TT Projection Curved Space v2

1 Maximally Symmetric Space TT

$$
\begin{aligned}
h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\
&+ \frac{2-D}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\nabla^\sigma W_\sigma - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')h
\end{aligned} \tag{1.1}
$$

$$
\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)D(x,x') = g^{-1/2}\delta^{(D)}(x-x') \tag{1.2}
$$

$$
\nabla^\mu h_{\mu\nu} = \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)W_\nu \tag{1.3}
$$

With the covariant operator $(\nabla^2 - R/D)$ mixing indices of W_ν, the particular integral solution for W_ν involves a bi-tensor Green's function $D_{\sigma\rho'}$ which obeys

$$
\left(\nabla^\alpha\nabla_\alpha - \frac{R}{D}\right)D_{\sigma\rho'}(x,x') = g_{\sigma\rho'}g^{-1/2}\delta^4(x-x'). \tag{1.4}
$$

Here $g_{\sigma\rho'}$ represents a parallel propagator, defined in terms of Vierbeins e^a_μ:

$$
g^{\alpha'}{}_\beta(x,x') = e^{\alpha'}_a(x')e^a_\beta(x), \qquad g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu. \tag{1.5}
$$

In terms of (1.4), W_ν has particular solution

$$
W_\nu = g^{1/2}\int D_\nu{}^{\rho'}(x,x')\nabla^{\sigma'}h_{\rho'\sigma'}. \tag{1.6}
$$

2 Curved Space TT

To generalize to curved space, we assume $h_{\mu\nu}$ to be of the form:

$$
h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \underbrace{\left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right)}_{W_{\mu\nu}} + \underbrace{\frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi}_{S_{\mu\nu}} \tag{2.1}
$$

Taking the trace of (2.1), we find the vector sector $W_{\mu\nu}$ is decoupled from the trace and Ψ can easily be inverted,

$$
g^{\mu\nu}W_{\mu\nu} = 0 \tag{2.2}
$$

$$
g^{\mu\nu}S_{\mu\nu} = \nabla_\alpha\nabla^\alpha\Psi = h \qquad \to \Psi = \int g^{1/2}D(x,x')h \tag{2.3}
$$

Taking the divergence of (2.1), we have

$$\nabla^\mu h_{\mu\nu} \quad = \quad \nabla^\mu W_{\mu\nu} + \nabla^\mu S_{\mu\nu}(h) \tag{2.4}$$

By substituting (2.3), the above serves to define an equation for W_μ in terms of h and $h_{\mu\nu}$, namely

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha \quad = \quad \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h \tag{2.5}$$

Commuting derivatives, (2.5) can be expressed in the equivalent forms,

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \nabla_\alpha\nabla_\nu - \frac{2}{D}\nabla_\nu\nabla_\alpha\right]W^\alpha \quad = \quad \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h, \tag{2.6}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha \quad = \quad \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}R_{\nu\alpha}\nabla^\alpha\int g^{1/2}D(x,x')h. \tag{2.7}$$

Similar to (1.4), the requisite Green's function that solves W_α is a bi-tensor defined as

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]D^{\alpha\gamma'} \quad = \quad g^{\alpha\gamma'}g^{-1/2}\delta^{(D)}(x,x'). \tag{2.8}$$

Hence, W_μ takes the form

$$W_\mu \quad = \quad \int g^{1/2}D_\mu{}^{\sigma'}\left[\nabla^{\rho'}h_{\sigma'\rho'} - \frac{1}{D-1}R_{\sigma'\rho'}\nabla^{\rho'}\int g^{1/2}D(x',x'')h\right]. \tag{2.9}$$

