Coordinate Transformations RW $k < 0$ v4

Summary

In the radiation dominated early universe with scale factor $L^2 a^2(t) = (d^2 + t^2)$, the leading order large time behavior for $K_{\mu\nu}$ as evaluated in the comoving $k < 0$ R.W. background takes the form:

$$
\begin{aligned}
K_{00}^{(cm)} &\sim 1 \\
K_{01}^{(cm)} &\sim \frac{d}{L}(u) \\
K_{02}^{(cm)} &\sim d(u) \\
K_{03}^{(cm)} &\sim d(u) \\
K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^2) \\
K_{22}^{(cm)} &\sim d^2(u^2) \\
K_{33}^{(cm)} &\sim d^2(u^2) \\
K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{23}^{(cm)} &\sim d^2(u^2),
\end{aligned}
\tag{1}
$$

where $u = t/d$. This result differs from APM3 perturbations by a u^{-1} suppression for each angular index. This is due to the Cartesian to polar coordinate transformation, where factors of $r'(t,r)$ or $R(t,r)$ in the transformation have non-negligible u dependence. The large time behavior for the new coordinate system of (T, R) was found to only match that of the old coordinate system of (p', r') when integrating the z-direction plane wave over the full solid angle.

It remains to look into the necessity (or non-necessity) of spatial averaging.

Notation

From the original form of the scale factor

$$a^2(t) = \frac{2AL^2}{S_0^2} + \frac{t^2}{L^2} \tag{2}$$

we see that for setting up a definition for large t, we should take

$$\frac{t^2}{L^2} \gg \frac{2AL^2}{S_0^2}. \tag{3}$$

This is equivalent to requiring $t \gg d$. If the scale behaves such that $2AL^2/S_0^2 \ll 1$, then $t \gg d$ does not necessarily imply $t \gg L$. Noting in addition the R.W. comoving geometry distance r/L, we introduce two scales of comparison

$$u \equiv \frac{t}{d}, \qquad v \equiv \frac{r}{L}. \tag{4}$$

Thus we define large t behavior as taking $u \gg 1$, holding v finite.

In terms of u and v, the scale factor takes the form

$$a^2(u) = \frac{d^2}{L^2}(1+u^2) \tag{5}$$

comoving R.W. metric takes the form

$$\begin{aligned} ds^2 &= dt^2 - a(t)^2\left(\frac{dr^2}{1+r^2/L^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &= d^2\left[du^2 - (1+u^2)\left(\frac{dv^2}{1+v^2} + v^2 d\Omega^2\right)\right] \end{aligned} \tag{6}$$

Coordinate Transformations

Cartesian to Polar

In going from the geometry of

$$ds^2 = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu \tag{7}$$

to

$$ds^2 = \Omega^2(dt^2 - dr^2 - r^2 d\Omega^2 + k^{(P)}_{\mu\nu} dx^\mu dx^\nu), \tag{8}$$

we must perform the appropriate coordinate transformation (given in the Appendix). Denoting the polar coordinate system as $x^{(P)}$, we find, after imposing the transverse and residual relations, the following:

$$\begin{aligned} k^{(P)}_{00} &= 0 \\ k^{(P)}_{01} &= k_{01}\sin(\theta)\cos(\phi) + k_{02}\sin(\theta)\sin(\phi) \\ k^{(P)}_{02} &= k_{01}r\cos(\theta)\cos(\phi) + k_{02}r\cos(\theta)\sin(\phi) \\ k^{(P)}_{03} &= -k_{01}r\sin(\theta)\sin(\phi) + k_{02}r\sin(\theta)\cos(\phi) \\ k^{(P)}_{11} &= k_{11}\sin^2(\theta)\cos(2\phi) + k_{12}\sin^2(\theta)\sin(2\phi) \\ k^{(P)}_{22} &= k_{11}r^2\cos^2(\theta)\cos(2\phi) + k_{12}r^2\cos^2(\theta)\sin(2\phi) \\ k^{(P)}_{33} &= -k_{11}r^2\sin^2(\theta)\cos(2\phi) - 2k_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) \\ k^{(P)}_{12} &= \frac{1}{2}k_{11}r\sin(2\theta)\cos(2\phi) + k_{12}r\sin(\theta)\cos(\theta)\sin(2\phi) \\ k^{(P)}_{13} &= -2k_{11}r\sin^2(\theta)\sin(\phi)\cos(\phi) + k_{12}r\sin^2(\theta)\cos(2\phi) \\ k^{(P)}_{23} &= -2k_{11}r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + k_{12}r^2\sin(\theta)\cos(\theta)\cos(2\phi) \end{aligned} \tag{9}$$

Since the $\Box^2 k_{\mu\nu} = 0$ is only valid in a conformal to Minkowski background, upon transforming the solution for $k_{\mu\nu}$ to polar coordinates, we must account for the factors of $R(t,r)$ and $r'(t,r)$ in regards to the asymptotic time behavior. As a rule, every angular index gets a power of r.

