Vector RW PDE's

1 Polar RW

$$\tilde{g}_{ij} = \mathrm{Diag}\left(\frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) \tag{1.1}$$

$$A^i = \tilde{\nabla}_a\tilde{\nabla}^a B^i \tag{1.2}$$

$$\tilde{\nabla}_a B^a = B_1(2r^{-1}-3kr) + B_2\cos\theta r^{-2}(\sin\theta)^{-1} + (1-kr^2)\partial_1 B_1 + r^{-2}\partial_2 B_2 + r^{-2}(\sin\theta)^{-2}\partial_3 B_3 \tag{1.3}$$

$$\begin{aligned} A^1 &= B_1(k-2r^{-2}+k^2r^2) + B_2\cos\theta(-2r^{-3}+2kr^{-1})(\sin\theta)^{-1} + (2r^{-1}-7kr+5k^2r^3)\partial_1 B_1 \\ &\quad +(1-2kr^2+k^2r^4)\partial_1\partial_1 B_1 + \cos\theta(-k+r^{-2})(\sin\theta)^{-1}\partial_2 B_1 + (-2r^{-3}+2kr^{-1})\partial_2 B_2 \\ &\quad +(-k+r^{-2})\partial_2\partial_2 B_1 + (-2r^{-3}+2kr^{-1})(\sin\theta)^{-2}\partial_3 B_3 + (-k+r^{-2})(\sin\theta)^{-2}\partial_3\partial_3 B_1 \end{aligned} \tag{1.4}$$

$$\text{Substituting } \partial_2 B_2 \tag{1.5}$$

$$\begin{aligned} A^1 &= B_1(-9k+2r^{-2}+7k^2r^2) + (4r^{-1}-11kr+7k^2r^3)\partial_1 B_1 + (1-2kr^2+k^2r^4)\partial_1\partial_1 B_1 \\ &\quad +\cos\theta(-k+r^{-2})(\sin\theta)^{-1}\partial_2 B_1 + (-k+r^{-2})\partial_2\partial_2 B_1 + (-k+r^{-2})(\sin\theta)^{-2}\partial_3\partial_3 B_1 \end{aligned} \tag{1.6}$$

$$\tag{1.7}$$

$$\begin{aligned} A^2 &= B_2\big(2kr^{-2} - r^{-4}(\sin\theta)^{-2}\big) - kr^{-1}\partial_1 B_2 + (-k+r^{-2})\partial_1\partial_1 B_2 + (2r^{-3}-2kr^{-1})\partial_2 B_1 \\ &\quad +\cos\theta r^{-4}(\sin\theta)^{-1}\partial_2 B_2 + r^{-4}\partial_2\partial_2 B_2 - 2\cos\theta r^{-4}(\sin\theta)^{-3}\partial_3 B_3 + r^{-4}(\sin\theta)^{-2}\partial_3\partial_3 B_2 \end{aligned} \tag{1.8}$$

$$\text{Substituting } \partial_3 B_3 \tag{1.9}$$

$$\begin{aligned} A^2 &= B_2\big(-2r^{-4}+2kr^{-2}+r^{-4}(\sin\theta)^{-2}\big) + B_1\cos\theta(4r^{-3}-6kr^{-1})(\sin\theta)^{-1} \\ &\quad +\cos\theta(-2k+2r^{-2})(\sin\theta)^{-1}\partial_1 B_1 - kr^{-1}\partial_1 B_2 + (-k+r^{-2})\partial_1\partial_1 B_2 + (2r^{-3}-2kr^{-1})\partial_2 B_1 \\ &\quad +3\cos\theta r^{-4}(\sin\theta)^{-1}\partial_2 B_2 + r^{-4}\partial_2\partial_2 B_2 + r^{-4}(\sin\theta)^{-2}\partial_3\partial_3 B_2 \end{aligned} \tag{1.10}$$

$$\tag{1.11}$$

$$\begin{aligned} A^3 &= 2kB_3r^{-2}(\sin\theta)^{-2} - kr^{-1}(\sin\theta)^{-2}\partial_1 B_3 + (-k+r^{-2})(\sin\theta)^{-2}\partial_1\partial_1 B_3 \\ &\quad -\cos\theta r^{-4}(\sin\theta)^{-3}\partial_2 B_3 + r^{-4}(\sin\theta)^{-2}\partial_2\partial_2 B_3 + (2r^{-3}-2kr^{-1})(\sin\theta)^{-2}\partial_3 B_1 \\ &\quad +2\cos\theta r^{-4}(\sin\theta)^{-3}\partial_3 B_2 + r^{-4}(\sin\theta)^{-4}\partial_3\partial_3 B_3 \end{aligned} \tag{1.12}$$

