

Cosmological Fluctuations in Standard and Conformal Gravity

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Doctoral Degree Final Examination

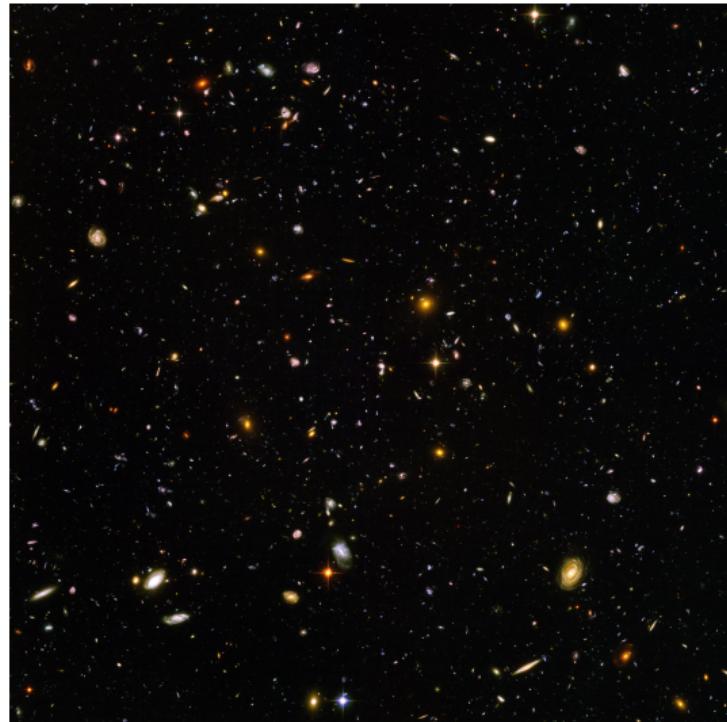


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- Introduction and Formalism
- Three Dimensional Scalar, Vector, Tensor Decomposition (SVT3)
- Four Dimensional Scalar, Vector, Tensor Decomposition (SVT4)
- Conformal Gravity (SVT and Conformal to Flat Backgrounds)
- Conformal Gravity Robertson-Walker Radiation Era Solution
- Conclusions
- Computational Methods

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker, de Sitter ($dS_4 \subset RW$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

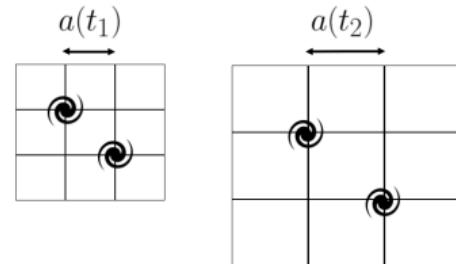
$$ds^2 = \Omega(x)^2 (-dt^2 + dx^2 + dy^2 + dz^2)$$



Hubble Ultra-Deep Field. NASA and the European Space Agency.

Comoving Robertson Walker geometry

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (1) \end{aligned}$$



- 3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \quad R_{ij} = -2k\tilde{g}_{ij}, \quad R = -6k, \quad k \in \{-1, 0, 1\} \quad (2)$$

- Define the conformal time $\tau = \int \frac{dt}{a(t)}$,

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (3)$$

- Conformal to flat form

$$ds^2 = a(\tau)^2 [-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2] \quad (k = 0) \quad (4)$$

$$ds^2 = \frac{4a^2(\tau)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2] \quad (k = 1) \quad (5)$$

$$ds^2 = \frac{4a^2(\tau)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2] \quad (k = -1) \quad (6)$$

- Einstein Hilbert action

$$I_{\text{EH}} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} g^{\mu\nu} R_{\mu\nu} \quad (7)$$

$$\frac{16\pi G}{(-g)^{1/2}} \frac{\delta I_{\text{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha, \quad \frac{2}{(-g)^{1/2}} \frac{\delta I_{\text{M}}}{\delta g_{\mu\nu}} = T_{\mu\nu} \quad (8)$$

- Einstein field equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha = -8\pi G T^{\mu\nu}, \quad (9)$$

subject to Bianchi identity

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R^\mu{}_\mu \implies \nabla_\mu G^{\mu\nu} = 0 \quad (10)$$

- Introduce fluctuation to background $g_{\mu\nu}^{(0)}$

$$g_{\mu\nu}(x) = g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x), \quad g_{(0)}^{\mu\nu} h_{\mu\nu} \equiv h \quad (11)$$

$$G_{\mu\nu} = G_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) + \delta G_{\mu\nu}(h_{\mu\nu}) \quad (12)$$

$$G_{\mu\nu}^{(0)} = R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R_{\alpha}^{(0)\alpha} \quad (13)$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R_{\alpha}^{(0)\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}_{\alpha}. \quad (14)$$

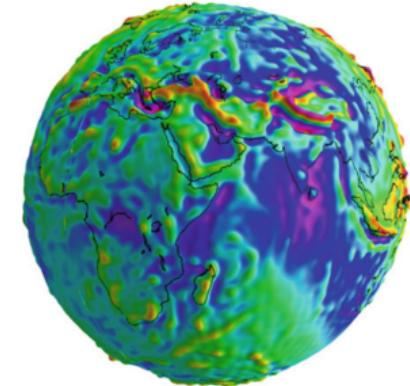
- Likewise perturb $T_{\mu\nu}$

$$T_{\mu\nu} = T_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta T_{\mu\nu}(h_{\mu\nu}) \quad (15)$$

- Form background and first order field equations

$$\Delta_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} + T_{\mu\nu}^{(0)} = 0 \quad (16)$$

$$\Delta_{\mu\nu} = \delta G_{\mu\nu}(h_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu}) = 0 \quad (17)$$



- Under $x^\mu \rightarrow x'^\mu$

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \quad (18)$$

- Under coordinate transformation $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$, with $\nabla_\mu \epsilon_\nu \sim \mathcal{O}(h)$

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) \equiv \Delta h_{\mu\nu}(x) \quad (19)$$

- Gauge transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \Delta h_{\mu\nu} \quad (20)$$

where $\Delta h_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$

- Transformation of fluctuation tensors themselves

$$\delta G_{\mu\nu} \rightarrow \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda G_{\mu\nu}^{(0)} \epsilon^\lambda \quad (21)$$

$$\delta T_{\mu\nu} \rightarrow \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda T_{\mu\nu}^{(0)} \epsilon^\lambda. \quad (22)$$

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} \quad (23)$$

- Spatial components of the perturbed Einstein tensor

$$\begin{aligned}
\delta G_{ij} = & -\frac{1}{2}\ddot{h}_{ij} + \frac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \frac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2 \tilde{g}_{ij}h_{00}\Omega^{-2} \\
& - \dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
& + 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
& - 4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
& - \tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h \tilde{\nabla}_b \Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
& + \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \frac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
& + \frac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \frac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
& - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \frac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
& + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \frac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h,
\end{aligned} \tag{24}$$

Decompose the metric perturbation $h_{\mu\nu}$ into a set of scalars, vectors, and tensors according to their transformation behavior under 3D rotations

- Define $h_{\mu\nu} = \Omega^2(x) f_{\mu\nu}$, perform 3 + 1 decomposition

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = (g_{\mu\nu}^{(0)} + h_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) (\tilde{g}_{\mu\nu}^{(0)} + f_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) [(-1 + f_{00}) dt^2 + 2f_{0i} dt dx^i + (\tilde{g}_{ij} + f_{ij})] dx^i dx^j \end{aligned} \quad (25)$$

- Decompose f_{00} , f_{0i} , and f_{ij} in terms of 3-dimensional scalars, vectors, and tensors

$$\begin{aligned} f_{00} &= -2\phi, \\ f_{0i} &= B_i + \tilde{\nabla}_i B, \\ f_{ij} &= -2\psi \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned} \quad (26)$$

with vectors and tensors obeying

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \tilde{g}^{ij} E_{ij} = 0. \quad (27)$$

$$ds^2 = \Omega^2(x) \left[-(1 + 2\phi) dt^2 + 2(B_i + \tilde{\nabla}_i B) dt dx^i + [(1 - 2\psi) \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}] dx^i dx^j \right] \quad (28)$$

