$\delta W_{\mu\nu}(h^{T\theta}_{\mu\nu})$ in $g^{(0)}_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ v1

1 Transverse Tracless Projection $h^{T\theta}_{\mu\nu}$

A general rank 2 scalar may be decomposed into transverse and longitudinal pieces, and further into transverse traceless, longitudinal traceless, and trace terms. The decomposition takes the form

$$\begin{aligned} h_{\mu\nu} &= h^T_{\mu\nu} + h^L_{\mu\nu} \\ &= h^{T\theta}_{\mu\nu} + h^{L\theta}_{\mu\nu} + \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h_{\sigma\tau} - \frac{1}{D-1} \nabla_\mu \nabla_\nu \int d^D y D^{(D)}(x-y) g^{\sigma\tau} h_{\sigma\tau} \\ &\equiv h^{T\theta}_{\mu\nu} + h^{L\theta}_{\mu\nu} + h^{tr}_{\mu\nu}, \end{aligned} \tag{1}$$

with the following properties:

$$\begin{aligned} &\nabla^\mu h^T_{\mu\nu} = 0, \quad \nabla^\mu h^L_{\mu\nu} = \nabla^\mu h_{\mu\nu}, \quad \nabla^\mu \nabla^\nu h^L_{\mu\nu} = \nabla^\mu \nabla^\nu h_{\mu\nu} \\ &g^{\mu\nu} h^T_{\mu\nu} \equiv h^T, \quad g^{\mu\nu} h^L_{\mu\nu} \equiv h^L, \quad h = h^T + h^L \\ &\nabla^\mu h^{T\theta}_{\mu\nu} = 0, \quad \nabla^\mu h^{L\theta}_{\mu\nu} = \nabla^\mu h_{\mu\nu}, \quad \nabla^\mu \nabla^\nu h^{L\theta}_{\mu\nu} = \nabla^\mu \nabla^\nu h_{\mu\nu} \\ &g^{\mu\nu} h^{T\theta}_{\mu\nu} = 0, \quad g^{\mu\nu} h^{L\theta}_{\mu\nu} = 0 \\ &\nabla^\mu h^{tr}_{\mu\nu} = 0, \quad g^{\mu\nu} h^{tr}_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = h. \end{aligned} \tag{2}$$

The decomposition can be carried about through projectors. The transverse projector (E24) is

$$\begin{aligned} T_{\mu\nu\sigma\tau} &= \eta_{\mu\sigma}\eta_{\nu\tau} - \nabla_\mu \int d^D y D^{(D)}(x-y)\eta_{\nu\tau} - \nabla_\nu \int d^D y D^{(D)}(x-y)\eta_{\mu\sigma}\nabla_\tau \\ &\quad + \nabla_\mu \nabla_\nu \int d^D y D^{(D)}(x-y)\nabla_\sigma \int d^D z D^{(D)}(y-z)\nabla_\tau, \end{aligned} \tag{3}$$

where ∇_i denotes the flat space derivative throughout this section. Together with a complementary trace projector (E29)

$$Q_{\mu\nu\sigma\tau} = \frac{1}{D-1}\left[\eta_{\mu\nu} - \nabla_\mu\nabla_\nu \int d^D y D^{(D)}(x-y)\right]\left[\eta_{\sigma\tau} - \nabla_\sigma \int d^D z D^{(D)}(y-z)\nabla_\tau\right], \tag{4}$$

a transverse traceless projector may be composed as

$$P_{\mu\nu\sigma\tau} = T_{\mu\nu\sigma\tau} - Q_{\mu\nu\sigma\tau}, \tag{5}$$

which may be used as

$$P_{\mu\nu\sigma\tau} h^{\sigma\tau} = h^{T\theta}_{\sigma\tau}. \tag{6}$$

As given in (C.72 *Brane Gravity*), the fluctuation of the flat $\delta W_{\mu\nu}$ may be expressed entirely in terms of the gauge invariant $h^{T\theta}_{\mu\nu}$.

