3-Space Einstein Tensor Gauge Dependence v5

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1 Covariant Decomposition

1.1 Geometry

Within the geometry of

$$ds^2 = (g_{ij}^{(0)} + h_{ij})dx^i dx^j \tag{1.1}$$

with maximally symmetric background

$$g_{ij}^{(0)} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.2}$$

assume the metric perturbation can be (covariant) SVT decomposed as

$$h_{ij} = -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}, \tag{1.3}$$

with 3-trace

$$h = -6\psi + 2\nabla^a\nabla_a E. \tag{1.4}$$

1.2 Background $G_{ij}^{(0)} = -\kappa_3^2 T_{ij}^{(0)}$

$$\begin{aligned} G_{ij}^{(0)} &= -\kappa_3^2 T_{ij}^{(0)} \\ k g_{ij} &= -\kappa_3^2 \Lambda g_{ij} \\ \rightarrow \quad \Lambda &= -\frac{k}{\kappa_3^2} \end{aligned} \tag{1.5}$$

1.3 Perturbed $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$

We choose to perturb only the background $T_{\mu\nu}^{(0)}$ to yield

$$-\kappa_3^2 \delta T_{ij} = k h_{ij}. \tag{1.6}$$

The perturbed Einstein equations $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ then take the form

$$\begin{aligned} &-\tfrac{1}{2}h_{ij}R + \tfrac{1}{2}g_{ij}h^{ab}R_{ab} + \tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h - \tfrac{1}{2}\nabla_a\nabla_i h_j{}^a - \tfrac{1}{2}\nabla_a\nabla_j h_i{}^a \\ &+\tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} + \tfrac{1}{2}\nabla_i\nabla_j h = k h_{ij}. \end{aligned} \tag{1.7}$$

In SVT terms this evaluates to:

$$\begin{aligned} &\nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi = \\ &k(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}), \end{aligned} \tag{1.8}$$

which may be simplified as

$$\boxed{(\nabla_a \nabla^a - 2k)E_{ij} + g_{ij}\nabla_a\nabla^a\psi - \nabla_j\nabla_i\psi + 2kg_{ij}\psi = 0.} \tag{1.9}$$

Taking the trace gives the solution for ψ

$$\boxed{(\nabla_a\nabla^a + 3k)\psi = 0} \tag{1.10}$$

Useful equations:

$$\nabla^j h_{ij} = -2\nabla_i\psi + 2\nabla_i\nabla^a\nabla_a E + 4k\nabla_i E + \nabla^a\nabla_a E_i + 2kE_i \tag{1.11}$$

$$\nabla^i\nabla^j h_{ij} = -2\nabla^i\nabla_i\psi + 2\nabla^i\nabla_i\nabla^j\nabla_j E + 4k\nabla_i\nabla^i E \tag{1.12}$$

$$\nabla^j \delta G_{ij} = -2k\nabla_i\psi + (2k^2 + k\nabla_a\nabla^a)E_i + 2k(\nabla_i\nabla^a\nabla_a E + 2k\nabla_i E) \tag{1.13}$$

$$\nabla^i\nabla^j\delta G_{ij} = -2k\nabla^a\nabla_a\psi + 2k\nabla^a\nabla_a\nabla^b\nabla_b E + 4k^2\nabla_a\nabla^a E \tag{1.14}$$

The covariant Bianchi identity $\delta(\nabla^j G_{ij}) = 0$ reduces to

$$\nabla^j\delta G_{ij} - k\nabla^j h_{ij} = 0. \tag{1.15}$$

1.4 Gauge Structure

Under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ in the RW geometry we decompose $\epsilon_i(x)$ into longitudinal and transverse components viz

$$\epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j\epsilon_j}_{L_i} + \nabla_i \underbrace{\int D\nabla^j\epsilon_j}_{L} \tag{1.16}$$

$$\nabla_i\epsilon_j = \nabla_i L_j + \nabla_i\nabla_j L \tag{1.17}$$

For the metric

$$\begin{aligned}\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\ &= \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.18}$$

Now form the gauge transformation equation

$$\begin{aligned}-2\bar{\psi}g_{ij} + 2\nabla_i\nabla_j\bar{E} + \nabla_i\bar{E}_j + \nabla_j\bar{E}_i + 2\bar{E}_{ij} &= -2\psi g_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij} \\ &\quad + \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.19}$$

The trace of (1.19) yields

$$-6\bar{\psi} + 2\nabla_a\nabla^a\bar{E} = -6\psi + 2\nabla_a\nabla^a E + 2\nabla_a\nabla^a L. \tag{1.20}$$

