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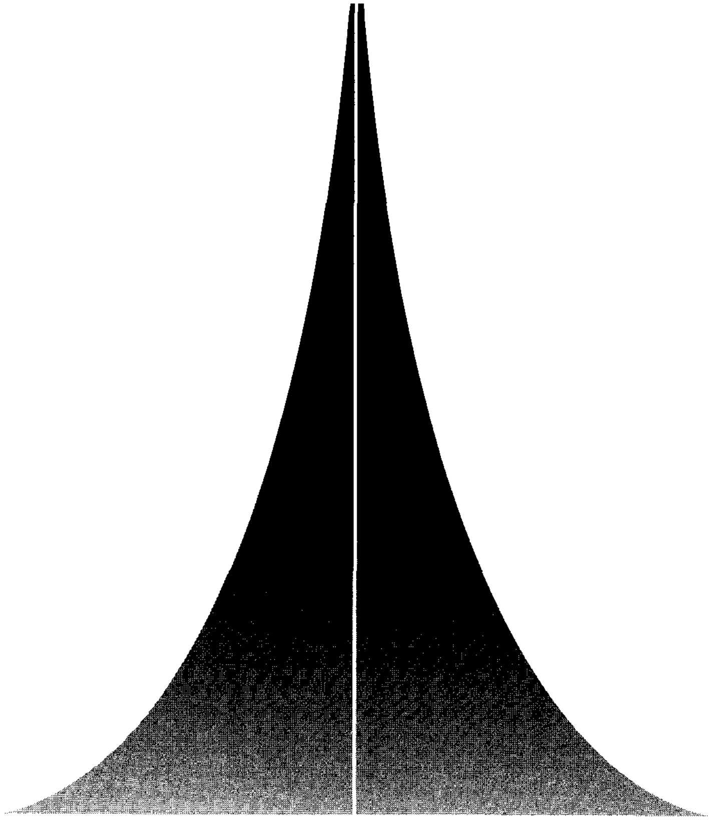

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World Scientific
NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

BRANE-LOCALIZED GRAVITY

ISBN 981-256-561-2

Printed in Singapore by B & JO Enterprise

To my parents Rupert and Gertrude Mannheim, to my wife Fay, and to my children Michael and Alexandra

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Foreword

The idea of extra spatial dimensions goes back at least to the 1920s, with the now-famous work of Theodor Kaluza and Oskar Klein. More recently, in the context of string theory, the concept of extra dimensions underwent a phase transition. It is no longer categorized as an intriguing speculation, but instead it has evolved into a persuasive theoretical construct. String theory requires for its consistency the existence of six or seven extra spatial dimensions, depending on which formulation is used.

But no one has seen any sign of these extra dimensions, and indeed the $1/r^2$ character of both gravity and electrostatics appears to be a clear signal that space is three-dimensional. For n space dimensions, the spreading of flux lines would produce a force proportional to $1/r^{n-1}$. If there are extra dimensions, they must be very effectively hidden from view.

For many years the only theoretical idea for hiding these dimensions was *compactification*: the extra dimensions were assumed to be curled up into a microscopic structure, with a size scale set by the Planck length, $\sqrt{\hbar G/c^3} \simeq 10^{-35}$ m. This assumption kept the extra dimensions impeccably sheltered from experimental challenge. The idea may of course be correct, but there is an obvious motivation to explore more potentially observable alternatives.

A key breakthrough was the 1998 paper by Nima Arkani-Hamed, Savas Dimopoulos, and Gia Dvali, in which they pointed out that — as far as gravity is concerned — the compactification scale could be vastly larger than the Planck scale. In fact, scales as large as 1 mm were consistent with all known observations. Shortly afterward these authors were joined by Ignatios Antoniadis in the proposal of a concrete perturbative model within type 1 string theory, in which the graviton was described as a closed string propagating in the higher-dimensional bulk, while the standard-model particles were open strings living on D3-branes. Thus, string theory was shown to provide a simple explanation for how the standard model could be localized tightly on a brane, while gravity remained free to explore the higher-dimensional bulk.

Within a year the idea of brane-world physics received a new boost, with the suggestion by Lisa Randall and Raman Sundrum that the extra dimensions could not only be large, but could perhaps even be infinite in extent. If a brane with Minkowski-space geometry is embedded in a higher-dimensional anti-de Sitter space, then Randall and Sundrum showed that gravitational field lines would not spread out into the bulk, but would instead be localized in the vicinity of the brane. The gravitational flux lines would therefore spread out only in three spatial dimensions, reproducing at large distances the familiar $1/r^2$ force law of Newtonian gravity.

While there is so far no direct experimental evidence for extra dimensions, the idea has certainly taken the theoretical physics community by storm. Extra dimensions appear to be required for the consistency of quantum gravity, and in any case they provide a rich context for theoretical model-building. The two Randall–Sundrum papers and the Arkani-Hamed–Dimopoulos–Dvali paper have each been cited well over 2000 times in the particle physics literature. Brane-world physics has certainly become a well-established approach to particle physics beyond the standard model.

In this book Philip Mannheim has presented a self-contained and fully-detailed description of brane-world physics. It is not just a summary of the literature, but it is truly an elaboration on the literature. All of the derivations are laid out in a consistent formalism and in complete detail. The book should be a valuable resource to anyone who is interested in following the progress of this exciting area of theoretical physics.

Alan H. Guth
August, 2005

Preface

While physicists have long entertained the notion that there might be extra spacetime dimensions beyond the four that are readily accessible to us, it had generally been thought that any such additional dimensions would be of microscopic size, and thus well out of experimental reach. However, in two seminal papers published in 1999, Lisa Randall and Raman Sundrum showed that under certain circumstances it was actually possible for extra dimensions to be large, and even macroscopically large, and yet not be in conflict with our everyday experience that only four large spacetime dimensions are manifest. Specifically, what they found was that if our ordinary four-dimensional world were to be of a very particular type, a so-called brane or membrane, which was embedded in a higher-dimensional, so-called bulk space which itself was also of a very particular type, viz. an anti-de Sitter space, then the curvature of the anti-de Sitter bulk could actually prevent gravitational signals from penetrating very far into it. In such a situation the bulk could be large, and even infinitely large, and yet an observer on the brane would only be able to receive gravitational signals which originated either on the brane or very close to it, with the bulk acting as a sort of a refractive medium which was essentially opaque to gravity. With the work of Randall and Sundrum the era of brane-localized gravity was thus ushered in, and it is the purpose of this monograph to describe it.

The emergence of the brane-localized gravity program came at a particularly propitious time for me, as I spent the Spring Semester of 2000 at the Massachusetts Institute of Technology as the guest of cosmologist Alan Guth while on sabbatic leave from the University of Connecticut. This sabbatic leave afforded me the time to dive into a new field and an ideal environment in which to do so. While at MIT it was my particular good fortune to get involved in a collaboration with Alan, his fellow MIT faculty member David Kaiser, and Ali Nayeri who was there at the time as a post-doctoral research associate. It was through our regular meetings and joint work that much of the material in this book was engendered, with much of it arising in response to Alan's invariably pointed and unfailingly pertinent questions, though Alan should not be held responsible for some of the answers that I give in the book. There were two issues in which our collaboration was particularly interested. One was how to extend Randall and Sundrum's original treatment of Minkowski branes to the cosmologically interesting case of Robertson-Walker branes. The other was how to deal with cases (such as a brane whose geometry was itself anti-de Sitter or whose tension was negative) where the bulk geometry did not inhibit the propagation of gravitational signals, since in such cases there would be no normalizable massless graviton. Our collaboration was totally mystified by the possibility that there might be gravitational theories without gravitons, and so we systematically went over every step of the standard treatment of brane-world fluctuations. The work of our collaboration on Robertson-Walker branes led to new insights into brane-world embeddings, while our analysis of brane-world gravitational fluctuations led to new insights in the treatment of the fluctuation equations, in the construction of an appropriate brane-world energy-momentum tensor, on the completeness of non-normalizable fluctuations and the construction of finite propagators out them, and in the causality of brane-world propagators and the central role played by gravitons in its enforcement. These insights together with some of my own form much of the core material of the book. The first twelve chapters of the book deal with the general background needed for navigating the brane world and present its known exact solutions. The remaining chapters deal primarily with brane-world fluctuations, providing a detailed treatment of fluctuations around Minkowski, de Sitter and anti-de Sitter branes of either positive or negative tension, with six appendices providing related material.

As a project the writing of this book took me far longer than I had anticipated (though not less than the time suggested by colleagues who already had experience with writing books), with the book growing to a size and scope much larger than I had ever envisaged. I am most indebted to the World Scientific Publishing Company for inviting me to write the book, and am particularly indebted to its editorial board for its patience in waiting for me to finish it. I am indebted to the University of Connecticut for providing me with a sabbatic leave and the travel funds needed to visit MIT both during the sabbatical and thereafter, and also for providing me with a semester teaching release during which much of the writing of the book was done. As a physicist I owe a great deal to Kurt Haller who was instrumental in bringing me to the University of Connecticut in 1979. He was always eager to support my career and was especially interested in my writing this book. It is very sad that he did not live to see its completion, and the book is a tribute to his memory. As a person I owe a great deal to my parents, Rupert and Gertrude Mannheim, who set my education as their highest priority and have given me nothing but support throughout my entire life. I also owe a great deal to my children, Michael and Alexandra, who have suffused my life with joy. Finally, to my wife Fay I owe ever so much, particularly for the strength and encouragement she has always given me, and for her remarkable forbearance during the writing of this book.

Philip D. Mannheim
Storrs Connecticut, May 2005

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Chapter 1

Why Higher Dimensions are Interesting

While there is currently no experimental evidence at all that supports the notion that our universe might possess any spacetime dimensions beyond the four established ones, it is a rather remarkable fact that we actually know of no compelling reason as to why the number of spacetime dimensions should actually be four. Certainly, a universe with four dimensions has some very special properties. For instance, it is only in four dimensions that the trilinear fermion, anti-fermion, gauge boson coupling constant is dimensionless, one of the key features which leads to the renormalizability of the strong, electromagnetic and weak interaction gauge theories. Nonetheless, it is equally possible to formulate such theories in spacetimes with dimension higher than four — they just would not be as easy to deal with — and even studying these theories in spacetimes with dimension lower than four has often been found to be extremely instructive. Consequently, it is legitimate to consider the possible existence of additional dimensions beyond four, with a view to either learning something theoretical about them which proves informative for 4-dimensional theories, or to being able to perform some experiment in four dimensions which might reveal their presence. What makes this enterprise so interesting is that the very entering into the issue of spacetime dimensions immediately entails the involvement of gravity, to thereby provide a possible road to a unification of gravity with the other fundamental forces.

In fact attempts at a higher-dimensional unification were initiated a long time ago, not long after Einstein's introduction of general relativity in 1916, with the most useful approach having been pioneered by Kaluza [Kaluza (1921)] and Klein [Klein (1926a); Klein (1926b)]. In their approach spacetime was taken to be a flat 5-dimensional $M(4,1)$ space with one timelike coordinate and four spacelike ones, so that the associated gravitational field then has 15 components. With respect to the standard 4-dimensional $M(3,1)$ spacetime these 15 degrees of freedom decompose into a 10-component rank-two tensor, a 4-component vector and a 1-component scalar. By identifying the rank-two tensor with the standard gravitational field $g_{\mu\nu}$ and the vector with the standard electromagnetic vector potential A_μ, Kaluza and Klein were able to achieve a purely geometrical unification of gravity with electromagnetism, one which long after their work then came back into prominence (see e.g. [Appelquist, Chodos and Freund (1985)]) following the twin realizations that spontaneously broken gauge theories had need for a scalar field as well, a Higgs field whose vacuum expectation value would serve to break the gauge symmetry, and that non–Abelian Yangs–Mills generalizations of Maxwell's theory could also be incorporated if the number of extra dimensions were to be increased even more [DeWitt (1964)]. To avoid the obvious fact that there was no apparent sign of any macroscopic fifth (or higher) dimension, (and to dispense with Kaluza's original assumption that none of the fields in the theory actually depend on the fifth coordinate at all), it was further presupposed in the Kaluza-Klein theory that any extra dimension be compactified into the topology of the 1-dimensional sphere S^1 (to give an overall geometry which was now $M(3,1)\times S^1$), with a compactification radius for S^1 which was taken to be microscopic. Because of the periodicity of this S^1 (whose implications Klein imported from Schrödinger theory which had just been developed at the time), the gravitational fluctuation modes, viz. the eigenmodes of the model, would then be discrete, having masses which would be the larger the smaller the compactification radius was taken to be. These modes, commonly referred to as KK modes, while characteristic of higher dimensional theories, could thus by hand phenomenologically be made heavier than the maximum energies available to high energy accelerators, with the same mechanism which would conceal any extra dimensions from detection thus also serving to conceal the KK modes too.

