Cosmological Fluctuations in Standard and Conformal Gravity

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Doctoral Degree Final Examination

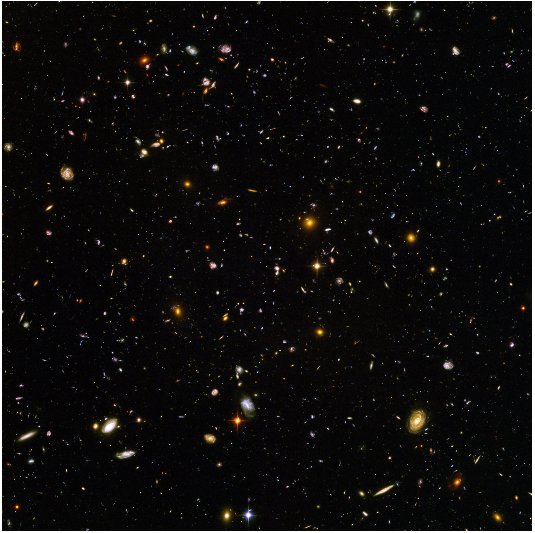
June 02, 2020

- Introduction and Formalism
- Three Dimensional Scalar, Vector, Tensor Decomposition (SVT3)
- Four Dimensional Scalar, Vector, Tensor Decomposition (SVT4)
- Conformal Gravity (SVT and Conformal to Flat Backgrounds)
- Conformal Gravity Robertson-Walker Radiation Era Solution
- Conclusions
- Computational Methods

Cosmological Geometries

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker, de Sitter ($dS_4 \subset \mathrm{RW}$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

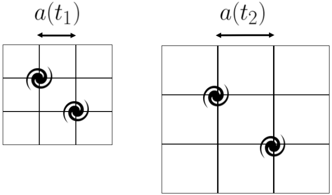
$$ds^2 = \Omega(x)^2 \big(- dt^2 + dx^2 + dy^2 + dz^2 \big)$$



Hubble Ultra-Deep Field. NASA and the European Space Agency.

Comoving Robertson Walker geometry

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \quad (1) \end{aligned}$$



- 3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -2k\tilde{g}_{ij}, \qquad R = -6k, \qquad k \in \{-1, 0, 1\} \tag{2}$$

- Define the conformal time $\tau = \int \frac{dt}{a(t)}$,

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \tag{3}$$

- Conformal to flat form

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \qquad (k=0) \tag{4}$$

$$ds^2 = \frac{4a^2(\tau)}{[1+(p'+r')^2][1+(p'-r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2] \qquad (k=1) \tag{5}$$

$$ds^2 = \frac{4a^2(\tau)}{[1-(p'+r')^2][1-(p'-r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2] \qquad (k=-1) \tag{6}$$

- Einstein Hilbert action

$$I_{\mathsf{EH}} = -\frac{1}{16\pi G}\int d^4x(-g)^{1/2}g^{\mu\nu}R_{\mu\nu} \tag{7}$$

$$\frac{16\pi G}{(-g)^{1/2}}\frac{\delta I_{\mathsf{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}, \qquad \frac{2}{(-g)^{1/2}}\frac{\delta I_{\mathsf{M}}}{\delta g_{\mu\nu}} = T_{\mu\nu} \tag{8}$$

- Einstein field equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha} = -8\pi G T^{\mu\nu}, \tag{9}$$

subject to Bianchi identity

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R^{\mu}{}_{\mu} \implies \nabla_\mu G^{\mu\nu} = 0 \tag{10}$$

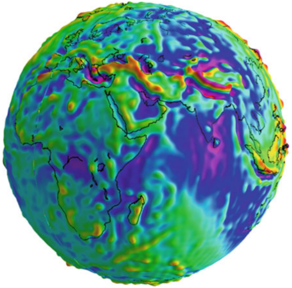
- Introduce fluctuation to background $g_{\mu\nu}^{(0)}$

$$g_{\mu\nu}(x) = g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x), \qquad g_{(0)}^{\mu\nu} h_{\mu\nu} \equiv h \tag{11}$$

$$G_{\mu\nu} = G_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) + \delta G_{\mu\nu}(h_{\mu\nu}) \tag{12}$$

$$G_{\mu\nu}^{(0)} = R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R_{\alpha}^{(0)\alpha} \tag{13}$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R_{\alpha}^{(0)\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}{}_{\alpha}. \tag{14}$$



- Likewise perturb $T_{\mu\nu}$

$$T_{\mu\nu} = T_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta T_{\mu\nu}(h_{\mu\nu}) \tag{15}$$

- Form background and first order field equations

$$\Delta_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} + T_{\mu\nu}^{(0)} = 0 \tag{16}$$
$$\Delta_{\mu\nu} = \delta G_{\mu\nu}(h_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu}) = 0 \tag{17}$$

Walter, U. (2019). Correction to: Astronautics. In Astronautics (pp. C1–C1). Springer International Publishing.

- Under $x^\mu \to x'^\mu$

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{18}$$

- Under coordinate transformation $x^\mu \to x^\mu - \epsilon^\mu(x)$, with $\nabla_\mu \epsilon_\nu \sim \mathcal{O}(h)$

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) \quad \equiv \quad \Delta h_{\mu\nu}(x) \tag{19}$$

- Gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu} \tag{20}$$

where $\Delta h_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$

- Transformation of fluctuation tensors themselves

$$\delta G_{\mu\nu} \quad \to \quad \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda G^{(0)}_{\mu\nu} \epsilon^\lambda \tag{21}$$

$$\delta T_{\mu\nu} \quad \to \quad \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda T^{(0)}_{\mu\nu} \epsilon^\lambda. \tag{22}$$

$$\Delta_{\mu\nu} \quad = \quad \delta G_{\mu\nu} + \delta T_{\mu\nu} \tag{23}$$

- Spatial components of the perturbed Einstein tensor

$$
\begin{aligned}
\delta G_{ij} \quad = \quad & -\tfrac{1}{2}\ddot{h}_{ij} + \tfrac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \tfrac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2\tilde{g}_{ij}h_{00}\Omega^{-2} \\
& -\dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
& +2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
& -4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
& -\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
& +\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& -\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
& +\tfrac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \tfrac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
& -\tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \tfrac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
& +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \tfrac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h, \qquad (24)
\end{aligned}
$$

Decompose the metric perturbation $h_{\mu\nu}$ into a set of scalars, vectors, and tensors

- Define $h_{\mu\nu} = \Omega^2(x) f_{\mu\nu}$, perform 3 + 1 decomposition

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = (g^{(0)}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x)(\tilde{g}^{(0)}_{\mu\nu} + f_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x)\big[(-1 + f_{00})dt^2 + 2 f_{0i} dt dx^i + (\tilde{g}_{ij} + f_{ij})\big] dx^i dx^j \end{aligned} \tag{25}$$

