QFT

Ch 2: The Klein-Gordon Field

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2.2 **The complex scalar field**

Consider the field theory of a complex-valued scalar field obeying the the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \; (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

(a) Find the conjugrate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \; (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

(c) Rewrite the conserved charge

$$Q = \int d^3x \; \frac{i}{2}(\phi^* \pi^* - \pi \phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q_i = \int d^3x \; \frac{i}{2} \left(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b \right),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

(a) With

$$\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})}$$

we form the conjugate momenta from our Lagrangian

$$\pi(\mathbf{x}) = \dot{\phi}^*; \qquad \pi^*(\mathbf{x}) = \dot{\phi}$$

From the canonical commutation relation of a field and its conjugate momenta,

$$[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}')$$

and taking the conjugate

$$[\phi(\mathbf{x})^*, \pi(\mathbf{x}')^*] = -[\phi(\mathbf{x}), \pi(\mathbf{x}')]^\dagger = i\delta(\mathbf{x} - \mathbf{x}').$$

The other usual commutation relations should also hold

$$[\phi(\mathbf{x}), \phi^*(\mathbf{x}')] = [\pi(\mathbf{x}), \pi^*(\mathbf{x}')] = [\phi(\mathbf{x}), \pi^*(\mathbf{x}')] = 0.$$

To find the Hamiltonian, we may use

$$H = \int d^3x \, \left(\sum_i \pi_i(\mathbf{x}) \dot{\phi}_i(\mathbf{x}) - \mathcal{L} \right) = \int d^3x \; \mathcal{H}$$

thus

$$\begin{aligned}
\mathcal{H} &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} \dot{\phi}^* - \mathcal{L} \\
&= \dot{\phi}^* \dot{\phi} + m^2 \phi \phi^* + \nabla\phi \cdot \nabla\phi^* \\
&= \pi^* \pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi
\end{aligned}$$

To find the operators' time dependence, we compute the Heisenberg equation of motion

$$\begin{aligned}
i\frac{\partial}{\partial t}\phi(x) &= [\phi(x), H] \\
&= \left[\phi(x), \int d^3x' \; \pi^*(x')\pi(x') + \nabla\phi^*(x') \cdot \nabla\phi(x') + m^2\phi^*(x')\phi(x') \right] \\
&= \int d^3x' \; \left(\pi^*(x')[\phi(x), \pi(x')] + \nabla\phi^*(x') \cdot [\phi(x), \nabla\phi(x')] + m^2\phi^*(x')[\phi(x), \phi(x')] \right) \\
&= \int d^3x' \; i\pi^*(x')\delta^3(\mathbf{x} - \mathbf{x}') \\
&= i\pi^*(\mathbf{x}, t).
\end{aligned}$$

Similarly

$$\frac{\partial}{\partial t}\phi^*(x) = \pi(x).$$

For the conjugate momenta

$$\begin{aligned}
[\pi(x), H] &= \left[\pi(x), \int d^3x' \; \pi^*(x')\pi(x') + \nabla\phi^*(x') \cdot \nabla\phi(x') + m^2\phi^*(x')\phi(x') \right] \\
&= \int d^3x' \; \left(\pi^*(x')[\pi(x), \pi(x')] + \nabla\phi^*(x') \cdot [\pi(x), \nabla\phi(x')] + m^2\phi^*(x')[\pi(x), \phi(x')] \right)
\end{aligned}$$

Looking at the commutator with the gradient

$$[\pi(x), \nabla\phi(x')] = \nabla_{x'}[\pi(x), \phi(x')] = -i\nabla\delta^3(\mathbf{x} - \mathbf{x}')$$

with the derivative defined as (for arbitrary $\mathbf{f}$)

$$\mathbf{f} \cdot \nabla \delta^3(\mathbf{x} - \mathbf{x}') = -(\nabla \cdot \mathbf{f}) \delta^3(\mathbf{x} - \mathbf{x}').$$

