Fluctuations in $W_{\mu\nu}$ and $G_{\mu\nu}$

Matthew Phelps

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Contents

1 Weyl Tensor

1.1 Fluctuations

$$
\begin{aligned}
W_{\mu\nu} &= W^2_{\mu\nu} - \frac{1}{3}W^1_{\mu\nu} \\
&= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R \\
&\quad + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\mu\nabla^\alpha R_{\nu\alpha} - \nabla_\nu\nabla^\alpha R_{\mu\alpha} + \tfrac{2}{3}\nabla_\nu\nabla_\mu R
\end{aligned}
\tag{1.1}
$$

where

$$
W^1_{\mu\nu} = \tfrac{1}{2}g_{\mu\nu}R^2 - 2RR_{\mu\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - 2\nabla_\nu\nabla_\mu R \tag{1.2}
$$

$$
W^2_{\mu\nu} = \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\mu\nabla^\alpha R_{\nu\alpha} - \nabla_\nu\nabla^\alpha R_{\mu\alpha} \tag{1.3}
$$

$$
\begin{aligned}
\delta W^1_{\mu\nu} &= \tfrac{1}{2}h_{\mu\nu}R^2 - g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta} - h_\nu{}^\alpha RR_{\mu\alpha} + 2h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} - h_\mu{}^\alpha RR_{\nu\alpha} + 2h^{\alpha\beta}RR_{\mu\alpha\nu\beta} - R\nabla_\alpha\nabla^\alpha h_{\mu\nu} \\
&\quad + 2h_{\mu\nu}\nabla_\alpha\nabla^\alpha R + g_{\mu\nu}\nabla_\alpha h^\beta{}_\beta\nabla^\alpha R - \nabla_\alpha h_{\mu\nu}\nabla^\alpha R - 2g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta - g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta} + 2R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&\quad - 2g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R + g_{\mu\nu}R\nabla_\beta\nabla^\beta h^\alpha{}_\alpha - 2R_{\mu\nu}\nabla_\beta\nabla^\beta h^\alpha{}_\alpha - 2g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} - 2g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
&\quad - 2g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} + 2g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta h^\alpha{}_\alpha - 4g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} + \nabla^\alpha R\nabla_\mu h_{\nu\alpha} \\
&\quad + R\nabla_\mu\nabla_\alpha h_\nu{}^\alpha + 2\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} + \nabla^\alpha R\nabla_\nu h_{\mu\alpha} + 2\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + R\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + 2R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} \\
&\quad - R\nabla_\nu\nabla_\mu h^\alpha{}_\alpha + 2h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + 2\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} - 2\nabla_\nu\nabla_\mu\nabla_\beta\nabla^\beta h^\alpha{}_\alpha
\end{aligned}
\tag{1.4}
$$

$$
\begin{aligned}
\delta W^2_{\mu\nu} &= \tfrac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}h^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} - h_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2h^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} \\
&\quad + 2h^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta - h_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \tfrac{1}{2}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\mu h^\beta{}_\beta - \\
&\quad \tfrac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\nu h^\beta{}_\beta + \tfrac{1}{4}g_{\mu\nu}\nabla_\alpha h^\beta{}_\beta\nabla^\alpha R - \tfrac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R - \tfrac{1}{2}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta \\
&\quad - \nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h^\gamma{}_\gamma + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha h_\mu{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu} \\
&\quad + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha h_\nu{}^\beta - \tfrac{1}{2}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R - h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \tfrac{1}{2}h_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} \\
&\quad + \tfrac{1}{2}h_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha h_\mu{}^\alpha \\
&\quad - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma h_\alpha{}^\gamma - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu h_\alpha{}^\beta - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu h_\alpha{}^\beta + \tfrac{1}{2}\nabla_\beta R_{\mu\nu}\nabla^\beta h^\alpha{}_\alpha \\
&\quad + \nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu{}^\alpha + \tfrac{1}{2}\nabla^\beta h^\alpha{}_\alpha\nabla_\gamma R_\mu{}^\gamma{}_{\nu\beta} - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta}
\end{aligned}
\tag{1.5}
$$

$$- \tfrac{1}{2} g_{\mu\nu} h^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} - h^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\mu\alpha\nu\beta} - \tfrac{1}{2} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha h^{\alpha\beta} + \tfrac{1}{2} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla^\beta h^\alpha{}_\alpha$$
$$- g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma h^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta} \nabla^\gamma h^{\alpha\beta} + R_{\mu\beta\nu\gamma} \nabla^\gamma \nabla_\alpha h^{\alpha\beta} + R_{\mu\gamma\nu\beta} \nabla^\gamma \nabla_\alpha h^{\alpha\beta} - \nabla_\beta R_{\nu\alpha} \nabla_\mu h^{\alpha\beta}$$
$$+ \tfrac{1}{2} \nabla^\alpha R \nabla_\mu h_{\nu\alpha} - \tfrac{1}{2} \nabla^\beta h^\alpha{}_\alpha \nabla_\mu R_{\nu\beta} + R^{\alpha\beta} \nabla_\mu \nabla_\beta h_{\nu\alpha} - \nabla_\beta R_{\mu\alpha} \nabla_\nu h^{\alpha\beta} + \nabla_\mu R_{\alpha\beta} \nabla_\nu h^{\alpha\beta} + \tfrac{1}{2} \nabla^\alpha R \nabla_\nu h_{\mu\alpha}$$
$$+ \nabla_\mu h^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + R^{\alpha\beta} \nabla_\nu \nabla_\beta h_{\mu\alpha} + h^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} + \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2} \nabla_\nu \nabla_\mu \nabla_\beta \nabla^\beta h^\alpha{}_\alpha$$

$$\begin{aligned}
\delta W_{\mu\nu} = {} & -\tfrac{1}{6} h_{\mu\nu} R^2 + \tfrac{1}{3} g_{\mu\nu} h^{\alpha\beta} R R_{\alpha\beta} + \tfrac{1}{2} h_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \tfrac{1}{3} h_\nu{}^\alpha R R_{\mu\alpha} - \tfrac{2}{3} h^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + \tfrac{1}{3} h_\mu{}^\alpha R R_{\nu\alpha} \\
& - g_{\mu\nu} h^{\alpha\beta} R^{\gamma\eta} R_{\alpha\gamma\beta\eta} - \tfrac{2}{3} h^{\alpha\beta} R R_{\mu\alpha\nu\beta} - h_\nu{}^\alpha R^{\beta\gamma} R_{\mu\beta\alpha\gamma} + 2h^{\alpha\beta} R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2h^{\alpha\beta} R_{\alpha\gamma\beta\eta} R_\mu{}^\gamma{}_\nu{}^\eta \\
& - h_\mu{}^\alpha R^{\beta\gamma} R_{\nu\beta\alpha\gamma} + \tfrac{1}{3} R \nabla_\alpha \nabla^\alpha h_{\mu\nu} - \tfrac{1}{6} h_{\mu\nu} \nabla_\alpha \nabla^\alpha R - \tfrac{1}{2} R_\nu{}^\alpha \nabla_\alpha \nabla_\mu h^\beta{}_\beta - \tfrac{1}{2} R_\mu{}^\alpha \nabla_\alpha \nabla_\nu h^\beta{}_\beta \\
& - \tfrac{1}{12} g_{\mu\nu} \nabla_\alpha h^\beta{}_\beta \nabla^\alpha R - \tfrac{1}{6} \nabla_\alpha h_{\mu\nu} \nabla^\alpha R + \tfrac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta h_\alpha{}^\beta - \nabla_\alpha h^{\alpha\beta} \nabla_\beta R_{\mu\nu} + \tfrac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla_\alpha h^{\alpha\beta} \\
& - \tfrac{2}{3} R_{\mu\nu} \nabla_\beta \nabla_\alpha h^{\alpha\beta} + \tfrac{1}{2} g_{\mu\nu} R^{\alpha\beta} \nabla_\beta \nabla_\alpha h^\gamma{}_\gamma + \tfrac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\alpha h_\mu{}^\beta - R^{\alpha\beta} \nabla_\beta \nabla_\alpha h_{\mu\nu} \\
& + \tfrac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\alpha h_\nu{}^\beta + \tfrac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_\beta \nabla_\alpha R - h^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} - \tfrac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla^\beta h^\alpha{}_\alpha + \tfrac{2}{3} R_{\mu\nu} \nabla_\beta \nabla^\beta h^\alpha{}_\alpha \\
& + \tfrac{1}{2} h_\nu{}^\alpha \nabla_\beta \nabla^\beta R_{\mu\alpha} + \tfrac{1}{2} h_\mu{}^\alpha \nabla_\beta \nabla^\beta R_{\nu\alpha} + \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha h_{\mu\nu} - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\mu \nabla_\alpha h_\nu{}^\alpha \\
& - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\nu \nabla_\alpha h_\mu{}^\alpha - g_{\mu\nu} R^{\alpha\beta} \nabla_\beta \nabla_\gamma h_\alpha{}^\gamma - \tfrac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\mu h_\alpha{}^\beta - \tfrac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\nu h_\alpha{}^\beta \\
& + \tfrac{1}{2} \nabla_\beta R_{\mu\nu} \nabla^\beta h^\alpha{}_\alpha + \nabla_\alpha R_{\nu\beta} \nabla^\beta h_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta} \nabla^\beta h_\nu{}^\alpha + \tfrac{1}{2} \nabla^\beta h^\alpha{}_\alpha \nabla_\gamma R_\mu{}^\gamma{}_{\nu\beta} + \tfrac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma h_{\alpha\beta} \\
& - 2R_{\mu\alpha\nu\beta} \nabla_\gamma \nabla^\gamma h^{\alpha\beta} + \tfrac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} - h^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\mu\alpha\nu\beta} + \tfrac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha h^{\alpha\beta} \\
& - \tfrac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla^\beta h^\alpha{}_\alpha + \tfrac{1}{3} g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma h^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta} \nabla^\gamma h^{\alpha\beta} + R_{\mu\beta\nu\gamma} \nabla^\gamma \nabla_\alpha h^{\alpha\beta} \\
& + R_{\mu\gamma\nu\beta} \nabla^\gamma \nabla_\alpha h^{\alpha\beta} - \nabla_\beta R_{\nu\alpha} \nabla_\mu h^{\alpha\beta} + \tfrac{1}{6} \nabla^\alpha R \nabla_\mu h_{\nu\alpha} - \tfrac{1}{2} \nabla^\beta h^\alpha{}_\alpha \nabla_\mu R_{\nu\beta} - \tfrac{1}{3} R \nabla_\mu \nabla_\alpha h_\nu{}^\alpha \\
& + R^{\alpha\beta} \nabla_\mu \nabla_\beta h_{\nu\alpha} - \nabla_\beta R_{\mu\alpha} \nabla_\nu h^{\alpha\beta} + \tfrac{1}{3} \nabla_\mu R_{\alpha\beta} \nabla_\nu h^{\alpha\beta} + \tfrac{1}{6} \nabla^\alpha R \nabla_\nu h_{\mu\alpha} + \tfrac{1}{3} \nabla_\mu h^{\alpha\beta} \nabla_\nu R_{\alpha\beta} \\
& - \tfrac{1}{3} R \nabla_\nu \nabla_\alpha h_\mu{}^\alpha + R^{\alpha\beta} \nabla_\nu \nabla_\beta h_{\mu\alpha} - \tfrac{2}{3} R^{\alpha\beta} \nabla_\nu \nabla_\mu h_{\alpha\beta} + \tfrac{1}{3} R \nabla_\nu \nabla_\mu h^\alpha{}_\alpha + \tfrac{1}{3} h^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} \\
& + \tfrac{1}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha h^{\alpha\beta} + \tfrac{1}{6} \nabla_\nu \nabla_\mu \nabla_\beta \nabla^\beta h^\alpha{}_\alpha
\end{aligned} \tag{1.6}$$

1.2 Conformal Properties

Under conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{1.7}$$

the Weyl tensor transforms as

$$W_{\mu\nu} \to \bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(g_{\mu\nu}) \tag{1.8}$$

and thus

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(\Omega^{-2}\bar{g}_{\mu\nu}). \tag{1.9}$$

Now expanding to first order in the gravitational perturbation

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \tag{1.10}$$

we have

$$\begin{aligned} \bar{W}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu}) &= \bar{W}^{(0)}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu}) + \delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) \\ &= \Omega^{-2}\left[W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}(h_{\mu\nu})\right]. \end{aligned} \tag{1.11}$$

i.e. the fluctuations transform according to conformal symmetry as

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(h_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(\Omega^{-2}\bar{h}_{\mu\nu}). \tag{1.12}$$

Hence if the fluctuations $\delta W_{\mu\nu}(h_{\mu\nu})$ are diagonal in $h_{\mu\nu}$, it immediately follows they will remain so under conformal transformations.

