RW $\delta G_{\mu\nu}$ Covariant Gauge Invariant v1

Contents

Summary

We calculate the perturbed $\delta G_{\mu\nu} = -\kappa_4^2 \delta T_{\mu\nu}$ in the RW geometry based on a covariant SVT decomposition of both the metric and the energy momentum tensor. The energy momentum tensor is based on a perfect fluid background and its perturbation. Behavior of $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ under infinitesimal coordinate transformation is determined, whereby gauge invariant quantities are constructed. As seen in the gauge transformation section 4, gauge invariant quantities separate into SO(3) sectors. Use of the background equations $G^{(0)}_{\mu\nu} = -\kappa_4^2 \delta T_{\mu\nu}$ allow us to express the perturbed Einstein equation in an entirely gauge invariant manner. If a covariant generalization of the external projection method is formulated, the Einstein equations themselves will separate into separate SO(3) representations.

1 $\delta G_{\mu\nu} = -\kappa_4^2 \delta T_{\mu\nu}$ Gauge Invariant Form

We evaluate in geometry

$$ds^2 = \Omega(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] = \Omega^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

$\delta G_{00} = -\kappa_4^2 \delta T_{00}$

$$6\mathcal{H}^2\Phi - 6k\Psi + 6\dot{\Psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi = -\Omega^2\kappa_4^2\delta\rho_\sigma \tag{1.2}$$

$\delta G_{0i} = -\kappa_4^2 \delta T_{0i}$

$$-2\tilde{\nabla}_i\dot{\Psi} - 2\mathcal{H}\tilde{\nabla}_i\Phi + k\mathcal{Q}_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\mathcal{Q}_i = (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i\mathcal{V} + (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\mathcal{B}_i \tag{1.3}$$

$\delta G_{ij} = -\kappa_4^2 \delta T_{ij}$

$$\begin{aligned}
&-\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\Phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\Psi + \tilde{\nabla}_i\tilde{\nabla}_j\Phi - \tilde{\nabla}_i\tilde{\nabla}_j\Psi - 2\tilde{g}_{ij}\ddot{\Psi} - 2\tilde{g}_{ij}\mathcal{H}\dot{\Phi} - 4\tilde{g}_{ij}\mathcal{H}\dot{\Psi} - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\Phi \\
&+\mathcal{H}\tilde{\nabla}_i\mathcal{Q}_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{\mathcal{Q}}_j + \mathcal{H}\tilde{\nabla}_j\mathcal{Q}_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{\mathcal{Q}}_i - \ddot{E}_{ij} - 2\mathcal{H}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} \\
&= \quad -2k\tilde{g}_{ij}\Psi - \Omega^2\kappa_4^2\tilde{g}_{ij}\delta p_\sigma + 2kE_{ij}
\end{aligned} \tag{1.4}$$

2 Perturbed Einstein Tensor

2.1 $\delta G_{\mu\nu}$ under Conformal Transformation

Under general conformal transformation $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, the Einstein tensor transforms as

$$\begin{aligned}
G_{\mu\nu} &\to G_{\mu\nu} + S_{\mu\nu} \\
&= G_{\mu\nu} + \Omega^{-1}\left(-2\tilde{g}_{\mu\nu}\tilde{\nabla}^\lambda\tilde{\nabla}_\lambda\Omega + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega\right) + \Omega^{-2}\left(\tilde{g}_{\mu\nu}\tilde{\nabla}_\lambda\Omega\tilde{\nabla}^\lambda\Omega - 4\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega\right).
\end{aligned} \tag{2.1}$$

Perturbing the above to first order yields the transformation of $\delta G_{\mu\nu}$:

$$\delta G_{\mu\nu} \to \delta G_{\mu\nu} + \delta S_{\mu\nu}, \tag{2.2}$$

where

$$\begin{aligned}
\delta S_{\mu\nu} = &-2h_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega + \Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha h_{\mu\nu} - \tilde{g}_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha h + h_{\mu\nu}\Omega^{-2}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega \\
&+ 2\tilde{g}_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\beta h^{\alpha\beta} - \tilde{g}_{\mu\nu}h^{\alpha\beta}\Omega^{-2}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\beta\Omega + 2\tilde{g}_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\tilde{\nabla}^\beta\tilde{\nabla}^\alpha\Omega \\
&- \Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\mu h_\nu{}^\alpha - \Omega^{-1}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}_\nu h_\mu{}^\alpha.
\end{aligned} \tag{2.3}$$

Note that in the transformation of $G_{\mu\nu}$, all curvature tensors ($R_{\mu\nu}$, R) are contained within $G_{\mu\nu}$ and not $S_{\mu\nu}$. Likewise, the first order perturbation $\delta S_{\mu\nu}$ does not include any zeroth order background curvature tensors and hence has no dependence upon the 3-space curvature k.