Using (1.12), the double transverse component of (1.19) yields

$$-2\nabla_a\nabla^a\bar{\psi} + 2\nabla_a\nabla^a\nabla_b\nabla^b\bar{E} + 4k\nabla_a\nabla^a\bar{E} = -2\nabla_a\nabla^a\psi + 2\nabla_a\nabla^a\nabla_b\nabla^b(E+L) + 4k\nabla_a\nabla^a(E+L). \tag{1.21}$$

Using (1.20) to eliminate ψ in the above, we arrive at an equation in terms of E and L

$$\tfrac{2}{3}\nabla^4\bar{E} + k\nabla^2\bar{E} = \tfrac{2}{3}\nabla^4(E+L) + k\nabla^2(E+L). \tag{1.22}$$

For quantities $\bar{E}$, E, and L that vanish on the boundary, we may integrate the associated Green's function by parts to show $\bar{E} = E + L$. Substitution into (1.20) then yields $\bar{\psi} = \psi$. The remaining transverse component of (1.19) is then

$$\nabla_a \nabla^a \bar{E}_i = \nabla_a \nabla^a (E_i + L_i). \tag{1.23}$$

With $\bar{E}_i$, E_i, and L_i vanishing on the boundary we have $\bar{E}_i = E_i + L_i$. In summary,

$$\begin{aligned} \bar{\psi} &= \psi \\ \bar{E} &= E + L \\ \bar{E}_i &= E_i + L_i \\ \bar{E}_{ij} &= E_{ij}. \end{aligned} \tag{1.24}$$

As E_i and E are not gauge invariant, the field equations $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ can only depend on ψ and E_{ij}, which agrees with (1.9). With the six components of h_{ij} we are free to make three coordinate transformation to reduce h_{ij} to three gauge invariant components, i.e. ψ and E_{ij}.

2 3-Space Scalar Eigenfunctions

In order for E_{ij} and ψ to decouple from (1.9), we assess whether every solution to

$$(\nabla_a \nabla^a + 3k)\psi = 0 \tag{2.1}$$

obeys

$$\nabla_i \nabla_j \psi \overset{!}{=} -k g_{ij} \psi. \tag{2.2}$$

Thus we first seek to find the general solution to curved space harmonic eigenfunction

$$(g^{ij} \nabla_i \nabla_j + \lambda^2)\psi = 0 \tag{2.3}$$

where $\lambda^2 = 3k$. Evaluating the above in the 3-space geometry, we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) - 3kr\frac{\partial \psi}{\partial r} - kr^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right] + \lambda^2 \psi = 0. \tag{2.4}$$

Noting the angular portion of the Laplacian, we take as solution $\psi = f_l(r) Y^l_m(\theta, \phi)$ to find a radial equation of

$$r^2 \frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - 3kr^3 \frac{df}{dr} - kr^4 \frac{d^2 f}{dr^2} + [\lambda^2 r^2 - l(l+1)] f_l(r) = 0. \tag{2.5}$$

We may rewrite the above as

$$\left[(1 - kr^2)\frac{d^2}{dr^2} + \left(\frac{2}{r} - 3kr\right)\frac{d}{dr} - \frac{l(l+1)}{r^2} + \lambda^2\right] f_l(r) = 0. \tag{2.6}$$

This coincides with [1] eq. (19), with the exception that λ^2 is not a separation constant in our case, but rather assumes the value of $\lambda^2 = 3k$.

To touch basis with [2] (which was used a reference within [1]), we substitute $r = \sinh \chi$, and use the following relations

$$\cosh^2\chi - \sinh^2\chi = 1, \qquad \cosh^2\chi = 1 + \sinh^2\chi, \qquad \sinh^2\chi = \cosh^2\chi - 1$$

$$r = \sinh\chi, \qquad \cosh^2\chi = 1 + r^2$$

$$\frac{d\cosh\chi}{d\sinh\chi} = \frac{\sinh\chi}{\cosh\chi}, \qquad \frac{d\sinh\chi}{d\cosh\chi} = \frac{\cosh\chi}{\sinh\chi}$$

$$\frac{d}{d\sinh\chi} = \frac{\sinh\chi}{\cosh\chi}\frac{d}{d\cosh\chi}$$

$$\frac{d^2}{d\sinh\chi^2} = \frac{1}{\cosh^3\chi}\frac{d}{d\cosh\chi} + \frac{\sinh^2\chi}{\cosh^2\chi}\frac{d^2}{d\cosh^2\chi}, \tag{2.7}$$

to show that (2.6) may be expressed as

$$\left(\sinh^2\chi\frac{d^2}{d\cosh^2\chi} + 3\cosh\chi\frac{d}{d\cosh\chi} - \frac{l(l+1)}{\sinh^2\chi} + \lambda^2\right) f_l(r) = 0. \tag{2.8}$$

