Weyl Tensor Simplifications

General, no Bianchi, no explicit covariant derivative commutation

$$W^{(1)}_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R^2 - 2RR_{\mu\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - 2\nabla_\nu\nabla_\mu R. \tag{1}$$

Here we apply the Bianchi only to $W^2_{\mu\nu}$

$$W^{(2)}_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta}R^{\alpha\beta} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \tfrac{1}{2} g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\nu\nabla_\mu R. \tag{2}$$

$$\begin{aligned} W_{\mu\nu} = & -\tfrac{1}{6} g_{\mu\nu} R^2 + \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3} RR_{\mu\nu} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \tfrac{1}{6} g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} \\ & - \tfrac{1}{3}\nabla_\nu\nabla_\mu R. \end{aligned} \tag{3}$$

where we use the perturbed quantities

$$\begin{aligned} \delta R_{\lambda\mu\nu\kappa} = & h^\alpha{}_\lambda R_{\alpha\mu\nu\kappa} - \tfrac{1}{2}\nabla_\kappa\nabla_\lambda h_{\mu\nu} + \tfrac{1}{2}\nabla_\kappa\nabla_\mu h_{\nu\lambda} + \tfrac{1}{2}\nabla_\kappa\nabla_\nu h_{\mu\lambda} - \tfrac{1}{2}\nabla_\nu\nabla_\kappa h_{\mu\lambda} + \tfrac{1}{2}\nabla_\nu\nabla_\lambda h_{\kappa\mu} \\ & - \tfrac{1}{2}\nabla_\nu\nabla_\mu h_{\kappa\lambda} \end{aligned} \tag{4}$$

$$\delta R_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(\nabla_\alpha\nabla_\beta h_{\mu\nu} - \nabla_\alpha\nabla_\mu h_{\beta\nu} - \nabla_\alpha\nabla_\nu h_{\beta\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta}). \tag{5}$$

60 Terms

$$\begin{aligned}\delta W_{\mu\nu}(h_{\mu\nu}) = & -\tfrac{1}{6}h_{\mu\nu}R^2+\tfrac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta}+\tfrac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}-g_{\mu\nu}h^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma}-\tfrac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}\\ & +2h^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta}-\tfrac{1}{6}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R+\tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R-h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu}\\ & +\tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}+\tfrac{1}{3}h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta}\\ \\ & +\tfrac{1}{3}R\nabla_\alpha\nabla^\alpha h_{\mu\nu}+R_{\mu\beta\nu\gamma}\nabla_\alpha\nabla^\gamma h^{\alpha\beta}+R_{\mu\gamma\nu\beta}\nabla_\alpha\nabla^\gamma h^{\alpha\beta}-\tfrac{1}{3}R\nabla_\alpha\nabla_\mu h_\nu{}^\alpha-\tfrac{1}{3}R\nabla_\alpha\nabla_\nu h_\mu{}^\alpha\\ & -\tfrac{1}{6}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R+\tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta-\nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu}+\tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta}\\ & -\tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta}+\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha h_\mu{}^\beta-R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu}+\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha h_\nu{}^\beta\\ & -\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla^\beta h_{\mu\alpha}-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla^\beta h_{\nu\alpha}+\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu}-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu h_\nu{}^\alpha\\ & -\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu h_\mu{}^\alpha-\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu h_\alpha{}^\beta+R^{\alpha\beta}\nabla_\beta\nabla_\mu h_{\nu\alpha}-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu h_\alpha{}^\beta\\ & +R^{\alpha\beta}\nabla_\beta\nabla_\nu h_{\mu\alpha}+\nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu{}^\alpha-\nabla_\beta R_{\nu\alpha}\nabla^\beta h_\mu{}^\alpha+\nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu{}^\alpha-\nabla_\beta R_{\mu\alpha}\nabla^\beta h_\nu{}^\alpha\\ & -g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta h_\alpha{}^\gamma+\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta}-R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta}+\tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta}\\ & +\tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta}-\nabla_\beta R_{\nu\alpha}\nabla_\mu h^{\alpha\beta}+\tfrac{1}{6}\nabla^\alpha R\nabla_\mu h_{\nu\alpha}-\nabla_\beta R_{\mu\alpha}\nabla_\nu h^{\alpha\beta}\\ & +\tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta}+\tfrac{1}{6}\nabla^\alpha R\nabla_\nu h_{\mu\alpha}+\tfrac{1}{3}\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta}-\tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta}\\ & +\tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}\\ \\ & -\tfrac{1}{3}g_{\mu\nu}R\nabla_\alpha\nabla^\alpha h+\tfrac{2}{3}R_{\mu\nu}\nabla_\alpha\nabla^\alpha h+\tfrac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h-\tfrac{1}{12}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R\\ & +\tfrac{1}{2}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h+\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h-\tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h-R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h\\ & +\tfrac{1}{3}R\nabla_\nu\nabla_\mu h-\tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \end{aligned} \tag{6}$$

