Gauge Invariant $\delta G_{\mu\nu} = \delta T_{\mu\nu}$ and $\delta W_{\mu\nu}$

Perturbed metric:

$$ds^2 = \Omega^2 \left\{ -(1+2\phi)d\tau^2 + 2(\nabla_i B + B_i)d\tau dx^i + \left[(1-2\psi)\delta_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}\right] dx^i dx^j \right\} \tag{1}$$

where

$$\nabla^i B_i = 0,\ \nabla^i E_i = 0,\ \nabla^i E_{ij} = 0,\ \delta^{ij} E_{ij} = 0.$$

Under coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu \tag{2}$$

where

$$\epsilon^\mu = (\epsilon^0, \nabla^i \epsilon + \epsilon^i), \qquad \nabla^i \epsilon_i = 0$$

the components of the metric transform as

$$\tilde{\phi} = \phi - H\epsilon^0 - \dot{\epsilon}^0 \tag{3}$$

$$\tilde{\psi} = \psi + H\epsilon^0 \tag{4}$$

$$\tilde{B} = B + \epsilon^0 - \dot{\epsilon} \tag{5}$$

$$\tilde{E} = E - \epsilon \tag{6}$$

$$\tilde{E}_i = E_i - \epsilon_i \tag{7}$$

$$\tilde{B}_i = B_i - \dot{\epsilon}_i \tag{8}$$

$$\tilde{E}_{ij} = E_{ij} \tag{9}$$

From the above, we may form gauge invariant combinations (adding to 6 DOF):

$$\Phi = \phi - H(\dot{E} - B) - (\ddot{E} - \dot{B}) \tag{10}$$

$$\Psi = \psi + H(\dot{E} - B) \tag{11}$$

$$\mathcal{Q}_i = B_i - \dot{E}_i \tag{12}$$

$$E_{ij} = E_{ij} \tag{13}$$

By orthogonal and parallel projections to the four velocity u^μ, a generic symmetric $T_{\mu\nu}$ may be decomposed as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} + u_\nu q_\mu + u_\mu q_\nu + \pi_{\mu\nu} \tag{14}$$

where

$$u^\mu q_\mu = 0,\ g^{\mu\nu}\pi_{\mu\nu} = 0,\ u^\mu u_\nu u^\rho u_\sigma \pi_{\nu\sigma} = 0.$$

The conditions on $\pi_{\mu\nu}$ specify that it is traceless and orthogonal to the four velocity u^μ, i.e. $\pi_{\mu\nu} = \pi_{ij}$. We may further decompose π_{ij} as

$$\pi_{ij} = \nabla_i\nabla_j\Pi - \frac{1}{3}\nabla^2\Pi\delta_{ij} + \frac{1}{2}\nabla_i\Pi_j + \frac{1}{2}\nabla_j\Pi_i + \Pi_{ij} \tag{15}$$

where as expected,

$$\nabla^i\Pi_i = 0,\ \nabla^i\Pi_{ij} = 0,\ \delta^{ij}\Pi_{ij} = 0.$$

We have 2 degrees of freedom from ρ and p, 3 from q_μ, and 5 from $\pi_{\mu\nu}$ adding to 10 in total. We decompose $T_{\mu\nu}$ into a background piece and first order fluctuations:

$$T_{\mu\nu} = {}^{(0)}T_{\mu\nu} + \delta T_{\mu\nu}.$$

The scalars, according to homogeneity and isotropy of the background, may only depend on τ,

$$\rho(x^\mu) = \bar{\rho}(\tau) + \delta\rho(x^\mu) \tag{16}$$
$$p(x^\mu) = \bar{p}(\tau) + \delta p(x^\mu). \tag{17}$$

The background $T_{\mu\nu}$ is given as a perfect fluid

$$T^{(0)}_{\mu\nu} = (\bar{\rho} + \bar{p})u^{(0)}_\mu u^{(0)}_\nu + \bar{p}g^{(0)}_{\mu\nu}.$$

The first order contribution goes as

$$\delta T_{\mu\nu} = (\delta\rho + \delta p)u^{(0)}_\mu u^{(0)}_\nu + (\bar{\rho} + \bar{p})\delta u_\mu u^{(0)}_\nu + (\bar{\rho} + \bar{p})u^{(0)}_\mu \delta u_\nu + \delta p g^{(0)}_{\mu\nu} + \bar{p}h_{\mu\nu} + u^{(0)}_\mu q_\nu + u^{(0)}_\nu q_\mu + \pi_{\mu\nu},$$

since the background of interest (FLRW) is homogeneous and isotropic, there is no anisotropic stress $\pi_{\mu\nu}$ or vector perturbation q_μ at zeroth order and so π_{ij} and q_μ are automatically first order. Perturbing the four velocity,