Within [1], the authors solve a radial equation of the form

$$\left[\sinh^2\theta\frac{d^2}{d\cosh^2\theta} + f\cosh\theta\frac{d}{d\cosh\theta} - \frac{\alpha(\alpha+f-2)}{\sinh^2\theta} + N^2 + \left(\frac{f-1}{2}\right)^2\right] Z^{(f)}_{N,\alpha}(\theta) = 0, \tag{2.9}$$

which corresponds to our (2.8) by

$$f = 3, \qquad \alpha = l, \qquad N^2 + 1 = \lambda^2. \tag{2.10}$$

2.1 Hyperbolic Geometry $k = -1$

However, in [2] Bander requires N to be real to meet their condition of continuity and a boundary condition of "smallest possible growth at infinity". This proves problematic for when we assume hyperbolic geometry viz.

$$\begin{aligned} \lambda^2 &= 3k = -3 \\ \rightarrow \quad N &= 2i. \end{aligned} \tag{2.11}$$

The lack of solutions for $\lambda^2 = -3$ is confirmed in a second manner in reference to [3], where it is shown that separable radial solutions to

$$(\nabla_a\nabla^a + \lambda^2)\psi = 0 \tag{2.12}$$

that are regular at the origin have an eigenspectrum of

$$\begin{aligned} k &= 0, \qquad \lambda^2 = \gamma^2, \qquad \gamma^2 \geq 0 \\ k &= 1, \qquad \lambda^2 = \gamma(\gamma+2), \qquad \gamma = 1, 2, 3, .. \\ k &= -1, \qquad \lambda^2 = \gamma^2 + 1, \qquad \gamma^2 \geq 0. \end{aligned} \tag{2.13}$$

As such, we require $\lambda^2 \geq 0$. As an aside, we also see the spectrum is discrete for the closed $k = 1$ geometry.

2.2 Spherical Geometry $k = 1$

For $\lambda^2 = 3$, we fall into the allowed range of eigenspectrum (2.13) and we may utilize the radial solutions given in [1] as

$$f_l(r) = \left[\frac{\pi}{2}2^2(2^2-1^2)...(2^2-l^2)\right]^{-1/2} \sin^l\chi\left(\frac{d}{d\cos\chi}\right)^{l+1}\cos(2\chi). \tag{2.14}$$

where $r = \sin\chi$. Since the geometry is bounded, the separation constant takes on discrete values, and the angular momentum may only take on the two values of $l = 0, 1$ [**mannheimgrg**]. Hence we may evaluate $f_l(r)$ for these two cases directly, in which we find:

$$\begin{aligned} f_0(r) &= \sqrt{\frac{2}{\pi}}(1-r^2)^{1/2} \\ f_1(r) &= \sqrt{\frac{8}{3\pi}}r. \end{aligned} \tag{2.15}$$

The relevant spherical harmonics are:

$$\begin{aligned} Y_0^0 &= \sqrt{\frac{1}{4\pi}}, \qquad m=0 \\ Y_{-1}^1 &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \\ Y_0^1 &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_1^1 &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \end{aligned} \tag{2.16}$$

Thus the general solution ψ may be expressed as

$$\begin{aligned} \psi &= \sum_l \sum_m C_{lm} f_l(r) Y_m^l(\theta,\phi) \\ &= C_{00}\frac{1}{\sqrt{2}\pi}(1-r^2)^{1/2} + \sqrt{\frac{8}{3\pi}}r \sum_{m=-1,0,1} C_{1m} Y_m^1(\theta,\phi) \end{aligned} \tag{2.17}$$

Recall that our purpose is to find if the solution (2.17) above also satisfies

$$(\nabla_i\nabla_j + k g_{ij})\psi = 0 \tag{2.18}$$

We evaluate (2.18) component by component, taking $k = 1$

$$\begin{aligned} &\left[\frac{1}{1-r^2} - \left(\frac{r}{1-r^2}\right)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right]\psi = 0 \\ &\left[r^2 + r(1-r^2)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right]\psi = 0 \\ &\left[r^2\sin^2\theta + r(1-r^2)\sin^2\theta\frac{\partial}{\partial r} + \sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}\right]\psi = 0 \\ &\left[\frac{\partial^2}{\partial\theta\partial r} - \frac{1}{r}\frac{\partial}{\partial\theta}\right]\psi = 0 \\ &\left[\frac{\partial^2}{\partial\phi\partial r} - \frac{1}{r}\frac{\partial}{\partial\phi}\right]\psi = 0 \\ &\left[\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right]\psi = 0 \end{aligned} \tag{2.19}$$

Finally we substitute solution (2.17) into the above to find in fact that every component vanishes.

