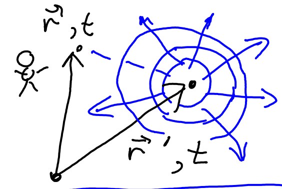
Lecture 4

02/01/2016

recap: Retarded Green Function

$$G(\vec r, t, \vec r\,', t')$$



$$G^{(r)}(\vec r, t \,|\, \vec r\,', t') = \frac{\delta\left(t - t' - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}$$

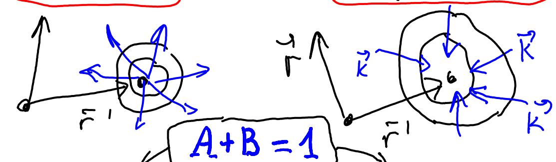
Fourier component $G_\omega(\vec r, \vec r\,')$:

$$G = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} G_\omega(R)$$

$$G_\omega = \frac{A e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} + \frac{B e^{-ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|}$$

retarded   advanced

$\vec R = \vec r - \vec r\,'$ and $\tau = t - t'$



$$A + B = 1$$

$A = 1,\ B = 0$ $A = 0,\ B = 1$

$k = \frac{\omega}{c} = 2\pi/\lambda$

$$\Psi_{non}(\vec r, t) = \int dt'\, d^3 r'\, G^{(r)}(\vec r, t \,|\, \vec r\,', t')\, f(\vec r\,', t') = \int d^3 r' \frac{f\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}$$

Retarded Potentials

Wave equations: $-4\pi f(\vec r, t)$

$$\begin{cases} \vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec j & \Rightarrow \\ \vec\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\rho/\varepsilon_0 & \Rightarrow \end{cases}$$

$$\left(\vec\nabla^2 \vec A + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0\right)$$

Lorenz gauge

$$\vec A_{non}(\vec r, t) = \frac{\mu_0}{4\pi} \int \frac{\vec j\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|} d^3 r'$$

$$\varphi_{non}(\vec r, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|} d^3 r'$$

The solution with the advanced Green function:

$$G_A^{(a)}(\vec r, t, \vec r\,', t') = \frac{\delta\left(t - t' + \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}$$

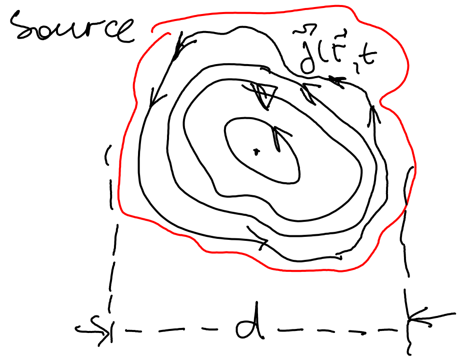
can be used for wave propagation in some specific physical conditions, for example, for standing waves!

Multipole expansion for Localized Sources.

The static zone: $\boxed{d \ll r \ll \lambda}$

The induction zone: $\boxed{d << r \sim \lambda}$

The radiation zone: $\boxed{d << \lambda << r}$



Symplest case: harmonic oscillations of the $\vec{j}$ current

$$\vec{j}(\vec{r},t) = \vec{j}(\vec{r}\,)\,e^{-i\omega t}$$

$$\Rightarrow \quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}\,e^{-i\omega(t-|\vec{r}-\vec{r}\,'|/c)} = e^{-i\omega t}\left(\frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'\right)$$

$$(*) \qquad \boxed{\vec{A}(\vec{r},t) = e^{-i\omega t}\,\frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$

harmonic $\vec{A}(\vec{r},t)$

$$(**) \qquad \boxed{\varphi(\vec{r},t) = e^{-i\omega t}\,\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$

harmonic $\varphi(\vec{r},t)$

We will evaluate $\vec{A}$ and φ potentials in the static and radiation zones. Equations $(*)$ and $(**)$ can be written as

$$\boxed{\begin{aligned} \vec{A}(\vec{r},t) &= e^{-i\omega t}\,\vec{A}_\omega(\vec{r}\,) \\ \varphi(\vec{r},t) &= e^{-i\omega t}\,\varphi_\omega(\vec{r}\,) \end{aligned}}$$

where

$$\boxed{\vec{A}_\omega(\vec{r}\,) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$

$$\boxed{\varphi_\omega(\vec{r}\,) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$

Static solutions:

$$\boxed{\begin{aligned} \vec{A}_s &= \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3r' \\ \varphi_s &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|} \end{aligned}}$$

Our "dynamic" solutions would be given the same formules as the static solution if $\boxed{c \to \infty}$:

$k = \frac{\omega}{c} \to 0 \;;\Rightarrow\; e^{ik|\vec{r}-\vec{r}\,'|} = 1$

Potentials in the static zone

$$d \ll r \ll \lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \; ; \quad k|\vec{r}-\vec{r}'| \simeq 2\pi \frac{|\vec{r}-\vec{r}'|}{\lambda} \ll 1$$

$$\Rightarrow \quad A_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}')\, e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} d^3r' = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|} \underbrace{\left(1 + ik|\cancel{\vec{r}}-\vec{r}'| + \dots\right)}^{e^{ik|\vec{r}-\vec{r}'|}}$$

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The quasi-static potentials are

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|} \cdot e^{-i\omega t} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}',t)\, d^3r'}{|\vec{r}-\vec{r}'|}}$$

where $\vec{j}(\vec{r},t) = \vec{j}(\vec{r})e^{-i\omega t}$

The quasi-static solution for the scalar potential:

$$\boxed{\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} d^3r'}$$

$\rho(\vec{r},t) = \rho(\vec{r})e^{-i\omega t}$