To see this, apply $\Box^2$ to both $T_{\mu\nu\sigma\tau}$ and $Q_{\mu\nu\sigma\tau}$:

$$\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta T_{\mu\nu\sigma\tau}h^{\sigma\tau} = \nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h_{\mu\nu} - \nabla_\alpha\nabla^\alpha\nabla_\mu\nabla_\sigma h^\sigma{}_\nu - \nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\sigma h^\sigma{}_\mu + \nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\tau h^{\sigma\tau} \tag{7}$$

$$\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta Q_{\mu\nu\sigma\tau}h^{\sigma\tau} = \frac{1}{3}\left(\eta_{\mu\nu}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h - \eta_{\mu\nu}\nabla_\alpha\nabla^\alpha\nabla_\sigma\nabla_\tau h^{\sigma\tau} + \nabla_\alpha\nabla^\alpha\nabla_\mu\nabla_\nu h + \nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\tau h^{\sigma\tau}\right) \tag{8}$$

Summing the result gives

$$\begin{aligned}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta(T_{\mu\nu\sigma\tau} - Q_{\mu\nu\sigma\tau})h^{\sigma\tau} &= \nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta P_{\mu\nu\sigma\tau}h^{\sigma\tau}\\ &= \nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h_{\mu\nu} - \nabla_\alpha\nabla^\alpha\nabla_\mu\nabla_\sigma h^\sigma{}_\nu - \nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\sigma h^\sigma{}_\mu\\ &+ \frac{2}{3}\nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\tau h^{\sigma\tau} - \frac{1}{3}\eta_{\mu\nu}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h - \frac{1}{3}\nabla_\alpha\nabla^\alpha\nabla_\mu\nabla_\nu h\\ &+ \frac{1}{3}\eta_{\mu\nu}\nabla_\alpha\nabla^\alpha\nabla_\sigma\nabla_\tau h^{\sigma\tau}.\end{aligned} \tag{9}$$

Comparison to $\delta W_{\mu\nu}$ evaluated in a flat background

$$\begin{aligned}\delta W_{\mu\nu} = \frac{1}{2}\bigg(&\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \frac{2}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}\\ &- \frac{1}{3}\eta_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \frac{1}{3}\eta_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta}\bigg),\end{aligned} \tag{10}$$

shows that $\delta W_{\mu\nu}$ may be expressed in terms of $\Box^2$ onto the transverse traceless projector,

$$\begin{aligned}\delta W_{\mu\nu} &= \frac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta(T_{\mu\nu\sigma\tau} - Q_{\mu\nu\sigma\tau})h^{\sigma\tau}\\ &= \frac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta P_{\mu\nu\sigma\tau}h^{\sigma\tau}\\ &= \frac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h^{T\theta}_{\mu\nu}.\end{aligned} \tag{11}$$

Based on the properties of $\delta W_{\mu\nu}$ under conformal transformation, namely $\delta\bar{W}_{\mu\nu} = \Omega^{-2}\delta W_{\mu\nu}$, it follows that within a conformal to flat background the fluctuation equations can still be expressed in terms of $\Box^2$ onto a transverse traceless projection, without an imposition of gauge.

2 $\delta W_{\mu\nu}(h_{\mu\nu})$

Starting with (APM 43), the direct perturbation of $W_{\mu\nu}$, we evaluate in a background $g^{(0)}_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ which yields the 153 term