Back to the EOM for $\pi(x)$

$$\begin{aligned} [\pi(x), H] &= \int d^3x' \ \left[i\nabla^2 \phi^*(x') \delta^3(\mathbf{x} - \mathbf{x}) - im^2 \phi^*(x') \delta^3(\mathbf{x} - \mathbf{x}') \right] \\ \frac{\partial \pi(x)}{\partial t} &= \nabla^2 \phi^*(x) - m^2 \phi^*(x) \end{aligned}$$

and likewise for the conjugate

$$\frac{\partial \pi^*(x)}{\partial t} = \nabla^2 \phi(x) - m^2 \phi(x).$$

Substituting $\pi = \dot{\phi}$ we arrive at the Klein-Gordon equation for each field

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0; \quad \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$$

(b) We can express $\phi(x)$ in momentum space as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

and then apply the Klein-Gordon equation to arrive at

$$\left(\frac{\partial^2}{\partial t^2} + \mathbf{p}\cdot\mathbf{p} + m^2\right)\phi(\mathbf{p}, t) = 0$$

which has solutions

$$\phi(\mathbf{p}, t) = a(\mathbf{p})e^{-i\omega_{\mathbf{p}}t} + b(\mathbf{p})e^{i\omega_{\mathbf{p}}t}; \qquad \omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}.$$

This applies to both fields $\phi(x)$ and $\phi^*(x)$. In the case of the real field, we choose the coefficients of momentum in analogy with the harmonic oscillator raising/lowering operators as the following

$$a(\mathbf{p}) \to \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}}$$

$$b(\mathbf{p}) \to \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} a^{\dagger}_{-\mathbf{p}}$$

thus

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} (a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t} + a^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t})$$

and from the Heisenberg equation of motion $\pi(x) = \dot{\phi}(x)$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t} - a^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t}\right).$$

Imposing $[\phi(x), \pi(x')]_{x_0 = x'_0} = i\delta^3(\mathbf{x} - \mathbf{x}')$ leads to

$$[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{p}'}] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}').$$

This last commutation relation aligns with our expectation of a harmonic oscillator. The specific form of the coefficients was chosen so that $\phi(x)$ is self adjoint i.e. $\phi^{\dagger}(\mathbf{p}, t) = \phi(-\mathbf{p}, t)$. This is important because for the complex field, $\phi(x)$ is no longer self adjoint. As such we choose coefficients as

$$a(\mathbf{p}) \to \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}}$$

$$b(\mathbf{p}) \to \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} b^{\dagger}_{-\mathbf{p}}.$$

Recalling that the Heisenberg e.o.m. are now $\pi = \dot{\phi}^{\dagger}$ we have the following:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} (a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t} + b^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t})$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t} - b_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t}\right).$$

$$\phi^{\dagger}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} (a^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t} + b_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t})$$

$$\pi^{\dagger}(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}t} - b^{\dagger}_{-\mathbf{p}} e^{i\omega_{\mathbf{p}}t}\right).$$

In maintaining $[\phi(\mathbf{x}), \pi(\mathbf{x}')] = [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')$ we find the following commutation relation (denote $c \equiv e^{i\omega_{\mathbf{p}} t_0}$)

$$\begin{aligned}
i\delta^3(\mathbf{x}-\mathbf{x}') &= \int \frac{d^3p d^3p'}{(2\pi)^6} \left(\frac{i}{2}\right) e^{i(\mathbf{p}\cdot x + \mathbf{p}'\cdot\mathbf{x}')} \Big[(ca_{\mathbf{p}} + c^* b^\dagger_{-\mathbf{p}}), (c'^* a^\dagger_{-\mathbf{p}'} - c' b_{\mathbf{p}'})\Big] \\
&= \int \frac{d^3p d^3p'}{(2\pi)^6} \left(\frac{i}{2}\right) e^{i(\mathbf{p}\cdot x + \mathbf{p}'\cdot\mathbf{x}')} \Big\{cc'^*[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}] + c'c^*[b_{\mathbf{p}'}, b^\dagger_{-\mathbf{p}}] - cc'[a_{\mathbf{p}}, b_{\mathbf{p}'}] + c^*c'^*[b^\dagger_{-\mathbf{p}}, a^\dagger_{-p'}]\Big\} \\
&= \int \frac{d^3p d^3p'}{(2\pi)^6} \left(\frac{i}{2}\right) e^{i(\mathbf{p}\cdot x + \mathbf{p}'\cdot\mathbf{x}')} \Big(cc'^*[a_{\mathbf{p}}, a^\dagger_{-\mathbf{p}'}] + c'c^*[b_{\mathbf{p}'}, b^\dagger_{-\mathbf{p}}]\Big)
\end{aligned}$$