Proceeding from a totally different direction, a role for extra dimensions also emerged from elementary particle physics. Specifically, studies of high energy hadron scattering in the 1960s led to the development of the dual resonance model [Veneziano (1968)], a model whose spectrum of states was then found to be reproduced by the spectrum of a vibrating string, a theory which for consistency would need to be formulated in 26 spacetime dimensions if bosonic and in 10 if fermionic. Such string theories (see e.g. [Green, Schwarz and Witten (1987); Polchinski (1998)] for a detailed history and bibliography) were initially thought to describe strong interactions and had a fundamental scale, the string tension, which would be a hadronic GeV scale. A shortcoming of these theories was the presence of a massless spin-two mode for which there was no known hadronic particle, and so it was suggested that this mode might instead be associated with the massless spin-two graviton of gravitational theory. With the replacement of the hadronic string scale by a quantum-gravitational $M_{PL}=10^{19}$ GeV Planck mass scale, string theory was then recast as a theory not of strong interactions at all but of quantum gravity instead, providing the first consistent melding of Einstein gravity with quantum mechanics. While this theory would still be a higher dimensional theory of quantum gravity rather than a 4-dimensional one, because the theory contained a naturally small length scale, viz. the 10^{-33} cm Planck length, in a compactification the extra dimensions would then naturally, i.e. without presupposition, be microscopic. Similarly, the masses of the excitation modes would automatically be of order M_{PL} and thus be totally beyond reach, a twin-edged aspect of the theory which simultaneously prevents it from being ruled out by experiment while making it almost impossible to be ruled in. Nonetheless, since quarks could also be introduced into the theory, such string theories could potentially provide for a unification of all of the fundamental forces into a "theory of everything", so that they nicely possess both the unification ideas of Kaluza and Klein and a consistent formulation of quantum gravity. With extensions of Kaluza-Klein theory having independently been developed to incorporate both Yang-Mills theory and supergravity theory (the supersymmetric extension of ordinary gravity), with the development of modern superstring theory it then became possible to incorporate all of these higher dimensional ideas into one comprehensive framework.

In parallel with these developments, work in Higgs-driven spontaneously broken non-Abelian gauge theories in four dimensions revealed that such theories possessed configurations in which the Higgs field expectation values could have a spatial dependence, leading to models in which elementary particles were to be extended, topological defects rather than point structures (see e.g. [Rebbi and Soliani (1984); Coleman (1985)]). As such, microscopic elementary particles could have structures analogous to superconducting vortices or ferromagnetic domains; with this last option leading by analog to the possibility [Rubakov and Shaposhnikov (1983)] that our entire 4-dimensional universe could itself be a macroscopic domain wall embedded in some higher dimensional world, a model in which the three fundamental strong, electromagnetic and weak interactions could then be confined to the wall. In such a picture, even while being associated with a long range force, it was possible for electromagnetic flux to be confined to the wall, with there being no emission of photons into the extra dimensions. While the extended structure idea could thus conceal higher dimensions from strong, electromagnetic or weak probes, it still would not do so for gravity which could not be confined in this manner, to thus still require the extra dimensions to be microscopic. Subsequently, the domain wall picture was also found to emerge in superstring theory where it was shown [Dai, Leigh and Polchinski (1989); Polchinski (1995)] that superstring theories had to possess analogous topological defects called D-branes (D denotes Dirichlet boundary conditions). Such branes (viz. membranes generalized to arbitrary dimension — with an N-brane possessing N spatial dimensions) would then provide the locations at which strings would terminate, with superstring theory thus becoming a theory of both strings and branes, though again with still microscopic extra dimensions.

While higher dimension theories thus characteristically enjoyed (or suffered from) having microscopic extra dimensions, research started to move in a very different direction when Arkani-Hamed, Dimopoulos and Dvali [Arkani-Hamed, Dimopoulos and Dvali (1998)] discussed a possible advantage to having altogether larger-sized extra dimensions, and posed the question as to how phenomenologically large such dimensions might actually be permitted to be. In an attempt to resolve the longstanding hierarchy problem of understanding why there was such a huge disparity between the $M_{EW} = 10^3$ GeV electroweak and the $M_{PL} = 10^{19}$ GeV gravitational mass scales, Arkani-Hamed, Dimopoulos and Dvali presupposed that we lived in a $(4+n)$ dimensional world in which there were to be n additional spacelike dimensions each one was of which was still to be compactified, but with some common radius R which was to no longer necessarily be microscopic. In such a world $(4+n)$ dimensional gravity was to be controlled by the electroweak scale rather than by a Planck one (to thus unify electroweak and gravitational interactions at the higher dimensional level), with a potential confinement of gravitational flux lines around our 4-dimensional world then converting the gravitational potential $V(r) = m_1 m_2 / M_{EW}^{n+2} r^{n+1}$ between two static masses m_1 and m_2 in $M(3+n,1)$ into the $V(r) = m_1 m_2 / M_{EW}^{n+2} R^n r$ one of $M(3,1) \times S_n$, leading to an effective 4-dimensional gravitational coupling given as

$$M_{EFF}^2 = M_{EW}^{n+2} R^n \tag{1.1}$$

instead of a fundamental M_{PL}^2 one. In order to not disagree with current gravitational phenomenology (in which M_{EFF} is to be of order 10^{19} GeV), a value of $n = 1$ was found to be excluded, with $n = 2$ leading to an R of order millimeters. While this value was no less than 10^{32} times larger than Planck length sized extra dimensions, what made this particular value so very interesting was that a millimeter was of order the smallest length scale on which the inverse square gravitational law had actually ever been tested; and while there were very tight constraints coming from precision electroweak measurements forbidding the $SU(3) \times SU(2) \times SU(1)$ interactions from spreading out very far into any extra dimension at all (Coulomb's law, for instance, is tested down to 10^{-16} cm), nonetheless there were no such tight constraints on gravity itself, and its flux lines could in principle spread out into any extra dimension and manifest themselves as modifications to standard gravity at the millimeter level. The original Arkani-Hamed, Dimopoulos and Dvali idea subsequently acquired greater generality when it was found [Antoniadis, Arkani-Hamed, Dimopoulos and Dvali (1998)] that it could explicitly be realized in string theory where our 4-dimensional universe would then be a 3-brane embedded in a higher dimensional bulk. Though no sign of any millimeter or sub-millimeter departure from standard Newtonian gravity has yet actually been detected [Adelberger, Heckel and Nelson (2003)], nonetheless the work of Arkani-Hamed, Dimopoulos and Dvali opened up the possibility that extra dimensions need not be microscopic, and showed that it was at least in principle possible to actually experimentally explore extra dimensions at ordinary (as opposed to Planckian) energies and distance scales.

Following on the work of Arkani-Hamed, Dimopoulos and Dvali, Randall and Sundrum [Randall and Sundrum (1999a)] found an alternate way to address the hierarchy problem based on three key ingredients which not only enabled them to construct a phenomenologically viable model with only one large extra dimension, but which also turned out to be seminal to the entire large extra dimension program. Firstly, they compactified this one extra dimension with an additional Z_2 orbifold symmetry in which opposite points in the compactified fifth coordinate were to be identified. Secondly, they located a 3-brane at each of the two orbifold fixed points, viz. at the two points at the end of a diameter of a circle which self-identify when points on opposite sides of the diameter are identified with other. And thirdly, the decisive step, they took the bulk geometry to no longer be flat, but to instead be the 5-dimensional anti-de Sitter geometry AdS_5. Given such a model, they then found that in it the effective 4-dimensional Planck mass would be given by

$$bM_{EFF}^2 = M_5^3[1 - e^{-2\pi bR}] \ , \tag{1.2}$$

where M_5 is the fundamental 5-dimensional gravitational mass scale, $-b^2$ is the negative AdS_5 5-curvature, and R is the compactification radius. A choice of M_5 and b both of order M_{EW}, could then through the $[1 - e^{-2\pi bR}]$ suppression factor lead, for an appropriate choice for R, to an M_{EFF} which could still be much smaller then M_{EW}. Thus instead of the suppression being provided by the volume of the extra dimension region as in Eq. (1.1), now the suppression was being supplied by the curvature in that region, with AdS_5 acting as a dynamical refractive type medium which sharply modifies the propagation of gravitational signals in it. Beyond possible millimeter region modifications to Newton's law, an extra feature of this AdS_5/Z_2 model[1] was that it predicted electroweakly coupled KK modes in the TeV region, a search for which could also provide a probe of extra dimensions.

However, quite the most dramatic aspect of Eq. (1.2) is that unlike the situation in Eq. (1.1), M_{EFF} would remain finite even in the event that R were allowed to become infinite. In such a case the 3-brane at the second orbifold fixed point could be discarded (without needing to undo the Z_2 symmetry), with the model turning into a model [Randall and Sundrum (1999b)] of just a single 3-brane embedded in an AdS_5/Z_2 bulk which would not be of finite radius at all. In such a model, even though the extra dimension would not merely be large, but as infinitely large as the three ordinary spatial dimensions, gravity would nonetheless still be localized to the brane, with it being the AdS_5/Z_2 curvature which would confine the gravitational flux to the brane. A large fifth dimension could then exist, with the exponential suppression of the exchange of gravitational information between brane and bulk concealing its presence. With the work of Randall and Sundrum then, it became possible to transit from microscopic extra dimensions to ones of unlimited size, to thus open up the entirely new field of brane-localized gravity.[2] It is the purpose of this monograph to describe some of the general aspects of this program, and we shall begin with a detailed discussion of the two Randall-Sundrum papers themselves.

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[1]The set-up consisting of an AdS_5 geometry and an orbifold symmetry is referred to as AdS_5/Z_2.

[2]While quite interested in the Kaluza theory because of its unification aspects, Einstein nonetheless commented [Einstein (1931)] that "Among the considerations which question this theory stands in the first place: It is anomalous to replace the four-dimensional continuum by a five-dimensional one and then subsequently to tie up artificially one of these five dimensions in order to account for the fact that it does not manifest itself". It is thus of interest to note that in brane-localized gravity theories the higher dimensions do not have to be tied up at all.

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Chapter 2

The Randall-Sundrum Set-Up

2.1 The Einstein tensor of pure AdS_5

The space AdS_5 is a 5-dimensional space of constant negative curvature $K = -b^2$, viz. one for which the Riemann and Ricci tensors and the Ricci scalar take the form

$$R_{LMNK} = -b^2[g_{MN}g_{LK} - g_{MK}g_{LN}] \quad , \quad R_{MN} = 4b^2 g_{MN} \quad , \quad R^M{}_M = 20b^2 \quad . \tag{2.1}$$

(Here and throughout we shall use $(M, N = 0, 1, 2, 3, 5)$ to denote 5-dimensional coordinates, $(\mu, \nu = 0, 1, 2, 3)$ to denote the standard 4-dimensional ones, and shall follow the signs and conventions of [Weinberg (1972)].) With AdS_5 being a maximally-symmetric space with 15 Killing vectors, its associated Weyl tensor

$$\begin{aligned} C_{LMNK} = \; & R_{LMNK} + \frac{1}{12}R^A{}_A(g_{LN}g_{MK} - g_{LK}g_{MN}) \\ & -\frac{1}{3}[g_{LN}R_{MK} - g_{LK}R_{MN} - g_{MN}R_{LK} + g_{MK}R_{LN}] \end{aligned} \tag{2.2}$$

vanishes identically. With its Einstein tensor taking the form

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R^A{}_A = -6b^2 g_{MN} \tag{2.3}$$

(the combination $R_{MN} - (1/2)g_{MN}R^A{}_A$ obeys the Bianchi identity in any dimension), an AdS_5 geometry will be a solution to the 5-dimensional Einstein equations

$$G_{MN} = -\kappa_5^2 T_{MN} \tag{2.4}$$

with coupling constant κ_5^2 if the energy-momentum tensor is given by $T_{MN} = -\Lambda_5 g_{MN}$, with the positively defined b then being given by

$$b = +\left(\frac{-\Lambda_5\kappa_5^2}{6}\right)^{1/2} \quad . \tag{2.5}$$

AdS_5 is thus supported by a negative 5-dimensional bulk cosmological constant.

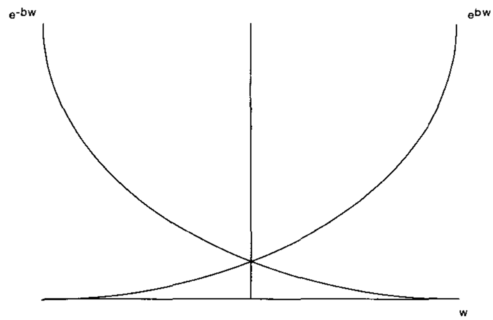

Fig. 2.1 The AdS_5 warp factor e^A for $A = bw$ and $A = -bw$.

The particular separable choice of metric of the form

$$ds^2 = dw^2 + e^{2A(w)}\eta_{\mu\nu}dx^\mu dx^\nu = dw^2 + e^{2A(w)}[dx^2 + dy^2 + dz^2 - dt^2] \ , \tag{2.6}$$

where w denotes the fifth coordinate (so that (w, x, y, z) are the four spacelike coordinates and t the timelike one) allows departures of the geometry from $M(4,1)$ to conveniently be described by the so-called "warp factor" $A(w)$, with respect to which the non-zero components of the associated Einstein and Riemann tensors are found to be given by

$$G_{55} = -6A'^2 \ , \quad G_{\mu\nu} = -e^{2A}\eta_{\mu\nu}[3A'' + 6A'^2] \ , \tag{2.7}$$

$$R_{5\mu 5\kappa} = e^{2A}[A'' + A'^2]\eta_{\mu\kappa} \ , \quad R_{\lambda\mu\nu\kappa} = e^{4A}A'^2[\eta_{\mu\kappa}\eta_{\lambda\nu} - \eta_{\mu\nu}\eta_{\lambda\kappa}] \ , \tag{2.8}$$

where the prime denotes differentiation with respect to w. Since the Bianchi identity $G^{MN}{}_{;N} = \partial_N G^{MN} + \Gamma^M_{NA}G^{NA} + \Gamma^N_{NA}G^{MA} = 0$ relates $\partial_5 G^{5M}$ to quantities which are second order derivative functions of w, as explicitly manifest in Eq. (2.7) G_{55} and $G_{5\mu}$ can be no more than first derivative functions of w, something that will prove crucial in the following. Comparison with Eqs. (2.1) and (2.3) then shows that the metric of Eq. (2.6) will be an AdS_5 metric if $A(w) = \pm bw$. With the coordinate transformation $v = e^{\mp bw}/b$ then transforming the metric into

$$ds^2 = \frac{1}{b^2v^2}[dv^2 + dx^2 + dy^2 + dz^2 - dt^2] \ , \tag{2.9}$$

we see that with $A = \pm bw$, the metric is indeed conformal to flat, so that the associated Weyl tensor then does indeed vanish.