- de Sitter geometry

$$ds^2 = \frac{1}{H^2 \tau^2} \left[-(1 + 2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i + [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right] \quad (29)$$

$$R_{\lambda\mu\nu\kappa} = H^2(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}), \quad R_{\mu\kappa} = -3H^2g_{\mu\kappa}, \quad R = -12H^2 \quad (30)$$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2h_{\mu\nu} = -3H^2\Omega(\tau)^2f_{\mu\nu} \quad (31)$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

SVT3 $\delta G_{\mu\nu}$ in a de Sitter Background

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2 \Omega(\tau)^2 f_{\mu\nu} \quad (29)$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

$$\begin{aligned} \delta G_{00} &= -\frac{6}{\tau}\dot{\psi} - \frac{2}{\tau}\tilde{\nabla}^2(\tau\psi + B - \dot{E}), \\ \delta G_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) + \frac{1}{\tau^2}\tilde{\nabla}_i(3B - 2\tau^2\dot{\psi} + 2\tau\phi) + \frac{3}{\tau^2}B_i, \\ \delta G_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[-2\tau^2\ddot{\psi} + 2\tau\dot{\phi} + 4\tau\dot{\psi} - 6\phi - 6\psi \right. \\ &\quad \left. + \tilde{\nabla}^2 \left(2\tau B - \tau^2\dot{B} + \tau^2\ddot{E} - 2\tau\dot{E} - \tau^2\phi + \tau^2\psi \right) \right] \\ &\quad + \frac{1}{\tau^2}\tilde{\nabla}_i\tilde{\nabla}_j \left[-2\tau B + \tau^2\dot{B} - \tau^2\ddot{E} + 2\tau\dot{E} + 6E + \tau^2\phi - \tau^2\psi \right] \\ &\quad + \frac{1}{2\tau^2}\tilde{\nabla}_i \left[-2\tau B_j + 2\tau\dot{E}_j + \tau^2\dot{B}_j - \tau^2\ddot{E}_j + 6E_j \right] \\ &\quad + \frac{1}{2\tau^2}\tilde{\nabla}_j \left[-2\tau B_i + 2\tau\dot{E}_i + \tau^2\dot{B}_i - \tau^2\ddot{E}_i + 6E_i \right] \\ &\quad - \ddot{E}_{ij} + \frac{6}{\tau^2}E_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij}, \end{aligned} \quad (30)$$

- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$

$$\begin{aligned}
 \Delta_{00} &= -\frac{6}{\tau^2}(\dot{\beta} - \alpha) - \frac{2}{\tau}\tilde{\nabla}^2\beta = 0, \\
 \Delta_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) - \frac{2}{\tau}\tilde{\nabla}_i(\dot{\beta} - \alpha) = 0, \\
 \Delta_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[-2\tau(\ddot{\beta} - \dot{\alpha}) + 6(\dot{\beta} - \alpha) + \tau\tilde{\nabla}^2(2\beta - \tau\alpha) \right] + \frac{1}{\tau}\tilde{\nabla}_i\tilde{\nabla}_j(-2\beta + \tau\alpha) \\
 &\quad + \frac{1}{2\tau}\tilde{\nabla}_i[-2(B_j - \dot{E}_j) + \tau(\dot{B}_j - \ddot{E}_j)] + \frac{1}{2\tau}\tilde{\nabla}_j[-2(B_i - \dot{E}_i) + \tau(\dot{B}_i - \ddot{E}_i)] \\
 &\quad - \ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} = 0, \\
 g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot{\beta} - \dot{\alpha}) + 24(\dot{\beta} - \alpha) + 6\tau\tilde{\nabla}^2\beta - 2\tau^2\tilde{\nabla}^2\alpha] = 0,
 \end{aligned} \tag{31}$$

where

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad \beta = \tau\psi + B - \dot{E}, \quad B_i - \dot{E}_i, \quad E_{ij}. \tag{32}$$

- Decouple scalar, vector, and tensor gauge invariants by applying higher derivatives

$$\tilde{\nabla}^4(\alpha + \dot{\beta}) = 0, \quad \tilde{\nabla}^4(\alpha - \dot{\beta}) = 0,$$

$$\tilde{\nabla}^4(B_i - \dot{E}_i) = 0,$$

$$\tilde{\nabla}^4 \left(-\ddot{E}_{ij} + \frac{2}{\tau} \dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} \right) = 0. \quad (33)$$

- Recap:

- Perturb $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$, evaluating in de Sitter background
- Decompose $h_{\mu\nu}$ into SVT3 components, inserting into fields equations
- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ to form evolution equations consisting entirely of gauge invariant quantities
- Apply higher derivatives to decouple SVT3 representations, solve

- What is the representation of the SVT3 basis in terms of $h_{\mu\nu}$? Let's take a Minkowski background,

$$\begin{aligned} h_{00} &= -2\phi, & h_{0i} &= B_i + \partial_i B \\ h_{ij} &= -2\psi\tilde{g}_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}, \end{aligned} \tag{34}$$

$$\partial^i B_i = \partial^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \partial^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0. \tag{35}$$

SVT3 Integral Formulation

Decomposition of $V_i = V_i^T + \partial_i V$

- Longitudinal decomposition does not hold for any scalar. $\partial^i V_i = \partial_i \partial^i V$
- Introduce a Green's function $\partial_i \partial^i D(x - x') = \delta^3(x - x')$ and use Green's identity

$$V(x') \partial_i \partial^i D(x - x') = D(x - x') \partial_i \partial^i V(x') + \partial_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')] \quad (36)$$

- Integrate

$$V(x) = \underbrace{\int d^3x' D(x - x') \partial_i \partial^i V(x')}_{\text{Non-Harmonic}} + \underbrace{\oint dS_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')]}_{\text{Harmonic}} \quad (37)$$

$$V = V^{NH} + V^H, \quad \partial_i \partial^i V = \partial_i \partial^i V^{NH}, \quad \partial_i \partial^i V^H = 0 \quad (38)$$

- Need a $\partial_i V$ which could never be transverse

$$\begin{aligned} V \equiv V^{NH} &= \int d^3x' D(x - x') \partial_i \partial^i V(x') = \int d^3x' D(x - x') \partial^i V_i(x') \\ &\Rightarrow \oint_{\partial V} dS_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')] = 0 \end{aligned} \quad (39)$$

- Transverse Longitudinal Decomposition

$$V_i = V_i^T + \partial_i V, \quad \partial_i V = \partial_i \int d^3x' D(x - x') \partial^j V_j(x'), \quad V_i^T = V_i - \partial_i \int d^3x' D(x - x') \partial^j V_j(x') \quad (40)$$

- Transverse Vector Decomposition

$$V_i = V_i^T + \partial_i V, \quad \partial_i V = \partial_i \int d^3x' D(x - x') \partial^j V_j(x'), \quad V_i^T = V_i - \partial_i \int d^3x' D(x - x') \partial^j V_j(x') \quad (41)$$

- Projector Formalism

$$\Pi_{ij} = \delta_{ij} - \frac{\partial}{\partial x^i} \int d^3x' D(x - x') \frac{\partial}{\partial x'^j}$$

$$\Pi_{ij} V^j = V_T^j$$

$$\Pi_{ij} \Pi^j{}_k = \Pi_{ik}, \quad \Pi_{ij} V_T^j = V_T^j, \quad \Pi_{ij} (\partial^j V) = 0 \quad (42)$$

- Hence, we can decompose h_{0i} as

$$h_{0i} = B_i + \partial_i B, \quad B = \int d^3x' D(x - x') \partial^j h_{0j}, \quad B_i = \Pi_{ij} h_0{}^j = h_{0i} - \partial_i \int d^3x' D(x - x') \partial^j h_{0j} \quad (43)$$

- Composed of non-local integrals
- B itself must vanish asymptotically (or decay sufficiently fast)

SVT3 Integral Formulation

$$h_{ij} = -2\psi\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij} \quad (44)$$