- Decompose f_{00}, f_{0i}, and f_{ij} in terms of 3-dimensional scalars, vectors, and tensors

$$\begin{aligned} f_{00} &= -2\phi, \\ f_{0i} &= B_i + \tilde{\nabla}_i B, \\ f_{ij} &= -2\psi \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned} \tag{26}$$

with vectors and tensors obeying

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \tilde{g}^{ij} E_{ij} = 0. \tag{27}$$

$$ds^2 = \Omega^2(x)\Big[-(1 + 2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B) dt dx^i + [(1 - 2\psi)\tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}] dx^i dx^j \Big] \tag{28}$$

- de Sitter geometry

$$ds^2 = \frac{1}{H^2\tau^2}\Big[-(1+2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i + [(1-2\psi)\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\Big] \quad (29)$$

$$R_{\lambda\mu\nu\kappa} = H^2(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}), \qquad R_{\mu\kappa} = -3H^2 g_{\mu\kappa}, \qquad R = -12H^2 \quad (30)$$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2\Omega(\tau)^2 f_{\mu\nu} \quad (31)$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2 \Omega(\tau)^2 f_{\mu\nu} \tag{29}$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

$$\begin{aligned}
\delta G_{00} &= -\frac{6}{\tau}\dot{\psi} - \frac{2}{\tau}\tilde{\nabla}^2(\tau\psi + B - \dot{E}), \\
\delta G_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) + \frac{1}{\tau^2}\tilde{\nabla}_i(3B - 2\tau^2\dot{\psi} + 2\tau\phi) + \frac{3}{\tau^2}B_i, \\
\delta G_{ij} &= \frac{\delta_{ij}}{\tau^2}\Big[- 2\tau^2\ddot{\psi} + 2\tau\dot{\phi} + 4\tau\dot{\psi} - 6\phi - 6\psi \\
&+ \tilde{\nabla}^2\left(2\tau B - \tau^2\dot{B} + \tau^2\ddot{E} - 2\tau\dot{E} - \tau^2\phi + \tau^2\psi\right)\Big] \\
&+ \frac{1}{\tau^2}\tilde{\nabla}_i\tilde{\nabla}_j\left[-2\tau B + \tau^2\dot{B} - \tau^2\ddot{E} + 2\tau\dot{E} + 6E + \tau^2\phi - \tau^2\psi\right] \\
&+ \frac{1}{2\tau^2}\tilde{\nabla}_i\left[-2\tau B_j + 2\tau\dot{E}_j + \tau^2\dot{B}_j - \tau^2\ddot{E}_j + 6E_j\right] \\
&+ \frac{1}{2\tau^2}\tilde{\nabla}_j\left[-2\tau B_i + 2\tau\dot{E}_i + \tau^2\dot{B}_i - \tau^2\ddot{E}_i + 6E_i\right] \\
&- \ddot{E}_{ij} + \frac{6}{\tau^2}E_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij},
\end{aligned} \tag{30}$$

- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$

$$
\begin{aligned}
\Delta_{00} &= -\frac{6}{\tau^2}(\dot\beta - \alpha) - \frac{2}{\tau}\tilde\nabla^2\beta = 0, \\
\Delta_{0i} &= \frac{1}{2}\tilde\nabla^2(B_i - \dot E_i) - \frac{2}{\tau}\tilde\nabla_i(\dot\beta - \alpha) = 0, \\
\Delta_{ij} &= \frac{\delta_{ij}}{\tau^2}\left[-2\tau(\ddot\beta - \dot\alpha) + 6(\dot\beta - \alpha) + \tau\tilde\nabla^2(2\beta - \tau\alpha)\right] + \frac{1}{\tau}\tilde\nabla_i\tilde\nabla_j(-2\beta + \tau\alpha) \\
&+ \frac{1}{2\tau}\tilde\nabla_i[-2(B_j - \dot E_j) + \tau(\dot B_j - \ddot E_j)] + \frac{1}{2\tau}\tilde\nabla_j[-2(B_i - \dot E_i) + \tau(\dot B_i - \ddot E_i)] \\
&- \ddot E_{ij} + \frac{2}{\tau}\dot E_{ij} + \tilde\nabla^2 E_{ij} = 0, \\
g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot\beta - \dot\alpha) + 24(\dot\beta - \alpha) + 6\tau\tilde\nabla^2\beta - 2\tau^2\tilde\nabla^2\alpha] = 0,
\end{aligned} \tag{31}
$$

where

$$
\alpha = \phi + \psi + \dot B - \ddot E, \quad \beta = \tau\psi + B - \dot E, \quad B_i - \dot E_i, \quad E_{ij}. \tag{32}
$$

- Decouple by applying higher derivatives

$$\tilde{\nabla}^4(\alpha + \dot{\beta}) = 0, \quad \tilde{\nabla}^4(\alpha - \dot{\beta}) = 0,$$

$$\tilde{\nabla}^4(B_i - \dot{E}_i) = 0,$$

$$\tilde{\nabla}^4\left(-\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij}\right) = 0. \tag{33}$$

- Recap:
  - Perturb $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$ around de Sitter background
  - Decompose $h_{\mu\nu}$ into SVT3 components
  - Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$
  - Apply higher derivatives to decouple SVT3 representations

- SVT3 in a Minskowski background,

$$\begin{aligned} h_{00} &= -2\phi, \\ h_{0i} &= B_i + \partial_i B \\ h_{ij} &= -2\psi\tilde{g}_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}, \end{aligned} \tag{34}$$

$$\partial^i B_i = \partial^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \partial^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0. \tag{35}$$

Decomposition of $V_i = V_i^T + \partial_i V$

- Longitudinal decomposition does not hold for any scalar. $\partial^i V_i = \partial_i \partial^i V = 0$
- Introduce a Green's function $\partial_i \partial^i D(x - x') = \delta^3(x - x')$ and use Green's identity

$$V(x')\partial_i\partial^i D(x - x') = D(x - x')\partial_i\partial^i V(x') + \partial_i[V(x')\partial^i D(x - x') - D(x - x')\partial^i V(x')] \tag{36}$$

- Integrate

$$V(x) = \underbrace{\int d^3x' D(x - x')\partial_i\partial^i V(x')}_{\text{Non-Harmonic}} + \underbrace{\oint dS_i[V(x')\partial^i D(x - x') - D(x - x')\partial^i V(x')]}_{\text{Harmonic}} \tag{37}$$

$$V = V^{NH} + V^H, \qquad \partial_i\partial^i V = \partial_i\partial^i V^{NH}, \qquad \partial_i\partial^i V^H = 0 \tag{38}$$

- Need a $\partial_i V$ which could never be transverse

$$\begin{aligned} V \equiv V^{NH} &= \int d^3x' D(x - x')\partial_i\partial^i V(x') = \int d^3x' D(x - x')\partial^i V_i(x') \\ &\implies \oint dS_i[V(x')\partial^i D(x - x') - D(x - x')\partial^i V(x')] = 0 \end{aligned} \tag{39}$$

