Astrophysics & Cosmology
HW 1

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18.3 From Newton's law of gravitation, the equations of motion for the sun of mass m_1 located at $\mathbf{r}_1$ and a planet of mass m_2 located at $\mathbf{r}_2$ are

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \frac{Gm_1m_2}{r^3}\mathbf{r} \tag{1}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = -\frac{Gm_1m_2}{r^3}\mathbf{r} \tag{2}$$

where

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

Adding these together

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} + m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \frac{Gm_1m_2}{r^3}\mathbf{r} - \frac{Gm_1m_2}{r^3}\mathbf{r} \tag{3}$$

$$\frac{d^2}{dt^2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) = 0 \tag{4}$$

Now define $m = m_1 + m_2$ and center of mass,

$$\mathbf{R} = \sum_i m_i\mathbf{r}_i = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m}$$

and we may rewrite (5) as

$$\frac{d^2}{dt^2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) = m\frac{d^2}{dt^2}\left(\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m}\right) = m\frac{d^2\mathbf{R}}{dt^2} = 0. \tag{5}$$

To arrive at an equation of motion in terms of $\mathbf{r}$, let us modify (1) and (2) as

$$\frac{d^2\mathbf{r}_1}{dt^2} - \frac{d^2\mathbf{r}_2}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{Gm_1m_2}{r^3}\mathbf{r} \tag{6}$$

$$\frac{d^2}{dt^2}\left(\mathbf{r}_1 - \mathbf{r}_2\right) = \left(\frac{m_1 + m_2}{m_1m_2}\right)\frac{Gm_1m_2}{r^3}\mathbf{r} \tag{7}$$

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G}{r^3}\mathbf{r}. \tag{8}$$

Lastly, we may find the euqation of motion relative to the center of mass $\mathbf{R}$ by defining vectors $\mathbf{R}_1 = \mathbf{r}_1 - \mathbf{R}$, $\mathbf{R}_2 = \mathbf{r}_2 - \mathbf{R}$. These can be expressed in terms of the separation vector $\mathbf{r}$ by

$$\begin{aligned}
\mathbf{R}_1 = \mathbf{r}_1 - \mathbf{R} &= \mathbf{r}_1 - \frac{1}{m_1 + m_2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) \\
&= \mathbf{r}_1\left(\frac{m_1 + m_2 - m_1}{m_1 + m_2}\right) - \mathbf{r}_2\left(\frac{m_2}{m_1 + m_2}\right) \\
&= -\left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{R}_2 = \mathbf{r}_2 - \mathbf{R} &= \mathbf{r}_2 - \frac{1}{m_1+m_2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) \\
&= \mathbf{r}_2\left(\frac{m_1+m_2-m_2}{m_1+m_2}\right) - \mathbf{r}_1\left(\frac{m_1}{m_1+m_2}\right) \\
&= \left(\frac{m_1}{m_1+m_2}\right)\mathbf{r}
\end{aligned}
$$

Looking at the above relation for $\mathbf{R}_1$ and $\mathbf{R}_2$, we see that the vectors are always opposite in direction and have magnitudes proportional to the opposite mass. For example if $m_1 > m_2$, $|\mathbf{R}_1| < |\mathbf{R}_2|$. This coincides with Figure 10.2 which illustrates a larger orbit for a small star than the large star (as seen by CM).

18.4 Start with the equation of motion (8),

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\mathbf{v} = -\frac{Gm}{r^3}\mathbf{r}$$

take the cross product with $\mathbf{r}$

$$\mathbf{r} \times \frac{d}{dt}\mathbf{v} = -\frac{Gm}{r^2}(\mathbf{r}\times\mathbf{r}) = 0.$$

Now differentiate the LHS

$$
\begin{aligned}
\frac{d}{dt}\left(\mathbf{r}\times\frac{d}{dt}\mathbf{v}\right) &= \frac{d}{dt}\left(\mathbf{r}\times\mathbf{v}\right) + \left(\mathbf{r}\times\frac{d}{dt}\mathbf{v}\right) = 0 \\
&= \frac{d}{dt}\left(\mathbf{r}\times\mathbf{v}\right) = 0 \\
&= \frac{d}{dt}\left(\mathbf{r}\times m\mathbf{v}\right) = 0 \\
&= \frac{d}{dt}\mathbf{J} = 0.
\end{aligned}
$$

From the above, we see that angular momentum $\mathbf{J}$ is conserved. Since the area swept out by vector $\mathbf{r}$ from the Sun to the planet is proportional to the area of the parallelogram, and since the magnitude of the cross product gives the area of the parallelogram, the conservation of angular momentum implies Kepler's law of equal areas of equal times. Moreover, $d/dt\mathbf{J} = 0$ implies the direction of $\mathbf{J}$ is constant and thus all motion takes place in a plane.

