RW Projections

Projection Method

Via the 3+1 splitting we may express a general $T_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)U_\nu U_\mu + pg_{\mu\nu} + U_\mu q_\nu + U_\nu q_\mu + \pi_{\mu\nu}. \tag{1}$$

Then we may deconstruct the curved space q_i as $q_i = Q_i + \nabla_i Q$, a procedure indicated below for a rank 1 tensor. Then, it only remains to decompose the spatial (traceless) π_{ij} in terms of curved space projectors. Since the RW background is maximally symmetric in the underyling 3-space, it suggests the possibility to construct longitudinal and transverse components based solely on the 3-space covariant derivatives. More precisely, spatial covariant derivatives $\tilde{\nabla}_i$ are defined solely with respect to the 3-space constant curvature background, γ_{ij}. In polar coordinates for example, this would be

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2. \tag{2}$$

Upon conformally transforming with a time dependent $\Omega(t)$, the transverse and longitudinal components preserve their decomposed structure. Below is an attempt at constructing projectors within a maximally symmetric space, first by looking at rank 1 tensors, then flat space rank 2 tensors, then finally constant curvature rank 2 tensors.

Longitudinal Decomposition

First we posit the form of the longitudinal component, project out any transverse components, then integrate to solve in terms of the original tensor.

Rank 1 Tensor In Curved Space

For a rank 1 tensor A_ν we express the longitudinal component in terms of a derivative onto a scalar A

$$A^L_\nu = \nabla_\nu A. \tag{3}$$

Now project out the transverse component,

$$\nabla^\nu A_\nu = \nabla_\nu \nabla^\nu A. \tag{4}$$

Solving for A, we have

$$A = \int d^D x' \sqrt{g}\ D(x, x') \nabla^\mu A_\mu, \tag{5}$$

where we have introduced the curved space propogator

$$\nabla_\mu \nabla^\mu D(x, x') = g^{-1/2} \delta^D (x - x'). \tag{6}$$

Thus we have

$$A^L_\nu = \nabla_\nu \int d^D x' \sqrt{g}\ D(x, x') \nabla^\mu A_\mu, \tag{7}$$

and the transverse component is just the remaining part,

$$A_\nu^T = A_\nu - A_\nu^L. \tag{8}$$

Lastly, we may construct a longitudinal projector $\Pi_{\mu\nu}^L$,

$$\Pi_{\mu\nu}^L = \nabla_\mu \int d^D x' \sqrt{g}\; D(x, x')\nabla_\nu \tag{9}$$

Rank 2 Tensor In Minkowski Space

For a rank 2 tensor (in Minkowski background), we posit

$$h_{\mu\nu}^L = \partial_\mu V_\nu + \partial_\nu V_\mu, \tag{10}$$

where V^μ remains to be determined in terms of $h^{\mu\nu}$. Now project out the transverse components of $h^{\mu\nu}$, noting $h_T^{\mu\nu}$ can make no contribution,

$$\partial_\nu h^{\mu\nu} = \partial_\nu \partial^\mu V^\nu + \partial_\nu \partial^\nu V^\mu, \tag{11}$$

$$\partial_\mu \partial_\nu h^{\mu\nu} = \partial_\mu \partial_\nu \partial^\mu V^\nu + \partial_\mu \partial_\nu \partial^\nu V^\mu = 2\partial_\mu \partial^\mu \partial_\nu V^\nu. \tag{12}$$

From $\partial_\mu \partial_\nu h^{\mu\nu}$, solve for $\partial_\nu V^\nu$,

$$\partial_\nu V^\nu = \frac{1}{2}\int d^3y D(x-y)\partial_\sigma \partial_\rho h^{\sigma\rho} = \frac{1}{2}\left[\partial_\sigma \int d^3y D(x-y)\partial_\rho h^{\sigma\rho} + \int dS_\sigma D(x-y)\partial_\rho h^{\sigma\rho}\right], \tag{13}$$

where we use the flat space propagator

$$\partial_\nu \partial^\nu D(x-x') = \delta(x-x'). \tag{14}$$

Insert $\partial_\nu V^\nu$ back into $\partial_\nu h^{\mu\nu}$

$$\partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu \left[\partial_\sigma \int d^3y D(x-y)\partial_\rho h^{\sigma\rho} + \int dS_\sigma D(x-y)\partial_\rho h^{\sigma\rho}\right] + \partial_\nu \partial^\nu V^\mu, \tag{15}$$

and solve for V^μ,

$$V^\mu = \int d^3y D(x-y)\partial_\sigma h^{\sigma\mu} - \frac{1}{2}\int d^3y D(x-y)\partial^\mu \left[\partial_\sigma \int d^3z D(y-z)\partial_\rho h^{\sigma\rho} + \int dS_\sigma D(y-z)\partial_\rho h^{\sigma\rho}\right] \tag{16}$$

$$= \int d^3y D(x-y)\partial_\sigma h^{\sigma\mu} - \frac{1}{2}\partial^\mu \int d^3y D(x-y)\left[\partial_\sigma \int d^3z D(y-z)\partial_\rho h^{\sigma\rho} + \int dS_\sigma D(y-z)\partial_\rho h^{\sigma\rho}\right] \tag{17}$$

$$- \frac{1}{2}\int dS^\mu D(x-y)\left[\partial_\sigma \int d^3z D(y-z)\partial_\rho h^{\sigma\rho} + \int dS_\sigma D(y-z)\partial_\rho h^{\sigma\rho}\right]. \tag{18}$$

Dropping surface terms, V^μ takes the form

$$V^\mu = \int d^3y D(x-y)\partial_\sigma h^{\sigma\mu} - \frac{1}{2}\partial^\mu \int d^3y D(x-y)\partial_\sigma \int d^3z D(y-z)\partial_\rho h^{\sigma\rho}. \tag{19}$$

Now using V^μ, we can construct $h_L^{\mu\nu} = \partial^\mu V^\nu + \partial^\nu V^\mu$,

$$h_L^{\mu\nu} = \partial^\mu \int d^3y D(x-y)\partial_\sigma h^{\sigma\nu} + \partial^\nu \int d^3y D(x-y)\partial_\sigma h^{\sigma\mu} \tag{20}$$

$$- \partial^\mu \partial^\nu \int d^3y D(x-y)\partial_\sigma \int d^3z D(y-z)\partial_\rho h^{\sigma\rho} \tag{21}$$

Lastly, we can express this in terms of the longitudinal projector

$$L_{\mu\nu\sigma\rho} = \partial_\mu \int d^4x'\; D(x-x')\eta_{\nu\rho}\partial_\sigma + \partial_\nu \int d^4x'\; D(x-x')\eta_{\mu\sigma}\partial_\tau \tag{22}$$

$$- \partial_\nu \partial_\mu \int d^4x'\; D(x-x')\partial_\sigma \int d^4x''\; D(x-x'')\partial_\rho. \tag{23}$$

Transverse and Longitudinal Decomposition: $h_{\mu\nu} = h^L_{\mu\nu} + h^T_{\mu\nu}$

In a maximally symmetric space of constant curvature, we have the curvature relations

$$R_{\lambda\mu\nu\kappa} = k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}), \qquad R_{\mu\nu} = -(D-1)kg_{\mu\nu}, \qquad R = -D(D-1)k. \tag{24}$$