- Rank 2 tensor transverse traceless decomposition

$$h_{ij}^{TT} = h_{ij} - \partial_i W_j - \partial_j W_i + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x') (\partial^k W_k + \delta^{kl} h_{kl}) + \frac{1}{2}\delta_{ij}(\partial^k W_k - \delta^{kl} h_{kl}) \quad (45)$$

where we introduce a W_k obeying

$$\partial^j h_{ij} = \partial_k \partial^k W_i \quad (46)$$

- Can further decompose W_i into transverse and longitudinal components

$$W_i^T = W_i - \partial_i \int d^3x' D(x-x') \partial^k W_k \quad (47)$$

- Make definitions,

$$\begin{aligned} h_{ij} &= \underbrace{\left[h_{ij} - \partial_i W_j - \partial_j W_i - \frac{1}{2}g_{ij}(\delta^{kl} h_{kl} - \partial^k W_k) + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x')(\delta^{kl} h_{kl} + \partial^k W_k) \right]}_{2E_{ij}} \\ &\quad + \underbrace{\partial_i \left(W_j - \partial_j \int d^3x' D(x-x') \partial^k W_k \right)}_{E_j} + \underbrace{\partial_j \left(W_i - \partial_i \int d^3x' D(x-x') \partial^k W_k \right)}_{E_i} \\ &\quad - 2\delta_{ij} \underbrace{\left(\frac{1}{4}\partial^k W_k - \frac{1}{4}\delta^{kl} h_{kl} \right)}_{\psi} + 2\partial_i\partial_j \underbrace{\int d^3x' D(x-x') \left(\frac{3}{4}\partial^k W_k - \frac{1}{4}\delta^{kl} h_{kl} \right)}_E \end{aligned} \quad (48)$$

- Transformation behavior under full 4D coordinate transformation $x^\mu \rightarrow x'^\mu$

$$h_{\mu\nu} \rightarrow \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} h_{\alpha\beta}, \quad h_{0\mu} = \begin{pmatrix} -2\phi \\ B_1 + \partial_1 B \\ B_2 + \partial_2 B \\ B_3 + \partial_3 B \end{pmatrix} \quad (49)$$

- We seek to

- Generalize to higher dimensions. $D = 4$ to match underlying GR transformation group
- Generalize to curved space backgrounds beyond Minkowski

$$[\nabla_\kappa, \nabla_\nu] V_\lambda = V^\sigma R_{\lambda\sigma\nu\kappa} \quad (50)$$

- SVT4 Decomposition

$$h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}, \quad (51)$$

subject to

$$\nabla^\mu F_\mu = 0, \quad F_{\mu\nu} = F_{\nu\mu}, \quad g^{\mu\nu} F_{\mu\nu} = 0, \quad \nabla^\mu F_{\mu\nu} = 0. \quad (52)$$

- Maximally Symmetric Space

$$R_{\lambda\mu\nu\kappa} = \frac{R}{D(1-D)}(g_{\lambda\nu}g_{\mu\kappa} - g_{\mu\nu}g_{\lambda\kappa}), \quad R_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa}, \quad R = H^2 D(1-D) \quad (53)$$

- Curved Space Green's Function

$$\left[\nabla_\alpha \nabla^\alpha - \frac{R}{D-1} \right] D^{(A)}(x, x') = (-g)^{-1/2} \delta^{(D)}(x - x') \quad (54)$$

- Transverse traceless decomposition of rank 2 tensor

$$\begin{aligned} h_{\mu\nu}^{TT} &= h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{2-D}{D-1} \left[\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)} \right] \int d^D x' (-g)^{1/2} D^{(A)}(x, x') \nabla^\sigma W_\sigma \\ &\quad + \frac{g_{\mu\nu}}{D-1} (\nabla^\sigma W_\sigma - h) + \frac{1}{D-1} \left[\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)} \right] \int d^D x' (-g)^{1/2} D^{(A)}(x, x') h \end{aligned} \quad (55)$$

- For transverse, use commutations

$$\begin{aligned} [\nabla^\sigma, \nabla_\nu] W_\sigma &= -\frac{R}{D} W_\nu, \quad [\nabla^\mu \nabla_\mu, \nabla_\nu] V = -\frac{R}{D} \nabla_\nu V, \\ [\nabla_\sigma \nabla^\sigma, \nabla_\mu \nabla_\nu] V &= g_{\mu\nu} \left[\frac{2R}{D(D-1)} \right] \nabla_\sigma \nabla^\sigma V - \frac{2R}{D-1} \nabla_\mu \nabla_\nu V, \end{aligned} \quad (56)$$

to obtain

$$\nabla^\mu h_{\mu\nu} = \left[\nabla_\alpha \nabla^\alpha - \frac{R}{D} \right] W_\nu \quad (57)$$

- Further decompose W_μ into longitudinal and transverse components

$$W_\mu^T = W_\mu - \nabla_\mu \int d^D x' (-g)^{1/2} D^{(B)}(x, x') \nabla^\sigma W_\sigma \quad (58)$$

where

$$\nabla_\alpha \nabla^\alpha D^{(B)}(x, x') = (-g)^{-1/2} \delta^{(D)}(x - x'). \quad (59)$$

- Make definitions

$$\begin{aligned} 2F_{\mu\nu} &= h_{\mu\nu}^{TT} \\ F_\mu &= W_\mu - \nabla_\mu \int d^D x' (-g)^{1/2} D^{(B)}(x, x') \nabla^\sigma W_\sigma \\ F &= \int d^D x' (-g)^{1/2} D^{(B)}(x, x') \nabla^\sigma W_\sigma + \frac{1}{2(D-1)} \int d^D x' (-g)^{1/2} D^{(A)}(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \\ \chi &= \frac{1}{2(D-1)} \left[\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)} \int d^D x' (-g)^{1/2} D^{(A)}(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \right] \end{aligned} \quad (60)$$

- SVTD

$$h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \quad (61)$$

- In taking $D \rightarrow 3$ and $g_{\mu\nu} \rightarrow \delta_{ij}$, we recover SVT3 of h_{ij}

- de Sitter geometry

$$ds^2 = (g_{\mu\nu}^{(0)} + h_{\mu\nu})dx^\mu dx^\nu, \quad h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \quad (62)$$

- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$, with $\delta T_{\mu\nu} = -3H^2 h_{\mu\nu}$

$$\Delta_{\mu\nu} = (\nabla_\alpha \nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu + 3H^2 g_{\mu\nu})\chi = 0 \quad (63)$$

- With trace and commutation relation

$$\nabla_\alpha \nabla^\alpha \nabla_\mu \nabla_\nu \chi = \nabla_\mu \nabla_\nu \nabla_\alpha \nabla^\alpha \chi - 2H^2 g_{\mu\nu} \nabla_\alpha \nabla^\alpha \chi + 8H^2 \nabla_\mu \nabla_\nu \chi, \quad (64)$$

we decouple $F_{\mu\nu}$ by applying derivatives

$$6(\nabla_\alpha \nabla^\alpha + 4H^2)\chi = 0, \quad (\nabla_\alpha \nabla^\alpha - 4H^2)(\nabla_\alpha \nabla^\alpha - 2H^2)F_{\mu\nu} = 0. \quad (65)$$

- SVT3

$$\begin{aligned}
 \Delta_{00} &= -\frac{6}{\tau^2}(\dot{\beta} - \alpha) - \frac{2}{\tau}\tilde{\nabla}^2\beta = 0, \\
 \Delta_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) - \frac{2}{\tau}\tilde{\nabla}_i(\dot{\beta} - \alpha) = 0, \\
 \Delta_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[-2\tau(\ddot{\beta} - \dot{\alpha}) + 6(\dot{\beta} - \alpha) + \tau\tilde{\nabla}^2(2\beta - \tau\alpha) \right] + \frac{1}{\tau}\tilde{\nabla}_i\tilde{\nabla}_j(-2\beta + \tau\alpha) \\
 &\quad + \frac{1}{2\tau}\tilde{\nabla}_i[-2(B_j - \dot{E}_j) + \tau(\dot{B}_j - \ddot{E}_j)] + \frac{1}{2\tau}\tilde{\nabla}_j[-2(B_i - \dot{E}_i) + \tau(\dot{B}_i - \ddot{E}_i)] \\
 &\quad - \ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} = 0, \\
 g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot{\beta} - \dot{\alpha}) + 24(\dot{\beta} - \alpha) + 6\tau\tilde{\nabla}^2\beta - 2\tau^2\tilde{\nabla}^2\alpha] = 0
 \end{aligned} \tag{66}$$