Taking the scalar product of (8) with $\mathbf{v}$

$$
\begin{aligned}
\mathbf{v}\cdot\frac{d}{dt}\mathbf{v} &= -\frac{Gm}{r^3}\left(\mathbf{v}\cdot\mathbf{r}\right) \\
\frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) &= -\frac{Gm}{r^3}\left(\frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})\right) \\
\frac{1}{2}\frac{d}{dt}|\mathbf{v}|^2 &= -\frac{Gm}{r^3}\left(\frac{1}{2}\frac{d}{dt}r^2\right) \\
\frac{1}{2}\frac{d}{dt}|\mathbf{v}|^2 &= -\frac{Gm}{r^3}\left(r\frac{dr}{dt}\right) \\
\frac{1}{2}\frac{d}{dt}|\mathbf{v}|^2 &= \frac{d}{dt}\frac{Gm}{r} \\
\frac{d}{dt}\left(\frac{1}{2}|\mathbf{v}|^2 - \frac{Gm}{r}\right) &= 0.
\end{aligned}
$$

Compared to Problem 3.1, here we have a conservation of energy for the effective one-body problem, defined in terms of the separation distance $\mathbf{r}$. The potential well is a central gravitational force of mass $m_1 + m_2$ (as opposed to $-Gm_1m_2/r$) and the translational energy will include an r_1r_2 cross term from $|\mathbf{v}|^2$. Both forms of conservation of energy should contain the same information, right?

Moving to polar coordinates r, θ, the velocity is

$$\mathbf{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}}.$$

Let look at $\dot{\hat{\mathbf{r}}}$:

$$\begin{aligned}
\frac{d}{dt}\hat{\mathbf{r}} &= \lim \epsilon \to 0 \quad \frac{\hat{\mathbf{r}}(t+\epsilon) - \hat{\mathbf{r}}(t)}{\epsilon} \\
&= \lim \epsilon \to 0 \quad \left.\frac{r\theta(t+\epsilon) - r\theta(t)}{\epsilon}\right|_{r=1} \hat{\theta} \\
&= \dot{\theta}\hat{\theta}.
\end{aligned}$$

We note that the change in the unit vector $\hat{\mathbf{r}}$ must lie in the $\hat{\theta}$ direction. Now using this for the conservation of angular momentum

$$\begin{aligned}
(r\hat{\mathbf{r}} \times \mathbf{v}) &= \left(r\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta})\right) \\
&= r^2\dot{\theta}(\hat{\mathbf{r}} \times \hat{\theta}) \\
&= r^2\dot{\theta}\hat{\mathbf{z}}
\end{aligned}$$

where we have taken $\hat{\mathbf{z}}$ to be normal to the r, θ plane. Thus

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0, \qquad J = r^2\dot{\theta}.$$

Now let us look at $|\mathbf{v}|^2$:

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}) \cdot (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}) = \dot{r}^2 + (r\dot{\theta})^2 = \dot{r}^2 + \frac{J^2}{r^2}. \tag{9}$$

Using (9), we form the conservation of energy in polar coordinates:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2r^2} - \frac{Gm}{r} = E. \tag{10}$$

In Keplar's law of equal areas of equal times, in a time dt, the radius vector sweeps out an area dA. Using the formula for the area of a triangle $A = \frac{1}{2}bh$ we have one leg h equal to r and the the base of the triangle b equal is the arc length $rd\theta$. Thus

$$dA = \frac{1}{2}r^2 d\theta, \qquad \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2}.$$

This solves Keplers proportionality constant.

In showing Kepler's first law, we first rewrite (10) as

$$\dot{r} = \left(2E + \frac{2Gm}{r} - \frac{J^2}{r^2}\right)^{1/2}.$$

Now divide by J

$$\begin{aligned}
\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} &= \left(\frac{2E}{J^2} + \frac{2Gm}{J^2 r} - \frac{1}{r^2}\right)^{1/2} \\
\frac{1}{r^2}\frac{dr}{d\theta} &= \left(\frac{2E}{J^2} + \frac{2Gm}{J^2 r} - \frac{1}{r^2}\right)^{1/2}.
\end{aligned}$$

Use a change in variable $u = 1/r$, $w = u - Gm/J^2$,

$$\frac{1}{r^2}\frac{dr}{d\theta} = \left(\frac{2E}{J^2} + \frac{2Gm}{J^2 r} - \frac{1}{r^2}\right)^{1/2}$$
$$-\frac{du}{d\theta} = \left(\frac{2E}{J^2} + u\frac{Gm}{J^2} - u^2\right)^{1/2}.$$

We can make another substitution $w = u - Gm/J^2$ in which it follows $dw = du$ (since J is a constant of motion). The above is then

$$\frac{dw}{d\theta} = -\left[\frac{2E}{J^2} + \left(\frac{Gm}{J^2}\right)^2 - \left(\frac{1}{r^2} + \left(\frac{Gm}{J^2}\right)^2 - \frac{2Gm}{J^2 r}\right)\right]^{1/2} \tag{11}$$
$$\frac{dw}{d\theta} = -(w_0^2 - w^2)^{1/2} \tag{12}$$

where $w_0 = \frac{2E}{J^2} + \left(\frac{Gm}{J^2}\right)^2$.

