# Coordinate Transformations RW $k < 0$ v8

## Summary

Since planes waves in the $(p', r')$ coordinate system behave much differently than those in the $(T, R)$ coordinate system, their asymptotic behavior may vary. After accounting for the leading $u$ behavior in the expansion of $e^{ik(R\cos\theta - T)}$ in the null and timelike configurations, all results are found to agree between both coordinate systems. However, there still seems to remain a mismatch with PRD 2012. This is summarized in the Plane Wave Expansion section. Null and timelike calculations are summarized below.

**Null**

$$\begin{aligned}
K_{00}^{(cm)} &\sim u^2 \\
K_{01}^{(cm)} &\sim u^2 \\
K_{02}^{(cm)} &\sim u^3 \\
K_{03}^{(cm)} &\sim u^3 \\
K_{11}^{(cm)} &\sim u^2 \\
K_{22}^{(cm)} &\sim u^4 \\
K_{33}^{(cm)} &\sim u^4 \\
K_{12}^{(cm)} &\sim u^3 \\
K_{13}^{(cm)} &\sim u^3 \\
K_{23}^{(cm)} &\sim u^4
\end{aligned} \tag{1}$$

**Timelike**

$$\begin{aligned}
K_{00}^{(cm)} &\sim 1 \\
K_{01}^{(cm)} &\sim u \\
K_{02}^{(cm)} &\sim u \\
K_{03}^{(cm)} &\sim u \\
K_{11}^{(cm)} &\sim u^2 \\
K_{22}^{(cm)} &\sim u^2 \\
K_{33}^{(cm)} &\sim u^2 \\
K_{12}^{(cm)} &\sim u^2 \\
K_{13}^{(cm)} &\sim u^2 \\
K_{23}^{(cm)} &\sim u^2
\end{aligned} \tag{2}$$

## Notation/Background

For $K < 1$ FRW cosmology with $L^2 a^2 = t^2 + d^2$, the line element takes the form

$$\begin{aligned}
ds^2 &= dt^2 - a(t)^2 \left( \frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \\
&= d^2 \left[ du^2 - (1 + u^2) \left( \frac{dv^2}{1 + v^2} + v^2 d\Omega^2 \right) \right],
\end{aligned} \tag{3}$$

where we have introduced

$$u = \frac{t}{d}, \qquad v = \frac{r}{L}. \tag{4}$$

## Original Coordinates $(p', r')$

Transformations and Asymptotics:

$$p' = \frac{u}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}}, \qquad r' = \frac{v}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}} \tag{5}$$

$$u^2 = \frac{4p'^2}{(1-(p'+r')^2)(1-(p'-r')^2)}, \qquad v = \left(\frac{r'}{p'}\right) u \tag{6}$$

$$\Omega^2(p',r') = \frac{4L^2a^2}{(1-(p'+r')^2)(1-(p'-r')^2)} = d^2(1+u^2)\left[(1+u^2)^{1/2} + (1+v^2)^{1/2}\right]^2 \tag{7}$$

$$r'\cos\theta - p' = \frac{v\cos\theta - u}{\sqrt{1+u^2}+\sqrt{1+v^2}} \tag{8}$$

## Null Trajectory

In the $u$, $v$ geometry, the condition for null separation (at fixed angle) is $u = v$. Inspection of coordinate transformation (5-8) shows the leading order ($u \gg 1$) contributions for null separation:

$$p' \sim 1, \qquad r' \sim 1, \qquad \Omega^2 \sim u^4. \tag{9}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{u} \qquad \frac{\partial p'}{\partial r} \sim \frac{1}{u}, \qquad \frac{\partial r'}{\partial t} \sim \frac{1}{u} \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{u}. \tag{10}$$

$$\exp[ik(r'\cos\theta - p')] \sim 1 \tag{11}$$

The leading behavior for the full $K^{(cm)}_{\mu\nu}$ behaves as

$$\begin{aligned}
K^{(cm)}_{00} &\sim u^2\\
K^{(cm)}_{01} &\sim u^2\\
K^{(cm)}_{02} &\sim u^3\\
K^{(cm)}_{03} &\sim u^3\\
K^{(cm)}_{11} &\sim u^2\\
K^{(cm)}_{22} &\sim u^4\\
K^{(cm)}_{33} &\sim u^4\\
K^{(cm)}_{12} &\sim u^3\\
K^{(cm)}_{13} &\sim u^3\\
K^{(cm)}_{23} &\sim u^4
\end{aligned} \tag{12}$$

The purely angular sector of this result coincides with the null configuration given in PRD 2012.