Taking $\Omega(\tau)$, i.e.

$$ds^2 = \Omega(\tau)^2 \left[-(1+h_{00})d\tau^2 + (\tilde{g}_{ij} + h_{ij})dx^i dx^j \right], \tag{2.4}$$

with overdots denoting $\partial/\partial\tau$, $\delta S_{\mu\nu}$ takes the form under the 3+1 splitting:

$$\delta S_{00} = -\dot{h}_{00}\dot{\Omega}\Omega^{-1} - \dot{h}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a h_0{}^a, \tag{2.5}$$

$$\delta S_{0i} = -\dot{\Omega}^2 h_{0i}\Omega^{-2} + 2\ddot{\Omega}h_{0i}\Omega^{-1} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{00}, \tag{2.6}$$

$$\begin{aligned}\delta S_{ij} = &-\dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2\tilde{g}_{ij}h_{00}\Omega^{-2} - \dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} \\ &+ 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} - 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h_0{}^a + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i}.\end{aligned} \tag{2.7}$$

2.2 SVT Basis

In terms of the SVT decomposition, $\delta S_{\mu\nu}$ takes the form

$$\begin{aligned}\delta S_{00} =\ &6\dot{\psi}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}, \\ \delta S_{0i} =\ &-\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} \\ \delta S_{ij} =\ &2\dot{\Omega}^2\tilde{g}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{g}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\phi\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\psi\Omega^{-1} \\ &- 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\ &- 2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j \\ &+ 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i \\ &- 2\dot{\Omega}^2 E_{ij}\Omega^{-2} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1}\end{aligned} \tag{2.8}$$

Finally, taking their sum $\delta\bar{G}_{\mu\nu} = \delta G_{\mu\nu} + \delta S_{\mu\nu}$ yields

$$\begin{aligned}\delta\bar{G}_{00} =\ &-6k\phi - 6k\psi + 6\dot{\psi}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi, \\ \delta\bar{G}_{0i} =\ &3k\tilde{\nabla}_i B - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi \\ &+ 2kB_i - k\dot{E}_i - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i. \\ \delta\bar{G}_{ij} =\ &-2\ddot{\psi}\tilde{g}_{ij} + 2\dot{\Omega}^2\tilde{g}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{g}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\phi\Omega^{-1} \\ &- 4\ddot{\Omega}\tilde{g}_{ij}\psi\Omega^{-1} - 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\ &- \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\ &+ 2k\tilde{\nabla}_j\tilde{\nabla}_i E - 2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\ &+ \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j + k\tilde{\nabla}_i E_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j \\ &+ \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i + k\tilde{\nabla}_j E_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i \\ &+ 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i - \ddot{E}_{ij} - 2\dot{\Omega}^2 E_{ij}\Omega^{-2} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}.\end{aligned} \tag{2.9}$$

3 Perturbed Energy Momentum Tensor

We perturb the background perfect fluid to obtain

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)U_\mu^{(0)}U_\nu^{(0)} + (\rho^{(0)} + p^{(0)})(\delta U_\mu U_\nu^{(0)} + U_\mu^{(0)}\delta U_\nu) + \delta p g_{\mu\nu}^{(0)} + p^{(0)}h_{\mu\nu} \tag{3.1}$$

According to homogenity and isotropy of the background, the scalars ρ and p only depend on the conformal time

$$\rho(x^\mu) = \rho^{(0)}(\tau) + \delta\rho(x^\mu), \qquad p(x^\mu) = p^{(0)}(\tau) + \delta p(x^\mu) \tag{3.2}$$

Regarding the four velocity, we first define a δu^i as

$$\delta U^i = \Omega^{-1}\delta u^i \tag{3.3}$$

and then decompose δu^i by defining the scalar

$$v = \int d^3y D(x-y)\tilde{\nabla}_i\delta u^i \tag{3.4}$$

to allow us to express δu^i as

$$\delta u^i = v^i + \tilde{\nabla}^i v. \tag{3.5}$$

Upon using $g^{\mu\nu}U_\mu U_\nu = -1$ it follows that

$$\begin{aligned} & g_{\mu\nu}(U^\mu \delta U^\nu + \delta U^\mu U^\nu) + h_{\mu\nu}U^\mu U^\nu = 0 \\ \rightarrow \qquad & \delta U^0 = -\Omega^{-1}\phi. \end{aligned} \tag{3.6}$$

Hence

$$\delta U^\mu = \Omega^{-1}(-\phi, v^i + \tilde{\nabla}^i v). \tag{3.7}$$

We do not lower with the metric directly, rather we must use

$$\delta U_\mu = \delta(g_{\mu\nu}U^\nu), \qquad \begin{aligned} \delta U_\mu &= \Omega(-\phi, \tilde{g}_{ij}v^i + \tilde{g}_{ij}\tilde{\nabla}^i v + B_i + \tilde{\nabla}_i B) \\ &\equiv \Omega(-\phi, v_i + \tilde{\nabla}_i v + B_i + \tilde{\nabla}_i B) \end{aligned} \tag{3.8}$$

Now we compose $\delta T_{\mu\nu}$:

$$\begin{aligned} \delta T_{00} &= \Omega^2(\delta\rho + 2\rho\phi) \\ \delta T_{0i} &= -\Omega^2\left[(\rho+p)(v_i + \tilde{\nabla}_i v) + \rho(B_i + \tilde{\nabla}_i B)\right] \\ \delta T_{ij} &= \Omega^2\left[\tilde{g}_{ij}\delta p + p(-2\psi\tilde{g}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij})\right] \end{aligned} \tag{3.9}$$

(Curiously, if we had instead started with a decomposition upon covariant δU_μ, the lack of B_i and $\nabla_i B$ causes a new problem with establishing the gauge transformation of v and v_i within $\delta\bar{T}_{0i} = \delta T_{0i} + \Delta_\epsilon T_{0i}$, as we lack a B term to cancel gauge term T and likewise for $\dot{L}_i$.)

