

RW SVT4 $k \neq 0$ v1

1 Background

1.1 Conformal $\Omega(\tau)$

$$ds^2 = \Omega^2(\tau) \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} = \text{diag} \left(-1, \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right) \quad (1.1)$$

$$G_{00} = -3k - 3\dot{\Omega}^2 \Omega^{-2} \quad G_{ij} = k\tilde{g}_{ij} - \dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} + 2\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij} \quad (1.2)$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\Omega^2 \tilde{g}_{\mu\nu}, \quad U_\mu = -\Omega \delta_\mu^0 \quad [\text{Evaluated in (1.1)}] \quad (1.3)$$

$$\Delta_{00}^{(0)} = -3k - 3\dot{\Omega}^2 \Omega^{-2} + \Omega^2 \rho \quad (1.4)$$

$$\rightarrow \boxed{\rho = 3k\Omega^{-2} + 3\dot{\Omega}^2 \Omega^{-4}} \quad (1.5)$$

$$\Delta_{ij}^{(0)} = k\tilde{g}_{ij} - \dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} + 2\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij} + \Omega^2 p \tilde{g}_{ij} \quad (1.6)$$

$$\rightarrow \boxed{p = -k\Omega^{-2} + \dot{\Omega}^2 \Omega^{-4} - 2\ddot{\Omega} \Omega^{-3}} \quad (1.7)$$

$$\begin{aligned} \nabla_\mu T^{\mu 0} &= \Omega^{-5} \left(\tilde{g}^{ab} T_{ab} \dot{\Omega} + T_{00} \dot{\Omega} + \dot{T}_{00} \Omega - \Omega \tilde{\nabla}_a T_0^a \right) \\ &= 3\dot{\Omega} \Omega^{-3} p + 3\dot{\Omega} \Omega^{-3} \rho + \Omega^{-2} \dot{\rho} \end{aligned} \quad (1.8)$$

$$\begin{aligned} \nabla_\mu T^{\mu i} &= \Omega^{-5} \left(-2T_0^i \dot{\Omega} - \dot{T}_0^i \Omega + \Omega \tilde{\nabla}_a T^{ia} \right) \\ &= 0 \end{aligned} \quad (1.9)$$

2 Fluctuations

$$ds^2 = \Omega^2(\tau) [\tilde{g}_{\mu\nu} + f_{\mu\nu}] dx^\mu dx^\nu \quad (2.1)$$

$$\tilde{g}_{\mu\nu} = \text{diag} \left(-1, \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right) \quad (2.2)$$

$$f_{\mu\nu} = -2\tilde{g}_{\mu\nu} \chi + 2\tilde{\nabla}_\mu \tilde{\nabla}_\nu F + \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu} \quad (2.3)$$

$$f_{00} = 2\chi + 2\ddot{F} + 2\dot{F}_0 + 2F_{00} \quad (2.4)$$

$$f_{0i} = 2\tilde{\nabla}_i \dot{F} + \dot{F}_i + \tilde{\nabla}_i F_0 + 2F_{0i} \quad (2.5)$$

$$f_{ij} = -2\tilde{g}_{ij}\chi + 2\tilde{\nabla}_i \tilde{\nabla}_j F + \tilde{\nabla}_i F_j + \tilde{\nabla}_j F_i + 2F_{ij} \quad (2.6)$$

$$\tilde{g}^{\mu\nu} F_{\mu\nu} = 0, \quad \tilde{\nabla}^\mu F_{\mu\nu} = 0, \quad \tilde{\nabla}^\mu F_\mu = 0 \quad (2.7)$$

$$\delta U_0 = \frac{1}{2}\Omega f_{00} \quad (2.8)$$

$$\begin{aligned} \delta T_{00} = & (-6k - 6\dot{\Omega}^2\Omega^{-2})\ddot{F} + \Omega^2\delta\rho \\ & + (-6k - 6\dot{\Omega}^2\Omega^{-2})\chi + (-6k - 6\dot{\Omega}^2\Omega^{-2})\dot{F}_0 + (-6k - 6\dot{\Omega}^2\Omega^{-2})F_{00} \end{aligned} \quad (2.9)$$

