\tilde{\nabla}_\ell\tilde{\nabla}^\ell - \partial_t^2\big)\partial_t\mathcal{Q}_i\\
\bar{\pi}_{ij}^{T\theta} &= \big(\tilde{\nabla}_\ell\tilde{\nabla}^\ell - \partial_t^2\big)^2E_{ij},
\end{aligned} \tag{34}$$

then $\delta W_{\mu\nu}$ takes the form

$$\begin{aligned}
\delta W_{00} &= \Omega^{-2}\delta\bar{\rho}, \\
\delta W_{0i} &= \Omega^{-2}\left[\tilde{\nabla}_k\tilde{\nabla}^k\int dt\ \bar{\pi}_i + \tilde{\nabla}_i\int d^3yD^3(\mathbf{x}-\mathbf{y})\partial_t\delta\bar{\rho}\right], \\
\delta W_{ij} &= \Omega^{-2}\left[\frac{1}{2}\delta_{ij}\delta\bar{\rho} - \frac{1}{2}\delta_{ij}\int d^3yD^3(\mathbf{x}-\mathbf{y})\partial_t^2\delta\bar{\rho}\right. \\
&\qquad \left. + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j\int d^3yD^3(\mathbf{x}-\mathbf{y})\left(\int d^3zD^3(\mathbf{y}-\mathbf{z})\partial_t^2\delta\bar{\rho} - \frac{1}{3}\delta\bar{\rho}\right) + \tilde{\nabla}_i\bar{\pi}_j + \tilde{\nabla}_j\bar{\pi}_i + \bar{\pi}_{ij}^{T\theta}\right],
\end{aligned} \tag{35}$$

which exactly parallels that of $\delta T_{\mu\nu}$. Solving in seqential order with $\delta W_{00} = \delta T_{00}$, then $\delta W_{0i} = \delta T_{0i}$, and finally $\delta W_{ij} = \delta T_{ij}$, it follows that

$$\delta\bar{\rho} = \delta\rho, \qquad \bar{\pi}_i = \pi_i, \qquad \bar{\pi}_{ij}^{T\theta} = \pi_{ij}^{T\theta}. \tag{36}$$

Scalars equate to scalars, vectors to vectors, and tensors to tensors, but here we did not make any assumptions in doing so - the equations themselves decouple exactly this way. Finally, we express the equations in their original definitions as

$$\begin{aligned}
\delta\rho &= -\frac{2}{3}\tilde{\nabla}_k\tilde{\nabla}^k\tilde{\nabla}_\ell\tilde{\nabla}^\ell(\phi + \psi + \dot{B} - \ddot{E}) \\
\pi_i &= \frac{1}{2}\left(\tilde{\nabla}_\ell\tilde{\nabla}^\ell - \partial_t^2\right)\partial_t(B_i - \dot{E}_i) \\
\pi_{ij}^{T\theta} &= \left(\tilde{\nabla}_\ell\tilde{\nabla}^\ell - \partial_t^2\right)^2 E_{ij}.
\end{aligned} \tag{37}$$