Cosmological Fluctuations in Standard and Conformal Gravity

Matthew Phelps

Doctoral Degree Final Examination

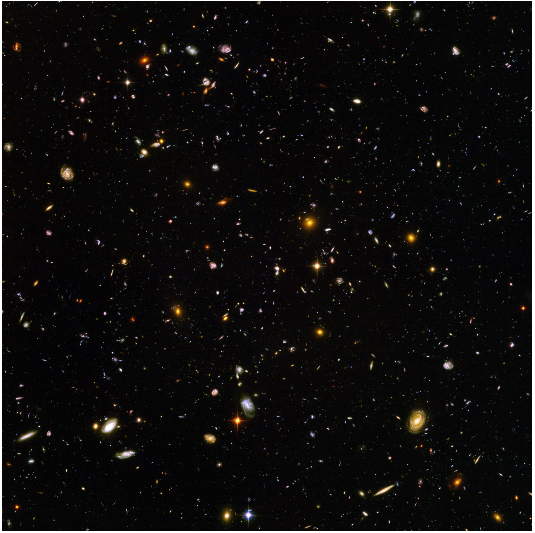
June 02, 2020

- Introduction and Formalism
- Three Dimensional Scalar, Vector, Tensor Decomposition (SVT3)
- Four Dimensional Scalar, Vector, Tensor Decomposition (SVT4)
- Conformal Gravity (SVT and Conformal to Flat Backgrounds)
- Conformal Gravity Robertson-Walker Radiation Era Solution
- Computational Methods
- Conclusions

- Introduction and Formalism
  - Cosmological Geometries
  - Einstein Gravity
  - Perturbation Theory
  - Gauge Transformations
  - Solution Methods

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker (flat, spherical, hyperbolic), de Sitter ($dS_4 \subset \mathrm{RW}$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

$$ds^2 = \Omega(x)^2\big(-dt^2 + dx^2 + dy^2 + dz^2\big)$$



Hubble Ultra-Deep Field. NASA and the European Space Agency.

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -3k\tilde{g}_{ij}, \qquad R = -6k$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$$

Comoving Robertson Walker geometry:

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with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

set $k = 0$ (flat), simple conformal to flat form

$$\boxed{ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]}$$

$k = 1$ (spherical)

$$ds^2 = a(\tau)^2\Big[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big]$$

Set $\sin\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\Big[-dp^2 + d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2\Big]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p+\chi)/2], \quad p' - r' = \tan[(p-\chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos\chi}, \quad r' = \frac{\sin\chi}{\cos p + \cos\chi} \end{aligned}$$

$$\Longrightarrow \boxed{ds^2 = \frac{4a^2(p)}{[1+(p'+r')^2][1+(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2]}$$

$k = -1$ (hyperbolic)

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

Set $\sinh\chi = r$, $p = \tau$,

$$ds^2 = a(p)^2\left[-dp^2 + d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi\sin^2\theta d\phi^2\right]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tanh[(p+\chi)/2], \quad p' - r' = \tanh[(p-\chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh\chi}, \quad r' = \frac{\sinh\chi}{\cosh p + \cosh\chi} \end{aligned}$$

$$\implies \boxed{ds^2 = \frac{4a^2(p)}{[1-(p'+r')^2][1-(p'-r')^2]}[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2]}$$

Einstein Hilbert action

$$I_{\mathsf{EH}} = -\frac{1}{16\pi G}\int d^4x(-g)^{1/2} g^{\mu\nu} R_{\mu\nu}.$$

Functional variation w.r.t $g_{\mu\nu}$ yields Einstein tensor,

$$\frac{16\pi G}{(-g)^{1/2}} \frac{\delta I_{\mathsf{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\alpha},$$

likewise, variation of matter action I_{M} w.r.t $g_{\mu\nu}$ yields Energy Momentum tensor

$$\frac{2}{(-g)^{1/2}} \frac{\delta I_{\mathsf{M}}}{\delta g_{\mu\nu}} = T_{\mu\nu}.$$