$$\begin{aligned} \delta T_{0i} = & (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i \dot{F} \\ & + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\dot{F}_i \\ & + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i \\ & + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_0 + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})F_{0i} \end{aligned} \quad (2.10)$$

$$\begin{aligned} \delta T_{ij} = & \Omega^2\delta p\tilde{g}_{ij} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\chi \\ & + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i \tilde{\nabla}_j F + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_j \\ & + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j F_i + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})F_{ij} \end{aligned} \quad (2.11)$$

$$\begin{aligned} g^{\mu\nu}\delta T_{\mu\nu} = & (6\dot{\Omega}^2\Omega^{-4} + 6k\Omega^{-2})\ddot{F} + 3\delta p - \delta\rho + (12\ddot{\Omega}\Omega^{-3} + 12k\Omega^{-2})\chi \\ & + (2\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} - 2k\Omega^{-2})\tilde{\nabla}_a \tilde{\nabla}^a F + (8\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} + 4k\Omega^{-2})\dot{F}_0 + (8\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} + 4k\Omega^{-2})F_{00} \end{aligned} \quad (2.12)$$

$$\begin{aligned} \delta G_{00} = & 6k\ddot{F} + 6\dot{\Omega}\Omega^{-1}\dot{\chi} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \dot{F} - 2\tilde{\nabla}_a \tilde{\nabla}^a \chi + 6k\dot{F}_0 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a F_0 - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00} \\ & + 8kF_{00} + \tilde{\nabla}_a \tilde{\nabla}^a F_{00} \end{aligned} \quad (2.13)$$

$$\begin{aligned} \delta G_{0i} = & 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i \ddot{F} + (4k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i \dot{F} - 2\tilde{\nabla}_i \dot{\chi} \\ & + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i \chi + (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\dot{F}_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i \dot{F}_0 \\ & + (3k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_0 - \ddot{F}_{0i} + (4k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})F_{0i} + \tilde{\nabla}_a \tilde{\nabla}^a F_{0i} \\ & + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00} \end{aligned} \quad (2.14)$$

$$\begin{aligned} \delta G_{ij} = & 2\dot{\Omega}\Omega^{-1}\ddot{\tilde{g}}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\ddot{F}\tilde{g}_{ij} - 2\dot{\chi}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\chi}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a \tilde{\nabla}^a \dot{F} \\ & + 2\tilde{g}_{ij}\tilde{\nabla}_a \tilde{\nabla}^a \chi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j \tilde{\nabla}_i \dot{F} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j \tilde{\nabla}_i F \\ & - 2\tilde{\nabla}_j \tilde{\nabla}_i \chi + 2\dot{\Omega}\Omega^{-1}\ddot{F}_{00}\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\dot{F}_0\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a \tilde{\nabla}^a F_0 \\ & + (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_j + (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j F_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j \tilde{\nabla}_i F_0 - \ddot{F}_{ij} \\ & - 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})F_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} \\ & + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i} \end{aligned} \quad (2.15)$$

$$\begin{aligned} g^{\mu\nu}\delta G_{\mu\nu} = & 6\dot{\Omega}\Omega^{-3}\ddot{\tilde{F}} + (-6\dot{\Omega}^2\Omega^{-4} + 12\ddot{\Omega}\Omega^{-3} - 6k\Omega^{-2})\ddot{F} - 6\Omega^{-2}\dot{\chi} - 12\dot{\Omega}\Omega^{-3}\dot{\chi} - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a \tilde{\nabla}^a \dot{F} \\ & + (-2\dot{\Omega}^2\Omega^{-4} + 4\ddot{\Omega}\Omega^{-3} + 2k\Omega^{-2})\tilde{\nabla}_a \tilde{\nabla}^a F + 6\Omega^{-2}\tilde{\nabla}_a \tilde{\nabla}^a \chi + 6\dot{\Omega}\Omega^{-3}\dot{F}_0 \end{aligned}$$