Original Coordinates

Performing coordinate transformations

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \qquad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{10}$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(p', r')(dp'^2 - dr'^2 - r'^2 d\Omega^2) \tag{11}$$

with conformal factor

$$\Omega^2(p',r') = \frac{4L^2a^2}{(1-(p'+r')^2)(1-(p'+r')^2)} = d^2(1+u^2)\left[(1+u^2)^{1/2}+(1+v^2)^{1/2}\right]^2. \tag{12}$$

We will soon make use of the coordinate relations

$$\begin{aligned}
\frac{\partial p'}{\partial t} &= \frac{1}{d}\frac{\partial p'}{\partial u} = \left(\frac{1}{d}\right)\frac{1+(1+u^2)^{1/2}(1+v^2)^{1/2}}{(1+u^2)^{1/2}((1+u^2)^{1/2}+(1+v^2)^{1/2})^2}\\
\frac{\partial p'}{\partial r} &= \frac{1}{L}\frac{\partial p'}{\partial v} = -\left(\frac{1}{L}\right)\frac{uv}{(1+v^2)^{1/2}((1+u^2)^{1/2}+(1+v^2)^{1/2})^2}\\
\frac{\partial r'}{\partial t} &= \frac{1}{d}\frac{\partial r'}{\partial u} = -\left(\frac{1}{d}\right)\frac{uv}{(1+u^2)^{1/2}((1+u^2)^{1/2}+(1+v^2)^{1/2})^2}\\
\frac{\partial r'}{\partial r} &= \frac{1}{L}\frac{\partial r'}{\partial v} = \left(\frac{1}{L}\right)\frac{1+(1+u^2)^{1/2}(1+v^2)^{1/2}}{(1+v^2)^{1/2}((1+u^2)^{1/2}+(1+v^2)^{1/2})^2}
\end{aligned} \tag{13}$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial p'}{\partial t}\frac{\partial r'}{\partial t}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial t}\right)^2 k_{11}^{(P)}\\
k_{01}^{(cm)} &= \frac{\partial p'}{\partial t}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial p'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial r'}{\partial r}k_{11}^{(P)}\\
k_{02}^{(cm)} &= \frac{\partial p'}{\partial t}k_{02}^{(P)} + \frac{\partial r'}{\partial t}k_{12}^{(P)}\\
k_{03}^{(cm)} &= \frac{\partial p'}{\partial t}k_{03}^{(P)} + \frac{\partial r'}{\partial t}k_{13}^{(P)}\\
k_{11}^{(cm)} &= 2\frac{\partial p'}{\partial r}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial r}\right)^2 k_{11}^{(P)}\\
k_{22}^{(cm)} &= k_{22}^{(P)}\\
k_{33}^{(cm)} &= k_{33}^{(P)}\\
k_{12}^{(cm)} &= \frac{\partial p'}{\partial r}k_{02}^{(P)} + \frac{\partial r'}{\partial r}k_{12}^{(P)}\\
k_{13}^{(cm)} &= \frac{\partial p'}{\partial r}k_{03}^{(P)} + \frac{\partial r'}{\partial r}k_{13}^{(P)}\\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{14}$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the z' axis is

$$K_{\mu\nu} = \Omega^2(p',r')\left[C_{\mu\nu}p'\cos(k(r'\cos\theta - p')) + D_{\mu\nu}\sin(k(r'\cos\theta - p'))\right] \tag{15}$$

where $k_\mu = (-k,0,0,k)$, $z' = r'\cos\theta$, $C_{\mu\nu} = B_{\mu\nu} + B^*_{\mu\nu}$, and $D_{\mu\nu} = i(B_{\mu\nu} - B^*_{\mu\nu})$.