- SVT4

$$\Delta_{\mu\nu} = (\nabla_\alpha\nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu + 3H^2g_{\mu\nu})\chi = 0 \tag{67}$$

Conformal Gravity Introduction

- Weyl Action, invariant under $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$

$$\begin{aligned} I_W &= -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\ &\equiv -2\alpha_g \int d^4x (-g)^{1/2} [R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} (R^\alpha_\alpha)^2], \end{aligned} \quad (68)$$

- Weyl Tensor

$$\begin{aligned} C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} - \frac{1}{2} (g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}) \\ &\quad + \frac{1}{6} R^\alpha_\alpha (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) \end{aligned} \quad (69)$$

- Bach Tensor (Einstein Analog)

$$\begin{aligned} -\frac{2}{(-g)^{1/2}} \frac{\delta I_W}{\delta g_{\mu\nu}} &= 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2\nabla_\kappa \nabla_\lambda C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} C^{\mu\lambda\nu\kappa} \right] \\ &\boxed{4\alpha_g \left[W_{(2)}^{\mu\nu} - \frac{1}{3} W_{(1)}^{\mu\nu} \right] = T^{\mu\nu}} \quad ^1 \end{aligned} \quad (70)$$

$$\begin{aligned} W_{(1)}^{\mu\nu} &= 2g^{\mu\nu} \nabla_\beta \nabla^\beta R^\alpha_\alpha - 2\nabla^\nu \nabla^\mu R^\alpha_\alpha - 2R^\alpha_\alpha R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (R^\alpha_\alpha)^2, \\ W_{(2)}^{\mu\nu} &= \frac{1}{2} g^{\mu\nu} \nabla_\beta \nabla^\beta R^\alpha_\alpha + \nabla_\beta \nabla^\beta R^{\mu\nu} - \nabla_\beta \nabla^\nu R^{\mu\beta} - \nabla_\beta \nabla^\mu R^{\nu\beta} \\ &\quad - 2R^{\mu\beta} R^\nu_\beta + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \end{aligned} \quad (71)$$

¹Compare to $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = T^{\mu\nu}$

- $W_{\mu\nu}$ obeys

$$g^{\mu\nu} W_{\mu\nu} = 0, \quad \nabla^\mu W_{\mu\nu} = 0 \quad (72)$$

- Under conformal transformation $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$,

$$C^\lambda{}_{\mu\nu\kappa} \rightarrow C^\lambda{}_{\mu\nu\kappa}, \quad W_{\mu\nu}(x) \rightarrow \Omega^{-2}(x)W_{\mu\nu}(x) \quad (73)$$

- Perturbed Bach tensor

$$W_{\mu\nu}(g_{\mu\nu}) = W_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) + \delta W_{\mu\nu}(h_{\mu\nu}) \quad (74)$$

- Field Equations

$$W_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) = T_{\mu\nu}^{(0)}, \quad \delta W_{\mu\nu}(h_{\mu\nu}) = \delta T_{\mu\nu}(h_{\mu\nu}) \quad (75)$$

- Perturbed Bach tensor under conformal transformation

$$\bar{h}_{\mu\nu}(x) = \Omega^2(x)h_{\mu\nu}(x), \quad \delta \bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2}(x)\delta W_{\mu\nu}(h_{\mu\nu}). \quad (76)$$

- Introduce $K_{\mu\nu}$

$$K_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{4}g_{\mu\nu}^{(0)}(x)g_{(0)}^{\alpha\beta}h_{\alpha\beta}, \quad (77)$$

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}\left(K_{\mu\nu} + \frac{h}{4}g_{\mu\nu}^{(0)}\right) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4}g_{\mu\nu}^{(0)}\right) \quad (78)$$

- From properties of conformal covariance we find

$$\delta W_{\mu\nu}\left(\frac{h}{4}g_{\mu\nu}^{(0)}\right) = -\frac{h}{4}W_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}), \quad g_{(0)}^{\mu\nu}\delta W_{\mu\nu}(h_{\mu\nu}) = h^{\mu\nu}W_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) \quad (79)$$

- For conformal to flat backgrounds, $W_{\mu\nu}^{(0)} = 0$

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}), \quad g_{(0)}^{\mu\nu}\delta W_{\mu\nu}(h_{\mu\nu}) = 0 \quad (80)$$