$$\begin{aligned}
& +(-8\dot{\Omega}^2\Omega^{-4} + 16\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})\dot{F}_0 \\
& -6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a F_0 + (-8\dot{\Omega}^2\Omega^{-4} + 16\ddot{\Omega}\Omega^{-3} - 8k\Omega^{-2})F_{00}
\end{aligned} \tag{2.16}$$

3 Field Equations

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} \tag{3.1}$$

$$\begin{aligned}
\Delta_{00} = & -6\dot{\Omega}^2\Omega^{-2}\ddot{F} + \Omega^2\delta\rho + 6\dot{\Omega}\Omega^{-1}\dot{\chi} + (-6k - 6\dot{\Omega}^2\Omega^{-2})\chi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} \\
& -2\tilde{\nabla}_a\tilde{\nabla}^a\chi - 6\dot{\Omega}^2\Omega^{-2}\dot{F}_0 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a F_0 - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00} + (2k - 6\dot{\Omega}^2\Omega^{-2})F_{00} \\
& + \tilde{\nabla}_a\tilde{\nabla}^a F_{00}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\Delta_{0i} = & 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\ddot{F} + 2k\tilde{\nabla}_i\dot{F} - 2\tilde{\nabla}_i\dot{\chi} + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V \\
& + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\chi + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{F}_0 + 2k\tilde{\nabla}_i F_0 - \ddot{F}_{0i} + 2kF_{0i} \\
& + \tilde{\nabla}_a\tilde{\nabla}^a F_{0i} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\Delta_{ij} = & 2\dot{\Omega}\Omega^{-1}\ddot{F}\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\ddot{F}\tilde{g}_{ij} - 2\ddot{\chi}\tilde{g}_{ij} + \Omega^2\delta p\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\chi}\tilde{g}_{ij} \\
& + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\chi - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} + 2\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\chi \\
& + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\tilde{\nabla}_j F + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{F} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\tilde{\nabla}_i F \\
& - 2\tilde{\nabla}_j\tilde{\nabla}_i\chi + 2\dot{\Omega}\Omega^{-1}\ddot{F}_0\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\dot{F}_0\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a F_0 \\
& + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i F_0 - \ddot{F}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} - 2kF_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{ij} \\
& + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i}
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} = & 6\dot{\Omega}\Omega^{-3}\ddot{F} + 12\ddot{\Omega}\Omega^{-3}\ddot{F} - 6\Omega^{-2}\ddot{\chi} + 3\delta p - \delta\rho - 12\dot{\Omega}\Omega^{-3}\dot{\chi} + (12\ddot{\Omega}\Omega^{-3} + 12k\Omega^{-2})\chi \\
& - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\chi + 6\dot{\Omega}\Omega^{-3}\dot{F}_0 + 12\ddot{\Omega}\Omega^{-3}\dot{F}_0 \\
& - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a F_0 + (12\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})F_{00}
\end{aligned} \tag{3.5}$$

4 Field Equations (G.I. Form)

$$\alpha = \chi - \dot{\Omega}\Omega^{-1}\dot{F} - \dot{\Omega}\Omega^{-1}F_0, \quad F_{\mu\nu}, \quad V_i \tag{4.1}$$

$$V^{GI} = V - \Omega^2\dot{\Omega}^{-1}\chi \tag{4.2}$$

$$\delta\rho^{GI} = \delta\rho - 12\dot{\Omega}^2\chi\Omega^{-4} + 6\ddot{\Omega}\chi\Omega^{-3} - 6k\chi\Omega^{-2} \tag{4.3}$$