1.2.1 Trace Considerations

We can continue to use conformal invariance to determine the trace depedendent properties of $W_{\mu\nu}$. Taking h as a first order perturbation in the metric and using the conformal invariance, we find up to first order

$$\begin{aligned} W_{\mu\nu}\left(g^{(0)}_{\mu\nu} + \frac{h}{4}g^{(0)}_{\mu\nu}\right) &= W_{\mu\nu}\left[\left(1+\frac{h}{4}\right)g^{(0)}_{\mu\nu}\right] = W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right) \\ &= \left(1-\frac{h}{4}\right)W_{\mu\nu}(g^{(0)}_{\mu\nu}), \end{aligned}$$

and hence

$$-\frac{h}{4}W_{\mu\nu}(g^{(0)}_{\mu\nu}) = \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right). \tag{1.13}$$

This is verified in Mathematica, which yields

$$g^{(0)\mu\nu}\delta W_{\mu\nu} = h^{\mu\nu}(-\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R \tag{1.14}$$

$$+ \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \nabla_\mu \nabla^\alpha R_{\nu\alpha} - \nabla_\nu \nabla^\alpha R_{\mu\alpha} + \tfrac{2}{3} \nabla_\nu \nabla_\mu R)$$
$$= h^{\mu\nu} W_{\mu\nu}(g^{(0)}_{\mu\nu})$$

Now, decomposing $h_{\mu\nu}$ into a trace and trace free components

$$h_{\mu\nu} = K_{\mu\nu} + g_{\mu\nu}\frac{h}{4} \tag{1.15}$$

(where $g^{(0)\mu\nu}K_{\mu\nu} = 0$, $h = g^{(0)\mu\nu}h_{\mu\nu}$), substitute the above in, again keeping only first order terms

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}\left(K_{\mu\nu} + \frac{h}{4}g^{(0)}_{\mu\nu}\right) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right). \tag{1.16}$$

If we work in a background that is conformal to flat, then 1.13 will vanish which implies from 1.16 that

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}). \tag{1.17}$$

We may also find a relationship in the trace of entire fluctuation $\delta W_{\mu\nu}$. The tracelessness of $W_{\mu\nu}$ implies

$$g^{\mu\nu}W_{\mu\nu}(g_{\mu\nu}) = \left(g^{(0)\mu\nu} - h^{\mu\nu}\right)\left(W^{(0)}_{\mu\nu} + \delta W_{\mu\nu}\right) = 0. \tag{1.18}$$

To first order,

$$- h^{\mu\nu}W^{(0)}_{\mu\nu} + g^{(0)\mu\nu}\delta W_{\mu\nu} = 0 \tag{1.19}$$

and thus

$$g^{(0)\mu\nu}\delta W_{\mu\nu}(h_{\mu\nu}) = h^{\mu\nu}W_{\mu\nu}(g^{(0)}_{\mu\nu}). \tag{1.20}$$

This is verified in mathematica, which yields

$$\begin{aligned} \delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) &= -\tfrac{1}{4}h(-\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &- \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\mu\nabla^\alpha R_{\nu\alpha} - \nabla_\nu\nabla^\alpha R_{\mu\alpha} + \tfrac{2}{3}\nabla_\nu\nabla_\mu R) \\ &= -\tfrac{h}{4}W_{\mu\nu}(g^{(0)}_{\mu\nu}) \end{aligned} \tag{1.21}$$

Once again, in a conformal to flat background, the trace of the fluctuations will vanish.

1.3 Conformal Transverse Gauge

In a conformal to flat geomerty with a cartesian coordinate system (this should be able to be generalized for any orthogonal flat coordinate system), the perturbed Weyl tensor computes as

$$
\begin{aligned}
\delta\bar{W}_{\mu\nu}(\bar{K}_{\mu\nu}) = &-\frac{48\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\overline{K}_{\mu\nu}}{\Omega^7}+\frac{24\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\eta\overline{K}_{\mu\nu}}{\Omega^6}+\frac{20\eta^{\alpha\beta}\eta^{\gamma\kappa}\eta^{\eta\lambda}\eta_{\mu\nu}\overline{K}_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega}{\Omega^8}\\
&+\frac{60\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\eta\Omega}{\Omega^8}-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta\overline{K}_{\mu\nu}}{\Omega^5}+\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega}{\Omega^6}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\mu\overline{K}_{\nu\gamma}}{\Omega^5}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\beta\partial_\nu\overline{K}_{\mu\gamma}}{\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\nu\partial_\mu\overline{K}_{\alpha\gamma}}{3\Omega^4}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\beta\partial_\nu\partial_\mu\Omega}{3\Omega^5}+\frac{12\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\overline{K}_{\mu\nu}}{\Omega^6}+\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\overline{K}_{\mu\nu}}{\Omega^6}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma\overline{K}_{\mu\nu}}{\Omega^5}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\overline{K}_{\mu\nu}\partial_\eta\partial_\gamma\Omega}{\Omega^6}-\frac{48\eta^{\alpha\gamma}\eta^{\beta\eta}\overline{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega}{\Omega^7}\\
&-\frac{24\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\gamma\Omega}{\Omega^7}+\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega}{\Omega^6}-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta\overline{K}_{\mu\nu}}{\Omega^5}\\
&-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\overline{K}_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\Omega}{\Omega^5}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\beta\Omega}{\Omega^6}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha\overline{K}_{\mu\nu}}{2\Omega^4}\\
&-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\partial_\alpha\Omega}{\Omega^5}-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\mu\overline{K}_{\nu\alpha}}{2\Omega^4}-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\partial_\nu\overline{K}_{\mu\alpha}}{2\Omega^4}\\
&+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\mu\overline{K}_{\nu\beta}}{\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\alpha}\partial_\eta\partial_\gamma\partial_\mu\partial_\beta\Omega}{\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\partial_\nu\overline{K}_{\mu\beta}}{\Omega^5}\\
&+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\alpha}\partial_\eta\partial_\gamma\partial_\nu\partial_\beta\Omega}{\Omega^5}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\mu\overline{K}_{\nu\beta}}{\Omega^5}-\frac{6\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\overline{K}_{\nu\gamma}}{\Omega^6}\\
&-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\mu\overline{K}_{\nu\gamma}}{\Omega^6}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\mu\overline{K}_{\nu\gamma}}{\Omega^5}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\gamma}\partial_\alpha\Omega\partial_\eta\partial_\mu\partial_\beta\Omega}{\Omega^6}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\overline{K}_{\nu\alpha}\partial_\eta\partial_\mu\partial_\beta\Omega}{\Omega^5}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\beta}\partial_\alpha\Omega\partial_\eta\partial_\mu\partial_\gamma\Omega}{\Omega^6}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\overline{K}_{\nu\alpha}\partial_\eta\partial_\mu\partial_\gamma\Omega}{\Omega^5}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\eta\partial_\nu\overline{K}_{\mu\beta}}{\Omega^5}-\frac{6\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\overline{K}_{\mu\gamma}}{\Omega^6}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\partial_\nu\overline{K}_{\mu\gamma}}{\Omega^6}\\
&+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\partial_\alpha\Omega\partial_\eta\partial_\nu\overline{K}_{\mu\gamma}}{\Omega^5}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\gamma}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\beta\Omega}{\Omega^6}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\overline{K}_{\mu\alpha}\partial_\eta\partial_\nu\partial_\beta\Omega}{\Omega^5}\\
&-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\beta}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\gamma\Omega}{\Omega^6}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\overline{K}_{\mu\alpha}\partial_\eta\partial_\nu\partial_\gamma\Omega}{\Omega^5}-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu\overline{K}_{\beta\gamma}}{3\Omega^5}\\
&+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\beta\gamma}\partial_\alpha\Omega\partial_\eta\partial_\nu\partial_\mu\Omega}{\Omega^6}-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\overline{K}_{\alpha\gamma}\partial_\eta\partial_\nu\partial_\mu\Omega}{3\Omega^5}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\overline{K}_{\eta\lambda}\partial_\kappa\partial_\gamma\Omega}{\Omega^6}\\
&+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\kappa\partial_\gamma\Omega\partial_\lambda\overline{K}_{\beta\eta}}{\Omega^6}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\lambda\overline{K}_{\beta\kappa}}{\Omega^6}-\frac{8\eta^{\alpha\eta}\eta^{\beta\kappa}\eta^{\gamma\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda\overline{K}_{\eta\kappa}}{\Omega^7}\\
&-\frac{8\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\lambda\overline{K}_{\eta\kappa}}{\Omega^7}+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\lambda\overline{K}_{\eta\kappa}}{\Omega^6}-\\
&\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\eta\overline{K}_{\beta\kappa}}{3\Omega^5}+\frac{4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\overline{K}_{\gamma\kappa}}{\Omega^6}+\\
&\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\overline{K}_{\gamma\kappa}}{\Omega^6}-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\beta\partial_\alpha\Omega\partial_\lambda\partial_\eta\overline{K}_{\gamma\kappa}}{3\Omega^5}-\\
&\frac{16\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\gamma\kappa}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega}{\Omega^7}-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\gamma\kappa}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega}{\Omega^7}
\end{aligned}
\tag{1.22}
$$

$$
\begin{aligned}
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\alpha\gamma}\partial_\kappa\partial_\beta\Omega\partial_\lambda\partial_\eta\Omega}{\Omega^6}-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta\overline{K}_{\gamma\kappa}}{3\Omega^5}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\gamma\kappa}\partial_\alpha\Omega\partial_\lambda\partial_\eta\partial_\beta\Omega}{\Omega^6}-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\kappa\overline{K}_{\alpha\gamma}\partial_\lambda\partial_\eta\partial_\beta\Omega}{3\Omega^5}\\
&-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\gamma\partial_\alpha\Omega\partial_\lambda\partial_\kappa\overline{K}_{\beta\eta}}{3\Omega^5}+\frac{\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa\overline{K}_{\gamma\eta}}{\Omega^6}\\
&-\frac{4\eta^{\alpha\gamma}\eta^{\beta\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega}{\Omega^7}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\beta\Omega\partial_\lambda\partial_\kappa\Omega}{\Omega^6}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta\overline{K}_{\beta\gamma}}{3\Omega^5}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\beta\gamma}\partial_\alpha\Omega\partial_\lambda\partial_\kappa\partial_\eta\Omega}{\Omega^6}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\beta\overline{K}_{\alpha\gamma}\partial_\lambda\partial_\kappa\partial_\eta\Omega}{3\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta\overline{K}_{\alpha\gamma}}{6\Omega^4}-\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\eta^{\kappa\lambda}\eta_{\mu\nu}\overline{K}_{\alpha\gamma}\partial_\lambda\partial_\kappa\partial_\eta\partial_\beta\Omega}{3\Omega^5}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\nu\partial_\beta\Omega\partial_\mu\overline{K}_{\alpha\gamma}}{3\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu\overline{K}_{\nu\alpha}}{\Omega^5}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu\overline{K}_{\nu\beta}}{\Omega^6}\\
&+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu\overline{K}_{\nu\eta}}{\Omega^7}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\mu\overline{K}_{\nu\eta}}{\Omega^6}-\frac{60\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\mu\Omega}{\Omega^8}\\
&-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\eta\overline{K}_{\nu\beta}\partial_\mu\Omega}{\Omega^6}+\frac{24\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\overline{K}_{\nu\gamma}\partial_\mu\Omega}{\Omega^7}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\overline{K}_{\nu\gamma}\partial_\mu\Omega}{\Omega^7}\\
&-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\partial_\alpha\Omega\partial_\eta\overline{K}_{\nu\gamma}\partial_\mu\Omega}{\Omega^6}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\beta\overline{K}_{\nu\gamma}\partial_\mu\Omega}{\Omega^6}+\frac{24\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\gamma}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\mu\Omega}{\Omega^7}\\
&-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\nu\overline{K}_{\alpha\gamma}\partial_\mu\Omega}{3\Omega^5}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\overline{K}_{\nu\beta}\partial_\mu\Omega}{\Omega^6}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\beta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\mu\Omega}{\Omega^7}\\
&+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\overline{K}_{\nu\alpha}\partial_\mu\Omega}{\Omega^5}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\alpha}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\mu\Omega}{\Omega^6}+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\nu\overline{K}_{\beta\gamma}\partial_\mu\Omega}{\Omega^6}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\nu\partial_\beta\Omega\partial_\mu\Omega}{\Omega^6}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\overline{K}_{\nu\gamma}\partial_\mu\partial_\alpha\Omega}{\Omega^5}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\overline{K}_{\nu\beta}\partial_\mu\partial_\alpha\Omega}{\Omega^5}\\
&-\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\nu\overline{K}_{\beta\gamma}\partial_\mu\partial_\alpha\Omega}{3\Omega^5}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\nu\gamma}\partial_\mu\partial_\beta\Omega}{\Omega^6}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\alpha}\partial_\eta\partial_\gamma\Omega\partial_\mu\partial_\beta\Omega}{\Omega^6}\\
&-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\overline{K}_{\nu\eta}\partial_\mu\partial_\gamma\Omega}{\Omega^6}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\nu\beta}\partial_\mu\partial_\gamma\Omega}{\Omega^6}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\alpha}\partial_\eta\partial_\beta\Omega\partial_\mu\partial_\gamma\Omega}{\Omega^6}\\
&+\frac{24\eta^{\alpha\gamma}\eta^{\beta\eta}\overline{K}_{\nu\gamma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\partial_\eta\Omega}{\Omega^7}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\nu\gamma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\partial_\eta\Omega}{\Omega^7}-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\mu\partial_\beta\Omega\partial_\nu\overline{K}_{\alpha\gamma}}{3\Omega^5}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu\overline{K}_{\beta\eta}}{\Omega^6}+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\mu\partial_\gamma\Omega\partial_\nu\overline{K}_{\beta\eta}}{\Omega^6}-\frac{8\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\overline{K}_{\gamma\eta}}{\Omega^7}\\
&+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu\overline{K}_{\mu\alpha}}{\Omega^5}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu\overline{K}_{\mu\beta}}{\Omega^6}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu\overline{K}_{\mu\eta}}{\Omega^7}\\
&-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\gamma\partial_\beta\Omega\partial_\nu\overline{K}_{\mu\eta}}{\Omega^6}-\frac{60\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\gamma\Omega\partial_\nu\Omega}{\Omega^8}-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\eta\overline{K}_{\mu\beta}\partial_\nu\Omega}{\Omega^6}\\
&+\frac{24\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\overline{K}_{\mu\gamma}\partial_\nu\Omega}{\Omega^7}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\eta\overline{K}_{\mu\gamma}\partial_\nu\Omega}{\Omega^7}-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\beta\partial_\alpha\Omega\partial_\eta\overline{K}_{\mu\gamma}\partial_\nu\Omega}{\Omega^6}\\
&-\frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\beta\overline{K}_{\mu\gamma}\partial_\nu\Omega}{\Omega^6}+\frac{24\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\gamma}\partial_\alpha\Omega\partial_\eta\partial_\beta\Omega\partial_\nu\Omega}{\Omega^7}-\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\partial_\mu\overline{K}_{\alpha\gamma}\partial_\nu\Omega}{3\Omega^5}\\
&-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\overline{K}_{\mu\beta}\partial_\nu\Omega}{\Omega^6}+\frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\beta}\partial_\alpha\Omega\partial_\eta\partial_\gamma\Omega\partial_\nu\Omega}{\Omega^7}+\frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\partial_\beta\overline{K}_{\mu\alpha}\partial_\nu\Omega}{\Omega^5}\\
&-\frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\alpha}\partial_\eta\partial_\gamma\partial_\beta\Omega\partial_\nu\Omega}{\Omega^6}+\frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\partial_\mu\overline{K}_{\beta\gamma}\partial_\nu\Omega}{\Omega^6}+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\mu\partial_\beta\Omega\partial_\nu\Omega}{\Omega^6}\\
&+\frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\mu\overline{K}_{\beta\eta}\partial_\nu\Omega}{\Omega^6}-\frac{8\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\overline{K}_{\gamma\eta}\partial_\nu\Omega}{\Omega^7}+\frac{40\eta^{\alpha\gamma}\eta^{\beta\eta}\overline{K}_{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega}{\Omega^8}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{16\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\beta\gamma}\partial_\mu\Omega\partial_\nu\Omega}{\Omega^7} + \frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\overline{K}_{\alpha\gamma}\partial_\mu\Omega\partial_\nu\Omega}{\Omega^6} - \frac{8\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\beta\Omega\partial_\mu\Omega\partial_\nu\Omega}{\Omega^7} \\
&+ \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\overline{K}_{\beta\gamma}\partial_\mu\partial_\alpha\Omega\partial_\nu\Omega}{\Omega^6} - \frac{16\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\beta\gamma}\partial_\alpha\Omega\partial_\mu\partial_\eta\Omega\partial_\nu\Omega}{\Omega^7} + \frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\overline{K}_{\mu\gamma}\partial_\nu\partial_\alpha\Omega}{\Omega^5} \\
&+ \frac{\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\gamma\overline{K}_{\mu\beta}\partial_\nu\partial_\alpha\Omega}{\Omega^5} - \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\mu\overline{K}_{\beta\gamma}\partial_\nu\partial_\alpha\Omega}{3\Omega^5} + \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\overline{K}_{\beta\gamma}\partial_\mu\Omega\partial_\nu\partial_\alpha\Omega}{\Omega^6} \\
&- \frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\mu\gamma}\partial_\nu\partial_\beta\Omega}{\Omega^6} - \frac{3\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\alpha}\partial_\eta\partial_\gamma\Omega\partial_\nu\partial_\beta\Omega}{\Omega^6} - \frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\overline{K}_{\mu\eta}\partial_\nu\partial_\gamma\Omega}{\Omega^6} \\
&- \frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\mu\beta}\partial_\nu\partial_\gamma\Omega}{\Omega^6} - \frac{6\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\alpha}\partial_\eta\partial_\beta\Omega\partial_\nu\partial_\gamma\Omega}{\Omega^6} + \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\mu\overline{K}_{\beta\eta}\partial_\nu\partial_\gamma\Omega}{\Omega^6} \\
&+ \frac{24\eta^{\alpha\gamma}\eta^{\beta\eta}\overline{K}_{\mu\gamma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\eta\Omega}{\Omega^7} \\
&+ \frac{12\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\mu\gamma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\eta\Omega}{\Omega^7} \\
&- \frac{16\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\beta\gamma}\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu\partial_\eta\Omega}{\Omega^7} + \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\mu\partial_\beta\Omega\partial_\nu\partial_\eta\Omega}{\Omega^6} - \frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\gamma\partial_\alpha\Omega\partial_\nu\partial_\mu\overline{K}_{\beta\eta}}{3\Omega^5} \\
&+ \frac{2\eta^{\alpha\gamma}\eta^{\beta\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu\overline{K}_{\gamma\eta}}{\Omega^6} - \frac{8\eta^{\alpha\gamma}\eta^{\beta\eta}\overline{K}_{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega\partial_\nu\partial_\mu\Omega}{\Omega^7} + \frac{4\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\alpha\Omega\partial_\eta\overline{K}_{\beta\gamma}\partial_\nu\partial_\mu\Omega}{\Omega^6} \\
&- \frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\partial_\eta\partial_\beta\overline{K}_{\alpha\gamma}\partial_\nu\partial_\mu\Omega}{3\Omega^5} + \frac{2\eta^{\alpha\beta}\eta^{\gamma\eta}\overline{K}_{\alpha\gamma}\partial_\eta\partial_\beta\Omega\partial_\nu\partial_\mu\Omega}{\Omega^6}
\end{aligned}
$$