Up to leading order in u, we have:

$$p' \sim 1, \qquad r' \sim \frac{1}{u}, \qquad \Omega^2(p',r') \sim d^2u^4. \tag{16}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{d}\left(\frac{1}{u^2}\right), \qquad \frac{\partial p'}{\partial r} \sim -\frac{1}{L}\left(\frac{1}{u}\right), \qquad \frac{\partial r'}{\partial t} \sim -\frac{1}{d}\left(\frac{1}{u^2}\right), \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{L}\left(\frac{1}{u}\right). \tag{17}$$

For the plane wave $\sin(k(z'-p'))$, the phase equates to

$$z' - p' = \frac{v\cos\theta - u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}. \tag{18}$$

For $u \to \infty$, the above converges and has asymptotic expansion

$$z' - p' \approx -1 + \frac{1}{u}(v + (1+v^2)^{1/2})\cos\theta - \frac{1}{u^2}\left(\frac{1}{2} + v^2 + v(1+v^2)^{1/2}\cos\theta\right) + O\left(\frac{1}{u^3}\right). \tag{19}$$

Hence, to second leading order, the (p', z') plane wave behave asymptotically as

$$\begin{aligned}\sin(k(z'-p')) &\approx -\sin(k) + \frac{k\cos(k)}{u}(v\cos\theta + (1+v^2)^{1/2})\\ \cos(k(z'-p')) &\approx \cos(k) + \frac{k\sin(k)}{u}(v\cos\theta + (1+v^2)^{1/2})\end{aligned} \tag{20}$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim r'$, the leading large u behavior of $B^{(cm)}_{\mu\nu}$ is calculated as:

$$\begin{aligned}&B^{(cm)}_{00} \sim \frac{1}{d^2}\left(\frac{1}{u^4}\right), \quad B^{(cm)}_{01} \sim \frac{1}{dL}\left(\frac{1}{u^3}\right), \quad B^{(cm)}_{02} \sim \frac{1}{d}\left(\frac{1}{u^3}\right), \quad B^{(cm)}_{03} \sim \frac{1}{d}\left(\frac{1}{u^3}\right)\\ &B^{(cm)}_{11} \sim \frac{1}{L^2}\left(\frac{1}{u^2}\right), \quad B^{(cm)}_{22} \sim \frac{1}{u^2}, \quad B^{(cm)}_{33} \sim \frac{1}{u^2}, \quad B^{(cm)}_{12} \sim \frac{1}{L}\left(\frac{1}{u^2}\right), \quad B^{(cm)}_{13} \sim \frac{1}{L}\left(\frac{1}{u^2}\right), \quad B^{(cm)}_{23} \sim \frac{1}{u^2}\end{aligned} \tag{21}$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K^{(cm)}_{\mu\nu}$, which follows

$$K^{(cm)}_{\mu\nu} = \Omega^2(p', r')B^{(cm)}_{\mu\nu} r' \sin(k(z'-p')) \sim d^2u^4 B^{(cm)}_{\mu\nu}. \tag{22}$$

$$\begin{aligned}K^{(cm)}_{00} &\sim 1\\ K^{(cm)}_{01} &\sim \frac{d}{L}(u)\\ K^{(cm)}_{02} &\sim d(u)\\ K^{(cm)}_{03} &\sim d(u)\\ K^{(cm)}_{11} &\sim \frac{d^2}{L^2}(u^2)\\ K^{(cm)}_{22} &\sim d^2(u^2)\\ K^{(cm)}_{33} &\sim d^2(u^2)\\ K^{(cm)}_{12} &\sim \frac{d^2}{L}(u^2)\\ K^{(cm)}_{13} &\sim \frac{d^2}{L}(u^2)\\ K^{(cm)}_{23} &\sim d^2(u^2)\end{aligned} \tag{23}$$

Angular Average Over Plane Wave

$$\int \sin(k(r\cos\theta - t))d\Omega = -4\pi\frac{\sin(kt)\sin(kr)}{kr} \tag{24}$$

In terms of the respective coordinates, this is

$$\langle\sin(k(z'-p'))\rangle = -4\pi\frac{\sin(kp')\sin(kr')}{kr'}. \tag{25}$$

Asymptotically, for large u, this behaves as

$$\langle\sin(k(z'-p'))\rangle \sim -\sin(k). \tag{26}$$

This in fact agrees with our asymptotic expansion of $\sin(k(z'-p'))$ and thus presents no change to the overall behavior.