$$\delta p^{GI} = \delta p - 4\dot{\Omega}^2\chi\Omega^{-4} + 8\ddot{\Omega}\chi\Omega^{-3} + 2k\chi\Omega^{-2} - 2\ddot{\Omega}\dot{\Omega}^{-1}\chi\Omega^{-2} \tag{4.4}$$

$$\begin{aligned}
\Delta_{00} = & 6\dot{\Omega}\Omega^{-1}\dot{\alpha} + \Omega^2\delta\rho^{GI} + (6\dot{\Omega}^2\Omega^{-2} - 6\ddot{\Omega}\Omega^{-1})\alpha - 2\tilde{\nabla}_a\tilde{\nabla}^a\alpha - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00} \\
& + (2k - 6\dot{\Omega}^2\Omega^{-2})F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{00}
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\Delta_{0i} = & -2\tilde{\nabla}_i\dot{\alpha} + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V^{GI} \\
& + (2\ddot{\Omega}\Omega^{-1} - 2\dot{\Omega}\Omega^{-1} - 2\dot{\Omega}^{-1}k\Omega)\tilde{\nabla}_i\alpha + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i - \ddot{F}_{0i} + 2kF_{0i}
\end{aligned}$$

$$+\tilde{\nabla}_a \tilde{\nabla}^a F_{0i} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00} \quad (4.6)$$

$$\begin{aligned} \Delta_{ij} = & -2\ddot{\alpha}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\alpha}\tilde{g}_{ij} + \Omega^2\tilde{g}_{ij}\delta p^{GI} + (2\ddot{\Omega}\dot{\Omega}^{-1} + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\alpha + 2\tilde{g}_{ij}\tilde{\nabla}_a \tilde{\nabla}^a \alpha \\ & - 2\tilde{\nabla}_j \tilde{\nabla}_i \alpha - \ddot{F}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} - 2kF_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} \\ & + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i} \end{aligned} \quad (4.7)$$

$$\begin{aligned} g^{\mu\nu}\Delta_{\mu\nu} = & -6\Omega^{-2}\ddot{\alpha} - 12\dot{\Omega}\Omega^{-3}\dot{\alpha} + 3\delta p^{GI} - \delta\rho^{GI} + (-6\ddot{\Omega}\Omega^{-3} + 6\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-2})\alpha \\ & + 6\Omega^{-2}\tilde{\nabla}_a \tilde{\nabla}^a \alpha + (12\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})F_{00} \end{aligned} \quad (4.8)$$

5 Conservation

Variations are with respect to background (1.1).

$$\delta(\nabla_\mu G^{\mu 0}) = 0 \quad (5.1)$$

$$\delta(\nabla_\mu G^{\mu i}) = 0 \quad (5.2)$$

$$\begin{aligned} \delta(\nabla_\mu T^{\mu 0}) = & \Omega^{-2}\dot{\delta\rho} + 3\dot{\Omega}\Omega^{-3}\delta p + 3\dot{\Omega}\Omega^{-3}\delta\rho + (-12\dot{\Omega}^2\Omega^{-6} + 6\ddot{\Omega}\Omega^{-5} - 6k\Omega^{-4})\dot{\chi} \\ & + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}_a \tilde{\nabla}^a \dot{F} \\ & + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a \tilde{\nabla}^a V + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}_a \tilde{\nabla}^a F_0 \\ & + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\dot{F}_{00} \end{aligned} \quad (5.3)$$

$$\begin{aligned} = & \Omega^{-2}\dot{\delta\rho}^{GI} + 3\dot{\Omega}\Omega^{-3}\delta p^{GI} + 3\dot{\Omega}\Omega^{-3}\delta\rho^{GI} + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a \tilde{\nabla}^a V^{GI} \\ & + (4\dot{\Omega}\Omega^{-5} - 2\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-4} + 2\dot{\Omega}^{-1}k\Omega^{-3})\tilde{\nabla}_a \tilde{\nabla}^a \alpha + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\dot{F}_{00} \end{aligned} \quad (5.4)$$