Requiring sum of actions to be stationary gives us Einstein field equations

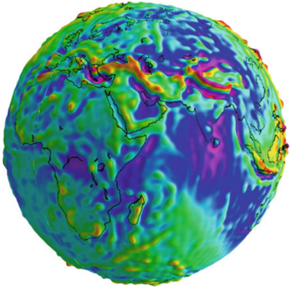
$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\alpha} = -8\pi G T^{\mu\nu},$$

subject to Bianchi identity

$$\nabla_{\mu} R^{\mu\nu} = \frac{1}{2} \nabla^{\nu} R^{\mu}{}_{\mu} \implies \nabla_{\mu} G^{\mu\nu} = 0.$$

Decompose metric into background and fluctuation, truncating at linear order

$$
\begin{aligned}
g_{\mu\nu}(x) &= g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad g^{\mu\nu}_{(0)} h_{\mu\nu} \equiv h \\
G_{\mu\nu} &= G_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) \\
G^{(0)}_{\mu\nu} &= R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(0)\alpha}_{\alpha} \\
\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R^{(0)\alpha}_{\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}{}_{\alpha}.
\end{aligned}
$$



Likewise perturb $T_{\mu\nu}$ around background

$$T_{\mu\nu} = T_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta T_{\mu\nu}(h_{\mu\nu})$$

Form background and first order equations of motion (upon setting $8\pi G = 1$)

$$
\begin{aligned}
\Delta^{(0)}_{\mu\nu} &= G^{(0)}_{\mu\nu} + T^{(0)}_{\mu\nu} = 0 \\
\Delta_{\mu\nu} &= \delta G^{(0)}_{\mu\nu} + \delta T^{(0)}_{\mu\nu} = 0
\end{aligned}
$$

Walter, U. (2019). Correction to: Astronautics. In Astronautics (pp. C1–C1). Springer International Publishing.

- Under coordinate transformation $x^\mu \to x^\mu - \epsilon^\mu(x)$, with $\epsilon^\mu \sim \mathcal{O}(h)$, the perturbed metric transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$$

- For every solution $h_{\mu\nu}$ to $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$, a transformed $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ will also serve as a solution
- Set of four $\epsilon^\mu(x)$ define gauge freedom under coordinate transformation
- 10 components in $h_{\mu\nu}$, 4 coordinate transformations, leads to 6 independent degrees of freedom
- Under $x^\mu \to x^\mu - \epsilon^\mu(x)$, the perturbed tensors transform as

$$\delta G_{\mu\nu} \to \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda G^{(0)}_{\mu\nu} \epsilon^\lambda$$
$$\delta T_{\mu\nu} \to \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\mu + \nabla_\lambda T^{(0)}_{\mu\nu} \epsilon^\lambda.$$

- If background $G^{(0)}_{\mu\nu} = 0$, then $\delta G_{\mu\nu}$ separately gauge invariant; likewise for vanishing background energy momentum tensor
- If $G^{(0)}_{\mu\nu} \neq 0$, then only the entire $\Delta_{\mu\nu} = \delta G_{\mu\nu} + T_{\mu\nu}$ is gauge invariant

- Perturbed field equations $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ form a rather complex and extensive set of coupled tensor PDE's
- Much effort involved in simplifying, decoupling, and solving them

$$
\begin{aligned}
\delta G_{ij} \quad = \quad & -\tfrac{1}{2}\ddot{h}_{ij} + \tfrac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \tfrac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2\tilde{g}_{ij}h_{00}\Omega^{-2} \\
& -\dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
& +2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
& -4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
& -\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
& +\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& -\tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \tfrac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
& +\tfrac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \tfrac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
& -\tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \tfrac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
& +\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \tfrac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h,
\end{aligned}
$$

- Three-dimensional Scalar, Vector, Tensor Basis (SVT3)
  - SVT3 Decomposition
  - Decouple Einstein Fluctuations in a de Sitter Background
  - Integral Formalism

Decompose the metric perturbation $h_{\mu\nu}$ into a set of scalars, vectors, and tensors according to their transformation behavior under 3D rotations

- Define $h_{\mu\nu} = \Omega^2(x) f_{\mu\nu}$, perform 3 + 1 decomposition

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = (g^{(0)}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \\
&= \Omega^2(x)(\tilde{g}^{(0)}_{\mu\nu} + f_{\mu\nu}) dx^\mu dx^\nu \\
&= \Omega^2(x)\big[(-1 + f_{00}) dt^2 + 2 f_{0i} dt dx^i + (\tilde{g}_{ij} + f_{ij})\big] dx^i dx^j
\end{aligned}
$$