General, no Bianchi, no explicit covariant derivative commutation

$$W^{(1)}_{\mu\nu}=\tfrac{1}{2}g_{\mu\nu}R^2-2RR_{\mu\nu}+2g_{\mu\nu}\nabla_\alpha\nabla^\alpha R-2\nabla_\nu\nabla_\mu R. \tag{7}$$

No Bianchi applied to $W^2_{\mu\nu}$

$$W^{(2)}_{\mu\nu}=\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}-2R^{\alpha\beta}R_{\alpha\mu\beta\nu}+\tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R+\nabla_\alpha\nabla^\alpha R_{\mu\nu}-\nabla_\mu\nabla^\alpha R_{\nu\alpha}-\nabla_\nu\nabla^\alpha R_{\mu\alpha}. \tag{8}$$

$$\begin{aligned}W_{\mu\nu}= & -\tfrac{1}{6}g_{\mu\nu}R^2+\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}+\tfrac{2}{3}RR_{\mu\nu}-2R^{\alpha\beta}R_{\alpha\mu\beta\nu}-\tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R+\nabla_\alpha\nabla^\alpha R_{\mu\nu}\\ & -\nabla_\mu\nabla^\alpha R_{\nu\alpha}-\nabla_\nu\nabla^\alpha R_{\mu\alpha}+\tfrac{2}{3}\nabla_\nu\nabla_\mu R. \end{aligned} \tag{9}$$