$$\begin{aligned} \delta(\nabla_\mu T^{\mu i}) = & (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i \ddot{F} + \Omega^{-2}\tilde{\nabla}^i \delta p + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}^i \dot{V} \\ & + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^i V \\ & + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i \chi + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\dot{V}^i \\ & + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})V^i \\ & + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i \dot{F}_0 + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i F_{00} \end{aligned} \quad (5.5)$$

$$\begin{aligned} = & (4\dot{\Omega}\Omega^{-5} - 2\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-4} + 2\dot{\Omega}^{-1}k\Omega^{-3})\tilde{\nabla}^i \dot{\alpha} + \Omega^{-2}\tilde{\nabla}^i \delta p^{GI} \\ & + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}^i \dot{V}^{GI} \\ & + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^i V^{GI} \\ & + (4\dot{\Omega}^2\Omega^{-6} - 6\ddot{\Omega}\Omega^{-5} + 2\ddot{\Omega}^2\dot{\Omega}^{-2}\Omega^{-4} + 2k\Omega^{-4} - 2\ddot{\Omega}\dot{\Omega}^{-2}k\Omega^{-3})\tilde{\nabla}^i \alpha + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\dot{V}^i \\ & + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})V^i + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i F_{00} \end{aligned} \quad (5.6)$$

$$\begin{aligned} \delta(\nabla_\mu \Delta^{\mu 0}) = & \Omega^{-2}\dot{\delta\rho}^{GI} + 3\dot{\Omega}\Omega^{-3}\delta p^{GI} + 3\dot{\Omega}\Omega^{-3}\delta\rho^{GI} + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a \tilde{\nabla}^a V^{GI} \\ & + (4\dot{\Omega}\Omega^{-5} - 2\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-4} + 2\dot{\Omega}^{-1}k\Omega^{-3})\tilde{\nabla}_a \tilde{\nabla}^a \alpha + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\dot{F}_{00} \end{aligned} \quad (5.7)$$

$$\begin{aligned} \delta(\nabla_\mu \Delta^{\mu i}) = & (4\dot{\Omega}\Omega^{-5} - 2\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-4} + 2\dot{\Omega}^{-1}k\Omega^{-3})\tilde{\nabla}^i \dot{\alpha} + \Omega^{-2}\tilde{\nabla}^i \delta p^{GI} \\ & + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}^i \dot{V}^{GI} \end{aligned}$$

$$\begin{aligned}
& +(-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^i V^{GI} \\
& + (4\dot{\Omega}^2\Omega^{-6} - 6\ddot{\Omega}\Omega^{-5} + 2\ddot{\Omega}^2\Omega^{-2}\Omega^{-4} + 2k\Omega^{-4} - 2\ddot{\Omega}\dot{\Omega}^{-2}k\Omega^{-3})\tilde{\nabla}^i \alpha + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\dot{V}^i \\
& + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})V^i + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i F_{00} \tag{5.8}
\end{aligned}$$

$$\begin{aligned}
\nabla_i \delta(\nabla_\mu \Delta^{\mu i}) &= (4\dot{\Omega}\Omega^{-5} - 2\ddot{\Omega}\dot{\Omega}^{-1}\Omega^{-4} + 2\dot{\Omega}^{-1}k\Omega^{-3})\tilde{\nabla}_a \tilde{\nabla}^a \dot{\alpha} + \Omega^{-2}\tilde{\nabla}_a \tilde{\nabla}^a \delta p^{GI} \\
& + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a \tilde{\nabla}^a \dot{V}^{GI} \\
& + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}_a \tilde{\nabla}^a V^{GI} \\
& + (4\dot{\Omega}^2\Omega^{-6} - 6\ddot{\Omega}\Omega^{-5} + 2\ddot{\Omega}^2\Omega^{-2}\Omega^{-4} + 2k\Omega^{-4} - 2\ddot{\Omega}\dot{\Omega}^{-2}k\Omega^{-3})\tilde{\nabla}_a \tilde{\nabla}^a \alpha \\
& + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}_a \tilde{\nabla}^a F_{00} \tag{5.9}
\end{aligned}$$