which leads to

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}').$$

All other combinations commute. Note that it seemed to helpful to single out which commutators are relevant by keeping the phase factors in; those that were not conjugates were discarded for many (hopefully self evident) reasons.

Now we form the Hamiltonian

$$\begin{aligned}
H &= \int d^3x \; (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) \\
&= \int \frac{d^3x}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \int \frac{d^3p d^3p'}{(2\pi)^3} \Bigg\{ \frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2} (a_{\mathbf{p}'} - b^\dagger_{-\mathbf{p}'})(a^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}) \\
&\qquad\qquad + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_p\omega_{p'}}} (a^\dagger_{-\mathbf{p}'} + b_{\mathbf{p}'})(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}) \Bigg\} \\
&= \int \frac{d^3p d^3p'}{(2\pi)^3} \delta^3(\mathbf{p}+\mathbf{p}') \Bigg\{ \frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2} (a_{\mathbf{p}'} - b^\dagger_{-\mathbf{p}'})(a^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}) \\
&\qquad\qquad + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_p\omega_{p'}}} (a^\dagger_{-\mathbf{p}'} + b_{\mathbf{p}'})(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}) \Bigg\} \\
&= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{2}\right) \Big\{(a_{-\mathbf{p}} - b^\dagger_{\mathbf{p}})(a^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}) + (a^\dagger_{\mathbf{p}} + b_{-\mathbf{p}})(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}})\Big\} \\
&= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{2}\right) \Big\{(a_{-\mathbf{p}} - b^\dagger_{\mathbf{p}})(a^\dagger_{-\mathbf{p}} - b_{\mathbf{p}}) + (a^\dagger_{-\mathbf{p}} + b_{\mathbf{p}})(a_{-\mathbf{p}} + b^\dagger_{\mathbf{p}})\Big\} \\
&= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{2}\right) \Big(a_{-\mathbf{p}}a^\dagger_{-\mathbf{p}} + a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] + \frac{1}{2}[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}]\right)
\end{aligned}$$

It appears we have creation/annihilation operators for particles of type a and b, with the same relativistic energy and thus same mass.

(c) The Lagrangian is invariant under the transformation

$$\phi \to e^{i\alpha}\phi; \qquad \phi^* \to e^{-i\alpha}\phi^*$$

which in infinitesimal form amounts to the variation $\Delta\phi = i\alpha\phi$. The conserved current is then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = i\alpha\left[(\partial_\mu\phi^*)\phi - (\partial_\mu\phi)\phi^*\right].$$

The conserved charge is then $Q = \int d^3x\ j^0$ and so (for $\alpha = 1/2$)

$$Q = \int d^3x\ \frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right)$$