It is convenient to express the curvature tensors in terms of R, via

$$R_{\mu\nu} = \frac{R}{D}g_{\mu\nu}, \qquad \nabla_\mu R = 0. \tag{25}$$

We posit the longitudinal component of $h^{\mu\nu}$ may be expressed as derivatives onto vectors,

$$h^{\mu\nu}_L = \nabla^\mu V^\nu + \nabla^\nu V^\mu, \tag{26}$$

where V^μ remains to be determined in terms of $h^{\mu\nu}$. Now project out the transverse components of $h^{\mu\nu}$,

$$\nabla_\nu h^{\mu\nu} = \nabla_\nu\nabla^\mu V^\nu + \nabla_\nu\nabla^\nu V^\mu = \left(\nabla_\nu\nabla^\nu - \frac{R}{D}\right)V^\mu + \nabla^\mu\nabla_\nu V^\nu \tag{27}$$

$$\begin{aligned}\nabla_\mu\nabla_\nu h^{\mu\nu} &= \nabla_\mu\nabla_\nu(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \\ &= 2\nabla_\mu\nabla^\mu\nabla_\nu V^\nu - 2(\nabla^\mu R_{\mu\nu})V^\nu - 2R_{\mu\nu}\nabla^\mu V^\nu \\ &= 2\left(\nabla_\mu\nabla^\mu - \frac{R}{D}\right)\nabla_\nu V^\nu. \end{aligned} \tag{28}$$

From $\nabla_\mu\nabla_\nu h^{\mu\nu}$, solve for $\nabla_\nu V^\nu$

$$\nabla_\nu V^\nu = \frac{1}{2}\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma\nabla_\rho h^{\sigma\rho}, \tag{29}$$

where we have introduced the curved space scalar propagator

$$\left(\nabla_\nu\nabla^\nu - \frac{R}{D}\right)D(x,x') = g^{-1/2}\delta^D(x-x'). \tag{30}$$

Now insert $\nabla_\nu V^\nu$ back into $\nabla_\nu h^{\mu\nu}$

$$\begin{aligned}\left(\nabla_\nu\nabla^\nu - \frac{R}{D}\right)V^\mu &= \nabla_\nu h^{\mu\nu} - \nabla^\mu\nabla_\nu V^\nu \\ &= \nabla_\nu h^{\mu\nu} - \frac{1}{2}\nabla^\mu\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma\nabla_\rho h^{\sigma\rho}. \end{aligned} \tag{31}$$

Solving for V^μ,

$$V^\mu = \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma h^{\mu\sigma} - \frac{1}{2}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\mu\int d^Dx''\sqrt{g}\ D(x',x'')\nabla_\sigma\nabla_\rho h^{\sigma\rho}. \tag{32}$$

Performing integration by parts and dropping the surface integrals (an action whos validity needs investigation), we can bring V^μ to the form

$$V^\mu = \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma h^{\mu\sigma} - \frac{1}{2}\nabla^\mu\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla_\rho h^{\sigma\rho}. \tag{33}$$

Now we can construct the longitudinal tensor $h^{\mu\nu}_L = \nabla^\mu V^\nu + \nabla^\nu V^\mu$,

$$h^{\mu\nu}_L = \nabla^\mu\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma h^{\sigma\nu} + \nabla^\nu\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma h^{\sigma\mu} \tag{34}$$

$$-\nabla^\mu\nabla^\nu\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla_\rho h^{\sigma\rho}. \tag{35}$$

To verify, let us confirm $\nabla_\nu h_L^{\mu\nu} = \nabla_\nu h^{\mu\nu}$,

$$\nabla_\nu h_L^{\mu\nu} = \nabla_\nu \nabla^\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\nu} + \nabla_\sigma h^{\sigma\mu} + \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\mu} \tag{36}$$

$$- \nabla_\nu \nabla^\mu \nabla^\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho h^{\sigma\rho}. \tag{37}$$

Noting the commutation relation

$$\nabla_\nu \nabla^\mu \nabla^\nu f(x) = \nabla^\mu \left[\left(\nabla_\nu \nabla^\nu - \frac{R}{D} \right) f(x) \right] \tag{38}$$

we can express the longitudinal tensor as

$$\begin{aligned} \nabla_\nu h_L^{\mu\nu} = \nabla_\nu \nabla^\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\nu} + \nabla_\sigma h^{\sigma\mu} + \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\mu} \\ - \nabla^\mu \nabla_\sigma \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho h^{\sigma\rho}. \end{aligned} \tag{39}$$

Taking another commutation relation

$$\nabla^\mu \nabla_\sigma A^\sigma(x) = \nabla_\sigma \nabla^\mu A^\sigma(x) + \frac{R}{D} A^\mu(x), \tag{40}$$

we are finally left with

$$\nabla_\nu h_L^{\mu\nu} = \nabla_\nu h^{\mu\nu}. \tag{41}$$

Lastly, we cast the longitudinal component into the form a projector

$$\begin{aligned} L_{\mu\nu\sigma\rho} = \nabla_\mu \int d^D x' \sqrt{g}\ D(x,x') g_{\sigma\nu} \nabla_\rho + \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') g_{\sigma\mu} \nabla_\rho \\ - \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho. \end{aligned} \tag{42}$$

It follows that the transverse projector is just what remains,

$$\begin{aligned} T_{\mu\nu\sigma\rho} = g_{\mu\sigma} g_{\nu\rho} - \nabla_\mu \int d^D x' \sqrt{g}\ D(x,x') g_{\sigma\nu} \nabla_\rho - \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') g_{\sigma\mu} \nabla_\rho \\ + \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho. \end{aligned} \tag{43}$$

Still need to confirm if the above actually behave as projectors, i.e. $L_{\mu\nu\sigma\rho} L^{\sigma\rho}{}_{\alpha\beta} = L_{\mu\nu\alpha\beta}$*, etc.*

Traceless Transverse and Traceless Longitudinal Decomposition: : $h_{\mu\nu} = h_{\mu\nu}^{L\theta} + h_{\mu\nu}^{T\theta} + h_{\mu\nu}^{tr}$

Following C.93 in *Brane Gravity*, we may construct the traceless longitudinal component via

$$h_{\mu\nu}^{L\theta} = h_{\mu\nu}^L - \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h_{\sigma\tau}^L + \frac{1}{D-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{D(D-1)} \right] \int d^D x' \sqrt{g}\ F(x,x') g^{\sigma\tau} h_{\sigma\tau}^L, \tag{44}$$

where we have introduced another scalar propogator obeying

$$\left(\nabla_\rho \nabla^\rho - \frac{R}{D-1} \right) F(x,x') = g^{-1/2} \delta^D(x-x'). \tag{45}$$

As written, the tensor $h_{\mu\nu}^{L\theta}$ obeys

$$g^{\mu\nu} h_{\mu\nu}^{L\theta} = 0, \qquad \nabla^\nu h_{\mu\nu}^{L\theta} = \nabla^\nu h_{\mu\nu}^L. \tag{46}$$