Computationally, we find that $\delta(\nabla_\mu G^{\mu\nu}) = 0$ evaluates to zero as expected from the Bianchi identity. In addition, we find $\delta(\nabla_\mu \Delta^{\mu\nu}) = \delta(\nabla_\mu T^{\mu\nu})$, providing the perturbed conservation condition for a RW perfect fluid, analogous to (1.8).

6 TT Projection

$$\begin{aligned}
(\nabla^2 - 2k)(\nabla^2 - 3k)\Delta_{ij}^{T\theta} &= (\nabla^2 - 2k)(\nabla^2 - 3k)\Delta_{ij} - \nabla^2 \nabla_i \nabla^l \Delta_{jl} - \nabla^2 \nabla_j \nabla^l \Delta_{il} + 3k \nabla_j \nabla^l \Delta_{il} + 3k \nabla_i \nabla^l \Delta_{jl} \\
& + \frac{1}{2} \nabla_i \nabla_j \nabla^k \nabla^l \Delta_{kl} + \frac{1}{2} g_{ij} \nabla^2 \nabla^k \nabla^l \Delta_{kl} - 2k g_{ij} \nabla^l \nabla^k \Delta_{kl} + \frac{1}{2} \nabla_i \nabla_j (\nabla^2 + 4k)\Delta \\
& - \frac{1}{2} g_{ij} \nabla^2 (\nabla^2 - 3k)\Delta - \frac{1}{2} g_{ij} k(\nabla^2 + 4k)\Delta \tag{6.1}
\end{aligned}$$

$$\begin{aligned}
&= -6k^2 \ddot{F}_{ij} - 12k^3 F_{ij} + 2k \ddot{F}_{00} \tilde{g}_{ij} + 6k^2 \ddot{F}_{00} \tilde{g}_{ij} + 4k^3 F_{00} \tilde{g}_{ij} - 12k^2 \dot{F}_{ij} \dot{\Omega} \Omega^{-1} + 4k \ddot{F}_{00} \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \\
& + 4k^2 \dot{F}_{00} \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} - \frac{1}{2} \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{F}_{00} - \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{F}_{00} + 5k \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} - 5k \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{F}_{00} \\
& + 10k \dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} - 4k \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} + 16k^2 \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} - 6k^2 \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a F_{00} \\
& + \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i \ddot{F}_{0j} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i \dot{F}_{0j} + 5k \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i \dot{F}_{0j} + \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \ddot{F}_{0i} \\
& + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \dot{F}_{0i} + 5k \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \dot{F}_{0i} - \dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \dot{F}_{00} - 2k \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \tilde{\nabla}_i F_{00} \\
& - \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} + \tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} + \dot{\Omega} \tilde{g}_{ij} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} \\
& - 7k \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} + 3k \tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{00} - \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i \dot{F}_{0j} - \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \dot{F}_{0i} \\
& + \frac{1}{2} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \tilde{\nabla}_i F_{00} + \tilde{\nabla}_c \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{\nabla}_c \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{00} - 3k \tilde{\nabla}_i \ddot{F}_{0j} \\
& - 6k \dot{\Omega} \Omega^{-1} \tilde{\nabla}_i \dot{F}_{0j} - 6k^2 \tilde{\nabla}_i \dot{F}_{0j} - 3k \tilde{\nabla}_j \ddot{F}_{0i} - 6k \dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \dot{F}_{0i} - 6k^2 \tilde{\nabla}_j \dot{F}_{0i} - \frac{1}{2} \tilde{\nabla}_j \tilde{\nabla}_i \ddot{F}_{00} \\
& - \dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i \ddot{F}_{00} + 2k \dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i \dot{F}_{00} + 2k^2 \tilde{\nabla}_j \tilde{\nabla}_i F_{00} \tag{6.2}
\end{aligned}$$

where Δ is the 3-trace $\Delta = g^{ab} \Delta_{ab}$.