## Timelike Trajectory

For coordinate separations which are timelike, $u \gg v$. In order to find the leading contribution in $u$, we take both $u \gg v$ and $u \gg 1$. These results yield a leading behavior of:

$$p' \sim 1, \qquad r' \sim \frac{1}{u}, \qquad \Omega^2 \sim u^4. \tag{13}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{u^2} \qquad \frac{\partial p'}{\partial r} \sim \frac{1}{u}, \qquad \frac{\partial r'}{\partial t} \sim \frac{1}{u^2} \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{u}. \tag{14}$$

$$\exp[ik(r'\cos\theta - p')] \sim 1 \tag{15}$$

The leading behavior for the full $K_{\mu\nu}^{(cm)}$ behaves as

$$
\begin{aligned}
&K_{00}^{(cm)} \sim 1 \\
&K_{01}^{(cm)} \sim u \\
&K_{02}^{(cm)} \sim u \\
&K_{03}^{(cm)} \sim u \\
&K_{11}^{(cm)} \sim u^2 \\
&K_{22}^{(cm)} \sim u^2 \\
&K_{33}^{(cm)} \sim u^2 \\
&K_{12}^{(cm)} \sim u^2 \\
&K_{13}^{(cm)} \sim u^2 \\
&K_{23}^{(cm)} \sim u^2
\end{aligned} \tag{16}
$$

# New Coordinates $(T, R)$

Transformations and Asymptotics:

$$T = \left[u + (1+u^2)^{1/2}\right](1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2}\right] v \tag{17}$$

$$\Omega^2(T, R) = \frac{L^2 a^2}{T^2 - R^2} = d^2 \frac{(1+u^2)}{(u + (1+u^2)^{1/2})^2} \tag{18}$$

$$R\cos\theta - T = (u + \sqrt{1+u^2})(v\cos\theta - \sqrt{1+v^2}) \sim 2u(v\cos\theta - \sqrt{1+v^2}) \tag{19}$$

## Null Trajectory

In the $u$, $v$ geometry, the condition for null separation (at fixed angle) is $u = v$. Inspection of coordinate transformation (17-19) shows the leading order ($u \gg 1$) contributions for null separation:

$$T \sim u^2, \qquad R \sim u^2, \qquad \Omega^2 \sim 1. \tag{20}$$

$$\frac{\partial T}{\partial t} \sim u, \qquad \frac{\partial T}{\partial r} \sim u, \qquad \frac{\partial R}{\partial t} \sim u, \qquad \frac{\partial R}{\partial r} \sim u. \tag{21}$$

$$\exp[R\cos\theta - T] \sim \frac{1}{u^2} \tag{22}$$

The leading behavior for the full $K^{(cm)}_{\mu\nu}$ behaves as

$$\begin{aligned}
&K^{(cm)}_{00} \sim u^2 \\
&K^{(cm)}_{01} \sim u^2 \\
&K^{(cm)}_{02} \sim u^3 \\
&K^{(cm)}_{03} \sim u^3 \\
&K^{(cm)}_{11} \sim u^2 \\
&K^{(cm)}_{22} \sim u^4 \\
&K^{(cm)}_{33} \sim u^4 \\
&K^{(cm)}_{12} \sim u^3 \\
&K^{(cm)}_{13} \sim u^3 \\
&K^{(cm)}_{23} \sim u^4
\end{aligned} \tag{23}$$

The purely angular sector of this result coincides with the null configuration given in PRD 2012.

### Timelike Trajectory

For coordinate separations which are timelike, $u \gg v$. In order to find the leading contribution in $u$, we take both $u \gg v$ and $u \gg 1$. These results yield a leading behavior of:

$$T \sim u, \qquad R \sim u, \qquad \Omega^2 \sim 1. \tag{24}$$

$$\frac{\partial T}{\partial t} \sim 1, \qquad \frac{\partial T}{\partial r} \sim u, \qquad \frac{\partial R}{\partial t} \sim 1, \qquad \frac{\partial R}{\partial r} \sim u. \tag{25}$$

$$\exp[R\cos\theta - T] \sim \frac{1}{u} \tag{26}$$

The leading behavior for the full $K^{(cm)}_{\mu\nu}$ behaves as

$$\begin{aligned}
&K^{(cm)}_{00} \sim 1 \\
&K^{(cm)}_{01} \sim u \\
&K^{(cm)}_{02} \sim u \\
&K^{(cm)}_{03} \sim u \\
&K^{(cm)}_{11} \sim u^2 \\
&K^{(cm)}_{22} \sim u^2 \\
&K^{(cm)}_{33} \sim u^2 \\
&K^{(cm)}_{12} \sim u^2 \\
&K^{(cm)}_{13} \sim u^2 \\
&K^{(cm)}_{23} \sim u^2
\end{aligned} \tag{27}$$

## Plane-Wave Expansions

The plane wave $e^{ik(Z-T)}$ behaves much differently from $e^{ik(z'-p')}$. In order to extract the asymptotic behavior as $u \to \infty$, we utilize the plane wave expansion in terms of Spherical Bessels, viz (30).