2.2 Introducing the brane

Now as such, the two choices $A(w) = \pm bw$ are coordinate equivalent to each other (under $w \to -w$), with, as shown in Fig. (2.1), $A = +bw$ describing a warp factor which is very small for $w \to -\infty$ and very large for $w \to \infty$, and with $A = -bw$ describing a warp factor which is very large for $w \to -\infty$ and very small for $w \to \infty$. If we were to try to form a hybrid out of the two warp factors which would have good convergence properties for all w, we would want to keep the $w \to -\infty$ region of $A = bw$ and the $w \to \infty$ region of $A = -bw$. Thus it is suggested [Randall and Sundrum (1999a); Randall and Sundrum (1999b)] to take as warp factor the w-even

$$e^A = e^{-b|w|} \quad , \tag{2.10}$$

a choice which would be bounded everywhere, and which is not coordinate equivalent to the unbounded $e^A = e^{b|w|}$. Noting that

$$\frac{d|w|}{dw} = \theta(w) - \theta(-w) = \epsilon(w) \quad , \quad \frac{d^2|w|}{dw^2} = 2\delta(w) \quad , \tag{2.11}$$

we see that in this solution the Einstein tensor of Eq. (2.7) evaluates to

$$G_{55} = -6b^2 \quad , \quad G_{\mu\nu} = -6b^2 e^{2A}\eta_{\mu\nu} + 6b\delta(w)\eta_{\mu\nu} \quad . \tag{2.12}$$

Consequently, what is needed to support this metric is not just a bulk cosmological constant but also an additional energy-momentum tensor of the form

$$T_{MN} = -\lambda\delta^{\mu}_{M}\delta^{\nu}_{N}\eta_{\mu\nu}\delta(w) \quad , \tag{2.13}$$

with a coefficient λ which would have to obey

$$\kappa_5^2\lambda = 6b \quad , \tag{2.14}$$

viz.

$$\kappa_5^2\lambda^2 + 6\Lambda_5 = 0 \quad . \tag{2.15}$$

The presence of this additional energy-momentum tensor entails the presence of a new matter field λ which is localized to the $w = 0$ region, one which could therefore be associated with a brane which was localized there. On this brane there would then be a brane energy-momentum tensor $T_{\mu\nu} = -\lambda\eta_{\mu\nu}$, one which would explicitly have no need for any $\delta(w)$ dependent T_{55} or $T_{5\mu}$ components precisely because G_{55} and $G_{5\mu}$ contain no second order w derivative terms, one which if built solely out of the $SU(3) \times SU(2) \times U(1)$ fields would precisely be expected to be confined to the brane. The matter field λ is a brane cosmological constant or tension, and because of Eq. (2.14) explicitly needs to have positive sign. The set-up which will support an AdS_5 bulk with an $A = -b|w|$ warp factor is thus a brane with a Minkowski geometry and positive tension embedded in it (hereafter the M_4^+ brane world), but with its tension having to be chosen to be fine-tuned to the bulk cosmological constant so as to precisely obey $\kappa_5^2\lambda^2 + 6\Lambda_5 = 0$.

As such the above M_4^+ example contains the key elements of brane-localized gravity with the metric falling exponentially fast away from the brane, the implications of which we will study below in detail. This convergent behavior is to be contrasted with the M_4^- brane-world set-up associated with a negative-tension $M(3,1)$ brane where the warp factor is given by the divergent $A = +b|w|$, though study of this case will also prove informative in the following. Additionally, the supporting of the $\delta(w)$ discontinuity term in Eq. (2.12) by the brane $T_{\mu\nu}$ is an example of the Israel junction conditions [Israel (1966)], a set of junction conditions whose generalization will enable us to construct other brane world set-ups below.

2.3 Gravitational Gauss's law in the context of the brane world

The above M_4^+ metric describes the basic Randall-Sundrum background set-up, with it being the modes associated with tensor fluctuations around this background which will yield the matter content of the theory. As we will show in detail below, the M_4^+ fluctuation modes associated with axial gauge transverse-traceless (TT) gravitational spin-two degrees of freedom which correspond to the typical perturbative line element

$$ds^2 = dw^2 + e^{-2b|w|}(\eta_{\mu\nu} + h_{\mu\nu}^{TT})dx^\mu dx^\nu \tag{2.16}$$

will be found (cf. Eq. (14.20) below) to obey

$$\left[\frac{\partial^2}{\partial w^2} - 4b^2 + 4b\delta(w) + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = -2\kappa_5^2\delta(w)S_{\mu\nu}^{TT} \tag{2.17}$$

when a perturbation $S_{\mu\nu}$ with TT piece $S_{\mu\nu}^{TT}$ is added to the brane. In this case an h_{00}^{TT} which is static then obeys an equation of motion which can be written purely in terms of spatial total derivatives as

$$e^{2b|w|}\frac{\partial}{\partial w}\left[e^{-2b|w|}\left(\frac{\partial}{\partial w} + 2b\epsilon(w)\right)h_{00}^{TT}\right] + e^{2b|w|}\frac{\partial}{\partial x^i}\left[\eta^{ij}\frac{\partial}{\partial x^j}h_{00}^{TT}\right] = -2\kappa_5^2\delta(w)S_{00}^{TT}\,. \tag{2.18}$$

Integrating Eq. (2.18) with $d|w|d^3xe^{-2b|w|}$ over the range $0 \leq w \leq W$ yields

$$\int_0^R d^3x\left[e^{-2b|w|}\left(\frac{\partial}{\partial w} + 2b\right)h_{00}^{TT}\right]\Bigg|_0^W + \int_0^W d|w|\left[\left(\frac{\partial}{\partial x^j}h_{00}^{TT}\right)\eta^{ij}n_i r^2 d\Omega\right]\Bigg|_{r=R}$$
$$\equiv S_W + S_R = -\kappa_5^2\int_0^R d^3x S_{00}^{TT}\;, \tag{2.19}$$

with Eq. (2.19) thus serving as the gravitational Gauss's law in the brane case. In particular, if S_{00}^{TT} is taken to be a point source at $\bar{x} = 0$, then in the region away

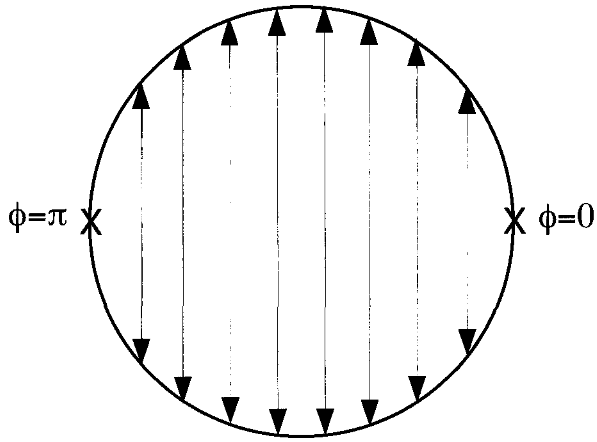

Fig. 2.2 Identifying opposite points across the diameter of a circle. The two points at $\phi = 0$, $\phi = \pi$ self-identify.

from the point source Eq. (2.17) will admit of a solution which obeys

$$h_{00}^{TT} = \frac{e^{-2b|w|}}{r} \quad , \quad \eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{00}^{TT} = 0 \ . \tag{2.20}$$

The insertion of this solution into Eq. (2.19) will cause S_W to vanish identically at any W, and yield an S_R which is independent of R with a coefficient $\int_0^W d|w|e^{-2b|w|}$ which, by being finite in the limit in which W goes to infinity, thereby recovers the finiteness of $-\kappa_5^2 \int_0^R d^3x S_{00}^{TT}$ to which $S_W + S_R$ is equal. As we thus see, in the event of a convergent warp factor, there is a massless graviton mode in the theory with a wave function which falls off exponentially fast away from the brane, a mode which has an S_W flux which vanishes, and a finite S_R flux which behaves just like the gravitational flux associated with a massless 4-dimensional graviton propagating on the brane. The localization of the gravitational flux to the brane thus enables the M_4^+ brane world to support standard 4-dimensional gravity on the brane despite the fact that the brane is embedded in an infinite-sized 5-dimensional space, with it being the curvature of the higher dimensional space rather than a compactification which prevents the gravitational flux from spreading out from the brane. This then is the essence of brane-localization of gravity.

Now as well as the w-even solution of Eq. (2.20), as an equation Eq. (2.17) can also admit of w-odd solutions, since a w-even equation can have solutions which are w-odd.[1] However, such solutions will be forbidden if an identification of points at $w = +\infty$ and $w = -\infty$ is made, since the entire geometry (both background and fluctuations) would then be Z_2 symmetric. The Randall-Sundrum brane world

[1] w-odd functions such as, for instance, $h_{\mu\nu}^{TT} = A_{\mu\nu}e^{i\bar{k}\cdot\bar{x}-i|\bar{k}|t}\epsilon(w)(e^{2b|w|} - e^{-2b|w|})$ where $A_{\mu\nu}$ is a constant tensor satisfy $[\partial_w^2 - 4b^2 + 4b\delta(w) + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta]h_{\mu\nu}^{TT} = 0$ identically.

is thus to be understood as being one in which precisely such a Z_2 symmetry is imposed.[2]

2.4 Compactified brane world with two branes

Now while having a finite-sized bulk is not needed for localization of gravity, one can nonetheless explore the above scenario in the compactified case [Randall and Sundrum (1999a)]. In such a case we may set $w = R\phi$ where ϕ is an angle which goes from 0 to 2π, with the above positive-tension brane being located at $\phi = 0$ for definitiveness. To implement a Z_2 symmetry we identify points ϕ with points $2\pi - \phi$ (see Fig. (2.2)), so that ϕ can now be considered to run from 0 to π, with the points at $\phi = 0$ and $\phi = \pi$ being identified with themselves. With there thus being two of such fixed points we can locate a second brane at $\phi = \pi$, and can look for a candidate solution to the theory in which $A = -bR|\phi|$. To determine $A'' = (1/R^2)d^2A/d\phi^2$ in this specific case, we note that because of the Z_2 symmetry, we can visualize ϕ as having a periodic sawtooth shape in which it takes the value $|\phi| = \pi$ at $\phi = -\pi$, drops linearly to $|\phi| = 0$ at $\phi = 0$, and then rises linearly to $|\phi| = \pi$ at $\phi = \pi$, and so on (see Fig. (2.3)). Thus in the range $-\pi < \phi < 0$ the first derivative $d|\phi|/d\phi$ is equal to -1, while in the range $0 < \phi < \pi$ the derivative is equal to $+1$. Consequently, the second derivative $d^2|\phi|/d\phi^2$ is given as $2\delta(\phi) - 2\delta(\pi - \phi)$, so that

$$A'' = \frac{d^2A}{dw^2} = -\frac{b}{R}\frac{d^2|\phi|}{d\phi^2} = -\frac{2b}{R}[\delta(\phi) - \delta(\pi - \phi)] \quad . \tag{2.21}$$

The 5-dimensional Einstein equations will thus be able to support such a solution if in addition to the positive-tension brane with $\lambda = 6b/\kappa_5^2$ located at $\phi = 0$ we add in a second brane at $\phi = \pi$ with a tension equal but opposite to that of brane at $\phi = 0$, viz. a negative-tension Minkowski brane with $\lambda = -6b/\kappa_5^2$. The Z_2 symmetric configuration of two branes with equal and opposite tension has an antecedent in string theory [Horava and Witten (1996a); Horava and Witten (1996b)], with the new feature of Randall and Sundrum being the introduction of an AdS_5 geometry in the region between the branes.

In terms of a perturbative Ricci scalar $\bar{R}$ $(= \eta^{\mu\nu}\bar{R}_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu} = e^{-2A}R)$ and a metric determinant $\bar{g}$ $(= e^{-4A}g)$ as evaluated to lowest order with the metric $ds_4^2 = h_{\mu\nu}^{TT}dx^\mu dx^\nu$, the first order change in the gravitational action due to the fluctuation is given by

$$-\frac{1}{2\kappa_5^2}\int d^5x\delta[g^{1/2}R] = -\frac{1}{\kappa_5^2}\int d^4x\int_0^\pi d|\phi|Re^{-2bR|\phi|}\bar{g}\bar{R} \quad . \tag{2.22}$$

[2] While we shall adhere to this requirement in this monograph, when we analyze the fluctuations in detail below, we shall find that the brane-world fluctuation causal propagator will itself be constructable from the w-even sector alone. Consequently, even without the imposition of an explict Z_2 symmetry for the fluctuations, the only fluctuations which perturbations on the brane could ever generate then would be ones which would be even in w anyway.