With the analogous decomposition following for $h^{T\theta}_{\mu\nu}$ taking the form

$$h^{T\theta}_{\mu\nu} = h^T_{\mu\nu} - \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}h^T_{\sigma\tau} + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\frac{R}{D(D-1)}\right]\int d^Dx'\sqrt{g}\ F(x,x')g^{\sigma\tau}h^T_{\sigma\tau}, \tag{47}$$

we may construct the full $h_{\mu\nu}$ by taking their sum:

$$h^{T\theta}_{\mu\nu} + h^{L\theta}_{\mu\nu} = h_{\mu\nu} - \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}h_{\sigma\tau} + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\frac{R}{D(D-1)}\right]\int d^Dx'\sqrt{g}\ F(x,x')g^{\sigma\tau}h_{\sigma\tau}. \tag{48}$$

Hence the full $h_{\mu\nu}$ takes the form

$$\begin{aligned} h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + h^{L\theta}_{\mu\nu} + \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}h_{\sigma\tau} - \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\frac{R}{D(D-1)}\right]\int d^Dx'\sqrt{g}\ F(x,x')g^{\sigma\tau}h_{\sigma\tau} \\ &\equiv h^{T\theta}_{\mu\nu} + h^{L\theta}_{\mu\nu} + h^{tr}_{\mu\nu}. \end{aligned} \tag{49}$$

The SVT Basis

Given the form for $h^{L\theta}_{\mu\nu}$, unlike the flat space case, I was unable to construct a vector V_μ such that

$$h^{L\theta}_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\sigma V_\sigma. \tag{50}$$

However, this intermediate step, though useful, is not required for an SVT decomposition. First, let us note the relation

$$\begin{aligned} h^{L\theta}_{\mu\nu} + h^{tr}_{\mu\nu} &= h^L_{\mu\nu} + \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}(h_{\sigma\tau} - h^L_{\sigma\tau}) \\ &\qquad - \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\frac{R}{D(D-1)}\right]\int d^Dx'\sqrt{g}\ F(x,x')g^{\sigma\tau}(h_{\sigma\tau} - h^L_{\sigma\tau}) \end{aligned} \tag{51}$$

Next, let us introduce the vector

$$W_\mu = \int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma h_{\sigma\mu}, \tag{52}$$

whereby the longitudinal component (ref) may be expressed as

$$h^L_{\mu\nu} = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \nabla_\mu\nabla_\nu\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma W_\sigma, \tag{53}$$

with a trace obeying

$$g^{\mu\nu}h^L_{\mu\nu} = \nabla^\sigma W_\sigma - \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma W_\sigma. \tag{54}$$

Now we elect to decompose W_μ into its transverse and longitudinal components viz.

$$W_\mu = W^T_\mu + \nabla_\mu W, \qquad W = \int d^Dx'\sqrt{g}\ A(x,x')\nabla^\sigma W_\sigma, \qquad \nabla_\rho\nabla^\rho W = \nabla^\sigma W_\sigma, \tag{55}$$

where we have introduced the scalar propagator which obeys

$$\nabla_\rho\nabla^\rho A(x,x') = g^{-1/2}\delta^D(x-x'). \tag{56}$$

In the scalar vector basis, $h^L_{\mu\nu}$ takes the form

$$h^L_{\mu\nu} = \nabla_\mu W^T_\nu + \nabla_\nu W^T_\mu + \nabla_\mu\nabla_\nu\left(2W - \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W\right), \tag{57}$$

with trace

$$g^{\mu\nu}h^L_{\mu\nu} = \nabla_\rho\nabla^\rho W - \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W. \tag{58}$$

For compactness, let us define the scalar

$$\begin{aligned}
M(x) &= g^{\mu\nu}h_{\mu\nu} - g^{\mu\nu}h^L_{\mu\nu}\\
&= g^{\sigma\tau}h_{\sigma\tau} - \nabla_\rho\nabla^\rho W + \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W\\
&= g^{\sigma\tau}h_{\sigma\tau} - \nabla^\sigma\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\rho h_{\sigma\rho} + \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\rho h_{\sigma\rho}.
\end{aligned} \tag{59}$$

Now we can express (ref) in terms of scalars and vectors as

$$\begin{aligned}
h^{L\theta}_{\mu\nu} + h^{tr}_{\mu\nu} &= \nabla_\mu W^T_\nu + \nabla_\nu W^T_\mu\\
&\quad + \nabla_\mu\nabla_\nu\left[2W - \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W - \frac{1}{D-1}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\right]\\
&\quad + \frac{1}{D-1}g_{\mu\nu}\left[M(x) + \frac{R}{D(D-1)}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\right].
\end{aligned} \tag{60}$$

The full $h_{\mu\nu}$ then may be written as

$$\begin{aligned}
h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + \nabla_\mu W^T_\nu + \nabla_\nu W^T_\mu\\
&\quad + \nabla_\mu\nabla_\nu\left[2W - \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W - \frac{1}{D-1}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\right]\\
&\quad + \frac{1}{D-1}g_{\mu\nu}\left[M(x) + \frac{R}{D(D-1)}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\right].
\end{aligned} \tag{61}$$

With the two scalars and the transverse vector

$$\begin{aligned}
M(x) &= g^{\sigma\tau}h_{\sigma\tau} - \nabla^\sigma\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\rho h_{\sigma\rho} + \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\rho h_{\sigma\rho}\\
W(x) &= \int d^Dx'\sqrt{g}\ A(x,x')\nabla^\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\rho h_{\sigma\rho}\\
W^T_\mu &= \int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma h_{\sigma\mu} - \nabla_\mu\int d^Dx'\sqrt{g}\ A(x,x')\nabla^\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\rho h_{\sigma\rho},
\end{aligned} \tag{62}$$

upon defining

$$\begin{aligned}
2\psi &= -\frac{1}{(D-1)}\left[M(x) + \frac{R}{D(D-1)}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\right]\\
2E &= 2W(x) - \int d^Dx'\sqrt{g}\ D(x,x')\nabla_\rho\nabla^\rho W(x') - \frac{1}{D-1}\int d^Dx'\sqrt{g}\ F(x,x')M(x')\\
E_\mu &= W^T_\mu\\
2E_{\mu\nu} &= h^{T\theta}_{\mu\nu},
\end{aligned} \tag{63}$$

the tensor takes the SVT form

$$h_{\mu\nu} = -2g_{\mu\nu}\psi + 2\nabla_\mu\nabla_\nu E + \nabla_\mu E_\nu + \nabla_\nu E_\mu + 2E_{\mu\nu}. \tag{64}$$

If we restrict to flat space, we have the following simplifications:

$$\begin{aligned}
&R = 0, \qquad A(x,x') = D(x,x') = F(x,x'), \qquad M(x) = g^{\sigma\tau}h_{\sigma\tau} - \nabla^\sigma\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\rho h_{\sigma\rho}\\
&\sqrt{g} = 1, \qquad W(x) = \int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\rho h_{\sigma\rho}.
\end{aligned} \tag{65}$$

According to (ref 63), the SVT components would then be reduce to

$$
\begin{aligned}
2\psi &= -\frac{1}{(D-1)}\left[g^{\sigma\tau}h_{\sigma\tau} - \nabla^{\sigma}\int d^Dx'\ D(x,x')\nabla^{\rho}h_{\sigma\rho}\right] \\
2E &= \frac{D}{D-1}\int d^Dx'\ D(x,x')\nabla^{\sigma}\int d^Dx'\ D(x,x')\nabla^{\rho}h_{\sigma\rho} - \frac{1}{D-1}\int d^Dx'\ D(x,x')g^{\sigma\tau}h_{\sigma\tau} \\
E_{\mu} &= \int d^Dx'\ D(x,x')\nabla^{\sigma}h_{\sigma\mu} - \nabla_{\mu}\int d^Dx'\ D(x,x')\nabla^{\sigma}\int d^Dx''\ D(x',x'')\nabla^{\rho}h_{\sigma\rho} \\
2E_{\mu\nu} &= h^{T\theta}_{\mu\nu}.
\end{aligned}
\tag{66}
$$

Follwing an integration by parts on E and ψ, the above equates to our prior paper results.