General, no Bianchi, no explicit covariant derivative commutation

80 Terms

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\begin{aligned}
\delta W_{\mu\nu}(h_{\mu\nu}) = & -\tfrac{1}{6}h_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}h^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\
& + 2h^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} - \tfrac{1}{6}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R - h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} \\
& + \tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} + h^{\alpha\beta}\nabla_\mu\nabla_\beta R_{\nu\alpha} + h^{\alpha\beta}\nabla_\nu\nabla_\beta R_{\mu\alpha} - \tfrac{2}{3}h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\
& \\
& + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha h_{\mu\nu} + R_{\mu\beta\nu\gamma}\nabla_\alpha\nabla^\gamma h^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla_\alpha\nabla^\gamma h^{\alpha\beta} - \tfrac{1}{3}R\nabla_\alpha\nabla_\mu h_\nu{}^\alpha - \tfrac{1}{3}R\nabla_\alpha\nabla_\nu h_\mu{}^\alpha \\
& + \tfrac{1}{3}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta - \nabla^\alpha h_{\mu\nu}\nabla_\beta R_\alpha{}^\beta - \nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu} \\
& + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha h_\mu{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu} \\
& + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha h_\nu{}^\beta - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla^\beta h_{\mu\alpha} - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla^\beta h_{\nu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} \\
& - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu h_\mu{}^\alpha - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu h_\alpha{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\mu h_{\nu\alpha} \\
& - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu h_\alpha{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\nu h_{\mu\alpha} + \nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu{}^\alpha - \nabla_\beta R_{\nu\alpha}\nabla^\beta h_\mu{}^\alpha \\
& + \nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha}\nabla^\beta h_\nu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta h_\alpha{}^\gamma + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} \\
& - R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} - \tfrac{1}{3}\nabla^\alpha R\nabla_\mu h_{\nu\alpha} \\
& + \nabla_\beta R_\alpha{}^\beta\nabla_\mu h_\nu{}^\alpha + \nabla_\alpha h^{\alpha\beta}\nabla_\mu R_{\nu\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha\nabla_\beta\nabla^\beta h_\nu{}^\alpha + R_\nu{}^\alpha\nabla_\mu\nabla_\beta h_\alpha{}^\beta \\
& + \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla^\beta h_\nu{}^\alpha + \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla_\nu h^{\alpha\beta} + \tfrac{1}{2}R^{\alpha\beta}\nabla_\mu\nabla_\nu h_{\alpha\beta} - \tfrac{1}{6}\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} \\
& - \tfrac{1}{3}\nabla^\alpha R\nabla_\nu h_{\mu\alpha} + \nabla_\beta R_\alpha{}^\beta\nabla_\nu h_\mu{}^\alpha - \tfrac{1}{6}\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + \nabla_\alpha h^{\alpha\beta}\nabla_\nu R_{\mu\beta} \\
& - \tfrac{1}{2}\nabla_\nu\nabla_\alpha\nabla_\beta\nabla^\beta h_\mu{}^\alpha + R_\mu{}^\alpha\nabla_\nu\nabla_\beta h_\alpha{}^\beta + \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla^\beta h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla_\mu h^{\alpha\beta} \\
& - \tfrac{7}{6}R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} - \tfrac{2}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& \\
& - \tfrac{1}{3}g_{\mu\nu}R\nabla_\alpha\nabla^\alpha h + \tfrac{2}{3}R_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h - \tfrac{1}{12}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R \\
& + \tfrac{1}{2}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h - R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h \\
& - \tfrac{1}{2}\nabla^\alpha h\nabla_\mu R_{\nu\alpha} - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\alpha h - \tfrac{1}{2}\nabla_\mu\nabla_\alpha\nabla^\alpha\nabla_\nu h - \tfrac{1}{2}\nabla^\alpha h\nabla_\nu R_{\mu\alpha} - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\alpha h \\
& - \tfrac{1}{2}\nabla_\nu\nabla_\alpha\nabla^\alpha\nabla_\mu h + \tfrac{1}{3}R\nabla_\nu\nabla_\mu h + \tfrac{2}{3}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \qquad (10)
\end{aligned}
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Now make subsitution

$$h_{\mu\nu} = K_{\mu\nu} + \tfrac{1}{4}hg^{(0)}_{\mu\nu} \qquad (11)$$

in which it follows

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu} + \tfrac{1}{4}hg^{(0)}_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}(\tfrac{1}{4}hg^{(0)}_{\mu\nu}). \qquad (12)$$