New Coordinates

Performing coordinate transformations

$$T = \left[u + (1+u^2)^{1/2}\right](1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2}\right]v, \qquad X^2 = T^2 - R^2, \tag{27}$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(T,R)(dT^2 - dR^2 - R^2 d\Omega^2) \tag{28}$$

with conformal factor

$$\Omega^2(T,R) = \frac{L^2 a^2}{T^2 - R^2} = d^2(1+u^2)((1+u^2)^{1/2} - u)^2. \tag{29}$$

We will soon make use of the coordinate relations

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{1}{d}\frac{\partial T}{\partial u} = \left(\frac{1}{d}\right)\frac{(u+(1+u^2)^{1/2})(1+v^2)^{1/2}}{(1+u^2)^{1/2}} \\
\frac{\partial T}{\partial r} &= \frac{1}{L}\frac{\partial T}{\partial v} = \left(\frac{1}{L}\right)\frac{(u+(1+u^2)^{1/2})v}{(1+v^2)^{1/2}} \\
\frac{\partial R}{\partial t} &= \frac{1}{d}\frac{\partial R}{\partial u} = \left(\frac{1}{d}\right)\frac{(u+(1+u^2)^{1/2})v}{(1+u^2)^{1/2}} \\
\frac{\partial R}{\partial r} &= \frac{1}{L}\frac{\partial R}{\partial v} = \left(\frac{1}{L}\right)(u+(1+u^2)^{1/2})
\end{aligned} \tag{30}$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial T}{\partial t}\frac{\partial R}{\partial t}k_{01}^{(P)} + \left(\frac{\partial R}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial T}{\partial t}\frac{\partial R}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial T}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial T}{\partial t}k_{02}^{(P)} + \frac{\partial R}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial T}{\partial t}k_{03}^{(P)} + \frac{\partial R}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial T}{\partial r}\frac{\partial R}{\partial r}k_{01}^{(P)} + \left(\frac{\partial R}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial T}{\partial r}k_{02}^{(P)} + \frac{\partial R}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial T}{\partial r}k_{03}^{(P)} + \frac{\partial R}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{31}$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the Z axis is

$$K_{\mu\nu} = \Omega^2(T,R)\left[C_{\mu\nu}T\cos(k(R\cos\theta - T)) + D_{\mu\nu}\sin(k(R\cos\theta - T))\right] \tag{32}$$

where $k_\mu = (-k, 0, 0, k)$, $Z = R\cos\theta$, $C_{\mu\nu} = B_{\mu\nu} + B^*_{\mu\nu}$, and $D_{\mu\nu} = i(B_{\mu\nu} - B^*_{\mu\nu})$.

Up to leading order in u, we have:

$$T \sim u, \qquad R \sim u, \qquad \Omega^2(T,R) \sim d^2 \tag{33}$$

$$\frac{\partial T}{\partial t} \sim \frac{1}{d}, \qquad \frac{\partial T}{\partial r} \sim \frac{u}{L}, \qquad \frac{\partial R}{\partial t} \sim \frac{1}{d}, \qquad \frac{\partial R}{\partial r} \sim \frac{u}{L} \tag{34}$$

For the plane wave $\sin(k(Z-T))$, the phase equates to

$$Z - T = \left[u + (1+u^2)^{1/2}\right]\left[v\cos\theta - (1+v^2)^{1/2}\right] \tag{35}$$

For $u \to \infty$, the above diverges and has asymptotic expansion

$$Z - T \approx 2u\left(v\cos\theta - (1+v^2)^{1/2}\right) + \frac{1}{2u}\left(v\cos\theta - (1+v^2)^{1/2}\right) + O\left(\frac{1}{u^3}\right) \tag{36}$$

