3-Space Einstein Tensor Gauge Dependence v4

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1 Covariant Decomposition

1.1 Geometry

Within the geometry of

$$ds^2 = (g^{(0)}_{ij} + h_{ij})dx^i dx^j \tag{1.1}$$

with maximally symmetric background

$$g^{(0)}_{ij} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.2}$$

assume the metric perturbation can be (covariant) SVT decomposed as

$$h_{ij} = -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}, \tag{1.3}$$

with 3-trace

$$h = -6\psi + 2\nabla^a\nabla_a E. \tag{1.4}$$

1.2 Background $G^{(0)}_{ij} = -\kappa_3^2 T^{(0)}_{ij}$

$$\begin{aligned} G^{(0)}_{ij} &= -\kappa_3^2 T^{(0)}_{ij} \\ kg_{ij} &= -\kappa_3^2\Lambda g_{ij} \\ \rightarrow \quad \Lambda &= -\frac{k}{\kappa_3^2} \end{aligned} \tag{1.5}$$

1.3 Perturbed $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$

We choose to perturb only the background $T^{(0)}_{\mu\nu}$ to yield

$$-\kappa_3^2\delta T_{ij} = kh_{ij}. \tag{1.6}$$

The perturbed Einstein equations $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$ then take the form

$$\delta G_{ij} = -\kappa_3^2\delta T_{ij} \tag{1.7}$$

$$\tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h + \tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} - \tfrac{1}{2}\nabla_i\nabla_a h_j{}^a - \tfrac{1}{2}\nabla_j\nabla_a h_i{}^a + \tfrac{1}{2}\nabla_j\nabla_i h = kh_{ij}. \tag{1.8}$$

In SVT terms this evaluates to:

$$\begin{aligned} &\nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi = \\ &k(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}), \end{aligned} \tag{1.9}$$

which may be simplified as

$$\boxed{(\nabla_a\nabla^a - 2k)E_{ij} + g_{ij}\nabla_a\nabla^a\psi - \nabla_j\nabla_i\psi + 2kg_{ij}\psi = 0.} \tag{1.10}$$

Taking the trace gives the solution for ψ

$$\boxed{(\nabla_a\nabla^a + 3k)\psi = 0} \tag{1.11}$$

As the above equation is traceless, it is not clear how to decouple ψ from the tensor mode E_{ij}.

Useful equations:

$$\nabla^j h_{ij} = -2\nabla_i\psi + 2\nabla_i\nabla^a\nabla_a E + 4k\nabla_i E + \nabla^a\nabla_a E_i \tag{1.12}$$

$$\nabla^i\nabla^j h_{ij} = -2\nabla^i\nabla_i\psi + 2\nabla^i\nabla_i\nabla^j\nabla_j E + 4k\nabla_i\nabla^i E \tag{1.13}$$

$$\nabla^j\delta G_{ij} = -2k\nabla_i\psi + (2k^2 + k\nabla_a\nabla^a)E_i + 2k(\nabla_i\nabla^a\nabla_a E + 2k\nabla_i E) \tag{1.14}$$

$$\nabla^i\nabla^j\delta G_{ij} = -2k\nabla^a\nabla_a\psi + 2k\nabla^a\nabla_a\nabla^b\nabla_b E + 4k^2\nabla_a\nabla^a E \tag{1.15}$$

1.4 Gauge Structure

Under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ in the RW geometry we decompose $\epsilon_i(x)$ into longitudinal and transverse components viz

$$\epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j\epsilon_j}_{L_i} + \nabla_i \underbrace{\int D\nabla^j\epsilon_j}_{L} \tag{1.16}$$

$$\nabla_i\epsilon_j = \nabla_i L_j + \nabla_i\nabla_j L \tag{1.17}$$

For the metric

$$\begin{aligned}\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\ &= \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.18}$$

Now form the gauge transformation equation

$$\begin{aligned}-2\bar{\psi}g_{ij} + 2\nabla_i\nabla_j\bar{E} + \nabla_i\bar{E}_j + \nabla_j\bar{E}_i + 2\bar{E}_{ij} &= -2\psi g_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij} \\ &\quad + \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.19}$$

The trace of (1.19) yields

$$-6\bar{\psi} + 2\nabla_a\nabla^a\bar{E} = -6\psi + 2\nabla_a\nabla^a E + 2\nabla_a\nabla^a L. \tag{1.20}$$