64 Terms

$$\begin{aligned}\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2+\tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta}+\tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}-g_{\mu\nu}K^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma}-\tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}\\ & +2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta}+\tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu}-\tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R+R_{\mu\beta\nu\gamma}\nabla_\alpha\nabla^\gamma K^{\alpha\beta}\\ & +R_{\mu\gamma\nu\beta}\nabla_\alpha\nabla^\gamma K^{\alpha\beta}-\tfrac{1}{3}R\nabla_\alpha\nabla_\mu K_\nu{}^\alpha-\tfrac{1}{3}R\nabla_\alpha\nabla_\nu K_\mu{}^\alpha+\tfrac{1}{3}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R\\ & +\tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta-\nabla^\alpha K_{\mu\nu}\nabla_\beta R_\alpha{}^\beta-\nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu}+\tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta}\\ & -\tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta}+\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha K_\mu{}^\beta-R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu}+\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha K_\nu{}^\beta\\ & +\tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R-K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu}-\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla^\beta K_{\mu\alpha}-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla^\beta K_{\nu\alpha}\\ & +\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu}-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu K_\nu{}^\alpha-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu K_\mu{}^\alpha-\tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu K_\alpha{}^\beta\\ & +R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha}-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu K_\alpha{}^\beta+R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha}+\nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha\\ & -\nabla_\beta R_{\nu\alpha}\nabla^\beta K_\mu{}^\alpha+\nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha-\nabla_\beta R_{\mu\alpha}\nabla^\beta K_\nu{}^\alpha-g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta K_\alpha{}^\gamma\\ & +\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta}-R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta}+\tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}\\ & +\tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta}+\tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta}-\tfrac{1}{3}\nabla^\alpha R\nabla_\mu K_{\nu\alpha}+\nabla_\beta R_\alpha{}^\beta\nabla_\mu K_\nu{}^\alpha\\ & +\nabla_\alpha K^{\alpha\beta}\nabla_\mu R_{\nu\beta}-\tfrac{1}{2}\nabla_\mu\nabla_\alpha\nabla_\beta\nabla^\beta K_\nu{}^\alpha+R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta+K^{\alpha\beta}\nabla_\mu\nabla_\beta R_{\nu\alpha}\\ & +\tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla^\beta K_\nu{}^\alpha+\tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla_\nu K^{\alpha\beta}+\tfrac{1}{2}R^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta}-\tfrac{1}{6}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta}\\ & -\tfrac{1}{3}\nabla^\alpha R\nabla_\nu K_{\mu\alpha}+\nabla_\beta R_\alpha{}^\beta\nabla_\nu K_\mu{}^\alpha-\tfrac{1}{6}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta}+\nabla_\alpha K^{\alpha\beta}\nabla_\nu R_{\mu\beta}\\ & -\tfrac{1}{2}\nabla_\nu\nabla_\alpha\nabla_\beta\nabla^\beta K_\mu{}^\alpha+R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta+K^{\alpha\beta}\nabla_\nu\nabla_\beta R_{\mu\alpha}+\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla^\beta K_\mu{}^\alpha\\ & +\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla_\mu K^{\alpha\beta}-\tfrac{7}{6}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta}-\tfrac{2}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta}-\tfrac{2}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta}. \end{aligned} \tag{13}$$

21 Terms

$$\begin{aligned}\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = & \tfrac{1}{24}g_{\mu\nu}R^2h-\tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h-\tfrac{1}{6}RR_{\mu\nu}h+\tfrac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h+\tfrac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R\\ & -\tfrac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu}+\tfrac{1}{4}h\nabla_\mu\nabla_\alpha R_\nu{}^\alpha+\tfrac{1}{4}h\nabla_\nu\nabla_\alpha R_\mu{}^\alpha-\tfrac{1}{6}h\nabla_\nu\nabla_\mu R\\ & +\tfrac{1}{4}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h+\tfrac{1}{8}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R-\tfrac{1}{4}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h-\tfrac{1}{4}g_{\mu\nu}\nabla^\alpha h\nabla_\beta R_\alpha{}^\beta\\ & -\tfrac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h+\tfrac{1}{4}\nabla_\alpha R_\nu{}^\alpha\nabla_\mu h-\tfrac{1}{4}\nabla_\mu\nabla_\alpha\nabla^\alpha\nabla_\nu h-\tfrac{1}{8}\nabla_\mu h\nabla_\nu R+\tfrac{1}{4}\nabla_\alpha R_\mu{}^\alpha\nabla_\nu h\\ & -\tfrac{1}{8}\nabla_\mu R\nabla_\nu h-\tfrac{1}{4}\nabla_\nu\nabla_\alpha\nabla^\alpha\nabla_\mu h+\tfrac{1}{4}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \end{aligned} \tag{14}$$