Hence, in the (T, Z) coordinate system, plane waves remain at least periodic with asymptotic form

$$\begin{aligned}
\sin(k(Z-T)) &\approx \sin\left[2ku\left(v\cos\theta - (1+v^2)^{1/2}\right)\right] \\
\cos(k(Z-T)) &\approx \cos\left[2ku\left(v\cos\theta - (1+v^2)^{1/2}\right)\right]
\end{aligned} \tag{37}$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim R$, the leading large u behavior of $B^{(cm)}_{\mu\nu}$ is calculated as:

$$\begin{aligned}
&B^{(cm)}_{00} \sim \frac{1}{d^2}, \quad B^{(cm)}_{01} \sim \frac{u}{dL}, \quad B^{(cm)}_{02} \sim \frac{u}{d}, \quad B^{(cm)}_{03} \sim \frac{u}{d}, \quad B^{(cm)}_{11} \sim \frac{u^2}{L^2} \\
&B^{(cm)}_{22} \sim u^2, \quad B^{(cm)}_{33} \sim u^2, \quad B^{(cm)}_{12} \sim \frac{u^2}{L}, \quad B^{(cm)}_{13} \sim \frac{u^2}{L}, \quad B^{(cm)}_{23} \sim u^2
\end{aligned} \tag{38}$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K^{(cm)}_{\mu\nu}$, which follows

$$K^{(cm)}_{\mu\nu} = \Omega^2(T,R) B^{(cm)}_{\mu\nu} T \sin(k(Z-T)) \sim d^2 u B^{(cm)}_{\mu\nu}. \tag{39}$$

$$\begin{aligned}
K^{(cm)}_{00} &\sim u \\
K^{(cm)}_{01} &\sim \frac{d}{L}u^2 \\
K^{(cm)}_{02} &\sim d(u^2) \\
K^{(cm)}_{03} &\sim d(u^2) \\
K^{(cm)}_{11} &\sim \frac{d^2}{L^2}(u^3) \\
K^{(cm)}_{22} &\sim d^2(u^3) \\
K^{(cm)}_{33} &\sim d^2(u^3) \\
K^{(cm)}_{12} &\sim \frac{d^2}{L}(u^3) \\
K^{(cm)}_{13} &\sim \frac{d^2}{L}(u^3) \\
K^{(cm)}_{23} &\sim d^2(u^3)
\end{aligned} \tag{40}$$

Angular Average Over Plane Wave

$$\int \sin(k(r\cos\theta - t))d\Omega = -4\pi \frac{\sin(kt)\sin(kr)}{kr} \tag{41}$$

In terms of the respective coordinates, this is

$$\langle \sin(k(Z-T)) \rangle = -4\pi \frac{\sin(kT)\sin(kR)}{kR}. \tag{42}$$

Asymptotically, for large u, this behaves as

$$\langle \sin(k(Z-T)) \rangle \sim \frac{1}{u}\sin(kT)\sin(kR). \tag{43}$$

The averaged plane wave angular behavior will thus reduce the large time behavior by an overall u^{-1}. As a result, we have the angular averaged asymptotic behavior

$$\begin{aligned}
K_{00}^{(cm)} &\sim 1 \\
K_{01}^{(cm)} &\sim \frac{d}{L}(u) \\
K_{02}^{(cm)} &\sim d(u) \\
K_{03}^{(cm)} &\sim d(u) \\
K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^2) \\
K_{22}^{(cm)} &\sim d^2(u^2) \\
K_{33}^{(cm)} &\sim d^2(u^2) \\
K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{23}^{(cm)} &\sim d^2(u^2)
\end{aligned} \tag{44}$$

Appendix

Early Universe Setup

Given the geometry

$$ds^2 = (g_{\mu\nu} + K_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu, \tag{45}$$

upon imposing the conformal gauge condition $\nabla_\nu K^{\mu\nu} - \frac{1}{2}K^{\mu\nu}g^{\alpha\beta}_{(0)}\partial_\nu g^{(0)}_{\alpha\beta} = 0$, solutions to the first order source free Bach tensor $\delta W_{\mu\nu} = 0$ are found to obey

$$\frac{1}{2}\Omega^{-2}\Box^2 k_{\mu\nu} = 0 \tag{46}$$

After performing residual gauge transformations to eliminate gauge degrees of freedom, the general momentum eigenstate solution to (46) for a given k-mode is

$$k_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \begin{pmatrix} 0 & B_{01} & B_{02} & 0 \\ B_{01} & B_{11} & B_{12} & 0 \\ B_{02} & B_{12} & -B_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} n_\alpha x^\alpha e^{ikx} \tag{47}$$

with timelike $n_\alpha = (1, 0, 0, 0)$. The full solution for $K_{\mu\nu}$ is then given as

$$K_{\mu\nu} = \Omega^2 k_{\mu\nu}. \tag{48}$$

The $k < 0$ R.W. line element is given in comoving coordinates as

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{49}$$

where $k = -1/L^2$ (with $k < 0$). By coordinate transformation, the hyperbolic R.W. background geometry may be expressed in the form of $g^{(0)}_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, with the general conformal factor Ω having time and spatial dependence in the Minkowski coordinates.