4 Gauge Transformations

Under coordinate transformation $x^\mu \rightarrow \bar{x}^\mu = x^\mu - \epsilon^\mu(x)$ a general tensor A (of arbitrary rank) will transform as

$$\bar{A} = A + \Delta_\epsilon A. \tag{4.1}$$

If A consists of a background and a perturbation, then to first order it follows that

$$\delta\bar{A} = \delta A + \Delta_\epsilon A^{(0)}. \tag{4.2}$$

In the RW geometry we take the general $\epsilon_\mu(x)$ as $\epsilon_\mu = \Omega^2 f_\mu$ with

$$f_0 = -T, \qquad f_i = L_i + \tilde{\nabla}_i L, \qquad \tilde{g}^{ij}\tilde{\nabla}_j L_i = \tilde{\nabla}^i L_i = 0. \tag{4.3}$$

It will be helpful to calculate $\nabla_\mu\epsilon_\nu$ in terms of f_μ,

$$\begin{aligned} \nabla_\mu\epsilon_\nu &= \partial_\mu\epsilon_\nu - \Gamma^\lambda_{\mu\nu}\epsilon_\lambda \\ &= \partial_\mu\epsilon_\nu - \epsilon_\lambda\left[\tilde{\Gamma}^\lambda_{\mu\nu} + \Omega^{-1}(\delta^\lambda_\mu\partial_\nu + \delta^\lambda_\nu\partial_\mu - \tilde{g}_{\mu\nu}\tilde{g}^{\lambda\rho}\partial_\rho)\Omega\right]) \\ &= \Omega^2\tilde{\nabla}_\mu f_\nu + (f_\nu\delta^0_\mu - f_\mu\delta^0_\nu + \tilde{g}_{\mu\nu}T)\Omega\dot{\Omega} \end{aligned} \tag{4.4}$$

Since $\tilde{\Gamma}^\lambda_{\mu\nu} = 0$ for any time index, we have

$$\tilde{\nabla}_0 f_0 = -\dot{T}, \qquad \tilde{\nabla}_0 f_i = \dot{L}_i + \tilde{\nabla}_i\dot{L}, \qquad \tilde{\nabla}_i f_0 = -\tilde{\nabla}_i T, \qquad \tilde{\nabla}_i f_j = \tilde{\nabla}_i L_j + \tilde{\nabla}_i\tilde{\nabla}_j L \tag{4.5}$$

and consequently

$$\nabla_0\epsilon_0 = -\dot{T}\Omega^2 - T\Omega\dot{\Omega}, \qquad \nabla_0\epsilon_i = \Omega^2(\dot{L}_i + \tilde{\nabla}_i\dot{L}) + (L_i + \tilde{\nabla}_i L)\Omega\dot{\Omega}, \qquad \nabla_i\epsilon_0 = -\Omega^2\tilde{\nabla}_i T - (L_i + \tilde{\nabla}_i L)\Omega\dot{\Omega} \tag{4.6}$$

$$\nabla_i\epsilon_j = \Omega^2(\tilde{\nabla}_i L_j + \tilde{\nabla}_i\tilde{\nabla}_j L) + \tilde{g}_{ij}T\Omega\dot{\Omega} \tag{4.7}$$

4.1 Metric

For the metric

$$\begin{aligned}\Delta_\epsilon g_{\mu\nu} &= \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \\ &= \Omega^2(\tilde{\nabla}_\mu f_\nu + \tilde{\nabla}_\nu f_\mu) + 2\tilde{g}_{\mu\nu} T\Omega\dot{\Omega} \end{aligned} \tag{4.8}$$

and thus

$$\begin{aligned}\Delta_\epsilon g_{00} &= -2\dot{T}\Omega^2 - 2T\Omega\dot{\Omega} \\ \Delta_\epsilon g_{0i} &= \Omega^2(\dot{L}_i + \tilde{\nabla}_i(\dot{L} - T)) \\ \Delta_\epsilon g_{ij} &= \Omega^2(\tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L) + 2\tilde{g}_{ij} T\Omega\dot{\Omega}. \end{aligned} \tag{4.9}$$

From the Lie derivatives we can now form three equations in terms of gauge transformed SVT quantities viz $\bar{h}_{\mu\nu} = h_{\mu\nu} + \Delta_\epsilon g_{\mu\nu}$. These are

$$\begin{aligned}\bar{\phi} &= \phi + \dot{T} + T\Omega^{-1}\dot{\Omega} \\ \bar{B}_i + \tilde{\nabla}_i \bar{B} &= B_i + \tilde{\nabla}_i B + \dot{L}_i + \tilde{\nabla}_i(\dot{L} - T) \\ -2\bar{\psi}\tilde{g}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j \bar{E} + \tilde{\nabla}_i \bar{E}_j + \tilde{\nabla}_j \bar{E}_i + 2\bar{E}_{ij} &= -2\psi\tilde{g}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \\ &\quad + \tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + 2\tilde{g}_{ij} T\Omega^{-1}\dot{\Omega} \end{aligned} \tag{4.10}$$

If surface terms are to vanish, then we may take the trace and longitudinal projections upon the above equations to yield direct relations among SVT quantities. This will involve covariant derivative commutation (see (B.3)) and consequently requires the use of a scalar and vector curved space Green's functions behaving as $(\tilde{\nabla}^2 + 3k)D(x-y) \sim \delta(x-y)$ and $(\tilde{\nabla}^2 + 2k)D_{ij'}(x-x') \sim \delta_{ij'}(x-x')$ (see Greens_Functions_Cov). Denoting $\mathcal{H} \equiv \dot{\Omega}/\Omega$ we have

$$\begin{aligned}\bar{\phi} &= \phi + \dot{T} + \mathcal{H}T \\ \bar{\psi} &= \psi - \mathcal{H}T \\ \bar{B} &= B + \dot{L} - T \\ \bar{E} &= E + L \\ \bar{B}_i &= B_i + \dot{L}_i \\ \bar{E}_i &= E_i + L_i \\ \bar{E}_{ij} &= E_{ij} \end{aligned} \tag{4.11}$$