General, with Bianchi, no explicit covariant derivative commutation

52 Terms

$$
\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\beta\gamma} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\
& + 2K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\mu\gamma\nu\beta} - \tfrac{1}{6}K_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R - K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} \\
& + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} + \tfrac{1}{2}K^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R_{\alpha\beta} - \tfrac{1}{6}K^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}R_{\alpha\beta} \\
& \\
& + \tfrac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} + R_{\mu\beta\nu\gamma}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} - \tfrac{1}{3}R\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} \\
& - \tfrac{1}{3}R\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \tfrac{1}{6}\nabla_{\alpha}K_{\mu\nu}\nabla^{\alpha}R + \tfrac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}K_{\alpha}{}^{\beta} - \nabla_{\alpha}K^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} \\
& + \tfrac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\mu}{}^{\beta} - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}K_{\mu\nu} \\
& + \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\nu}{}^{\beta} - \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\mu\alpha} - \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\nu\alpha} + \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} \\
& - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\mu}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\mu}K_{\nu\alpha} \\
& - \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\nu}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\nu}K_{\mu\alpha} + \nabla_{\alpha}R_{\nu\beta}\nabla^{\beta}K_{\mu}{}^{\alpha} - \nabla_{\beta}R_{\nu\alpha}\nabla^{\beta}K_{\mu}{}^{\alpha} \\
& + \nabla_{\alpha}R_{\mu\beta}\nabla^{\beta}K_{\nu}{}^{\alpha} - \nabla_{\beta}R_{\mu\alpha}\nabla^{\beta}K_{\nu}{}^{\alpha} - g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla_{\beta}K_{\alpha}{}^{\gamma} + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}K_{\alpha\beta} \\
& - R_{\mu\alpha\nu\beta}\nabla_{\gamma}\nabla^{\gamma}K^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_{\gamma}R_{\alpha\beta}\nabla^{\gamma}K^{\alpha\beta} - \nabla_{\beta}R_{\nu\alpha}\nabla_{\mu}K^{\alpha\beta} \\
& + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\mu}K_{\nu\alpha} + \tfrac{1}{2}R^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}K_{\alpha\beta} - \nabla_{\beta}R_{\mu\alpha}\nabla_{\nu}K^{\alpha\beta} + \tfrac{1}{3}\nabla_{\mu}R_{\alpha\beta}\nabla_{\nu}K^{\alpha\beta} \\
& + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\nu}K_{\mu\alpha} + \tfrac{1}{3}\nabla_{\mu}K^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta} - \tfrac{7}{6}R^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}K_{\alpha\beta} + \tfrac{1}{3}\nabla_{\nu}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta}. \qquad (15)
\end{aligned}
$$

15 Terms

$$
\begin{aligned}
\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = & \tfrac{1}{24}g_{\mu\nu}R^2h - \tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \tfrac{1}{6}RR_{\mu\nu}h + \tfrac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h + \tfrac{1}{24}g_{\mu\nu}h\nabla_{\alpha}\nabla^{\alpha}R \\
& - \tfrac{1}{4}h\nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} + \tfrac{1}{12}h\nabla_{\nu}\nabla_{\mu}R + \tfrac{1}{4}\nabla_{\alpha}\nabla^{\alpha}\nabla_{\nu}\nabla_{\mu}h - \tfrac{1}{4}\nabla_{\alpha}R_{\mu\nu}\nabla^{\alpha}h - \tfrac{1}{2}R_{\mu\alpha\nu\beta}\nabla^{\beta}\nabla^{\alpha}h \\
& + \tfrac{1}{4}\nabla^{\alpha}h\nabla_{\mu}R_{\nu\alpha} + \tfrac{1}{4}R_{\nu}{}^{\alpha}\nabla_{\mu}\nabla_{\alpha}h + \tfrac{1}{4}\nabla^{\alpha}h\nabla_{\nu}R_{\mu\alpha} + \tfrac{1}{4}R_{\mu}{}^{\alpha}\nabla_{\nu}\nabla_{\alpha}h - \tfrac{1}{4}\nabla_{\nu}\nabla_{\mu}\nabla_{\alpha}\nabla^{\alpha}h. \qquad (16)
\end{aligned}
$$