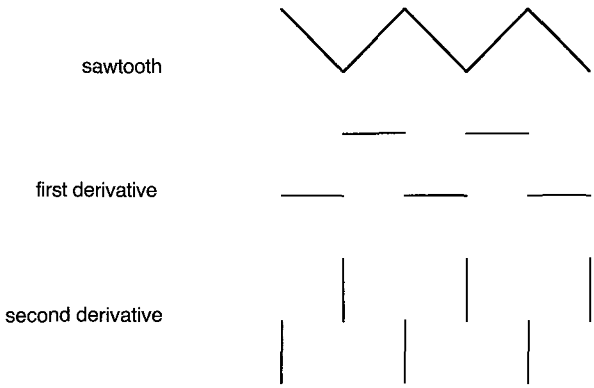

Fig. 2.3 A sawtooth and its first two derivatives.

Equation (2.22) thus describes an effective 4-dimensional gravitational theory with an effective coupling constant which is given by

$$\frac{1}{\kappa_4^2} = \frac{2}{\kappa_5^2} \int_0^{\pi} d|\phi| e^{-2bR|\phi|} = \frac{1}{\kappa_5^2 b}(1 - e^{-2\pi bR}) \ , \tag{2.23}$$

with the identification $M_5^3 = 1/\kappa_5^2$ then recovering Eq. (1.2). With the warp factor $e^{-2bR|\phi|}$ getting smaller as one goes from the positive-tension to the negative-tension brane, the ensuing suppression enabled Randall and Sundrum to address the hierarchy problem provided our 4-dimensional universe was located on the negative-tension brane in a 5-dimensional world in which R was expressly not infinite (so that there would then only be a finite amount of suppression). However, their single brane gravity localization program requires our 4-dimensional universe to be located on the positive-tension brane instead (so that gravity falls rather than rises away from the brane); and since the prospect of localization in an infinite-sized bulk is so attractive, it is this latter option which will be explored in this monograph. In fact as a byproduct of Eq. (2.23), we see that if do remove the negative-tension brane to infinity, the effective 4-dimensional coupling on an isolated positive tension Minkowski brane is then directly determinable, being given by $\kappa_4^2 = \kappa_5^2 b$, i.e. by

$$\kappa_4^2 = \frac{\kappa_5^4 \lambda}{6} \ , \tag{2.24}$$

a relation which we will encounter often in this monograph.

2.5 Removing the second brane

In order to illustrate how the above limit of removing the negative tension brane to infinity is to explicitly be taken, it is useful to consider an intermediate model consisting of two branes, a positive λ one at $w = 0$ and a negative $\hat{\lambda}$ one at $w =$

$w_0 > 0$, both of which are embedded in a geometry in which w can range between $w = -\infty$ and $w = \infty$. For such a set-up we try as candidate warp factor

$$A = -b|w| + a|w - w_0| \ , \tag{2.25}$$

where b is as before and a is to be determined. With A'^2 and A'' being given by

$$\begin{aligned} A'^2 &= [-b[\theta(w) - \theta(-w)] + a[\theta(w - w_0) - \theta(-w + w_0)]]^2 \\ &= b^2 + a^2 - 2ab[\theta(w - w_0) + \theta(-w) - \theta(w)\theta(-w + w_0)] \\ &= b^2 + a^2 - 2ab[1 - 2\theta(w)\theta(-w + w_0)] \\ &= (b - a)^2 + 4ab\theta(w)\theta(w_0 - w) \ , \\ A'' &= -2b\delta(w) + 2a\delta(w - w_0) \ , \end{aligned} \tag{2.26}$$

the Einstein equations will thus be able to support this specific warp factor provided the energy-momentum tensor is of the form

$$\begin{aligned} T_{MN} &= -\Lambda_5 g_{MN}\left[\left(1 - \frac{a}{b}\right)^2 + \frac{4a}{b}\theta(w)\theta(w_0 - w)\right] \\ &\quad - \lambda\delta^\mu_M\delta^\nu_N\eta_{\mu\nu}e^{2aw_0}\delta(w) + \hat{\lambda}\delta^\mu_M\delta^\nu_N\eta_{\mu\nu}e^{-2bw_0}\delta(w - w_0) \ , \end{aligned} \tag{2.27}$$

where

$$\kappa_5^2\Lambda_5 = -6b^2 \ , \quad \kappa_5^2\lambda = 6b \ , \quad \kappa_5^2\hat{\lambda} = 6a \ . \tag{2.28}$$

This hybrid model has two interesting limits, $\hat{\lambda} = -\lambda$ and $\hat{\lambda} = 0$. In the $\hat{\lambda} = -\lambda$ (viz. $a = b$) limit, the model becomes a model of two branes with equal and opposite tension in which there is only bulk cosmological constant in the region $0 < w < w_0$ between the two branes, with the geometry being AdS_5 between the branes, and flat $M(4,1)$ elsewhere (with $A = -b|w| + b|w - w_0|$, $A(w < 0) = bw_0$, $A(0 < w < w_0) = -2bw + bw_0$, $A(w > w_0) = -bw_0$).[3] In such a model, when we allow w_0 to become larger and larger, we see that the $M(4,1)$ region with $w > w_0$ becomes smaller and smaller, while the $M(4,1)$ region with $w < w_0$ remains semi-infinite. The compactified model of Fig. (2.2) is then obtained by letting w_0 become very large, and replacing the $w < 0$ region by a copy of the $w > 0$ one, since in such a limit all $M(4,1)$ regions are shrunk to zero. The alternative limit is to set $\hat{\lambda} = 0$, a limit in which we do not actually remove the second brane to infinity, rather, we simply eliminate it altogether, with the model no longer depending on the parameter w_0. Then, with there no longer being any need to satisfy a delta function constraint at the second brane, the model becomes the infinite-sized bulk, single brane Randall-Sundrum model with both the $w > 0$ and $w < 0$ regions being described by one and the same warp factor $A = e^{-b|w|}$. Having now given the gist of the Randall-Sundrum program, we embark on a detailed exploration of its various components.

[3]This model is analogous to a parallel plate capacitor in ordinary electrostatics, where there is an electric field between the plates and none outside (in the limit of infinite area of the plates).

Chapter 3

General Structure of Anti-de Sitter Spacetimes

3.1 Topology of anti-de Sitter space

As a spacetime the general N-dimensional anti-de Sitter space $AdS(N-1,1)$ has some very unusual properties [Hawking and Ellis (1973); Avis, Isham and Storey (1978)]. It has a topology $R^{N-1} \times S^1$ which contains closed timelike lines (since S^1 is periodic in time) and an asymptotic spatial infinity from which null geodesics can come in in a finite time.[1] It is possible to avoid the closed timelike line problem by unwrapping S^1 into R^1, and it is the unwrapped universal covering space with topology $R^{N-1,1}$ which is commonly referred to as anti-de Sitter space in the literature, with it being the unwrapped one which is of relevance to brane gravity. The problem of new information being able to come in from spatial infinity in a finite time is not evaded by the unwrapping, since for the typical $AdS(N-1,1)$ metric

$$ds^2 = dw^2 + e^{-2bw}[dx_1^2 + dx_2^2 + ... + dx_{N-2}^2 - dt^2] \tag{3.1}$$

with a now unrestricted time coordinate, the null geodesic $t = e^{bw}/b$ which leaves $w = -\infty$ at time $t = 0$ can reach $w = 0$ in the finite time $t = 1/b$. Thus for a general anti-de Sitter spacetime it is not possible to predict the Cauchy evolution of initial data on a spacelike hypersurface of compact support at times beyond which new information from spatial infinity could enter the forward lightcone (the Cauchy development) of the initial hypersurface. For instance, for the typical case of an initial region with compact support at $w = 0$, any new information originating from the boundary at $w = -\infty$ would reach the forward lightcone of the initial region at a time $t = 1/2b$ (see Fig. (3.1)). And even in cases where no such new information does in fact come in, in determining the late time response at $w = 0$ to an initial disturbance at $w = 0$, one still has to allow for the fact that the null signal which originates from $w = 0$ at $t = 0$ and which arrives at $w = -\infty$ at a time $t = 1/b$ can then reflect back off the boundary and return to $w = 0$ at a time $t = 2/b$.

Noting that for the metric of Eq. (3.1) the $w \geq 0$ region null signal $t = (e^{bw}-1)/b$ would actually take an infinite amount of time to travel from $w = 0$ to $w = +\infty$, we

[1] Spatially infinite spacetimes which can be traversed in a finite time are said to be globally non-hyperbolic, with globally hyperbolic spacetimes only being traversable in an infinite time.

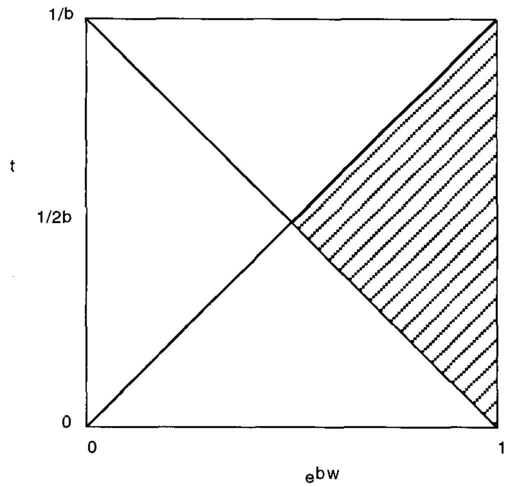

Fig. 3.1 The shaded region denotes that part of the Cauchy development of $w = 0$, $t = 0$ which is not affected by information coming in from $w = -\infty$.

see that if we only keep the $w \geq 0$ region and augment it with a Z_2 copy of itself, to realize the convergent warp factor metric

$$ds^2 = dw^2 + e^{-2b|w|}[dx_1^2 + dx_2^2 + ... + dx_{N-2}^2 - dt^2] \ , \tag{3.2}$$

in the resulting AdS_5/Z_2 spacetime we then would have a spacetime in which no new information could actually come in from its boundary in a finite time at all, with the boundary now being a horizon. Remarkably then, we see that it is precisely in the Randall-Sundrum convergent warp factor M_4^+ brane-world set-up that the troublesome global properties of pure AdS spaces are completely avoided, with brane-localization of gravity thus securing a fully predictive initial value Cauchy problem. Now while we do get this nice result for M_4^+, we of course get completely the opposite for M_4^- where the metric is given by the divergent warp factor metric

$$ds^2 = dw^2 + e^{2b|w|}[dx_1^2 + dx_2^2 + ... + dx_{N-2}^2 - dt^2] \ , \tag{3.3}$$

and thus in brane-world set-ups with divergent warp factors asymptotic boundary issues cannot be ignored, and the treatment of causal propagation in such spaces is not straightforward.

3.2 Coordinate system choices for anti-de Sitter space

In order to explore the structure of AdS spaces in detail, it is very convenient to take advantage of the fact that any space of constant curvature K can be embedded in a flat space with one higher dimension. We shall thus follow the discussion of [Balasubramanian, Kraus and Albion (1999)] which constructs the typical AdS_3

geometry via its embedding in a flat $M(2,2)$. We define AdS_3 as the hyperboloid

$$U^2 + V^2 - X^2 - Y^2 = \ell^2 \tag{3.4}$$

embedded in a space with flat line element

$$ds^2 = dX^2 + dY^2 - dU^2 - dV^2 \quad , \tag{3.5}$$

with the $M(2,2)$ coordinates (U, V, X, Y) ranging over all values between $-\infty$ and $+\infty$ which are compatible with Eq. (3.4). Global coordinates which cover (just once) the whole of this range of AdS_3 coordinates are defined by

$$U = \ell\cosh\mu\ \cos\tau \quad , \quad V = \ell\cosh\mu\ \sin\tau \quad , \quad X = \ell\sinh\mu\ \cos\theta \quad , \quad Y = \ell\sinh\mu\ \sin\theta \quad , \tag{3.6}$$

where $0 \leq \mu \leq \infty$, $0 \leq \theta \leq 2\pi$, $-\pi \leq \tau \leq \pi$, and yield the line element

$$ds^2 = \ell^2[d\mu^2 + \sinh^2\mu\ d\theta^2 - \cosh^2\mu\ d\tau^2] \quad . \tag{3.7}$$

Explicit evaluation of the Riemann tensor for this line element then shows that it is given by $R_{\lambda\mu\nu\kappa} = (-1/\ell^2)[g_{\mu\nu}g_{\lambda\kappa} - g_{\mu\kappa}g_{\lambda\nu}]$, so that it does indeed describe a space with constant negative curvature $K = -1/\ell^2$. As we see, in these global coordinates $\tau = \arctan(V/U)$ is confined to a finite range, a range which is then unwrapped by adding in repeated copies of the space for the time intervals $-3\pi \leq \tau \leq -\pi$, $\pi \leq \tau \leq 3\pi$ and so on, so that τ can then range between $-\infty$ and ∞. The redefinition $\sinh\mu = \tan\rho$ allows us to compactify the coordinate μ into the range $0 \leq \rho \leq \pi/2$, with the line element becoming

$$ds^2 = \ell^2\sec^2\rho[d\rho^2 + \sin^2\rho\ d\theta^2 - d\tau^2] \quad . \tag{3.8}$$

The space thus has the topology of a (ρ, θ) disk times a line in τ; and with $\rho = \tau$ being a null geodesic,[2] information can travel from $\rho = 0$ to the spatial boundary at $\rho = \pi/2$ in the finite time $\tau = \pi/2$.