I suspect we write it in this form so that $Q^\dagger = Q$. Continuing,

$$\begin{aligned}
Q &= \int \frac{d^3x}{(2\pi)^3}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\int\frac{d^3p d^3p'}{(2\pi)^3}\left(\frac{1}{4}\right)\sqrt{\frac{\omega'_{\mathbf{p}}}{\omega_{\mathbf{p}}}}\Big\{\left(a^\dagger_{-\mathbf{p}}e^{i\omega_{\mathbf{p}}t} + b_{\mathbf{p}}e^{-i\omega_{\mathbf{p}}t}\right)\left(a_{\mathbf{p}'}e^{-i\omega'_{\mathbf{p}}t} - b^\dagger_{-\mathbf{p}'}e^{i\omega'_{\mathbf{p}}t}\right) \\
&\qquad + \left(a^\dagger_{-\mathbf{p}'}e^{i\omega'_{\mathbf{p}}t} - b_{\mathbf{p}'}e^{-i\omega'_{\mathbf{p}}t}\right)\left(a_{\mathbf{p}}e^{-i\omega_{\mathbf{p}}t} + b^\dagger_{-\mathbf{p}}e^{i\omega_{\mathbf{p}}t}\right)\Big\} \\
&= \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{4}\right)\Big\{\left(a^\dagger_{-\mathbf{p}}e^{i\omega_{\mathbf{p}}t} + b_{\mathbf{p}}e^{-i\omega_{\mathbf{p}}t}\right)\left(a_{-\mathbf{p}}e^{-i\omega_{\mathbf{p}}t} - b^\dagger_{\mathbf{p}}e^{i\omega_{\mathbf{p}}t}\right) \\
&\qquad + \left(a^\dagger_{\mathbf{p}}e^{i\omega_{\mathbf{p}}t} - b_{-\mathbf{p}}e^{-i\omega_{\mathbf{p}}t}\right)\left(a_{\mathbf{p}}e^{-i\omega_{-\mathbf{p}}t} + b^\dagger_{-\mathbf{p}}e^{i\omega_{\mathbf{p}}t}\right)\Big\} \\
&= \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{2}\right)\left\{a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right\} \\
&= \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{2}\right)\left\{a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + \delta^3(0)\right\}
\end{aligned}$$

Not accounting for the infinite constant, we see that the corresponding charges are $\pm\frac{1}{2}q$, where $\alpha = \frac{1}{2}q$ is some charge constant.

(d) To maintain the Klein-Gordon equation for two complex fields (4 independent fields), we may add on a similar Lagrangian so that

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi_1\partial^\mu\phi_1^* + \partial_\mu\phi_2\partial^\mu\phi_2^* - m^2(\phi_1\phi_1^* + \phi_2\phi_2^*)\\ &= |\partial_\mu\phi_1|^2 + |\partial_\mu\phi_2|^2 - m^2\left(|\phi_1|^2 + |\phi_2|^2\right).\end{aligned}$$

Based on the form of the conserved current, we need to write this in terms of vectors. Denote

$$\vec{\Phi} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}; \qquad \Phi_i = \phi_i$$

and now

$$\mathcal{L} = \partial^\mu\Phi_i^\dagger\partial_\mu\Phi^i - m^2\left(\Phi_i^\dagger\Phi^i\right).$$

The Lagrangian is invariant under $SU(2)$ transformation

$$\vec{\Phi} \to e^{i\vec{\sigma}\cdot\vec{\alpha}}\vec{\Phi}$$

infinitesimally

$$\alpha^i\Delta\Phi_j = i\alpha^i(\sigma_{jk})^i\Phi^k.$$

We may also include the identity matrix, which amounts to the same transformation as earlier ($U(1)$)

$$\phi_i \to e^{i\alpha}\phi_i.$$

Then we may combine everything into a conserved tensor of four currents (define $\sigma^0 = \mathbb{1}$)

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^i)}\alpha^\nu\Delta\Phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^{\dagger i})}\alpha^\nu\Delta\Phi_i^\dagger\\ &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^i)}i\alpha^\nu(\sigma_{ij})^\nu\Phi^j - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^{\dagger i})}i\alpha^\nu(\sigma_{ij}^\dagger)^\nu\Phi^{\dagger j}\end{aligned}$$

The four conserved charges are then (with $\alpha = -1/2$)

$$Q^\nu = \int d^3x\; T^{0\nu} = \int d^3x\;\frac{i}{2}\left[\pi_a^\dagger(\sigma_{ab}^\dagger)^\nu\phi_b^\dagger - \pi_a(\sigma_{ab})^\nu\phi_b\right]$$

