Cosmological Fluctuations in Standard and Conformal Gravity

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Overview

- Introduction and Formalism
- SVT3 Decomposition
- SVTD Decomposition
- Conformal Gravity
- Computational Methods

- Introduction and Formalism
  - Cosmological Geometries
  - Einstein Gravity
  - Perturbation Theory
  - Gauge Transformations

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker (flat, spherical, hyperbolic), de Sitter ($dS_4 \subset \mathrm{RW}$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

$$ds^2 = \Omega(x)^2 \big(- dt^2 + dx^2 + dy^2 + dz^2 \big)$$

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -3k\tilde{g}_{ij}, \qquad R = -6k$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$$

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with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

set $k = 0$ (flat), simple conformal to flat form

$$ds^2 = a(\tau)^2 \Big[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \Big]$$

$k = 1$ (spherical)

$$ds^2 = a(\tau)^2\Big[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\Big]$$

Set $\sin\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\Big[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\Big]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p+\chi)/2], \quad p' - r' = \tan[(p-\chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos\chi}, \quad r' = \frac{\sin\chi}{\cos p + \cos\chi} \end{aligned}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1+(p'+r')^2][1+(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]$$

$k=-1$ (hyperbolic)

$$ds^2 = a(\tau)^2\Big[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\Big]$$

Set $\sin\chi = r$, $p=\tau$,

$$ds^2 = a(p)^2\Big[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2\Big]$$

Introduce coordinates

$$\begin{aligned} p'+r' &= \tanh[(p+\chi)/2], \quad p'-r' = \tanh[(p-\chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh\chi}, \quad r' = \frac{\sinh\chi}{\cosh p + \cosh\chi} \end{aligned}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1-(p'+r')^2][1-(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]$$

Einstein Hilbert action

$$I_{\mathsf{EH}} = -\frac{1}{16\pi G}\int d^4x(-g)^{1/2} g^{\mu\nu} R_{\mu\nu}.$$

Functional variation w.r.t $g_{\mu\nu}$ yields Einstein tensor,

$$\frac{16\pi G}{(-g)^{1/2}}\frac{\delta I_{\mathsf{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\alpha},$$

likewise, variation of matter action I_{M} w.r.t $g_{\mu\nu}$ yields Energy Momentum tensor

$$\frac{2}{(-g)^{1/2}}\frac{\delta I_{\mathsf{M}}}{\delta g_{\mu\nu}} = T_{\mu\nu}.$$

Requiring sum of actions to be stationary gives us Einstein field equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\alpha} = -8\pi G T^{\mu\nu},$$

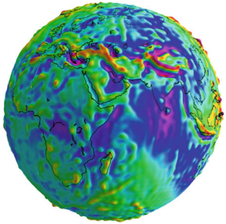
subject to Bianchi identity

$$\nabla_{\mu} R^{\mu\nu} = \frac{1}{2}\nabla^{\nu} R^{\mu}{}_{\mu} \implies \nabla_{\mu} G^{\mu\nu} = 0.$$

Cosmological Perturbation Theory

Decompose metric into background and fluctuation, truncating at linear order

$$\begin{aligned} g_{\mu\nu}(x) &= g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad g^{\mu\nu}_{(0)} h_{\mu\nu} \equiv h \\ G_{\mu\nu} &= G_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) \\ G^{(0)}_{\mu\nu} &= R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(0)\alpha}_{\alpha} \\ \delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R^{(0)\alpha}_{\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}{}_{\alpha}. \end{aligned}$$



[1]

Likewise perturb $T_{\mu\nu}$ around background

$$T_{\mu\nu} = T_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu})$$

Form background and first order equations of motion (upon setting $8\pi G = 1$)

$$\begin{aligned} \Delta^0_{\mu\nu} &= G^{(0)}_{\mu\nu} + T^{(0)}_{\mu\nu} = 0 \\ \Delta_{\mu\nu} &= \delta G^{(0)}_{\mu\nu} + \delta T^{(0)}_{\mu\nu} = 0 \end{aligned}$$

[1] Walter, U. (2019). Correction to: Astronautics. In Astronautics (pp. C1–C1). Springer International Publishing.