- General $\delta W_{\mu\nu}(K_{\mu\nu})$

$$\begin{aligned}
 \delta W_{\mu\nu}(K_{\mu\nu}) = & \frac{1}{2} K_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{2} K_{\nu}{}^{\alpha} R_{\alpha\beta} R_{\mu}{}^{\beta} - \frac{2}{3} K^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + K^{\alpha\beta} R_{\mu\alpha} R_{\nu\beta} - \frac{1}{2} K_{\mu}{}^{\alpha} R_{\alpha\beta} R_{\nu}{}^{\beta} \\
 & + \frac{1}{3} g_{\mu\nu} K^{\alpha\beta} R_{\alpha\beta} R + \frac{1}{3} K_{\nu}{}^{\alpha} R_{\mu\alpha} R + \frac{1}{3} K_{\mu}{}^{\alpha} R_{\nu\alpha} R - \frac{1}{6} K_{\mu\nu} R^2 - g_{\mu\nu} K^{\alpha\beta} R^{\gamma\kappa} R_{\alpha\gamma\beta\kappa} - \frac{2}{3} K^{\alpha\beta} R R_{\mu\alpha\nu\beta} \\
 & - K_{\nu}{}^{\alpha} R^{\beta\gamma} R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta} R_{\alpha}{}^{\gamma} R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta} R_{\alpha\gamma\beta\kappa} R_{\mu}{}^{\gamma}{}_{\nu}{}^{\kappa} - K_{\mu}{}^{\alpha} R^{\beta\gamma} R_{\nu\beta\alpha\gamma} + \frac{1}{3} R \nabla_{\alpha} \nabla^{\alpha} K_{\mu\nu} \\
 & - \frac{1}{6} K_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha} R + \frac{1}{2} R_{\nu}{}^{\alpha} \nabla_{\alpha} \nabla_{\beta} K_{\mu}{}^{\beta} + \frac{1}{2} R_{\mu}{}^{\alpha} \nabla_{\alpha} \nabla_{\beta} K_{\nu}{}^{\beta} - \frac{1}{6} \nabla_{\alpha} K_{\mu\nu} \nabla^{\alpha} R + \frac{1}{6} g_{\mu\nu} \nabla^{\alpha} R \nabla_{\beta} K_{\alpha}{}^{\beta} \\
 & - \nabla_{\alpha} K^{\alpha\beta} \nabla_{\beta} R_{\mu\nu} - \frac{2}{3} R_{\mu\nu} \nabla_{\beta} \nabla_{\alpha} K^{\alpha\beta} + \frac{1}{3} g_{\mu\nu} R \nabla_{\beta} \nabla_{\alpha} K^{\alpha\beta} - R^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} K_{\mu\nu} - K^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} R_{\mu\nu} \\
 & + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_{\beta} \nabla_{\alpha} R + \frac{1}{2} K_{\nu}{}^{\alpha} \nabla_{\beta} \nabla^{\beta} R_{\mu\alpha} + \frac{1}{2} K_{\mu}{}^{\alpha} \nabla_{\beta} \nabla^{\beta} R_{\nu\alpha} + \frac{1}{2} \nabla_{\beta} \nabla^{\beta} \nabla_{\alpha} \nabla^{\alpha} K_{\mu\nu} - \frac{1}{2} \nabla_{\beta} \nabla^{\beta} \nabla_{\mu} \nabla_{\alpha} K_{\nu}{}^{\alpha} \\
 & - \frac{1}{2} \nabla_{\beta} \nabla^{\beta} \nabla_{\nu} \nabla_{\alpha} K_{\mu}{}^{\alpha} - g_{\mu\nu} R^{\alpha\beta} \nabla_{\beta} \nabla_{\gamma} K_{\alpha}{}^{\gamma} + \nabla_{\alpha} R_{\nu\beta} \nabla^{\beta} K_{\mu}{}^{\alpha} + \nabla_{\alpha} R_{\mu\beta} \nabla^{\beta} K_{\nu}{}^{\alpha} + \frac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_{\gamma} \nabla^{\gamma} K_{\alpha\beta} \\
 & - 2R_{\mu\alpha\nu\beta} \nabla_{\gamma} \nabla^{\gamma} K^{\alpha\beta} + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_{\gamma} \nabla^{\gamma} R_{\alpha\beta} - K^{\alpha\beta} \nabla_{\gamma} \nabla^{\gamma} R_{\mu\alpha\nu\beta} + \frac{1}{6} g_{\mu\nu} \nabla_{\gamma} \nabla^{\gamma} \nabla_{\beta} \nabla_{\alpha} K^{\alpha\beta} \\
 & + \frac{1}{3} g_{\mu\nu} \nabla_{\gamma} R_{\alpha\beta} \nabla^{\gamma} K^{\alpha\beta} - 2\nabla_{\gamma} R_{\mu\alpha\nu\beta} \nabla^{\gamma} K^{\alpha\beta} + R_{\mu\beta\nu\gamma} \nabla^{\gamma} \nabla_{\alpha} K^{\alpha\beta} + R_{\mu\gamma\nu\beta} \nabla^{\gamma} \nabla_{\alpha} K^{\alpha\beta} - \nabla_{\beta} R_{\nu\alpha} \nabla_{\mu} K^{\alpha\beta} \\
 & + \frac{1}{6} \nabla^{\alpha} R \nabla_{\mu} K_{\nu\alpha} - \frac{1}{3} R \nabla_{\mu} \nabla_{\alpha} K_{\nu}{}^{\alpha} - \frac{1}{2} R_{\nu}{}^{\alpha} \nabla_{\mu} \nabla_{\beta} K_{\alpha}{}^{\beta} + R^{\alpha\beta} \nabla_{\mu} \nabla_{\beta} K_{\nu\alpha} - \nabla_{\beta} R_{\mu\alpha} \nabla_{\nu} K^{\alpha\beta} \\
 & + \frac{1}{3} \nabla_{\mu} R_{\alpha\beta} \nabla_{\nu} K^{\alpha\beta} + \frac{1}{6} \nabla^{\alpha} R \nabla_{\nu} K_{\mu\alpha} + \frac{1}{3} \nabla_{\mu} K^{\alpha\beta} \nabla_{\nu} R_{\alpha\beta} - \frac{1}{3} R \nabla_{\nu} \nabla_{\alpha} K_{\mu}{}^{\alpha} - \frac{1}{2} R_{\mu}{}^{\alpha} \nabla_{\nu} \nabla_{\beta} K_{\alpha}{}^{\beta} \\
 & + R^{\alpha\beta} \nabla_{\nu} \nabla_{\beta} K_{\mu\alpha} - \frac{2}{3} R^{\alpha\beta} \nabla_{\nu} \nabla_{\mu} K_{\alpha\beta} + \frac{1}{3} K^{\alpha\beta} \nabla_{\nu} \nabla_{\mu} R_{\alpha\beta} + \frac{1}{3} \nabla_{\nu} \nabla_{\mu} \nabla_{\beta} \nabla_{\alpha} K^{\alpha\beta}
 \end{aligned} \tag{81}$$

$\delta W_{\mu\nu}$ in Conformal to Flat Backgrounds

- Evaluate (81) in a conformal to Minkowski background

$$\begin{aligned}\delta W_{\mu\nu} = & \Omega^{-5} \partial_\alpha \partial_\nu \partial^\alpha \Omega \partial_\beta K_\mu{}^\beta + \Omega^{-5} \partial_\alpha \partial_\mu \partial^\alpha \Omega \partial_\beta K_\nu{}^\beta + 2\Omega^{-5} \partial^\alpha \partial_\nu \Omega \partial_\beta \partial_\alpha K_\mu{}^\beta \\& + 2\Omega^{-5} \partial^\alpha \partial_\mu \Omega \partial_\beta \partial_\alpha K_\nu{}^\beta + 2\Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial_\alpha \partial_\mu K_\nu{}^\beta + 2\Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial_\alpha \partial_\nu K_\mu{}^\beta \\& + \frac{1}{3} \Omega^{-4} \partial_\beta \partial_\alpha \partial_\nu \partial_\mu K^{\alpha\beta} - \frac{2}{3} K^{\alpha\beta} \Omega^{-5} \partial_\beta \partial_\alpha \partial_\nu \partial_\mu \Omega + \Omega^{-5} \partial^\alpha \partial_\nu \Omega \partial_\beta \partial^\beta K_{\mu\alpha} \\& - 2\Omega^{-5} \partial_\alpha \partial^\alpha \Omega \partial_\beta \partial^\beta K_{\mu\nu} + 6\Omega^{-6} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta \partial^\beta K_{\mu\nu} + \Omega^{-5} \partial^\alpha \partial_\mu \Omega \partial_\beta \partial^\beta K_{\nu\alpha} \\& + 3K_{\mu\nu} \Omega^{-6} \partial_\alpha \partial^\alpha \Omega \partial_\beta \partial^\beta \Omega + 12\Omega^{-6} \partial_\alpha K_{\mu\nu} \partial^\alpha \Omega \partial_\beta \partial^\beta \Omega - 24K_{\mu\nu} \Omega^{-7} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta \partial^\beta \Omega \\& - 4\Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial^\beta \partial_\alpha K_{\mu\nu} + 12K_{\mu\nu} \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial^\beta \partial_\alpha \Omega + \frac{1}{2} \Omega^{-4} \partial_\beta \partial^\beta \partial_\alpha \partial^\alpha K_{\mu\nu} \\& - K_{\mu\nu} \Omega^{-5} \partial_\beta \partial^\beta \partial_\alpha \partial^\alpha \Omega - \frac{1}{2} \Omega^{-4} \partial_\beta \partial^\beta \partial_\alpha \partial_\mu K_\nu{}^\alpha - \frac{1}{2} \Omega^{-4} \partial_\beta \partial^\beta \partial_\alpha \partial_\nu K_\mu{}^\alpha \\& - 4\Omega^{-5} \partial_\alpha K_{\mu\nu} \partial_\beta \partial^\beta \partial^\alpha \Omega + \Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial^\beta \partial_\mu K_{\nu\alpha} + \Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial^\beta \partial_\nu K_{\mu\alpha} \\& - \frac{4}{3} \Omega^{-5} \partial^\alpha \partial_\nu \Omega \partial_\beta \partial_\mu K_\alpha{}^\beta + \Omega^{-5} \partial_\alpha \partial^\alpha \Omega \partial_\beta \partial_\mu K_\nu{}^\beta - 3\Omega^{-6} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta \partial_\mu K_\nu{}^\beta \\& - 6K_\nu{}^\beta \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\mu \partial_\alpha \Omega - 3K_{\nu\alpha} \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\mu \partial^\beta \Omega - \frac{4}{3} \Omega^{-5} \partial^\alpha \partial_\mu \Omega \partial_\beta \partial_\nu K_\alpha{}^\beta \\& + \Omega^{-5} \partial_\alpha \partial^\alpha \Omega \partial_\beta \partial_\nu K_\mu{}^\beta - 3\Omega^{-6} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta \partial_\nu K_\mu{}^\beta - 6K_\mu{}^\beta \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\nu \partial_\alpha \Omega \\& - 3K_{\mu\alpha} \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\nu \partial^\beta \Omega - \frac{4}{3} \Omega^{-5} \partial^\alpha \Omega \partial_\beta \partial_\nu \partial_\mu K_\alpha{}^\beta - \frac{4}{3} \Omega^{-5} \partial_\alpha K^{\alpha\beta} \partial_\beta \partial_\nu \partial_\mu \Omega \\& + 4K_\alpha{}^\beta \Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\nu \partial_\mu \Omega - 48\Omega^{-7} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta K_{\mu\nu} \partial^\beta \Omega + 60K_{\mu\nu} \Omega^{-8} \partial_\alpha \Omega \partial^\alpha \Omega \partial_\beta \Omega \partial^\beta \Omega \\& + 12\Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\alpha K_{\mu\nu} \partial^\beta \Omega - 48K_{\mu\nu} \Omega^{-7} \partial^\alpha \Omega \partial_\beta \partial_\alpha \Omega \partial^\beta \Omega - 6\Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\mu K_{\nu\alpha} \partial^\beta \Omega \\& - 6\Omega^{-6} \partial^\alpha \Omega \partial_\beta \partial_\nu K_{\mu\alpha} \partial^\beta \Omega + 24\Omega^{-6} \partial^\alpha \Omega \partial_\beta K_{\mu\nu} \partial^\beta \partial_\alpha \Omega + K_{\nu\beta} \Omega^{-5} \partial^\beta \partial_\alpha \partial_\mu \partial^\alpha \Omega \\& + K_{\mu\beta} \Omega^{-5} \partial^\beta \partial_\alpha \partial_\nu \partial^\alpha \Omega + 2\Omega^{-5} \partial_\alpha \partial_\mu K_{\nu\beta} \partial^\beta \partial^\alpha \Omega + 2\Omega^{-5} \partial_\alpha \partial_\nu K_{\mu\beta} \partial^\beta \partial^\alpha \Omega\end{aligned}$$

