SVT Decomposition from Orthogonal Projections

Fundamental observers are locally at rest with respect to the matter fluid. Motivated by these "preferred" frames, we seek to split a given rank 2 tensor T_{ab} into components parallel and orthogonal to a velocity vector u_μ. The rest frames locally define surfaces of constant t. The induced metric for the surfaces of simultaneity is

$$h_{ab} = g_{ab} + u_a u_b.$$

That this acts like a 3-space metric can verified by

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^c{}_c = 3, \qquad h^a{}_b u^b = 0.$$

Note that the last relation uses $h^a{}_b$ to project the components orthogonal to u^a. Likewise $U^a{}_b \equiv -u^a u_b$ projects components parallel to u_a.

We can use these projectors to decompose a tensor into components parallel and orthogonal to the local velocity. Take arbitrary symmetric rank 2 tensor T_{ab}

$$\begin{aligned} T_{ab} &= g^c{}_a g^d{}_b T_{cd} && (1)\\ &= (h^c{}_a + U^c{}_a)(h^d{}_b + U^d{}_b) T_{cd} && (2)\\ &= h^c{}_a h^d{}_b T_{cd} - u_a(u^c h^d{}_b T_{cd}) - u_b(u^d h^c{}_a T_{cd}) + u_a u_b (u^c u^d T_{cd}). && (3) \end{aligned}$$

This can be re-expressed as:

$$\begin{aligned} T_{ab} = \frac{1}{3} h_{ab} h^{cd} T_{cd} + \left[\frac{1}{2} h^c{}_a h^d{}_b + \frac{1}{2} h^c{}_b h^d{}_a - \frac{1}{3} h_{ab} h^{cd}\right] T_{cd} + \left[\frac{1}{2} h^c{}_a h^d{}_b - \frac{1}{2} h^d{}_a h^c{}_b\right] T_{cd} && (4)\\ - u_a(h^d{}_b T_{cd} u^c) - u_b(h^c{}_a T_{cd} u^d) + u_a u_b (u^c u^d T_{cd}). && (5) \end{aligned}$$

In this form, we note the quantity

$$\left[\frac{1}{2} h^c{}_a h^d{}_b + \frac{1}{2} h^c{}_b h^d{}_a - \frac{1}{3} h_{ab} h^{cd}\right] T_{cd}$$

is symmetric, trace-free, and orthogonal to u^a and u^b, i.e. the projected symmetric tracefree (PTSF) part. We also note that the quantity

$$\left[\frac{1}{2} h^c{}_a h^d{}_b - \frac{1}{2} h^d{}_a h^c{}_b\right] T_{cd} \qquad (6)$$

vanishes given a symmetric T_{cd}. Let us relabel the following quantities:

$$\begin{aligned} \rho &= u^c u^d T_{cd} && (7)\\ p &= \frac{1}{3} h^{cd} T_{cd} && (8)\\ q_a &= -h^b{}_a u^c T_{bc} && (9)\\ \pi_{ab} &= \left[\frac{1}{2} h^c{}_a h^d{}_b + \frac{1}{2} h^c{}_b h^d{}_a - \frac{1}{3} h_{ab} h^{cd}\right] T_{cd}. && (10) \end{aligned}$$

Now the energy momentum tensor may be expressed as

$$T_{ab} = u_a u_b \rho + h_{ab} p + u_a q_b + u_b q_a + \pi_{ab} \qquad (11)$$

$$= (\rho + p)u_a u_b + p g_{ab} + u_a q_b + u_b q_a + \pi_{ab} \tag{12}$$

Note that $u^a q_a = 0$, and that π_{ab} (projected symmetric traceless) has 5 components. Thus 2 scalars, 3 vector components, and 5 from the tensor give us 10 in total.

In comoving coordinates in FLRW space, the velocity vector is

$$u^a = \delta^a_0, \qquad u_a = -\delta^0_a$$

and thus the only non-zero components of π_{ab} are π_{ij} (spatial). According to York (1973) we may decompose a symmetric tensor on a positive definite Riemannian space as

$$\pi_{ab} = \pi^{TT}_{ab} + \pi^{L}_{ab} + \pi^{Tr}_{ab}$$

where

$$\pi^{Tr}_{ab} = \frac{1}{3} g_{ab} g^{cd} \pi_{cd}$$

and (restricting to 3 dimensions)

$$\pi^{L}_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c.$$

By construction

$$g^{ab} \pi^{TT}_{ab} = 0.$$

The transverse requirement leads to an equation for the vector W_a

$$-\nabla_b \pi^{ab(L)} = -\nabla_b (\pi^{ab} - \frac{1}{3} g^{ab} g_{cd} \pi^{cd}).$$

York shows that such a vector W_a must exists and is unique, up to conformal Killing vectors. Moreover, he also shows that decomposition actually holds its form under conformal transformation on the metric.

Going back to the symmetric traceless tensor π_{ab}, we may write this as

$$\pi_{ab} = \pi^{TT}_{ab} + \pi^{L}_{ab} \tag{13}$$

$$= \pi^{TT}_{ab} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c. \tag{14}$$

Going back to the general symmetric tensor, if we instead make the substitutions

$$\rho = -2\phi \tag{15}$$

$$p = \psi \tag{16}$$

$$q_a = -(B_a + \nabla_a B); \qquad \nabla_a B^a = 0 \tag{17}$$

$$W_a = (E_a + \nabla_a E); \qquad \nabla_a E^a \tag{18}$$

$$\pi^{TT}_{ab} = E_{ab} \tag{19}$$

then

$$\pi_{ab} = \pi^{TT}_{ab} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c \tag{20}$$

$$= E_{ab} + \nabla_a E_b + \nabla_b E_a + 2\nabla_a \nabla_b E - \frac{2}{3} h_{ab} \nabla^2 E. \tag{21}$$

It follow that we end up with the same form of the perturbation metric as given in the standard SVT decomposition:

$$T_{ab} = -2\phi u_a u_b - (B_b + \nabla_b B)u_a - (B_a + \nabla_a B)u_b - 2\gamma_{ab}\psi + \nabla_a E_b + \nabla_b E_a + +2E_{ab}.$$

In flat space the spacetime interval is

$$ds^2 = -(1 + 2\phi)dt^2 + 2(B_i + \nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i \nabla_j E + 2E_{ij}\right] dx^i dx^j.$$

Thus the SVT decomposition can be achieved first by orthogonal decomposition of a symmetric tensor relative to the four velocity, and then decomposing the projected symmetric trace-free portion into transverse and longitudinal components.