Thus we conclude that for any ψ that obeys

$$(\nabla_a\nabla^a + 3)\psi = 0 \tag{2.20}$$

within a spherical geometry given by the $k = 1$

$$ds^2 = \frac{1}{1 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{2.21}$$

must also obey

$$(\nabla_i \nabla_j + k g_{ij})\psi = 0, \tag{2.22}$$

to thereby yield a field equation for Δ_{ij} only involving the tensor component E_{ij} via

$$(\nabla_a \nabla^a - 2)E_{ij} = 0. \tag{2.23}$$

3 Conformal to Flat

The 3-space of constant curvature can be expressed in the conformal flat form as in (A.1)

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{4}{\left(1+k\rho^2\right)^2}\left(d\rho^2+\rho^2 d\Omega^2\right) \end{aligned} \tag{3.1}$$

3.1 Background $G^{(0)}_{ij} = -\kappa_3^2 T^{(0)}_{ij}$

From (B.3) we see since $G_{\mu\nu}$ vanishes in a flat geometry, the background equation is given as

$$g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = -\kappa_3^2\Lambda\Omega^2 g_{ij}. \tag{3.2}$$

Taking the trace

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = -3\kappa_3^2\Lambda\Omega^2. \tag{3.3}$$

In the covariant formulation, we saw from (1.5) that $-\kappa_3^2\Lambda = k$, a constant relation independent of choice of coordinate system. As such we expect the above to obey

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = 3\Omega^2 k \tag{3.4}$$

Calculation of the above indeed yields

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \frac{12k}{(1+k\rho^2)^2} = 3\Omega^2 k \tag{3.5}$$

The two background equations that will prove useful are:

$$-\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \Omega^2 k \tag{3.6}$$

$$g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = k\Omega^2 g_{ij}. \tag{3.7}$$

3.2 $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$

Within geometry

$$ds^2 = \Omega^2(\rho)(g_{ij}+f_{ij})dx^i dx^j, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E, +\tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{3.8}$$

the perturbed Einstein tensor takes the form (with ∇ denoting flat space derivative)

$$
\begin{aligned}
\delta G_{ij} \;=\;& g_{ij}\nabla_a\nabla^a\psi + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E - 2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega \\
&+2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega - 2\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\
&+2\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\
& \\
&+g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\
&-\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j - \Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i \\
&-\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a \\
& \\
&+\nabla_a\nabla^a E_{ij} - 2E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + 2E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
&+2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}. \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
-\kappa_3^2\delta T_{ij} &= -\kappa_3^2\Lambda\Omega^2 h_{ij} \\
&= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\
& \\
-\kappa_3^2 g^{ij}\delta T_{ij} &= k\Omega^2(-6\psi + 2\nabla_a\nabla^a E) \qquad (3.10)
\end{aligned}
$$

3.3 Gauge Structure

Within the conformal flat geometry of (3.1) under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ we take the general $\epsilon_i(x)$ as $\epsilon_i = \Omega^2 f_i$ with

$$
f_i = \underbrace{f_i - \tilde{\nabla}_i \int D\tilde{\nabla}^j f_j}_{L_i} + \tilde{\nabla}_i \underbrace{\int D\tilde{\nabla}^j f_j}_{L} \qquad (3.11)
$$

It will be helpful to calculate $\nabla_i\epsilon_j$ in terms of f_i,

$$
\begin{aligned}
\nabla_i\epsilon_j &= \partial_i\epsilon_j - \Gamma^k_{ij}\epsilon_k \\
&= \partial_i\epsilon_j - \epsilon_k\left[\tilde{\Gamma}^k_{ij} + \Omega^{-1}(\delta^k_i\partial_j + \delta^k_j\partial_i - g_{ij}g^{kl}\partial_l)\Omega\right]) \\
&= \Omega^2\nabla_i f_j - \Omega(f_i\tilde{\nabla}_j\Omega - f_j\tilde{\nabla}_i\Omega - \tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega) \qquad (3.12)
\end{aligned}
$$

It then follows

$$
\begin{aligned}
\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + 2\Omega^{-1}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega) \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + \Omega^{-2}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega^2). \qquad (3.13)
\end{aligned}
$$

The transformation of f_{ij} is then

$$
\bar{f}_{ij} = f_{ij} + \tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + \Omega^{-2}\tilde{g}_{ij}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega^2 \qquad (3.14)
$$