- Transverse Longitudinal Decomposition

$$V_i = V_i^T + \partial_i V, \qquad \partial_i V = \partial_i \int d^3x' D(x - x')\partial^j V_j(x'), \qquad V_i^T = V_i - \partial_i \int d^3x' D(x - x')\partial^j V_j(x') \tag{40}$$

- Transverse Vector Decomposition

$$V_i = V_i^T + \partial_i V, \qquad \partial_i V = \partial_i \int d^3x' D(x-x')\partial^j V_j(x'), \qquad V_i^T = V_i - \partial_i \int d^3x' D(x-x')\partial^j V_j(x') \quad (41)$$

- Projector Formalism

$$\Pi_{ij} = \delta_{ij} - \frac{\partial}{\partial x^i} \int d^3x' D(x-x') \frac{\partial}{\partial x'^j}$$

$$\Pi_{ij} V^j = V_T^j$$

$$\Pi_{ij}\Pi^j{}_k = \Pi_{ik}, \quad \Pi_{ij} V_T^j = V_T^j, \quad \Pi_{ij}(\partial^j V) = 0 \quad (42)$$

- Hence, we can decompose h_{0i} as

$$h_{0i} = B_i + \partial_i B, \qquad B = \int d^3x' D(x-x')\partial^j h_{0j}, \qquad B_i = \Pi_{ij} h_0{}^j = h_{0i} - \partial_i \int d^3x' D(x-x')\partial^j h_{0j} \quad (43)$$

  - Composed of non-local integrals
  - B itself must vanish asymptotically (or decay sufficiently fast)

$$h_{ij} \quad = \quad -2\psi\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij} \tag{44}$$

- Rank 2 tensor transverse traceless decomposition

$$h_{ij}^{TT} = h_{ij} - \partial_i W_j - \partial_j W_i + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x') \left(\partial^k W_k + \delta^{kl} h_{kl}\right) + \frac{1}{2}\delta_{ij}\left(\partial^k W_k - \delta^{kl} h_{kl}\right) \tag{45}$$

where we introduce a W_k obeying

$$\partial^j h_{ij} = \partial_k \partial^k W_i \tag{46}$$

- Can further decompose W_i into transverse and longitudinal components

$$W_i^T = W_i - \partial_i \int d^3x' D(x-x') \partial^k W_k \tag{47}$$

- Make definitions,

$$\begin{aligned} h_{ij} \quad = \quad & \underbrace{\left[h_{ij} - \partial_i W_j - \partial_j W_i - \frac{1}{2} g_{ij}(\delta^{kl} h_{kl} - \partial^k W_k) + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x')(\delta^{kl} h_{kl} + \partial^k W_k)\right]}_{2E_{ij}} \\ & + \partial_i \underbrace{\left(W_j - \partial_j \int d^3x' D(x-x') \partial^k W_k\right)}_{E_j} + \partial_j \underbrace{\left(W_i - \partial_i \int d^3x' D(x-x') \partial^k W_k\right)}_{E_i} \\ & -2\delta_{ij} \underbrace{\left(\tfrac{1}{4}\partial^k W_k - \tfrac{1}{4}\delta^{kl} h_{kl}\right)}_{\psi} + 2\partial_i\partial_j \underbrace{\int d^3x' D(x-x') \left(\tfrac{3}{4}\partial^k W_k - \tfrac{1}{4}\delta^{kl} h_{kl}\right)}_{E} \end{aligned} \tag{48}$$

- Transformation behavior under full 4D coordinate transformation $x^\mu \to x'^\mu$

$$h_{\mu\nu} \to \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} h_{\alpha\beta}, \qquad h_{00} = -2\phi \tag{49}$$

- We seek to
  (A) Generalize to $D = 4$
  (B) Generalize to curved space backgrounds

$$[\nabla_\kappa, \nabla_\nu] V_\lambda = V^\sigma R_{\lambda\sigma\nu\kappa} \tag{50}$$

- SVT4 Decomposition

$$h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}, \tag{51}$$

subject to

$$\nabla^\mu F_\mu = 0, \qquad F_{\mu\nu} = F_{\nu\mu}, \qquad g^{\mu\nu} F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0. \tag{52}$$

- Maximally Symmetric Space

$$R_{\lambda\mu\nu\kappa} = \frac{R}{D(1-D)}(g_{\lambda\nu}g_{\mu\kappa} - g_{\mu\nu}g_{\lambda\kappa}), \qquad R_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa}, \qquad R = H^2D(1-D) \tag{53}$$

- Curved Space Green's Function

$$\left[\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right] D^{(A)}(x,x') = (-g)^{-1/2}\delta^{(D)}(x-x') \tag{54}$$

- Transverse traceless decomposition of rank 2 tensor

$$\begin{aligned} h^{TT}_{\mu\nu} &= h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{2-D}{D-1}\left[\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right]\int d^Dx'(-g)^{1/2}D^{(A)}(x,x')\nabla^\sigma W_\sigma \\ &\quad + \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right]\int d^Dx'(-g)^{1/2}D^{(A)}(x,x')h \end{aligned} \tag{55}$$

$$\nabla^\mu h_{\mu\nu} = \left[\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right] W_\nu \tag{56}$$

- Commutations

$$\begin{aligned} [\nabla^\sigma, \nabla_\nu]W_\sigma &= -\frac{R}{D}W_\nu, \qquad [\nabla^\mu\nabla_\mu, \nabla_\nu]V = -\frac{R}{D}\nabla_\nu V, \\ [\nabla_\sigma\nabla^\sigma, \nabla_\mu\nabla_\nu]V &= g_{\mu\nu}\left[\frac{2R}{D(D-1)}\right]\nabla_\sigma\nabla^\sigma V - \frac{2R}{D-1}\nabla_\mu\nabla_\nu V, \end{aligned} \tag{57}$$

- Further decompose W_μ into longitudinal and transverse components

$$W_\mu^T = W_\mu - \nabla_\mu \int d^D x' (-g)^{1/2} D^{(B)}(x,x') \nabla^\sigma W_\sigma \tag{58}$$

where

$$\nabla_\alpha \nabla^\alpha D^{(B)}(x,x') = (-g)^{-1/2} \delta^{(D)}(x-x'). \tag{59}$$

- Make definitions

$$\begin{aligned}
2F_{\mu\nu} &= h_{\mu\nu}^{TT} \\
F_\mu &= W_\mu - \nabla_\mu \int d^D x' (-g)^{1/2} D^{(B)}(x,x') \nabla^\sigma W_\sigma \\
F &= \int d^D x' (-g)^{1/2} D^{(B)}(x,x') \nabla^\sigma W_\sigma + \frac{1}{2(D-1)} \int d^D x' (-g)^{1/2} D^{(A)}(x,x') [(2-D)\nabla^\sigma W_\sigma - h] \\
\chi &= \frac{1}{2(D-1)} \left[\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)} \int d^D x' (-g)^{1/2} D^{(A)}(x,x') [(2-D)\nabla^\sigma W_\sigma - h] \right]
\end{aligned} \tag{60}$$