In choosing a gauge, it is important to note that in a flat geometry, the equations of motion are diagonal in the transverse gauge. By conformally transforming the transverse gauge condition, i.e.

$$
\bar{\nabla}_\nu \bar{K}^{\mu\nu} = 4\Omega^{-1}\bar{K}^{\mu\nu}\partial_\nu\Omega \tag{1.23}
$$

we anticipate that the equations of motion will retain diagonalization but now in a conformal to flat geometry. The equivalent gauge covariant in $K_{\mu\nu}$ is

$$
\eta^{\alpha\beta}\partial_\alpha\bar{K}_{\mu\beta} = 2\Omega^{-1}\eta^{\alpha\beta}\bar{K}_{\mu\beta}\partial_\alpha\Omega. \tag{1.24}
$$

Working within this gauge, we find the Weyl tensor reduces beautifully as

$$
\begin{aligned}
\delta\bar{W}_{\mu\nu}(\bar{K}_{\mu\nu}) = \Omega^{-5}\Bigg(&- \frac{48\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma\bar{K}_{\mu\nu}}{\Omega^2} + \frac{24\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\rho\partial_\beta\Omega\partial_\sigma\bar{K}_{\mu\nu}}{\Omega} \\
&+ \frac{60\eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma\Omega}{\Omega^3} - 4\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta\bar{K}_{\mu\nu} + \frac{6\eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta\Omega}{\Omega} \\
&+ \frac{12\eta^{\alpha\rho}\eta^{\beta\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\bar{K}_{\mu\nu}}{\Omega} + \frac{6\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\bar{K}_{\mu\nu}}{\Omega} - 2\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho\bar{K}_{\mu\nu} \\
&+ \frac{12\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\bar{K}_{\mu\nu}\partial_\sigma\partial_\rho\Omega}{\Omega} - \frac{48\eta^{\alpha\rho}\eta^{\beta\sigma}\bar{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega}{\Omega^2} - \frac{24\eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega}{\Omega^2} \\
&+ \frac{3\eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho\Omega}{\Omega} - 4\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta\bar{K}_{\mu\nu} - 4\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\bar{K}_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\Omega \\
&+ \frac{12\eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta\Omega}{\Omega} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\rho\sigma}\Omega\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha\bar{K}_{\mu\nu} - \eta^{\alpha\beta}\eta^{\rho\sigma}\bar{K}_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha\Omega\Bigg)
\end{aligned} \tag{1.25}
$$

$$= \frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta(\Omega^{-2}\bar{K}_{\mu\nu})$$

1.4 Gauge Invariance

Under conformal transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $W_{\mu\nu}$ transforms as

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2}W_{\mu\nu}(g_{\mu\nu}). \tag{1.26}$$

Perturbing the metric,

$$\bar{g}_{\mu\nu} = \bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu} = \Omega^2 g^{(0)}_{\mu\nu} + \Omega^2 h_{\mu\nu} \tag{1.27}$$

it follows that to first order

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(h_{\mu\nu}). \tag{1.28}$$

Under an infinitesimal oordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, the perturbed tensor $\delta W_{\mu\nu}$ transforms as

$$\delta W_{\mu\nu}(h_{\mu\nu}) \to \delta W'_{\mu\nu}(h'_{\mu\nu}) = \delta W_{\mu\nu}(h_{\mu\nu}) - \delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}) \tag{1.29}$$

At the same time, we also consider the transformation of the entire $W_{\mu\nu}$ under the infinitesimal coordinate transformation

$$W_{\mu\nu} \to W'_{\mu\nu} = W_{\mu\nu} - \mathcal{L}_e(W_{\mu\nu}) \tag{1.30}$$

where the Lie derivative $\mathcal{L}_e$ for the rank 2 tensor is

$$\mathcal{L}_e(W_{\mu\nu}) = W^\lambda{}_\mu \epsilon_{\lambda;\nu} + W^\lambda{}_\nu \epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda. \tag{1.31}$$

Defining $\delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}) \equiv \delta W_{\mu\nu}(\epsilon)$, if we expand 1.30 to first order (that is $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$), we get

$$W_{\mu\nu} \to W'_{\mu\nu} = W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}(h_{\mu\nu}) - \mathcal{L}_e(W_{\mu\nu}) \tag{1.32}$$

and conclude that

$$\delta W_{\mu\nu}(\epsilon) = \mathcal{L}_e(W_{\mu\nu}) = W^\lambda{}_\mu \epsilon_{\lambda;\nu} + W^\lambda_\nu \epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda. \tag{1.33}$$

Hence, in any background that is conformal to flat, the Lie derivative vanishes and thus $\delta W_{\mu\nu}$ must be gauge invariant. As such, it must always be possible to express $\delta W_{\mu\nu}$ in terms of 5 gauge invariant quantities (10 symmetric components - 4 coordinate transformation - 1 traceless condition = 5). This is shown below. Alternatively, we may also fix the gauge, as we have done to make $\delta W_{\mu\nu}$ diagonal in its indicies.

1.5 S.V.T. Decomposition

We decompose $h_{\mu\nu}$ according to

$$ds^2 = \Omega^2 \left\{ -(1+2\phi)d\tau^2 + (\tilde{\nabla}_i B + B_i)dx^i d\tau + [(1-2\psi)\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right\}. \quad (1.34)$$

The covariant derivatives are defined with respect to the 3-space background γ_{ij} and are indicated as $\tilde{\nabla}_i$. We have in conformal flat space $\delta W_{\mu\nu}(h_{\mu\nu})$ in arbitrary coordinate system

Scalars:

$$\delta W_{00} = -\frac{2}{3\Omega^2}\gamma^{mn}\gamma^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k(\phi+\psi-(E'-B)') \quad (1.35)$$

$$\delta W_{0i} = -\frac{2}{3\Omega^2}\gamma^{mn}\gamma^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k(\phi'+\psi'-(E'-B)'') \quad (1.36)$$

$$\begin{aligned}\delta W_{ij} = \frac{1}{3\Omega^2}\big[&g_{ij}\gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k(\phi''+\psi''-(E'-B)''') + \gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\tilde{\nabla}_i\tilde{\nabla}_j(\phi+\psi-(E'-B)') \\ &- g_{ij}\gamma^{mn}\gamma^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k(\phi+\psi-(E'-B)') - 3\tilde{\nabla}_i\tilde{\nabla}_j(\phi''+\psi''-(E'-B)''')\big]\end{aligned} \quad (1.37)$$

Vectors:

$$\delta W_{0i} = \frac{1}{2\Omega^2}\left[\gamma^{mn}\gamma^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k(B_i-E_i') - \gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k(B_i-E_i')''\right] \quad (1.38)$$

$$\delta W_{ij} = \frac{1}{2\Omega^2}\left[\gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\tilde{\nabla}_i(B_j-E_j')' + \gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\tilde{\nabla}_j(B_i-E_i')' - \tilde{\nabla}_i(B_j-E_j')''' - \tilde{\nabla}_j(B_i-E_i')'''\right] \quad (1.39)$$

Tensors:

$$\delta W_{ij} = \frac{1}{\Omega^2}\left(E_{ij}'''' - 2\gamma^{lk}\tilde{\nabla}_l\tilde{\nabla}_k E_{ij}'' + \gamma^{mn}\gamma^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k E_{ij}\right) \quad (1.40)$$

According to 1.12, we may find $\delta W_{\mu\nu}$ based on a conformal to flat background by simply multiplying the above by a factor of Ω^{-2}.

Under coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu$ where $\epsilon^\mu = (T, \partial^i L + L^i)$ the SVT quantities in the RW $K = 0$ background transform as ($\mathcal{H} = \frac{\dot{\Omega}}{\Omega}$)

$$\tilde{\phi} = \phi - T' - \mathcal{H}T \tag{1.41}$$

$$\tilde{\psi} = \psi + \mathcal{H}T \tag{1.42}$$

$$\tilde{E} = E - L \tag{1.43}$$

$$\tilde{B} = B + T - L' \tag{1.44}$$

$$\tilde{B}_i = B_i - L_i' \tag{1.45}$$

$$\tilde{E}_i = E_i - L_i \tag{1.46}$$

$$\tilde{E}_{ij} = E_{ij} \tag{1.47}$$

in which the gauge invariant combinations are

$$\Phi = \phi - \mathcal{H}(E' - B) - (E' - B)' \tag{1.48}$$

$$\Psi = \psi + \mathcal{H}(E' - B) \tag{1.49}$$

$$\mathcal{Q}_i = B_i - E_i' \tag{1.50}$$

$$E_{ij} = E_{ij}. \tag{1.51}$$

and, importantly for the Weyl tensor

$$\Sigma = \Phi + \Psi = \phi + \psi - (E' - B)'. \tag{1.52}$$

Now, if we generalize the conformal factor $\Omega(\tau) \to \Omega(x)$ we can calculate the gauge transformations by effectively sending

$$T\mathcal{H} \to \tilde{H} = \frac{\epsilon^\mu \partial_\mu \Omega}{\Omega} = T\mathcal{H} + (\partial^i L + L^i)\frac{\partial_i \Omega}{\Omega}. \tag{1.53}$$

That this is true can be seen from the first order contrubtion of $\Omega(x^\mu + \epsilon^\mu)$. As such, the analogous SVT quantities under the coordinate transformation are

$$\tilde{\phi} = \phi - T' - \tilde{H} \tag{1.54}$$

$$\tilde{\psi} = \psi + \tilde{H} \tag{1.55}$$

$$\tilde{E} = E - L \tag{1.56}$$

$$\tilde{B} = B + T - L' \tag{1.57}$$

$$\tilde{B}_i = B_i - L_i' \tag{1.58}$$

$$\tilde{E}_i = E_i - L_i \tag{1.59}$$

$$\tilde{E}_{ij} = E_{ij} \tag{1.60}$$

The gauge invariant combinations can then only possibly differ from that of RW for those involving ψ and ϕ and in the Weyl case we only care about

$$\Sigma = \phi + \psi - (E' - B)'. \tag{1.61}$$

But we note that the $\tilde{H}$ terms drop out identically, and thus in the general conformal case and thus the same form for Σ remains invariant. Thus the gauge invariant quantities for any conformal factor are:

$$\Sigma = \phi + \psi - (E' - B)' \tag{1.62}$$

$$\mathcal{Q}_i = B_i - E_i' \tag{1.63}$$

$$E_{ij} = E_{ij}. \tag{1.64}$$

This brings us to 5 independent components in total, as mentioned above, and the gauge invariant combinations within $\delta W_{\mu\nu}$ drop out very clearly.