- Decompose f_{00}, f_{0i}, and f_{ij} in terms of 3-dimensional scalars, vectors, and tensors

$$
\begin{aligned}
f_{00} &= -2\phi, \qquad f_{0i} = B_i + \tilde{\nabla}_i B \\
f_{ij} &= -2\psi \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij},
\end{aligned}
$$

with vectors and tensors obeying

$$
\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \tilde{g}^{ij} E_{ij} = 0.
$$

$$
ds^2 = \Omega^2(x)\Big[-(1+2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B) dt dx^i + [(1-2\psi)\tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}] dx^i dx^j\Big]
$$

- de Sitter geometry

$$ds^2 = \frac{1}{H^2\tau^2}\Big[-(1+2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i + [(1-2\psi)\delta_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\Big]$$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2\Omega(\tau)^2 f_{\mu\nu}$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2\Omega(\tau)^2 f_{\mu\nu}$$

- Insert the SVT3 decomposed $h_{\mu\nu}$ into a 3+1 $\delta G_{\mu\nu}$

$$\begin{aligned}
\delta G_{00} &= -\frac{6}{\tau}\dot{\psi} - \frac{2}{\tau}\tilde{\nabla}^2(\tau\psi + B - \dot{E}), \\
\delta G_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) + \frac{1}{\tau^2}\tilde{\nabla}_i(3B - 2\tau^2\dot{\psi} + 2\tau\phi) + \frac{3}{\tau^2}B_i, \\
\delta G_{ij} &= \frac{\delta_{ij}}{\tau^2}\Big[- 2\tau^2\ddot{\psi} + 2\tau\dot{\phi} + 4\tau\dot{\psi} - 6\phi - 6\psi \\
&+ \tilde{\nabla}^2\left(2\tau B - \tau^2\dot{B} + \tau^2\ddot{E} - 2\tau\dot{E} - \tau^2\phi + \tau^2\psi\right)\Big] \\
&+ \frac{1}{\tau^2}\tilde{\nabla}_i\tilde{\nabla}_j\left[-2\tau B + \tau^2\dot{B} - \tau^2\ddot{E} + 2\tau\dot{E} + 6E + \tau^2\phi - \tau^2\psi\right] \\
&+ \frac{1}{2\tau^2}\tilde{\nabla}_i\left[-2\tau B_j + 2\tau\dot{E}_j + \tau^2\dot{B}_j - \tau^2\ddot{E}_j + 6E_j\right] \\
&+ \frac{1}{2\tau^2}\tilde{\nabla}_j\left[-2\tau B_i + 2\tau\dot{E}_i + \tau^2\dot{B}_i - \tau^2\ddot{E}_i + 6E_i\right] \\
&- \ddot{E}_{ij} + \frac{6}{\tau^2}E_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij},
\end{aligned}$$

- Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$

$$
\begin{aligned}
\Delta_{00} &= -\frac{6}{\tau^2}(\dot\beta - \alpha) - \frac{2}{\tau}\tilde\nabla^2\beta = 0, \\
\Delta_{0i} &= \frac{1}{2}\tilde\nabla^2(B_i - \dot E_i) - \frac{2}{\tau}\tilde\nabla_i(\dot\beta - \alpha) = 0, \\
\Delta_{ij} &= \frac{\delta_{ij}}{\tau^2}\left[-2\tau(\ddot\beta - \dot\alpha) + 6(\dot\beta - \alpha) + \tau\tilde\nabla^2(2\beta - \tau\alpha)\right] + \frac{1}{\tau}\tilde\nabla_i\tilde\nabla_j(-2\beta + \tau\alpha) \\
&+ \frac{1}{2\tau}\tilde\nabla_i[-2(B_j - \dot E_j) + \tau(\dot B_j - \ddot E_j)] + \frac{1}{2\tau}\tilde\nabla_j[-2(B_i - \dot E_i) + \tau(\dot B_i - \ddot E_i)] \\
&- \ddot E_{ij} + \frac{2}{\tau}\dot E_{ij} + \tilde\nabla^2 E_{ij} = 0, \\
g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot\beta - \dot\alpha) + 24(\dot\beta - \alpha) + 6\tau\tilde\nabla^2\beta - 2\tau^2\tilde\nabla^2\alpha] = 0,
\end{aligned}
$$

where

$$
\alpha = \phi + \psi + \dot B - \ddot E, \quad \beta = \tau\psi + B - \dot E, \quad B_i - \dot E_i, \quad E_{ij}.
$$