Gauge invariant quantities are

$$\begin{aligned}\Phi &= \phi + \mathcal{H}(B - \dot{E}) + (\dot{B} - \ddot{E}) \\ \Psi &= \psi - \mathcal{H}(B - \dot{E}) \\ \mathcal{Q}_i &= B_i - \dot{E}_i \\ E_{ij} &= E_{ij} \end{aligned} \tag{4.12}$$

(This presents a slightly different, but computationally more conveinent form from those defined in APM-CPII. Instead we take combinations of)

$$\begin{aligned}\Phi &= \phi + (\dot{B} - \ddot{E}) + \Omega^{-1}\dot{\Omega}(B - \dot{E}) - \Omega^{-1}\tilde{\nabla}^i\Omega(E_i + \tilde{\nabla}_i E) \\ \Psi &= \psi - \Omega^{-1}\dot{\Omega}(B - \dot{E}) + \Omega^{-1}\tilde{\nabla}^i\Omega(E_i + \tilde{\nabla}_i E) \end{aligned} \tag{4.13}$$

4.2 Energy Momentum Tensor

According to (4.2) the perturbed Energy Momentum tensor will transform as a Lie derivative

$$\bar{\delta}T_{\mu\nu} = \delta T_{\mu\nu} + \Delta_\epsilon T_{\mu\nu} \tag{4.14}$$

where $T_{\mu\nu}$ implies the background. For a perfect fluid we find in Weinberg pg. 292

$$\begin{aligned}\Delta_\epsilon T_{\mu\nu} &= T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + T^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \epsilon_\lambda \nabla^\lambda T_{\mu\nu} \\ &= p\Delta_\epsilon g_{\mu\nu} + g_{\mu\nu}\Delta_\epsilon p + (\rho + p)\left[U^{(0)}_\mu \Delta_\epsilon U^{(0)}_\nu + \Delta_\epsilon U^{(0)}_\mu U^{(0)}_\nu\right] + U^{(0)}_\mu U^{(0)}_\nu(\Delta_\epsilon\rho + \Delta_\epsilon p). \end{aligned} \tag{4.15}$$

For scalars

$$\Delta_\epsilon p = \epsilon_\lambda \nabla^\lambda p = g^{\mu\nu}\epsilon_\mu \nabla_\nu p = T\dot{p} \tag{4.16}$$

For the four velocity, the only non-zero covariant derivative $\nabla_\mu U_\nu^{(0)}$ is

$$\nabla_i U_j^{(0)} = \tilde{g}_{ij}\dot{\Omega} \quad \text{because} \quad \Gamma^0_{00} = \frac{\dot{\Omega}}{\Omega}, \quad \Gamma^0_{ij} = \frac{\dot{\Omega}}{\Omega}\tilde{g}_{ij} \tag{4.17}$$

Thus

$$\Delta_\epsilon U_0^{(0)} = -\Omega\dot{T} - T\dot{\Omega}, \qquad \Delta_\epsilon U_i^{(0)} = -\Omega\tilde{\nabla}_i T. \tag{4.18}$$

Calculating the components of the gauge sector:

$$\begin{aligned}
\Delta_\epsilon T_{00} &= 2\rho(\Omega^2\dot{T} + T\Omega\dot{\Omega}) + \Omega^2 T\dot{\rho} \\
\Delta_\epsilon T_{0i} &= \Omega^2\rho\tilde{\nabla}_i T + \Omega^2 p(\dot{L}_i + \tilde{\nabla}_i \dot{L}) \\
\Delta_\epsilon T_{ij} &= \Omega^2 p(\tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + 2\tilde{g}_{ij}T\Omega^{-1}\dot{\Omega}) + \Omega^2\tilde{g}_{ij}T\dot{p}
\end{aligned} \tag{4.19}$$

From the Lie derivatives we can now form three equations in terms of gauge transformed SVT quantities viz $\bar{\delta}T_{\mu\nu} = \delta T_{\mu\nu} + \Delta_\epsilon T_{\mu\nu}$. Using the gauge transformations from the metric, the gauge dependence for $\delta T_{\mu\nu}$ is straightforward.

$$\begin{aligned}
-\Omega^2\kappa_4^2(\delta\bar{\rho} + 2\rho\bar{\phi}) &= -\Omega^2\kappa_4^2\left[\delta\rho + 2\rho\phi + 2\rho\dot{T} + 2\rho T\mathcal{H} + T\dot{\rho}\right] \\
-\Omega^2\kappa_4^2\left[-\rho(\bar{v}_i + \bar{B}_i + \nabla_i\bar{v} + \nabla_i\bar{B}) - p(\bar{v}_i + \nabla_i\bar{v})\right] &= -\Omega^2\kappa_4^2\Big[-\rho(v_i + B_i + \nabla_i v + \nabla_i B - \nabla_i T) \\
&\quad -p(v_i - \dot{L}_i + \nabla_i v - \tilde{\nabla}_i\dot{L})\Big] \\
-\Omega^2\kappa_4^2\Big[\gamma_{ij}\delta\bar{p} + p(-2\bar{\psi}\tilde{g}_{ij} + 2\nabla_i\nabla_j\bar{E} + \nabla_i\bar{E}_j + \nabla_j\bar{E}_i + 2\bar{E}_{ij})\Big] &= -\Omega^2\kappa_4^2\Big[\tilde{g}_{ij}\delta p + p(-2\psi\gamma_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i \\
&\quad +2E_{ij}) + p(\tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + 2\tilde{g}_{ij}T\mathcal{H}) \\
&\quad +\tilde{g}_{ij}T\dot{p}\Big]
\end{aligned} \tag{4.20}$$