Traceless $\pi_{\mu\nu}$ Decomposition

After the 3+1 splitting of $T_{\mu\nu}$, we are left with a traceless $\pi_{\mu\nu}$ of which we would like to decompose into scalars, vectors tensors. Taking $\pi_{\mu\nu}$ to be of the same SVT form as $h_{\mu\nu}$, namely

$$
\pi_{\mu\nu} = -2g_{\mu\nu}\psi + 2\nabla_{\mu}\nabla_{\nu}E + \nabla_{\mu}E_{\nu} + \nabla_{\nu}E_{\mu} + 2E_{\mu\nu}. \tag{67}
$$

From the tracelessness of $\pi_{\mu\nu}$ it follows

$$
2D\psi = 2\nabla_{\rho}\nabla^{\rho}E \tag{68}
$$

(expressing ψ and E in their projected integral form, the above holds identically when $g^{\mu\nu}\pi_{\mu\nu} = 0$, as anticipated). Substituting

$$
\psi = \frac{1}{D}\nabla_{\rho}\nabla^{\rho}E, \tag{69}
$$

the tensor becomes

$$
\pi_{\mu\nu} = -\frac{2}{D}g_{\mu\nu}\nabla_{\rho}\nabla^{\rho}E + 2\nabla_{\mu}\nabla_{\nu}E + \nabla_{\mu}E_{\nu} + \nabla_{\nu}E_{\mu} + 2E_{\mu\nu}. \tag{70}
$$

Finally, upon defining

$$
\pi = E, \qquad \pi_{\mu} = E_{\mu}, \qquad 2E_{\mu\nu} = \pi^{T\theta}_{\mu\nu}, \tag{71}
$$

we may write $\pi_{\mu\nu}$ in the desired form

$$
\pi_{\mu\nu} = -\frac{2}{D}g_{\mu\nu}\nabla_{\rho}\nabla^{\rho}\pi + 2\nabla_{\mu}\nabla_{\nu}\pi + \nabla_{\mu}\pi_{\nu} + \nabla_{\nu}\pi_{\mu} + \pi^{T\theta}_{\mu\nu}. \tag{72}
$$

For reference, the components in their projected form are

$$
\begin{aligned}
2\pi &= 2W(x) - \int d^Dx'\sqrt{g}\ D(x,x')\nabla_{\rho}\nabla^{\rho}W(x') - \frac{1}{D-1}\int d^Dx'\sqrt{g}\ F(x,x')M(x') \\
\pi_{\mu} &= \int d^Dx'\sqrt{g}\ D(x,x')\nabla^{\sigma}h_{\sigma\mu} - \nabla_{\mu}\int d^Dx'\sqrt{g}\ A(x,x')\nabla^{\sigma}\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^{\rho}h_{\sigma\rho},
\end{aligned}
\tag{73}
$$

where

$$
\begin{aligned}
M(x) &= -\nabla^{\sigma}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^{\rho}h_{\sigma\rho} + \frac{R}{D}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^{\sigma}\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^{\rho}h_{\sigma\rho} \\
W(x) &= \int d^Dx'\sqrt{g}\ A(x,x')\nabla^{\sigma}\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^{\rho}h_{\sigma\rho}.
\end{aligned}
\tag{74}
$$

the longitudinal traceless component is expressed as

$$
\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu \\
&\quad + \nabla_\mu \nabla_\nu \Bigg[- \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \\
&\qquad + \frac{1}{D-1} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \right) \Bigg] \\
&\quad + \frac{g_{\mu\nu}}{D-1} \Bigg[- \nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \\
&\qquad - \frac{R}{D(D-1)} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \right) \Bigg]. \qquad (75)
\end{aligned}
$$

At this point, we may elect to decompose the W_μ into its transverse and longitudinal components viz.

$$
W_\mu = W^T_\mu + \nabla_\mu W, \qquad \nabla_\rho \nabla^\rho W = \nabla^\sigma W_\sigma = \nabla^\sigma \int d^D x'' \sqrt{g}\ (x',x'') \nabla^\rho h_{\sigma\rho}. \qquad (76)
$$

Now $h^{L\theta}_{\mu\nu}$ may be expressed in terms of transverse vectors W^T_μ and scalars W,

$$
\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W^T_\nu + \nabla_\nu W^T_\mu \\
&\quad + \nabla_\mu \nabla_\nu \Bigg[2W - \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \\
&\qquad + \frac{1}{D-1} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \right) \Bigg] \\
&\quad + \frac{g_{\mu\nu}}{D-1} \Bigg[- \nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \\
&\qquad - \frac{R}{D(D-1)} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \right) \Bigg]. \qquad (77)
\end{aligned}
$$

Remark

We are at somewhat of an impasse since 1) The vector decomposition involved a propagator different from the tensor decomposition and 2) We have not fully decomposed $h^L_{\mu\nu}$ in terms of scalars and vectors, since the integral relation still remains.

$$
h^{L\theta}_{\mu\nu} = h^L_{\mu\nu} - \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h^L_{\sigma\tau} + \frac{1}{D-1} \left[\nabla_\mu \nabla_\nu + g_{\mu\nu} \frac{R}{D} - g_{\mu\nu} \frac{R}{D-1} \right] \int d^D x' \sqrt{g}\ F(x,x') g^{\sigma\tau} h^L_{\sigma\tau} \qquad (78)
$$

asl;dkfjsl;kdfj

$$
h^{L\theta}_{\mu\nu} = h^L_{\mu\nu} - \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h^L_{\sigma\tau} + \frac{1}{D-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{D(D-1)} \right] \int d^D x' \sqrt{g}\ F(x,x') g^{\sigma\tau} h^L_{\sigma\tau} \qquad (79)
$$

$$
\begin{aligned}
h^{\mu\nu}_L &= \nabla^\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\nu} + \nabla^\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma h^{\sigma\mu} \qquad (80) \\
&\quad - \nabla^\nu \nabla^\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho h^{\sigma\rho}, \qquad (81)
\end{aligned}
$$

where

$$
\left(\nabla_\nu \nabla^\nu - \frac{R}{D} \right) D(x,x') = g^{-1/2} \delta^D (x-x'), \qquad \left(\nabla_\nu \nabla^\nu - \frac{R}{D-1} \right) F(x,x') = g^{-1/2} \delta^F (x-x'). \qquad (82)
$$