SVT3 $\delta G_{\mu\nu}$ in a de Sitter Background

- Decouple scalar, vector, and tensor gauge invariants by applying higher derivatives

$$\tilde{\nabla}^4(\alpha + \dot{\beta}) = 0, \quad \tilde{\nabla}^4(\alpha - \dot{\beta}) = 0,$$

$$\tilde{\nabla}^4(B_i - \dot{E}_i) = 0,$$

$$\tilde{\nabla}^4\left(-\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij}\right) = 0.$$

- Recap:
  - Perturb $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$, evaluating in de Sitter background
  - Decompose $h_{\mu\nu}$ into SVT3 components, inserting into fields equations
  - Compose $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ to form evolution equations consisting entirely of gauge invariant quantities
  - Apply higher derivatives to decouple SVT3 representations, solve

- How can we ensure such an SVT3 decomposition exists for the general $h_{\mu\nu}$? Let's take a Minskowski background,

$$\begin{aligned} h_{00} &= -2\phi, \qquad h_{0i} = B_i + \partial_i B \\ h_{ij} &= -2\psi\tilde{g}_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}, \end{aligned}$$

$$\partial^i B_i = \partial^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \partial^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0.$$

Decomposition of $V_i = V_i^T + \partial_i V$

- Longitudinal decomposition does not hold for any scalar. $\partial^i V_i = \partial_i \partial^i V$
- Introduce a Green's function $\partial_i \partial^i D(x - x') = \delta^3(x - x')$ and use Green's identity

$$V(x')\partial_i\partial^i D(x-x') = D(x-x')\partial_i\partial^i V(x') + \partial_i[V(x')\partial^i D(x-x') - D(x-x')\partial^i V(x')]$$

- Integrate

$$V(x) = \underbrace{\int_V d^3x' D(x-x')\partial_i\partial^i V(x')}_{\text{Non-Harmonic}} + \underbrace{\oint_{\partial V} dS_i[V(x')\partial^i D(x-x') - D(x-x')\partial^i V(x')]}_{\text{Harmonic}}$$

$$V = V^{NH} + V^H, \qquad \partial_i\partial^i V = \partial_i\partial^i V^{NH}, \qquad \partial_i\partial^i V^H = 0$$

- Need a $\partial_i V$ which could never be transverse

$$\begin{aligned} V \equiv V^{NH} \quad &= \quad \int d^3x' D(x-x')\partial_i\partial^i V(x') = \int d^3x' D(x-x')\partial^i V_i(x') \\ \implies \quad & \oint_{\partial V} dS_i[V(x')\partial^i D(x-x') - D(x-x')\partial^i V(x')] = 0 \end{aligned}$$

- Transverse Longitudinal Decomposition

$$V_i = V_i^T + \partial_i V, \qquad \partial_i V = \partial_i \int d^3x' D(x-x')\partial^j V_j(x'), \qquad V_i^T = V_i - \partial_i \int d^3x' D(x-x')\partial^j V_j(x')$$

- Transverse Vector Decomposition

$$V_i = V_i^T + \partial_i V, \qquad \partial_i V = \partial_i \int d^3x' D(x-x') \partial^j V_j(x'), \qquad V_i^T = V_i - \partial_i \int d^3x' D(x-x') \partial^j V_j(x')$$

- Projector Formalism

$$\Pi_{ij} = \delta_{ij} - \frac{\partial}{\partial x^i} \int d^3x' D(x-x') \frac{\partial}{\partial x'^j} \tag{1}$$

$$\Pi_{ij} V^j = V_T^j \tag{2}$$

$$\Pi_{ij} \Pi^j{}_k = \Pi_{ik}, \quad \Pi_{ij} V_T^j = V_T^j, \quad \Pi_{ij}(\partial^j V) = 0 \tag{3}$$