2 Einstein

2.1 Fluctuations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.1}$$

$$\begin{aligned}\delta G_{\mu\nu} = &-\tfrac{1}{2} h_{\mu\nu} R + \tfrac{1}{2} g_{\mu\nu} h^{\alpha\beta} R_{\alpha\beta} + \tfrac{1}{2} h_{\nu}{}^{\alpha} R_{\mu\alpha} + \tfrac{1}{2} h_{\mu}{}^{\alpha} R_{\nu\alpha} - h^{\alpha\beta} R_{\mu\alpha\nu\beta} + \tfrac{1}{2} \nabla_\alpha \nabla^\alpha h_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} \nabla_\beta \nabla_\alpha h^{\alpha\beta} \\ &- \tfrac{1}{2} g_{\mu\nu} \nabla_\beta \nabla^\beta h^{\alpha}{}_{\alpha} - \tfrac{1}{2} \nabla_\mu \nabla_\alpha h_{\nu}{}^{\alpha} - \tfrac{1}{2} \nabla_\nu \nabla_\alpha h_{\mu}{}^{\alpha} + \tfrac{1}{2} \nabla_\nu \nabla_\mu h^{\alpha}{}_{\alpha}\end{aligned} \tag{2.2}$$

2.2 Conformal Properties

Under conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.3}$$

the Ricci tensor transforms as

$$R_{\mu\nu}(g_{\mu\nu}) \to \bar{R}_{\mu\nu}(\bar{g}_{\mu\nu}) = R_{\mu\nu}(g_{\mu\nu}) + \tilde{S}_{\mu\nu}(g_{\mu\nu}) \tag{2.4}$$

where $\tilde{S}_{\mu\nu}$ involves terms with covariant derivatives of Ω. It follows that the Ricci scalar transforms as

$$g^{\alpha\beta} R_{\alpha\beta}(g_{\mu\nu}) \to \bar{R}(\bar{g}_{\mu\nu}) = \Omega^{-2}[R(g_{\mu\nu}) + g^{\alpha\beta} \tilde{S}_{\alpha\beta}(g_{\mu\nu})] \tag{2.5}$$

and thus

$$g_{\mu\nu} R \to \bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} R + S'_{\mu\nu}. \tag{2.6}$$

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ must then transform as

$$G_{\mu\nu}(g_{\mu\nu}) \to \bar{G}_{\mu\nu}(\bar{g}_{\mu\nu}) = G_{\mu\nu}(g_{\mu\nu}) + S_{\mu\nu}(g_{\mu\nu}) \tag{2.7}$$

where again $S_{\mu\nu}$ is some arbitrary tensor of Ω and $g_{\mu\nu}$. Now expanding to first order in the gravitational perturbation

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \tag{2.8}$$

we have

$$\begin{aligned}\bar{G}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu}) &= \bar{G}^{(0)}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu}) + \delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) \\ &= G^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) + S^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta S_{\mu\nu}(h_{\mu\nu}).\end{aligned} \tag{2.9}$$

Now looking at the first order contribution,

$$\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) = \delta G_{\mu\nu}(h_{\mu\nu}) + \delta S_{\mu\nu}(h_{\mu\nu}), \tag{2.10}$$

we note that diagonality in $\bar{h}_{\mu\nu}$ of $\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu})$ requires the sum of $\delta G_{\mu\nu}(h_{\mu\nu})$ and $\delta S_{\mu\nu}(h_{\mu\nu})$ to be diagonal in $h_{\mu\nu}$.

Specifically, we may calculate $S_{\mu\nu}$ to be

$$S_{\mu\nu} = \Omega^{-1}(g_{\mu\nu}\nabla_\alpha\nabla^\alpha\Omega + 2\nabla_\nu\nabla_\mu\Omega) + \Omega^{-2}(g_{\mu\nu}\nabla_\alpha\Omega\nabla^\alpha\Omega - 4\nabla_\mu\Omega\nabla_\nu\Omega) \tag{2.11}$$

and expanding to first order (here $g_{\mu\nu} = g^{(0)}_{\mu\nu}$)

$$\begin{aligned}\delta S_{\mu\nu} = \Omega^{-1}[&-g_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta h^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta h^\gamma{}_\gamma + g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta h_{\mu\nu} - g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha\Omega \\ &+ g^{\alpha\beta}h_{\mu\nu}\nabla_\beta\nabla_\alpha\Omega - \nabla_\alpha\Omega\nabla_\mu h^\alpha{}_\nu - \nabla_\alpha\Omega\nabla_\nu h^\alpha{}_\mu] \\ &+ \Omega^{-2}[g^{\alpha\beta}h_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta\Omega - g_{\mu\nu}h^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega].\end{aligned} \tag{2.12}$$

In the conformal to flat case, $\delta S_{\mu\nu}$ simplifies to

$$\begin{aligned}\delta S_{\mu\nu} = \Omega^{-1}[&\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta h_{\gamma\eta} + \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} + \eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega \\ &- \eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\eta h_{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\eta\partial_\beta\Omega - \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\nu h_{\mu\beta}] \\ &+ \Omega^{-2}[\eta^{\alpha\beta}h_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega - \eta^{\alpha\gamma}\eta^{\beta\eta}\eta_{\mu\nu}h^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega].\end{aligned} \tag{2.13}$$

In the harmonic gauge, the extra term $\delta S_{\mu\nu}(g_{\mu\nu})$ does not vanish, and thus does not yield

$$\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) = \delta G_{\mu\nu}(h_{\mu\nu}). \tag{2.14}$$

If the harmonic gauge did in fact cause $\delta S_{\mu\nu}$ to vanish, then we would be able to use the conformally transformed harmonic condition directly within $\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu})$ to obain (nearly) diagonal equations of motion (or just as diagonal as can be found using harmonic in the flat fluctuations).

2.3 Special Gauge

The perturbed Einstein tensor $\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu})$ evaluated in a $K=0$ RW background (in cartesian coordinates) takes the general form (dropping the bar on $\delta G_{\mu\nu}$):

$$\begin{aligned}\delta G_{00} = {} & 2\Omega'^2\overline{h}_{00} + \Omega\Omega'\partial_0\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{00} - 2\Omega\Omega'\partial_1\overline{h}_{01} - 2\Omega\Omega'\partial_2\overline{h}_{02} \\ & - 2\Omega\Omega'\partial_3\overline{h}_{03} - \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{11} - \Omega^2\partial_2\partial_1\overline{h}_{12} - \Omega^2\partial_3\partial_1\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{22} - \Omega^2\partial_3\partial_2\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{33} \\ & + \frac{\Omega'\partial_0 h}{\Omega} + \tfrac{1}{2}\partial_1\partial_1 h + \tfrac{1}{2}\partial_2\partial_2 h + \tfrac{1}{2}\partial_3\partial_3 h\end{aligned} \tag{2.15}$$

$$\begin{aligned}\delta G_{11} = {} & -\Omega\Omega''\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{00} - \Omega^2\partial_2\partial_0\overline{h}_{02} - \Omega^2\partial_3\partial_0\overline{h}_{03} + 4\Omega'^2\overline{h}_{11} - 3\Omega\Omega''\overline{h}_{11} - \Omega\Omega'\partial_0\overline{h}_{11} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{11} \\ & + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{11} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{11} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{22} + \Omega^2\partial_3\partial_2\overline{h}_{23} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{33} - \frac{\Omega'\partial_0 h}{\Omega} + \tfrac{1}{2}\partial_0\partial_0 h \\ & - \tfrac{1}{2}\partial_2\partial_2 h - \tfrac{1}{2}\partial_3\partial_3 h\end{aligned} \tag{2.16}$$

$$\begin{aligned}\delta G_{22} = {} & -\Omega\Omega''\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{00} - \Omega^2\partial_1\partial_0\overline{h}_{01} - \Omega^2\partial_3\partial_0\overline{h}_{03} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{11} + \Omega^2\partial_3\partial_1\overline{h}_{13} + 4\Omega'^2\overline{h}_{22} \\ & - 3\Omega\Omega''\overline{h}_{22} - \Omega\Omega'\partial_0\overline{h}_{22} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{22} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{22} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{22} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{33} - \frac{\Omega'\partial_0 h}{\Omega} \\ & + \tfrac{1}{2}\partial_0\partial_0 h - \tfrac{1}{2}\partial_1\partial_1 h - \tfrac{1}{2}\partial_3\partial_3 h\end{aligned} \tag{2.17}$$

$$\begin{aligned}\delta G_{33} = {} & -\Omega\Omega''\overline{h}_{00} + \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{00} - \Omega^2\partial_1\partial_0\overline{h}_{01} - \Omega^2\partial_2\partial_0\overline{h}_{02} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{11} + \Omega^2\partial_2\partial_1\overline{h}_{12} + \\ & \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{22} + 4\Omega'^2\overline{h}_{33} - 3\Omega\Omega''\overline{h}_{33} - \Omega\Omega'\partial_0\overline{h}_{33} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{33} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{33} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{33} \\ & - \frac{\Omega'\partial_0 h}{\Omega} + \tfrac{1}{2}\partial_0\partial_0 h - \tfrac{1}{2}\partial_1\partial_1 h - \tfrac{1}{2}\partial_2\partial_2 h\end{aligned} \tag{2.18}$$

$$\begin{aligned}\delta G_{01} = {} & \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{00} + 3\Omega'^2\overline{h}_{01} - 2\Omega\Omega''\overline{h}_{01} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{01} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{01} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{02} \\ & - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{03} - \Omega\Omega'\partial_1\overline{h}_{11} - \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{11} - \Omega\Omega'\partial_2\overline{h}_{12} - \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{12} - \Omega\Omega'\partial_3\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{13} \\ & + \tfrac{1}{2}\partial_1\partial_0 h\end{aligned} \tag{2.19}$$

$$\begin{aligned}\delta G_{02} = {} & \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{00} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{01} + 3\Omega'^2\overline{h}_{02} - 2\Omega\Omega''\overline{h}_{02} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{02} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{02} \\ & - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{03} - \Omega\Omega'\partial_1\overline{h}_{12} - \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{12} - \Omega\Omega'\partial_2\overline{h}_{22} - \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{22} - \Omega\Omega'\partial_3\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{23} \\ & + \tfrac{1}{2}\partial_2\partial_0 h\end{aligned} \tag{2.20}$$

$$\delta G_{03} = \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{00} - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{01} - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{02} + 3\Omega'^2\overline{h}_{03} - 2\Omega\Omega''\overline{h}_{03} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{03} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{03} - \Omega\Omega'\partial_1\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{13} - \Omega\Omega'\partial_2\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{23} - \Omega\Omega'\partial_3\overline{h}_{33} - \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{33} + \tfrac{1}{2}\partial_3\partial_0 h \tag{2.21}$$

$$\delta G_{12} = \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{01} + \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{02} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{11} + 4\Omega'^2\overline{h}_{12} - 3\Omega\Omega''\overline{h}_{12} - \Omega\Omega'\partial_0\overline{h}_{12} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{12} + \tfrac{1}{2}\Omega^2\partial_3\partial_3\overline{h}_{12} - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{22} - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{23} + \tfrac{1}{2}\partial_2\partial_1 h \tag{2.22}$$

$$\delta G_{13} = \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{01} + \tfrac{1}{2}\Omega^2\partial_1\partial_0\overline{h}_{03} - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{11} - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{12} + 4\Omega'^2\overline{h}_{13} - 3\Omega\Omega''\overline{h}_{13} - \Omega\Omega'\partial_0\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{13} + \tfrac{1}{2}\Omega^2\partial_2\partial_2\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{33} + \tfrac{1}{2}\partial_3\partial_1 h \tag{2.23}$$

$$\delta G_{23} = \tfrac{1}{2}\Omega^2\partial_3\partial_0\overline{h}_{02} + \tfrac{1}{2}\Omega^2\partial_2\partial_0\overline{h}_{03} - \tfrac{1}{2}\Omega^2\partial_3\partial_1\overline{h}_{12} - \tfrac{1}{2}\Omega^2\partial_2\partial_1\overline{h}_{13} - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{22} + 4\Omega'^2\overline{h}_{23} - 3\Omega\Omega''\overline{h}_{23} - \Omega\Omega'\partial_0\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_0\partial_0\overline{h}_{23} + \tfrac{1}{2}\Omega^2\partial_1\partial_1\overline{h}_{23} - \tfrac{1}{2}\Omega^2\partial_3\partial_2\overline{h}_{33} + \tfrac{1}{2}\partial_3\partial_2 h \tag{2.24}$$

We see that each tensor component is far away from being diagonal in the perturbation components $h_{\mu\nu}$. In order to solve these equations, we seek to find a gauge that allows the equations to become diagonalized. To this end, we may impose the most general gauge as

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + P\Omega^2\partial_\nu h + R\Omega h\partial_\nu h \tag{2.25}$$

where J, P, and R are constant coefficients that we vary. Upon taking $J = 0$, $P = 1$, and $R = -1$, the fluctation equations take a form diagonal in $h_{\mu\nu}$ up to its trace. Indeed other combinations do exist, but deviation from this configuration will result in a trace conditions that involve derivatives of the trace, where as the above choice allows us to solve the trace explicity in terms of h_{00}. To be precise, given the special gauge choice, the trace of the Einstein tensor evaulates to

$$g^{\mu\nu}\delta G_{\mu\nu} = -\frac{10\overline{h}_{00}\Omega'^2}{\Omega^6} + \frac{2h\Omega'^2}{\Omega^4} + \frac{6\overline{h}_{00}\frac{\partial^2\Omega(t)}{\partial t^2}}{\Omega^5} + \frac{3h\frac{\partial^2\Omega(t)}{\partial t^2}}{\Omega^3}. \tag{2.26}$$