2 Polar Conformal RW

$$\tilde{g}_{ij} = (1+\rho^2 k/4)^{-2}\mathrm{Diag}\left(1,\rho^2,\rho^2\sin^2\theta\right) \tag{2.1}$$

$$A^i = \tilde{\nabla}_a\tilde{\nabla}^a B^i \tag{2.2}$$

$$\begin{aligned}\tilde{\nabla}_a B^a &= B_1(2\rho^{-1}+\tfrac{1}{2}k\rho)+B_2\cos\theta(\tfrac{1}{2}k+\rho^{-2}+\tfrac{1}{16}k^2\rho^2)(\sin\theta)^{-1}+(1+\tfrac{1}{2}k\rho^2+\tfrac{1}{16}k^2\rho^4)\partial_1 B_1\\ &\quad +(\tfrac{1}{2}k+\rho^{-2}+\tfrac{1}{16}k^2\rho^2)\partial_2 B_2+(\tfrac{1}{2}k+\rho^{-2}+\tfrac{1}{16}k^2\rho^2)(\sin\theta)^{-2}\partial_3 B_3\end{aligned} \tag{2.3}$$

$$\begin{aligned}A^1 &= B_1(\tfrac{3}{2}k-2\rho^{-2}+\tfrac{9}{8}k^2\rho^2+\tfrac{5}{32}k^3\rho^4)+B_2\cos\theta(-2\rho^{-3}-k\rho^{-1}+\tfrac{1}{16}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)(\sin\theta)^{-1}\\ &\quad +(2\rho^{-1}+\tfrac{5}{2}k\rho+\tfrac{9}{8}k^2\rho^3+\tfrac{7}{32}k^3\rho^5+\tfrac{1}{64}k^4\rho^7)\partial_1 B_1\\ &\quad +(1+k\rho^2+\tfrac{3}{8}k^2\rho^4+\tfrac{1}{16}k^3\rho^6+\tfrac{1}{256}k^4\rho^8)\partial_1\partial_1 B_1\\ &\quad +\cos\theta(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)(\sin\theta)^{-1}\partial_2 B_1\\ &\quad +(-2\rho^{-3}-k\rho^{-1}+\tfrac{1}{16}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)\partial_2 B_2\\ &\quad +(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)\partial_2\partial_2 B_1\\ &\quad +(-2\rho^{-3}-k\rho^{-1}+\tfrac{1}{16}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)(\sin\theta)^{-2}\partial_3 B_3\\ &\quad +(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)(\sin\theta)^{-2}\partial_3\partial_3 B_1\end{aligned} \tag{2.4}$$

$$\text{Substituting } \partial_2 B_2 \tag{2.5}$$

$$\begin{aligned}A^1 &= B_1(\tfrac{5}{2}k+2\rho^{-2}+\tfrac{7}{8}k^2\rho^2+\tfrac{3}{32}k^3\rho^4)+(4\rho^{-1}+\tfrac{7}{2}k\rho+\tfrac{9}{8}k^2\rho^3+\tfrac{5}{32}k^3\rho^5+\tfrac{1}{128}k^4\rho^7)\partial_1 B_1\\ &\quad +(1+k\rho^2+\tfrac{3}{8}k^2\rho^4+\tfrac{1}{16}k^3\rho^6+\tfrac{1}{256}k^4\rho^8)\partial_1\partial_1 B_1\\ &\quad +\cos\theta(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)(\sin\theta)^{-1}\partial_2 B_1\\ &\quad +(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)\partial_2\partial_2 B_1\\ &\quad +(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)(\sin\theta)^{-2}\partial_3\partial_3 B_1\end{aligned} \tag{2.6}$$

