# Special Gauge Trace Matthew v7

## Summary

Metric decomposed to first order:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}). \tag{1}$$

We then split $h_{\mu\nu}$ into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \tag{2}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. We impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega. \tag{3}$$

For arbitrary $\Omega(\tau)$, we calculate for $J = -3$, $P = \frac{1}{4}$, and $R = -\frac{1}{2}$:

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-10\Omega^{-4}\dot{\Omega}^2 + 5\Omega^{-3}\ddot{\Omega})K_{00} + (\Omega^{-4}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-3}\ddot{\Omega} - \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{5}{4}\Omega^{-3}\dot{\Omega}\partial_0)h. \tag{4}$$

$$\begin{aligned}\delta G_{00} = {} & (3\Omega^{-2}\dot{\Omega}^2 - \tfrac{3}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{3}{8}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{5}$$

$$\delta G_{0i} = -\tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_i K_{00} + (\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}\Omega^{-1}\dot{\Omega}\partial_0)K_{0i}. \tag{6}$$

$$\begin{aligned}\delta G_{ij} = {} & \delta_{ij}(-\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-1}\ddot{\Omega})K_{00} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_j K_{0i} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_i K_{0j} \\ & + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{ij} + \delta_{ij}(\tfrac{1}{4}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{8}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{7}$$

For arbitrary $\Omega(\tau) = \frac{1}{H\tau}$, we calculate for $J = -3$, $P = \frac{1}{4}$, and $R = -\frac{1}{2}$:

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (2H^2 - \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{5}{4}H^2\tau\partial_0)h. \tag{8}$$

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{8}\tau^{-1}\partial_0)h. \tag{9}$$

$$\delta G_{0i} = \tfrac{1}{2}\tau^{-1}\partial_i K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{0i}. \tag{10}$$

$$\delta G_{ij} = \tfrac{1}{2}\tau^{-1}\partial_j K_{0i} + \tfrac{1}{2}\tau^{-1}\partial_i K_{0j} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{ij} + \delta_{ij}(\tfrac{1}{2}\tau^{-2} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{8}\tau^{-1}\partial_0)h. \tag{11}$$

## Special $K_{\mu\nu}$ Gauge for Trace

The perturbed Einstein tensor $\delta G_{\mu\nu}(h_{\mu\nu})$ evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{12}$$

is calculated as

$$\begin{aligned}\delta G_{\mu\nu} = {}& \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{13}$$

Now we split $h_{\mu\nu}$ into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \tag{14}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. With this substitution, (2) takes the form

$$\begin{aligned}\delta G_{\mu\nu} = {}& -2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\alpha\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta K_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \eta^{\alpha\beta}K_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha K_{\mu\nu} + 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ & - 2\eta^{\alpha\beta}K_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta K_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta K_{\alpha\gamma} \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\mu\partial_\beta K_{\nu\alpha} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} + 2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega \\ & - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\nu\partial_\beta K_{\mu\alpha} + \tfrac{3}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & - \tfrac{1}{8}\partial_\mu\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu h\partial_\nu\Omega + \tfrac{3}{8}\partial_\nu\partial_\mu h.\end{aligned} \tag{15}$$

Now we impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega. \tag{16}$$

For a strictly time dependent conformal factor $\Omega(\tau)$, we find the flucuation trace takes the form

$$\begin{aligned}\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = {}& (-4\Omega^{-4}\dot{\Omega}^2 + 5J\Omega^{-4}\dot{\Omega}^2 + J^2\Omega^{-4}\dot{\Omega}^2 + 8\Omega^{-3}\ddot{\Omega} + J\Omega^{-3}\ddot{\Omega})K_{00} + (-5R\Omega^{-4}\dot{\Omega}^2 \\ & - JR\Omega^{-4}\dot{\Omega}^2 - R\Omega^{-3}\ddot{\Omega} - \tfrac{3}{4}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\Omega^{-3}\dot{\Omega}\partial_0 - 6P\Omega^{-3}\dot{\Omega}\partial_0 \\ & - JP\Omega^{-3}\dot{\Omega}\partial_0 - R\Omega^{-3}\dot{\Omega}\partial_0)h.\end{aligned} \tag{17}$$

In the deSitter background, we take $\Omega(\tau) = \frac{1}{H\tau}$, in which the trace reduces to

$$\begin{aligned}\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = {}& (12H^2 + 7H^2J + H^2J^2)K_{00} + (-7H^2R - H^2JR - \tfrac{3}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{3}{2}H^2\tau\partial_0 + 6H^2P\tau\partial_0 + H^2JP\tau\partial_0 + H^2R\tau\partial_0)h.\end{aligned} \tag{18}$$