In the gauge

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = \Omega^2\partial_\nu h - \Omega h\partial_\nu h, \tag{2.27}$$

the perturbed Einstein tensor has been calculated as:

$$\delta G_{00} = (2\Omega'^2 + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{00} + (-\frac{\Omega'^2}{2\Omega^2} - \frac{\Omega''}{2\Omega} - \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_0\partial_0)h \tag{2.28}$$

$$\delta G_{11} = -\Omega\Omega''\overline{h}_{00} + (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{11} + (\frac{3\Omega'^2}{2\Omega^2} - \frac{\Omega''}{2\Omega} - \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_1\partial_1)h \tag{2.29}$$

$$\delta G_{22} = -\Omega\Omega''\overline{h}_{00} + (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{22} + (\frac{3\Omega'^2}{2\Omega^2} - \frac{\Omega''}{2\Omega} - \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_2\partial_2)h \tag{2.30}$$

$$\delta G_{33} = -\Omega\Omega''\overline{h}_{00} + (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{33} + (\frac{3\Omega'^2}{2\Omega^2} - \frac{\Omega''}{2\Omega} - \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_3\partial_3)h \tag{2.31}$$

$$\delta G_{01} = (3\Omega'^2 - 2\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{01} + (-\frac{\Omega'\partial_1}{2\Omega} - \tfrac{1}{2}\partial_1\partial_0)h \tag{2.32}$$

$$\delta G_{02} = (3\Omega'^2 - 2\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{02} + (-\frac{\Omega'\partial_2}{2\Omega} - \tfrac{1}{2}\partial_2\partial_0)h \tag{2.33}$$

$$\delta G_{03} = (3\Omega'^2 - 2\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{03} + (-\frac{\Omega'\partial_3}{2\Omega} - \tfrac{1}{2}\partial_3\partial_0)h \tag{2.34}$$

$$\delta G_{12} = (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{12} - \tfrac{1}{2}\partial_2\partial_1 h \tag{2.35}$$

$$\delta G_{13} = (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{13} - \tfrac{1}{2}\partial_3\partial_1 h \tag{2.36}$$

$$\delta G_{23} = (4\Omega'^2 - 3\Omega\Omega'' + \tfrac{1}{2}\Omega^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega\Omega'\partial_0)\overline{h}_{23} - \tfrac{1}{2}\partial_3\partial_2 h \tag{2.37}$$

In the deSitter background geometry $\Omega(t) = \frac{1}{Ht}$ there exists a similar gauge that simplifies the result even further. That is, upon taking $J = 0$, $P = \frac{1}{2}$, and $R = 1$ we have

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = \tfrac{1}{2}\Omega^2\partial_\nu h + \Omega h\partial_\nu h. \tag{2.38}$$

The trace of the Einstein tensor evaluates to

$$g^{\mu\nu}\delta G_{\mu\nu} = 2H^4\overline{h}_{00}t^2 + (-2H^2 - \tfrac{1}{2}H^2\eta^{\mu\nu}\partial_\mu\partial_\nu t^2)h. \tag{2.39}$$

The tensor perturbations are then:

$$\delta G_{00} = (2H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{00} + (\frac{1}{2t^2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{\partial_0}{t})h \tag{2.40}$$

$$\delta G_{11} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{11} + (-\frac{3}{2t^2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h \tag{2.41}$$

$$\delta G_{22} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{22} + (-\frac{3}{2t^2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h \tag{2.42}$$

$$\delta G_{33} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{33} + (-\frac{3}{2t^2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h \tag{2.43}$$

$$\delta G_{01} = (3H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{01} + \frac{\partial_1 h}{2t} \tag{2.44}$$

$$\delta G_{02} = (3H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{02} + \frac{\partial_2 h}{2t} \tag{2.45}$$

$$\delta G_{03} = (3H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{03} + \frac{\partial_3 h}{2t} \tag{2.46}$$

$$\delta G_{12} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{12} \tag{2.47}$$

$$\delta G_{13} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{13} \tag{2.48}$$

$$\delta G_{23} = (4H^2 + \tfrac{1}{2}H^2t^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2t\partial_0)\overline{h}_{23} \tag{2.49}$$

2.4 Gauge Invariance

Under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, both the Einstein and energy momentum tensor transform as

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} - \mathcal{L}_e(G_{\mu\nu}) \tag{2.50}$$

$$T_{\mu\nu} \to T'_{\mu\nu} = T_{\mu\nu} - \mathcal{L}_e(T_{\mu\nu}). \tag{2.51}$$

If we then perturb the above quantities, the gravitational sector via the metric perturbation $\bar{g}_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ and the energy sector via flucations in a perfect fluid background, we arrive at

$$G'_{\mu\nu} = G^{(0)}_{\mu\nu} - \mathcal{L}_e(G^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) \tag{2.52}$$

$$T'_{\mu\nu} = T^{(0)}_{\mu\nu} - \mathcal{L}_e(T^{(0)}_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu}). \tag{2.53}$$

where the Lie derivative goes as

$$\mathcal{L}_e G_{\mu\nu} = G^\lambda{}_\mu \epsilon_{\lambda;\nu} + G^\lambda{}_\nu \epsilon_{\lambda;\mu} + G_{\mu\nu;\lambda}\epsilon^\lambda. \tag{2.54}$$

Note that all the gauge terms reside in the Lie derivative (pure gauge).In addition using the background equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(0)}_{\mu\nu}$, we may express the Lie derivative acting on the EM tensor as (Brane Localized Gravity Appendix A)

$$\mathcal{L}_e(T^{(0)}_{\mu\nu}) = R^\alpha{}_\nu \epsilon_{\alpha;\mu} + R^\alpha{}_\mu \epsilon_{\alpha;\nu} + R_{\mu\nu;\alpha}\epsilon^\alpha - \frac{1}{2}R\epsilon_{\mu;\nu} - \frac{1}{2}R\epsilon_{\nu;\mu} + g_{\mu\nu}R_{;\alpha}\epsilon^\alpha. \tag{2.55}$$

If we take the Einstein field equations to first order, we have

$$\delta G_{\mu\nu}(h_{\mu\nu}) - \left(G^\lambda{}_\mu \epsilon_{\lambda;\nu} + G^\lambda{}_\nu \epsilon_{\lambda;\mu} + G_{\mu\nu;\lambda}\epsilon^\lambda\right) \tag{2.56}$$

$$= \delta T_{\mu\nu}(h_{\mu\nu}) - \left(T^\lambda{}_\mu \epsilon_{\lambda;\nu} + T^\lambda{}_\nu \epsilon_{\lambda;\mu} + T_{\mu\nu;\lambda}\epsilon^\lambda\right). \tag{2.57}$$

It is important to observe that $\delta G_{\mu\nu}$ is not a gauge invariant quantity on its own - that is the Lie derivative which contains all gauge dependence does not vanish in the general geometry. However, if the geometry is flat the background $G^{(0)}_{\mu\nu}$ vanishes and so $\delta G_{\mu\nu}$ becomes gauge invariant on its own. But in any general geometry such as the RW metric of interest, we see that it is in fact

$$\delta G_{\mu\nu} - \delta T_{\mu\nu} \equiv \Delta G_{\mu\nu} = 0 \tag{2.58}$$

that is gauge invariant. That this is true can be seen by applying a coordinate transformation to the above, which yields

$$\Delta G_{\mu\nu} \to \Delta G_{\mu\nu} = \delta G_{\mu\nu} - \delta T_{\mu\nu} - \mathcal{L}_e(G^{(0)}_{\mu\nu} - T^{(0)}_{\mu\nu}) \tag{2.59}$$

$$= \delta G_{\mu\nu} - \delta T_{\mu\nu} \tag{2.60}$$

2.5 S.V.T. Decomposition

In perturbing the metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, we may express $h_{\mu\nu}$ as an SVT decomposition as indicated in the Appendix (decomposition by orthogonal projects), which combined with the background yields the line element

$$ds^2 = \Omega^2 \left\{ -(1+2\phi)d\tau^2 + 2(\tilde{\nabla}_i B + B_i)d\tau dx^i + \left[(1-2\psi)\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}\right] dx^i dx^j \right\} \tag{2.61}$$

where

$$\gamma^{ij}\tilde{\nabla}_i B_j = 0, \gamma^{ij}\tilde{\nabla}_i E_j = 0,\ \gamma^{ij}\tilde{\nabla}_i E_{kj} = 0,\ \gamma^{ij}E_{ij} = 0. \tag{2.62}$$

Again, the covariant derivatives are defined with respect to the 3-space background γ_{ij} and are indicated as $\tilde{\nabla}_i$.

2.5.1 General Covariant Form

In the RW background, the SVT decomposed perturbation equation take the general form (arbitrary orthogonal coordinate system, where $\mathcal{H} = \frac{\Omega'}{\Omega}$)

$$\delta G_{00} = 6\mathcal{H}\psi' + 2\mathcal{H}\gamma_{ij}\tilde{\nabla}_i\tilde{\nabla}_j(B - E') - 2\gamma^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\psi \tag{2.63}$$

$$\delta G^{(S)}_{0i} = -2\tilde{\nabla}_i\psi' - 2\mathcal{H}\tilde{\nabla}_i\phi - \mathcal{H}^2\tilde{\nabla}_i B + 2\Omega''\Omega\tilde{\nabla}_i B \tag{2.64}$$

$$\delta G^{(V)}_{0i} = \frac{1}{2}\gamma ij\tilde{\nabla}_i\tilde{\nabla}_j(B_i - E'_i) - \mathcal{H}^2 B_i + 2\frac{\Omega''}{\Omega}B_i \tag{2.65}$$

$$\begin{aligned}\delta G^{(S)}_{ij} &= -2\gamma_{ij}\psi'' + 2\mathcal{H}^2\gamma_{ij}\phi + 2\mathcal{H}^2\gamma_{ij}\psi - 4\frac{\Omega''}{\Omega}\gamma_{ij}\phi - 2\mathcal{H}\gamma_{ij}\phi' - 4\frac{\Omega''}{\Omega}\gamma_{ij}\psi - 4\mathcal{H}\gamma_{ij}\psi' \\ &\quad - 2\mathcal{H}\gamma_{ij}\gamma^{kl}\tilde{\nabla}_k\tilde{\nabla}_l(B-E)' - \gamma_{ij}\gamma^{kl}\tilde{\nabla}_k\tilde{\nabla}_l(B-E')' - \gamma_{ij}\gamma^{kl}\tilde{\nabla}_k\tilde{\nabla}_l(\phi+\psi) + 2\mathcal{H}\tilde{\nabla}_i\tilde{\nabla}_j B + \tilde{\nabla}_i\tilde{\nabla}_j B' \\ &\quad - 2\mathcal{H}^2\tilde{\nabla}_i\tilde{\nabla}_j E + 4\frac{\Omega''}{\Omega}\tilde{\nabla}_i\tilde{\nabla}_j E - 2\mathcal{H}\tilde{\nabla}_i\tilde{\nabla}_j E' - \tilde{\nabla}_i\tilde{\nabla}_j E'' + \tilde{\nabla}_i\tilde{\nabla}_j(\phi-\psi)\end{aligned} \tag{2.66}$$

$$\begin{aligned}\delta G^{(V)}_{ij} &= \mathcal{H}\tilde{\nabla}_i(B_j - E'_j) + \mathcal{H}\tilde{\nabla}_j(B_i - E'_i) + \frac{1}{2}\tilde{\nabla}_i(B_j - E'_j)' + \frac{1}{2}\tilde{\nabla}_j(B_i - E'_i)' - \mathcal{H}^2\tilde{\nabla}_i E_j - \mathcal{H}^2\tilde{\nabla}_j E_i \\ &\quad + 2\frac{\Omega''}{\Omega}\tilde{\nabla}_i E_j + 2\frac{\Omega''}{\Omega}\tilde{\nabla}_j E_i\end{aligned} \tag{2.67}$$

$$\delta G_{ij}^{(T)} = E''_{ij} + \gamma^{kl}\tilde{\nabla}_k\tilde{\nabla}_l E_{ij} - 2\mathcal{H}^2 E_{ij} - 2\mathcal{H}E'_{ij} + 4\frac{\Omega''}{\Omega}E_{ij} \tag{2.68}$$

2.5.2 Polar

In the flat background $\Omega = 1$ polar coordinate system, we choose to set $\sin\theta = p$ such that the metric takes the form

$$g_{\mu\nu}^{(0)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-p^2} & 0 \\ 0 & 0 & 0 & p^2 r^2 \end{pmatrix} \tag{2.69}$$

In this polar coodinate system, the Einstein perturbations are calculated as

$$\text{WILL INSERT IF NEEDED} \tag{2.70}$$

2.5.3 Flat Space Solutions

Refer to Mannheim SVTSolution.pdf (Sep 12) for additional details

Here we are working in flat space $\Omega = 1$, $\gamma_{ij} = \delta_{ij}$, where $\dot{A} \equiv \frac{\partial A}{\partial \tau}$. For simplicity in this section, we note that

$$\nabla^4 = \delta^{mn}\delta^{lk}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\nabla}_l\tilde{\nabla}_k$$

$$\nabla^2 = \delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k$$

The covariant derivatives are defined with respect to the 3-space background γ_{ij} and are indicated as $\tilde{\nabla}_i$.