- Hence, we can decompose h_{0i} as

$$h_{0i} = B_i + \partial_i B, \qquad B = \int d^3x' D(x-x') \partial^j h_{0j}, \qquad B_i = \Pi_{ij} h_0{}^j = h_{0i} - \partial_i \int d^3x' D(x-x') \partial^j h_{0j}$$

  - Composed of non-local integrals
  - B itself must vanish asymptotically (or decay sufficiently fast)

$$h_{ij} \quad = \quad -2\psi\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij} \tag{4}$$

- Rank 2 tensor transverse traceless decomposition

$$h_{ij}^{TT} = h_{ij} - \partial_i W_j - \partial_j W_i + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x') \left(\partial^k W_k + \delta^{kl} h_{kl}\right) + \frac{1}{2}\delta_{ij}\left(\partial^k W_k - \delta^{kl} h_{kl}\right) \tag{5}$$

where we introduce a W_k obeying

$$\partial^j h_{ij} = \partial_k \partial^k W_i \tag{6}$$

- Can further decompose W_i into transverse and longitudinal components

$$W_i^T = W_i - \partial_i \int d^3x' D(x-x')\partial^k W_k \tag{7}$$

- Make definitions, $D \equiv D(x-x')$

$$\begin{aligned} h_{ij} \quad = \quad & \underbrace{\left[h_{ij} - \partial_i W_j - \partial_j W_i - \frac{1}{2} g_{ij}(\delta^{kl} h_{kl} - \partial^k W_k) + \frac{1}{2}\partial_i\partial_j \int d^3x' D(x-x')(\delta^{kl} h_{kl} + \partial^k W_k)\right]}_{2E_{ij}} \\ & +\partial_i \underbrace{\left(W_j - \partial_j \int d^3x' D(x-x')\partial^k W_k\right)}_{E_j} + \partial_j \underbrace{\left(W_i - \partial_i \int d^3x' D(x-x')\partial^k W_k\right)}_{E_i} \\ & -2\delta_{ij} \underbrace{\left(\tfrac{1}{4}\partial^k W_k - \tfrac{1}{4}\delta^{kl} h_{kl}\right)}_{\psi} + 2\partial_i\partial_j \underbrace{\int d^3x' D(x-x') \left(\tfrac{3}{4}\partial^k W_k - \tfrac{1}{4}\delta^{kl} h_{kl}\right)}_{E} \end{aligned} \tag{8}$$

- 3D S.V.T. components do not close under general 4D coordinate transformations

$$h_{\mu\nu} \to \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} h_{\alpha\beta}, \qquad h_{0\mu} = \begin{pmatrix} -2\phi \\ B_1 + \partial_1 B \\ B_2 + \partial_2 B \\ B_3 + \partial_3 B \end{pmatrix} \tag{9}$$

- We seek to
  (A) Generalize to higher dimensions. $D = 4$ to match underlying GR transformation group
  (B) Generalize to curved space backgrounds beyond Minkowski

$$[\nabla_\kappa, \nabla_\nu] V_\lambda = V^\sigma R_{\lambda\sigma\nu\kappa} \tag{10}$$

- SVT4 Decomposition

$$h_{\mu\nu} = -2\chi g_{\mu\nu} + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}, \tag{11}$$

subject to

$$\nabla^\mu F_\mu = 0, \qquad F_{\mu\nu} = F_{\nu\mu}, \qquad g^{\mu\nu} F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0. \tag{12}$$

$$
\begin{aligned}
I_{\mathrm{W}} &= -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\
&\equiv -2\alpha_g \int d^4x(-g)^{1/2} \left[R_{\mu\nu} R^{\mu\nu} - \tfrac{1}{3} (R^\alpha{}_\alpha)^2 \right],
\end{aligned}
$$

$$
\begin{aligned}
C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} - \frac{1}{2} \left(g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right) \\
&\quad + \frac{1}{6} R^\alpha{}_\alpha \left(g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right)
\end{aligned}
$$

$$
-\frac{2}{(-g)^{1/2}} \frac{\delta I_{\mathrm{W}}}{\delta g_{\mu\nu}} = 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2\nabla_\kappa \nabla_\lambda C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} C^{\mu\lambda\nu\kappa} \right].
$$