$$\begin{aligned}
& -4\Omega^{-5}\partial_\beta\partial_\alpha K_{\mu\nu}\partial^\beta\partial^\alpha\Omega + 6K_{\mu\nu}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial^\beta\partial^\alpha\Omega - 6\Omega^{-6}\partial_\alpha K_{\nu\beta}\partial^\alpha\Omega\partial^\beta\partial_\mu\Omega \\
& + 2\Omega^{-5}\partial_\alpha K_{\nu\beta}\partial^\beta\partial_\mu\partial^\alpha\Omega - 6\Omega^{-6}\partial_\alpha K_{\mu\beta}\partial^\alpha\Omega\partial^\beta\partial_\nu\Omega + 2\Omega^{-5}\partial_\alpha K_{\mu\beta}\partial^\beta\partial_\nu\partial^\alpha\Omega \\
& + 2\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\gamma K_\alpha{}^\gamma - 8\eta_{\mu\nu}\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma K_\beta{}^\gamma + 4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\gamma K_\beta{}^\gamma \\
& - \frac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\gamma K_\beta{}^\gamma + 2\eta_{\mu\nu}K_\beta{}^\gamma\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial_\alpha\Omega + 4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial_\beta K_\alpha{}^\gamma \\
& - \frac{4}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial_\beta K_\alpha{}^\gamma - \frac{1}{3}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha\partial^\alpha\Omega\partial_\gamma\partial_\beta K^{\beta\gamma} + \eta_{\mu\nu}\Omega^{-6}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\gamma\partial_\beta K^{\beta\gamma} \\
& + \eta_{\mu\nu}K^{\beta\gamma}\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\gamma\partial_\beta\Omega - 4\eta_{\mu\nu}K^{\beta\gamma}\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\gamma\partial_\beta\Omega - 16\eta_{\mu\nu}K_\alpha{}^\gamma\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial_\beta\Omega \\
& - \frac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\alpha\Omega\partial_\gamma\partial_\beta\partial_\alpha K^{\beta\gamma} + 2\eta_{\mu\nu}K^{\beta\gamma}\Omega^{-6}\partial^\alpha\Omega\partial_\gamma\partial_\beta\partial_\alpha\Omega + \eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial^\gamma K_{\alpha\beta} \\
& - \frac{1}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\gamma\partial^\gamma K_{\alpha\beta} - 4\eta_{\mu\nu}K_{\alpha\beta}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma\partial^\gamma\Omega - \frac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial^\alpha\Omega\partial_\gamma\partial^\gamma\partial_\beta K_\alpha{}^\beta \\
& + \frac{1}{6}\eta_{\mu\nu}\Omega^{-4}\partial_\gamma\partial^\gamma\partial_\beta\partial_\alpha K^{\alpha\beta} + 20\eta_{\mu\nu}K_\beta{}^\gamma\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial^\gamma\Omega - 8\eta_{\mu\nu}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\gamma K_{\alpha\beta}\partial^\gamma\Omega \\
& + 2\eta_{\mu\nu}K_{\alpha\gamma}\Omega^{-6}\partial^\alpha\Omega\partial^\gamma\partial_\beta\partial^\beta\Omega + 2\eta_{\mu\nu}\Omega^{-6}\partial_\alpha K_{\beta\gamma}\partial^\alpha\Omega\partial^\gamma\partial^\beta\Omega + 4\eta_{\mu\nu}\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\alpha\gamma}\partial^\gamma\partial^\beta\Omega \\
& - \frac{1}{3}\eta_{\mu\nu}K_\beta{}^\gamma\Omega^{-5}\partial^\gamma\partial^\beta\partial_\alpha\partial^\alpha\Omega - \frac{2}{3}\eta_{\mu\nu}\Omega^{-5}\partial_\alpha K_{\beta\gamma}\partial^\gamma\partial^\beta\partial^\alpha\Omega + 4\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\nu\Omega\partial_\mu K_{\alpha\beta} \\
& - \frac{2}{3}\Omega^{-5}\partial_\beta\partial_\nu\partial_\alpha\Omega\partial_\mu K^{\alpha\beta} - 3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\mu K_{\nu\alpha} + 12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu K_{\nu\beta} \\
& - 6\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\mu K_{\nu\beta} + \Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\mu K_{\nu\beta} + 4\Omega^{-6}\partial^\alpha\partial_\nu\Omega\partial_\beta K_\alpha{}^\beta\partial_\mu\Omega \\
& - 3\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta\partial_\mu\Omega + 12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta K_\nu{}^\beta\partial_\mu\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\alpha K_\nu{}^\beta\partial_\mu\Omega \\
& + 24K_\nu{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial_\alpha\Omega\partial_\mu\Omega - \frac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha\partial_\nu K^{\alpha\beta}\partial_\mu\Omega - 3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\nu\alpha}\partial_\mu\Omega \\
& + 12K_{\nu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\mu\Omega + \Omega^{-5}\partial_\beta\partial^\beta\partial_\alpha K_\nu{}^\alpha\partial_\mu\Omega + 4\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\nu K_\alpha{}^\beta\partial_\mu\Omega
\end{aligned}$$