Bianchi Identities:

First lets see how they decouple using only Bianchi identities and the decomposition of $\delta G_{\mu\nu}$. The $\nu = 0$ component of the Bianchi identity $\eta^{\mu\alpha}\partial_\alpha \delta G_{\mu\nu}$ gives

$$-\partial_0 \delta G_{00} + \delta^{ij}\nabla_j \delta G_{0i} = 0. \tag{2.71}$$

Expressed in terms of the $\delta G_{\mu\nu}$ decomposition, this is

$$2\dot{\bar{\phi}} + \nabla^2 \bar{B} = 0. \tag{2.72}$$

Time Bianchi:

$$\boxed{\nabla^2 \bar{B} = -2\dot{\bar{\phi}}}. \tag{2.73}$$

Now set $\nu = i$, the Bianchi identity gives

$$-\partial_0 \delta G_{0i} + \delta^{jk} \tilde{\nabla}_j \delta G_{ik} = 0. \tag{2.74}$$

Space Bianchi:

$$\boxed{-(\tilde{\nabla}_i \dot{\bar{B}} + \dot{\bar{B}}_i) - 2\tilde{\nabla}_i \bar{\psi} + 2\nabla_i \nabla^2 \bar{E} + \nabla^2 \bar{E}_i = 0} \tag{2.75}$$

For later use we will also take the divergence of the above:

$$-\nabla^2 (2\bar{\psi} + \dot{\bar{B}} - 2\nabla^2 \bar{E}) = 0. \tag{2.76}$$

Decompositions:

Now let us equate decompositions in $\delta G_{\mu\nu}$ to that of $h_{\mu\nu}$. Start with δG_{00}

$$\delta G_{00} : \quad \boxed{\bar{\phi} = \nabla^2 \psi} \quad \text{or} \quad \boxed{\nabla^2 \bar{B} = -2\nabla^2 \dot{\psi}} \tag{2.77}$$

Another scalar to solve for is the trace $\eta^{\mu\nu} \delta G_{\mu\nu}$

$$2\bar{\phi} - 6\bar{\psi} + 2\nabla^2 \bar{E} = 2\nabla^2 \psi + 2(\nabla^2 - 3\partial_0 \partial_0)\psi - 2\nabla^2 (\phi + \dot{B} - \ddot{E}). \tag{2.78}$$

Substituting 2.77 we arrive at

$$\boxed{-6\bar{\psi} + 2\nabla^2 \bar{E} = -6\ddot{\psi} + 2\nabla^2 \psi - 2\nabla^2 (\phi + \dot{B} - \ddot{E})} \tag{2.79}$$

To relate vectors to vectors, we look at δG_{0i} and its Laplacian,

$$\delta G_{0i} : \quad \tilde{\nabla}_i \bar{B} + \bar{B}_i = -2\tilde{\nabla}_i \dot{\psi} + \frac{1}{2} \nabla^2 (B_i - \dot{E}_i) \tag{2.80}$$

$$\tilde{\nabla}_i \nabla^2 \bar{B} + \nabla^2 \bar{B}_i = -2\tilde{\nabla}_i \nabla^2 \dot{\psi} + \frac{1}{2} \nabla^4 (B_i - \dot{E}_i). \tag{2.81}$$

Now substitute 2.77 into the above, giving

$$\boxed{\nabla^2 \bar{B}_i = \frac{1}{2}\nabla^4(B_i - \dot{E}_i)} \tag{2.82}$$

Now for the spatial components δG_{ij} we have

$$\begin{aligned} &-2\bar{\psi}\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j\bar{E} + \tilde{\nabla}_i\bar{E}_j + \tilde{\nabla}_j\bar{E}_i + 2\bar{E}_{ij} = \\ &\quad -2\ddot{\psi}\delta_{ij} - (\tilde{\nabla}_i\tilde{\nabla}_j - \delta_{ij}\nabla^2)\psi + (\tilde{\nabla}_i\tilde{\nabla}_j - \nabla^2\delta_{ij})(\phi + \dot{B} - \ddot{E}) + \frac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) + \Box E_{ij}. \end{aligned} \tag{2.83}$$

We may form a scalar by taking $\delta^{ij}\delta G_{ij}$ or $\tilde{\nabla}^i\tilde{\nabla}^j\delta G_{ij}$. The former is the trace equation, while the later is

$$\boxed{-2\nabla^2\bar{\psi} + 2\nabla^4\bar{E} = -2\nabla^2\ddot{\psi}}. \tag{2.84}$$

This is equivalent to the equation we found for the divergence of the Bianchi space identity. We can try to form a vector by taking the divergence $\nabla^j\delta G_{ij}$

$$-2\tilde{\nabla}_i\bar{\psi} + 2\tilde{\nabla}_i\nabla^2\bar{E} + \nabla^2\bar{E}_i = -2\ddot{\tilde{\nabla}}_i\psi + \frac{1}{2}\nabla^2(\dot{B}_i - \ddot{E}_i) \tag{2.85}$$

If we further take the Laplacian we can subtitute 2.84 in to give

$$\boxed{\nabla^4 \bar{E}_i = \frac{1}{2}\nabla^4(\dot{B}_i - \ddot{E}_i)} \tag{2.86}$$

In order to equate tensor components, scalar and vector parts must cancel. For the vector portions to cancel according to eq. (7), we must take $\nabla^4\delta G_{ij}$. This gives

$$\begin{aligned} &-2\nabla^4\bar{\psi}\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j\nabla^4\bar{E} + \tilde{\nabla}_i\nabla^4\bar{E}_j + \tilde{\nabla}_j\nabla^4\bar{E}_i + 2\nabla^4\bar{E}_{ij} = \\ &-2\nabla^4\ddot{\psi}\delta_{ij} - (\tilde{\nabla}_i\tilde{\nabla}_j - \delta_{ij}\nabla^2)\nabla^4\psi + (\tilde{\nabla}_i\tilde{\nabla}_j - \nabla^2\delta_{ij})\nabla^4(\phi + \dot{B} - \ddot{E}) \\ &+ \frac{1}{2}\tilde{\nabla}_i\nabla^4(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\tilde{\nabla}_j\nabla^4(\dot{B}_i - \ddot{E}_i) + \nabla^4\Box E_{ij} \end{aligned} \tag{2.87}$$

in which the vector portion of the equation is

$$\tilde{\nabla}_i\nabla^4\bar{E}_j + \tilde{\nabla}_j\nabla^4\bar{E}_i = \frac{1}{2}\tilde{\nabla}_i\nabla^4(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\tilde{\nabla}_j\nabla^4(\dot{B}_i - \ddot{E}_i). \tag{2.88}$$

This is satisfied by 2.86 and hence the vectors will drop out. However, the scalar portions

$$-2\nabla^4\bar{\psi}\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j\nabla^4\bar{E} = -2\nabla^4\ddot{\psi}\delta_{ij} - (\tilde{\nabla}_i\tilde{\nabla}_j - \delta_{ij}\nabla^2)\nabla^4\psi + (\tilde{\nabla}_i\tilde{\nabla}_j - \nabla^2\delta_{ij})\nabla^4(\phi + \dot{B} - \ddot{E}) \tag{2.89}$$

will not cancel using substitutions eq. 2.79 and 2.84. Thus I am not sure how to relate tensor components to each other.

2.5.4 Gauge Invariant Equations in RW K=0

(Notation: $\dot{A} \equiv \frac{\partial A}{\partial \tau}$, covariant derivatives with respect to γ_{ij}). Under coordinate transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu \tag{2.90}$$

where

$$\epsilon^\mu = (\epsilon^0, \tilde{\nabla}^i\epsilon + \epsilon^i), \qquad \tilde{\nabla}^i\epsilon_i = 0 \tag{2.91}$$

the components of the metric transform as

$$\tilde{\phi} = \phi - H\epsilon^0 - \dot{\epsilon}^0 \tag{2.92}$$

$$\tilde{\psi} = \psi + H\epsilon^0 \tag{2.93}$$

$$\tilde{B} = B + \epsilon^0 - \dot{\epsilon} \tag{2.94}$$

$$\tilde{E} = E - \epsilon \tag{2.95}$$

$$\tilde{E}_i = E_i - \epsilon_i \tag{2.96}$$

$$\tilde{B}_i = B_i - \dot{\epsilon}_i \tag{2.97}$$

$$\tilde{E}_{ij} = E_{ij} \tag{2.98}$$

From the above, we may form gauge invariant combinations (adding to 6 DOF):

$$\Phi = \phi - H(\dot{E} - B) - (\ddot{E} - \dot{B}) \tag{2.99}$$

$$\Psi = \psi + H(\dot{E} - B) \tag{2.100}$$

$$\mathcal{Q}_i = B_i - \dot{E}_i \tag{2.101}$$

$$E_{ij} = E_{ij} \tag{2.102}$$

By orthogonal and parallel projections to the four velocity u^μ, a generic symmetric $T_{\mu\nu}$ may be decomposed as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} + u_\nu q_\mu + u_\mu q_\nu + \pi_{\mu\nu} \tag{2.103}$$

where

$$u^\mu q_\mu = 0,\ g^{\mu\nu}\pi_{\mu\nu} = 0,\ u^\mu u_\nu u^\rho u_\sigma \pi_{\nu\sigma} = 0. \tag{2.104}$$

The conditions on $\pi_{\mu\nu}$ specify that it is traceless and orthogonal to the four velocity u^μ, i.e. $\pi_{\mu\nu} = \pi_{ij}$. We may further decompose π_{ij} as

$$\pi_{ij} = \tilde{\nabla}_i\tilde{\nabla}_j\Pi - \frac{1}{3}\nabla^2\Pi\delta_{ij} + \frac{1}{2}\tilde{\nabla}_i\Pi_j + \frac{1}{2}\tilde{\nabla}_j\Pi_i + \Pi_{ij} \tag{2.105}$$

where as expected,

$$\delta^{ij}\tilde{\nabla}_i\Pi_j = 0,\ \delta^{ij}\tilde{\nabla}_i\Pi_{jk} = 0,\ \delta^{ij}\Pi_{ij} = 0. \tag{2.106}$$

We have 2 degrees of freedom from ρ and p, 3 from q_μ, and 5 from $\pi_{\mu\nu}$ adding to 10 in total. We decompose $T_{\mu\nu}$ into a background piece and first order fluctuations:

$$T_{\mu\nu} = {}^{(0)}T_{\mu\nu} + \delta T_{\mu\nu}. \tag{2.107}$$

The scalars, according to homogeneity and isotropy of the background, may only depend on τ,

$$\rho(x^\mu) = \bar{\rho}(\tau) + \delta\rho(x^\mu) \tag{2.108}$$

$$p(x^\mu) = \bar{p}(\tau) + \delta p(x^\mu). \tag{2.109}$$

The background $T_{\mu\nu}$ is given as a perfect fluid

$$T^{(0)}_{\mu\nu} = (\bar{\rho} + \bar{p})u^{(0)}_\mu u^{(0)}_\nu + \bar{p}g^{(0)}_{\mu\nu}. \tag{2.110}$$

The first order contribution goes as

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)u^{(0)}_\mu u^{(0)}_\nu + (\bar{\rho} + \bar{p})\delta u_\mu u^{(0)}_\nu + (\bar{\rho} + \bar{p})u^{(0)}_\mu \delta u_\nu + \delta p g^{(0)}_{\mu\nu} + \bar{p}h_{\mu\nu} + u^{(0)}_\mu q_\nu + u^{(0)}_\nu q_\mu + \pi_{\mu\nu}, \tag{2.111}$$

since the background of interest (FLRW) is homogeneous and isotropic, there is no anisotropic stress $\pi_{\mu\nu}$ or vector perturbation q_μ at zeroth order and so π_{ij} and q_μ are automatically first order. Perturbing the four velocity,

$$u^\mu = \frac{1}{a}\frac{dx^\mu}{d\tau} = \bar{u}^\mu + \delta u^\mu \tag{2.112}$$

where $\bar{u}^\mu = a^{-1}\delta^\mu{}_0$ and $\delta u^i = \tilde{\nabla}^i v + v^i$ with $\tilde{\nabla}_i v^i = 0$. By normalization of the four velocity $-1 = g_{\mu\nu}u^\mu u^\nu$, we may derive the background and perturbed components of u^μ:

$$u^\mu = \frac{1}{\Omega}\left(1-\phi,\ \tilde{\nabla}^i v + v^i\right), \qquad u_\mu = \Omega\left(-1-\phi,\ \tilde{\nabla}_i v + v_i + \tilde{\nabla}_i B - B_i\right). \tag{2.113}$$

$$\delta T_{00} = \Omega^2(2\rho\phi + \delta\rho) \tag{2.114}$$

$$\delta T_{0i} = -\Omega^2\rho(\tilde{\nabla}_i v + \tilde{\nabla}_i B + v_i + B_i) - \Omega^2 p(\tilde{\nabla}_i v + v_i) \tag{2.115}$$

$$\delta T_{ij} = \Omega^2 p h_{ij} + \Omega^2\delta_{ij}\delta p + \pi_{ij} \tag{2.116}$$

Under gauge transformation 2.90, scalars transform as

$$\delta\tilde{\rho} = \delta\rho - \epsilon^0\dot{\bar{\rho}} \tag{2.117}$$

$$\delta\tilde{p} = \delta p - \epsilon^0\dot{\bar{p}} \tag{2.118}$$

and the velocity transforms as

$$\tilde{v} = v + \dot{\epsilon}, \quad \tilde{v}^i = v^i + \dot{\epsilon}^i. \tag{2.119}$$

The components of π_{ij}, that is Π, Π_i and Π_{ij} are all gauge invariant since they vanish in the background. From these transformation laws, we may form many gauge invariant quantities (omitting the bars on all background quantities now and denoting $\sigma \equiv \dot{E} - B$):

$$\delta\rho_\sigma = \delta\rho - \dot{\rho}\sigma \tag{2.120}$$

$$\delta p_\sigma = \delta p - \dot{p}\sigma \tag{2.121}$$

$$\mathcal{V} = v + \dot{E} \tag{2.122}$$

$$\mathcal{B}_i = B_i + v_i \tag{2.123}$$

$$\pi_{ij} = \pi_{ij} \tag{2.124}$$

and Π, Π_i, Π_{ij}.

