Special Gauge Matthew

The perturbed Einstein tensor $\delta G_{\mu\nu}(h_{\mu\nu})$ evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{1}$$

is calculated as

$$\begin{aligned}\delta G_{\mu\nu} = {} & \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{2}$$

$$\begin{aligned}\delta G_{\mu\nu} = {} & \frac{\eta^{\alpha\beta}\partial_\alpha\Omega\partial_\beta h_{\mu\nu}}{\Omega} - \frac{\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha h\partial_\beta\Omega}{\Omega} - \frac{\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\partial_\alpha\Omega\partial_\beta\Omega}{\Omega^2} + \frac{\eta^{\alpha\beta}h_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega}{\Omega^2} \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h - \frac{2\eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega}{\Omega} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} \\ & - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} + \frac{2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma}}{\Omega} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} \\ & + \frac{2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\zeta\partial_\beta\Omega}{\Omega} - \frac{\eta^{\alpha\beta}\partial_\alpha\Omega\partial_\mu h_{\nu\beta}}{\Omega} - \frac{\eta^{\alpha\beta}\partial_\alpha\Omega\partial_\nu h_{\mu\beta}}{\Omega} + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{3}$$

When calculated explicitly in the Cartesian coordinate system, we see that each tensor component is far away from being diagonal in the perturbation components $h_{\mu\nu}$. In order to solve these equations, we seek to find a gauge that allows the equations to become diagonalized. To this end, we may impose the most general gauge as

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega \tag{4}$$

where J, P, and R are constant coefficients that we vary. Upon taking $J=-2$, $P=1$, and $R=-1$, the fluctuation equations take a form diagonal in $h_{\mu\nu}$ up to its trace. Indeed other combinations do exist, but deviation from this configuration will result in a trace conditions that involve derivatives of the trace, where as the above choice allows us to solve the trace explicity in terms of h_{00}. To be precise, given the special gauge choice, the trace of the Einstein tensor evaluates to

$$g^{\mu\nu}\delta G_{\mu\nu} = \Bigg(-\frac{10\Omega'^2}{\Omega^4} + \frac{6\Omega''}{\Omega^3}\Bigg)h_{00} + \Bigg(\frac{2\Omega'^2}{\Omega^4} + \frac{3\Omega''}{\Omega^3}\Bigg)h. \tag{5}$$

In the gauge

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + \partial_\nu h - \Omega^{-1}h\partial_\nu\Omega \tag{6}$$

the perturbed Einstein tensor has been calculated as :

$$\delta G_{00} = \Bigg(\frac{3\Omega'^2}{\Omega^2} - \frac{\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg)h_{00} + \Bigg(-\frac{3\Omega'^2}{2\Omega^2} + \frac{\Omega''}{2\Omega} + \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_0\partial_0\Bigg)h. \tag{7}$$

$$\begin{aligned}123 = {} & (-2\Omega^{-2}\dot\Omega^2\eta_{11} + \Omega^{-1}\ddot\Omega\eta_{11})h_{00} + (-\Omega^{-2}\dot\Omega^2 + 2\Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot\Omega\partial_0)h_{11} \\ & + (-\tfrac{1}{2}\Omega^{-2}\dot\Omega^2\eta_{11} - \tfrac{1}{2}\Omega^{-1}\ddot\Omega\eta_{11} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu\eta_{11})h.\end{aligned} \tag{8}$$

$$\delta G_{11} = \Bigg(-\frac{2\Omega'^2}{\Omega^2} + \frac{\Omega''}{\Omega}\Bigg) h_{00} + \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{11} + \Bigg(\frac{\Omega'^2}{2\Omega^2} + \frac{\Omega''}{2\Omega} + \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_1\partial_1\Bigg) h. \tag{9}$$

$$\delta G_{22} = \Bigg(-\frac{2\Omega'^2}{\Omega^2} + \frac{\Omega''}{\Omega}\Bigg) h_{00} + \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{22} + \Bigg(\frac{\Omega'^2}{2\Omega^2} + \frac{\Omega''}{2\Omega} + \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_2\partial_2\Bigg) h. \tag{10}$$

