$g^{\mu\nu}\delta G_{\mu\nu}$ Radiation v1

1 Conformal Flat $\Omega(x)$

$$
\begin{aligned}
ds^2 &= (g_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(x)(\tilde{g}_{\mu\nu}+f_{\mu\nu})dx^\mu dx^\nu && (1.1)\\
\tilde{g}_{\mu\nu} &= \mathrm{diag}\left(-1,1,r^2,r^2\sin^2\theta\right) \qquad \tilde{\Gamma}^\lambda_{\alpha\beta} = \delta^\lambda_i\delta^j_\alpha\delta^k_\beta\tilde{\Gamma}^i_{jk} && (1.2)
\end{aligned}
$$

All subsequent equations hold for any flat $\tilde{g}_{\mu\nu}$, i.e. any $\tilde{g}_{\mu\nu}$ such that the corresponding curvature tensors vanish.

1.1 $G_{\mu\nu}$

$$
G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \qquad (1.3)
$$

$$
\begin{aligned}
g^{\mu\nu}G_{\mu\nu} &= -R\\
&= -\tilde{R}\Omega^{-2} - 6\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\\
&= 6\ddot{\Omega}\Omega^{-3} - 6\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega && (1.4)
\end{aligned}
$$

1.2 $\delta(g^{\mu\nu}G_{\mu\nu})$

We calculate $\delta(g^{\mu\nu}G_{\mu\nu}) = -h^{\mu\nu}G^{(0)}_{\mu\nu} + g^{\mu\nu}\delta G_{\mu\nu}$ as this is the perturbed equation that follows directly from (1.4). Additional remarks on the trace are in Trace Gauge Invariance.

$$
\begin{aligned}
\delta(g^{\mu\nu}G_{\mu\nu}) = {}& -6\dot{\phi}\dot{\Omega}\Omega^{-3} - 18\dot{\psi}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\phi\Omega^{-3} - 6\ddot{\psi}\Omega^{-2} - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B}\\
&+2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi - 12\psi\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\\
&-12\Omega^{-3}\tilde{\nabla}_a\dot{\Omega}\tilde{\nabla}^a B - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\dot{B} - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\phi + 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\psi\\
&+6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E + 12\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E - 12B^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{B}^a\Omega^{-3}\tilde{\nabla}_a\Omega\\
&+6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a + 12\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \qquad (1.5)
\end{aligned}
$$

We substitute the gauge invariants and null trace condition ($g^{\mu\nu}G_{\mu\nu}=0 \implies \tilde{\nabla}_a\tilde{\nabla}^a\Omega = \ddot{\Omega}$),

$$
\begin{aligned}
\alpha &= \phi+\psi+\dot{B}-\ddot{E}, \qquad \gamma = \psi - \Omega^{-1}[(B-\dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega],\\
Q_i &= B_i - \dot{E}_i, \qquad E_{ij}. && (1.6)
\end{aligned}
$$

The perturbed trace $\delta(g^{\mu\nu}G_{\mu\nu})$ is then expressed entirely in terms of the gauge invariants as

$$
\begin{aligned}
\delta(g^{\mu\nu}G_{\mu\nu}) = {}& -6\dot{\alpha}\dot{\Omega}\Omega^{-3} - 12\dot{\gamma}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\alpha\Omega^{-3} - 6\ddot{\gamma}\Omega^{-2} - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\gamma\\
&-6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\alpha + 12\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\gamma - 12Q^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{Q}^a\Omega^{-3}\tilde{\nabla}_a\Omega + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega. \quad (1.7)
\end{aligned}
$$

Defining the gauge invariants instead as

$$
\alpha = \phi+\psi+\dot{B}-\ddot{E}, \qquad \gamma = \phi-\psi+\dot{B}-\ddot{E}+2\Omega^{-1}[(B-\dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega],
$$

$$Q_i = B_i - \dot{E}_i, \qquad E_{ij}. \tag{1.8}$$

then (1.5) becomes

$$\begin{aligned} g^{\mu\nu}\delta G_{\mu\nu} &= -12\dot{\alpha}\dot{\Omega}\Omega^{-3} + 6\dot{\gamma}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\alpha\Omega^{-3} - 3\ddot{\alpha}\Omega^{-2} + 3\ddot{\gamma}\Omega^{-2} + \Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 3\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \\ &\quad -6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\gamma - 12Q^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{Q}^a\Omega^{-3}\tilde{\nabla}_a\Omega + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega. \end{aligned} \tag{1.9}$$

1.3 Trace Gauge Invariance

With the transformation of the first order $\delta G_{\mu\nu}$ behaving as

$$\Delta_\epsilon[\delta G_{\mu\nu}] = G^\lambda{}_\mu\nabla_\nu\epsilon_\lambda + G^\lambda{}_\nu\nabla_\mu\epsilon_\lambda + \nabla_\lambda G_{\mu\nu}\epsilon^\lambda, \tag{1.10}$$

upon taking the trace, we have

$$g^{\mu\nu}\Delta_\epsilon[\delta G_{\mu\nu}] = 2G^{\lambda\mu}\nabla_\mu\epsilon_\lambda + \nabla_\lambda G^\mu{}_\mu\epsilon^\lambda. \tag{1.11}$$

The above indicates that a vanishing $G^\mu{}_\mu$ alone does not ensure $g^{\mu\nu}\delta G_{\mu\nu}$ is gauge invariant. However, we may subtract from (1.10) the contribution $h^{\mu\nu}G_{\mu\nu}$, which transforms as

$$G_{\mu\nu}\Delta_\epsilon[h^{\mu\nu}] = 2G^{\lambda\mu}\nabla_\mu\epsilon_\lambda. \tag{1.12}$$

Thus the quantity that is invariant under gauge transformation (assuming only that $G^\lambda{}_\lambda = 0$) is in fact

$$\Delta_\epsilon[\delta(g^{\mu\nu}G_{\mu\nu})] = \Delta_\epsilon[-h^{\mu\nu}G_{\mu\nu} + g^{\mu\nu}\delta G_{\mu\nu}] = 0. \tag{1.13}$$

If the background trace vanishes, it must be the perturbation of the full trace (and not just $g^{\mu\nu}\delta G_{\mu\nu}$) that is then gauge invariant.