Here in deSitter we may take $J = -3$ or $J = -4$ to allow the trace of the perturbation to be proportional to $h$. The rest of $\delta G_{\mu\nu}$ in deSitter is:

$$\begin{aligned}\delta G_{00} = {}& (-\tfrac{3}{2}J\tau^{-2} - \tfrac{1}{2}J^2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tau^{-1}\partial_0 - J\tau^{-1}\partial_0)K_{00} + (\tfrac{3}{2}R\tau^{-2} + \tfrac{1}{2}JR\tau^{-2} \\ & + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{4}\tau^{-1}\partial_0 - 2P\tau^{-1}\partial_0 - \tfrac{1}{2}JP\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_0\partial_0 \\ & - P\partial_0\partial_0)h.\end{aligned} \tag{19}$$

$$\begin{aligned}\delta G_{01} = {}& (-\tau^{-1}\partial_1 - \tfrac{1}{2}J\tau^{-1}\partial_1)K_{00} + (3\tau^{-2} + \tfrac{1}{2}J\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}J\tau^{-1}\partial_0)K_{01} + (\tfrac{1}{4}\tau^{-1}\partial_1 \\ & + \tfrac{1}{2}R\tau^{-1}\partial_1 + \tfrac{1}{4}\partial_1\partial_0 - P\partial_1\partial_0)h.\end{aligned} \tag{20}$$

$$\delta G_{11} = (3\tau^{-2} + \tfrac{5}{2}J\tau^{-2} + \tfrac{1}{2}J^2\tau^{-2})K_{00} + (-2\tau^{-1}\partial_1 - J\tau^{-1}\partial_1)K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (-\tfrac{5}{2}R\tau^{-2} - \tfrac{1}{2}JR\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{4}\tau^{-1}\partial_0 + 2P\tau^{-1}\partial_0 + \tfrac{1}{2}JP\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_1\partial_1 - P\partial_1\partial_1)h. \tag{21}$$

$$\delta G_{12} = (-\tau^{-1}\partial_2 - \tfrac{1}{2}J\tau^{-1}\partial_2)K_{01} + (-\tau^{-1}\partial_1 - \tfrac{1}{2}J\tau^{-1}\partial_1)K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} + (\tfrac{1}{4}\partial_2\partial_1 - P\partial_2\partial_1)h. \tag{22}$$

Since the trace simplifies completely particularly in deSitter, we will try to simplify the equations most in deSitter, and then find their form in the general $\Omega(\tau)$. Within deSitter, for $J = -3$ and arbitrary $P$, $R$, we have

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-4H^2R - \tfrac{3}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}H^2\tau\partial_0 + 3H^2P\tau\partial_0 + H^2R\tau\partial_0)h. \tag{23}$$

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{4}\tau^{-1}\partial_0 - \tfrac{1}{2}P\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_0\partial_0 - P\partial_0\partial_0)h. \tag{24}$$

$$\delta G_{01} = \tfrac{1}{2}\tau^{-1}\partial_1 K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{01} + (\tfrac{1}{4}\tau^{-1}\partial_1 + \tfrac{1}{2}R\tau^{-1}\partial_1 + \tfrac{1}{4}\partial_1\partial_0 - P\partial_1\partial_0)h. \tag{25}$$

$$\delta G_{11} = \tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (-R\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{4}\tau^{-1}\partial_0 + \tfrac{1}{2}P\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_1\partial_1 - P\partial_1\partial_1)h. \tag{26}$$

$$\delta G_{12} = \tfrac{1}{2}\tau^{-1}\partial_2 K_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} + (\tfrac{1}{4}\partial_2\partial_1 - P\partial_2\partial_1)h. \tag{27}$$

Within deSitter, for $J = -4$ and arbitrary $P$, $R$, we have

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-3H^2R - \tfrac{3}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}H^2\tau\partial_0 + 2H^2P\tau\partial_0 + H^2R\tau\partial_0)h. \tag{28}$$

$$\delta G_{00} = (-2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-1}\partial_0)K_{00} + (-\tfrac{1}{2}R\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{4}\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_0\partial_0 - P\partial_0\partial_0)h. \tag{29}$$

$$\delta G_{01} = \tau^{-1}\partial_1 K_{00} + (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{01} + (\tfrac{1}{4}\tau^{-1}\partial_1 + \tfrac{1}{2}R\tau^{-1}\partial_1 + \tfrac{1}{4}\partial_1\partial_0 - P\partial_1\partial_0)h. \tag{30}$$

$$\delta G_{11} = \tau^{-2}K_{00} + 2\tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (-\tfrac{1}{2}R\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{4}\tau^{-1}\partial_0 + \tfrac{1}{2}R\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_1\partial_1 - P\partial_1\partial_1)h. \tag{31}$$

$$\delta G_{12} = \tau^{-1}\partial_2 K_{01} + \tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} + (\tfrac{1}{4}\partial_2\partial_1 - P\partial_2\partial_1)h. \tag{32}$$