$$\delta G_{33} = \Bigg(-\frac{2\Omega'^2}{\Omega^2} + \frac{\Omega''}{\Omega}\Bigg) h_{00} + \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{33} + \Bigg(\frac{\Omega'^2}{2\Omega^2} + \frac{\Omega''}{2\Omega} + \frac{\Omega'\partial_0}{2\Omega} - \tfrac{1}{2}\partial_3\partial_3\Bigg) h. \tag{11}$$

$$\delta G_{01} = \Bigg(\frac{\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{01} + \Bigg(\frac{\Omega'\partial_1}{2\Omega} - \tfrac{1}{2}\partial_1\partial_0\Bigg) h. \tag{12}$$

$$\delta G_{02} = \Bigg(\frac{\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{02} + \Bigg(\frac{\Omega'\partial_2}{2\Omega} - \tfrac{1}{2}\partial_2\partial_0\Bigg) h. \tag{13}$$

$$\delta G_{03} = \Bigg(\frac{\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{03} + \Bigg(\frac{\Omega'\partial_3}{2\Omega} - \tfrac{1}{2}\partial_3\partial_0\Bigg) h. \tag{14}$$

$$\delta G_{12} = \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{12} - \tfrac{1}{2}\partial_2\partial_1 h. \tag{15}$$

$$\delta G_{13} = \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{13} - \tfrac{1}{2}\partial_3\partial_1 h. \tag{16}$$

$$\delta G_{23} = \Bigg(-\frac{\Omega'^2}{\Omega^2} + \frac{2\Omega''}{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{\Omega'\partial_0}{\Omega}\Bigg) h_{23} - \tfrac{1}{2}\partial_3\partial_2 h. \tag{17}$$

In the deSitter background geometry $\Omega(t) = \frac{1}{Ht}$ there exists a similar gauge that simplifies the result even further. That is, upon taking $J = -2$, $P = \frac{1}{2}$, and $R = 1$ we have

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + \tfrac{1}{2}\partial_\nu h + \Omega^{-1}h\partial_\nu\Omega \tag{18}$$

The trace of the Einstein tensor evaluates to

$$g^{\mu\nu}\delta G_{\mu\nu} = 2H^2 h_{00} + (-2H^2 - \tfrac{1}{2}H^2\eta^{\mu\nu}\partial_\mu\partial_\nu\tau^2)h. \tag{19}$$

The tensor perturbations are then:

$$123 = h_{11}(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-2} + \partial_0\tau^{-1}) + (-\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu\eta_{11} - \tfrac{3}{2}\eta_{11}\tau^{-2})h. \tag{20}$$

$$\delta G_{00} = \Bigg(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-2} + \frac{\partial_0}{\tau}\Bigg) h_{00} + \Bigg(\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{2\tau^2} + \frac{\partial_0}{\tau}\Bigg) h. \tag{21}$$

$$\delta G_{11} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{11} + \left(-\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{3}{2\tau^2}\right)h. \tag{22}$$

$$\delta G_{22} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{22} + \left(-\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{3}{2\tau^2}\right)h. \tag{23}$$

$$\delta G_{33} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{33} + \left(-\tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{3}{2\tau^2}\right)h. \tag{24}$$

$$\delta G_{01} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{2}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{01} + \frac{\partial_1 h}{2\tau}. \tag{25}$$

$$\delta G_{02} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{2}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{02} + \frac{\partial_2 h}{2\tau}. \tag{26}$$

$$\delta G_{03} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{2}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{03} + \frac{\partial_3 h}{2\tau}. \tag{27}$$

$$\delta G_{12} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{12}. \tag{28}$$

$$\delta G_{13} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{13}. \tag{29}$$

$$\delta G_{23} = \left(\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{\tau^2} + \frac{\partial_0}{\tau}\right)h_{23}. \tag{30}$$