Quantum Mechanics III
HW 2

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Due: Feb. 3

1.12 Consider measurements of a spin 1/2 particle in the direction $\hat{\mathbf{n}} = \cos\alpha\hat{\mathbf{e}}_z + \sin\alpha\hat{\mathbf{e}}_x$, so that we are measuring the operator $S(\alpha)$ with the corresponding unit-normalized eigenvectors $\chi_\pm(\alpha)$ corresponding to the eigenvalues $\pm\hbar/2$:

$$S(\alpha) = \frac{\hbar}{2}\begin{bmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{bmatrix};\ \chi_+(\alpha) = \begin{bmatrix}\cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}\end{bmatrix},\ \chi_-(\alpha) = \begin{bmatrix}-\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\end{bmatrix}$$

(a) Suppose we prepare the system initially in the state $\chi_+(0)$ (along the z axis), then measure successively in the directions α, 2α, ..., $n\alpha$. Show that the probability that the result is $+\hbar/2$ everytime is $\left[\cos\frac{\alpha}{2}\right]^{2n}$.

(b) Now make the angle α smaller and the number of measurements larger in such a way that $n\alpha = \pi$ remains constant. What do you achieve in the limit $n \to \infty$?

(a) After each measurement, the state collapses into the eigenstate of $S(\alpha)$ with eigenvalue $\hbar/2$. In order to find probabilities of measuring these eigenvalues, we expand the state in question in the eigenbasis of our operator $S(\alpha)$. An arbitrary state $\xi(\alpha')$ can be expanded in said basis as

$$\xi(\alpha') = \chi_+^\dagger(\alpha)\xi(\alpha')\chi_+(\alpha) + \chi_-^\dagger(\alpha)\xi(\alpha')\chi_-(\alpha).$$

with probability of measuring eigenvalue $\hbar/2$

$$|\chi_+^\dagger(\alpha)\xi(\alpha')|^2.$$

Now, if we start off with the initial state $\chi_+(0)$ and perform a measurement at angle α, the probabilty is

$$\begin{aligned}|\chi_+^\dagger(\alpha)\chi(0)|^2 &= (\chi_+^\dagger(\alpha)\chi_+(0))^2 \\ &= \left(\begin{bmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}\right)^2 \\ &= \left(\cos\frac{\alpha}{2}\right)^2\end{aligned}$$

The system is now in state $\chi_+(\alpha)$. If we perform a measurement at 2α we have probability

$$(\chi_+^\dagger(2\alpha)\chi_+(\alpha))^2 = \left(\begin{bmatrix}\cos\frac{2\alpha}{2} & \sin\frac{2\alpha}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}\end{bmatrix}\right)^2.$$

If we perform n measurements in intervals of α the probability is

$$\left[\chi_+^\dagger(n\alpha)\chi_+((n-1)\alpha)\right]^2 = \left(\begin{bmatrix}\cos\frac{n\alpha}{2} & \sin\frac{n\alpha}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{(n-1)\alpha}{2} \\ \sin\frac{(n-1)\alpha}{2}\end{bmatrix}\right)^2$$

$$
\begin{aligned}
&= \left[\cos\left(\frac{n\alpha}{2}\right)\cos\left(\frac{(n-1)\alpha}{2}\right) + \sin\left(\frac{n\alpha}{2}\right)\sin\left(\frac{(n-1)\alpha}{2}\right)\right]^2 \\
&= \left\{\cos\left(\frac{n\alpha}{2} - \frac{(n-1)\alpha}{2}\right)\right\}^2 \\
&= \left(\cos\frac{\alpha}{2}\right)^2
\end{aligned}
$$

The final probability is the product of individual probabilities, thus

$$
P(n) = \left(\cos\frac{\alpha}{2}\right)^{2n} .
$$

(b) For $n\alpha = \pi$ we may write our probability as

$$
\begin{aligned}
P &= \lim_{n\to\infty}\left[\cos\left(\frac{\pi}{2n}\right)\right]^{2n} \\
&= \lim_{n\to\infty} \exp\left[2n\ln\left(\cos\left(\frac{\pi}{2n}\right)\right)\right] \\
&= \exp\left\{\lim_{n\to\infty}\left(\frac{\ln\left(\cos\left(\frac{\pi}{2n}\right)\right)}{1/2n}\right)\right\} \\
&\Rightarrow \exp\left\{\lim_{n\to\infty}\left(\frac{-\tan\left(\frac{\pi}{2n}\right)\frac{\pi}{2n^2}}{1/2n^2}\right)\right\} \\
&= \exp\left\{\lim_{n\to\infty}\tan\left(\frac{-\pi}{2n}\right)\pi\right\} \\
&= 1
\end{aligned}
$$

So it seems that given this type of successive measurement, our state will always be in a spin up eigenstate.

2.1 Verify the completeness relation in Eq. (2.3) from the notions of orthonormality and completeness as discussed in Ch. 1.