$\delta G_{\mu\nu} = -8\pi G\delta T_{\mu\nu}$

Scalars:

$\underline{\delta G_{00} = -8\pi G \delta T_{00}}$:

$$\begin{aligned}\delta G_{00} &= -2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\psi - 2\mathcal{H}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\sigma + 6\mathcal{H}\dot{\psi} \\ &= -2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\Psi + 6\mathcal{H}\dot{\Psi} - 6\mathcal{H}\dot{\mathcal{H}}\sigma - 6\mathcal{H}^2\dot{\sigma}\end{aligned}$$

$$\begin{aligned}-8\pi G\delta T_{00} &= -8\pi G\Omega^2(2\rho\phi + \delta\rho) \\ &= -6\mathcal{H}^2\Phi - 6\mathcal{H}^2\dot{\sigma} - 6\mathcal{H}\dot{\mathcal{H}}\sigma - 8\pi G\Omega^2\delta\rho_\sigma\end{aligned}$$

This leads to field equation

$$\boxed{\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\Psi - 3\mathcal{H}\dot{\Psi} - 3\mathcal{H}^2\Phi - 4\pi G\Omega^2\delta\rho_\sigma = 0} \tag{2.125}$$

$\underline{\delta G_{0i} = -8\pi G \delta T_{0i}}$:

$$\begin{aligned}\delta G_{0i} &= \tilde{\nabla}_i\left(-2\dot{\psi} - 2\mathcal{H}\phi - \mathcal{H}^2 B + 2\frac{\ddot{\Omega}}{\Omega}B\right) \\ &= \tilde{\nabla}_i\left(2\dot{\Psi} + 2\dot{\mathcal{H}}\sigma + 2\dot{\mathcal{H}}B + \mathcal{H}^2 B - 2\mathcal{H}\Phi - 2\mathcal{H}^2\sigma\right)\end{aligned}$$

$$\begin{aligned}-8\pi G\delta T_{0i} &= -8\pi G\Omega^2\tilde{\nabla}_i(\rho(v+B) + pv) \\ &= \tilde{\nabla}_i\left(-v(2\dot{\mathcal{H}} + \mathcal{H}^2) + 3\mathcal{H}^2(v+B)\right)\end{aligned}$$

This leads to field equation

$$\boxed{\dot{\Psi} - \dot{\mathcal{H}}\mathcal{B} + \mathcal{H}\Phi + \mathcal{H}^2\mathcal{V} = 0} \tag{2.126}$$

$\underline{\delta G_{ij} = -8\pi G \delta T_{ij}, \quad i \neq j}$:

$$\begin{aligned}\delta G_{ij} &= \tilde{\nabla}_i\tilde{\nabla}_j\left(-\dot{\sigma} + \phi - \psi - 2\mathcal{H}^2 E - 2\mathcal{H}\sigma + 4\dot{\mathcal{H}}E + 4\mathcal{H}^2 E\right) \\ &= \tilde{\nabla}_i\tilde{\nabla}_j\left(\Phi - \Psi + 2\mathcal{H}^2 E + 4\dot{\mathcal{H}}E\right)\end{aligned}$$

$$
\begin{aligned}
-8\pi G\delta T_{0i} &= -8\pi G\Omega^2\tilde{\nabla}_i\tilde{\nabla}_j(2pE)\\
&= \tilde{\nabla}_i\tilde{\nabla}_j(2(2\dot{\mathcal{H}}+\mathcal{H}^2)E)
\end{aligned}
$$

This leads to field equation (need to recheck this result)

$$
\square \tag{2.127}
$$

$\underline{\delta^{ij}\delta G_{ij} = -8\pi G\delta^{ij}\delta T_{ij},:}$

$$
\begin{aligned}
\delta^{ij}\delta G_{ij} &= 2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\psi - 2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\mathcal{H} + 2\delta\dot{\sigma} - 2\mathcal{H}^2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE - 6\ddot{\psi} - 6\mathcal{H}\dot{\phi} + 6\mathcal{H}^2(\phi+\psi) + 4\mathcal{H}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\sigma\\
&\quad + 4(\mathcal{H}^2+\dot{\mathcal{H}})\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE - 12(\mathcal{H}^2+\dot{\mathcal{H}})(\phi+\psi) - 12\mathcal{H}\dot{\psi}\\
&= 2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k(\Psi-\Phi) - 6\ddot{\Psi} - 6\mathcal{H}^2(\Psi+\Phi) + 2\mathcal{H}^2\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE + 6\ddot{\mathcal{H}}\sigma + 6\mathcal{H}\dot{\mathcal{H}}\sigma + 4\dot{\mathcal{H}}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE - 6\mathcal{H}\dot{\Phi}\\
&\quad - 12\dot{\mathcal{H}}(\Psi+\Phi) - 12\mathcal{H}\dot{\Psi}
\end{aligned}
$$

$$
\begin{aligned}
-8\pi G\delta^{ij}\delta T_{ij} &= -8\pi G\Omega^2(-6\psi p + 2p\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE + 3\delta p)\\
&= 2(2\dot{\mathcal{H}}+\mathcal{H}^2)\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE + 6\mathcal{H}\dot{\mathcal{H}}\sigma + 6\ddot{\mathcal{H}}\sigma - 12\dot{\mathcal{H}}\Psi - 12\mathcal{H}^2\psi - 24\pi G\Omega^2\delta\rho_\sigma
\end{aligned}
$$

This leads to field equation (need to recheck this result)

$$
\square \tag{2.128}
$$

Vectors:

$\underline{\delta G_{0i} = -8\pi G\delta T_{0i}:}$

$$
\begin{aligned}
\delta G_{0i} &= -\frac{1}{2}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k(\dot{E}_i - B_i) - \mathcal{H}^2B_i + 2B_i(\dot{\mathcal{H}}+\mathcal{H}^2)\\
&= \frac{1}{2}\nabla^2\mathcal{Q}_i + B_i(2\dot{\mathcal{H}}+\mathcal{H}^2)
\end{aligned}
$$

$$
-8\pi G\delta T_{0i} = -8\pi G\Omega^2(-\rho B_i - (\rho+p)v_i)
$$

$$= 3\mathcal{H}^2(v_i + B_i) - (2\dot{\mathcal{H}} + \mathcal{H}^2)v_i$$
$$= 3\mathcal{H}^2\mathcal{B}_i - 2(\dot{\mathcal{H}} + \mathcal{H}^2)v_i$$

This leads to field equation

$$\boxed{\frac{1}{2}\nabla^2\mathcal{Q}_i - 3\mathcal{H}^2\mathcal{B}_i + (2\dot{\mathcal{H}} + \mathcal{H}^2)\mathcal{B}_i = 0} \tag{2.129}$$

<u>$\delta G_{ii} = -8\pi G\delta T_{ii}$:</u>

$$\delta G_{ii} = \tilde{\nabla}_i\left[\dot{\mathcal{Q}}_i + 2\mathcal{H}^2E_i + 2\mathcal{H}\mathcal{Q}_i + 4\dot{\mathcal{H}}E_i\right]$$
$$= \frac{1}{2}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\mathcal{Q}_i + B_i(2\dot{\mathcal{H}} + \mathcal{H}^2)$$

$$-8\pi G\delta T_{ii} = -8\pi G\Omega^2\tilde{\nabla}_i(2E_ip)$$
$$= (4\dot{\mathcal{H}} + 2\mathcal{H}^2)\tilde{\nabla}_iE_i$$

This leads to field equation

$$\boxed{\frac{1}{2}\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_k\mathcal{Q}_i - 3\mathcal{H}^2\mathcal{B}_i + (2\dot{\mathcal{H}} + \mathcal{H}^2)\mathcal{B}_i = 0} \tag{2.130}$$

Tensors:

<u>$\delta G_{ij} = -8\pi G\delta T_{ij}$:</u>

$$\delta G_{ij} = \delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE_{ij} - \ddot{E}_{ij} - 2\mathcal{H}^2E_{ij} - 2\mathcal{H}\dot{E}_{ij} + 4(\dot{\mathcal{H}} + \mathcal{H}^2)E_{ij}$$

$$-8\pi G\delta T_{ij} = -8\pi G\Omega^2(pE_{ij} + \pi_{ij})$$
$$= (2\dot{\mathcal{H}} + \mathcal{H}^2)(E_{ij} + \pi_{ij})$$

This leads to field equation

$$\boxed{\delta^{lk}\tilde{\nabla}_l\tilde{\nabla}_kE_{ij} - \ddot{E}_{ij} - 2\mathcal{H}\dot{E}_{ij} + (2\dot{\mathcal{H}} + \mathcal{H}^2)(E_{ij} - \pi_{ij})} \tag{2.131}$$

Current question is how many gauge degrees of freedom are we allowed in the entire $\delta G_{\mu\nu} - \delta T_{\mu\nu}$? It appears we have a total of 6 from the $h_{\mu\nu}$ side, and 7 from perfect fluid perturbations. In addition, we have not yet determined equations for q_i nonzero. Every literature reference sets $q_i = 0$ without explanation (and in fact they simply add in $\pi_{\mu\nu}$ purely by hand)

Friedman Equations:

Useful forms of the Friedman equations required for $\delta G_{\mu\nu} = \delta T_{\mu\nu}$:

$$\rho\frac{8\pi G\Omega}{3} = \mathcal{H}^2 \tag{2.132}$$

$$8\pi G\Omega^2\dot{\rho} = 6(\dot{\mathcal{H}}\mathcal{H} - \mathcal{H}^3) \tag{2.133}$$

$$\frac{4\pi G\Omega}{3}(\rho - 3p) = \frac{\ddot{\Omega}}{\Omega} \tag{2.134}$$

$$-(2\dot{\mathcal{H}} + \mathcal{H}^2) = 8\pi G\Omega^2 p \tag{2.135}$$

$$8\pi G\Omega^2\dot{p} = 2\mathcal{H}(2\dot{\mathcal{H}} + \mathcal{H}^2) - (2\ddot{\mathcal{H}} + 2\mathcal{H}\dot{\mathcal{H}}) \tag{2.136}$$

Pure Gauge:

After decomposing the metric into SVT as

$$ds^2 = -(1+2\phi)d\tau^2 + (\partial_i B + B_i)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}]dx^i dx^j \tag{2.137}$$

there exist transformations that leave the metric invariant. These are

$$B \to B + p \tag{2.138}$$

$$B_i \to B_i - \tilde{\nabla}_i p \tag{2.139}$$

$$E \to E + q \tag{2.140}$$

$$E_i \to E_i - \tilde{\nabla}_i q + 2r_i \tag{2.141}$$

$$E_{ij} \to E_{ij} - \nabla_i r_j - \tilde{\nabla}_j r_i. \tag{2.142}$$

In order to preserve transverness in B_i, E_i and E_{ij} we require

$$\nabla^2 p = 0 \tag{2.143}$$

$$\nabla^2 q = 0 \tag{2.144}$$

$$\nabla^2 r_i = 0 \tag{2.145}$$

$$\tilde{\nabla}^i r_i = 0. \tag{2.146}$$

The above total to four indepedent components, as expected from the four gauge depedent coordinate transformations. In forming the decomposition of the metric, we choose scalars, vectors, and tensors such that q, p, and r_i are zero, or in other words $B' = B + p$ etc.

2.5.5 Fourier Solution

Refer to Mannheim SVTSolution.pdf "Solving the SVT Puzzle for additional Fourier details

In a flat background of $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, Lie derivatives of $\delta G_{\mu\nu}$ vanish thus making $\delta G_{\mu\nu}$ gauge invariant all on its own. We will decompose $\delta G_{\mu\nu}$ by S.V.T. via the metric

$$ds^2 = -(1+2\phi)d\tau^2 + 2(B_i + \partial_i B)d\tau dx^i + \left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}\right] dx^i dx^j. \tag{2.147}$$

The perturbed Einstein tensor can be expressed in terms of the following gauge invariant quantities:

$$\begin{aligned}
\Psi &= \psi \\
\Phi &= \phi + \dot{B} - \ddot{E} \\
Q_i &= \dot{E}_i - B_i \\
E_{ij} &= E_{ij}.
\end{aligned}$$

The perturbed tensor is then

$$\delta G_{00} = -2\delta^{lk}\partial_l\partial_k\Psi \tag{2.148}$$

$$\delta G_{0i} = -2\partial_i\dot{\Psi} - \frac{1}{2}\delta^{lk}\partial_l\partial_k Q_i \tag{2.149}$$

$$\delta G_{ij} = -2\ddot{\Psi}\delta_{ij} + (\delta_{ij}\delta^{lk}\partial_l\partial_k - \partial_i\partial_j)(\Psi - \Phi) - \frac{1}{2}\left(\partial_i\dot{Q}_j + \partial_j\dot{Q}_i\right) + \Box E_{ij}. \tag{2.150}$$

The gauge invariant variables are subject to the constraints

$$\partial^i Q_i = 0, \qquad \partial^i E_{ij} = 0, \qquad \delta^{ij}E_{ij} = 0. \tag{2.151}$$

We may also decompose $\delta T_{\mu\nu}$ in a manner exactly analogous to $\delta g_{\mu\nu}$, where perturbed variables are denoted with bars. Since the zeroth order terms in $T_{\mu\nu}$ vanish, all first order terms are all automatically gauge invariant. In addition to the Einstein equation, we have the conservation of energy

$$\partial^\mu \delta T_{\mu\nu} = 0 \tag{2.152}$$

yielding the two equations

$$-2\dot{\bar{\phi}} - \delta^{lk}\partial_l\partial_k\bar{B} = 0 \tag{2.153}$$

$$-(\dot{\bar{B}}_i + \partial_i\dot{\bar{B}}) - 2\partial_i\bar{\psi} + 2\delta^{lk}\partial_l\partial_k\partial_i\bar{E} + \delta^{lk}\partial_l\partial_k\bar{E}_i = 0 \tag{2.154}$$

Let us represent each of the perturbed variables in terms of its Fourier decomposition, i.e.