$$\begin{aligned}\delta W_{\mu\nu}(h_{\mu\nu}) = &-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta h_{\nu\eta}\partial_\gamma\partial_\mu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta h_{\mu\eta}\partial_\gamma\partial_\nu\Omega\\ &-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\nu\Omega\partial_\eta h_{\mu\beta} - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\partial_\nu\Omega\partial_\eta h_{\mu\gamma}\\ &-48\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta h_{\mu\nu} + 24\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\eta h_{\mu\nu}\\ &-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\mu\Omega\partial_\eta h_{\nu\beta} - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\partial_\mu\Omega\partial_\eta h_{\nu\gamma}\\ &+20\eta^{\alpha\beta}\eta^{\gamma\kappa}\eta^{\eta\lambda}\eta_{\mu\nu}h_{\kappa\lambda}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega + 60\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega\\ &+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\beta h_{\mu\gamma} - 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta h_{\mu\nu}\\ &+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\beta h_{\nu\gamma} + 6\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\\ &-6\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\alpha}\Omega^{-6}\partial_\gamma\partial_\mu\Omega\partial_\eta\partial_\beta\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\alpha}\Omega^{-6}\partial_\gamma\partial_\nu\Omega\partial_\eta\partial_\beta\Omega\end{aligned}$$

$$
\begin{aligned}
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\mu h_{\nu\gamma}-6\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\mu\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma h_{\nu\alpha}\partial_\eta\partial_\beta\partial_\mu\Omega+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\nu h_{\mu\gamma}\\
&-6\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\nu\Omega+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma h_{\mu\alpha}\partial_\eta\partial_\beta\partial_\nu\Omega\\
&+\tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\beta\partial_\nu\partial_\mu h_{\alpha\gamma}-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu\partial_\mu\Omega\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\gamma h_{\mu\beta}+12\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma h_{\mu\nu}\\
&+6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma h_{\mu\nu}-2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma h_{\mu\nu}\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\gamma h_{\nu\beta}+12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta h_{\mu\nu}\partial_\eta\partial_\gamma\Omega\\
&-48\eta^{\alpha\gamma}\eta^{\beta\eta}h_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega-24\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega\\
&+3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\alpha}\Omega^{-6}\partial_\beta\partial_\mu\Omega\partial_\eta\partial_\gamma\Omega\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\alpha}\Omega^{-6}\partial_\beta\partial_\nu\Omega\partial_\eta\partial_\gamma\Omega-4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta h_{\mu\nu}\\
&-4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha h_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\Omega+12\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta\Omega\\
&+\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha h_{\mu\nu}-\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\nu}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha\Omega\\
&-\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu h_{\nu\alpha}+\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\alpha}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu\Omega\\
&-\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu h_{\mu\alpha}+\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\alpha}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu\Omega\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\mu h_{\nu\beta}-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\beta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\mu\Omega\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta h_{\nu\alpha}\partial_\eta\partial_\gamma\partial_\mu\Omega+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\nu h_{\mu\beta}\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\beta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\nu\Omega+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta h_{\mu\alpha}\partial_\eta\partial_\gamma\partial_\nu\Omega\\
&-\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\mu h_{\beta\gamma}+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\mu h_{\nu\beta}\\
&-6\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu h_{\nu\gamma}-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu h_{\nu\gamma}\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\mu h_{\nu\gamma}+24\eta^{\alpha\gamma}\eta^{\beta\eta}h_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\Omega\\
&+12\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\Omega-\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\nu h_{\beta\gamma}\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\nu h_{\mu\beta}-6\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu h_{\mu\gamma}\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu h_{\mu\gamma}+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\nu h_{\mu\gamma}\\
&+24\eta^{\alpha\gamma}\eta^{\beta\eta}h_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\Omega+12\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\Omega\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-6}\partial_\beta\partial_\mu\Omega\partial_\eta\partial_\nu\Omega-\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu h_{\beta\gamma}\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\beta\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu\Omega-\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta h_{\alpha\gamma}\partial_\eta\partial_\nu\partial_\mu\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta h_{\eta\lambda}\partial_\kappa\partial_\gamma\Omega+4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\kappa\partial_\gamma\Omega\partial_\lambda h_{\beta\eta}\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\lambda h_{\beta\kappa}-8\eta^{\alpha\eta}\eta^{\beta\kappa}\eta^{\gamma\lambda}\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda h_{\eta\kappa}\\
&-8\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda h_{\eta\kappa}+4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\lambda h_{\eta\kappa}\\
&-\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\eta h_{\beta\kappa}+4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta h_{\gamma\kappa}\\
&+\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta h_{\gamma\kappa}-\tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\lambda\partial_\eta h_{\gamma\kappa}\\
&-16\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\gamma\kappa}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega-4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\gamma\kappa}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-6}\partial_\kappa\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta h_{\gamma\kappa}\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\gamma\kappa}\Omega^{-6}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\kappa h_{\alpha\gamma}\partial_\lambda\partial_\eta\partial_\beta\Omega\\
&-\tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\kappa h_{\beta\eta}+\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa h_{\gamma\eta}\\
&-4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega+\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega\\
&-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta h_{\beta\gamma}+2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\beta\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta\Omega\\
&-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\beta h_{\alpha\gamma}\partial_\lambda\partial_\kappa\partial_\eta\Omega+\tfrac{1}{6}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-4}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta h_{\alpha\gamma}\\
&-\tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-5}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu\Omega\partial_\mu h_{\alpha\gamma}
\end{aligned}
$$