Instead of taking the trace and transverse components of (3.14) as we did for the covariant case, since we know the projectors in flat space, we can instead make use of the defining conditions for SVT quantities and find their gauge structure. Enforcing the SVT quantities vanish on the spatial boundary at infinity, we use the decomposition defined in APM-CPII (66) and solve the gauge transformation of ψ

$$
\bar{\psi} = \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega. \qquad (3.15)
$$

Substituting the above into $\tilde{\nabla}^i \tilde{\nabla}^j \bar{f}_{ij}$ yields the relation for $\bar{E}$

$$\bar{E} \quad = E + L. \tag{3.16}$$

Then substitution of $\bar{E}$ into the $\tilde{\nabla}^j \bar{f}_{ij}$ yields the expression for $\bar{E}_i$

$$\bar{E}_i \quad = \quad E_i + L_i. \tag{3.17}$$

In summary

$$\begin{aligned} \bar{\psi} &= \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k \Omega \\ \bar{E} &= E + L \\ \bar{E}_i &= E_i + L_i \\ \bar{E}_{ij} &= E_{ij} \end{aligned} \tag{3.18}$$

We find two gauge invariant quantities

$$\begin{aligned} \bar{\psi} + \Omega^{-1}(\tilde{\nabla}_k \bar{E} + \bar{E}_k)\tilde{\nabla}^k \Omega &= \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k \Omega \\ \bar{E}_{ij} &= E_{ij} \end{aligned} \tag{3.19}$$

We will denote

$$\Psi \equiv \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k \Omega. \tag{3.20}$$

3.4 $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ Simplification to Gauge Invariant Form

First we transform $\delta T_{\mu\nu}$ using the background equations

$$\begin{aligned} -\kappa_3^2 \Lambda \delta T_{ij} &= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\ &= -2g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega)\psi - 2\Omega^{-1}\nabla_i\nabla_j\Omega\psi + 4\Omega^{-2}\nabla_i\Omega\nabla_j\Omega\psi \\ &\quad -\tfrac{4}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i\nabla_j E + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i\nabla_j E \\ &\quad -\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\ &\quad -\tfrac{4}{3}\Omega^{-1}\nabla_a\nabla^a\Omega E_{ij} + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega E_{ij} \end{aligned} \tag{3.21}$$

Now forming $\Delta_{ij} \equiv \delta G_{ij} + \kappa_3^2 \delta T_{ij} = 0$

$$\begin{aligned} \Delta_{ij} &= g_{ij}\nabla_a\nabla^a\psi - 2g_{ij}\psi\Omega^{-1}\nabla_a\nabla^a\Omega + 2g_{ij}\psi\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E \\ &\quad -2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega \\ &\quad -4\psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E + \tfrac{4}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi \\ &\quad +2\psi\Omega^{-1}\nabla_j\nabla_i\Omega - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\ &\quad +g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\ &\quad -\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\ &\quad +\tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\ &\quad +\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\ &\quad +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}. \end{aligned} \tag{3.22}$$

To facilitate simplification into gauge invariant components, we make substitution

$$\psi \quad = \quad \Psi - \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k \Omega \tag{3.23}$$

in which Δ_{ij} then becomes

$$\begin{aligned}\Delta_{ij} &= g_{ij}\nabla_a\nabla^a\Psi - 2\Psi g_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Psi g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + \Omega^{-1}\nabla_i\Omega\nabla_j\Psi + \Omega^{-1}\nabla_i\Psi\nabla_j\Omega \\ &\quad -4\Psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \nabla_j\nabla_i\Psi + 2\Psi\Omega^{-1}\nabla_j\nabla_i\Omega \qquad (3.24)\end{aligned}$$

$$\begin{aligned}&+3g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a E\nabla_b\nabla^b\Omega - g_{ij}\Omega^{-1}\nabla^a E\nabla_b\nabla^b\nabla_a\Omega - 4g_{ij}\Omega^{-3}\nabla_a\Omega\nabla^a E\nabla_b\Omega\nabla^b\Omega \\ &+2g_{ij}\Omega^{-2}\nabla^a E\nabla_b\nabla_a\Omega\nabla^b\Omega + \Omega^{-1}\nabla^a\nabla_j E\nabla_i\nabla_a\Omega + 8\Omega^{-3}\nabla_a\Omega\nabla^a E\nabla_i\Omega\nabla_j\Omega \\ &-2\Omega^{-2}\nabla^a\Omega\nabla_i\nabla_a E\nabla_j\Omega - 2\Omega^{-2}\nabla^a E\nabla_i\nabla_a\Omega\nabla_j\Omega - 2\Omega^{-2}\nabla^a\Omega\nabla_i\Omega\nabla_j\nabla_a E \\ &+\Omega^{-1}\nabla^a\nabla_i E\nabla_j\nabla_a\Omega - 2\Omega^{-2}\nabla^a E\nabla_i\Omega\nabla_j\nabla_a\Omega - \tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\ &+\tfrac{4}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - 3\Omega^{-2}\nabla_a\Omega\nabla^a E\nabla_j\nabla_i\Omega + \Omega^{-1}\nabla^a E\nabla_j\nabla_i\nabla_a\Omega \qquad (3.25)\end{aligned}$$