- SVTD

$$h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{61}$$

- In taking $D \to 3$ and $g_{\mu\nu} \to \delta_{ij}$, we recover SVT3 of h_{ij}

- de Sitter geometry

$$ds^2 = (g^{(0)}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \qquad h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{62}$$

- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$, with $\delta T_{\mu\nu} = -3H^2 h_{\mu\nu}$

$$\Delta_{\mu\nu} = (\nabla_\alpha\nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu + 3H^2 g_{\mu\nu})\chi = 0 \tag{63}$$

- With trace and commutation relation

$$\nabla_\alpha\nabla^\alpha\nabla_\mu\nabla_\nu\chi = \nabla_\mu\nabla_\nu\nabla_\alpha\nabla^\alpha\chi - 2H^2 g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + 8H^2\nabla_\mu\nabla_\nu\chi, \tag{64}$$

we decouple $F_{\mu\nu}$ by applying derivatives

$$6(\nabla_\alpha\nabla^\alpha + 4H^2)\chi = 0, \qquad (\nabla_\alpha\nabla^\alpha - 4H^2)(\nabla_\alpha\nabla^\alpha - 2H^2)F_{\mu\nu} = 0. \tag{65}$$

- SVT3

$$\begin{aligned}
\Delta_{00} &= -\frac{6}{\tau^2}(\dot{\beta}-\alpha) - \frac{2}{\tau}\tilde{\nabla}^2\beta = 0, \\
\Delta_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) - \frac{2}{\tau}\tilde{\nabla}_i(\dot{\beta}-\alpha) = 0, \\
\Delta_{ij} &= \frac{\delta_{ij}}{\tau^2}\left[-2\tau(\ddot{\beta}-\dot{\alpha}) + 6(\dot{\beta}-\alpha) + \tau\tilde{\nabla}^2(2\beta - \tau\alpha)\right] + \frac{1}{\tau}\tilde{\nabla}_i\tilde{\nabla}_j(-2\beta+\tau\alpha) \\
&+ \frac{1}{2\tau}\tilde{\nabla}_i[-2(B_j - \dot{E}_j) + \tau(\dot{B}_j - \ddot{E}_j)] + \frac{1}{2\tau}\tilde{\nabla}_j[-2(B_i - \dot{E}_i) + \tau(\dot{B}_i - \ddot{E}_i)] \\
&- \ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} = 0, \\
g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot{\beta}-\dot{\alpha}) + 24(\dot{\beta}-\alpha) + 6\tau\tilde{\nabla}^2\beta - 2\tau^2\tilde{\nabla}^2\alpha] = 0 \qquad (66)
\end{aligned}$$

- SVT4

$$\Delta_{\mu\nu} = (\nabla_\alpha\nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu + 3H^2 g_{\mu\nu})\chi = 0 \qquad (67)$$

- Weyl Action, invariant under $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$

$$I_{\mathrm{W}} = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \tag{68}$$

- Weyl Tensor

$$\begin{aligned} C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} - \frac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) + \frac{1}{6}R^{\alpha}{}_{\alpha}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right) \\ \text{under } g_{\mu\nu} &\to \Omega^2(x) g_{\mu\nu}, \quad C^{\lambda}{}_{\mu\nu\kappa} \to C^{\lambda}{}_{\mu\nu\kappa} \end{aligned} \tag{69}$$

- Bach Tensor (Einstein Analog)

$$-\frac{2}{(-g)^{1/2}}\frac{\delta I_{\mathrm{W}}}{\delta g_{\mu\nu}} = 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2\nabla_\kappa \nabla_\lambda C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} C^{\mu\lambda\nu\kappa}\right]$$

$$\boxed{4\alpha_g \left[W^{\mu\nu}_{(2)} - \frac{1}{3} W^{\mu\nu}_{(1)}\right] = T^{\mu\nu}} \quad [1] \tag{70}$$

$$\begin{aligned} W^{\mu\nu}_{(1)} &= 2g^{\mu\nu}\nabla_\beta\nabla^\beta R^\alpha{}_\alpha - 2\nabla^\nu\nabla^\mu R^\alpha{}_\alpha - 2R^\alpha{}_\alpha R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(R^\alpha{}_\alpha)^2, \\ W^{\mu\nu}_{(2)} &= \frac{1}{2}g^{\mu\nu}\nabla_\beta\nabla^\beta R^\alpha{}_\alpha + \nabla_\beta\nabla^\beta R^{\mu\nu} - \nabla_\beta\nabla^\nu R^{\mu\beta} - \nabla_\beta\nabla^\mu R^{\nu\beta} \\ &\quad -2R^{\mu\beta}R^\nu{}_\beta + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} \end{aligned} \tag{71}$$

[1] Compare to $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}$

- $W_{\mu\nu}$ obeys

$$g^{\mu\nu}W_{\mu\nu} = 0, \qquad \nabla^{\mu}W_{\mu\nu} = 0 \tag{72}$$

- Under conformal transformation $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$,

$$C^{\lambda}{}_{\mu\nu\kappa} \to C^{\lambda}{}_{\mu\nu\kappa}, \qquad W_{\mu\nu}(x) \to \Omega^{-2}(x)W_{\mu\nu}(x) \tag{73}$$

- Perturbed Bach tensor

$$W_{\mu\nu}(g_{\mu\nu}) = W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}(h_{\mu\nu}) \tag{74}$$

- Field Equations

$$W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) = T^{(0)}_{\mu\nu}, \qquad \delta W_{\mu\nu}(h_{\mu\nu}) = \delta T_{\mu\nu}(h_{\mu\nu}) \tag{75}$$

- Introduce $K_{\mu\nu}$

$$K_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{4} g^{(0)}_{\mu\nu}(x) g^{\alpha\beta}_{(0)} h_{\alpha\beta}, \qquad h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4} h g_{\mu\nu} \tag{76}$$

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}\left(K_{\mu\nu} + \frac{h}{4} g^{(0)}_{\mu\nu}\right) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4} g^{(0)}_{\mu\nu}\right) \tag{77}$$

- From properties of conformal covariance we find

$$\delta W_{\mu\nu}\left(\frac{h}{4} g^{(0)}_{\mu\nu}\right) = -\frac{h}{4} W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}), \qquad g^{\mu\nu}_{(0)} \delta W_{\mu\nu}(h_{\mu\nu}) = h^{\mu\nu} W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) \tag{78}$$

- For conformal to flat backgrounds, $W^{(0)}_{\mu\nu} = 0$

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}), \qquad g^{\mu\nu}_{(0)} \delta W_{\mu\nu}(h_{\mu\nu}) = 0 \tag{79}$$