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\begin{aligned}
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\nu\Omega\partial_\mu h_{\beta\eta}+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu h_{\nu\alpha}\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu h_{\nu\beta}+12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu h_{\nu\eta}\\
&-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\mu h_{\nu\eta}-60\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu\Omega\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\partial_\nu\Omega\partial_\eta h_{\beta\gamma}\partial_\mu\Omega-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta h_{\nu\beta}\partial_\mu\Omega\\
&+24\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta h_{\nu\gamma}\partial_\mu\Omega+12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta h_{\nu\gamma}\partial_\mu\Omega\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta h_{\nu\gamma}\partial_\mu\Omega-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta h_{\nu\gamma}\partial_\mu\Omega\\
&+24\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\mu\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu h_{\alpha\gamma}\partial_\mu\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\partial_\nu\Omega\partial_\mu\Omega-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma h_{\nu\beta}\partial_\mu\Omega\\
&+12\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\beta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu\Omega+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta h_{\nu\alpha}\partial_\mu\Omega\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\nu\alpha}\Omega^{-6}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu\Omega+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\nu h_{\beta\gamma}\partial_\mu\Omega\\
&-16\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\nu\Omega\partial_\mu\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\mu\Omega\partial_\nu h_{\alpha\gamma}\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\mu\Omega\partial_\nu h_{\beta\eta}+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu h_{\beta\eta}\\
&-8\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu h_{\gamma\eta}+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu h_{\mu\alpha}\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu h_{\mu\beta}+12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu h_{\mu\eta}\\
&-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\nu h_{\mu\eta}-60\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu\Omega\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\partial_\mu\Omega\partial_\eta h_{\beta\gamma}\partial_\nu\Omega-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta h_{\mu\beta}\partial_\nu\Omega\\
&+24\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta h_{\mu\gamma}\partial_\nu\Omega+12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta h_{\mu\gamma}\partial_\nu\Omega\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta h_{\mu\gamma}\partial_\nu\Omega-6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta h_{\mu\gamma}\partial_\nu\Omega\\
&+24\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\nu\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\mu h_{\alpha\gamma}\partial_\nu\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\partial_\mu\Omega\partial_\nu\Omega-3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma h_{\mu\beta}\partial_\nu\Omega\\
&+12\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\beta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu\Omega+\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta h_{\mu\alpha}\partial_\nu\Omega\\
&-3\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\mu\alpha}\Omega^{-6}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu\Omega+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\mu h_{\beta\gamma}\partial_\nu\Omega\\
&-16\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\mu\Omega\partial_\nu\Omega+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\mu h_{\beta\eta}\partial_\nu\Omega\\
&-8\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu h_{\gamma\eta}\partial_\nu\Omega+40\eta^{\alpha\gamma}\eta^{\beta\eta}h_{\gamma\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega\\
&-16\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta h_{\beta\gamma}\partial_\mu\Omega\partial_\nu\Omega+2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\eta\partial_\beta h_{\alpha\gamma}\partial_\mu\Omega\partial_\nu\Omega\\
&-8\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-7}\partial_\eta\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\nu\partial_\mu h_{\beta\eta}\\
&+2\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu h_{\gamma\eta}-8\eta^{\alpha\gamma}\eta^{\beta\eta}h_{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu\Omega\\
&+4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta h_{\beta\gamma}\partial_\nu\partial_\mu\Omega-\tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta h_{\alpha\gamma}\partial_\nu\partial_\mu\Omega\\
&+2\eta^{\alpha\beta}\eta^{\gamma\eta}h_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\Omega\partial_\nu\partial_\mu\Omega\\
&+\tfrac{1}{6}\eta^{\alpha\beta}\Omega^{-2}\partial_\beta\partial_\alpha\partial_\nu\partial_\mu h-\tfrac{1}{6}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\Omega^{-2}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha h. \qquad (12)
\end{aligned}
$$