$$\begin{aligned}&+3E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega - E^a g_{ij}\Omega^{-1}\nabla_b\nabla^b\nabla_a\Omega - 4E^a g_{ij}\Omega^{-3}\nabla_a\Omega\nabla_b\Omega\nabla^b\Omega \\ &+2E^a g_{ij}\Omega^{-2}\nabla_b\nabla_a\Omega\nabla^b\Omega - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j \\ &-2\Omega^{-2}\nabla_a\Omega\nabla_i\Omega\nabla_j E^a + \Omega^{-1}\nabla_i\nabla_a\Omega\nabla_j E^a - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i \\ &+\tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i - 2\Omega^{-2}\nabla_a\Omega\nabla_i E^a\nabla_j\Omega + 8E^a\Omega^{-3}\nabla_a\Omega\nabla_i\Omega\nabla_j\Omega \\ &-2E^a\Omega^{-2}\nabla_i\nabla_a\Omega\nabla_j\Omega + \Omega^{-1}\nabla_i E^a\nabla_j\nabla_a\Omega - 2E^a\Omega^{-2}\nabla_i\Omega\nabla_j\nabla_a\Omega \\ &-3E^a\Omega^{-2}\nabla_a\Omega\nabla_j\nabla_i\Omega + E^a\Omega^{-1}\nabla_j\nabla_i\nabla_a\Omega. \qquad (3.26)\end{aligned}$$

$$\begin{aligned}&+\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\ &+2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}. \qquad (3.27)\end{aligned}$$

Inputting the explicit form of $\Omega(\rho)$, expanding covariant derivatives, and evaluating component by component we find that (3.25) and (3.26) vanish identically. To show an example of the interplay between contracted vector quantities like $E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega$ and free vectors such as $\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i$, we first isolate the free vector contribution to Δ_{ij}:

$$\begin{aligned}\Delta_{ij}^{(V_1)} &= \left(-\tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\right)(\nabla_i E_j + \nabla_j E_i) \\ &= \left(\frac{2k}{1-k\rho^2}\right)(\nabla_i E_j + \nabla_j E_i) \qquad (3.28)\end{aligned}$$

The remaining contracted vector contribution to $\Delta_{ij}^{(V)} = \Delta_{ij}^{(V_1)} + \Delta_{ij}^{(V_2)}$ is

$$\begin{aligned}\Delta_{ij}^{(V_2)} &= 3E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega - E^a g_{ij}\Omega^{-1}\nabla_b\nabla^b\nabla_a\Omega - 4E^a g_{ij}\Omega^{-3}\nabla_a\Omega\nabla_b\Omega\nabla^b\Omega + 2E^a g_{ij}\Omega^{-2}\nabla_b\nabla_a\Omega\nabla^b\Omega \\ &\quad -2\Omega^{-2}\nabla_a\Omega\nabla_i\Omega\nabla_j E^a + \Omega^{-1}\nabla_i\nabla_a\Omega\nabla_j E^a - 2\Omega^{-2}\nabla_a\Omega\nabla_i E^a\nabla_j\Omega + 8E^a\Omega^{-3}\nabla_a\Omega\nabla_i\Omega\nabla_j\Omega \\ &\quad -2E^a\Omega^{-2}\nabla_i\nabla_a\Omega\nabla_j\Omega + \Omega^{-1}\nabla_i E^a\nabla_j\nabla_a\Omega - 2E^a\Omega^{-2}\nabla_i\Omega\nabla_j\nabla_a\Omega - 3E^a\Omega^{-2}\nabla_a\Omega\nabla_j\nabla_i\Omega \\ &\quad +E^a\Omega^{-1}\nabla_j\nabla_i\nabla_a\Omega. \qquad (3.29)\end{aligned}$$