Within the early universe radiation era, the perfect fluid energy momentum tensor obeys $\rho = 3p$, $\rho = A/a^4(t)$, $A > 0$, with $a(t)$ following the evolution equation

$$\begin{aligned} \dot{a}^2 - \frac{1}{L^2} &= \alpha a^2 - \frac{2A}{S_0^2 a^2} \\ &= -2\frac{a^2}{S_0^2}\left(\lambda_S S_0^4 + \frac{A}{a^4}\right) \end{aligned} \tag{50}$$

With the radiation dominating over the cosmological constant in the early universe (since $a(t)$ is small), i.e.

$$\frac{A}{a^4} \gg \lambda_S S_0^4, \tag{51}$$

the evolution equation can then be brought to the form

$$L^2\dot{a}^2 = 1 - \frac{d^2}{L^2}\left(\frac{1}{a^2}\right), \tag{52}$$

in which the solution $a(t)$ is

$$a^2(t) = \frac{1}{L^2}(d^2 + t^2) \tag{53}$$

where we have defined

$$d^2 \equiv \frac{2AL^4}{S_0^2}. \tag{54}$$

(With $A \sim [L]^{-4}$ and $S_0 \sim [L]^{-1}$ fixed early on, the relevant quantities to compare in the radiation dominated era should be the dimensionless $a(t)$ and λ_S).

Cartesian to Polar

Transformation Matrices

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \tag{55}$$

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{56}$$

Time-Time

$$K'_{00} = K_{00} \tag{57}$$

Time-Space

$$K'_{0i} = \frac{\partial x^j}{\partial x'^i} K_{0j} \tag{58}$$

$$\begin{pmatrix} K'_{01} \\ K'_{02} \\ K'_{03} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{01} \\ K_{02} \\ K_{03} \end{pmatrix} \tag{59}$$

$$K'_{01} = K_{01}\sin(\theta)\cos(\phi) + K_{02}\sin(\theta)\sin(\phi) + K_{03}\cos(\theta) \tag{60}$$

$$K'_{02} = K_{01}r\cos(\theta)\cos(\phi) + K_{02}r\cos(\theta)\sin(\phi) - K_{03}r\sin(\theta) \tag{61}$$

$$K'_{03} = -K_{01}r\sin(\theta)\sin(\phi) + K_{02}r\sin(\theta)\cos(\phi) \tag{62}$$

Space-Space

$$K'_{ij} = \frac{\partial x^k}{\partial x'^i} K_{kl} \frac{\partial x^l}{\partial x'^j} \tag{63}$$

$$\begin{pmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{21} & K'_{22} & K'_{23} \\ K'_{31} & K'_{32} & K'_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}^T \tag{64}$$

$$\begin{aligned} K'_{11} = {} & K_{11}\sin^2(\theta)\cos^2(\phi) + K_{12}\sin^2(\theta)\sin(2\phi) + K_{13}\sin(2\theta)\cos(\phi) + K_{22}\sin^2(\theta)\sin^2(\phi) \\ & + K_{23}\sin(2\theta)\sin(\phi) + K_{33}\cos^2(\theta) \end{aligned} \tag{65}$$

$$\begin{aligned} K'_{22} = {} & K_{11}r^2\cos^2(\theta)\cos^2(\phi) + K_{12}r^2\cos^2(\theta)\sin(2\phi) - K_{13}r^2\sin(2\theta)\cos(\phi) + K_{22}r^2\cos^2(\theta)\sin^2(\phi) \\ & - K_{23}r^2\sin(2\theta)\sin(\phi) + K_{33}r^2\sin^2(\theta) \end{aligned} \tag{66}$$

$$K'_{33} = K_{11}r^2\sin^2(\theta)\sin^2(\phi) - 2K_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{22}r^2\sin^2(\theta)\cos^2(\phi) \tag{67}$$

$$K'_{12} = K_{11} r\sin(\theta)\cos(\theta)\cos^2(\phi) + K_{12} r\sin(\theta)\cos(\theta)\sin(2\phi) + K_{13} r\cos(2\theta)\cos(\phi) + K_{22} r\sin(\theta)\cos(\theta)\sin^2(\phi)$$
$$+ K_{23} r\cos(2\theta)\sin(\phi) - K_{33} r\sin(\theta)\cos(\theta) \tag{68}$$

