3Space Conformal Transformations v1

1 Conformal Transformation of Curvature Tensors

For $D = 3$ with $\mu, \nu = 1, 2, 3$ the Ricci tensor and scalar transform under conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ as

$$
\begin{aligned}
R_{\mu\nu} &\to R_{\mu\nu} + g_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega + \Omega^{-1}\nabla_\mu\nabla_\nu\Omega - 2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \\
R &\to \Omega^{-2}R + 4\Omega^{-3}\nabla_\alpha\nabla^\alpha\Omega - 2\Omega^{-4}\nabla_\alpha\Omega\nabla^\alpha\Omega
\end{aligned} \tag{1.1}
$$

and thus the Einstein tensor transforms as

$$
G_{\mu\nu} \to G_{\mu\nu} + g_{\mu\nu}(\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega - \Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega) + \Omega^{-1}\nabla_\mu\nabla_\nu\Omega - 2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \tag{1.2}
$$

Perturbing the above we find

$$
\delta G_{\mu\nu} \to \delta G_{\mu\nu} + \delta S_{\mu\nu} \tag{1.3}
$$

$$
\begin{aligned}
\delta S_{\mu\nu} = &-h_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega + \tfrac{1}{2}\Omega^{-1}\nabla_\alpha h_{\mu\nu}\nabla^\alpha\Omega - \tfrac{1}{2}g_{\mu\nu}\Omega^{-1}\nabla_\alpha h\nabla^\alpha\Omega + h_{\mu\nu}\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega \\
&+ g_{\mu\nu}\Omega^{-1}\nabla^\alpha\Omega\nabla_\beta h_\alpha{}^\beta - g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha\Omega\nabla^\beta\Omega + g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta\nabla^\alpha\Omega \\
&- \tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\mu h_{\nu\alpha} - \tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\nu h_{\mu\alpha}.
\end{aligned} \tag{1.4}
$$

2 Background $G^{(0)}_{ij} = -\kappa_3^2 T^{(0)}_{ij}$

Since $G_{\mu\nu}$ vanishes in a flat geometry, the background equation is given as

$$
g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = -\kappa_3^2\Lambda\Omega^2 g_{ij}. \tag{2.1}
$$

In the covariant formulation, the background equation fixed $k = -\kappa_3^2\Lambda$. Taking the trace of (2.1)

$$
-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \frac{12k}{(1+k\rho^2)^2} = 3\Omega^2 k = -3\kappa_3^2\Lambda\Omega^2. \tag{2.2}
$$

We also find

$$
\begin{aligned}
G^{(0)}_{\rho\rho} &= k\Omega^2 g_{\rho\rho} \\
\frac{4k}{(1+k\rho^2)^2} &= k\Omega^2 g_{\rho\rho}
\end{aligned} \tag{2.3}
$$

However, for $\theta\theta$ we find

$$
\begin{aligned}
G^{(0)}_{\theta\theta} &= k\Omega^2 g_{\theta\theta} \\
\frac{2k\rho^2(3+k\rho^2)}{(1+k\rho^2)^2} &\neq \frac{k\rho^2}{(1+k\rho^2)^2}
\end{aligned} \tag{2.4}
$$

Similarly for $\phi\phi$

$$
\begin{aligned}
G^{(0)}_{\phi\phi} &= k\Omega^2 g_{\phi\phi} \\
\frac{2k\rho^2(3+k\rho^2)\sin^2\theta}{(1+k\rho^2)^2} &\neq \frac{k\rho^2\sin^2\theta}{(1+k\rho^2)^2}
\end{aligned} \tag{2.5}
$$

3 Perturbation $\delta G_{ij} = -\kappa_3^2 \delta T_{ij}$

The perturbed energy momentum tensor takes the form

$$\begin{aligned}
-\kappa_3^2 \delta T_{ij} &= -\kappa_3^2 \Lambda \Omega^2 h_{ij} \\
&= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\
-\kappa_3^2 g^{ij}\delta T_{ij} &= k\Omega^2(-6\psi + 2\nabla_a\nabla^a E)
\end{aligned} \tag{3.1}$$

If we evaluate (1.3) and (1.4) we find

$$\begin{aligned}
\delta G_{ij} = \; & g_{ij}\nabla_a\nabla^a\psi + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E - 2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega \\
& +2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega - 2\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\
& +2\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\
& \\
& +g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\
& -\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j - \Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i \\
& -\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a \\
& \\
& +\nabla_a\nabla^a E_{ij} - 2E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + 2E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
& +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{3.2}$$

The only two gauge invariant quantities are

$$\begin{aligned}
\bar{\psi} + \Omega^{-1}(\tilde{\nabla}_k\bar{E} + \bar{E}_k)\tilde{\nabla}^k\Omega &= \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k\Omega \\
\bar{E}_{ij} &= E_{ij}.
\end{aligned} \tag{3.3}$$

Hence any E_i or E_j (specifically with index i or j) term must vanish identically in the full $\delta G_{ij} = -\kappa_2^3\delta T_{ij}$.

Looking only at the relevant vector pieces, we see

$$\begin{aligned}
= \delta G_{ij}^{(V)} &= -\kappa_3^2\delta T_{ij}^{(V)} \\
-\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j + (i\leftrightarrow j) &= k\Omega^2\nabla_i E_j + (i\leftrightarrow j)
\end{aligned} \tag{3.4}$$

which implies

$$\left(-\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\right)\nabla_i E_j = \left(-\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\right)\nabla_i E_j. \tag{3.5}$$

The above equation along with (2.4) and (2.5) hints at the necessary form for δS_{ij} and thus $G_{ij}^{(0)}$ and serves as another check upon the conformal flat form of G_{ij}.

$$\delta T_{\mu\nu} = \delta T^{T\theta}_{\mu\nu} + \frac{g_{\mu\nu}}{D-1}\delta T - \frac{1}{D-1}\nabla_\mu\nabla_\nu \int D\delta T. \tag{3.6}$$

$$\begin{aligned} \delta G^{T\theta}_{\mu\nu} &= \delta T^{T\theta}_{\mu\nu} \\ \delta G &= \delta T \end{aligned} \tag{3.7}$$

$$\begin{aligned} \delta G_{\mu\nu} &= \nabla^2 E_{\mu\nu} + (D-2)(g_{\mu\nu}\nabla^2\psi - \nabla_\mu\nabla_\nu\psi) \\ \delta G &= (D-2)(D-1)\nabla^2\psi \end{aligned} \tag{3.8}$$

$$\delta G^{T\theta}_{\mu\nu} = \nabla^2 E_{\mu\nu} - (D-2)\nabla_\mu\nabla_\nu\left[\psi - \int D\nabla^2\psi\right] \tag{3.9}$$

$$\begin{aligned} h^{T\theta}_{\mu\nu} &= h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \frac{g_{\mu\nu}}{D-1}(\nabla^\alpha W_\alpha - h) \\ &\quad + \frac{D-2}{D-1}\nabla_\mu\nabla_\nu \int D\nabla^\alpha W_\alpha + \frac{\nabla_\mu\nabla_\nu}{D-1}\int Dh \end{aligned} \tag{3.10}$$

$$\mathcal{N}(h^{T\theta}_{\mu\nu}) = \int D\nabla^2 h^{T\theta}_{\mu\nu} \tag{3.11}$$

$$(\mathcal{N}h_{\mu\nu})^{T\theta} = \int D\nabla^2 h^{T\theta}_{\mu\nu} \tag{3.12}$$