There is an implied summation over a, b here. I suppose as we require $Q = Q^\dagger$ we rewrite this as

$$Q^\nu = \int d^3x\;\frac{i}{2}\left[\phi_a^\dagger(\sigma_{ab})^\nu\pi_b^\dagger - \pi_a(\sigma_{ab})^\nu\phi_b\right].$$

Now forming the commutator

$$[Q^i, Q^j] = \int d^3x\; d^3x'\left(\frac{i}{2}\right)^2\left[\left(\phi_a^\dagger(\sigma_{ab})^i\pi_b^\dagger - \pi_a(\sigma_{ab})^i\phi_b\right), \left(\phi_{a'}^\dagger(\sigma_{a'b'})^j\pi_{b'}^\dagger - \pi_{a'}(\sigma_{a'b'})^j\phi_{b'}\right)\right]$$

$$\begin{aligned}&\left[\left(\phi_a^\dagger(\sigma_{ab})^i\pi_b^\dagger - \pi_a(\sigma_{ab})^i\phi_b\right), \left(\phi_{a'}^\dagger(\sigma_{a'b'})^j\pi_{b'}^\dagger - \pi_{a'}(\sigma_{a'b'})^j\phi_{b'}\right)\right]\\ &= \delta_{aa',bb'}\left[\phi_a^\dagger(\sigma_{ab})^i\pi_b^\dagger, \phi_{a'}^\dagger(\sigma_{a'b'})^j\pi_{b'}^\dagger\right] + \delta_{aa',bb'}\left[\pi_a(\sigma_{ab})^i\phi_b, \pi_{a'}(\sigma_{a'b'})^j\phi_{b'}\right]\\ &= \left[\phi_a^\dagger(\mathbf{x})(\sigma_{ab})^i\pi_b^\dagger(\mathbf{x}), \phi_a^\dagger(\mathbf{x}')(\sigma_{ab})^j\pi_b^\dagger(\mathbf{x}')\right] + \left[\pi_a(\mathbf{x})(\sigma_{ab})^i\phi_b(\mathbf{x}), \pi_a(\mathbf{x}')(\sigma_{ab})^j\phi_b(\mathbf{x}')\right]\\ &= \phi_a^\dagger\pi_b^\dagger[(\sigma_{ab})^i, (\sigma_{ab})^j] + \pi_a\phi_b[(\sigma_{ab})^i, (\sigma_{ab}^j)]\\ &= \left(\phi_a^\dagger\pi_b^\dagger + \pi_a\phi_b\right)[(\sigma_{ab})^i, (\sigma_{ab})^j](wrong)\end{aligned}$$

Looking at part of the commutator

$$[(\sigma_{ab})^i\phi_a(\mathbf{x})\pi_b(\mathbf{x}),(\sigma_{ab})^j\phi_a(\mathbf{x}')\pi_b(\mathbf{x}')] = 2i\delta^3(\mathbf{x}-\mathbf{x}')\phi_a\pi_b\sigma^i_{ab}\sigma^j_{ab}.$$

The conjugate expression is then

$$[(\sigma_{ab})^i\phi_a(\mathbf{x})\pi_b(\mathbf{x}),(\sigma_{ab})^j\phi_a(\mathbf{x}')\pi_b(\mathbf{x}')]^\dagger = -2i\delta^3(\mathbf{x}-\mathbf{x}')\pi^\dagger_a\phi^\dagger_b\sigma^j_{ab}\sigma^i_{ab}.$$

Back to the integral

$$[Q^i,Q^j] = \int d^3x\left(\frac{i}{2}\right)\left(\phi^\dagger_a\pi^\dagger_b - \pi_a\phi_b\right)[(\sigma_{ab})^i,(\sigma_{ab})^j] = \epsilon_{ijk}Q^k.$$

In this form it should be easy to generalize to n independent fields by replacing the sigma matrices by the general $SU(n)$ commutation relation

$$[\lambda_i,\lambda_j] = i\sum_{k=1}^{n^2-1} f_{ijk}\lambda_k\,.$$

where f_{ijk} are the structure constants. As such, there will be n^2 conserved charges associated with n independent complex fields (at least for those due to the $SU(n)$) symmetry.