We can represent the AdS_3 space pictorially in Fig. (3.2) as the interior of a cylinder. In the picture we can break the cylinder up into two 3-dimensional wedges obtained by making the indicated cuts through the cylinder along the $U + X = 0$ null surface which contains the $\theta = 0$ null geodesics $\tau = \pm(\rho + \pi/2)$ and the $\theta = \pi$ null geodesics $\tau = \pm(\rho - \pi/2)$. One wedge is anchored at A, B and C, and the second one is obtained by identifying the top and bottom ends of the cylinder. In an unwrapping of the τ coordinate the wedges would then repeat indefinitely. The significance of these particular wedges is that each such wedge is an independent causal region in which the Cauchy development of initial data on the earlier part of the bounding null surface is contained entirely within the wedge.[3] A $Y = 0$

[2]Because of its simplicity, the general null geodesics can readily be extracted from Eq. (3.8), with the fixed θ ones being of the form $\tau = \pm\rho +$ constant, and the fixed ρ ones being of the form $\tau = \pm\theta\sin\rho +$ constant.

[3]The null geodesics which originate on AC are along AC itself or parallel to AB.

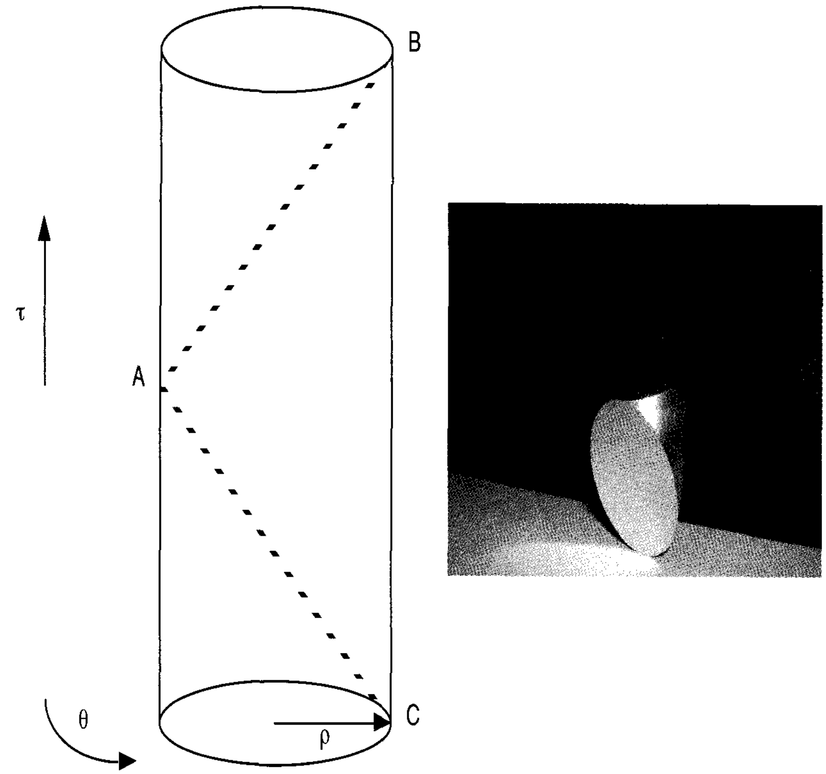

Fig. 3.2 The left figure shows the topology of AdS_3. The right figure illustrates the sectioning of a cylinder by two diagonal planes.

slice across the cylinder yields the Penrose diagram exhibited in the left panel of Fig. (3.3). In the right-half of the $Y = 0$ slice θ is equal to zero (so that $\rho\cos\theta$ is positive), and in the left-half θ is equal to π (so that $\rho\cos\theta$ is negative).

3.3 "Poincaré patch" coordinates

Even though a single wedge does not cover the full AdS_3 space, nonetheless, a particularly convenient set of coordinates, known as Poincaré coordinates, can be found to parameterize each such "Poincaré patch" wedge. Specifically, we define

$$U = \frac{1}{2v}[\ell^2 + v^2 + x^2 - t^2] \ , \quad V = \ell\frac{t}{v} \ , \quad X = \frac{1}{2v}[\ell^2 - v^2 - x^2 + t^2] \ , \quad Y = \ell\frac{x}{v} \ , \tag{3.9}$$

where v is required to be positive only, with v then being found to range between 0 and ∞, and with x and t each being found to range between $-\infty$ and $+\infty$. With

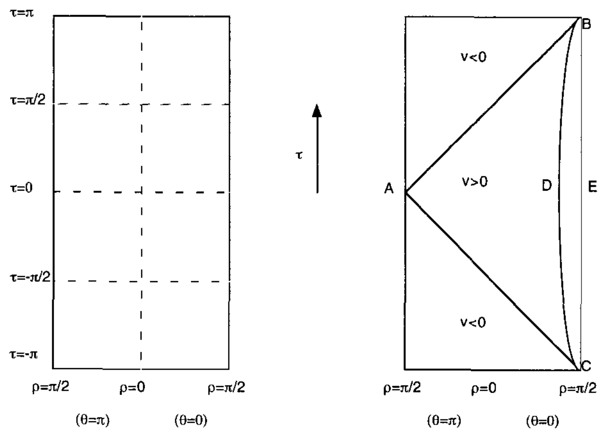

Fig. 3.3 Global coordinate Penrose diagram with axes ρ, τ for $Y = 0$ slice of AdS_3. In the right panel the Poincaré patch is shown superimposed.

this particular choice of coordinates we obtain the line element

$$ds^2 = \frac{\ell^2}{v^2}[dv^2 + dx^2 - dt^2] \ , \tag{3.10}$$

with the additional transformation $v = \ell e^{w/\ell}$ bringing this line element to the form

$$ds^2 = dw^2 + e^{-2w/\ell}[dx^2 - dt^2] \tag{3.11}$$

used earlier. Similarly, the second wedge in the AdS_3 cylinder can be covered by an analogous set of coordinates in which v is taken to be negative. Comparison of Eqs. (3.6) and (3.9) shows that the global and Poincaré coordinates are related by

$$v = \frac{\ell\cos\rho}{(\cos\tau + \cos\theta\ \sin\rho)} \ , \ x = \frac{\ell\sin\rho\ \sin\theta}{(\cos\tau + \cos\theta\ \sin\rho)} \ , \ t = \frac{\ell\sin\tau}{(\cos\tau + \cos\theta\ \sin\rho)} \ . \tag{3.12}$$

In the $\theta = 0$ sector of the $Y = 0$ slice these coordinates are related by

$$v = \frac{\ell\cos\rho}{(\cos\tau - \sin\rho)} \ , \ x = 0 \ , \ t = \frac{\ell\sin\tau}{(\cos\tau - \sin\rho)} \ , \tag{3.13}$$

while being related by

$$v = \frac{\ell\cos\rho}{(\cos\tau + \sin\rho)} \ , \ x = 0 \ , \ t = \frac{\ell\sin\tau}{(\cos\tau + \sin\rho)} \tag{3.14}$$

in the $\theta = \pi$ sector. The diagonal AB null geodesic given by $\tau = \pi/2 - \rho$ (when $\theta = \pi$), $\tau = \rho + \pi/2$ (when $\theta = 0$) corresponds to $v = \infty$ (viz. to $U + X = 0$). Similarly, the $\rho = \pi/2$ boundary corresponds to $v = 0$. Additionally, in the $\theta = 0$

region the curve $v = \ell$, viz. $w = 0$, corresponds to $\cos\rho - \sin\rho = \cos\tau$, i.e. to

$$2\sin\rho = (1 + \sin^2\tau)^{1/2} - \cos\tau \quad , \quad 2\cos\rho = (1 + \sin^2\tau)^{1/2} + \cos\tau \quad . \tag{3.15}$$

The location of the brane is thus given very simply in Poincaré coordinates, and is shown as curve BDC in the right panel of Fig. (3.3) where the Poincaré patch is plotted on the global coordinate Penrose diagram. As we see, the $v = \ell$ curve breaks up the Poincaré patch into two distinct regions, regions which segregate the $v = \infty$ Poincaré patch horizon (curves CA and AB) from the $v = 0$ AdS_3 boundary (curve BEC).[4] The Randall-Sundrum M_4^+ brane-world set-up thus keeps only the region between $v = \ell$ and $v = \infty$, while the M_4^- brane-world set-up keeps only the region between $v = \ell$ and $v = 0$. Consequently, the M_4^+ brane-world set-up beautifully excludes the AdS_3 boundary from which new information is able to come in, so that forward propagation in M_4^+ is fully predictable.

3.4 Other coordinate sectionings of anti-de Sitter space

As such, the metric of Eq. (3.1) has sectioned the $AdS(N-1,1)$ according to the Minkowski $M(N-2,1)$. However many other such sectionings are possible, one for every subspace which can be embedded in $AdS(N-1,1)$. Since dS_4, AdS_4 and the comoving Robertson-Walker (RW) geometries with expansion radius $a(t)$ and 3-curvature k can all be embedded in AdS_5, pure AdS_5 thus admits of the following coordinate equivalent sectionings

$$ds^2(dS_4) = dw^2 + \frac{H^2}{b^2}\sinh^2(bw)[e^{2Ht}(dx^2 + dy^2 + dz^2) - dt^2] \quad , \tag{3.16}$$

$$ds^2(AdS_4) = dw^2 + \frac{H^2}{b^2}\cosh^2(bw)[dx^2 + e^{2Hx}(dy^2 + dz^2 - dt^2)] \quad , \tag{3.17}$$

$$\begin{aligned} ds^2(RW) = dw^2 &- [\cosh(bw) - F(t)\sinh(bw)]^2 dt^2 \\ &+ a^2(t)[\cosh(bw) - G(t)\sinh(bw)]^2 d\Omega_3 \quad , \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} G(t) &= \frac{[\dot{a}^2(t) + k + b^2a^2(t)]^{1/2}}{ba(t)} \quad , \quad d\Omega_3 = \frac{dr^2}{(1-kr^2)} + r^2 d\Omega_2 \quad , \\ F(t) &= \frac{\ddot{a}(t) + b^2 a(t)}{b[\dot{a}^2(t) + k + b^2a^2(t)]^{1/2}} = \frac{a(t)\dot{G}}{\dot{a}(t)} + G \quad . \end{aligned} \tag{3.19}$$

Below we shall show how all of these metrics are actually derived, though one can check immediately (either by brute force, or by using a general relativity software package) that for all of them the Riemann tensor is indeed given by $R_{LMNK} =$

[4] In terms of the Poincaré patch coordinate t, it takes the $v = t + \ell$ null signal an infinite amount of time to travel from $v = \ell$ to $v = \infty$.

$-b^2[g_{MN}g_{LK} - g_{MK}g_{LN}]$.[5] While all of these metrics are coordinate equivalent in pure AdS_5 (since there is only one AdS_5), they become inequivalent in the AdS_5/Z_2 brane-world case, where, as we shall see, each one will require its own specific matter field content on the brane.

While we can embed these particular highly symmetric (10 or 6 Killing vector) 4-dimensional spaces in the 15 Killing vector AdS_5, we note that we cannot embed the much less symmetric 3 Killing vector 4-dimensional Schwarzschild type metric in AdS_5, with the metric

$$ds^2 = dw^2 + e^{2bw}\left[\frac{dr^2}{(1-\beta/r)} + r^2 d\Omega_2 - \left(1 - \frac{\beta}{r}\right)dt^2\right] \tag{3.20}$$

for instance not being an AdS_5 metric. Even though this particular metric does happen to obey the standard AdS_5 Einstein equation $G_{MN} = \kappa_5^2 \Lambda_5 g_{MN}$ away from the $\delta^3(x)$ singularities (Eq. (3.20) corresponds to a $\delta^3(x)$ point source singularity on every w slice and not just to one at $w = 0$), the metric does not possess a vanishing Weyl tensor, with components such as C_{0101} for instance being found to evaluate to the non-zero $\beta e^{2bw}/r^3$. As this example indicates, in and of themselves, imposing the Einstein equations (even in the brane case) is not sufficient to force the bulk to be AdS_5, since for that to be the case, as well as having an Einstein tensor which is proportional to the metric tensor, in addition the bulk Weyl tensor is required to vanish. With some brane sources being able to set up a gravitational field in the bulk which is AdS_5 symmetric, and some not, we turn now to a general discussion of embeddings.

[5] While the above sectioning of AdS_5 in RW coordinates is achievable for an arbitrarily assigned $a(t)$ in the $k > 0$ and $k = 0$ cases, the reality of the functions $G(t)$ and $F(t)$ requires that quantity $[\dot{a}^2(t) + k + b^2 a^2(t)]$ not be negative, a condition which could be violated in some $k < 0$ *RW* geometries. Technically, such $k < 0$ geometries would then not section AdS_5 itself, but rather a continuation of it in which the spacelike w is replaced by the timelike $v = iw$ in the metric of Eq. (3.18).

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Chapter 4

General Properties of Embeddings

4.1 Gravitational versus purely geometric embeddings

In differential geometry there are two complementary facets to surfaces. First, a surface can be thought of as existing in and of itself without reference to anything else, with the surface then defining its own (intrinsic) curvature which describes its shape via properties intrinsic to it. Beyond this though, a surface also has the property that it can be embedded in certain specific higher dimensional spaces, in the sense that it can "fit into" them. In general, any arbitrary N-dimensional surface can always be locally embedded in an appropriately signatured higher-dimensional flat space with dimensionality no higher than $N(N+1)$, with the needed flat space itself requiring fewer dimensions the larger the number of isometries of the N-dimensional surface, isometries which themselves can number up to $N(N+1)$.