$$K'_{13} = -K_{11} r\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{12} r\sin^2(\theta)\cos(2\phi) - K_{13} r\sin(\theta)\cos(\theta)\sin(\phi) + K_{22} r\sin^2(\theta)\sin(\phi)\cos(\phi)$$
$$+ K_{23} r\sin(\theta)\cos(\theta)\cos(\phi) \tag{69}$$

$$K'_{23} = -K_{11} r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + K_{12} r^2\sin(\theta)\cos(\theta)\cos(2\phi) + K_{13} r^2\sin^2(\theta)\sin(\phi)$$
$$+ K_{22} r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) - K_{23} r^2\sin^2(\theta)\cos(\phi) \tag{70}$$

Polar to Comoving

$$K'_{\mu\nu}(t,r,\theta,\phi) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} k_{\alpha\beta}(T,R,\theta,\phi) \tag{71}$$

$$J_{\mu\nu} = \frac{\partial x^\nu}{\partial x'^\mu}, \qquad \text{where} \quad x(T,R,\theta,\phi) \quad x'(t,r,\theta,\phi) \tag{72}$$

$$J_{\mu\nu} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{73}$$

$$k^{(cm)}_{\mu\nu} = \frac{\partial x^k_{(P)}}{\partial x^i_{(cm)}} k^{(P)}_{kl} \frac{\partial x^l_{(P)}}{\partial x^j_{(cm)}} \tag{74}$$

$$\begin{pmatrix} k^{(cm)}_{00} & k^{(cm)}_{01} & k^{(cm)}_{02} & k^{(cm)}_{03} \\ k^{(cm)}_{10} & k^{(cm)}_{11} & k^{(cm)}_{12} & k^{(cm)}_{13} \\ k^{(cm)}_{20} & k^{(cm)}_{21} & k^{(cm)}_{22} & k^{(cm)}_{23} \\ k^{(cm)}_{30} & k^{(cm)}_{31} & k^{(cm)}_{32} & k^{(cm)}_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^{(P)}_{00} & k^{(P)}_{01} & k^{(P)}_{02} & k^{(P)}_{03} \\ k^{(P)}_{10} & k^{(P)}_{11} & k^{(P)}_{12} & k^{(P)}_{13} \\ k^{(P)}_{20} & k^{(P)}_{21} & k^{(P)}_{22} & k^{(P)}_{23} \\ k^{(P)}_{30} & k^{(P)}_{31} & k^{(P)}_{32} & k^{(P)}_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \tag{75}$$

$$\begin{aligned}
k^{(cm)}_{00} &= 2\frac{\partial T}{\partial t}\frac{\partial R}{\partial t} k^{(P)}_{01} + \left(\frac{\partial R}{\partial t}\right)^2 k^{(P)}_{11} \\
k^{(cm)}_{01} &= \frac{\partial T}{\partial t}\frac{\partial R}{\partial r} k^{(P)}_{01} + \frac{\partial R}{\partial t}\frac{\partial T}{\partial r} k^{(P)}_{01} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r} k^{(P)}_{11} \\
k^{(cm)}_{02} &= \frac{\partial T}{\partial t} k^{(P)}_{02} + \frac{\partial R}{\partial t} k^{(P)}_{12} \\
k^{(cm)}_{03} &= \frac{\partial T}{\partial t} k^{(P)}_{03} + \frac{\partial R}{\partial t} k^{(P)}_{13} \\
k^{(cm)}_{11} &= 2\frac{\partial T}{\partial r}\frac{\partial R}{\partial r} k^{(P)}_{01} + \left(\frac{\partial R}{\partial r}\right)^2 k^{(P)}_{11} \\
k^{(cm)}_{22} &= k^{(P)}_{22} \\
k^{(cm)}_{33} &= k^{(P)}_{33} \\
k^{(cm)}_{12} &= \frac{\partial T}{\partial r} k^{(P)}_{02} + \frac{\partial R}{\partial r} k^{(P)}_{12} \\
k^{(cm)}_{13} &= \frac{\partial T}{\partial r} k^{(P)}_{03} + \frac{\partial R}{\partial r} k^{(P)}_{13} \\
k^{(cm)}_{23} &= k^{(P)}_{23}
\end{aligned} \tag{76}$$