Requiring only v to vanish on the surface, the gauge dependence is then:

$$\begin{aligned}
\delta\bar{\rho} &= \delta\rho + T\dot{\rho} \\
\bar{\delta}p &= \delta p + T\dot{p} \\
\bar{v} &= v - \dot{L} \\
\bar{v}_i &= v_i - \dot{L}_i
\end{aligned} \tag{4.21}$$

Gauge invariant forms may be solved as:

$$\begin{aligned}
\delta\rho_\sigma &= \delta\rho + \dot{\rho}(B - \dot{E}) \\
\delta p_\sigma &= \delta p + \dot{p}(B - \dot{E}) \\
\mathcal{V} &= v + \dot{E} \\
\mathcal{B}_i &= v_i + B_i
\end{aligned} \tag{4.22}$$

Appendix A Friedmann Equations

RW geometry:

$$ds^2 = \Omega^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] = \Omega^2(\tau)\left[-d\tau^2 + \tilde{g}_{ij}dx^i dx^j\right] \tag{A.1}$$

Einstein Tensor:

$$G_{00} = -\left(3k + 3\left(\frac{\dot{\Omega}}{\Omega}\right)^2\right), \qquad G_{0i} = 0, \qquad G_{ij} = \tilde{g}_{ij}\left[k - \left(\frac{\dot{\Omega}}{\Omega}\right)^2 + 2\frac{\ddot{\Omega}}{\Omega}\right] \tag{A.2}$$

EM Tensor (by conditions of homogeneity and isotropy $\rightarrow$ perfect fluid):

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{A.3}$$

$$T_{00} = \Omega^2 \rho, \qquad T_{0i} = 0, \qquad T_{ij} = \Omega^2 \tilde{g}_{ij} p \tag{A.4}$$

** As an aside, to find the background U_μ we note the proper time $d\tilde{\tau}$

$$d\tilde{\tau}^2 = -ds^2 = \Omega^2(\tau)d\tau^2\left(1 - \frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right). \tag{A.5}$$

In the RW background, coordinates are co-moving, i.e. $\frac{dx^i}{dt} = 0$ and thus $\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = 0$. This gives us the proper time as related to the conformal time

$$d\tilde{\tau} = \Omega(\tau)d\tau. \tag{A.6}$$

By definition of four velocity

$$U^\mu = \frac{dx^\mu}{d\tau}\frac{d\tau}{d\tilde{\tau}} = \Omega^{-1}\delta^\mu_0, \qquad U_\mu = -\Omega(\tau)\delta^0_\mu \quad ** \tag{A.7}$$

Friedmann equations:

$$\mathcal{H} \equiv \frac{\dot{\Omega}}{\Omega}, \qquad \dot{\mathcal{H}} = \frac{\ddot{\Omega}}{\Omega} - \left(\frac{\dot{\Omega}}{\Omega}\right)^2, \qquad \frac{\ddot{\Omega}}{\Omega} = \mathcal{H}^2 + \dot{\mathcal{H}} \tag{A.8}$$

$$G_{\mu\nu} = -\kappa_4^2 T_{\mu\nu}, \qquad \Delta_{\mu\nu} \equiv G_{\mu\nu} + \kappa_4^2 T_{\mu\nu} = 0 \tag{A.9}$$

The 00, ij, and trace are respectively

$$3k + 3\mathcal{H}^2 = \kappa_4^2 \Omega^2 \rho \tag{A.10}$$

$$\tilde{g}_{ij}(k + \mathcal{H}^2 + 2\dot{\mathcal{H}}) = -\kappa_4^2 \tilde{g}_{ij}\Omega^2 p \tag{A.11}$$

$$\frac{6}{\Omega^2}(k + \mathcal{H}^2 + \dot{\mathcal{H}}) = -\kappa_4^2(-\rho + 3p) \tag{A.12}$$

Under gauge transformations, we will find $\delta T_{\mu\nu}$ will depend on derivatives of its background quantities, and to this end we give five useful forms of the Friedmann equations:

$$\begin{aligned} 3k + 3\mathcal{H}^2 &= \kappa_4^2\Omega^2\rho \\ (k + \mathcal{H}^2 + 2\dot{\mathcal{H}}) &= -\kappa_4^2\Omega^2 p \\ 6(k + \mathcal{H}^2 + \dot{\mathcal{H}}) &= \kappa_4^2\Omega^2(\rho - 3p) \\ 6\mathcal{H}\dot{\mathcal{H}} - 6k\mathcal{H} - 6\mathcal{H}^3 &= \kappa_4^2\Omega^2\dot{\rho} \\ 2\mathcal{H}k + 2\mathcal{H}^3 + 2\mathcal{H}\dot{\mathcal{H}} - 2\ddot{\mathcal{H}} &= \kappa_4^2\Omega^2\dot{p} \end{aligned} \tag{A.13}$$