Gauge Transformations

- Under coordinate transformation $x^\mu \to x^\mu - \epsilon^\mu(x)$, with $\epsilon^\mu \sim \mathcal{O}(h)$, the perturbed metric transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$$

- For every solution $h_{\mu\nu}$ to $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$, a transformed $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ will also serve as a solution
- Set of four $\epsilon^\mu(x)$ define gauge freedom under coordinate transformation
- 10 components in $h_{\mu\nu}$, 4 coordinate transformations, leads to 6 independent degrees of freedom
- Under $x^\mu \to x^\mu - \epsilon^\mu(x)$, the perturbed tensors transform as

$$\delta G_{\mu\nu} \to \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda G^{(0)}_{\mu\nu} \epsilon^\lambda$$
$$\delta T_{\mu\nu} \to \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda T^{(0)}_{\mu\nu} \epsilon^\lambda.$$

- If background $G^{(0)}_{\mu\nu} = 0$, then $\delta G_{\mu\nu}$ separately gauge invariant; likewise for vanishing background energy momentum tensor
- If $G^{(0)}_{\mu\nu} \neq 0$, then only the entire $\Delta_{\mu\nu} = \delta G_{\mu\nu} + T_{\mu\nu}$ is gauge invariant

- Perturbed field equations $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ form a rather complex and extensive set of coupled non-linear tensor PDE's
- Much effort involved in simplifying, decoupling, and solving them

$$\begin{aligned}
\delta G_{ij} \quad = \quad & -\tfrac{1}{2}\ddot{h}_{ij} + \tfrac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \tfrac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2\tilde{g}_{ij}h_{00}\Omega^{-2} \\
& -\dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
& +2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
& -4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
& -\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
& +\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& -\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
& +\tfrac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \tfrac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
& -\tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \tfrac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
& +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \tfrac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h,
\end{aligned}$$

Two main approaches

- Fix the gauge by constraining $h_{\mu\nu}$, e.g. transverse gauge $\nabla^{\mu} h_{\mu\nu} = 0$, then solve fluctuation equations directly in terms of $h_{\mu\nu}$
  - Simplification usually not effective in more general curved backgrounds
  - Some exceptions for maximally symmetry spacetimes and in conformal gravity
- Decompose $h_{\mu\nu}$ into a basis of scalars, vectors, and tensors, express in terms of gauge invariant combinations, and solve fluctuation equations with possible decoupling between modes
  - SVT Decomposition, de facto approach in modern cosmology

- Three-dimensional Scalar, Vector, Tensor Basis
  - SVT3 Decomposition
  - Gauge Invariants in Minkowski background
  - de Sitter Solution

- Decompose the metric perturbation $h_{\mu\nu}$ into a set of scalars, vectors, and tensors according to their transformation behavior under 3D rotations. Assuming a Minkowski background, for instance

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = (g^{(0)}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= (-1 + h_{00})dt^2 + 2h_{0i}dtdx^i + (\tilde{g}_{ij} + h_{ij})dx^i dx^j \\ &= -(1+2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i \\ &\quad +[(1-2\psi)\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j, \end{aligned}$$

$$\begin{aligned} h_{00} &= -2\phi, \qquad h_{0i} = B_i + \tilde{\nabla}_i B \\ h_{ij} &= -2\psi\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned}$$

with vectors and tensors obeying

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0.$$

- 10 components in total

- Insert the SVT3 decomposed $h_{\mu\nu}$ into $\delta G_{\mu\nu}$

$$\begin{aligned}
\delta G_{00} &= -2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi, \\
\delta G_{0i} &= -2\tilde{\nabla}_i\dot{\psi} + \tfrac{1}{2}\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(B_i - \dot{E}_i), \\
\delta G_{ij} &= -2\delta_{ij}\ddot{\psi} - \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}) + \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a\psi + \tilde{\nabla}_j\tilde{\nabla}_i(\phi + \dot{B} - \ddot{E}) \\
&\quad -\tilde{\nabla}_j\tilde{\nabla}_i\psi + \tfrac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \tfrac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) - \ddot{E}_{ij} + \delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a E_{ij}, \\
g^{\mu\nu}\delta G_{\mu\nu} &= -\delta G_{00} + \delta^{ij}\delta G_{ij} = 4\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi - 6\ddot{\psi} - 2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}),
\end{aligned}$$