Using (1.13), the double transverse component of (1.19) yields

$$-2\nabla_a\nabla^a\bar{\psi} + 2\nabla_a\nabla^a\nabla_b\nabla^b\bar{E} + 4k\nabla_a\nabla^a\bar{E} = -2\nabla_a\nabla^a\psi + 2\nabla_a\nabla^a\nabla_b\nabla^b(E+L) + 4k\nabla_a\nabla^a(E+L). \tag{1.21}$$

Using (1.20) to eliminate ψ in the above, we arrive at an equation in terms of E and L

$$\tfrac{2}{3}\nabla^4\bar{E} + k\nabla^2\bar{E} = \tfrac{2}{3}\nabla^4(E+L) + k\nabla^2(E+L). \tag{1.22}$$

For quantities $\bar{E}$, E, and L that vanish on the boundary, we may integrate the associated Green's function by parts to show $\bar{E} = E + L$. Substitution into (1.20) then yields $\bar{\psi} = \psi$. The remaining transverse component of (1.19) is then

$$\nabla_a \nabla^a \bar{E}_i = \nabla_a \nabla^a (E_i + L_i). \tag{1.23}$$

With $\bar{E}_i$, E_i, and L_i vanishing on the boundary we have $\bar{E}_i = E_i + L_i$. In summary,

$$\begin{aligned} \bar{\psi} &= \psi \\ \bar{E} &= E + L \\ \bar{E}_i &= E_i + L_i \\ \bar{E}_{ij} &= E_{ij}. \end{aligned} \tag{1.24}$$

Interestingly, as E_i and E are not gauge invariant, the field equations $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ can only depend on ψ and E_{ij}, which agrees with (1.10). With the six components of h_{ij} we are free to make three coordinate transformation to reduce h_{ij} to three gauge invariant components, i.e. ψ and E_{ij}.

2 Conformal to Flat

The 3-space of constant curvature can be expressed in the conformal flat form as in (A.1)

$$\begin{aligned} ds^2 &= \Omega^2(\rho) \left(d\rho^2 + \rho^2 d\Omega^2 \right) \\ &= \frac{4}{\left(1 + k\rho^2\right)^2} \left(d\rho^2 + \rho^2 d\Omega^2 \right) \end{aligned} \tag{2.1}$$

2.1 Background $G_{ij}^{(0)} = -\kappa_3^2 T_{ij}^{(0)}$

From (B.3) we see since $G_{\mu\nu}$ vanishes in a flat geometry, the background equation is given as

$$g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = -\kappa_3^2\Lambda\Omega^2 g_{ij}. \tag{2.2}$$

Taking the trace

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = -3\kappa_3^2\Lambda\Omega^2. \tag{2.3}$$

In the covariant formulation, we saw from (1.5) that $-\kappa_3^2\Lambda = k$, a constant relation independent of choice of coordinate system. As such we expect the above to obey

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = 3\Omega^2 k \tag{2.4}$$

Calculation of the above indeed yields

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \frac{12k}{(1+k\rho^2)^2} = 3\Omega^2 k \tag{2.5}$$

The two background equations that will prove useful are:

$$\begin{aligned} -\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega &= \Omega^2 k \\ g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega &= k\Omega^2 g_{ij}. \end{aligned} \tag{2.6}$$

2.2 $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$

Within geometry

$$ds^2 = \Omega^2(\rho)(g_{ij} + f_{ij})dx^i dx^j, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E, +\tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{2.7}$$

the perturbed Einstein tensor takes the form (with ∇ denoting flat space derivative)

$$\begin{aligned}
\delta G_{ij} \;=\;& g_{ij}\nabla_a\nabla^a\psi + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E - 2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega \\
& +2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega - 2\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\
& +2\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\
& \\
& +g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\
& -\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j - \Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i \\
& -\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a \\
& \\
& +\nabla_a\nabla^a E_{ij} - 2E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + 2E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
& +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
-\kappa_3^2\delta T_{ij} &= -\kappa_3^2\Lambda\Omega^2 h_{ij} \\
&= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij})
\end{aligned}$$

$$-\kappa_3^2 g^{ij}\delta T_{ij} = k\Omega^2(-6\psi + 2\nabla_a\nabla^a E) \tag{2.9}$$

2.3 Gauge Structure

Within the conformal flat geometry of (2.1) under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ we take the general $\epsilon_i(x)$ as $\epsilon_i = \Omega^2 f_i$ with

$$f_i = \underbrace{f_i - \tilde{\nabla}_i \int D\tilde{\nabla}^j f_j}_{L_i} + \tilde{\nabla}_i \underbrace{\int D\tilde{\nabla}^j f_j}_{L} \tag{2.10}$$