$$\begin{aligned}
& + 2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\nu\partial_\alpha\Omega\partial_\mu\Omega - 60K_{\nu\beta}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega + 24\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_{\nu\alpha}\partial^\beta\Omega\partial_\mu\Omega \\
& - 3K_{\nu\beta}\Omega^{-6}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\mu\Omega - 6\Omega^{-6}\partial_\alpha K_{\nu\beta}\partial^\beta\partial^\alpha\Omega\partial_\mu\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\nu}{}^\beta\partial_\mu\partial_\alpha\Omega \\
& - 6K_{\nu\beta}\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu\partial_\alpha\Omega - 3K_{\nu}{}^\beta\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega + 12K_{\nu}{}^\beta\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega \\
& + 24K_{\nu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\partial_\beta\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\nu\alpha}\partial_\mu\partial^\beta\Omega + 4\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\mu\Omega\partial_\nu K_{\alpha\beta} \\
& - 8\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega\partial_\nu K_{\alpha\beta} + 2\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu\Omega\partial_\nu K_{\alpha\beta} - \frac{2}{3}\Omega^{-5}\partial_\beta\partial_\mu\partial_\alpha\Omega\partial_\nu K^{\alpha\beta} \\
& - 3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\nu K_{\mu\alpha} + 12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu K_{\mu\beta} - 6\Omega^{-6}\partial^\alpha\Omega\partial^\beta\partial_\alpha\Omega\partial_\nu K_{\mu\beta} \\
& + \Omega^{-5}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\nu K_{\mu\beta} + 4\Omega^{-6}\partial^\alpha\partial_\mu\Omega\partial_\beta K_{\alpha}{}^\beta\partial_\nu\Omega - 3\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\beta K_{\mu}{}^\beta\partial_\nu\Omega \\
& + 12\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\beta K_{\mu}{}^\beta\partial_\nu\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\alpha K_{\mu}{}^\beta\partial_\nu\Omega + 24K_{\mu}{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial_\alpha\Omega\partial_\nu\Omega \\
& - \frac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha\partial_\mu K^{\alpha\beta}\partial_\nu\Omega - 3\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial^\beta K_{\mu\alpha}\partial_\nu\Omega + 12K_{\mu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial_\beta\partial^\beta\Omega\partial_\nu\Omega \\
& + \Omega^{-5}\partial_\beta\partial^\beta\partial_\alpha K_{\mu}{}^\alpha\partial_\nu\Omega + 4\Omega^{-6}\partial^\alpha\Omega\partial_\beta\partial_\mu K_{\alpha}{}^\beta\partial_\nu\Omega + 2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\mu\partial_\alpha\Omega\partial_\nu\Omega \\
& - 60K_{\mu\beta}\Omega^{-8}\partial_\alpha\Omega\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\Omega + 24\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_{\mu\alpha}\partial^\beta\Omega\partial_\nu\Omega - 3K_{\mu\beta}\Omega^{-6}\partial^\beta\partial_\alpha\partial^\alpha\Omega\partial_\nu\Omega \\
& - 6\Omega^{-6}\partial_\alpha K_{\mu\beta}\partial^\beta\partial^\alpha\Omega\partial_\nu\Omega - 8\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu K_{\alpha\beta}\partial_\nu\Omega + 2\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\mu K_{\alpha\beta}\partial_\nu\Omega \\
& - 16\Omega^{-7}\partial^\alpha\Omega\partial_\beta K_{\alpha}{}^\beta\partial_\mu\Omega\partial_\nu\Omega + 2\Omega^{-6}\partial_\beta\partial_\alpha K^{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega - 8K^{\alpha\beta}\Omega^{-7}\partial_\beta\partial_\alpha\Omega\partial_\mu\Omega\partial_\nu\Omega \\
& + 40K_{\alpha\beta}\Omega^{-8}\partial^\alpha\Omega\partial^\beta\Omega\partial_\mu\Omega\partial_\nu\Omega - 16K_{\alpha}{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\mu\partial_\beta\Omega\partial_\nu\Omega - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\mu}{}^\beta\partial_\nu\partial_\alpha\Omega \\
& - 6K_{\mu\beta}\Omega^{-6}\partial^\beta\partial^\alpha\Omega\partial_\nu\partial_\alpha\Omega - 3K_{\mu}{}^\beta\Omega^{-6}\partial_\alpha\partial^\alpha\Omega\partial_\nu\partial_\beta\Omega + 12K_{\mu}{}^\beta\Omega^{-7}\partial_\alpha\Omega\partial^\alpha\Omega\partial_\nu\partial_\beta\Omega \\
& + 24K_{\mu\alpha}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\beta\Omega - 16K_{\alpha}{}^\beta\Omega^{-7}\partial^\alpha\Omega\partial_\mu\Omega\partial_\nu\partial_\beta\Omega + 4K^{\alpha\beta}\Omega^{-6}\partial_\mu\partial_\alpha\Omega\partial_\nu\partial_\beta\Omega
\end{aligned}$$

$$\begin{aligned}
 & - 6\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\mu\alpha}\partial_\nu\partial^\beta\Omega + 2\Omega^{-6}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\mu K_{\alpha\beta} - \frac{2}{3}\Omega^{-5}\partial^\beta\partial^\alpha\Omega\partial_\nu\partial_\mu K_{\alpha\beta} \\
 & + 4\Omega^{-6}\partial^\alpha\Omega\partial_\beta K_{\alpha}{}^\beta\partial_\nu\partial_\mu\Omega - \frac{2}{3}\Omega^{-5}\partial_\beta\partial_\alpha K^{\alpha\beta}\partial_\nu\partial_\mu\Omega + 2K^{\alpha\beta}\Omega^{-6}\partial_\beta\partial_\alpha\Omega\partial_\nu\partial_\mu\Omega \\
 & - 8K_{\alpha\beta}\Omega^{-7}\partial^\alpha\Omega\partial^\beta\Omega\partial_\nu\partial_\mu\Omega
 \end{aligned} \tag{82}$$

- (82) can be expressed as

$$\begin{aligned}
 \delta W_{\mu\nu} = & \frac{1}{2}\Omega^{-2}\left(\partial_\sigma\partial^\sigma\partial_\tau\partial^\tau[\Omega^{-2}K_{\mu\nu}] - \partial_\sigma\partial^\sigma\partial_\mu\partial^\alpha[\Omega^{-2}K_{\alpha\nu}] - \partial_\sigma\partial^\sigma\partial_\nu\partial^\alpha[\Omega^{-2}K_{\alpha\mu}] \right. \\
 & \left. + \frac{2}{3}\partial_\mu\partial_\nu\partial^\alpha\partial^\beta[\Omega^{-2}K_{\alpha\beta}] + \frac{1}{3}\eta_{\mu\nu}\partial_\sigma\partial^\sigma\partial^\alpha\partial^\beta[\Omega^{-2}K_{\alpha\beta}]\right).
 \end{aligned} \tag{83}$$

- Consider $h_{\mu\nu}^{TT}$ in terms of $K_{\mu\nu}$

$$\begin{aligned}
 h_{\mu\nu}^{TT} = & K_{\mu\nu} - \int d^4x' D^{(4)}(x-x')\partial_\mu\partial^\alpha K_{\alpha\nu} - \int d^4x' D^{(4)}(x-x')\partial_\nu\partial^\alpha K_{\alpha\mu} \\
 & + \frac{2}{3}\partial_\mu\partial_\nu \int d^4x' D^{(4)}(x-x') \int d^4x'' D^{(4)}(x'-x'')\partial^\alpha\partial^\beta K_{\alpha\beta} + \frac{1}{3}g_{\mu\nu} \int d^4x D^{(4)}(x-x')\partial^\alpha\partial^\beta K_{\alpha\beta}
 \end{aligned} \tag{84}$$

- (83) can thus be expressed as

$$\delta W_{\mu\nu} = \frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta[\Omega^{-2}h_{\mu\nu}]^{TT}$$

(85)

- SVT4 Decomposition

$$h_{\mu\nu} = \Omega^2(x) \left[-2g_{\mu\nu}\chi + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu F + \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu} \right] \quad (86)$$

- Fluctuation Equations $\delta W_{\mu\nu} = 0$

$$\boxed{\delta W_{\mu\nu} = \Omega^{-2}(x)\tilde{\nabla}_\sigma\tilde{\nabla}^\sigma\tilde{\nabla}_\tau\tilde{\nabla}^\tau F_{\mu\nu} = 0} \quad (87)$$

Conformal Gravity Robertson-Walker Radiation Era

- Robertson Walker $k = -1$ comoving geometry

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (88)$$

- Robertson Walker $k = -1$ conformal flat geometry

$$ds^2 = \frac{4a^2(p', r')}{[1 - (p' + r')^2][1 - (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2] \quad (89)$$

- Conformal Gravity field equation (in basis of (89))

$$\delta W_{\mu\nu} = \frac{1}{2} \Omega^{-2} \eta^{\sigma\rho} \eta^{\alpha\beta} \partial_\sigma \partial_\rho \partial_\alpha \partial_\beta [\Omega^{-2} h_{\mu\nu}]^{TT} \quad (90)$$