- General $\delta W_{\mu\nu}(K_{\mu\nu})$

$$\delta W_{\mu\nu}(K_{\mu\nu}) = \frac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}K_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta - \frac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \frac{1}{2}K_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta$$
$$+\frac{1}{3}g_{\mu\nu}K^{\alpha\beta}R_{\alpha\beta}R + \frac{1}{3}K_\nu{}^\alpha R_{\mu\alpha}R + \frac{1}{3}K_\mu{}^\alpha R_{\nu\alpha}R - \frac{1}{6}K_{\mu\nu}R^2 - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\kappa}R_{\alpha\gamma\beta\kappa} - \frac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta}$$
$$-K_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\kappa}R_\mu{}^\gamma{}_\nu{}^\kappa - K_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \frac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu}$$
$$-\frac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \frac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\beta K_\mu{}^\beta + \frac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\beta K_\nu{}^\beta - \frac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \frac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta$$
$$-\nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} - \frac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \frac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu}$$
$$+\frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R + \frac{1}{2}K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} + \frac{1}{2}K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha K_\nu{}^\alpha$$
$$-\frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta}$$
$$-2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta}$$
$$+\frac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta} + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta}$$
$$+\frac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} - \frac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha - \frac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta + R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta}$$
$$+\frac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \frac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \frac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \frac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - \frac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta$$
$$+R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha} - \frac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} + \frac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta} \tag{80}$$

$$\delta W_{\mu\nu}(h) = -\frac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \frac{1}{6}R_{\mu\nu}Rh + \frac{1}{24}g_{\mu\nu}R^2h + \frac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h - \frac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} + \frac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R$$
$$+\frac{1}{12}h\nabla_\nu\nabla_\mu R + \frac{1}{4}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h - \frac{1}{4}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h - \frac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h + \frac{1}{4}\nabla_\mu R_{\nu\alpha}\nabla^\alpha h - \frac{1}{4}\nabla_\alpha R_\nu{}^\alpha\nabla_\mu h$$
$$+\frac{1}{4}R_\nu{}^\alpha\nabla_\mu\nabla_\alpha h + \frac{1}{4}\nabla_\nu R_{\mu\alpha}\nabla^\alpha h + \frac{1}{8}\nabla_\nu R\nabla_\mu h - \frac{1}{4}\nabla_\alpha R_\mu{}^\alpha\nabla_\nu h + \frac{1}{8}\nabla_\mu R\nabla_\nu h + \frac{1}{4}R_\mu{}^\alpha\nabla_\nu\nabla_\alpha h$$
$$-\frac{1}{4}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \tag{81}$$

- Evaluate (80) in a conformal to Minkowski background $ds^2 = (\Omega^2(x)\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$

$$
\begin{aligned}
\delta W_{\mu\nu} = {} & \Omega^{-5}\partial_\alpha\partial_\nu\partial^\alpha\Omega\partial_\beta K_\mu{}^\beta + \Omega^{-5}\partial_\alpha\partial_\mu\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta + 2\Omega^{-5}\partial^\alpha\partial_\nu\Omega\partial_\beta\partial_\alpha K_\mu{}^\beta \\
& + 2\Omega^{-5}\partial^\alpha\partial_\mu\Omega\partial_\beta\partial_\alpha K_\nu{}^\beta + 2\Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial_\alpha\partial_\mu K_\nu{}^\beta + 2\Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial_\alpha\partial_\nu K_\mu{}^\beta \\
& + \tfrac{1}{3}\Omega^{-4}\partial_\beta\partial_\alpha\partial_\nu\partial_\mu K^{\alpha\beta} - \tfrac{2}{3}K^{\alpha\beta}\Omega^{-5}\partial_\beta\partial_\alpha\partial_\nu\partial_\mu\Omega + \Omega^{-5}\partial^\alpha\partial_\nu\Omega\partial_\beta\partial^\beta K_{\mu\alpha} \\
& - 2\Omega^{-5}\partial_\alpha\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\mu\nu} + 6\Omega^{-6}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\mu\nu} + \Omega^{-5}\partial^\alpha\partial_\mu\Omega\partial_\beta\partial^\beta K_{\nu\alpha} \\
& + 3K_{\mu\nu}\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega + 12\Omega^{-6}\partial_\alpha K_{\mu\nu}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega - 24K_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega \\
& - 4\Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial^\beta\partial_\alpha K_{\mu\nu} + 12K_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\partial_\alpha\Omega + \tfrac{1}{2}\Omega^{-4}\partial_\beta\partial^\beta\partial_\alpha\partial^\alpha K_{\mu\nu} \\
& - K_{\mu\nu}\Omega^{-5}\partial_\beta\partial^\beta\partial_\alpha\partial^\alpha\Omega - \tfrac{1}{2}\Omega^{-4}\partial_\beta\partial^\beta\partial_\alpha\partial_\mu K_\nu{}^\alpha - \tfrac{1}{2}\Omega^{-4}\partial_\beta\partial^\beta\partial_\alpha\partial_\nu K_\mu{}^\alpha \\
& - 4\Omega^{-5}\partial_\alpha K_{\mu\nu}\partial_\beta\partial^\beta\partial^\alpha\Omega + \Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial^\beta\partial_\mu K_{\nu\alpha} + \Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial^\beta\partial_\nu K_{\mu\alpha} \\
& - \tfrac{4}{3}\Omega^{-5}\partial^\alpha\partial_\nu\Omega\partial_\beta\partial_\mu K_\alpha{}^\beta + \Omega^{-5}\partial_\alpha\partial^\alpha\Omega\partial_\beta\partial_\mu K_\nu{}^\beta - 3\Omega^{-6}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta\partial_\mu K_\nu{}^\beta \\
& - 6K_\nu{}^\beta\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\mu\partial_\alpha\Omega - 3K_{\nu\alpha}\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\mu\partial^\beta\Omega - \tfrac{4}{3}\Omega^{-5}\partial^\alpha\partial_\mu\Omega\partial_\beta\partial_\nu K_\alpha{}^\beta \\
& + \Omega^{-5}\partial_\alpha\partial^\alpha\Omega\partial_\beta\partial_\nu K_\mu{}^\beta - 3\Omega^{-6}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta\partial_\nu K_\mu{}^\beta - 6K_\mu{}^\beta\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu\partial_\alpha\Omega \\
& - 3K_{\mu\alpha}\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu\partial^\beta\Omega - \tfrac{4}{3}\Omega^{-5}\partial^\alpha\Omega\partial_\beta\partial_\nu\partial_\mu K_\alpha{}^\beta - \tfrac{4}{3}\Omega^{-5}\partial_\alpha K^{\alpha\beta}\partial_\beta\partial_\nu\partial_\mu\Omega \\
& + 4K_\alpha{}^\beta\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu\partial_\mu\Omega - 48\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta K_{\mu\nu}\partial^\beta\Omega + 60K_{\mu\nu}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta\Omega\partial^\beta\Omega \\
& + 12\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\alpha K_{\mu\nu}\partial^\beta\Omega - 48K_{\mu\nu}\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial_\alpha\Omega\partial^\beta\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\mu K_{\nu\alpha}\partial^\beta\Omega \\
& - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu K_{\mu\alpha}\partial^\beta\Omega + 24\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\mu\nu}\partial^\beta\partial_\alpha\Omega + K_{\nu\beta}\Omega^{-5}\partial^\beta\partial_\alpha\partial_\mu\partial^\alpha\Omega \\
& + K_{\mu\beta}\Omega^{-5}\partial^\beta\partial_\alpha\partial_\nu\partial^\alpha\Omega + 2\Omega^{-5}\partial_\alpha\partial_\mu K_{\nu\beta}\partial^\beta\partial^\alpha\Omega + 2\Omega^{-5}\partial_\alpha\partial_\nu K_{\mu\beta}\partial^\beta\partial^\alpha\Omega
\end{aligned}
$$