$$
\nabla_\mu \nabla^\mu \nabla^\nu \phi = \nabla^\nu \nabla_\mu \nabla^\mu \phi - \frac{R}{D} \nabla^\nu \phi, \quad \nabla_\nu \nabla^\mu \nabla^\nu \phi = \nabla^\mu \nabla_\nu \nabla^\nu \phi - \frac{R}{D} \nabla^\mu \phi, \quad \nabla_\nu \nabla^\sigma W_\sigma = \nabla^\sigma \nabla_\nu W_\sigma + \frac{R}{D} W_\nu \qquad (83)
$$

$$\nabla^\nu h^L_{\mu\nu} = \nabla^\nu \left[\nabla_\mu W_\nu + \nabla_\nu W_\mu - \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma W^\sigma \right]$$

$$= \left(\nabla_\alpha \nabla^\alpha - \frac{R}{D} \right) W_\mu$$

$$= \nabla^\nu h_{\mu\nu} \tag{84}$$

$$\nabla^\nu h^L_{\mu\nu} = \left(\nabla_\alpha \nabla^\alpha - \frac{R}{D} \right) W_\mu, \qquad \nabla^\mu \nabla^\nu h^L_{\mu\nu} = \left(\nabla_\alpha \nabla^\alpha - 2\frac{R}{D} \right) \nabla^\sigma W_\sigma \tag{85}$$

$$g^{\mu\nu} h^L_{\mu\nu} = \nabla^\sigma W_\sigma - \frac{R}{D} \int dx' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \tag{86}$$

$$\nabla^\nu h^L_{\mu\nu} = \left(\nabla_\nu \nabla^\nu - \frac{R}{D} \right) V_\mu + \nabla_\mu \nabla_\nu V^\nu, \qquad \nabla^\mu \nabla^\nu h^L_{\mu\nu} = 2\left(\nabla_\alpha \nabla^\alpha - \frac{R}{D} \right) \nabla_\nu V^\nu, \qquad g^{\mu\nu} h^L_{\mu\nu} = 2\nabla^\sigma V_\sigma \tag{87}$$

$$h^{L\theta}_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{D-1} g_{\mu\nu} \nabla^\sigma V_\sigma + \frac{2}{D-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{D(D-1)} \right] \int d^D x' \sqrt{g}\ F(x,x') \nabla^\sigma V_\sigma \tag{88}$$

$$h^{L\theta}_{\mu\nu} = \nabla_\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma h_{\sigma\nu} + \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma h_{\sigma\mu} \tag{89}$$

$$- \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho h^{\sigma\rho},$$

$$- \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h^L_{\sigma\tau} + \frac{1}{D-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{D(D-1)} \right] \int d^D x' \sqrt{g}\ F(x,x') g^{\sigma\tau} h^L_{\sigma\tau} \tag{90}$$

$$h^{L\theta}_{\mu\nu} = \nabla_\mu \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma h_{\sigma\nu} + \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma h_{\sigma\mu} \tag{91}$$

$$- \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') \nabla_\sigma \int d^D x'' \sqrt{g}\ D(x',x'') \nabla_\rho h^{\sigma\rho},$$

$$- \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h^L_{\sigma\tau} + \frac{1}{D-1} \nabla_\mu \nabla_\nu \int d^D x' \sqrt{g}\ D(x,x') g^{\sigma\tau} h^L_{\sigma\tau} \tag{92}$$

$$h_{\mu\nu}^{L\theta} = \nabla_\mu V_\nu + \nabla_\nu V_\mu + \alpha g_{\mu\nu} \nabla^\sigma V_\sigma + \beta R g_{\mu\nu} \nabla^\sigma V_\sigma \tag{93}$$

$$g^{\mu\nu} h_{\mu\nu}^{L\theta} = (2 + \alpha D + \beta R D) \nabla^\sigma V_\sigma \tag{94}$$

$$\begin{aligned} \nabla^\nu h_{\mu\nu}^{L\theta} &= \nabla^\sigma \nabla_\sigma V_\mu + \nabla_\mu \nabla^\sigma V_\sigma - \frac{R}{D} V_\mu + \alpha \nabla_\mu \nabla^\sigma V_\sigma + \beta R \nabla_\mu \nabla^\sigma V_\sigma \\ &= (1 + \alpha + \beta R) \nabla_\mu \nabla^\sigma V_\sigma + \left(\nabla^\sigma \nabla_\sigma - \frac{R}{D} \right) V_\mu \end{aligned} \tag{95}$$

$$\begin{aligned} \nabla^\mu \nabla^\nu h_{\mu\nu}^{L\theta} &= 2 \nabla^\sigma \nabla_\sigma \nabla^\mu V_\mu - 2 \frac{R}{D} \nabla^\mu V_\mu + \alpha \nabla^\sigma \nabla_\sigma \nabla^\mu V_\mu + \beta R \nabla^\sigma \nabla_\sigma \nabla^\mu V_\mu \\ &= \left[(2 + \alpha + \beta R) \nabla_\sigma \nabla^\sigma - 2 \frac{R}{D} \right] \nabla^\mu V_\mu \end{aligned} \tag{96}$$

$$\nabla^\mu V_\mu = \int dx' \sqrt{g}\ D(x, x') \nabla^\sigma \nabla^\tau h_{\sigma\tau} \tag{97}$$

where

$$\left[(2 + \alpha + \beta R) \nabla_\sigma \nabla^\sigma - 2 \frac{R}{D} \right] D(x, x') = \sqrt{g} \delta(x - x') \tag{98}$$

Substitute this into $\nabla^\nu h_{\mu\nu}^{L\theta}$,

$$h_{\mu\nu}^{L\theta} = \nabla_\mu V_\nu + \nabla_\nu V_\mu + \alpha g_{\mu\nu} \nabla^\sigma V_\sigma + \beta R g_{\mu\nu} \nabla^\sigma V_\sigma + [\gamma \nabla_\mu \nabla_\nu + \rho g_{\mu\nu} + \kappa R g_{\mu\nu}] \int D(x, x') \nabla^\sigma V_\sigma \tag{99}$$

$$\nabla_\nu \nabla^\nu D(x, x') - A(x) D(x, x') = \sqrt{g} \delta^D (x - x') \tag{100}$$