$$u^\mu = \frac{1}{a}\frac{dx^\mu}{d\tau} = \bar{u}^\mu + \delta u^\mu \tag{18}$$

where $\bar{u}^\mu = a^{-1}\delta^\mu{}_0$ and $\delta u^i = \nabla^i v + v^i$ with $\nabla_i v^i = 0$. By normalization of the four velocity $-1 = g_{\mu\nu}u^\mu u^\nu$, we may derive the background and perturbed components of u^μ:

$$u^\mu = \frac{1}{\Omega}\left(1 - \phi,\ \nabla^i v + v^i\right), \qquad u_\mu = \Omega\left(-1 - \phi,\ \nabla_i v + v_i + \nabla_i B - B_i\right). \tag{19}$$

$$\delta T_{00} = \Omega^2(2\rho\phi + \delta\rho) \tag{20}$$
$$\delta T_{0i} = -\Omega^2\rho(\nabla_i v + \nabla_i B + v_i + B_i) - \Omega^2 p(\nabla_i v + v_i) \tag{21}$$
$$\delta T_{ij} = \Omega^2 p h_{ij} + \Omega^2\delta_{ij}\delta p + \pi_{ij} \tag{22}$$

Under gauge transformation (2), scalars transform as (see A.1)

$$\delta\tilde{\rho} = \delta\rho - \epsilon^0\dot{\bar{\rho}} \tag{23}$$
$$\delta\tilde{p} = \delta p - \epsilon^0\dot{\bar{p}} \tag{24}$$

and the velocity transforms as (see A.2)

$$\tilde{v} = v + \dot{\epsilon}, \quad \tilde{v}^i = v^i + \dot{\epsilon}^i. \tag{25}$$

The components of π_{ij}, that is Π, Π_i and Π_{ij} are all gauge invariant since they vanish in the background (A.3). From these transformation laws, we may form many gauge invariant quantities (omitting the bars on all background quantities now and denoting $\sigma \equiv \dot{E} - B$):

$$\delta\rho_\sigma = \delta\rho - \dot{\rho}\sigma \tag{26}$$
$$\delta p_\sigma = \delta p - \dot{p}\sigma \tag{27}$$
$$\mathcal{V} = v + \dot{E} \tag{28}$$
$$\mathcal{B}_i = B_i + v_i \tag{29}$$
$$\pi_{ij} = \pi_{ij} \tag{30}$$

and Π, Π_i, Π_{ij}.

Einstein equations

$\delta G_{\mu\nu} = -8\pi G \delta T_{\mu\nu}$

Scalars:

$\underline{\delta G_{00} = -8\pi G \delta T_{00}}$:

$$\begin{aligned}\delta G_{00} &= -2\nabla^2\psi - 2\mathcal{H}\nabla^2\sigma + 6\mathcal{H}\dot{\psi} \\ &= -2\nabla^2\Psi + 6\mathcal{H}\dot{\Psi} - 6\mathcal{H}\dot{\mathcal{H}}\sigma - 6\mathcal{H}^2\dot{\sigma}\end{aligned}$$

$$\begin{aligned}-8\pi G\delta T_{00} &= -8\pi G\Omega^2(2\rho\phi + \delta\rho) \\ &= -6\mathcal{H}^2\Phi - 6\mathcal{H}^2\dot{\sigma} - 6\mathcal{H}\dot{\mathcal{H}}\sigma - 8\pi G\Omega^2\delta\rho_\sigma\end{aligned}$$

This leads to field equation

$$\boxed{\nabla^2\Psi - 3\mathcal{H}\dot{\Psi} - 3\mathcal{H}^2\Phi - 4\pi G\Omega^2\delta\rho_\sigma = 0}$$

$\underline{\delta G_{0i} = -8\pi G \delta T_{0i}}$:

$$\begin{aligned}\delta G_{0i} &= \nabla_i\left(-2\dot{\psi} - 2\mathcal{H}\phi - \mathcal{H}^2 B + 2\frac{\ddot{\Omega}}{\Omega}B\right) \\ &= \nabla_i\left(2\dot{\Psi} + 2\dot{\mathcal{H}}\sigma + 2\dot{\mathcal{H}}B + \mathcal{H}^2 B - 2\mathcal{H}\Phi - 2\mathcal{H}^2\sigma\right)\end{aligned}$$

$$\begin{aligned}-8\pi G\delta T_{0i} &= -8\pi G\Omega^2\nabla_i(\rho(v+B) + pv) \\ &= \nabla_i\left(-v(2\dot{\mathcal{H}} + \mathcal{H}^2) + 3\mathcal{H}^2(v+B)\right)\end{aligned}$$