$$ \tag{2.7}$$

$$\begin{aligned}A^2 &= B_2\big(k^2+2k\rho^{-2}+\tfrac{1}{8}k^3\rho^2+(-\tfrac{3}{8}k^2-\rho^{-4}-k\rho^{-2}-\tfrac{1}{16}k^3\rho^2-\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\big)\\ &\quad +(\tfrac{1}{2}k\rho^{-1}+\tfrac{3}{8}k^2\rho+\tfrac{3}{32}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)\partial_1 B_2\\ &\quad +(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)\partial_1\partial_1 B_2+(2\rho^{-3}+k\rho^{-1}-\tfrac{1}{16}k^3\rho^3-\tfrac{1}{128}k^4\rho^5)\partial_2 B_1\\ &\quad +\cos\theta(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-1}\partial_2 B_2\\ &\quad +(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)\partial_2\partial_2 B_2\\ &\quad +\cos\theta(-\tfrac{3}{4}k^2-2\rho^{-4}-2k\rho^{-2}-\tfrac{1}{8}k^3\rho^2-\tfrac{1}{128}k^4\rho^4)(\sin\theta)^{-3}\partial_3 B_3\\ &\quad +(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\partial_3\partial_3 B_2\end{aligned} \tag{2.8}$$

$$\text{Substituting } \partial_3 B_3 \tag{2.9}$$

$$\begin{aligned}A^2 &= B_2\big(\tfrac{1}{4}k^2-2\rho^{-4}-\tfrac{1}{128}k^4\rho^4+(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\big)\\ &\quad +B_1\cos\theta(4\rho^{-3}+3k\rho^{-1}+\tfrac{3}{4}k^2\rho+\tfrac{1}{16}k^3\rho^3)(\sin\theta)^{-1}\\ &\quad +\cos\theta(2k+2\rho^{-2}+\tfrac{3}{4}k^2\rho^2+\tfrac{1}{8}k^3\rho^4+\tfrac{1}{128}k^4\rho^6)(\sin\theta)^{-1}\partial_1 B_1\\ &\quad +(\tfrac{1}{2}k\rho^{-1}+\tfrac{3}{8}k^2\rho+\tfrac{3}{32}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)\partial_1 B_2\end{aligned}$$

$$
\begin{aligned}
&+(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)\partial_1\partial_1 B_2+(2\rho^{-3}+k\rho^{-1}-\tfrac{1}{16}k^3\rho^3-\tfrac{1}{128}k^4\rho^5)\partial_2 B_1\\
&+\cos\theta(\tfrac{9}{8}k^2+3\rho^{-4}+3k\rho^{-2}+\tfrac{3}{16}k^3\rho^2+\tfrac{3}{256}k^4\rho^4)(\sin\theta)^{-1}\partial_2 B_2\\
&+(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)\partial_2\partial_2 B_2\\
&+(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\partial_3\partial_3 B_2
\end{aligned}
\tag{2.10}
$$

$$
\tag{2.11}
$$

$$
\begin{aligned}
A^3 \;=\;& B_3(k^2+2k\rho^{-2}+\tfrac{1}{8}k^3\rho^2)(\sin\theta)^{-2}+(\tfrac{1}{2}k\rho^{-1}+\tfrac{3}{8}k^2\rho+\tfrac{3}{32}k^3\rho^3+\tfrac{1}{128}k^4\rho^5)(\sin\theta)^{-2}\partial_1 B_3\\
&+(k+\rho^{-2}+\tfrac{3}{8}k^2\rho^2+\tfrac{1}{16}k^3\rho^4+\tfrac{1}{256}k^4\rho^6)(\sin\theta)^{-2}\partial_1\partial_1 B_3\\
&+\cos\theta(-\tfrac{3}{8}k^2-\rho^{-4}-k\rho^{-2}-\tfrac{1}{16}k^3\rho^2-\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-3}\partial_2 B_3\\
&+(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-2}\partial_2\partial_2 B_3\\
&+(2\rho^{-3}+k\rho^{-1}-\tfrac{1}{16}k^3\rho^3-\tfrac{1}{128}k^4\rho^5)(\sin\theta)^{-2}\partial_3 B_1\\
&+\cos\theta(\tfrac{3}{4}k^2+2\rho^{-4}+2k\rho^{-2}+\tfrac{1}{8}k^3\rho^2+\tfrac{1}{128}k^4\rho^4)(\sin\theta)^{-3}\partial_3 B_2\\
&+(\tfrac{3}{8}k^2+\rho^{-4}+k\rho^{-2}+\tfrac{1}{16}k^3\rho^2+\tfrac{1}{256}k^4\rho^4)(\sin\theta)^{-4}\partial_3\partial_3 B_3
\end{aligned}
\tag{2.12}
$$