With Covariant Derivative Commutation and Bianchi

71 Terms

$$
\begin{aligned}
\delta W_{\mu\nu}(h_{\mu\nu}) = & -\tfrac{1}{6}h_{\mu\nu}R^2+\tfrac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta}+\tfrac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}h_\nu{}^\alpha RR_{\mu\alpha}-\tfrac{1}{2}h_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta \\
& -\tfrac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}+\tfrac{1}{3}h_\mu{}^\alpha RR_{\nu\alpha}+h^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta}-\tfrac{1}{2}h_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta-g_{\mu\nu}h^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\
& -\tfrac{2}{3}h^{\alpha\beta}RR_{\mu\alpha\nu\beta}-h_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma}+2h^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta}+2h^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta \\
& -h_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma}-\tfrac{1}{6}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R+\tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R-h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} \\
& +\tfrac{1}{2}h_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha}+\tfrac{1}{2}h_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha}+\tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}-h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} \\
& +\tfrac{1}{3}h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\
\\
& +\tfrac{1}{3}R\nabla_\alpha\nabla^\alpha h_{\mu\nu}+\tfrac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\beta h_\mu{}^\beta+\tfrac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\beta h_\nu{}^\beta-\tfrac{1}{6}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R \\
& +\tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta-\nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu}+\tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta}-\tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& -R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu}+\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu}-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha h_\nu{}^\alpha-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha h_\mu{}^\alpha \\
& -g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma h_\alpha{}^\gamma+\nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu{}^\alpha+\nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu{}^\alpha+\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} \\
& -2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta}+\tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta}+\tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} \\
& -2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma h^{\alpha\beta}+R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha h^{\alpha\beta}+R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha h^{\alpha\beta}-\nabla_\beta R_{\nu\alpha}\nabla_\mu h^{\alpha\beta} \\
& +\tfrac{1}{6}\nabla^\alpha R\nabla_\mu h_{\nu\alpha}-\tfrac{1}{3}R\nabla_\mu\nabla_\alpha h_\nu{}^\alpha-\tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta h_\alpha{}^\beta+R^{\alpha\beta}\nabla_\mu\nabla_\beta h_{\nu\alpha}-\nabla_\beta R_{\mu\alpha}\nabla_\nu h^{\alpha\beta} \\
& +\tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta}+\tfrac{1}{6}\nabla^\alpha R\nabla_\nu h_{\mu\alpha}+\tfrac{1}{3}\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta}-\tfrac{1}{3}R\nabla_\nu\nabla_\alpha h_\mu{}^\alpha \\
& -\tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta h_\alpha{}^\beta+R^{\alpha\beta}\nabla_\nu\nabla_\beta h_{\mu\alpha}-\tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta}+\tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
\\
& -\tfrac{1}{3}g_{\mu\nu}R\nabla_\alpha\nabla^\alpha h+\tfrac{2}{3}R_{\mu\nu}\nabla_\alpha\nabla^\alpha h-\tfrac{1}{12}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R+\tfrac{1}{2}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h \\
& +\tfrac{1}{2}\nabla^\alpha h\nabla_\beta R_\mu{}^\beta{}_{\nu\alpha}+\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h-\tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h-\tfrac{1}{2}\nabla^\alpha h\nabla_\mu R_{\nu\alpha} \\
& -\tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\alpha h-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\alpha h+\tfrac{1}{3}R\nabla_\nu\nabla_\mu h+\tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \qquad (17)
\end{aligned}
$$

59 Terms

$$\begin{aligned}\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2+\tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta}+\tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha}-\tfrac{1}{2}K_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta \\ & -\tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}+\tfrac{1}{3}K_\mu{}^\alpha RR_{\nu\alpha}+K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta}-\tfrac{1}{2}K_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta-g_{\mu\nu}K^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\ & -\tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta}-K_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma}+2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta}+2K^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta \\ & -K_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma}-\tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R+\tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R-K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} \\ & +\tfrac{1}{2}K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha}+\tfrac{1}{2}K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha}+\tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}-K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} \\ & +\tfrac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\ \\ & +\tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu}+\tfrac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\beta K_\mu{}^\beta+\tfrac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\beta K_\nu{}^\beta-\tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R \\ & +\tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta-\nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu}+\tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta}-\tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\ & -R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu}+\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu}-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha K_\nu{}^\alpha-\tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha K_\mu{}^\alpha \\ & -g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma+\nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha+\nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha+\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} \\ & -2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta}+\tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta}+\tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} \\ & -2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta}+R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta}+R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta}-\nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} \\ & +\tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha}-\tfrac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha-\tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta+R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} \\ & -\nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta}+\tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta}+\tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha}+\tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} \\ & -\tfrac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha-\tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta+R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha}-\tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} \\ & +\tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta}. \end{aligned} \tag{18}$$

$$\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = -\frac{1}{4}hW_{\mu\nu}(g^{(0)}_{\mu\nu}) \tag{19}$$