It will be helpful to calculate $\nabla_i\epsilon_j$ in terms of f_i,

$$\begin{aligned}
\nabla_i\epsilon_j &= \partial_i\epsilon_j - \Gamma^k_{ij}\epsilon_k \\
&= \partial_i\epsilon_j - \epsilon_k\left[\tilde{\Gamma}^k_{ij} + \Omega^{-1}(\delta^k_i\partial_j + \delta^k_j\partial_i - g_{ij}g^{kl}\partial_l)\Omega\right]) \\
&= \Omega^2\nabla_i f_j - \Omega(f_i\tilde{\nabla}_j\Omega - f_j\tilde{\nabla}_i\Omega - \tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega)
\end{aligned} \tag{2.11}$$

It then follows

$$\begin{aligned}
\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + 2\Omega^{-1}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega) \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + \Omega^{-2}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega^2).
\end{aligned} \tag{2.12}$$

The transformation of f_{ij} is then

$$\bar{f}_{ij} = f_{ij} + \tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + \Omega^{-2}\tilde{g}_{ij}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega^2 \tag{2.13}$$

Instead of taking the trace and transverse components of (2.13) as we did for the covariant case, since we know the projectors in flat space, we can instead make use of the defining conditions for SVT quantities and find their gauge structure. Enforcing the SVT quantities vanish on the spatial boundary at infinity, we use the decomposition defined in APM-CPII (66) and solve the gauge transformation of ψ

$$\bar{\psi} = \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega. \tag{2.14}$$

Substituting the above into $\tilde{\nabla}^i\tilde{\nabla}^j\bar{f}_{ij}$ yields the relation for $\bar{E}$

$$\bar{E} = E + L. \tag{2.15}$$

Then substitution of $\bar{E}$ into the $\tilde{\nabla}^j\bar{f}_{ij}$ yields the expression for $\bar{E}_i$

$$\bar{E}_i = E_i + L_i. \tag{2.16}$$

In summary

$$\begin{aligned}
\bar{\psi} &= \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega \\
\bar{E} &= E + L \\
\bar{E}_i &= E_i + L_i \\
\bar{E}_{ij} &= E_{ij}
\end{aligned} \tag{2.17}$$

We find two gauge invariant quantities

$$\begin{aligned}
\bar{\psi} + \Omega^{-1}(\tilde{\nabla}_k\bar{E} + \bar{E}_k)\tilde{\nabla}^k\Omega &= \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k\Omega \\
\bar{E}_{ij} &= E_{ij}
\end{aligned} \tag{2.18}$$

2.4 $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$ Simplifications

First we transform $\delta T_{\mu\nu}$ using the background equations

$$\begin{aligned}
-\kappa_3^2\Lambda\delta T_{\mu\nu} &= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\
&= -2g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega)\psi - 2\Omega^{-1}\nabla_i\nabla_j\Omega\psi + 4\Omega^{-2}\nabla_i\Omega\nabla_j\Omega\psi
\end{aligned} \tag{2.19}$$

Appendix A Conformal to Flat Maximal 3-Space

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right) \\ &= \frac{4}{\left(1+k\rho^2\right)^2}\left(d\rho^2 + \rho^2 d\Omega^2\right) \\ &= \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \end{aligned} \tag{A.1}$$

The relevant transformations are:

$$\begin{aligned} \rho(r) &= \frac{r}{1+\left(1-kr^2\right)^{1/2}}, \qquad \Omega^2(r) = \left[1+\left(1-kr^2\right)^{1/2}\right] \\ r(\rho) &= \frac{2\rho}{1+k\rho^2}, \qquad \Omega^2(\rho) = \frac{4}{\left(1+k\rho^2\right)^2} \end{aligned} \tag{A.2}$$

Appendix B δG_{ij} Under Conformal Transformation

Although the Riemann tensor transforms the same under conformal transformation, viz.

$$\begin{aligned} R_{\lambda\mu\nu\kappa} \rightarrow\ & \Omega^2 R_{\lambda\mu\nu\kappa} + \Omega\left(-g_{\mu\nu}\nabla_\lambda\nabla_\kappa\Omega + g_{\lambda\nu}\nabla_\mu\nabla_\kappa\Omega + g_{\mu\kappa}\nabla_\nu\nabla_\lambda\Omega - g_{\lambda\kappa}\nabla_\mu\nabla_\nu\Omega\right) \\ & +2g_{\mu\nu}\nabla_\kappa\Omega\nabla_\lambda\Omega - 2g_{\lambda\nu}\nabla_\kappa\Omega\nabla_\mu\Omega - 2g_{\mu\kappa}\nabla_\lambda\Omega\nabla_\nu\Omega + 2g_{\lambda\kappa}\nabla_\mu\Omega\nabla_\nu\Omega \\ & +(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})\nabla^\rho\Omega\nabla_\rho\Omega \end{aligned} \tag{B.1}$$