In the above the trace is defined as

$$h=g_{(0)}^{\sigma\tau}h_{\sigma\tau}=\Omega^{-2}\eta^{\sigma\tau}h_{\sigma\tau}. \qquad (13)$$

Now taking $h_{\mu\nu}=\Omega^2 f_{\mu\nu}$ with trace

$$f=\eta^{\sigma\tau}f_{\sigma\tau}=\Omega^{-2}\eta^{\sigma\tau}h_{\sigma\tau}=h, \qquad (14)$$

(**??**) evaluates to

$$
\begin{aligned}
\delta W_{\mu\nu}(f_{\mu\nu})=\frac{1}{2}\Omega^{-2}\Big(&\frac{1}{3}\eta^{\alpha\beta}\partial_\beta\partial_\alpha\partial_\nu\partial_\mu f+\frac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\nu\partial_\mu f_{\alpha\gamma}+\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha f_{\mu\nu}\\
&-\frac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha f-\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu f_{\nu\alpha}-\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu f_{\mu\alpha}
\end{aligned}
$$

$$+\frac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta f_{\alpha\gamma}\Big). \tag{15}$$

3 $\delta W_{\mu\nu}(K_{\mu\nu})$

In terms of $K_{\mu\nu}$, we may start with (**??**) and substitute $h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\Omega^2\eta_{\mu\nu}h$, which evaluates to the 151 term