This leads to field equation

$$\boxed{\dot{\Psi} - \dot{\mathcal{H}}\mathcal{B} + \mathcal{H}\Phi + \mathcal{H}^2\mathcal{V} = 0}$$

$\underline{\delta G_{ij} = -8\pi G \delta T_{ij}, \quad i \neq j}$:

$$\begin{aligned}\delta G_{ij} &= \nabla_i\nabla_j\left(-\dot{\sigma} + \phi - \psi - 2\mathcal{H}^2 E - 2\mathcal{H}\sigma + 4\dot{\mathcal{H}}E + 4\mathcal{H}^2 E\right) \\ &= \nabla_i\nabla_j\left(\Phi - \Psi + 2\mathcal{H}^2 E + 4\dot{\mathcal{H}}E\right)\end{aligned}$$

$$\begin{aligned}-8\pi G\delta T_{0i} &= -8\pi G\Omega^2\nabla_i\nabla_j(2pE) \\ &= \nabla_i\nabla_j(2(2\dot{\mathcal{H}} + \mathcal{H}^2)E)\end{aligned}$$

This leads to field equation (need to recheck this result)

$$\Box$$

$\underline{\delta^{ij}\delta G_{ij} = -8\pi G\delta^{ij}\delta T_{ij},}$:

$$\delta^{ij}\delta G_{ij} = 2\nabla^2\psi - 2\nabla^2\mathcal{H} + 2\delta\dot{\sigma} - 2\mathcal{H}^2\nabla^2 E - 6\ddot{\psi} - 6\mathcal{H}\dot{\phi} + 6\mathcal{H}^2(\phi+\psi) + 4\mathcal{H}\nabla^2\sigma$$

$$
\begin{aligned}
&+ 4(\mathcal{H}^2 + \dot{\mathcal{H}})\nabla^2 E - 12(\mathcal{H}^2 + \dot{\mathcal{H}})(\phi + \psi) - 12\mathcal{H}\dot{\psi} \\
= {}& 2\nabla^2(\Psi - \Phi) - 6\ddot{\Psi} - 6\mathcal{H}^2(\Psi + \Phi) + 2\mathcal{H}^2\nabla^2 E + 6\ddot{\mathcal{H}}\sigma + 6\mathcal{H}\dot{\mathcal{H}}\sigma + 4\dot{\mathcal{H}}\nabla^2 E - 6\mathcal{H}\dot{\Phi} \\
&- 12\dot{\mathcal{H}}(\Psi + \Phi) - 12\mathcal{H}\dot{\Psi}
\end{aligned}
$$

$$
\begin{aligned}
-8\pi G\delta^{ij}\delta T_{ij} &= -8\pi G\Omega^2(-6\psi p + 2p\nabla^2 E + 3\delta p) \\
&= 2(2\dot{\mathcal{H}} + \mathcal{H}^2)\nabla^2 E + 6\mathcal{H}\dot{\mathcal{H}}\sigma + 6\ddot{\mathcal{H}}\sigma - 12\dot{\mathcal{H}}\Psi - 12\mathcal{H}^2\psi - 24\pi G\Omega^2\delta\rho_\sigma
\end{aligned}
$$

This leads to field equation (need to recheck this result)

$\square$

Vectors:

$\underline{\delta G_{0i} = -8\pi G\delta T_{0i}}$:

$$
\begin{aligned}
\delta G_{0i} &= -\frac{1}{2}\nabla^2(\dot{E}_i - B_i) - \mathcal{H}^2 B_i + 2B_i(\dot{\mathcal{H}} + \mathcal{H}^2) \\
&= \frac{1}{2}\nabla^2\mathcal{Q}_i + B_i(2\dot{\mathcal{H}} + \mathcal{H}^2)
\end{aligned}
$$

$$
\begin{aligned}
-8\pi G\delta T_{0i} &= -8\pi G\Omega^2(-\rho B_i - (\rho + p)v_i) \\
&= 3\mathcal{H}^2(v_i + B_i) - (2\dot{\mathcal{H}} + \mathcal{H}^2)v_i \\
&= 3\mathcal{H}^2\mathcal{B}_i - 2(\dot{\mathcal{H}} + \mathcal{H}^2)v_i
\end{aligned}
$$

This leads to field equation

$$
\boxed{\frac{1}{2}\nabla^2\mathcal{Q}_i - 3\mathcal{H}^2\mathcal{B}_i + (2\dot{\mathcal{H}} + \mathcal{H}^2)\mathcal{B}_i = 0}
$$