- Plane Wave Solutions

$$\Omega^{-2} h'^{TT}_{\mu\nu} = A'_{\mu\nu} e^{ik' \cdot x'} + (n' \cdot x') B'_{\mu\nu} e^{ik' \cdot x'} + A'^*_{\mu\nu} e^{-ik' \cdot x'} + (n' \cdot x') B'^*_{\mu\nu} e^{-ik' \cdot x'}, \quad (91)$$

where $k'_\mu k'_\nu \eta^{\mu\nu} = 0$ and $n' \cdot x' = p'$

- Following coordinate transformations,

$$h_{\mu\nu}^{TT} \sim \Omega^2(p', r') p' \sim t^4 \quad (92)$$

- SVT3 Decomposition
 - Non-local integrals, incorporates asymptotic boundary conditions
 - Decouple gauge-invariant fluctuations with higher derivatives

- SVTD Decomposition
 - Generalize to dimension D
 - Generalize to curved backgrounds (maximally symmetric)
 - Einstein de Sitter $D = 4$

$$\Delta_{\mu\nu} = (\nabla_\alpha \nabla^\alpha - 2H^2)F_{\mu\nu} + 2(g_{\mu\nu}\nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu + 3H^2 g_{\mu\nu})\chi$$

- Conformal Gravity
 - Fluctuation equation in conformal to flat backgrounds

$$\delta W_{\mu\nu} = \frac{1}{2}\Omega^{-2}\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{\nabla}_\beta \tilde{\nabla}^\beta [\Omega^{-2}h_{\mu\nu}]^{TT} = 0$$

- Robertson Walker $k = -1$ Radiation era

$$h_{\mu\nu}^{TT} \sim t^4$$

- Mathematica + xAct
- <https://github.com/phelpsmatthew/Cosmological-Fluctuations>
- (Notebook demonstration)

phelpsmatthew / Cosmological-Fluctuations

Code Issues Pull requests Actions Projects Wiki Security Insights

Source code for PhD research pertaining to cosmological fluctuations in standard and conformal gravity

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README.md

Cosmological-Fluctuations

Source code for PhD research pertaining to cosmological fluctuations in standard and conformal gravity

- Mathematica
 - Calculations performed in Mathematica include generating equations, varying geometries, decomposing tensors, and extensive simplification methods
 - Simplifying equations in general relativity requires software capable of symbolic tensor calculus
 - The third party package called **xAct** allows one to take special derivatives (covariant derivatives), refactor dummy indices on tensors, expand metric connections, etc
 - SVT3.RW.nb**, **SVT4.General.nb**, and **3_1_Splitting.nb** are the most heavily used modules, continually modified over multiple years
 - Unfortunately GitHub does not render Mathematica files; even worse, Mathematica is not open-source; in the future I may try to host them within jupyter notebooks on a server

- [1] Matthew G. Phelps, Asanka Amarasinghe, and Philip D. Mannheim. Three-dimensional and four-dimensional scalar, vector, tensor cosmological fluctuations and the cosmological decomposition theorem. *General Relativity and Gravitation*, under review 2020. URL arxiv.org/abs/1912.10448.
- [2] Asanka Zmarasinghe, Matthew G. Phelps, and Philip D. Mannheim. Cosmological perturbations in conformal gravity. ii. *Physical Review D*, 99(8), 2019. URL dx.doi.org/10.1103/revmodphys.54.729.



The End

$$V(x) = \int_V d^3x' D(x - x') \partial_i \partial^i V(x') + \oint_{\partial V} dS_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')] \quad (93)$$

$$\partial_k \partial^k V(x) = \partial_k \partial^k V(x) + \oint_{\partial V} dS_i [V(x') \partial^i \delta^3(x - x') - \delta^3(x - x') \partial^i V(x')] \quad (94)$$

Using

$$\int d^3x' V(x') \partial^i \delta^3(x - x') = \int d^3x' \partial^i V(x') \delta^3(x - x'), \quad (95)$$

the surface term vanishes identically.

$$ds^2 = \Omega^2(x) \left[-(1 + 2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dt dx^i + [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right] \quad (96)$$

- Fluctuation Equations $\delta W_{\mu\nu} = 0$, $\alpha = \phi + \psi + \dot{B} - \ddot{E}$

$$\begin{aligned} \delta W_{00} &= -\frac{2}{3\Omega^2} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \alpha, \\ \delta W_{0i} &= -\frac{2}{3\Omega^2} \tilde{\nabla}_i \tilde{\nabla}^a \tilde{\nabla}_a \partial_0 \alpha + \frac{1}{2\Omega^2} \left[\tilde{\nabla}_b \tilde{\nabla}^b (\tilde{\nabla}_a \tilde{\nabla}^a - \partial_0^2)(B_i - \dot{E}_i) \right], \\ \delta W_{ij} &= \frac{1}{3\Omega^2} \left[\delta_{ij} \tilde{\nabla}_b \tilde{\nabla}^b (\partial_0^2 - \tilde{\nabla}_a \tilde{\nabla}^a) + (\tilde{\nabla}_a \tilde{\nabla}^a - 3\partial_0^2) \tilde{\nabla}_i \tilde{\nabla}_j \right] \alpha \\ &\quad + \frac{1}{2\Omega^2} \left[[\tilde{\nabla}_a \tilde{\nabla}^a - \partial_0^2] [\tilde{\nabla}_i \partial_0 (B_j - \dot{E}_j) + \tilde{\nabla}_j \partial_0 (B_i - \dot{E}_i)] \right] + \frac{1}{\Omega^2} [\tilde{\nabla}_a \tilde{\nabla}^a - \partial_0^2]^2 E_{ij}. \end{aligned} \quad (97)$$

- Decouple by applying higher derivatives to vector and tensor components

$$\begin{aligned} -\frac{2}{3} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \alpha &= 0, \quad \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^b [\tilde{\nabla}_c \tilde{\nabla}^c - \partial_0^2]^2 E_{ij} = 0, \\ -\frac{1}{2} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a (\ddot{B}_i - \ddot{E}_i) + \frac{1}{2} \tilde{\nabla}_c \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a (B_i - \dot{E}_i) &= 0 \end{aligned} \quad (98)$$

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$

$$\begin{aligned}\delta W_{\mu\nu} &= \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \frac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\ &\quad - \frac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \frac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}\end{aligned}$$

$$\delta G_{\mu\nu} = \frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \frac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \frac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \frac{1}{2}\nabla_\nu\nabla_\mu h$$

$$\delta G = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h$$

$$\delta G_{\mu\nu}^{T\theta} = \delta G_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\delta G + \frac{1}{3}\nabla_\mu\nabla_\nu \int D\delta G$$

$$\nabla^2\delta G_{\mu\nu}^{T\theta} = \nabla^2\delta G_{\mu\nu} + \frac{1}{3}[\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2]\delta G$$

$$\begin{aligned}\nabla^2\delta G_{\mu\nu}^{T\theta} &= \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \frac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\ &\quad - \frac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \frac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\ &= \delta W_{\mu\nu}\end{aligned}\tag{99}$$

- $k = 1$ (spherical)

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (100)$$

- Set $\sin \chi = r$, $p = \tau$,

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right] \quad (101)$$

- Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p + \chi)/2], & p' - r' &= \tan[(p - \chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos \chi}, & r' &= \frac{\sin \chi}{\cos p + \cos \chi} \end{aligned} \quad (102)$$

$$\implies ds^2 = \frac{4a^2(p)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2] \quad (103)$$

- $k = -1$ (hyperbolic)

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (104)$$

- Set $\sinh \chi = r$, $p = \tau$,

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sinh^2 \chi d\theta^2 + \sinh^2 \chi \sin^2 \theta d\phi^2 \right] \quad (105)$$

- Introduce coordinates

$$\begin{aligned} p' + r' &= \tanh[(p + \chi)/2], & p' - r' &= \tanh[(p - \chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh \chi}, & r' &= \frac{\sinh \chi}{\cosh p + \cosh \chi} \end{aligned} \quad (106)$$

$$\implies ds^2 = \frac{4a^2(p)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2] \quad (107)$$