Evaluating (3.29) component by component we find

$$
\begin{aligned}
\Delta^{(V_2)}_{rr} &= -\left(\frac{4k}{1+k\rho^2}\right)\partial_r E_r \\
\Delta^{(V_2)}_{\theta\theta} &= -\left(\frac{4k}{1+k\rho^2}\right)(rE_r + \partial_\theta E_\theta) \\
\Delta^{(V_2)}_{\phi\phi} &= -\left(\frac{4k}{1+k\rho^2}\right)(r\sin^2\theta E_r + \sin\theta\cos\theta E_\theta + \partial_\phi E_\phi) \\
\Delta^{(V_2)}_{r\theta} &= \left(\frac{2k}{1+k\rho^2}\right)\left(\frac{2E_\theta}{r} - \partial_r E_\theta - \partial_\theta E_r\right) \\
\Delta^{(V_2)}_{r\phi} &= \left(\frac{2k}{1+k\rho^2}\right)\left(\frac{2E_\phi}{r} - \partial_r E_\phi - \partial_\phi E_r\right) \\
\Delta^{(V_2)}_{\theta\phi} &= \left(\frac{2k}{1+k\rho^2}\right)\left(2\sin^{-1}\theta\cos\theta E_\phi - \partial_\theta E_\phi - \partial_\phi E_\theta\right)
\end{aligned} \tag{3.30}
$$

Comparison of (3.30) to (3.28) illustrates the vanishing of the entire vector portion $\Delta^{(V)}_{ij}$. When evaluated component by component, the remaining scalar piece (3.25) similarly vanishes, and thus we are left with the gauge invariant form

$$
\begin{aligned}
\Delta_{ij} = {} & g_{ij}\nabla_a\nabla^a\Psi - 2\Psi g_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Psi g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + \Omega^{-1}\nabla_i\Omega\nabla_j\Psi + \Omega^{-1}\nabla_i\Psi\nabla_j\Omega \\
& -4\Psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \nabla_j\nabla_i\Psi + 2\Psi\Omega^{-1}\nabla_j\nabla_i\Omega \\
& +\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
& +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{3.31}
$$

Taking the trace of the above, we find

$$
\begin{aligned}
g^{ij}\Delta_{ij} = {} & 3\nabla_a\nabla^a\Psi - 6\Psi\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Omega^{-1}\nabla_a\Omega\nabla^a\Psi + 2\Psi\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \nabla^a\nabla_a\Psi \\
& +2\Psi\Omega^{-1}\nabla^a\nabla_a\Omega - 6E_{ab}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega + 6E_{ab}\Omega^{-1}\nabla^b\nabla^a\Omega.
\end{aligned} \tag{3.32}
$$

As (3.31) is already transverse, we may form the transverse traceless component of Δ_{ij} by

$$
\Delta^{T\theta}_{ij} = \Delta_{ij} - \frac{1}{2}g_{ij}g^{kl}\Delta_{kl} + \frac{1}{2}\left[\nabla_i\nabla_j + kg_{ij}\right]\int F g^{kl}\Delta_{kl} \tag{3.33}
$$

where

$$
(\nabla_a\nabla^a + 3k)F(x,x') = g^{-1/2}\delta^3(x-x') \tag{3.34}
$$

Appendix A Conformal to Flat Maximal 3-Space

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{4}{\left(1+k\rho^2\right)^2}\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{dr^2}{1-kr^2}+r^2 d\Omega^2 \end{aligned} \tag{A.1}$$

The relevant transformations are:

$$\begin{aligned} \rho(r) &= \frac{r}{1+\left(1-kr^2\right)^{1/2}}, \qquad \Omega^2(r)=\left[1+\left(1-kr^2\right)^{1/2}\right] \\ r(\rho) &= \frac{2\rho}{1+k\rho^2}, \qquad \Omega^2(\rho)=\frac{4}{\left(1+k\rho^2\right)^2} \end{aligned} \tag{A.2}$$

Appendix B δG_{ij} Under Conformal Transformation

Although the Riemann tensor transforms the same under conformal transformation, viz.

$$\begin{aligned} R_{\lambda\mu\nu\kappa} \;\rightarrow\;& \Omega^2 R_{\lambda\mu\nu\kappa}+\Omega\left(-g_{\mu\nu}\nabla_\lambda\nabla_\kappa\Omega+g_{\lambda\nu}\nabla_\mu\nabla_\kappa\Omega+g_{\mu\kappa}\nabla_\nu\nabla_\lambda\Omega-g_{\lambda\kappa}\nabla_\mu\nabla_\nu\Omega\right) \\ &+2g_{\mu\nu}\nabla_\kappa\Omega\nabla_\lambda\Omega-2g_{\lambda\nu}\nabla_\kappa\Omega\nabla_\mu\Omega-2g_{\mu\kappa}\nabla_\lambda\Omega\nabla_\nu\Omega+2g_{\lambda\kappa}\nabla_\mu\Omega\nabla_\nu\Omega \\ &+(g_{\lambda\nu}g_{\mu\kappa}-g_{\lambda\kappa}g_{\mu\nu})\nabla^\rho\Omega\nabla_\rho\Omega \end{aligned} \tag{B.1}$$