$$
\begin{aligned}
&-4\Omega^{-5}\partial_\beta\partial_\alpha K_{\mu\nu}\partial^\beta\partial^\alpha\Omega+6K_{\mu\nu}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial^\beta\partial^\alpha\Omega-6\Omega^{-6}\partial_\alpha K_{\nu\beta}\partial^\alpha\Omega\partial^\beta\partial_\mu\Omega\\
&+2\Omega^{-5}\partial_\alpha K_{\nu\beta}\partial^\beta\partial_\mu\partial^\alpha\Omega-6\Omega^{-6}\partial_\alpha K_{\mu\beta}\partial^\alpha\Omega\partial^\beta\partial_\nu\Omega+2\Omega^{-5}\partial_\alpha K_{\mu\beta}\partial^\beta\partial_\nu\partial^\alpha\Omega\\
&+2\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\gamma K_\alpha{}^\gamma-8\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma K_\beta{}^\gamma+4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\gamma K_\beta{}^\gamma\\
&-\tfrac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\gamma K_\beta{}^\gamma+2\eta_{\mu\nu}K_\beta{}^\gamma\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial_\alpha\Omega+4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial_\beta K_\alpha{}^\gamma\\
&-\tfrac{4}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial_\beta K_\alpha{}^\gamma-\tfrac{1}{3}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\partial^\alpha\Omega\partial_\gamma\partial_\beta K^{\beta\gamma}+\eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\gamma\partial_\beta K^{\beta\gamma}\\
&+\eta_{\mu\nu}K^{\beta\gamma}\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\gamma\partial_\beta\Omega-4\eta_{\mu\nu}K^{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\gamma\partial_\beta\Omega-16\eta_{\mu\nu}K_\alpha{}^\gamma\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial_\beta\Omega\\
&-\tfrac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\alpha\Omega\partial_\gamma\partial_\beta\partial_\alpha K^{\beta\gamma}+2\eta_{\mu\nu}K^{\beta\gamma}\Omega^{-6}\partial^\alpha\Omega\partial_\gamma\partial_\beta\partial_\alpha\Omega+\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial^\gamma K_{\alpha\beta}\\
&-\tfrac{1}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial^\gamma K_{\alpha\beta}-4\eta_{\mu\nu}K_{\alpha\beta}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial^\gamma\Omega-\tfrac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\alpha\Omega\partial_\gamma\partial^\gamma\partial_\beta K_\alpha{}^\beta\\
&+\tfrac{1}{6}\eta_{\mu\nu}\Omega^{-4}\partial_\gamma\partial^\gamma\partial_\beta\partial_\alpha K^{\alpha\beta}+20\eta_{\mu\nu}K_{\beta\gamma}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial^\gamma\Omega-8\eta_{\mu\nu}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma K_{\alpha\beta}\partial^\gamma\Omega\\
&+2\eta_{\mu\nu}K_{\alpha\gamma}\Omega^{-6}\partial^\alpha\Omega\partial^\gamma\partial_\beta\partial^\beta\Omega+2\eta_{\mu\nu}\Omega^{-6}\partial_\alpha K_{\beta\gamma}\partial^\alpha\Omega\partial^\gamma\partial^\beta\Omega+4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\alpha\gamma}\partial^\gamma\partial^\beta\Omega\\
&-\tfrac{1}{3}\eta_{\mu\nu}K_{\beta\gamma}\Omega^{-5}\partial^\gamma\partial^\beta\partial_\alpha\partial^\alpha\Omega-\tfrac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha K_{\beta\gamma}\partial^\gamma\partial^\beta\partial^\alpha\Omega+4\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\nu\Omega\partial_\mu K_{\alpha\beta}\\
&-\tfrac{2}{3}\Omega^{-5}\partial_\beta\partial_\nu\partial_\alpha\Omega\partial_\mu K^{\alpha\beta}-3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\mu K_{\nu\alpha}+12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu K_{\nu\beta}\\
&-6\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\mu K_{\nu\beta}+\Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\mu K_{\nu\beta}+4\Omega^{-6}\partial^\alpha\partial_\nu\Omega\partial_\beta K_\alpha{}^\beta\partial_\mu\Omega\\
&-3\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta\partial_\mu\Omega+12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta\partial_\mu\Omega-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\alpha K_\nu{}^\beta\partial_\mu\Omega\\
&+24K_\nu{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial_\alpha\Omega\partial_\mu\Omega-\tfrac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha\partial_\nu K^{\alpha\beta}\partial_\mu\Omega-3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\nu\alpha}\partial_\mu\Omega\\
&+12K_{\nu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\mu\Omega+\Omega^{-5}\partial_\beta\partial^\beta\partial_\alpha K_\nu{}^\alpha\partial_\mu\Omega+4\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu K_\alpha{}^\beta\partial_\mu\Omega
\end{aligned}
$$