Given an arbitrary vector $|v\rangle$ in $\mathscr{H}$ space, if we have a complete orthonormal basis $\{|n\rangle\}$, then $|v\rangle$ may be expanded as a linear combination of $\{|n\rangle\}$ with coefficients $c_n = (n, v)$

$$
|v\rangle = \sum_n c_n |n\rangle = \sum_n \langle n|v\rangle |n\rangle = \left(\sum_n |n\rangle\langle n|\right)|v\rangle = \mathbb{1}\,|v\rangle .
$$

Or perhaps if we apply the completeness relation the vector $|v\rangle$ (which we expand in the same basis labeled by $\{|m\rangle\}$)

$$
\left(\sum_n |n\rangle\langle n|\right)|v\rangle = \sum_{n,m} |n\rangle\langle n|m\rangle\langle m|v\rangle = \sum_m |m\rangle\langle m|v\rangle = |v\rangle .
$$

Thus we verify that $\sum_n |n\rangle\langle n| = \mathbb{1}$.

2.3 Any operator with the property that $P^2 = P$ is called a projection operator or projector. This property, in fact, does not guarantee that P is hermitian, but in quantum mechanics only hermitian projection operators really figure. Thus, assume that $P^2 = P$ and $P = P^\dagger$. Show that such an operator is always an orthogonal projection to some subspace of the Hilbert space.

As always, take is as given that any orthonormal set in the quantum mechanical Hilbert space can be completed into an orthonormal basis.

Since $P = P^\dagger$, its eigenvectors form an orthogonal set, which may be completed to an orthonormal basis $\{|n\rangle\}$ for a particular subspace $\mathscr{S}$. We apply the completeness relation

$$\sum_{n\in\mathscr{S}} |n\rangle\langle n| = \mathbb{1}$$

to the operator P

$$\sum_{n,m\in\mathscr{S}} |n\rangle\langle m|\langle n|P|m\rangle = \sum_{n,m\in\mathscr{S}} \lambda_n |n\rangle\langle m|\langle n|m\rangle = \sum_{n\in\mathscr{S}} \lambda_n |n\rangle\langle n|$$

where λ_n denote the eigenvalues $P|n\rangle = \lambda_n |n\rangle$. Now we take P^2

$$\begin{aligned} P^2 &= \left(\sum_{n\in\mathscr{S}} \lambda_n |n\rangle\langle n|\right)\left(\sum_{m\in\mathscr{S}} \lambda_m |m\rangle\langle m|\right) \\ &= \sum_{n,m\in\mathscr{S}} \lambda_n\lambda_m\delta_{nm} |n\rangle\langle m| \\ &= \sum_{n\in\mathscr{S}} \lambda_n^2 |n\rangle\langle n| \stackrel{!}{=} \sum_{n\in\mathscr{S}} \lambda_n |n\rangle\langle n| = P \end{aligned}$$

We have the equality $P^2 = P$ only if the eigenvalues $\lambda_n^2 = \lambda_n$. Hence $\lambda_n = 1$ (or trivially 0). Therefore

$$P = \sum_{n\in\mathscr{S}} |n\rangle\langle n|$$

which is the definition of the orthogonal projection on a subspace in $\mathscr{H}$.

2.4 An alternative approach to normal operators: Take it as given that two hermitian operators may be diagonalized simultaneously if and only if they commute.

(a) As already noted, every operator A may be decomposed trivially in the form $A = A_1 + iA_2$, where A_1 and A_2 are hermitian. Suppose we have a normal operator N with the corresponding components N_1 and N_2. Verify the following items: (i) $[N_1, N_2] = 0$. (ii) N may be diagonalized.

(b) Conversely, suppose that an operator N can be diagonalilzed, with the eigenvalues c_n (not necessarily real) and the orthonormal eigenvectors u_n. Verify the following items: (i) $(u_n, N^\dagger u_m) = c_n^*\delta_{nm}$. (ii) $N^\dagger u_m = c_m^* u_m$. Therefore, $N^\dagger$ can also be diagonalized, eigenvalues and eigenvectors c_n^* and u_n. (iii) N is normal.

We have again, the result that normal, and only normal, operators can be diagonalized.

(a)

$$N = N_1 + iN_2; \quad N^\dagger = N_1 - iN_2$$

$$[N, N^\dagger] = [N_1 + iN_2, N_1 - iN_2] = 2i[N_2, N_1] = 0$$

$$\Rightarrow [N_1, N_2] = 0.$$

Since $[N_1, N_2] = 0$ both N_1 and N_2 (hermitian operators) can be simultaneously diagonalized. As $N = N_1 + iN_2$ it is the sum of two diagonalized operators and thus is itself diagonal (but not strictly hermitian). Therefore normal operators are diagonalizable.

(b) i).

$$(u_n, N^\dagger u_m) = (Nu_n, u_m) = (c_n u_n, u_m) = c_n^* \delta_{nm}$$

ii).

$$N^\dagger = \sum_{n,m} |n\rangle \langle n|N^\dagger|m\rangle \langle m| = \sum_n c_n^* |n\rangle \langle n|$$

$$N^\dagger |n\rangle = \left(\sum_n c_n^* |n\rangle \langle n| \right) |n\rangle = c_n^* |n\rangle .$$

iii).

$$[N, N^\dagger] |n\rangle = (|c_n|^2 - |c_n|^2) |n\rangle = 0$$

$$\Rightarrow [N, N^\dagger] = 0$$