its contractions do depend on the dimension under consideration. For $D=3$ $\mu,\nu=1,2,3$ the Ricci tensor and scalar transform as

$$\begin{aligned} R_{\mu\nu} &\rightarrow R_{\mu\nu}+g_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega+\Omega^{-1}\nabla_\mu\nabla_\nu\Omega-2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \\ R &\rightarrow \Omega^{-2}R+4\Omega^{-3}\nabla_\alpha\nabla^\alpha\Omega-2\Omega^{-4}\nabla_\alpha\Omega\nabla^\alpha\Omega \end{aligned} \tag{B.2}$$

and thus the Einstein tensor transforms as

$$G_{\mu\nu} \rightarrow G_{\mu\nu}+g_{\mu\nu}(\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega-\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega)+\Omega^{-1}\nabla_\mu\nabla_\nu\Omega-2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \tag{B.3}$$

$$\begin{aligned} \delta\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2}g^{\lambda\rho}\left[\nabla_\mu h_{\nu\rho}+\nabla_\nu h_{\mu\rho}-\nabla_\rho h_{\mu\nu}\right] \\ \nabla_\mu\nabla_\nu\Omega &= \partial_\mu\nabla_\nu\Omega-\Gamma^\lambda_{\mu\nu}\nabla_\lambda\Omega \\ \delta(\nabla_\mu\nabla_\nu\Omega) &= -\tfrac{1}{2}\nabla^\rho\Omega(\nabla_\mu h_{\rho\nu}+\nabla_\nu h_{\mu\rho}-\nabla_\rho h_{\mu\nu}) \end{aligned} \tag{B.4}$$

$$\delta G_{\mu\nu}\rightarrow\delta G_{\mu\nu}+\delta S_{\mu\nu} \tag{B.5}$$

$$\begin{aligned} \delta G_{\mu\nu} =\;& \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h+\tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta}-\tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha-\tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha+\tfrac{1}{2}\nabla_\nu\nabla_\mu h \\ \delta S_{\mu\nu} =\;& -h_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega+\tfrac{1}{2}\Omega^{-1}\nabla_\alpha h_{\mu\nu}\nabla^\alpha\Omega-\tfrac{1}{2}g_{\mu\nu}\Omega^{-1}\nabla_\alpha h\nabla^\alpha\Omega+h_{\mu\nu}\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega \\ &+g_{\mu\nu}\Omega^{-1}\nabla^\alpha\Omega\nabla_\beta h_\alpha{}^\beta-g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha\Omega\nabla^\beta\Omega+g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta\nabla^\alpha\Omega \\ &-\tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\mu h_{\nu\alpha}-\tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\nu h_{\mu\alpha}. \end{aligned} \tag{B.6}$$

Appendix C Maximal 3-Space Geometric Quantities

Geometry

$$ds^2 = g_{ij}dx^i dx^j = \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right): \tag{C.1}$$

$$R_{ijkl} = k(g_{jk}g_{il} - g_{ik}g_{jl}), \qquad R_{ij} = -2kg_{ij}, \qquad R = -6k \tag{C.2}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R = -2kg_{ij} - \frac{1}{2}g_{ij}(-6k) = kg_{ij} \\ g^{ij}G_{ij} &= R - \frac{3}{2}R = -\frac{1}{2}R = 3k \end{aligned} \tag{C.3}$$

$$[\nabla_i, \nabla_j]V_k = -V_l R^l{}_{jki} = -V_l(k(g_{jk}g^l{}_i - g^l{}_k g_{ij})) = k(g_{ij}V_k - g_{jk}V_i) \tag{C.4}$$

$$\begin{aligned} [\nabla_a\nabla^a, \nabla_i]E &= 2k\nabla_i E \\ [\nabla^j, \nabla_i]\nabla_j E &= 2k\nabla_i E \\ [\nabla_a\nabla^a, \nabla_i\nabla_j]E &= -2kg_{ij}\nabla_a\nabla^a E + 6k\nabla_i\nabla_j E \\ [\nabla_a\nabla^a, \nabla_i]E_j &= 2k(\nabla_i E_j + \nabla_j E_i) \\ [\nabla^i, \nabla_j]E_i &= 2kE_j \\ [\nabla^i, \nabla_a\nabla^a]E_{ij} &= 0 \end{aligned} \tag{C.5}$$

$$\begin{aligned} \Gamma^r_{rr} &= \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\ \Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \quad \text{with all others zero} \end{aligned} \tag{C.6}$$

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