$$\begin{aligned} \nabla^\mu \nabla^\nu h_{\mu\nu}^{L\theta} &= 2 \left(\nabla_\rho \nabla^\rho - \frac{R}{D} \right) \nabla^\sigma V_\sigma + (\alpha + \beta R) \nabla_\rho \nabla^\rho \nabla^\sigma V_\sigma + \gamma \left(\nabla_\rho \nabla^\rho - \frac{R}{D} \right) \nabla^\sigma V_\sigma \\ &\quad + \gamma \left(\nabla_\rho \nabla^\rho - \frac{R}{D} \right) \left(A(x) \int D(x, x') \nabla^\sigma V_\sigma \right) \\ &\quad + (\rho + \kappa R) \nabla^\sigma V_\sigma + (\rho + \kappa R) \left(A(x) \int D(x, x') \nabla^\sigma V_\sigma \right) \\ &= \left[\left(\nabla_\rho \nabla^\rho - \frac{R}{D} \right) (2 + \gamma) + (\alpha + \beta R) \nabla_\rho \nabla^\rho + \rho + \kappa R \right] \nabla^\sigma V_\sigma \\ &\quad + \left[\gamma \left(\nabla_\rho \nabla^\rho - \frac{R}{D} \right) + \rho + \kappa R \right] A(x) \int D(x, x') \nabla^\sigma V_\sigma \\ &= \left[(\alpha + \beta R + 2 + \gamma) \nabla_\rho \nabla^\rho - (2 + \gamma) \frac{R}{D} + \rho + \kappa R + \gamma A(x) \right] \nabla^\sigma V_\sigma \\ &\quad + \left[\left(\rho + \kappa R - \gamma \frac{R}{D} + \gamma A(x) \right) A(x) + \gamma \nabla_\rho \nabla^\rho A(x) \right] \int D(x, x') \nabla^\sigma V_\sigma \end{aligned} \tag{101}$$

$$g^{\mu\nu} h_{\mu\nu}^{L\theta} = [2 + D(\alpha + \beta R) + \gamma] \nabla^\sigma V_\sigma + \gamma A(x) \int D(x, x') \nabla^\sigma V_\sigma + D(\rho + \kappa R) \int D(x, x') \nabla^\sigma V_\sigma \tag{102}$$

$$\begin{aligned} \nabla^\mu \nabla^\nu h_{\mu\nu}^{L\theta} &= \left[(\alpha + 2 + \gamma) \nabla_\rho \nabla^\rho + R \left(\frac{-2 - \gamma}{D} + \kappa + \gamma q \right) + \rho + \gamma p \right] \nabla^\sigma V_\sigma \\ &\quad + \left\{ p(\rho + \gamma p) + R \left[p(\kappa - \frac{\gamma}{D} + \gamma q) + q(\rho + \gamma p) \right] + R^2 q \left(k - \frac{\gamma}{D} + \gamma q \right) \right\} \int D(x, x') \nabla^\sigma V_\sigma \end{aligned} \tag{103}$$

Hence we require

$$2+\gamma+D(\alpha+\beta R)=0, \qquad \gamma A(x)+D(\rho+\kappa R)=0. \tag{104}$$

Taking $A(x)=p+qR$, the conditions are then (holding for each power of R),

$$2+\gamma+D\alpha=0, \qquad D\beta R=0, \qquad \gamma p+D\rho=0, \qquad \gamma q+D\kappa=0 \tag{105}$$

For convenience, we would like the integral relation in $\nabla^\mu\nabla^\nu h^{L\theta}_{\mu\nu}$ to vanish, and thus we set

$$0=p(\rho+\gamma p)+R\left[p(\kappa-\frac{\gamma}{D}+\gamma q)+q(\rho+\gamma p)\right]+R^2 q\left(k-\frac{\gamma}{D}+\gamma q\right). \tag{106}$$

Hence all together we have

$$2+\gamma+D\alpha=0, \qquad \beta=0, \qquad \gamma p+D\rho=0, \qquad \gamma q+D\kappa=0 \tag{107}$$

$$p(\rho+\gamma p)=0, \qquad p(\kappa-\frac{\gamma}{D}+\gamma q)+q(\rho+\gamma p)=0, \qquad q\left(\kappa-\frac{\gamma}{D}+\gamma q\right)=0 \tag{108}$$

Six equations, six unknowns. Start with the relations

$$\kappa=-\gamma\frac{q}{D}, \qquad \rho=-\gamma\frac{p}{D}, \tag{109}$$

which leads to

$$\gamma p\left(p-\frac{p}{D}\right)=0 \tag{110}$$

Now either $p=0$, $\gamma=0$, or $p-\frac{p}{D}=0$.

Helpful covariant commutations (within maximally symmetric space):

$$\nabla^\nu\nabla_\mu V_\nu=\nabla_\mu\nabla^\nu V_\nu-\frac{R}{D}V_\mu, \qquad \nabla^\mu\nabla_\rho\nabla^\rho V_\mu=\nabla_\rho\nabla^\rho\nabla^\mu V_\mu-\frac{R}{D}\nabla^\sigma V_\sigma. \tag{111}$$

Let us posit $h^{L\theta}_{\mu\nu}$ to be of the follwing form (see A.1)

$$h^{L\theta}_{\mu\nu}=\nabla_\mu V_\nu+\nabla_\nu V_\mu-\frac{2}{D}g_{\mu\nu}\nabla^\sigma V_\sigma. \tag{112}$$

It follows that

$$g^{\mu\nu}h^{L\theta}_{\mu\nu}=0, \tag{113}$$

$$\nabla^\nu h^{L\theta}_{\mu\nu}=\nabla^\nu h_{\mu\nu}=\left(\nabla_\rho\nabla^\rho-\frac{R}{D}\right)V_\mu+\frac{D-2}{D}\nabla_\mu\nabla^\sigma V_\sigma, \tag{114}$$

$$\begin{aligned}\nabla^\mu\nabla^\nu h^{L\theta}_{\mu\nu}=\nabla^\mu\nabla^\nu h_{\mu\nu}&=2\left(\frac{D-1}{D}\nabla_\rho\nabla^\rho-\frac{R}{D}\right)\nabla^\sigma V_\sigma\\ \to\quad \frac{D}{2(D-1)}\nabla^\mu\nabla^\nu h_{\mu\nu}&=\left(\nabla_\rho\nabla^\rho-\frac{R}{D-1}\right)\nabla^\sigma V_\sigma,\end{aligned} \tag{115}$$

where we have imposed $\nabla^\nu h^{L\theta}_{\mu\nu}=\nabla^\nu h_{\mu\nu}$, $\nabla^\mu\nabla^\nu h^{L\theta}_{\mu\nu}=\nabla^\mu\nabla^\nu h_{\mu\nu}$. Now introduce a scalar propagator $D(x,x')$ which obeys

$$\left(\nabla_\rho\nabla^\rho-\frac{R}{D-1}\right)D(x,x')=g^{-1/2}\delta^D(x-x'), \tag{116}$$

and solve for $\nabla^\sigma V_\sigma$, viz.

$$\nabla^\sigma V_\sigma = \frac{D}{2(D-1)}\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{117}$$

Next, substitute $\nabla^\sigma V_\sigma$ into $\nabla^\nu h^{L\theta}_{\mu\nu}$, to yield

$$\nabla^\nu h_{\mu\nu} = \left(\nabla_\rho\nabla^\rho - \frac{R}{D}\right)V_\mu + \frac{D-2}{2(D-1)}\nabla_\mu\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{118}$$

or

$$\left(\nabla_\rho\nabla^\rho - \frac{R}{D}\right)V_\mu = \nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\nabla_\mu\int d^Dx'\sqrt{g}\ D(x,x')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{119}$$

Introduce another scalar propogator $F(x,x')$, which obeys

$$\left(\nabla_\rho\nabla^\rho - \frac{R}{D}\right)F(x,x') = g^{-1/2}\delta^D(x-x'), \tag{120}$$

whereby V_μ is solved as

$$V_\mu = \int d^Dx'\sqrt{g}\ F(x,x')\nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\int d^Dx'\sqrt{g}\ F(x,x')\nabla^{x'}_\mu\int d^Dx''\sqrt{g}\ D(x',x'')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{121}$$