To summarize, in curved space $h_{\mu\nu}$ may be decomposed according to

$$h_{\mu\nu} \quad = \quad h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \tag{2.10}$$

with Ψ and W_μ obeying

$$\Psi \quad = \quad \int g^{1/2}D(x,x')h \tag{2.11}$$

$$W_\mu \quad = \quad \int g^{1/2}D_\mu{}^{\sigma'}\left[\nabla^{\rho'}h_{\sigma'\rho'} - \frac{1}{D-1}R_{\sigma'\rho'}\nabla^{\rho'}\int g^{1/2}D(x',x'')h\right] \tag{2.12}$$

$$\nabla_\alpha\nabla^\alpha D(x,x') \quad = \quad g^{-1/2}\delta^{(D)}(x-x') \tag{2.13}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]D^{\alpha\gamma'} \quad = \quad g^{\alpha\gamma'}g^{-1/2}\delta^{(D)}(x,x'). \tag{2.14}$$

2.1 SVTD Decomposition

Starting with

$$h_{\mu\nu} \quad = \quad h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi, \tag{2.15}$$

we decompose W_μ into transverse and longitudinal components viz.

$$W_\mu \quad = \quad \underbrace{W_\mu - \nabla_\mu\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{F_\mu} + \nabla_\mu\underbrace{\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{H}. \tag{2.16}$$

Setting $h^{T\theta}_{\mu\nu} = 2F_{\mu\nu}$, (2.28) becomes

$$h_{\mu\nu} = 2F_{\mu\nu} + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2\nabla_\mu\nabla_\nu H - \frac{2}{D} g_{\mu\nu}\nabla_\alpha\nabla^\alpha H + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi. \tag{2.17}$$

Upon further defining

$$F = H - \frac{1}{2(D-1)}\Psi \tag{2.18}$$

$$\chi = \frac{1}{D}\nabla_\alpha\nabla^\alpha H - \frac{1}{2(D-1)}\nabla_\alpha\nabla^\alpha\Psi, \tag{2.19}$$

we may express (2.28) as the desired SVTD form:

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{2.20}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{2.21}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x') h \tag{2.22}$$

$$F_\mu = W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \tag{2.23}$$

$$2F_{\mu\nu} = 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \tag{2.24}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{2.25}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{2.26}$$

2.2 Curved TT in Max. Symmetric Space (Incomplete)

In a space of maximal symmetry defined by

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D), \end{aligned} \tag{2.27}$$

the defining equation for W_μ reduces to

$$\left[g_{\nu\alpha}\left(\nabla_\beta\nabla^\beta - \frac{R}{D}\right) + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{R}{D-1}\nabla_\nu\int g^{1/2}D(x,x')h \tag{2.28}$$

2.3 Curved TT in Minkowski

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi, \tag{2.29}$$

In a Minkowski geometry, the defining equation for W_μ reduces to

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} \tag{2.30}$$

$$W_\mu = \int g^{1/2} D_\mu{}^{\sigma'} \nabla^{\rho'} h_{\sigma'\rho'} \tag{2.31}$$

$$\left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \left(\frac{D-2}{D} \right) \nabla_\nu \nabla_\alpha \right] D^{\alpha\gamma'} = g^{\alpha\gamma'} \delta^{(D)}(x, x') \tag{2.32}$$

$$\nabla_\alpha \nabla^\alpha \Psi = h \tag{2.33}$$

$$\Psi = \int g^{1/2} D(x, x') h \tag{2.34}$$

$$\nabla_\alpha \nabla^\alpha D(x, x') = g^{-1/2} \delta^{(D)}(x - x') \tag{2.35}$$

Decompose W_μ into transverse and longitudinal components viz.

$$W_\mu = \underbrace{W_\mu - \nabla_\mu \int g^{1/2} D(x, x') \nabla^\sigma W_\sigma}_{F_\mu} + \nabla_\mu \underbrace{\int g^{1/2} D(x, x') \nabla^\sigma W_\sigma}_{H} . \tag{2.36}$$

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \tag{2.37}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{2.38}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{2.39}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x') h \tag{2.40}$$

$$F_\mu = W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \tag{2.41}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right] W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{2.42}$$

$$\nabla_\alpha\nabla^\alpha W_\nu + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla^\alpha W_\alpha - R_{\nu\alpha}W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{2.43}$$

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{2.44}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{2.45}$$