its contractions do depend on the dimension under consideration. For $D = 3$ $\mu,\nu = 1,2,3$ the Ricci tensor and scalar transform as

$$\begin{aligned} R_{\mu\nu} &\rightarrow R_{\mu\nu} + g_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega + \Omega^{-1}\nabla_\mu\nabla_\nu\Omega - 2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \\ R &\rightarrow \Omega^{-2}R + 4\Omega^{-3}\nabla_\alpha\nabla^\alpha\Omega - 2\Omega^{-4}\nabla_\alpha\Omega\nabla^\alpha\Omega \end{aligned} \tag{B.2}$$

and thus the Einstein tensor transforms as

$$G_{\mu\nu} \rightarrow G_{\mu\nu} + g_{\mu\nu}(\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega - \Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega) + \Omega^{-1}\nabla_\mu\nabla_\nu\Omega - 2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \tag{B.3}$$

$$\begin{aligned} \delta\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2}g^{\lambda\rho}\left[\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\mu\rho} - \nabla_\rho h_{\mu\nu}\right] \\ \nabla_\mu\nabla_\nu\Omega &= \partial_\mu\nabla_\nu\Omega - \Gamma^\lambda_{\mu\nu}\nabla_\lambda\Omega \\ \delta(\nabla_\mu\nabla_\nu\Omega) &= -\tfrac{1}{2}\nabla^\rho\Omega(\nabla_\mu h_{\rho\nu} + \nabla_\nu h_{\mu\rho} - \nabla_\rho h_{\mu\nu}) \end{aligned} \tag{B.4}$$

$$\delta G_{\mu\nu} \rightarrow \delta G_{\mu\nu} + \delta S_{\mu\nu} \tag{B.5}$$

$$\begin{aligned} \delta S_{\mu\nu} = & -h_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega + \tfrac{1}{2}\Omega^{-1}\nabla_\alpha h_{\mu\nu}\nabla^\alpha\Omega - \tfrac{1}{2}g_{\mu\nu}\Omega^{-1}\nabla_\alpha h\nabla^\alpha\Omega + h_{\mu\nu}\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega \\ & +g_{\mu\nu}\Omega^{-1}\nabla^\alpha\Omega\nabla_\beta h_\alpha{}^\beta - g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha\Omega\nabla^\beta\Omega + g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta\nabla^\alpha\Omega \\ & -\tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\mu h_{\nu\alpha} - \tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\nu h_{\mu\alpha}. \end{aligned} \tag{B.6}$$

Appendix C Maximal 3-Space Geometric Quantities

Geometry

$$ds^2 = g_{ij}dx^i dx^j = \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) : \tag{C.1}$$

$$R_{ijkl} = k(g_{jk}g_{il} - g_{ik}g_{jl}), \qquad R_{ij} = -2kg_{ij}, \qquad R = -6k \tag{C.2}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R = -2kg_{ij} - \frac{1}{2}g_{ij}(-6k) = kg_{ij} \\ g^{ij}G_{ij} &= R - \frac{3}{2}R = -\frac{1}{2}R = 3k \end{aligned} \tag{C.3}$$

$$[\nabla_i, \nabla_j]V_k = -V_l R^l{}_{jki} = -V_l(k(g_{jk}g^l{}_i - g^l{}_k g_{ij})) = k(g_{ij}V_k - g_{jk}V_i) \tag{C.4}$$

$$\begin{aligned} [\nabla_a\nabla^a, \nabla_i]E &= 2k\nabla_i E \\ [\nabla^j, \nabla_i]\nabla_j E &= 2k\nabla_i E \\ [\nabla_a\nabla^a, \nabla_i\nabla_j]E &= -2kg_{ij}\nabla_a\nabla^a E + 6k\nabla_i\nabla_j E \\ [\nabla_a\nabla^a, \nabla_i]E_j &= 2k(\nabla_i E_j + \nabla_j E_i) \\ [\nabla^i, \nabla_j]E_i &= 2kE_j \\ [\nabla^i, \nabla_a\nabla^a]E_{ij} &= 0 \end{aligned} \tag{C.5}$$

$$\begin{aligned} \Gamma^r_{rr} &= \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\ \Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \quad \text{with all others zero} \end{aligned} \tag{C.6}$$