$$\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) = & -6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta K_{\nu\eta}\partial_\gamma\partial_\mu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta K_{\mu\eta}\partial_\gamma\partial_\nu\Omega \\
& -6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\nu\Omega\partial_\eta K_{\mu\beta} - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\partial_\nu\Omega\partial_\eta K_{\mu\gamma} \\
& -48\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta K_{\mu\nu} + 24\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\eta K_{\mu\nu} \\
& -6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\mu\Omega\partial_\eta K_{\nu\beta} - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\partial_\mu\Omega\partial_\eta K_{\nu\gamma} \\
& +20\eta^{\alpha\beta}\eta^{\gamma\kappa}\eta^{\eta\lambda}\eta_{\mu\nu}K_{\kappa\lambda}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega + 60\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega \\
& +2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\beta K_{\mu\gamma} - 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta K_{\mu\nu} \\
& +2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\beta K_{\nu\gamma} + 6\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega \\
& -6\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\alpha}\Omega^{-6}\partial_\gamma\partial_\mu\Omega\partial_\eta\partial_\beta\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\alpha}\Omega^{-6}\partial_\gamma\partial_\nu\Omega\partial_\eta\partial_\beta\Omega \\
& +2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\mu K_{\nu\gamma} - 6\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\mu\Omega \\
& +2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma K_{\nu\alpha}\partial_\eta\partial_\beta\partial_\mu\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\nu K_{\mu\gamma} \\
& -6\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\nu\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma K_{\mu\alpha}\partial_\eta\partial_\beta\partial_\nu\Omega \\
& +\tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\beta\partial_\nu\partial_\mu K_{\alpha\gamma} - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu\partial_\mu\Omega \\
& +\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\gamma K_{\mu\beta} + 12\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma K_{\mu\nu} \\
& +6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma K_{\mu\nu} - 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma K_{\mu\nu} \\
& +\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\gamma K_{\nu\beta} + 12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta K_{\mu\nu}\partial_\eta\partial_\gamma\Omega \\
& -48\eta^{\alpha\gamma}\eta^{\beta\eta}K_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega - 24\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega \\
& +3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega - 3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\alpha}\Omega^{-6}\partial_\beta\partial_\mu\Omega\partial_\eta\partial_\gamma\Omega \\
& -3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\alpha}\Omega^{-6}\partial_\beta\partial_\nu\Omega\partial_\eta\partial_\gamma\Omega - 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta K_{\mu\nu} \\
& -4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha K_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\Omega + 12\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta\Omega \\
& +\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha K_{\mu\nu} - \eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\nu}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha\Omega \\
& -\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu K_{\nu\alpha} + \eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\alpha}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu\Omega \\
& -\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-4}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu K_{\mu\alpha} + \eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\alpha}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu\Omega \\
& +\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\mu K_{\nu\beta} - 3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\beta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\mu\Omega \\
& +\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta K_{\nu\alpha}\partial_\eta\partial_\gamma\partial_\mu\Omega + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\nu K_{\mu\beta} \\
& -3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\beta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\nu\Omega + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta K_{\mu\alpha}\partial_\eta\partial_\gamma\partial_\nu\Omega \\
& -\tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\nu\Omega\partial_\eta\partial_\mu K_{\beta\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\mu K_{\nu\beta} \\
& -6\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu K_{\nu\gamma} - 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu K_{\nu\gamma} \\
& +\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\mu K_{\nu\gamma} + 24\eta^{\alpha\gamma}\eta^{\beta\eta}K_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\Omega \\
& +12\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\Omega - \tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\partial_\mu\Omega\partial_\eta\partial_\nu K_{\beta\gamma} \\
& +2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\nu K_{\mu\beta} - 6\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu K_{\mu\gamma} \\
& -3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu K_{\mu\gamma} + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\nu K_{\mu\gamma} \\
& +24\eta^{\alpha\gamma}\eta^{\beta\eta}K_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\Omega + 12\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\Omega
\end{aligned}$$