$\underline{\delta G_{ii} = -8\pi G\delta T_{ii}}$:

$$
\begin{aligned}
\delta G_{ii} &= \nabla_i\left[\dot{\mathcal{Q}}_i + 2\mathcal{H}^2 E_i + 2\mathcal{H}\mathcal{Q}_i + 4\dot{\mathcal{H}}E_i\right] \\
&= \frac{1}{2}\nabla^2\mathcal{Q}_i + B_i(2\dot{\mathcal{H}} + \mathcal{H}^2)
\end{aligned}
$$

$$
\begin{aligned}
-8\pi G\delta T_{ii} &= -8\pi G\Omega^2\nabla_i(2E_i p) \\
&= (4\dot{\mathcal{H}} + 2\mathcal{H}^2)\nabla_i E_i
\end{aligned}
$$

This leads to field equation

$$
\boxed{\frac{1}{2}\nabla^2\mathcal{Q}_i - 3\mathcal{H}^2\mathcal{B}_i + (2\dot{\mathcal{H}} + \mathcal{H}^2)\mathcal{B}_i = 0}
$$

Tensors:

$\underline{\delta G_{ij} = -8\pi G\delta T_{ij}}$:

$$
\delta G_{ij} = \nabla^2 E_{ij} - \ddot{E}_{ij} - 2\mathcal{H}^2 E_{ij} - 2\mathcal{H}\dot{E}_{ij} + 4(\dot{\mathcal{H}} + \mathcal{H}^2)E_{ij}
$$

$$
\begin{aligned}
-8\pi G\delta T_{ij} &= -8\pi G\Omega^2(pE_{ij} + \pi_{ij}) \\
&= (2\dot{\mathcal{H}} + \mathcal{H}^2)(E_{ij} + \pi_{ij})
\end{aligned}
$$

This leads to field equation

$$
\boxed{\nabla^2 E_{ij} - \ddot{E}_{ij} - 2\mathcal{H}\dot{E}_{ij} + (2\dot{\mathcal{H}} + \mathcal{H}^2)(E_{ij} - \pi_{ij})}
$$

Weyl equations

In conformal gravity, there are only 5 independent degrees of freedom (via the traceless $K_{\mu\nu}$):

$$\begin{aligned}\Sigma &= \Psi + \Psi = \psi + \phi - \dot{\sigma} \\ \mathcal{Q}_i &= B_i - \dot{E}_i \\ E_{ij} &= E_{ij}.\end{aligned}$$

The gauge invariance of the SVT decomposition $\delta W_{\mu\nu}$ is immediate, and here we write $\delta W_{\mu\nu}$ in terms of the gauge invariant quanties for arbitrary Ω.

Scalars:

$$\begin{aligned}\delta W_{00} &= -\frac{2}{3\Omega^2}\nabla^4\Sigma \\ \delta W_{0i} &= -\frac{2}{3\Omega^2}\nabla^4\dot{\Sigma} \\ \delta W_{ij} &= \frac{1}{3\Omega^2}\left(g_{ij}\nabla^2\ddot{\Sigma} + \nabla^2\nabla_i\nabla_j\Sigma - g_{ij}\nabla^4\Sigma - 3\nabla_i\nabla_j\ddot{\Sigma}\right)\end{aligned}$$

Vectors:

$$\begin{aligned}\delta W_{0i} &= \frac{1}{2\Omega^2}\left(\nabla^4\mathcal{Q}_i - \nabla^2\ddot{\mathcal{Q}}_i\right) \\ \delta W_{ij} &= \frac{1}{2\Omega^2}\left(\nabla^2\nabla_i\dot{\mathcal{Q}}_j + \nabla^2\nabla_j\dot{\mathcal{Q}}_i - \nabla_i\ddot{\mathcal{Q}}_j - \nabla_j\ddot{\mathcal{Q}}_i\right)\end{aligned}$$

Tensors:

$$\delta W_{ij} = \frac{1}{\Omega^2}\left(E_{ij} - 2\nabla^2\ddot{E}_{ij} + \nabla^4 E_{ij}\right)$$

Appendix

Useful forms of the Friedman equations required for $\delta G_{\mu\nu} = \delta T_{\mu\nu}$:

$$\rho\frac{8\pi G\Omega}{3} = \mathcal{H}^2$$

$$\begin{aligned}8\pi G\Omega^2\dot{\rho} &= 6(\dot{\mathcal{H}}\mathcal{H} - \mathcal{H}^3) \\ \frac{4\pi G\Omega}{3}(\rho - 3p) &= \frac{\ddot{\Omega}}{\Omega} \\ -(2\dot{\mathcal{H}} + \mathcal{H}^2) &= 8\pi G\Omega^2 p \\ 8\pi G\Omega^2\dot{p} &= 2\mathcal{H}(2\dot{\mathcal{H}} + \mathcal{H}^2) - (2\ddot{\mathcal{H}} + 2\mathcal{H}\dot{\mathcal{H}})\end{aligned}$$