In gravitational physics, while the above remarks all of course hold true, there is a further effect which also needs to be taken into consideration. Specifically, when there are matter fields on the N-dimensional surface, they then set up a gravitational field outside the surface, a gravitational field which then serves to modify the external geometry. Thus while a brane might be able to geometrically embed (i.e. fit) into some given bulk, from the point of view of gravity the fields on the brane could also serve to modify the geometry of the bulk. Thus while an empty $M(3,1)$ brane can geometrically embed in AdS_5, the addition of matter fields to the brane will, other than in some highly restricted circumstances where the matter field configurations have very high isometry, generally serve to lower the symmetry of the bulk and force it to no longer be AdS_5. It is those very special cases where the bulk geometry does in fact remain AdS_5 even in the presence of matter fields on the brane that are then so central to the gravity brane-localization program, with cases other than those typically only being treatable perturbatively.

4.2 Gauss embedding formula

In discussing the embedding of a surface $\mathcal{S}_1$ in some larger surface $\mathcal{S}_2$ with metric g_{AB}, it most convenient to introduce the extrinsic curvature K_{AB}. In terms of a spacelike vector n^A (the case of relevance to brane gravity) of unit length $g_{AB}n^An^B = 1$ which is orthogonal to every vector in $\mathcal{S}_1$, one can introduce an induced metric on $\mathcal{S}_1$, viz.

$$i_{AB} = g_{AB} - n_A n_B \quad , \tag{4.1}$$

in terms of which the extrinsic curvature K_{MN} is then given as

$$K_{MN} = i^A{}_M i^B{}_N n_{B;A} = i^A{}_M i^B{}_N [\partial_A n_B - \Gamma^C_{AB} n_C] \quad , \tag{4.2}$$

with the covariant derivative $n_{B;A}$ being calculated with respect to the full metric g_{AB}. The extrinsic curvature obeys the kinematic properties $K_{MN} = K_{NM}$, $n^M K_{MN} = 0$, and its utility is that it enables us to express the Riemann tensor on $\mathcal{S}_1$ as calculated with the induced metric i_{AB} in terms of the Riemann tensor on $\mathcal{S}_2$ as calculated with the full metric g_{AB} according to the Gauss embedding formula [Hawking and Ellis (1973)], a relation which takes the form

$$^{(4)}R^E{}_{MFN} = R^A{}_{BCD} i_A{}^E i^B{}_M i^C{}_F i^D{}_N - K^E{}_F K_{MN} + K^E{}_N K_{MF} \tag{4.3}$$

in the specific case where $\mathcal{S}_1$ has four dimensions and $\mathcal{S}_2$ has five.[1] Contraction of indices with $i^F{}_E$ in Eq. (4.3) and use of Eq. (4.1) then yields for the 4-space Ricci tensor

$$^{(4)}R_{MN} = R_{BD} i^B{}_M i^D{}_N - R^A{}_{BCD} n_A i^B{}_M n^C i^D{}_N - K^E{}_E K_{MN} + K^E{}_M K_{NE} \quad . \tag{4.4}$$

Using Eq. (2.2) to re-express the $\mathcal{S}_2$ Riemann tensor R_{ABCD} in terms of the $\mathcal{S}_2$ Weyl tensor C_{ABCD} enables us to also obtain [Shiromizu, Maeda and Sasaki (2000)] a compact expression for the $\mathcal{S}_1$ Einstein tensor as calculated with i_{ab}, viz.

$$\begin{aligned} ^{(4)}G_{MN} &= \frac{2}{3} G_{AB} i^A{}_M i^B{}_N + \frac{1}{3} G_{AB} n^A n^B [2 i_{MN} + 3 n_M n_N] - \frac{1}{6} G^A{}_A i_{MN} \\ &-K^A{}_A K_{MN} + K^A{}_M K_{AN} + \frac{1}{2}[(K^A{}_A)^2 - K_{AB}K^{AB}](i_{MN} + n_M n_N) - E_{MN} \ , \end{aligned} \tag{4.5}$$

where E_{MN} denotes

$$E_{MN} = C^A{}_{BCD} n_A n^C i^B{}_M i^D{}_N \quad . \tag{4.6}$$

As we see from Eq. (4.5), differences between the $\mathcal{S}_1$ and $\mathcal{S}_2$ Einstein tensors are thus characterized by the extrinsic curvature K_{MN} and the E_{MN} components of

[1] While $^{(4)}R^E{}_{MFN}$ has the standard $D^2(D^2 - 1)/12 = 20$ independent components required of a 4-dimensional Riemann tensor, its indices will in general range over all five allowed values for E, M, F, N unless $i_{\mu\nu} = g_{\mu\nu}$, i.e. unless g_{MN} and n^A are such that $g_{5\mu} = 0$, $n^A = (0, 0, 0, 0, (g_{55})^{-1/2})$.

the 5-dimensional Weyl tensor, with the 4-dimensional Einstein tensor ${}^{(4)}G_{\mu\nu}$ not in any way looking like the $(\mu\nu)$ component of the 5-dimensional G_{AB}.

4.3 Evaluation of the induced Einstein tensor in a specific case

As such, Eqs. (4.3) – (4.5) are quite remarkable. Specifically, since quantities such as ${}^{(4)}G_{00}$ are calculated with the induced metric alone, all derivatives with respect to the fifth coordinate must identically drop out of the right-hand side of Eq. (4.5) even though they appear in the individual terms contained therein. To illustrate the point we consider the typical metric $ds^2 = dw^2 + e^{2A(w)}[dx^2 + dy^2 + dz^2 - dt^2]$ associated with an embedding of $M(3,1)$ in a 5-space, a metric for which the non-vanishing Christoffel symbols are given by $\Gamma^5_{\mu\kappa} = -A' g_{\mu\kappa}$, $\Gamma^\lambda_{5\kappa} = A'\delta^\lambda_\kappa$. With the normal to the $M(3,1)$ subspace being simply given by the spacelike

$$n_A = (0,0,0,0,1) \tag{4.7}$$

in this case, the non-vanishing components of the extrinsic curvature evaluate to

$$K_{\mu\nu} = A' e^{2A} \eta_{\mu\nu} \quad . \tag{4.8}$$

Evaluating the various terms in Eq. (4.5) yields [Mannheim (2001c)]

$$\begin{aligned}
&\frac{2}{3} G_{AB} i^A{}_0 i^B{}_0 = 2e^{2A}[A'' + 2A'^2] \ , \quad \frac{1}{3} G_{AB} n^A n^B (2i_{00} + 3n_0^2) = 4e^{2A} A'^2 \ , \\
&-\frac{1}{6} G^A{}_A i_{00} = -e^{2A}[2A'' + 5A'^2] \ , \quad -K^\alpha{}_\alpha K_{00} = 4e^{2A} A'^2 \ , \\
&K^\alpha{}_0 K_{\alpha 0} = -e^{2A} A'^2 \ , \quad \frac{1}{2}(K^\alpha{}_\alpha)^2 (i_{00} + n_0^2) = -8e^{2A} A'^2 \ , \\
&-\frac{1}{2} K_{\alpha\beta} K^{\alpha\beta} (i_{00} + n_0^2) = 2e^{2A} A'^2 \ , \quad -E_{00} = 0 \ ,
\end{aligned} \tag{4.9}$$

from which we infer that not only do all the w-derivative dependent terms indeed drop out of ${}^{(4)}G_{00}$, additionally we find that ${}^{(4)}G_{00}$ evaluates to zero, i.e. to precisely the value the Einstein tensor does in fact take when calculated on any w slice with induced metric $ds^2 = e^{2A(w)}[dx^2 + dy^2 + dz^2 - dt^2]$, a calculation in which w is treated as an external parameter with respect to which there is no differentiation.

With the 4-dimensional Einstein tensor departing radically from the 5-dimensional one in Eq. (4.5), we can anticipate that gravitational dynamics on a brane embedded in a higher-dimensional bulk can be quite different from dynamics on the selfsame S_1 surface with the selfsame matter fields and no external bulk at all. A monitoring of the contributions of the $K_{\mu\nu}$ and $E_{\mu\nu}$ terms in Eq. (4.5) to dynamics on the brane can thus serve as a potential window on extra dimensions. In fact it can serve as such even in the absence of any Z_2 symmetry or any non-compactified AdS_5 bulk, since the derivation of Eq. (4.5) is independent of dynamics and makes no assumptions at all as to the matter content of either S_1

or $\mathcal{S}_2$. In fact, as such, it does not even require that G_{AB} obey the 5-dimensional Einstein equations. However, to actually use Eq. (4.5) in a brane-world set-up (one where the bulk Einstein equations then will of course hold) requires determining the extrinsic curvature. And with the extrinsic curvature at the brane being fixed by the matter content on the brane via the Israel junction conditions, it is to these conditions that we now turn.

Chapter 5

The Israel Junction Conditions

5.1 Junction conditions in the Newtonian and M_4^+ cases

In crossing a boundary layer of gravitating material there will typically be a change in the gravitational field. A familiar example from Newtonian gravity is a static 2-dimensional flat sheet of infinite extent containing matter of uniform surface density σ lying in the $z = 0$ plane of a flat 3-dimensional space. While the matter on the sheet produces a gravitational potential which grows linearly with z and a gravitational force $F(z)$ per unit mass whose magnitude is independent of z, the Newtonian gravitational force always points toward the sheet no matter which side we consider, with Gauss' Law yielding

$$F(z > 0) - F(z < 0) = 4\pi G\sigma \quad , \tag{5.1}$$

and thus

$$F(z > 0) = -F(z < 0) = 2\pi G\sigma \quad , \tag{5.2}$$

i.e.

$$F(z) = \theta(z)2\pi G\sigma - \theta(-z)2\pi G\sigma = \epsilon(z)2\pi G\sigma \quad , \tag{5.3}$$

so that the sheet sets up a gravitational potential

$$\phi = 2\pi G\sigma|z| \quad . \tag{5.4}$$

As we thus see, the gravitational force produced by the sheet is discontinuous across the surface of the sheet, with it being the matter density on the sheet which is the cause of the discontinuity. Since $F(z)$ is independent of the magnitude of z we can also write Eq. (5.1) in the form

$$F(z = 0^+) - F(z = 0^-) = 4\pi G\sigma \quad , \tag{5.5}$$

a form whose generalization will prove to be of significance in the following. (While this Newtonian model allows us to extract and motivate the fully covariant junction conditions which we will present below, the physical status of Eq. (5.5) itself needs to be analyzed carefully since a constant Newtonian gravitational force is equivalent to a uniform acceleration and thus a pure gauge artifact, and we will address this issue below when we provide a covariant description of an embedded 2-dimensional sheet.)

A dependence of the gravitational potential on the modulus of the coordinate of the extra dimension associated with the space in which the sheet is embedded is of course familiar from the Randall-Sundrum single-brane model discussed earlier, and so for it as well there should be a relation of the form of Eq. (5.5). Thus, with the M_4^+ brane-world model warp factor $A(w)$ of Eq. (2.6) being given as $A(w) = -b|w| = -\kappa_5^2\lambda|w|/6$, in the Randall-Sundrum single-brane model the extrinsic curvature $K_{\mu\nu} = A'e^{2A}\eta_{\mu\nu}$ of Eq. (4.8) will take the value

$$K_{\mu\nu} = -[\theta(w=0^+) - \theta(w=0^-)]\frac{\kappa_5^2\lambda}{6}\eta_{\mu\nu} \tag{5.6}$$

at the brane. Now to support the M_4^+ brane-world model we found that we needed an energy-momentum tensor $T_{\mu\nu} = -\lambda\eta_{\mu\nu}\delta(w)$ on the brane. Writing this energy-momentum tensor in the more generic form $T_{\mu\nu} = \tau_{\mu\nu}\delta(w)$ where $\tau_{\mu\nu} = -\lambda\eta_{\mu\nu}$, we see that we can write

$$K_{\mu\nu}(w=0^+) = \frac{\kappa_5^2}{6}\tau_{\mu\nu} \quad , \quad K_{\mu\nu}(w=0^-) = -\frac{\kappa_5^2}{6}\tau_{\mu\nu} \quad . \tag{5.7}$$

As such, Eq. (5.7) is a particular example of the fully covariant Israel junction conditions [Israel (1966)], whose generic form in this particular case can be written as

$$K_{\mu\nu}(w=0^+) - K_{\mu\nu}(w=0^-) = -\kappa_5^2\left[\tau_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\tau^{\alpha}{}_{\alpha}\right] \quad , \tag{5.8}$$

with $K_{\mu\nu}(w=0^+)$ being given by

$$K_{\mu\nu}(w=0^+) = -\frac{\kappa_5^2}{2}\left[\tau_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\tau^{\alpha}{}_{\alpha}\right] \tag{5.9}$$

due to the Z_2 symmetry. A delta function singularity on the brane thus supports an extrinsic curvature in the space immediately adjacent to it (one has to go slightly off the brane in order to define the covariant derivative of the normal to the brane), with there being a jump in the extrinsic curvature as the brane is crossed from one side to the other.