The solution to (12) is given by $w = w_0 \cos\theta$, which we show below satisifies the differential equation:

$$-w_0 \sin\theta = -(w_0^2 - w_0^2 \cos^2\theta)^{1/2}$$
$$-w_0 \sin\theta = -w_0(1 - cos^2\theta)^{1/2}$$
$$\sin\theta = \sin\theta.$$

Reverting our substitutions, we may express the solution to (12) in terms of r:

$$w = w_0 \cos\theta$$
$$\frac{1}{r} - \frac{Gm}{J^2} = \left(\frac{2E}{J^2} + \left(\frac{Gm}{J^2}\right)^2\right)^{1/2} \cos\theta$$
$$\frac{1}{r} = \frac{Gm}{J^2} + \frac{Gm}{J^2}\left(\frac{2E}{J^2}\left(\frac{J^2}{Gm}\right)^2 + 1\right)^{1/2} \cos\theta$$
$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos\theta)$$

where $r_0 = J^2/Gm$ and $\epsilon = \left(1 + 2EJ^2/G^2m^2\right)^{1/2}$.

The semi-major axis may be found by summing the the maxmimum radius r_{max} with the minimum r_{min} and dividing the results by 2. These locations occuar at $\theta = 0$ and $\theta = \pi$

$$a = \frac{1}{2}\left(r_{max} + r_{min}\right)$$
$$= \frac{1}{2}\left(\frac{r_0}{1+\epsilon} + \frac{r_0}{1-\epsilon}\right)$$
$$= \frac{r_0}{1-\epsilon^2}.$$

With the eccentricity defined as $\epsilon^2 = 1 - \frac{b^2}{a^2}$, we find the semi-minor axis to be

$$\epsilon^2 = 1 - \frac{b^2}{r_0^2}(1-\epsilon^2)^2$$
$$b^2 = \frac{r_0^2}{(1-\epsilon^2)}$$

$$b = \frac{r_0}{(1-\epsilon^2)^{1/2}}.$$

With the substitutions

$$x = r\cos\theta + \epsilon a, \qquad y = r\sin\theta$$

we transform the equation of an ellipse (relative to the origin)

$$x^2/a^2 + y^2/b^2 = 1$$

into

$$\begin{aligned}
\frac{r^2\cos^2\theta + 2\epsilon a r\cos\theta + \epsilon^2 a^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} &= 1\\
r^2\cos^2\theta\frac{(1-\epsilon^2)^2}{r_0^2} + 2r\cos\theta\frac{\epsilon(1-\epsilon^2)}{r_0} + \epsilon^2 + r^2\sin^2\theta\frac{(1-\epsilon^2)}{r_0^2} &= 1\\
\frac{r^2}{r_0^2}(\cos^2\theta+\sin^2\theta) + \epsilon\left(2\frac{r}{r_0}\cos\theta\right) + \epsilon^2\left(1 - 2\frac{r^2}{r_0^2}\cos^2\theta - \frac{r^2}{r_0^2}\sin^2\theta\right) - \epsilon^3\frac{2r}{r_0}\cos\theta + \epsilon^4\frac{r^2}{r_0^2}\cos^2\theta &= 1\\
\frac{r^2}{r_0^2}(\cos^2\theta+\sin^2\theta) + \epsilon\left(2\frac{r}{r_0}\cos\theta\right) + \epsilon^2\left(1 - 2\frac{r^2}{r_0^2}\cos^2\theta - \frac{r^2}{r_0^2}(1-\cos^2\theta)\right) - \epsilon^3\frac{2r}{r_0}\cos\theta + \epsilon^4\frac{r^2}{r_0^2}\cos^2\theta &= 1\\
\frac{r^2}{r_0^2} + \epsilon\left(2\frac{r}{r_0}\cos\theta\right) + \epsilon^2\left(1 - \frac{r^2}{r_0^2}\cos^2\theta - \frac{r^2}{r_0^2}\right) - \epsilon^3\frac{2r}{r_0}\cos\theta + \epsilon^4\frac{r^2}{r_0^2}\cos^2\theta &= 1\\
\frac{r^2}{r_0^2}(1-\epsilon^2) + 2\frac{r}{r_0}\cos\theta\epsilon(1-\epsilon^2) - \epsilon^2(1-\epsilon^2)\frac{r^2}{r_0^2}\cos^2\theta &= 1-\epsilon^2\\
\frac{r^2}{r_0^2} + 2\frac{r}{r_0}\cos\theta\epsilon - \epsilon^2\frac{r^2}{r_0^2}\cos^2\theta &= 1\\
\frac{r_0}{r} &= (1+\epsilon\cos\theta)\\
\frac{1}{r} &= \frac{1}{r_0}(1+\epsilon\cos\theta).
\end{aligned}$$

Thus by starting with the cartesion form and doing the substitution, we arrive at $\frac{1}{r} = \frac{1}{r_0}(1+\epsilon\cos\theta)$.