(6.2) can be factorized to the form

$$\begin{aligned}
(\nabla^2 - 2k)(\nabla^2 - 3k)\Delta_{ij}^{T\theta} &= (\tilde{\nabla}_a \tilde{\nabla}^a - 2k)(\tilde{\nabla}_b \tilde{\nabla}^b - 3k) \left[-\ddot{F}_{ij} - 2k F_{ij} - 2\dot{\Omega} \Omega^{-1} \dot{F}_{ij} + \tilde{\nabla}_c \tilde{\nabla}^c F_{ij} - \tilde{\nabla}_i \dot{F}_{0j} - \tilde{\nabla}_j \dot{F}_{0i} \right. \\
& + \tilde{g}_{ij} \ddot{F}_{00} + \frac{2}{3} k \tilde{g}_{ij} F_{00} - \frac{1}{2} \tilde{g}_{ij} \tilde{\nabla}_c \tilde{\nabla}^c F_{00} + \frac{1}{2} \tilde{\nabla}_i \tilde{\nabla}_j F_{00} \Big] \\
& + (2k \tilde{g}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a - \frac{1}{2} \tilde{\nabla}_i \tilde{\nabla}_j) \left[\ddot{F}_{00} + 2\dot{\Omega} \Omega^{-1} (\dot{F}_{00} + k F_{00}) + \frac{1}{3} k \tilde{\nabla}_c \tilde{\nabla}^c F_{00} \right] \\
& + (\tilde{\nabla}_a \tilde{\nabla}^a - 3k) \left[\tilde{\nabla}_i \ddot{F}_{0j} + \tilde{\nabla}_j \ddot{F}_{0i} + 2\dot{\Omega} \Omega^{-1} (\tilde{\nabla}_i \dot{F}_{0j} + \tilde{\nabla}_j \dot{F}_{0i}) + \dot{\Omega} \Omega^{-1} (\tilde{g}_{ij} \tilde{\nabla}_c \tilde{\nabla}^c \dot{F}_{00} - \tilde{\nabla}_i \tilde{\nabla}_j \dot{F}_{00}) \right. \\
& + \frac{2}{3} k \tilde{\nabla}_i \tilde{\nabla}_j F_{00} \Big] \tag{6.3}
\end{aligned}$$

There are other possible factorizations for F_{00} in specific.

Appendix A Possibly Useful Relations

$$\begin{aligned}\tilde{\nabla}^i \Delta_{0i} &= -2\tilde{\nabla}_a \tilde{\nabla}^a \dot{\alpha} + (-4\dot{\Omega}^2 \Omega^{-3} + 2\ddot{\Omega} \Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_a \tilde{\nabla}^a V^{GI} \\ &\quad + (2\ddot{\Omega} \dot{\Omega}^{-1} - 2\dot{\Omega} \Omega^{-1} - 2\dot{\Omega}^{-1} k\Omega)\tilde{\nabla}_a \tilde{\nabla}^a \alpha - \ddot{F}_{00} + 4k\dot{F}_{00} + \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a F_{00}\end{aligned}\quad (\text{A.1})$$

$$\begin{aligned}\tilde{g}^{ij} \Delta_{ij} &= -6\ddot{\alpha} - 6\dot{\Omega} \Omega^{-1} \dot{\alpha} + 3\Omega^2 \delta p^{GI} + (6\ddot{\Omega} \dot{\Omega}^{-1} + 6\dot{\Omega}^2 \Omega^{-2} - 12\ddot{\Omega} \Omega^{-1})\alpha + 4\tilde{\nabla}_a \tilde{\nabla}^a \alpha - \ddot{F}_{00} \\ &\quad + 2\dot{\Omega} \Omega^{-1} \dot{F}_{00} + (-2k - 6\dot{\Omega}^2 \Omega^{-2} + 12\ddot{\Omega} \Omega^{-1})F_{00} + \tilde{\nabla}_a \tilde{\nabla}^a F_{00}\end{aligned}\quad (\text{A.2})$$