Appendix B Maximal 3-Space Geometric Quantities

With the geometry of

$$ds^2 = \tilde{g}_{ij}dx^i dx^j = \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right): \tag{B.1}$$

we have the following geometric quantities

$$\tilde{R}_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad \tilde{R}_{ij} = -2k\tilde{g}_{ij}, \qquad \tilde{R} = -6k \tag{B.2}$$

$$[\tilde{\nabla}_i, \tilde{\nabla}_j]V_k = V_m \tilde{R}^m{}_{kij} = k(\tilde{g}_{ki}\tilde{g}^m{}_j - \tilde{g}^m{}_i\tilde{g}_{kj})V_m = k(\tilde{g}_{ik}V_j - \tilde{g}_{jk}V_i) \tag{B.3}$$

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\
\Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \quad \text{with all others zero}
\end{aligned} \tag{B.4}$$

Appendix C $\delta T_{\mu\nu}$ Simplified Form

Here we insert the background equations and partially form gauge invariant quantities for $\delta T_{\mu\nu}$.

$$\begin{aligned}
-\kappa_4^2\delta T_{00} &= -\Omega^2\kappa_4^2\left[\delta\rho + 2\rho\phi\right] \\
&= -\Omega^2\kappa_4^2\delta\rho - (6k + 6\mathcal{H}^2)\phi \\
\\
-\kappa_4^2\delta T_{0i} &= -\Omega^2\kappa_4^2\left[-(\rho+p)(v_i + \tilde{\nabla}_i v) - \rho(B_i + \tilde{\nabla}_i B)\right] \\
&= -\Omega^2\kappa_4^2\left[-\rho(v_i + B_i + \tilde{\nabla}_i v + \tilde{\nabla}_i B) - p(v_i + \tilde{\nabla}_i v)\right] \\
&= (3k + 3\mathcal{H}^2)(\mathcal{B}_i + \tilde{\nabla}_i v + \tilde{\nabla}_i B) - (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})(v_i + \tilde{\nabla}_i v) \\
\\
-\kappa_4^2\delta T_{ij} &= -\Omega^2\kappa_4^2\left[\gamma_{ij}\delta p + p(-2\psi\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij})\right] \\
&= -\Omega^2\kappa_4^2\tilde{g}_{ij}\delta p + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})(-2\psi\tilde{g}_{ij} + 2\nabla_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij})
\end{aligned} \tag{C.1}$$

Appendix D $\delta G_{\mu\nu}$ Simplified Form

Here we insert the background equations and partially form gauge invariant quantities for $\delta G_{\mu\nu}$.

$$
\begin{aligned}
\delta G_{00} &= -6k\phi - 6k\psi + 6\dot{\psi}\mathcal{H} + 2\mathcal{H}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\mathcal{H}\tilde{\nabla}_a\tilde{\nabla}^a \dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi, \\
&= -6k\phi - 6k\psi + 6\dot{\psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi \\
\\
\delta G^{(S)}_{0i} &= 3k\tilde{\nabla}_i B - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi \\
&= 3k\tilde{\nabla}_i B - \mathcal{H}^2\tilde{\nabla}_i B + 2(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi \\
&= (3k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi \\
\\
\delta G^{(V)}_{0i} &= 2kB_i - k\dot{E}_i - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \\
&= 2kB_i - k\dot{E}_i - \mathcal{H}^2 B_i + 2(\mathcal{H}^2 + \dot{\mathcal{H}})B_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \\
&= k\mathcal{Q}_i + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})B_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\mathcal{Q}_i \\
\\
\delta G^{(S)}_{ij} &= -2\ddot{\psi}\tilde{g}_{ij} + 2\dot{\Omega}^2\tilde{g}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{g}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\phi\Omega^{-1} \\
&\quad -4\ddot{\Omega}\tilde{g}_{ij}\psi\Omega^{-1} - 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\
&\quad +2k\tilde{\nabla}_j\tilde{\nabla}_i E - 2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
&= -2\ddot{\psi}\tilde{g}_{ij} + 2\tilde{g}_{ij}\mathcal{H}^2\phi + 2\mathcal{H}^2\tilde{g}_{ij}\psi - 2\mathcal{H}\dot{\phi}\tilde{g}_{ij} - 4\mathcal{H}\dot{\psi}\tilde{g}_{ij} - 4(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{g}_{ij}\phi \\
&\quad -4(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{g}_{ij}\psi - 2\mathcal{H}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\mathcal{H}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + 2\mathcal{H}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\mathcal{H}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\
&\quad +2k\tilde{\nabla}_j\tilde{\nabla}_i E - 2\mathcal{H}^2\tilde{\nabla}_j\tilde{\nabla}_i E + 4(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
&= -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\Phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\Psi + \tilde{\nabla}_i\tilde{\nabla}_j\Phi - \tilde{\nabla}_i\tilde{\nabla}_j\Psi + (2k + 2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{\nabla}_i\tilde{\nabla}_j E \\
&\quad -2\tilde{g}_{ij}\ddot{\psi} - 2\tilde{g}_{ij}\mathcal{H}\dot{\phi} - 4\tilde{g}_{ij}\mathcal{H}\dot{\psi} - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\phi - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\psi \\
\\
\delta G^{(V)}_{ij} &= \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j + k\tilde{\nabla}_i E_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j \\
&\quad +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i + k\tilde{\nabla}_j E_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i \\
&\quad +2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i \\
&= \mathcal{H}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \mathcal{H}\tilde{\nabla}_i\dot{E}_j + k\tilde{\nabla}_i E_j - \mathcal{H}^2\tilde{\nabla}_i E_j + 2(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{\nabla}_i E_j \\
&\quad +\mathcal{H}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \mathcal{H}\tilde{\nabla}_j\dot{E}_i + k\tilde{\nabla}_j E_i - \mathcal{H}^2\tilde{\nabla}_j E_i \\
&\quad +2(\mathcal{H}^2 + \dot{\mathcal{H}})\tilde{\nabla}_j E_i \\
&= \mathcal{H}\nabla_i\mathcal{Q}_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{\mathcal{Q}}_j + \mathcal{H}\tilde{\nabla}_j\mathcal{Q}_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{\mathcal{Q}}_i + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i E_j + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_j E_i \\
\\
\delta G^{(T)}_{ij} &= -\ddot{E}_{ij} - 2\dot{\Omega}^2 E_{ij}\Omega^{-2} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}. \\
&= -\ddot{E}_{ij} - 2\mathcal{H}^2 E_{ij} - 2\mathcal{H}\dot{E}_{ij} + 4(\mathcal{H}^2 + \dot{H})E_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}.
\end{aligned}
\tag{D.1}
$$