To get the area of an ellipse, we integrate

$$A = \int dxdy.$$

With the substitution $x' = x/a$ and $y' = y/b$, we get the equation of a circle

$$x'^2 + y'^2 = 1$$

which is easily integrated

$$\begin{aligned}
A = ab\int dx'dy' &= ab\int_{-1}^{1} dx' \int_{-\sqrt{1-x'^2}}^{\sqrt{1-x'^2}} dy'\\
&= ab\int_{-1}^{1} 2\sqrt{1-x^2}\ dx'\\
&= 2ab\int_0^\pi cos^2\theta\ d\theta\\
&= \pi ab.
\end{aligned}$$

Now we find the period

$$\tau = \int dt = \int \frac{dt}{dA}dA = \frac{2}{J}\int dA = \frac{2A}{J} = 2\pi\frac{ab}{J}$$

and square it

$$\begin{aligned}\tau^2 &= 4\pi^2\frac{a^4(1-\epsilon^2)}{J^2}\\ &= 4\pi^2\frac{a^3 r_0}{J^2}\\ &= 4\pi^2\frac{a^3}{Gm}\end{aligned}$$

Now we calculate the periods for the following planets' semi-major axes: 0.387, 0.723, 1.00, 1.52, 5.20, 9.54, 19.2, 30.1, 39.4.

$$\frac{4\pi^2}{Gm_{sun}} = \frac{4\pi^2}{(6.67\times10^{-8})(1.99\times10^{33})(6.685\times10^{-14})^3} = 9.96\times10^{14}\ [\mathrm{s}^2\mathrm{AU}^{-3}].$$

Now we multiply by

$$\frac{1}{3600*24*365} = 3.17\times10^{-8}$$

to get to time units in years. Thus

$$\tau = 3.17\times10^{-8}\sqrt{a^3(9.96\times10^{14})}.$$

The resulting periods in Earth years are

$$\begin{aligned}&(0.387, 0.723, 1.00, 1.52, 5.20, 9.54, 19.2, 30.1, 39.4)\ [\mathrm{AU}]\\ \rightarrow\ &(0.240, 0.615, 1.00, 1.874, 11.863, 29.478, 84.166, 165.211, 247.419)\ [\mathrm{yr}].\end{aligned}$$

18.5 Starting with the total conserved energy of an orbit (in the Newtonian theory), we have

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2r^2} - \frac{Gm}{r}. \tag{13}$$

We recognize the first term as the kinetic energy, and we may combine the last two terms into an effective potential

$$V_{eff} = \frac{J^2}{2r^2} - \frac{Gm}{r}.$$

V_{eff} includes the contribution due to a "centrifugal barrier" and the gravitation potential. If we plot V_{eff} against r for various energies, we can graphically see the range of energies which allow elliptic, hyperbolic, and circular orbits. Only the elliptical and circular orbits are bound. In the special case of circular orbits, the radius must be constant, which only occurs at the minimum of the effective potential. Thus, the condition for closed orbits is that

$$\begin{aligned}\frac{dV_{eff}}{dr} &= 0\\ \frac{J^2}{r^3} &= \frac{Gm}{r^2}\\ r &= \frac{J^2}{Gm}.\end{aligned}$$

Now, since the radius is fixed in a circular orbit we may write this as the defining condition for the radius given angular momentum J

$$r_0 = \frac{J^2}{Gm}.$$

Looking back at (13), we see that since $\dot{r} = 0$ must hold, the energy for circular orbits is

$$E_0 = \frac{J^2}{2r_0^2} - \frac{Gm}{r_0} = -\frac{Gm}{2r_0}.$$

With the eccentricity defined as

$$\epsilon = \left(1 + \frac{2EJ^2}{G^2m^2}\right)^{1/2}$$

we see that

$$\begin{aligned}
\frac{\epsilon^2}{2} &= \frac{1}{2} + \frac{EJ^2}{G^2m^2} \\
&= \frac{J^2}{G^2m^2}\left(E + \frac{G^2m^2}{2J^2}\right) \\
&= \frac{r_0}{Gm}\left(E - E_0\right).
\end{aligned}$$