$$
\begin{aligned}
&+2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\nu\partial_\alpha\Omega\partial_\mu\Omega-60K_{\nu\beta}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega+24\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_{\nu\alpha}\partial^\beta\Omega\partial_\mu\Omega\\
&-3K_{\nu\beta}\Omega^{-6}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\mu\Omega-6\Omega^{-6}\partial_\alpha K_{\nu\beta}\partial^\beta\partial^\alpha\Omega\partial_\mu\Omega-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta\partial_\mu\partial_\alpha\Omega\\
&-6K_{\nu\beta}\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu\partial_\alpha\Omega-3K_\nu{}^\beta\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega+12K_\nu{}^\beta\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega\\
&+24K_{\nu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\partial_\beta\Omega-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\nu\alpha}\partial_\mu\partial^\beta\Omega+4\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\mu\Omega\partial_\nu K_{\alpha\beta}\\
&-8\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega\partial_\nu K_{\alpha\beta}+2\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu\Omega\partial_\nu K_{\alpha\beta}-\tfrac{2}{3}\Omega^{-5}\partial_\beta\partial_\mu\partial_\alpha\Omega\partial_\nu K^{\alpha\beta}\\
&-3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\nu K_{\mu\alpha}+12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu K_{\mu\beta}-6\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\nu K_{\mu\beta}\\
&+\Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\nu K_{\mu\beta}+4\Omega^{-6}\partial^\alpha\partial_\mu\Omega\partial_\beta K_\alpha{}^\beta\partial_\nu\Omega-3\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\beta K_\mu{}^\beta\partial_\nu\Omega\\
&+12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta K_\mu{}^\beta\partial_\nu\Omega-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\alpha K_\mu{}^\beta\partial_\nu\Omega+24K_\mu{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial_\alpha\Omega\partial_\nu\Omega\\
&-\tfrac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha\partial_\mu K^{\alpha\beta}\partial_\nu\Omega-3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\mu\alpha}\partial_\nu\Omega+12K_{\mu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\nu\Omega\\
&+\Omega^{-5}\partial_\beta\partial^\beta\partial_\alpha K_\mu{}^\alpha\partial_\nu\Omega+4\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\mu K_\alpha{}^\beta\partial_\nu\Omega+2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\mu\partial_\alpha\Omega\partial_\nu\Omega\\
&-60K_{\mu\beta}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\Omega+24\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_{\mu\alpha}\partial^\beta\Omega\partial_\nu\Omega-3K_{\mu\beta}\Omega^{-6}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\nu\Omega\\
&-6\Omega^{-6}\partial_\alpha K_{\mu\beta}\partial^\beta\partial^\alpha\Omega\partial_\nu\Omega-8\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu K_{\alpha\beta}\partial_\nu\Omega+2\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu K_{\alpha\beta}\partial_\nu\Omega\\
&-16\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_\alpha{}^\beta\partial_\mu\Omega\partial_\nu\Omega+2\Omega^{-6}\partial_\beta\partial_\alpha K^{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega-8K^{\alpha\beta}\Omega^{-7}\partial_\beta\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu\Omega\\
&+40K_{\alpha\beta}\Omega^{-8}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega\partial_\nu\Omega-16K_\alpha{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega\partial_\nu\Omega-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_\mu{}^\beta\partial_\nu\partial_\alpha\Omega\\
&-6K_{\mu\beta}\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\nu\partial_\alpha\Omega-3K_\mu{}^\beta\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\nu\partial_\beta\Omega+12K_\mu{}^\beta\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\nu\partial_\beta\Omega\\
&+24K_{\mu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\beta\Omega-16K_\alpha{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\mu\Omega\partial_\nu\partial_\beta\Omega+4K^{\alpha\beta}\Omega^{-6}\partial_\mu\partial_\alpha\Omega\partial_\nu\partial_\beta\Omega
\end{aligned}
$$

$$
\begin{aligned}
&-6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\mu\alpha}\partial_\nu\partial^\beta\Omega+2\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\mu K_{\alpha\beta}-\tfrac{2}{3}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\nu\partial_\mu K_{\alpha\beta}\\
&+4\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_\alpha{}^\beta\partial_\nu\partial_\mu\Omega-\tfrac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha K^{\alpha\beta}\partial_\nu\partial_\mu\Omega+2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\nu\partial_\mu\Omega\\
&-8K_{\alpha\beta}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\mu\Omega
\end{aligned}
\tag{82}
$$

- (82) can be expressed as

$$
\begin{aligned}
\delta W_{\mu\nu} &= \frac{1}{2}\Omega^{-2}\Big(\partial_\sigma\partial^\sigma\partial_\tau\partial^\tau[\Omega^{-2}K_{\mu\nu}]-\partial_\sigma\partial^\sigma\partial_\mu\partial^\alpha[\Omega^{-2}K_{\alpha\nu}]-\partial_\sigma\partial^\sigma\partial_\nu\partial^\alpha[\Omega^{-2}K_{\alpha\mu}]\\
&+ \frac{2}{3}\partial_\mu\partial_\nu\partial^\alpha\partial^\beta[\Omega^{-2}K_{\alpha\beta}]+\frac{1}{3}\eta_{\mu\nu}\partial_\sigma\partial^\sigma\partial^\alpha\partial^\beta[\Omega^{-2}K_{\alpha\beta}]\Big).
\end{aligned}
\tag{83}
$$

- Consider $h_{\mu\nu}^{TT}$ in terms of $K_{\mu\nu}$

$$
\begin{aligned}
h_{\mu\nu}^{TT} &= K_{\mu\nu}-\int d^4x' D^{(4)}(x-x')\partial_\mu\partial^\alpha K_{\alpha\nu}-\int d^4x' D^{(4)}(x-x')\partial_\nu\partial^\alpha K_{\alpha\mu} \qquad (84)\\
&+ \frac{2}{3}\partial_\mu\partial_\nu\int d^4x' D^{(4)}(x-x')\int d^4x'' D^{(4)}(x'-x'')\partial^\alpha\partial^\beta K_{\alpha\beta}+\frac{1}{3}g_{\mu\nu}\int d^4x D^{(4)}(x-x')\partial^\alpha\partial^\beta K_{\alpha\beta}
\end{aligned}
$$

- (83) can thus be expressed as

$$
\boxed{\delta W_{\mu\nu}=\frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta[\Omega^{-2}h_{\mu\nu}]^{TT}}
\tag{85}
$$

SVT4 $\delta W_{\mu\nu}$ in Conformal to Flat Backgrounds

- SVT4 Decomposition

$$h_{\mu\nu} = \Omega^2(x)\left[-2g_{\mu\nu}\chi + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu F + \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu}\right] \tag{86}$$

- Fluctuation Equations $\delta W_{\mu\nu} = 0$

$$\boxed{\delta W_{\mu\nu} = \Omega^{-2}(x)\tilde{\nabla}_\sigma\tilde{\nabla}^\sigma\tilde{\nabla}_\tau\tilde{\nabla}^\tau F_{\mu\nu} = 0} \tag{87}$$

- Robertson Walker $k = -1$ comoving geometry

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \tag{88}$$

- Robertson Walker $k = -1$ conformal flat geometry

$$ds^2 = \frac{4a^2(p', r')}{[1-(p'+r')^2][1-(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2] \tag{89}$$

- Conformal Gravity field equation (in basis of (89))

$$\delta W_{\mu\nu} = \frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta[\Omega^{-2}h_{\mu\nu}]^{TT} \tag{90}$$