Appendix E $\delta T_{\mu\nu}$ Full Form

$$
\begin{aligned}
\delta T_{00} &= \Omega^2(\delta\rho + 2\rho\phi) \\
\delta T_{0i} &= -\Omega^2\left[(\rho + p)(v_i + \tilde{\nabla}_i v) + \rho(B_i + \tilde{\nabla}_i B)\right] \\
\delta T_{ij} &= \Omega^2\left[\tilde{g}_{ij}\delta p + p(-2\psi\tilde{g}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij})\right]
\end{aligned}
\tag{E.1}
$$

Appendix F $\delta G_{\mu\nu}$ Full Form

Mathematica output.

$$\begin{aligned}
\delta G_{00} &= -6k\phi - 6k\psi + 6\dot{\psi}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a \dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi, \\
\delta G_{0i} &= 3k\tilde{\nabla}_i B - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi \\
&\quad +2kB_i - k\dot{E}_i - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i. \\
\delta G_{ij} &= -2\ddot{\psi}\tilde{g}_{ij} + 2\dot{\Omega}^2\tilde{g}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{g}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\phi\Omega^{-1} \\
&\quad -4\ddot{\Omega}\tilde{g}_{ij}\psi\Omega^{-1} - 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + g_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\
&\quad +2k\tilde{\nabla}_j\tilde{\nabla}_i E - 2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
&\quad +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j + k\tilde{\nabla}_i E_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j \\
&\quad +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i + k\tilde{\nabla}_j E_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i \\
&\quad +2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i - \ddot{E}_{ij} - 2\dot{\Omega}^2 E_{ij}\Omega^{-2} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}.
\end{aligned} \tag{F.1}$$

Appendix G $\delta G_{\mu\nu} = -\kappa_4^2\delta T_{\mu\nu}$ Algebra

This section entails the steps required to bring $\delta G_{\mu\nu} = -\kappa_4^2\delta T_{\mu\nu}$ into its entirely gauge invariant form. Scalars, vectors, and tensors are only separated for convenience (they are still coupled in the total $\Delta_{\mu\nu} = 0$).

$$\begin{aligned}
\delta G_{00} &= -\kappa_4^2\delta T_{00} \\
-6k\phi - 6k\psi + 6\dot{\psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi &= -\Omega^2\kappa_4^2\delta\rho - (6k + 6\mathcal{H}^2)\phi \\
6\mathcal{H}^2\phi - 6k\psi + 6\dot{\psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi &= -\Omega^2\kappa_4^2\delta\rho \\
\kappa_4^2\Omega^2\dot{\rho} + 6\mathcal{H}^2\Phi - 6k\Psi + 6\dot{\Psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi &= -\Omega^2\kappa_4^2\delta\rho \\
6\mathcal{H}^2\Phi - 6k\Psi + 6\dot{\Psi}\mathcal{H} - 2\tilde{\nabla}_a\tilde{\nabla}^a\Psi &= -\Omega^2\kappa_4^2\delta\rho_\sigma
\end{aligned} \tag{G.1}$$

$$\begin{aligned}
\delta G_{0i}^{(S)} &= -\kappa_4^2\delta T_{0i}^{(S)} \\
(3k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi &= (3k + 3\mathcal{H}^2)(\tilde{\nabla}_i v + \tilde{\nabla}_i B) - (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i v \\
(-2\mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i v \\
(-2\mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i(\mathcal{V} - \dot{E}) \\
(-2\mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i B + (2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i\mathcal{V} \\
(-2\mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i(B - \dot{E}) - 2\tilde{\nabla}_i\dot{\psi} - 2\mathcal{H}\tilde{\nabla}_i\phi &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i\mathcal{V} \\
-2\tilde{\nabla}_i\dot{\Psi} - 2\mathcal{H}\tilde{\nabla}_i\Phi &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\tilde{\nabla}_i\mathcal{V}
\end{aligned} \tag{G.2}$$

$$\begin{aligned}
\delta G_{0i}^{(V)} &= -\kappa_4^2\delta T_{0i}^{(V)} \\
k\mathcal{Q}_i + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})B_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\mathcal{Q}_i &= (3k + 3\mathcal{H}^2)\mathcal{B}_i - (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})v_i \\
k\mathcal{Q}_i + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\mathcal{B}_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\mathcal{Q}_i &= (3k + 3\mathcal{H}^2)\mathcal{B}_i \\
k\mathcal{Q}_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\mathcal{Q}_i &= (2k + 2\mathcal{H}^2 - 2\dot{\mathcal{H}})\mathcal{B}_i
\end{aligned} \tag{G.3}$$