It appears that a simplifying choice for $J = -3$ would be $P = \frac{1}{4}$ and $R = -\frac{1}{2}$. With these coefficients $\delta G_{\mu\nu}$ becomes

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (2H^2 - \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{5}{4}H^2\tau\partial_0)h. \tag{33}$$

$$\delta G_{00} = (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{00} + (\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{8}\tau^{-1}\partial_0)h. \tag{34}$$

$$\delta G_{01} = \tfrac{1}{2}\tau^{-1}\partial_1 K_{00} + (\tfrac{3}{2}\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)K_{01}. \tag{35}$$

$$\delta G_{11} = \tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{2}\tau^{-2} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{8}\tau^{-1}\partial_0)h. \tag{36}$$

$$\delta G_{12} = \tfrac{1}{2}\tau^{-1}\partial_2 K_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12}. \tag{37}$$

It appears that a simplifying choice for $J = -4$ would be again $P = \frac{1}{4}$ and $R = -\frac{1}{2}$. With these coefficients $\delta G_{\mu\nu}$ becomes

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (\tfrac{3}{2}H^2 - \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}H^2\tau\partial_0)h. \tag{38}$$

$$\delta G_{00} = (-2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-1}\partial_0)K_{00} + (\tfrac{1}{4}\tau^{-2} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}\tau^{-1}\partial_0)h. \tag{39}$$

$$\delta G_{01} = \tau^{-1}\partial_1 K_{00} + (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)K_{01}. \tag{40}$$

$$\begin{aligned}\delta G_{11} = {}& \tau^{-2}K_{00} + 2\tau^{-1}\partial_1 K_{01} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{4}\tau^{-2} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{1}{2}\tau^{-1}\partial_0)h.\end{aligned} \tag{41}$$

$$\delta G_{12} = \tau^{-1}\partial_2 K_{01} + \tau^{-1}\partial_1 K_{02} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12}. \tag{42}$$

Going back to arbitrary $\Omega(\tau)$, we calculate for $J = -3$, $P = \frac{1}{4}$, and $R = -\frac{1}{2}$:

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-10\Omega^{-4}\dot{\Omega}^2 + 5\Omega^{-3}\ddot{\Omega})K_{00} + (\Omega^{-4}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-3}\ddot{\Omega} - \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{5}{4}\Omega^{-3}\dot{\Omega}\partial_0)h. \tag{43}$$

$$\begin{aligned}\delta G_{00} = {}& (3\Omega^{-2}\dot{\Omega}^2 - \tfrac{3}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{3}{8}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{44}$$

$$\delta G_{01} = -\tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_1 K_{00} + (\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}\Omega^{-1}\dot{\Omega}\partial_0)K_{01}. \tag{45}$$

$$\begin{aligned}\delta G_{11} = {}& (-\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-1}\ddot{\Omega})K_{00} - \Omega^{-1}\dot{\Omega}\partial_1 K_{01} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \Omega^{-1}\dot{\Omega}\partial_0)K_{11} + (\tfrac{1}{4}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{8}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{46}$$

$$\delta G_{12} = -\tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_2 K_{01} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_1 K_{02} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{12}. \tag{47}$$

For arbitrary $\Omega(\tau)$, we calculate for $J = -4$, $P = \frac{1}{4}$, and $R = -\frac{1}{2}$:

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-8\Omega^{-4}\dot{\Omega}^2 + 4\Omega^{-3}\ddot{\Omega})K_{00} + (\tfrac{1}{2}\Omega^{-4}\dot{\Omega}^2 + \tfrac{1}{2}\Omega^{-3}\ddot{\Omega} - \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\Omega^{-3}\dot{\Omega}\partial_0)h. \tag{48}$$

$$\begin{aligned}\delta G_{00} = {}& (2\Omega^{-2}\dot{\Omega}^2 - 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 3\Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{1}{4}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{49}$$

$$\delta G_{01} = -\Omega^{-1}\dot{\Omega}\partial_1 K_{00} + (\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\Omega^{-1}\dot{\Omega}\partial_0)K_{01}. \tag{50}$$

$$\begin{aligned}\delta G_{11} = {}& \Omega^{-2}\dot{\Omega}^2 K_{00} - 2\Omega^{-1}\dot{\Omega}\partial_1 K_{01} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{11} \\ & + (-\tfrac{1}{4}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_0)h.\end{aligned} \tag{51}$$

$$\delta G_{12} = -\Omega^{-1}\dot{\Omega}\partial_2 K_{01} - \Omega^{-1}\dot{\Omega}\partial_1 K_{02} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{12}. \tag{52}$$