## Spherical Bessels

$$e^{ikr\cos\theta} = \sum_{n=0}^{\infty}(2n+1)i^n j_n(kr)P_n(\cos\theta) \tag{28}$$

where $j_n$ are the spherical Bessels and $P_n$ are the Legendre polynomials. The asymptotic form for $j_n(z)$ is (NIST 10.52.3)

$$j_n(z) = z^{-1}\sin(z - \frac{1}{2}n\pi) + e^{\mathbb{I}(z)}\mathcal{O}(z^{-2}). \tag{29}$$

Hence for $r \to \infty$, we have

$$e^{ikz} \to \sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\left(\frac{\sin(kr - \frac{1}{2}n\pi)}{kr}\right) \sim \frac{1}{kr} \tag{30}$$

since the Legendre polynomials and sin function are bounded between $[-1, 1]$.

## Null

Here we express the phase in terms of $u$ and $v$, set $u = v$ for the null condition, and find the asymptotic behavior for $u \gg 1$:

$$r'\cos\theta - p' = \frac{1}{2}\left(\frac{u(\cos\theta - 1)}{\sqrt{1+u^2}}\right) \sim \frac{1}{2}(\cos\theta - 1) \tag{31}$$

$$\implies \exp[ik(r'\cos\theta - p')] \approx \exp[ik(\tfrac{1}{2}\cos\theta - \tfrac{1}{2})] \sim 1 \tag{32}$$

$$R\cos\theta - T = (u + \sqrt{1+u^2})(u\cos\theta - \sqrt{1+u^2}) \sim 2u^2(\cos\theta - 1) \tag{33}$$

$$\implies \exp[ik(R\cos\theta - T)] \approx \exp[-2iku^2]\exp[2iku^2\cos\theta] \tag{34}$$

$$\sim \exp[-2iku^2]\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\left(\frac{\sin(2ku^2 - \frac{1}{2}n\pi)}{2ku^2}\right) \tag{35}$$

$$\sim \frac{1}{u^2} \tag{36}$$

## Timelike

Here we express the phase in terms of $u$ and $v$, hold $v$ finite, and take $u \gg 1$ and $u \gg v$:

$$r'\cos\theta - p' = \frac{v\cos\theta - u}{\sqrt{1+u^2} + \sqrt{1+v^2}} \sim \frac{v\cos\theta - u}{u} \sim -1 \tag{37}$$

$$\implies \exp[ik(r'\cos\theta - p')] \approx \exp[-ik] \sim 1 \tag{38}$$

$$R\cos\theta - T = (u + \sqrt{1+u^2})(v\cos\theta - \sqrt{1+v^2}) \sim 2u(v\cos\theta - \sqrt{1+v^2}) \tag{39}$$

$$\implies \exp[R\cos\theta - T] \approx \exp[-2iku\sqrt{1+v^2}]\exp[2ikuv\cos\theta] \tag{40}$$

$$\sim \exp[-2iku\sqrt{1+v^2}]\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\left(\frac{\sin(2kuv - \frac{1}{2}n\pi)}{2kuv}\right) \tag{41}$$

$$\sim \frac{1}{u} \tag{42}$$

## PRD 2012

**Null**:
$u = v$ implies $p' = r'$ and thus

$$\Omega^2 p' r' e^{ik(r'-p')} = \Omega^2 p' r' \sim u^4 \tag{43}$$

**Timelike**:
In the timelike configuration, $r' \sim \frac{1}{u}$, and so large $u$ implies small $r'$. Thus we have

$$\Omega^2 p' r' e^{ikr'} e^{-ikp'} \sim u^4 e^{-ikp'} r'^2 \left( \frac{e^{ikr'}}{r'} \right). \tag{44}$$

We note that as $r' \to 0$,

$$\frac{e^{ikr'}}{r'} \sim \frac{1}{r'}, \tag{45}$$

and thus it would seem

$$\Omega^2 p' r' e^{ik(r'-p')} \approx \Omega^2 p' r' \sim u^3. \tag{46}$$

Asymptotically, $e^{ikr'\cos\theta}$ behaves as $\sin(kr')/r'$. However, since $r'$ is not small, we cannot use the asymptotic form of the spherical Bessels, so it not clear for me how to arrive at $\sim u^2$ in PRD.