$$\delta G_{ij}^{(S)} = -\kappa_4^2 \delta T_{ij}^{(S)}$$

$$-\tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Phi + \tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Psi + \tilde{\nabla}_i\nabla_j\Phi - \tilde{\nabla}_i\nabla_j\Psi + (2k + 2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{\nabla}_i\nabla_j E$$
$$-2\tilde{g}_{ij}\ddot{\psi} - 2\tilde{g}_{ij}\mathcal{H}\dot{\phi} - 4\tilde{g}_{ij}\mathcal{H}\dot{\psi} - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\phi - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\psi$$
$$= -\Omega^2\kappa_4^2\tilde{g}_{ij}\delta p + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})(-2\psi\tilde{g}_{ij} + 2\nabla_i\tilde{\nabla}_j E))$$

$$-\tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Phi + \tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Psi + \tilde{\nabla}_i\nabla_j\Phi - \tilde{\nabla}_i\nabla_j\Psi - 2\tilde{g}_{ij}\ddot{\psi} - 2\tilde{g}_{ij}\mathcal{H}\dot{\phi}$$
$$-4\tilde{g}_{ij}\mathcal{H}\dot{\Psi} - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\phi = -2k\tilde{g}_{ij}\psi - \Omega^2\kappa_4^2\tilde{g}_{ij}\delta p$$

$$-\tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Phi + \tilde{g}_{ij}\tilde{\nabla}_a\nabla^a\Psi + \tilde{\nabla}_i\nabla_j\Phi - \tilde{\nabla}_i\nabla_j\Psi - 2\tilde{g}_{ij}\ddot{\Psi} - 2\tilde{g}_{ij}\mathcal{H}\dot{\Phi}$$
$$-4\tilde{g}_{ij}\mathcal{H}\dot{\Psi} - (2\mathcal{H}^2 + 4\dot{\mathcal{H}})\tilde{g}_{ij}\Phi = -2k\tilde{g}_{ij}\Psi - \Omega^2\kappa_4^2\tilde{g}_{ij}\delta p_\sigma \tag{G.4}$$

$$\delta G_{ij}^{(V)} = -\kappa_4^2\delta T_{ij}^{(V)}$$
$$(\mathcal{H} + \tfrac{1}{2})\tilde{\nabla}_i\mathcal{Q}_j + (\mathcal{H} + \tfrac{1}{2})\tilde{\nabla}_j\mathcal{Q}_i + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_i E_j + (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})\tilde{\nabla}_j E_i$$
$$= (k + \mathcal{H}^2 + 2\dot{\mathcal{H}})(\tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i)$$
$$(\mathcal{H} + \tfrac{1}{2})\tilde{\nabla}_i\mathcal{Q}_j + (\mathcal{H} + \tfrac{1}{2})\tilde{\nabla}_j\mathcal{Q}_i = 0 \tag{G.5}$$

$$\begin{aligned}
\delta G_{ij}^{(T)} &= -\kappa_4^2\delta T_{ij}^{(T)} \\
-\ddot{E}_{ij} - 2\mathcal{H}^2 E_{ij} - 2\mathcal{H}\dot{E}_{ij} + 4(\mathcal{H}^2 + \dot{H})E_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= (2k + 2\mathcal{H}^2 + 4\dot{\mathcal{H}})E_{ij} \\
-2kE_{ij} - \ddot{E}_{ij} - 2\mathcal{H}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= 0
\end{aligned} \tag{G.6}$$

Appendix H Summarized Gauge Dependence

Transformations:

$$\begin{aligned}
\bar{\phi} &= \phi + \dot{T} + \mathcal{H}T \\
\bar{\psi} &= \psi - \mathcal{H}T \\
\bar{B} &= B + \dot{L} - T \\
\bar{E} &= E + L \\
\bar{B}_i &= B_i + \dot{L}_i \\
\bar{E}_i &= E_i + L_i \\
\bar{E}_{ij} &= E_{ij} \\
\delta\bar{\rho} &= \delta\rho + T\dot{\rho} \\
\bar{\delta}p &= \delta p + T\dot{p} \\
\bar{v} &= v - \dot{L} \\
\bar{v}_i &= v_i - \dot{L}_i
\end{aligned} \tag{H.1}$$

$$\begin{aligned}
\dot{\rho} &= (\kappa_4^2\Omega^2)^{-1}(6\mathcal{H}\dot{\mathcal{H}} - 6k\mathcal{H} - 6\mathcal{H}^3) \\
\dot{p} &= (\kappa_4^2\Omega^2)^{-1}(2\mathcal{H}k + 2\mathcal{H}^3 + 2\mathcal{H}\dot{\mathcal{H}} - 2\ddot{\mathcal{H}})
\end{aligned} \tag{H.2}$$

Invariants:

$$
\begin{aligned}
\Phi &= \phi + \mathcal{H}(B - \dot{E}) + (\dot{B} - \ddot{E}) \\
\Psi &= \psi - \mathcal{H}(B - \dot{E}) \\
\mathcal{Q}_i &= B_i - \dot{E}_i \\
E_{ij} &= E_{ij} \\
\delta\rho_\sigma &= \delta\rho + \dot{\rho}(B - \dot{E}) \\
\delta p_\sigma &= \delta p + \dot{p}(B - \dot{E}) \\
\mathcal{V} &= v + \dot{E} \\
\mathcal{B}_i &= v_i + B_i
\end{aligned}
\tag{H.3}
$$