$$
\begin{aligned}
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-6}\partial_\beta\partial_\mu\Omega\partial_\eta\partial_\nu\Omega - \tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu K_{\beta\gamma}\\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\beta\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu\Omega - \tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\beta K_{\alpha\gamma}\partial_\eta\partial_\nu\partial_\mu\Omega\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta K_{\eta\lambda}\partial_\kappa\partial_\gamma\Omega + 4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\kappa\partial_\gamma\Omega\partial_\lambda K_{\beta\eta}\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\lambda K_{\beta\kappa} - 8\eta^{\alpha\eta}\eta^{\beta\kappa}\eta^{\gamma\lambda}\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda K_{\eta\kappa}\\
&- 8\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda K_{\eta\kappa} + 4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\lambda K_{\eta\kappa}\\
&- \tfrac{4}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\eta K_{\beta\kappa} + 4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta K_{\gamma\kappa}\\
&+ \eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta K_{\gamma\kappa} - \tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\beta\partial_\alpha\Omega\partial_\lambda\partial_\eta K_{\gamma\kappa}\\
&- 16\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\gamma\kappa}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega - 4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\gamma\kappa}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\alpha\gamma}\Omega^{-6}\partial_\kappa\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta K_{\gamma\kappa}\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\gamma\kappa}\Omega^{-6}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\kappa K_{\alpha\gamma}\partial_\lambda\partial_\eta\partial_\beta\Omega\\
&- \tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\kappa K_{\beta\eta} + \eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa K_{\gamma\eta}\\
&- 4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega + \eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega\\
&- \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta K_{\beta\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\beta\gamma}\Omega^{-6}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta\Omega\\
&- \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-5}\partial_\beta K_{\alpha\gamma}\partial_\lambda\partial_\kappa\partial_\eta\Omega + \tfrac{1}{6}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\Omega^{-4}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta K_{\alpha\gamma}\\
&- \tfrac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}K_{\alpha\gamma}\Omega^{-5}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu\Omega\partial_\mu K_{\alpha\gamma}\\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\nu\Omega\partial_\mu K_{\beta\eta} + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu K_{\nu\alpha}\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu K_{\nu\beta} + 12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu K_{\nu\eta}\\
&- 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\mu K_{\nu\eta} - 60\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu\Omega\\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\partial_\nu\Omega\partial_\eta K_{\beta\gamma}\partial_\mu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta K_{\nu\beta}\partial_\mu\Omega\\
&+ 24\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta K_{\nu\gamma}\partial_\mu\Omega + 12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta K_{\nu\gamma}\partial_\mu\Omega\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta K_{\nu\gamma}\partial_\mu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta K_{\nu\gamma}\partial_\mu\Omega\\
&+ 24\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\mu\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\nu K_{\alpha\gamma}\partial_\mu\Omega\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\partial_\nu\Omega\partial_\mu\Omega - 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma K_{\nu\beta}\partial_\mu\Omega\\
&+ 12\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\beta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu\Omega + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta K_{\nu\alpha}\partial_\mu\Omega\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\nu\alpha}\Omega^{-6}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu\Omega + 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\nu K_{\beta\gamma}\partial_\mu\Omega\\
&- 16\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\nu\Omega\partial_\mu\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\mu\Omega\partial_\nu K_{\alpha\gamma}\\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\mu\Omega\partial_\nu K_{\beta\eta} + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu K_{\beta\eta}\\
&- 8\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu K_{\gamma\eta} + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu K_{\mu\alpha}\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu K_{\mu\beta} + 12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu K_{\mu\eta}\\
&- 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\nu K_{\mu\eta} - 60\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu\Omega\\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\partial_\mu\Omega\partial_\eta K_{\beta\gamma}\partial_\nu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\eta K_{\mu\beta}\partial_\nu\Omega\\
&+ 24\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta K_{\mu\gamma}\partial_\nu\Omega + 12\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta K_{\mu\gamma}\partial_\nu\Omega\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\eta K_{\mu\gamma}\partial_\nu\Omega - 6\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\beta K_{\mu\gamma}\partial_\nu\Omega\\
&+ 24\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\nu\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta\partial_\mu K_{\alpha\gamma}\partial_\nu\Omega\\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\partial_\mu\Omega\partial_\nu\Omega - 3\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\gamma K_{\mu\beta}\partial_\nu\Omega\\
&+ 12\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\beta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu\Omega + \eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\gamma\partial_\beta K_{\mu\alpha}\partial_\nu\Omega\\
&- 3\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\mu\alpha}\Omega^{-6}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu\Omega + 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta\partial_\mu K_{\beta\gamma}\partial_\nu\Omega\\
&- 16\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial_\eta\partial_\mu\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\gamma\partial_\alpha\Omega\partial_\mu K_{\beta\eta}\partial_\nu\Omega\\
&- 8\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu K_{\gamma\eta}\partial_\nu\Omega + 40\eta^{\alpha\gamma}\eta^{\beta\eta}K_{\gamma\eta}\Omega^{-8}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega
\end{aligned}
$$