Weyl Tensor Flat

$$\begin{aligned}\delta C_{\lambda\mu\nu\kappa} = & \tfrac{1}{4}\eta_{\mu\nu}\partial_\alpha\partial^\alpha K_{\kappa\lambda}-\tfrac{1}{4}\eta_{\lambda\nu}\partial_\alpha\partial^\alpha K_{\kappa\mu}-\tfrac{1}{4}\eta_{\kappa\mu}\partial_\alpha\partial^\alpha K_{\lambda\nu}+\tfrac{1}{4}\eta_{\kappa\lambda}\partial_\alpha\partial^\alpha K_{\mu\nu} \\ & -\tfrac{1}{6}\eta_{\kappa\mu}\eta_{\lambda\nu}\partial_\beta\partial_\alpha K^{\alpha\beta}+\tfrac{1}{6}\eta_{\kappa\lambda}\eta_{\mu\nu}\partial_\beta\partial_\alpha K^{\alpha\beta}-\tfrac{1}{4}\eta_{\mu\nu}\partial_\kappa\partial_\alpha K_\lambda{}^\alpha+\tfrac{1}{4}\eta_{\lambda\nu}\partial_\kappa\partial_\alpha K_\mu{}^\alpha \\ & +\tfrac{1}{2}\partial_\kappa\partial_\mu K_{\lambda\nu}-\tfrac{1}{4}\eta_{\mu\nu}\partial_\lambda\partial_\alpha K_\kappa{}^\alpha+\tfrac{1}{4}\eta_{\kappa\mu}\partial_\lambda\partial_\alpha K_\nu{}^\alpha-\tfrac{1}{2}\partial_\lambda\partial_\kappa K_{\mu\nu}+\tfrac{1}{2}\partial_\lambda\partial_\nu K_{\kappa\mu} \\ & +\tfrac{1}{4}\eta_{\lambda\nu}\partial_\mu\partial_\alpha K_\kappa{}^\alpha-\tfrac{1}{4}\eta_{\kappa\lambda}\partial_\mu\partial_\alpha K_\nu{}^\alpha-\tfrac{1}{2}\partial_\mu\partial_\nu K_{\kappa\lambda}+\tfrac{1}{4}\eta_{\kappa\mu}\partial_\nu\partial_\alpha K_\lambda{}^\alpha-\tfrac{1}{4}\eta_{\kappa\lambda}\partial_\nu\partial_\alpha K_\mu{}^\alpha. \end{aligned} \tag{20}$$

Applying Gauge

Now we apply the gauge condition

$$\nabla_\nu K^{\mu\nu} = 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \tag{21}$$

or the equivalent gauge covariant in $K_{\mu\nu}$

$$\eta^{\alpha\beta}\partial_\alpha K_{\mu\beta} = 2\Omega^{-1}\eta^{\alpha\beta}K_{\mu\beta}\partial_\alpha\Omega. \tag{22}$$

and $\delta W_{\mu\nu}$ reduces to

$$\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) &= -48\Omega^{-7}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma K_{\mu\nu} + 24\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\rho\partial_\beta\Omega\partial_\sigma K_{\mu\nu} \\
&+ 60\Omega^{-8}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma\Omega - 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta K_{\mu\nu} \\
&+ 6\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta\Omega + 12\Omega^{-6}\eta^{\alpha\rho}\eta^{\beta\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho K_{\mu\nu} \\
&+ 6\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho K_{\mu\nu} - 2\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho K_{\mu\nu} \\
&+ 12\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta K_{\mu\nu}\partial_\sigma\partial_\rho\Omega - 48\Omega^{-7}\eta^{\alpha\rho}\eta^{\beta\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega \\
&- 24\Omega^{-7}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega + 3\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho\Omega \\
&- 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta K_{\mu\nu} - 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha K_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\Omega \\
&+ 12\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta\Omega + \tfrac{1}{2}\Omega^{-4}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha K_{\mu\nu} \\
&- \Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha\Omega \\
&= \frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta(\Omega^{-2}K_{\mu\nu})
\end{aligned} \tag{23}$$