5.2 General Israel junction conditions

To generalize the junction conditions to the case of the embedding of an arbitrary 4-dimensional surface $\mathcal{S}_1$ in an arbitrary 5-dimensional $\mathcal{S}_2$, we can, without any loss of generality, either work with a simple form for the normal n_A or with a simple form for the metric g_{AB}. We shall choose the former, and take the covariant n_A to be purely in the fifth direction, viz.

$$n_A = (0,0,0,0,N) \ , \tag{5.10}$$

where N is a completely general function of the five coordinates. To maintain full generality we must simultaneously take the metric to have non-vanishing $g_{5\mu}$ components, and so we conveniently set (see e.g. [Stephani, Kramer, Maccallum, Hoenselaers and Herlt (2003)])

$$ds^2 = (N^2 + f_{\mu\nu}N^\mu N^\nu)dw^2 + 2f_{\mu\nu}N^\mu dx^\nu dw + f_{\mu\nu}dx^\mu dx^\nu \ , \tag{5.11}$$

where the four N^μ and the 10 $f_{\mu\nu}$ are also arbitrary functions of the coordinates, so that the metric thus has a full 15-fold complement of independent components. Since the contravariant N^μ only has four components we shall define a covariant N_μ according to $N_\mu = f_{\mu\nu}N^\nu$, and also we shall define $f^{\mu\nu}$ to be such that $f^{\mu\alpha}f_{\alpha\nu} = \delta^\mu_\nu$. Given Eq. (5.11) we see that associated with the induced metric $i_{AB} = g_{AB} - n_A n_B$ is the line element $i_{AB}dx^A dx^B = f_{\mu\nu}N^\mu N^\nu dw^2 + 2f_{\mu\nu}N^\mu dx^\nu dw + f_{\mu\nu}dx^\mu dx^\nu$, with a simple form for n_A thus in general being accompanied by a complicated form for the induced metric, despite which the induced metric still lies entirely in $\mathcal{S}_1$ since $i_{AB}n^B = 0$. The inverse of the metric of Eq. (5.11) is given by

$$g^{55} = \frac{1}{N^2} \ , \quad g^{5\mu} = -\frac{N^\mu}{N^2} \ , \quad g^{\mu\nu} = f^{\mu\nu} + \frac{N^\mu N^\nu}{N^2} \ , \tag{5.12}$$

and with the normal n_A having unit length $n_A n^A = 1$ (so that $n^A n_{A;B} = 0$), its contravariant components are given by

$$n^A = \left(-\frac{N^\mu}{N}, \frac{1}{N}\right) \ . \tag{5.13}$$

With the extrinsic curvature being given in general by

$$\begin{aligned} K_{MN} &= (g_M{}^A - n_M n^A)(g_N{}^B - n_N n^B)n_{B;A} \\ &= (g_M{}^A - n_M n^A)n_{N;A} = n_{N;M} - n_M n^A n_{N;A} \ , \end{aligned} \tag{5.14}$$

we see that the extrinsic curvature depends on the acceleration vector $a_N = n^A n_{N;A}$, a vector which by obeying $n^N a_N = 0$ must thus lie entirely in $\mathcal{S}_1$. Explicitly evaluating the covariant derivatives in Eq. (5.14) with the metric of Eq. (5.11) yields

$$\begin{aligned} K_{55} = N^\alpha n_{5;\alpha} \ , \quad K_{5\mu} &= K_{\mu 5} = \frac{N^\alpha}{2N}[\partial_w f_{\mu\alpha} - f_{\nu\alpha}\partial_\mu N^\nu - \partial_\alpha(f_{\mu\nu}N^\nu)] \ , \\ K_{\mu\nu} &= -N\Gamma^5_{\mu\nu} = \frac{1}{2N}\left(\frac{\partial f_{\mu\nu}}{\partial w} - N_{\mu;;\nu} - N_{\nu;;\mu}\right) \ , \end{aligned} \tag{5.15}$$

where the symbol $;;$ indicates that the covariant derivative is to be calculated in a geometry with metric $ds^2 = f_{\mu\nu}dx^\mu dx^\nu$. From Eq. (5.15) we confirm that K_{MN} is indeed symmetric. As well as being writable in the non-manifestly symmetric form

$K_{MN} = n_{N;M} - n_M a_N$, the extrinsic curvature can also be written in a form in which it is manifestly symmetric. Specifically, with the Lie derivative of a rank-two tensor with respect to a vector n^A being defined as

$$\mathcal{L}_n T_{\mu\nu} = n^A T_{\mu\nu;A} + T_{\mu A} n^A{}_{;\nu} + T_{A\nu} n^A{}_{;\mu} \quad , \tag{5.16}$$

we see that K_{MN} can be written as the Lie derivative of the induced metric with respect to the normal vector since $(1/2)\mathcal{L}_n i_{MN} = (n_{N;M} - n_M n^A n_{N;A} + n_{M;N} - n_N n^A n_{M;A})/2 = K_{MN}$, a form in which K_{MN} now is manifestly symmetric.

As well as being able to use the Gauss embedding formula of Eq. (4.4) to express the 4-dimensional Ricci tensor in terms of the 5-dimensional one, an alternate relation between these two tensors can also be found, one which involves not just the extrinsic curvature but also its covariant derivative. Specifically, with the covariant derivatives of a vector obeying

$$n_{M;N;K} - n_{M;K;N} = -n^S R_{SMNK} \quad , \tag{5.17}$$

we can set

$$\begin{aligned} n^C K_{AB;C} &= n^C [n_{B;A} - n_A a_B]_{;C} = n^C [n_{B;C;A} - n^D R_{DBAC}] - a_A a_B - n^C n_A a_{B;C} \\ &= a_{B;A} - n^C{}_{;A} n_{B;C} + n^C n^D R_{DBCA} - a_A a_B - n^C n_A a_{B;C} \quad , \end{aligned} \tag{5.18}$$

from which it follows (on repeated use of $n_M a^M = 0$) that

$$\begin{aligned} &{}^{(4)}R_{\mu\nu} - R_{BD} i^B{}_\mu i^D{}_\nu + K^A{}_A K_{\mu\nu} - K^A{}_\nu K_{\mu A} \\ &= -R_{DBCA} n^D i^B{}_\mu n^C i^A{}_\nu \\ &= i^B{}_\mu i^A{}_\nu (-n^C K_{AB;C} + a_{B;A} - n^C{}_{;A} n_{B;C} - a_A a_B - n^C n_A a_{B;C}) \\ &= -n^C K_{\nu\mu;C} - n_\nu a^A K_{A\mu} - n_\mu a^A K_{\nu A} + a_{\mu;\nu} - n_\mu n^A a_{A;\nu} \\ &\quad - n_\nu n^A a_{\mu;A} - n_\mu n_\nu a^B a_B + n_\nu a^C n_{\mu;C} - n^C{}_{;\nu} n_{\mu;C} - a_\nu a_\mu \\ &= -n^C K_{\nu\mu;C} + a_{\mu;\nu} - n_\nu n^A a_{\mu;A} - n^A{}_{;\nu} n_{\mu;A} - a_\nu a_\mu \\ &= -n^C K_{\nu\mu;C} + a_{\mu;\nu} - n_\nu n^A a_{\mu;A} - K_\nu{}^A K_{A\mu} - n_\nu a^A K_{A\mu} - a_\nu a_\mu \quad . \end{aligned} \tag{5.19}$$

On recalling that $n_\mu = 0$ (only $n_5 \neq 0$), we thus obtain

$$n^C K_{\mu\nu;C} - R_{BD} i^B{}_\mu i^D{}_\nu = -{}^{(4)}R_{\mu\nu} + \frac{1}{2}(a_{\mu;\nu} + a_{\nu;\mu}) - K^A{}_A K_{\mu\nu} - a_\mu a_\nu \tag{5.20}$$

as written here in a manifestly symmetric form. As such Eq. (5.20) is a completely general, dynamics independent, purely geometric relation which requires only that there in fact be an embedding.

The remarkable feature of Eq. (5.20) is that the part which is of relevance to the Israel junction conditions is contained in just two terms, viz. $n^C K_{\nu\mu;C}$ and $R_{BD} i^B{}_\mu i^D{}_\nu$, with all other terms being incapable of generating a discontinuous $\delta(w)$

(=$(1/2)d^2|w|/dw^2$) type singularity at the surface $\mathcal{S}_1$.[1] Now in a D-dimensional theory the Einstein equations would take the form $G_{MN} = -\kappa_D^2 T_{MN}$, i.e.

$$R_{MN} = -\kappa_D^2 \left(T_{MN} - \frac{1}{(D-2)} g_{MN} T^A{}_A \right) \quad . \tag{5.21}$$

Thus on introducing a source term $T_{MN} = \delta^\mu_M \delta^\nu_N \tau_{\mu\nu} \delta(w)$ at a junction at $w = 0$, we see that this delta function term would be supported by the $n^C K_{\nu\mu;C}$ term alone.[2] Since $R_{BD} i^B{}_\mu i^D{}_\nu = R_{BD}(g^B{}_\mu - n^B n_\mu)(g^D{}_\nu - n^D n_\nu)$ is equal to $R_{\mu\nu}$ when $n_\mu = 0$, for our $D = 5$ case Eq. (5.20) yields

$$\int_{0^-}^{0^+} dw \left[\frac{1}{N} \partial_w K_{\mu\nu} + \kappa_5^2 \left(\tau_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \tau^\alpha{}_\alpha \right) \delta(w) \right] = 0 \quad . \tag{5.22}$$

Since without loss of generality we can take N to be given by $N = 1$ at the junction (we can always set a metric coefficient equal to one at one point), the fully covariant Israel junction conditions

$$K_{\mu\nu}(w = 0^+) - K_{\mu\nu}(w = 0^-) = -\kappa_5^2 \left[\tau_{\mu\nu} - \frac{1}{3} g_{\mu\nu}(w = 0) \tau^\alpha{}_\alpha \right] \quad , \tag{5.23}$$

thus emerge.[3]

[1]The acceleration a_μ lies completely in $\mathcal{S}_1$ and actually vanishes identically in the Gaussian normal coordinate system (viz. $g_{55} = 1$, $g_{5\mu} = 0$) often used in brane-world set-ups. Moreover, even though the extrinsic curvature could itself be proportional to the discontinuous $\epsilon(w)$ ($K_{\mu\nu} = \mathcal{L}_n i_{\mu\nu}/2$ is a first derivative function of the metric), terms quadratic in $K_{\mu\nu}$ would then be proportional to $\epsilon(w)^2 = 1$ and still be continuous. Finally, ${}^{(4)}R_{\mu\nu}$ is evaluated with the induced metric alone and thus contains no w-derivative dependence at all.

[2]Since the metric itself is continuous at the surface $\mathcal{S}_1$ (which could still allow the metric to depend on $|w|$), it would take two derivatives with respect to w to generate a discontinuous $\delta(w)$ term.

[3]The discussion of the Israel junctions given here parallels that given in [Binetruy, Deffayet and Langlois (2000)], with these relations also having been obtained [Chamblin and Reall (1999); Davidson and Karasik (1999); Karasik and Davidson (2002)] very elegantly using an action principle.

Chapter 6

The Newtonian Limit of an Embedded Brane

6.1 Covariant analysis of an embedded Newtonian sheet of matter

To see how all these embedding and junction condition ideas work we consider as an extremely simple example the covariant generalization of the familiar Newtonian sheet model we studied in Chapter 5. We shall thus take a static sheet of infinite extent in the (x, y) plane (viz. a static 2-brane) embedded in a bulk space with one additional spatial dimension, with there being a perfect fluid on the sheet and no matter fields in the bulk. To maintain maximal 2-symmetry on the sheet we shall take the fluid on it to have constant energy density ρ and pressure p with non-vanishing energy-momentum tensor components $T^{\rm sheet}_{\mu\nu} = \tau_{\mu\nu}\delta(z)$ where

$$\tau_{00} = \rho \quad , \quad \tau_{11} = \tau_{22} = p \quad . \tag{6.1}$$

For this case there is no bulk energy-momentum tensor, and the Einstein equations take the form

$$G_{\mu\nu} = -\kappa_4^2 \tau_{\mu\nu}\delta(z) \quad . \tag{6.2}$$

The most general metric which satisfies the symmetry of this set-up is given as

$$ds^2 = dz^2 + a^2(z)[dx^2 + dy^2] - n^2(z)dt^2 \quad , \tag{6.3}$$

where without loss of generality we set $a(0) = 1$, $n(0) = 1$. For such a metric the non-trivial components of the Einstein tensor are readily given as

$$\begin{aligned} G_{11} = G_{22} &= -\frac{a}{n}[a''n + a'n' + an''] \quad , \\ G_{33} &= -\frac{a'}{a^2 n}[a'n + 2an'] \quad , \quad G_{00} = \frac{n^2}{a^2}[2aa'' + a'^2] \quad , \end{aligned} \tag{6.4}$$

where the prime denotes differentiation with respect to z. From the $G_{33} = 0$ equation we infer that $n = 1/a^{1/2}$. Revaluating the components of the remaining components of the Einstein tensor in this case yields, on setting $a(z) = f^{2/3}(z)$,

$$G_{11} = G_{22} = -\frac{f^{1/3}}{3}f'' \quad , \quad G_{00} = \frac{4}{3f^{5/3}}f'' \quad . \tag{6.5}$$