- Plane Wave Solutions

$$\Omega^{-2}h'^{TT}_{\mu\nu} = A'_{\mu\nu}e^{ik'\cdot x'} + (n'\cdot x')B'_{\mu\nu}e^{ik'\cdot x'} + A'^{*}_{\mu\nu}e^{-ik'\cdot x'} + (n'\cdot x')B'^{*}_{\mu\nu}e^{-ik'\cdot x'}, \tag{91}$$

where $k'_\mu k'_\nu \eta^{\mu\nu} = 0$ and $n'\cdot x' = p'$

- Following coordinate transformations,

$$h^{TT}_{\mu\nu} \sim \Omega^2(p', r')p' \sim t^4 \tag{92}$$

- SVT3 Decomposition
  - Non-local integrals, incorporates asymptotic boundary conditions
  - Decouple gauge-invariant fluctuations with higher derivatives
- SVTD Decomposition
  - Generalize to dimension D
  - Generalize to curved backgrounds (maximally symmetric)
  - Einstein de Sitter $D = 4$

$$\Delta_{\mu\nu} = (\nabla_\alpha \nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu + 3H^2 g_{\mu\nu})\chi$$

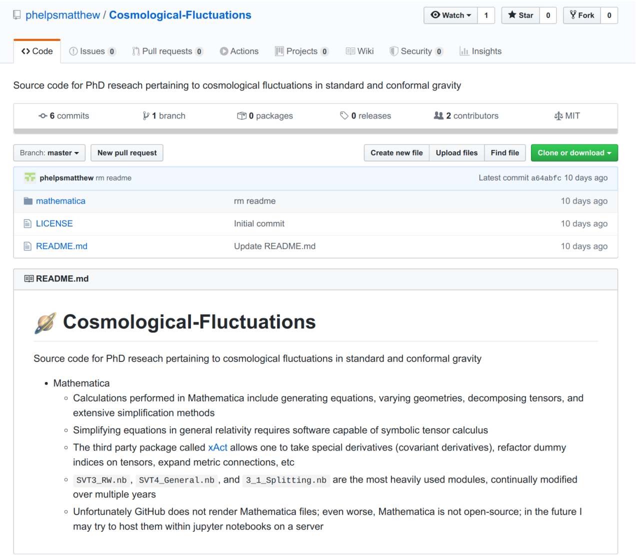
- Conformal Gravity
  - Fluctuation equation in conformal to flat backgrounds

$$\delta W_{\mu\nu} = \frac{1}{2}\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta[\Omega^{-2}h_{\mu\nu}]^{TT} = 0$$

  - Roberston Walker $k = -1$ Radiation era

$$h^{TT}_{\mu\nu} \sim t^4$$

- Mathematica + xAct
- https://github.com/phelpsmatthew/Cosmological-Fluctuations
- (Notebook demonstration)



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Acknowledgments

@bmunst https://link.medium.com/XQIAClsqJ6

The End

$$V(x) = \int_V d^3x' D(x-x')\partial_i\partial^i V(x') + \oint_{\partial V} dS_i[V(x')\partial^i D(x-x') - D(x-x')\partial^i V(x')] \quad (93)$$

$$\partial_k\partial^k V(x) = \partial_k\partial^k V(x) + \oint_{\partial V} dS_i[V(x')\partial^i \delta^3(x-x') - \delta^3(x-x')\partial^i V(x')] \quad (94)$$

Using

$$\int d^3x' V(x')\partial^i \delta^3(x-x') = \int d^3x' \partial^i V(x')\delta^3(x-x'), \quad (95)$$

the surface term vanishes identically.

$$ds^2 = \Omega^2(x)\Big[-(1+2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i + [(1-2\psi)\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\Big] \quad (96)$$

- Fluctuation Equations $\delta W_{\mu\nu} = 0$, $\alpha = \phi + \psi + \dot{B} - \ddot{E}$

$$\begin{aligned}
\delta W_{00} &= -\frac{2}{3\Omega^2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha, \\
\delta W_{0i} &= -\frac{2}{3\Omega^2}\tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}_a\partial_0\alpha + \frac{1}{2\Omega^2}\left[\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{\nabla}_a\tilde{\nabla}^a - \partial_0^2)(B_i - \dot{E}_i)\right], \\
\delta W_{ij} &= \frac{1}{3\Omega^2}\Big[\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}^b(\partial_0^2 - \tilde{\nabla}_a\tilde{\nabla}^a) + (\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_0^2)\tilde{\nabla}_i\tilde{\nabla}_j\Big]\alpha \\
&\quad + \frac{1}{2\Omega^2}\left[\left[\tilde{\nabla}_a\tilde{\nabla}^a - \partial_0^2\right]\left[\tilde{\nabla}_i\partial_0(B_j - \dot{E}_j) + \tilde{\nabla}_j\partial_0(B_i - \dot{E}_i)\right]\right] + \frac{1}{\Omega^2}\left[\tilde{\nabla}_a\tilde{\nabla}^a - \partial_0^2\right]^2 E_{ij}.
\end{aligned} \quad (97)$$

- Decouple by applying higher derivatives to vector and tensor components

$$\begin{gathered}
-\tfrac{2}{3}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha = 0, \qquad \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b\left[\tilde{\nabla}_c\tilde{\nabla}^c - \partial_0^2\right]^2 E_{ij} = 0, \\
-\tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a(\ddot{B}_i - \dddot{E}_i) + \tfrac{1}{2}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a(B_i - \dot{E}_i) = 0
\end{gathered} \quad (98)$$

$$
\begin{aligned}
ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
\delta W_{\mu\nu} &= \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\
&\quad -\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
\delta G_{\mu\nu} &= \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\mu h \\
\delta G &= \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h \\
\delta G^{T\theta}_{\mu\nu} &= \delta G_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\delta G + \frac{1}{3}\nabla_\mu\nabla_\nu\int D\delta G \\
\nabla^2\delta G^{T\theta}_{\mu\nu} &= \nabla^2\delta G_{\mu\nu} + \frac{1}{3}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2\right]\delta G \\
\nabla^2\delta G^{T\theta}_{\mu\nu} &= \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\
&\quad -\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&= \delta W_{\mu\nu} \qquad (99)
\end{aligned}
$$

- $k = 1$ (spherical)

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{100}$$

- Set $\sin\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\left[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\right] \tag{101}$$

- Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p+\chi)/2], \quad p' - r' = \tan[(p-\chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos\chi}, \quad r' = \frac{\sin\chi}{\cos p + \cos\chi} \end{aligned} \tag{102}$$

$$\Longrightarrow \boxed{ds^2 = \frac{4a^2(p)}{[1+(p'+r')^2][1+(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]} \tag{103}$$

- $k = -1$ (hyperbolic)

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \tag{104}$$

- Set $\sinh\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2 \right] \tag{105}$$

- Introduce coordinates

$$\begin{aligned} p' + r' &= \tanh[(p+\chi)/2], \quad p' - r' = \tanh[(p-\chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh\chi}, \quad r' = \frac{\sinh\chi}{\cosh p + \cosh\chi} \end{aligned} \tag{106}$$

$$\Longrightarrow \boxed{ds^2 = \frac{4a^2(p)}{[1-(p'+r')^2][1-(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2]} \tag{107}$$