$$\begin{aligned}\tilde{\nabla}^i \Delta_{ij} &= -2\tilde{\nabla}_j \ddot{\alpha} - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \dot{\alpha} + \Omega^2 \tilde{\nabla}_j \delta p^{GI} + (2\ddot{\Omega} \dot{\Omega}^{-1} - 4k + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1})\tilde{\nabla}_j \alpha - \ddot{F}_{0j} \\ &\quad - 2\dot{\Omega} \Omega^{-1} \ddot{F}_{0j} + 2k\dot{F}_{0j} + 4\dot{\Omega} k \Omega^{-1} F_{0j} + \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{0j} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a F_{0j} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \dot{F}_{00} \\ &\quad + (-2k - 2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1})\tilde{\nabla}_j F_{00}\end{aligned}\quad (\text{A.3})$$

$$\begin{aligned}\tilde{\nabla}^i \tilde{\nabla}^j \Delta_{ij} &= -2\tilde{\nabla}_a \tilde{\nabla}^a \ddot{\alpha} - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{\alpha} + \Omega^2 \tilde{\nabla}_a \tilde{\nabla}^a \delta p^{GI} + (2\ddot{\Omega} \dot{\Omega}^{-1} - 4k + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1})\tilde{\nabla}_a \tilde{\nabla}^a \alpha \\ &\quad - \ddot{F}_{00} - 2\dot{\Omega} \Omega^{-1} \ddot{F}_{00} + 4k\dot{F}_{00} + 8\dot{\Omega} k \Omega^{-1} \dot{F}_{00} + \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} + 4\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{00} \\ &\quad + (-2k - 2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1})\tilde{\nabla}_a \tilde{\nabla}^a F_{00}\end{aligned}\quad (\text{A.4})$$

$$\begin{aligned}\tilde{\nabla}_a \tilde{\nabla}^a \Delta_{ij} &= -2\tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{\alpha} - 2\dot{\Omega} \Omega^{-1} \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \dot{\alpha} + \Omega^2 \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \delta p^{GI} \\ &\quad + (2\ddot{\Omega} \dot{\Omega}^{-1} + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1})\tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \alpha - 2\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \tilde{\nabla}_i \alpha + 2\tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \alpha - \tilde{\nabla}_a \tilde{\nabla}^a \ddot{F}_{ij} \\ &\quad - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} - 2k\tilde{\nabla}_a \tilde{\nabla}^a F_{ij} + (-2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1})\tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a F_{00} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i F_{0j} \\ &\quad + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j F_{0i} + \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{ij}\end{aligned}\quad (\text{A.5})$$

$$\begin{aligned}\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^b \Delta_{ij} &= -2\tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \ddot{\alpha} - 2\dot{\Omega} \Omega^{-1} \tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{\alpha} + \Omega^2 \tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \delta p^{GI} \\ &\quad + (2\ddot{\Omega} \dot{\Omega}^{-1} + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1})\tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \alpha - 2\tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j \tilde{\nabla}_i \alpha \\ &\quad + 2\tilde{g}_{ij} \tilde{\nabla}_c \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \alpha - \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \ddot{F}_{ij} - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \dot{F}_{ij} - 2k\tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{ij} \\ &\quad + (-2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1})\tilde{g}_{ij} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{00} + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_i F_{0j} \\ &\quad + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_j F_{0i} + \tilde{\nabla}_c \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a \tilde{\nabla}^a F_{ij}\end{aligned}\quad (\text{A.6})$$