Let us now introduce a tensor $h^{tr}_{\mu\nu}$, to facilitate expressing the entire $h_{\mu\nu}$ as

$$h_{\mu\nu} = h^{L\theta}_{\mu\nu} + h^{T\theta}_{\mu\nu} + h^{tr}_{\mu\nu}. \tag{122}$$

For $h_{\mu\nu}$ to take this form, such a $h^{tr}_{\mu\nu}$ must obey

$$g^{\mu\nu}h^{tr}_{\mu\nu} = g^{\mu\nu}h_{\mu\nu}, \qquad \nabla^\nu h^{tr}_{\mu\nu} = 0. \tag{123}$$

With $h^{L\theta}_{\mu\nu}$ already obeying $\nabla^\nu h^{L\theta}_{\mu\nu} = \nabla^\nu h_{\mu\nu}$, $g^{\mu\nu}h^{L\theta}_{\mu\nu} = 0$, we see that $h^{T\theta}_{\mu\nu} = h_{\mu\nu} - h^{L\theta}_{\mu\nu} - h^{tr}_{\mu\nu}$ will be transverse and traceless as desired. As constructed in (C.93), the tensor that satisfies our requirements is

$$h^{tr}_{\mu\nu} = \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}h_{\sigma\tau} - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{1}{D(D-1)}g_{\mu\nu}R\right)\int d^Dx'\sqrt{g}\ D(x,x')g^{\sigma\tau}h_{\sigma\tau}. \tag{124}$$

Consequently, we may express the entire $h_{\mu\nu}$ as

$$\begin{aligned} h_{\mu\nu} = h^{T\theta}_{\mu\nu} &+ \nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\sigma V_\sigma + \frac{1}{D-1}g_{\mu\nu}g^{\sigma\tau}h_{\sigma\tau} \\ &- \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{1}{D(D-1)}g_{\mu\nu}R\right)\int d^Dx'\sqrt{g}\ D(x,x')g^{\sigma\tau}h_{\sigma\tau}. \end{aligned} \tag{125}$$

To match the desired form for SVT decomposition, we will need to decompose the vectors V_μ into transverse and longitudinal components (denoted here as W_μ and W). This is achieved by introducing the scalar propagator

$$\nabla_\rho\nabla^\rho A(x,x') = g^{-1/2}\delta^D(x-x'), \tag{126}$$

whereby V_μ is deconstructed as

$$V_\mu = W_\mu + \nabla_\mu W, \tag{127}$$

with

$$W = \int d^Dx'\sqrt{g}\ A(x,x')\nabla^\sigma V_\sigma, \qquad W_\mu = V_\mu - \nabla_\mu W. \tag{128}$$

The full $h_{\mu\nu}$ then takes the form

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu + 2\nabla_\mu\nabla_\nu W - \frac{2}{D} g_{\mu\nu}\nabla_\sigma\nabla^\sigma W + \frac{1}{D-1} g_{\mu\nu} g^{\sigma\tau} h_{\sigma\tau}$$
$$- \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{1}{D(D-1)} g_{\mu\nu} R\right)\int d^D x' \sqrt{g}\, D(x,x') g^{\sigma\tau} h_{\sigma\tau}. \tag{129}$$

Upon defining

$$2\psi = \frac{2}{D}\nabla_\sigma\nabla^\sigma W - \frac{1}{D-1} g^{\sigma\tau} h_{\sigma\tau} - \frac{R}{D(D-1)^2}\int d^D x' \sqrt{g}\, D(x,x') g^{\sigma\tau} h_{\sigma\tau}$$
$$2E = 2W - \frac{1}{D-1}\int d^D x' \sqrt{g}\, D(x,x') g^{\sigma\tau} h_{\sigma\tau}$$
$$E_\mu = W_\mu$$
$$2E_{\mu\nu} = h^{T\theta}_{\mu\nu}, \tag{130}$$

$h_{\mu\nu}$ may be written in the SVT form

$$h_{\mu\nu} = -2\psi g_{\mu\nu} + 2\nabla_\mu\nabla_\nu E + \nabla_\mu E_\nu + \nabla_\nu E_\mu + 2E_{\mu\nu}. \tag{131}$$

$$V_\mu = \int d^D x' \sqrt{g}\, F(x,x')\nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\int d^D x' \sqrt{g}\, F(x,x')\nabla^{x'}_\mu \int d^D x'' \sqrt{g}\, D(x',x'')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{132}$$

$$\nabla^\sigma V_\sigma = \tag{133}$$

$$V_\mu = W_\mu + \nabla_\mu W, \qquad h_{\mu\nu} = \nabla_\mu W_\nu + \nabla_\nu W_\mu + 2\nabla_\mu\nabla_\nu W \tag{134}$$

$$\nabla^\nu h_{\mu\nu} = \nabla_\sigma\nabla^\sigma W_\mu + 2\nabla_\sigma\nabla^\sigma\nabla_\mu W \tag{135}$$

$$\nabla^\mu\nabla^\nu h_{\mu\nu} = 2\nabla_\rho\nabla^\rho\nabla_\sigma\nabla^\sigma W \tag{136}$$

$$\nabla_\sigma\nabla^\sigma W = \frac{1}{2}\int D(x,x')\nabla^\mu\nabla^\nu h_{\mu\nu} = \nabla^\sigma V_\sigma \tag{137}$$

$$\nabla_\sigma\nabla^\sigma W_\mu = \nabla^\nu h_{\mu\nu} - 2\nabla_\mu\nabla_\sigma\nabla^\sigma W \tag{138}$$

$$W_\mu = \int D(x,x')\nabla^\nu h_{\mu\nu} - \int D(x,x')\nabla_\mu \int D(x',x'')\nabla^\mu\nabla^\nu h_{\mu\nu} \tag{139}$$

$$V_\mu = W_\mu + \nabla_\mu W \tag{140}$$
$$= \int D(x,x')\nabla^\sigma h_{\mu\sigma} - \int D(x,x')\nabla_\mu \int D(x',x'')\nabla^\sigma\nabla^\rho h_{\sigma\rho} + \frac{1}{2}\nabla_\mu \int D(x,x') \int D(x',x'')\nabla^\sigma\nabla^\rho h_{\sigma\rho} \tag{141}$$

$$V_\mu = \int d^D x' \sqrt{g}\, F(x,x')\nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\nabla_\mu \int d^D x' \sqrt{g}\, F(x,x')\nabla^\sigma \int d^D x'' \sqrt{g}\, D(x',x'')\nabla^\tau h_{\sigma\tau}. \tag{142}$$

$$\nabla^\nu h^{L\theta}_{\mu\nu} = \nabla^\nu h_{\mu\nu} = \left(\nabla_\rho\nabla^\rho - \frac{R}{D}\right) V_\mu + \frac{D-2}{D}\nabla_\mu\nabla^\sigma V_\sigma, \tag{143}$$