$$\Psi(x,t) = \int d^3k\ e^{ikx}\hat{\Psi}(k,t) \tag{2.155}$$

$$\Phi(x,t) = \int d^3k\ e^{ikx}\hat{\Phi}(k,t) \tag{2.156}$$

$$Q_i(x,t) = \int d^3k\ e^{ikx}\hat{Q}_i(k,t) \tag{2.157}$$

$$E_{ij}(x,t) = \int d^3k\ e^{ikx}\hat{E}_{ij}(k,t) \tag{2.158}$$

where the transformed quantities are defined as usual, for example

$$\hat{\Psi}(k,t) = \int d^3x\ e^{-ikx}\Psi(x,t). \tag{2.159}$$

Now if we substitute 2.157 and 2.158 into the constraint equations we have

$$\int d^3k\ e^{ikx} ik^i \hat{Q}_i(k,t) = 0, \qquad \int d^3k\ e^{ikx} ik^i \hat{E}_{ij}(k,t) = 0, \qquad \int d^3k\ e^{ikx} \delta^{ij} \hat{E}_{ij}(k,t) = 0 \tag{2.160}$$

For arbitrary k, it should then follow that the constraints can be expressed as

$$k^i \hat{Q}_i = 0, \qquad k^i \hat{E}_{ij} = 0, \qquad \delta^{ij} \hat{E}_{ij} = 0. \tag{2.161}$$

Next we will substitute 2.155 - 2.158 into $\delta G_{\mu\nu} = \delta T_{\mu\nu}$. This yields

$$\delta G_{00} - \delta T_{00} = \int d^3k\ e^{ikx} \left[2k^2 \hat{\Psi} - \delta \hat{T}_{00} \right] = 0 \tag{2.162}$$

$$\delta G_{0i} - \delta T_{0i} = \int d^3k\ e^{ikx} \left[-2ik_i \dot{\hat{\Psi}} + \frac{1}{2} k^2 \hat{Q}_i - \delta \hat{T}_{0i} \right] = 0 \tag{2.163}$$

$$\delta G_{ij} - \delta T_{ij} = \int d^3k\ e^{ikx} \left[-2\ddot{\hat{\Psi}} \delta_{ij} - (k^2 \delta_{ij} - k_i k_j)(\hat{\Psi} - \hat{\Phi}) - \frac{1}{2} \left(ik_i \dot{\hat{Q}}_j + ik_j \dot{\hat{Q}}_i \right) - k^2 \hat{E}_{ij} - \ddot{\hat{E}}_{ij} - \delta \hat{T}_{ij} \right] = 0 \tag{2.164}$$

Again, operating under the assumption that the inverse Fourier transform of zero is zero, we directly evaluate the integrand to zero, yielding the following new set of equations

$$2k^2 \hat{\Psi} = -2\hat{\bar{\phi}} \tag{2.165}$$

$$-2ik_i \dot{\hat{\Psi}} + \frac{1}{2} k^2 \hat{Q}_i = \hat{\bar{B}}_i + ik_i \hat{\bar{B}} \tag{2.166}$$

$$-2\ddot{\hat{\Psi}} \delta_{ij} - (k^2 \delta_{ij} - k_i k_j)(\hat{\Psi} - \hat{\Phi}) - \frac{1}{2} \left(ik_i \dot{\hat{Q}}_j + ik_j \dot{\hat{Q}}_i \right) - k^2 \hat{E}_{ij} - \ddot{\hat{E}}_{ij} = -2\hat{\bar{\psi}} \delta_{ij} - 2k_i k_j \hat{\bar{E}} + ik_i \hat{\bar{E}}_j \tag{2.167}$$

$$+ ik_j \hat{\bar{E}}_i + 2\hat{\bar{E}}_{ij} - 6\ddot{\hat{\Psi}} - 2k^2(\Psi - \Phi) = -6\bar{\psi} - 2k^2 \bar{E}$$

where we have included the spatial trace as the last equation. We also have the k-space conservation equations

$$-2\dot{\hat{\bar{\phi}}} + k^2 \hat{\bar{B}} = 0 \tag{2.168}$$

$$-(\dot{\hat{\bar{B}}}_i + ik_i \dot{\hat{\bar{B}}}) - 2ik_i \hat{\bar{\psi}} - 2ik_i k^2 \hat{\bar{E}} - k^2 \hat{\bar{E}}_i = 0. \tag{2.169}$$

When looking to decompose the equations in k *space* in terms of S.V.T., we first look at δG_{0i} and must assess whether $k_i \hat{\Psi}$ is orthogonal to $\hat{Q}_i$ and $\hat{\bar{B}}_i$. The most straightforward test is to take their scalar product

$$k^i \hat{\Psi} \hat{Q}_i = 0 \tag{2.170}$$

where we have used the constraint 2.161. Clearly then $k_i\hat{\Psi}$ lies along k_i and Q_i is orthogonal to it. Since $\hat{\bar{B}}_i$ follows the same constraint equation as $\hat{Q}_i$, it is also orthogonal to $k_i\hat{\Psi}$. Alternatively, we may choose to apply k^i to 2.166 in which we arrive at the same decomposition. The result is the decomposition of scalar and vector equations:

$$-2\dot{\hat{\Psi}} = \hat{\bar{B}} \tag{2.171}$$

$$\dot{\hat{\bar{B}}} + 2\hat{\bar{\psi}} + 2k^2\hat{\bar{E}} = 0 \tag{2.172}$$

$$\frac{1}{2}k^2\hat{Q}_i = \hat{\bar{B}}_i \tag{2.173}$$

$$\dot{\hat{\bar{B}}}_i + k^2\hat{\bar{E}}_i = 0 \tag{2.174}$$

Before looking at the spatial piece δG_{ij}, we can try to solve 2.171 and compare it to the solution obtained in Mannheim SVTsolution.pdf. The solution to 2.171 is

$$-2\hat{\Psi} = \int dt\ \hat{\bar{B}} + \hat{h}(k). \tag{2.175}$$

Having solved for $\hat{\Psi}$ we can now construct $\Psi(x,t)$ as

$$-2\Psi(x,t) = -2\int d^3k\ e^{ikx}\hat{\Psi} = \int d^3k e^{ikx}\left[\int dt\ \hat{\bar{B}} + \hat{h}(k)\right] = \int dt\ \bar{B} + h(x) \tag{2.176}$$

thus

$$-2\Psi(x,t) = \int dt\ \bar{B} + h(x). \tag{2.177}$$

Compare this to the equation calculated in SVTsolution.pdf (recall $\Psi = \psi$)

$$-2\psi(x,t) = \int dt\ \bar{B} + \alpha_j x_j \int dt\ f(t) + \int dt\ g(t) + h(x) \tag{2.178}$$

where $\delta^{lk}\partial_l\partial_k h(x) = 0$.

To try to make the discrepancy more transparent, we note that the equation one obtains from solving in position space is

$$-2\delta^{lk}\partial_l\partial_k\dot{\psi} = \delta^{lk}\partial_l\partial_k\bar{B} \tag{2.179}$$

in which it follows

$$- 2\dot{\psi} = \bar{B} + A(x,t) \tag{2.180}$$

where $\delta^{lk}\partial_l\partial_k A(x,t) = 0$. The solution is then

$$- 2\psi(x,t) = \int dt\ \bar{B} + \int dt\ A(x,t) + h(x). \tag{2.181}$$

However, if we transform 2.179 into Fourier components we get

$$- 2k^2\hat{\Psi} = k^2\hat{\bar{B}} \tag{2.182}$$

which reduces to

$$- 2\hat{\Psi} = \hat{\bar{B}} \tag{2.183}$$

with the solution given in 2.177.

Related to this problem is that there seems to reside an ambiguity when we consider a vector that is both longitudinal and transverse at the same time, as in the vector $\partial_i A$ where

$$\delta^{lk}\partial_l\partial_k A = 0. \tag{2.184}$$

Decomposing A into its Fourier transform we see

$$\delta^{lk}\partial_l\partial_k \int d^3k\ e^{ikx}\hat{A}(k,t) = -\int d^3k\ e^{ikx}k^2\hat{A}(k,t) = 0 \tag{2.185}$$

and hence

$$k^2\hat{A} = 0 \tag{2.186}$$

which for arbitrary k implies that $\hat{A} = 0$. The problem then is that if we try to construct $A(x,t)$ via

$$A(x,t) = \int d^3k\ e^{ikx}\hat{A}(k,t) \tag{2.187}$$

we find that $A(x,t) = 0$ which we know is not the general solution of Laplace's equation.

My mistake here is assuming that the Fourier transform of zero is zero, for instead, it could be a delta function, *for example $\delta(x)$. Hence Fourier treatement must be done more carefully.*

A Appendix

A.1 SVT Decomposition from Orthogonal Projections

Fundamental observers are locally at rest with respect to the matter fluid. Motivated by these "preferred" frames, we seek to split a given rank 2 tensor T_{ab} into components parallel and orthogonal to a velocity vector u_μ. The rest frames locally define surfaces of constant t. The induced metric for the surfaces of simultaneity is

$$h_{ab} = g_{ab} + u_a u_b. \tag{A.1}$$

That this acts like a 3-space metric can verified by

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^c{}_c = 3, \qquad h^a{}_b u^b = 0. \tag{A.2}$$

Note that the last relation uses $h^a{}_b$ to project the components orthogonal to u^a. Likewise $U^a{}_b \equiv -u^a u_b$ projects components parallel to u_a.

We can use these projectors to decompose a tensor into components parallel and orthogonal to the local velocity. Take arbitrary symmetric rank 2 tensor T_{ab}

$$\begin{aligned}
T_{ab} &= g_a{}^c g_b{}^d T_{cd} \\
&= (h_a{}^c + U_a{}^c)(h_b{}^d + U_b{}^d) T_{cd} \\
&= h_a{}^c h_b{}^d T_{cd} - u_a (u^c h_b{}^d T_{cd}) - u_b (u^d h_a{}^c T_{cd}) + u_a u_b (u^c u^d T_{cd}).
\end{aligned}$$

This can be expressed in terms of symmetric trace free quantities

$$\begin{aligned}
T_{ab} = \frac{1}{3} h_{ab} h^{cd} T_{cd} + \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h_{ab} h^{cd} \right] T_{cd} + h^a{}_b h^c{}_d T_{[cd]} \\
- u_a (h_b{}^d T_{cd} u^c) - u_b (h_a{}^c T_{cd} u^d) + u_a u_b (u^c u^d T_{cd}).
\end{aligned}$$

where

$$T_{<ab>} \equiv \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h_{ab} h^{cd} \right] T_{cd} \tag{A.3}$$

is the symmetric trace free projection orthogonal to u_a. For a vector, we have orthogonal projection

$$V_{<a>} \equiv h_a{}^b V_a. \tag{A.4}$$

Let us take T_{ab} to be symmetric and relabel the following quantities:

$$\begin{aligned}
\rho &= u^a u^b T_{ab} \\
p &= \frac{1}{3} h^{ab} T_{ab} \\
q_a &= q_{<a>} = -h_a{}^b T_{bc} u^c = -T_{<a>b} u^b \\
\pi_{ab} &= \pi_{<ab>} = T_{<ab>}.
\end{aligned}$$

Now the energy momentum tensor may be expressed as

$$\begin{aligned}
T_{ab} &= u_a u_b \rho + h_{ab} p + u_a q_b + u_b q_a + \pi_{ab} \\
&= (\rho + p) u_a u_b + p g_{ab} + u_a q_b + u_b q_a + \pi_{ab}
\end{aligned}$$

Note that $u^a q_a = 0$, and that π_{ab} (projected symmetric traceless) has 5 components. Thus 2 scalars, 3 vector components, and 5 from the tensor give us 10 in total.

In comoving coordinates in FLRW space, the velocity vector is

$$u^a = \delta_0^a, \qquad u_a = -\delta_a^0 \tag{A.5}$$

and the only non-zero components of π_{ab} are π_{ij} (spatial). According to York (1973) we may decompose a symmetric tensor on a positive definite Riemannian space as

$$\pi_{ab} = \pi_{ab}^{TT} + \pi_{ab}^{L} + \pi_{ab}^{Tr} \tag{A.6}$$

where

$$\pi_{ab}^{Tr} = \frac{1}{3} g_{ab} g^{cd} \pi_{cd} \tag{A.7}$$

and

$$\pi_{ab}^{L} = \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c. \tag{A.8}$$

By construction

$$g^{ab} \pi_{ab}^{TT} = 0. \tag{A.9}$$

The transverse requirement leads to an equation for the vector W_a

$$-\nabla_b \pi^{ab(L)} = -\nabla_b(\pi^{ab} - \frac{1}{3}g^{ab}g_{cd}\pi^{cd}). \tag{A.10}$$

York shows that such a vector W_a must exists and is unique, up to conformal Killing vectors. Moreover, he also shows that decomposition actually holds its form under conformal transformation on the metric.

Going back to the symmetric traceless tensor π_{ab}, we may write this as

$$\begin{aligned}\pi_{ab} &= \pi_{ab}^{TT} + \pi_{ab}^{L} \\ &= \pi_{ab}^{TT} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3}g_{ab}\nabla_c W^c.\end{aligned}$$

Going back to the general symmetric tensor, if we instead make the substitutions

$$\begin{aligned}\rho &= -2\phi \\ p &= -2\left(\psi' - \frac{1}{3}\nabla^2 E\right) = \psi \\ q_a &= -(B_a + \nabla_a B); \qquad \nabla_a B^a = 0 \\ W_a &= (E_a + \nabla_a E); \qquad \nabla_a E^a = 0\end{aligned}$$

and

$$\begin{aligned}\pi_{ab} &= \pi_{ab}^{TT} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3}g_{ab}\nabla_c W^c \\ &= E_{ab} + \nabla_a E_b + \nabla_b E_a + 2\nabla_a\nabla_b E - \frac{2}{3}h_{ab}\nabla^2 E\end{aligned}$$

we then end up with the same form of the perturbation metric as given in the standard SVT decomposition:

$$T_{ab} = -2\phi u_a u_b - (B_b + \nabla_b B)u_a - (B_a + \nabla_a B)u_b - 2\gamma_{ab}\psi + \nabla_a E_b + \nabla_b E_a + +2E_{ab}. \tag{A.11}$$

In flat space the spacetime interval is

$$ds^2 = -(1+2\phi)dt^2 + 2(B_i + \nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i\nabla_j E + 2E_{ij}\right]dx^i dx^j. \tag{A.12}$$

Thus the SVT decomposition can be achieved first by orthogonal decomposition of a symmetric tensor relative to the four velocity, and then decomposing the projected symmetric trace-free portion into transverse and longitudinal components.