$$
\begin{aligned}
&- 16\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\eta K_{\beta\gamma}\partial_\mu\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\eta\partial_\beta K_{\alpha\gamma}\partial_\mu\Omega\partial_\nu\Omega \\
&- 8\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-7}\partial_\eta\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\gamma\partial_\alpha\Omega\partial_\nu\partial_\mu K_{\beta\eta} \\
&+ 2\eta^{\alpha\gamma}\eta^{\beta\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu K_{\gamma\eta} - 8\eta^{\alpha\gamma}\eta^{\beta\eta}K_{\gamma\eta}\Omega^{-7}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu\Omega \\
&+ 4\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-6}\partial_\alpha\Omega\partial_\eta K_{\beta\gamma}\partial_\nu\partial_\mu\Omega - \tfrac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\Omega^{-5}\partial_\eta\partial_\beta K_{\alpha\gamma}\partial_\nu\partial_\mu\Omega \\
&+ 2\eta^{\alpha\beta}\eta^{\gamma\eta}K_{\alpha\gamma}\Omega^{-6}\partial_\eta\partial_\beta\Omega\partial_\nu\partial_\mu\Omega. \qquad (16)
\end{aligned}
$$

The dependence upon h drops out as expected since $W^{(0)}_{\mu\nu} = 0$.

Now taking $K_{\mu\nu} = \Omega^2 k_{\mu\nu}$ with trace

$$
k = \eta^{\sigma\tau}k_{\sigma\tau} = \Omega^{-2}\eta^{\sigma\tau}K_{\sigma\tau} = 0, \qquad (17)
$$

(**??**) evaluates to

$$
\begin{aligned}
\delta W_{\mu\nu}(k_{\mu\nu}) = \frac{1}{2}\Omega^{-2}\bigg(&\frac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\nu\partial_\mu k_{\alpha\gamma} + \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha k_{\mu\nu} - \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu k_{\nu\alpha} \\
&- \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu k_{\mu\alpha} + \frac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta k_{\alpha\gamma}\bigg). \qquad (18)
\end{aligned}
$$

Summary

Evaluated within a conformal to Minkowski background

$$
ds^2 = -(g^{(0)}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = -\Omega^2(x)(\eta_{\mu\nu} + f_{\mu\nu})dx^\mu dx^\nu, \qquad (19)
$$

the first order $\delta W_{\mu\nu}$ may be expressed entirely in terms of the transverse traceless $f^{T\theta}_{\mu\nu}$ as

$$
\delta W_{\mu\nu} = \frac{1}{2}\Omega^{-2}\partial_\alpha\partial^\alpha\partial_\beta\partial^\beta f^{T\theta}_{\mu\nu}. \qquad (20)
$$

I have not checked, but such an $f^{T\theta}_{\mu\nu}$ should be gauge invariant.

Alternatively, $\delta W_{\mu\nu}$ may be expressed within the same geometry of (**??**) in terms of the traceless $k_{\mu\nu} = \Omega^{-2}K_{\mu\nu}$ as

$$
\begin{aligned}
\delta W_{\mu\nu}(k_{\mu\nu}) = \frac{1}{2}\Omega^{-2}\bigg(&\frac{2}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\nu\partial_\mu k_{\alpha\gamma} + \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha k_{\mu\nu} - \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu k_{\nu\alpha} \\
&- \eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu k_{\mu\alpha} + \frac{1}{3}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta k_{\alpha\gamma}\bigg). \qquad (21)
\end{aligned}
$$

Imposing the conformal gauge such that $\partial^\nu k_{\mu\nu} = 0$ reduces the fluctuation to

$$
\delta W_{\mu\nu}(k_{\mu\nu}) = \frac{1}{2}\Omega^{-2}\partial_\alpha\partial^\alpha\partial_\beta\partial^\beta k_{\mu\nu}. \qquad (22)
$$

These results coincide with the analogous (E37) and (E38).