3 $\sinh\chi$ RW

$$
\tilde{g}_{ij} = \mathrm{Diag}\left(1, \sinh^2\chi, \sinh^2\chi\sin^2\theta\right) \tag{3.1}
$$

$$
A^i = \tilde{\nabla}_a\tilde{\nabla}^a B^i \tag{3.2}
$$

3.1 $B_i = g_i(\chi)$

$$
\tilde{\nabla}_a B^a = \dot{g}_1 + g_2\cos\theta(\sin\theta)^{-1}(\sinh\chi)^{-2} + 2g_1\cosh\chi(\sinh\chi)^{-1} = 0 \tag{3.3}
$$

$$
A^1 = -2g_1 + \ddot{g}_1 - 2g_2\cos\theta\cosh\chi(\sin\theta)^{-1}(\sinh\chi)^{-3} - 2g_1(\sinh\chi)^{-2} + 2\cosh\chi\dot{g}_1(\sinh\chi)^{-1} \tag{3.4}
$$

$$
\text{Substituting } g_2 \tag{3.5}
$$

$$
A^1 = \ddot{g}_1 + g_1\big(2 + 2(\sinh\chi)^{-2}\big) + 4\cosh\chi\dot{g}_1(\sinh\chi)^{-1} \tag{3.6}
$$

$$
A^2 = -g_2(\sin\theta)^{-2}(\sinh\chi)^{-4} - 2g_2(\sinh\chi)^{-2} + \ddot{g}_2(\sinh\chi)^{-2} \tag{3.7}
$$

$$
A^3 = -2g_3(\sin\theta)^{-2}(\sinh\chi)^{-2} + \ddot{g}_3(\sin\theta)^{-2}(\sinh\chi)^{-2} \tag{3.8}
$$

3.2 $B_i = h_i(\chi)\cos\theta$

$$
\tilde{\nabla}_a B^a = \cos\theta\dot{h}_1 + h_2(\sin\theta)^{-1}(\sinh\chi)^{-2} - 2h_2\sin\theta(\sinh\chi)^{-2} + 2h_1\cos\theta\cosh\chi(\sinh\chi)^{-1} \tag{3.9}
$$

$$
\begin{aligned}
A^1 \;=\;& \cos\theta\ddot{h}_1 + h_2\big(-2\cosh\chi(\sin\theta)^{-1}(\sinh\chi)^{-3} + 4\cosh\chi\sin\theta(\sinh\chi)^{-3}\big)\\
&+h_1\big(-2\cos\theta - 4\cos\theta(\sinh\chi)^{-2}\big) + 2\cos\theta\cosh\chi\dot{h}_1(\sinh\chi)^{-1}
\end{aligned}
\tag{3.10}
$$

$$
\text{Substituting } h_2 \tag{3.11}
$$

$$A^1 = 2h_1 \cos\theta + \cos\theta \ddot{h}_1 + 4\cos\theta \cosh\chi \dot{h}_1 (\sinh\chi)^{-1} \tag{3.12}$$

$$\begin{aligned} A^2 = {} & h_2\big(-2\cos\theta(\sinh\chi)^{-4} - \cos\theta(\sin\theta)^{-2}(\sinh\chi)^{-4} - 2\cos\theta(\sinh\chi)^{-2}\big) \\ & -2h_1 \cosh\chi \sin\theta (\sinh\chi)^{-3} + \cos\theta \ddot{h}_2 (\sinh\chi)^{-2} \end{aligned} \tag{3.13}$$

$$A^3 = -2h_3 \cos\theta (\sin\theta)^{-2}(\sinh\chi)^{-2} + \cos\theta \ddot{h}_3 (\sin\theta)^{-2}(\sinh\chi)^{-2} \tag{3.14}$$