$$\nabla^\mu\nabla^\nu h^{L\theta}_{\mu\nu} = \nabla^\mu\nabla^\nu h_{\mu\nu} = 2\left(\frac{D-1}{D}\nabla_\rho\nabla^\rho - \frac{R}{D}\right)\nabla^\sigma V_\sigma$$
$$\to \quad \frac{D}{2(D-1)}\nabla^\mu\nabla^\nu h_{\mu\nu} = \left(\nabla_\rho\nabla^\rho - \frac{R}{D-1}\right)\nabla^\sigma V_\sigma, \tag{144}$$

$$\nabla^\sigma V_\sigma = \nabla^\sigma \int F(x,x')\nabla^\rho h_{\sigma\rho} - \frac{D-2}{2(D-1)}\nabla^\sigma \int D(x,x')\nabla^\tau h_{\sigma\tau} + \frac{D-2}{2(D-1)}\frac{R}{D}\int F(x,x')\nabla^\sigma \int D(x',x'')\nabla^\tau h_{\sigma\tau} \tag{145}$$

$$\left(\nabla_\rho\nabla^\rho - \frac{R}{D}\right)V_\mu = \nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\nabla_\mu\nabla^\sigma \int d^Dx'\sqrt{g}\; D(x,x')\nabla^\tau h_{\sigma\tau}. \tag{146}$$

$$V^\mu = \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma h^{\mu\sigma} - \frac{1}{2}\nabla^\mu \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma \int d^Dx''\sqrt{g}\; D(x',x'')\nabla_\rho h^{\sigma\rho}. \tag{147}$$

Now we can construct the longitudinal tensor $h_L^{\mu\nu} = \nabla^\mu V^\nu + \nabla^\nu V^\mu$,

$$h_L^{\mu\nu} = \nabla^\mu \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma h^{\sigma\nu} + \nabla^\nu \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma h^{\sigma\mu} \tag{148}$$

$$- \nabla^\mu\nabla^\nu \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma \int d^Dx''\sqrt{g}\; D(x',x'')\nabla_\rho h^{\sigma\rho}. \tag{149}$$

$$W^\mu = \int d^Dx'\sqrt{g}\; D(x,x')\nabla_\sigma h^{\sigma\mu} \tag{150}$$

$$h^L_{\mu\nu} = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \nabla_\mu\nabla_\nu \int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma \tag{151}$$

$$h^{L\theta}_{\mu\nu} = h^L_{\mu\nu} - \frac{1}{D-1}g_{\mu\nu}g^{\sigma\rho}h^L_{\sigma\rho} + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - \frac{1}{D(D-1)}Rg_{\mu\nu}\right]\int d^Dx'\sqrt{g}\; F(x,x')g^{\sigma\rho}h^L_{\sigma\rho} \tag{152}$$

$$g^{\mu\nu}h^L_{\mu\nu} = \nabla^\sigma W_\sigma - \frac{R}{D}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma \tag{153}$$

$$\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu \\
&\quad + \nabla_\mu\nabla_\nu\left[-\int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma + \frac{1}{D-1}\int d^Dx'\sqrt{g}\; F(x,x')\left(\nabla^\sigma W_\sigma - \frac{R}{D}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma\right)\right] \\
&\qquad + \frac{g_{\mu\nu}}{D-1}\Bigg[-\nabla^\sigma W_\sigma - \frac{R}{D}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma \\
&\qquad - \frac{1}{D(D-1)}R\int d^Dx'\sqrt{g}\; F(x,x')\left(\nabla^\sigma W_\sigma - \frac{R}{D}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu - \nabla_\mu\nabla_\nu \int d^Dx'\sqrt{g}\; D(x,x')\nabla^\sigma W_\sigma \\
&\qquad - \frac{1}{D-1}g_{\mu\nu}\left(g^{\sigma\rho}h^L_{\sigma\rho} + \frac{1}{D(D-1)}R\int d^Dx'\sqrt{g}\; F(x,x')g^{\sigma\rho}h^L_{\sigma\rho}\right) \\
&\qquad + \frac{1}{D-1}\nabla_\mu\nabla_\nu \int d^Dx'\sqrt{g}\; F(x,x')g^{\sigma\rho}h^L_{\sigma\rho}
\end{aligned} \tag{154}$$

$$V_\mu = W_\mu - \frac{D-2}{2(D-1)}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^{x'}_\mu \int d^Dx''\sqrt{g}\; F(x',x'')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{155}$$

$$W_\mu = V_\mu + \frac{D-2}{2(D-1)}\int d^Dx'\sqrt{g}\; D(x,x')\nabla^{x'}_\mu \int d^Dx''\sqrt{g}\; F(x',x'')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{156}$$

$$V_\mu = \int d^Dx'\sqrt{g}\; F(x,x')\nabla^\nu h_{\mu\nu} - \frac{D-2}{2(D-1)}\int d^Dx'\sqrt{g}\; F(x,x')\nabla^{x'}_\mu \int d^Dx''\sqrt{g}\; D(x',x'')\nabla^\sigma\nabla^\tau h_{\sigma\tau}. \tag{157}$$

$$
\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu \\
&\quad + \nabla_\mu \nabla_\nu \left[- \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma + \frac{1}{D-1} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \right) \right] \\
&\qquad + \frac{g_{\mu\nu}}{D-1} \Big[- \nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \\
&\qquad - \frac{1}{D(D-1)} R \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla^\sigma W_\sigma - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla^\sigma W_\sigma \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
W_\mu &= W^T_\mu + \nabla_\mu W && (158) \\
&= W^T_\mu + \nabla_\mu \int d^D x' \sqrt{g}\ A(x,x') \nabla^\sigma \int d^D x'' \sqrt{g}\ (x',x'') \nabla^\rho h_{\sigma\rho} && (159)
\end{aligned}
$$

$$
\nabla_\rho \nabla^\rho W = \nabla^\sigma W_\sigma = \nabla^\sigma \int d^D x'' \sqrt{g}\ (x',x'') \nabla^\rho h_{\sigma\rho} \tag{160}
$$

$$
\begin{aligned}
h^{L\theta}_{\mu\nu} &= \nabla_\mu W^T_\nu + \nabla_\nu W^T_\mu \\
&\quad + \nabla_\mu \nabla_\nu \Big[2W - \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \\
&\qquad + \frac{1}{D-1} \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \right) \Big] \\
&\qquad + \frac{g_{\mu\nu}}{D-1} \Big[- \nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \\
&\qquad - \frac{1}{D(D-1)} R \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \right) \Big]
\end{aligned}
$$

Upon defining

$$
\begin{aligned}
2\psi &= \frac{1}{D-1} \Big[\nabla_\rho \nabla^\rho W + \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \\
&\qquad + \frac{1}{D(D-1)} R \int d^D x' \sqrt{g}\ F(x,x') \left(\nabla_\rho \nabla^\rho W - \frac{R}{D} \int d^D x' \sqrt{g}\ D(x,x') \nabla_\rho \nabla^\rho W \right) \Big] \\
2E &= 2W - \frac{1}{D-1} \int d^D x' \sqrt{g}\ D(x,x') g^{\sigma\tau} h_{\sigma\tau} \\
E_\mu &= W_\mu \\
2E_{\mu\nu} &= h^{T\theta}_{\mu\nu}, && (161)
\end{aligned}
$$