In source-free bulk away from the $\delta(z)$ singularity the most general solution to the bulk Einstein equation $G_{\mu\nu} = 0$ is thus given by an f which is linear in z, with the delta function singularity then requiring f to possess a term linear in $|z|$. From the $(0,0)$ component of Eq. (6.2) we thus obtain

$$a = \left(1 - \frac{3\kappa_4^2\rho|z|}{8}\right)^{2/3} \quad , \tag{6.6}$$

while from its $(1,1)$ component we obtain

$$a = \left(1 + \frac{3\kappa_4^2 p|z|}{2}\right)^{2/3} \quad . \tag{6.7}$$

While we thus obtain an exact solution for the model,[1] and while we see how a dependence on $|z|$ can nicely emerge in the brane case, nonetheless, consistency between these two conditions is not automatic, being in fact achievable for one and only one equation of state, viz. [Mannheim (2001a)]

$$p = -\frac{\rho}{4} \quad , \tag{6.8}$$

one without which the brane could not be static.[2]

To ensure that we made no mistake in obtaining this particular equation of state we rederive it using the junction conditions which take the form

$$K_{\mu\nu}(0^+) - K_{\mu\nu}(0^-) = -\kappa_4^2\left(\tau_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(z=0)\tau^{\alpha}{}_{\alpha}\right) \quad , \tag{6.9}$$

in the 2-brane case. With the normal to the brane being given by $n^\mu = (0,0,0,1)$, with the induced metric being given by $ds^2 = a^2(z)[dx^2 + dy^2] - n^2(z)dt^2$, and with the needed Christoffel symbols associated with the metric of Eq. (6.3) being given by $\Gamma^3_{11} = -aa'$, $\Gamma^3_{00} = -nn'$, in a solution where $a = (1 - 3\kappa_4^2\rho|z|/8)^{1/2} = 1/n^2$, the extrinsic curvature at the brane is found to evaluate to

$$K_{11} = K_{22} = -\frac{\kappa_4^2\rho}{4}\epsilon(z) \quad , \quad K_{00} = -\frac{\kappa_4^2\rho}{8}\epsilon(z) \quad , \tag{6.10}$$

i.e. to

$$K_{11}(0^+) = -K_{11}(0^-) = -\frac{\kappa_4^2\rho}{4} \quad , \quad K_{00}(0^+) = -K_{00}(0^-) = -\frac{\kappa_4^2\rho}{8} \quad . \tag{6.11}$$

[1] This is not actually the most general solution to the theory since the function f could also contain a term which is linear in z itself with a coefficient which is not fixed by the matter fields at all, with the full f then being given by $f = 1 - 3\kappa_4^2\rho|z|/8 + \alpha z$ where α is arbitrary.

[2] That this model possessed no static solution in the particular case of $p = -\rho$ was shown in [Vilenkin (1981)].

Then, with the quantity $\tau_{\mu\nu} - (1/2)g_{\mu\nu}\tau^{\alpha}{}_{\alpha}$ evaluating to

$$\tau_{11} - \frac{1}{2}g_{11}\tau^{\alpha}{}_{\alpha} = \frac{\rho}{2} \quad , \quad \tau_{00} - \frac{1}{2}g_{00}\tau^{\alpha}{}_{\alpha} = p + \frac{\rho}{2} \quad , \tag{6.12}$$

we see that the junction conditions also require $p = -\rho/4$.

As such, our analysis shows that it is only possible to obtain a static solution to this model if $p = -\rho/4$, to thus require a quite unusual form of matter on the brane, viz. matter with negative pressure, with such matter (more generally with equation of state $p = w\rho$ where w is some negative constant) being thought to potentially be of relevance to cosmology [Caldwell, Dave and Steinhardt (1998)] where it then goes under the generic name of quintessence. Now we had noted earlier that in the Randall-Sundrum static M_4^+ brane-world model the bulk and brane fields needed to be related according to Eq. (2.15), with it actually being a characteristic of brane-world set-ups that static solutions can generally only be achieved when there are some very specific relationships between the matter fields, and in Chapter 7 we shall show that the embedding of a static Robertson-Walker (RW) brane in an empty bulk is also only achievable for a quintessence fluid with a very specific equation of state. When these very special conditions do not hold it is possible for the solution to be time dependent.[3] However, of more interest to the brane world is that in certain circumstances it is also possible for the brane geometry to readjust into some new static solution, with this being found to be the case for both de Sitter (dS_4) and anti-de Sitter (AdS_4) branes, cases we will analyze in detail below in Chapter 8.

6.2 The embedded sheet Riemann tensor and its weak gravity limit

Even though the static 2-brane model discussed above is one in which the bulk is devoid of matter fields, it does not follow that in it the bulk is flat. In fact, explicit evaluation shows that associated with the metric of Eq. (6.3) are six non-vanishing Riemann tensor components, viz.

$$\begin{aligned} R_{1212} &= a^2 a'^2 \quad , \quad R_{1313} = R_{2323} = aa'' \quad , \\ R_{1414} &= R_{2424} = -aa'nn' \quad , \quad R_{3434} = nn'' \quad , \end{aligned} \tag{6.13}$$

so that in the solution R_{3434} for instance takes the value

$$R_{3434} = -\frac{\kappa_4^4\rho^2}{16}\left(1 - \frac{3\kappa_4^2\rho|z|}{8}\right)^{-8/3} + \frac{1}{4}\kappa_4^2\rho\delta(z) \quad , \tag{6.14}$$

and we especially note the presence here of a delta function term. This example thus explicitly shows how an embedding of brane matter fields in an empty bulk sets up a gravitational field in the bulk.

[3] A time dependent $p = -\rho$ 2-brane model was explicitly constructed in [Vilenkin (1983)] and we shall construct a time-dependent RW model below in Chapter 9.

The structure of the Riemann tensor given in Eqs. (6.13) and (6.14) is also of interest for a separate reason. Specifically, in the weak gravity limit in which κ_4^2 is small, we find that in the bulk, i.e. away from the $\delta(z)$ singularity, the Riemann tensor only begins in order κ_4^4. To lowest order in κ_4^2 then we thus recover the standard result that in the bulk the Newtonian gravity of an infinite sheet is a pure gauge artifact. However, while the true gravity of the sheet only begins in order κ_4^4 in the bulk, at the sheet itself we see that the Riemann tensor has a non-trivial piece of order κ_4^2 given by

$$R_{1313} = R_{2323} = -\frac{1}{2}\kappa_4^2\rho\delta(z) \quad , \quad R_{3434} = \frac{1}{4}\kappa_4^2\rho\delta(z) \quad . \tag{6.15}$$

In lowest order then the geometry associated with the embedding of a sheet in an empty bulk is not in fact flat at every point of the spacetime. Noting that the Israel junction conditions of Eq. (6.9) are also of order κ_4^2, we see that in lowest order a delta function distribution of matter on the sheet supports both a non-trivial jump in the extrinsic curvature at the sheet and a non-trivial Riemann tensor on it, with both then being physical, and with neither being a gauge artifact. It is in this way then that the Newtonian Gauss's Law of Eq. (5.5) is able to acquire physical significance — it is supported by a non-Newtonian Riemann tensor on the sheet even when gravity is weak.

While we thus see why Gauss's Law is not a gauge artifact, we note that there are differences in form between it and the weak gravity limit of Eq. (6.9). Specifically, if we define $g_{00} = -1 - 2\phi$, we find that in the solution of Eq. (6.6) the weak gravity potential is given by $\phi = \kappa_4^2\rho|z|/8$, whereas according to Eq. (5.4) the weak gravity Gauss' Law potential is given by $\phi = \kappa_4^2\sigma|z|/4$. While there is a lack of agreement between these two definitions of the weak gravity potential, we note that there is not any actual conflict since the two potentials correspond to different physical situations and can therefore not be compared. Specifically, in the covariant case the fluid on the sheet is to required to obey $p = -\rho/4$, while the Newtonian fluid associated with Gauss' Law is one which is to have an energy density σ and a pressure which is zero. Despite this difference, it is nonetheless still possible to relate the two cases if in the covariant case we endow the sheet with this pressureless σ fluid and an additional 2-dimensional network of cosmic strings, viz. one for which the energy and density are related by $p_s = -\rho_s/2$,[4] as this would yield for the full energy-momentum tensor on the sheet

$$\rho = \rho_s + \sigma \quad , \quad p = p_s = -\frac{1}{2}\rho_s = -\frac{1}{4}(\rho_s + \sigma) \quad , \tag{6.16}$$

[4]A general cosmic string is a vortex line pointing in a given spatial direction with a pressure p_s equal to the negative of its energy density. In a spacetime with d spatial dimensions a network of such strings would consist of an isotropic set of such strings each with this same pressure, with the net energy-momentum tensor associated with the network then being given by $T_{11} = T_{22} = \ldots = T_{dd} = p_s$, $T_{00} = -dp_s$, viz. one whose net energy density and pressure obey $p_s = -\rho_s/d$.

i.e.

$$\rho_s = \sigma \quad , \quad \rho = 2\sigma \quad . \tag{6.17}$$

The addition to a relativistic sheet of a cosmic string with an energy given by $\rho_s = \sigma$ thus enables us to secure stability of the static sheet in general while recovering the standard Newtonian gravity result when gravity is weak.

6.3 Analogy to the static Einstein universe

While it is initially surprising to find that the standard Newtonian result can only be recovered when the σ fluid on the sheet is augmented by a cosmic string network with the very special fine-tuned energy density $\rho_s = \sigma$, such a fine-tuning is actually familiar to physics in a somewhat different context, namely a similar such fine-tuning is required in the venerable static Einstein universe in which standard gravitational attraction is counter-balanced by the introduction of a repulsive cosmological constant. In the model the metric is given by the static RW metric $ds^2 = dr^2/(1-kr^2) + r^2 d\Omega - dt^2$ and the full energy-momentum tensor is given by $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$. The Einstein equations then take the form

$$G^0{}_0 = 3k = \kappa_4^2 \rho \quad , \quad G^i{}_j = k\delta^i_j = -\kappa_4^2 p \delta^i_j \quad , \tag{6.18}$$

to thus yield

$$p = -\frac{\rho}{3} \quad , \quad k = \frac{\kappa_4^2 \rho}{3} \quad . \tag{6.19}$$

Taking ρ to be positive thus entails that k would have to be positive and that p would have to be negative, with the Einstein universe thus being supportable by a fluid with $p = -\rho/3$ (see e.g. [Mannheim (2001a)]), viz. a 3-dimensional network of cosmic strings . However, rather than consider the source to be such a network of cosmic strings, Einstein instead decomposed the energy-momentum tensor into a two-fluid model, one an ordinary non-relativistic matter fluid with a density ρ_m and zero pressure, and the other a vacuum cosmological constant with energy density λ and pressure $-\lambda$, so that the full fluid was then given by

$$\rho = \rho_m + \lambda \quad , \quad p = -\lambda \quad . \tag{6.20}$$

The requirement that the full fluid obey $p = -\rho/3$ then requires the two fluids to be related according to the very specific fine-tuned relation

$$\lambda = \frac{\rho_m}{2} \quad . \tag{6.21}$$

As we thus see, static models typically require negative pressures and fine-tuning relations, and so we turn now to the study of the embedding of a static RW cosmology into a 5-dimensional bulk where an analogous such outcome will be found to occur.

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Chapter 7

Embedding of Static Robertson-Walker Branes and Negative Pressure

7.1 General structure of the Einstein equations in the RW case

For the embedding of a (not necessarily static) maximally 3-symmetric geometry in a 5-dimensional bulk, the most general metric which respects the maximal 3-symmetry is given by

$$ds^2 = b^2(w,t)dw^2 + 2c(w,t)dwdt + a^2(w,t)\left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega\right] - n^2(w,t)dt^2 \ . \tag{7.1}$$

With the two general coordinate transformations[1]

$$g^{ww} = g^{ij}\frac{\partial w}{\partial x^i}\frac{\partial w}{\partial x^j} = 1 \ , \ g^{tw} = g^{ij}\frac{\partial t}{\partial x^i}\frac{\partial w}{\partial x^j} = 0 \ , \tag{7.2}$$

in the 2-dimensional (x^1, x^2) space bringing the metric $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2$ to the form $ds^2 = dw^2 - n^2(w,t)dt^2$, without any loss of generality we can therefore rewrite the metric of Eq. (7.1) as

$$ds^2 = dw^2 + a^2(w,t)\left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega\right] - n^2(w,t)dt^2 \ , \tag{7.3}$$

with the most general embedded RW metric thus possessing only two independent metric coefficients.

A rewriting of these two coefficients according to

$$ds^2 = dw^2 + f(w,t)\left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega\right] - \frac{e^2(w,t)}{f(w,t)}dt^2 \tag{7.4}$$

will prove to be very convenient in the following since for this metric the components

[1]These relations are analogous to Hamilton-Jacobi equations in the (x^1, x^2) 2-space.

of the Einstein tensor are then given as

$$
\begin{aligned}
G^0{}_0 &= -\frac{3f''}{2f} + \frac{3k}{f} + \frac{3\dot{f}^2}{4e^2 f} \ , \\
G^1{}_1 = G^2{}_2 = G^3{}_3 &= -\frac{f''}{2f} - \frac{e''}{e} + \frac{k}{f} + \frac{\ddot{f}}{e^2} + \frac{\dot{f}^2}{4e^2 f} - \frac{\dot{e}\dot{f}}{e^3} \ , \\
G^5{}_5 &= -\frac{3f'e'}{2fe} + \frac{3k}{f} + \frac{3\ddot{f}}{2e^2} + \frac{3\dot{f}^2}{4e^2 f} - \frac{3\dot{e}\dot{f}}{2e^3} \ , \\
G^5{}_0 &= \frac{3e'\dot{f}}{2e^3} - \frac{3\dot{f}'}{2e^2} \ ,
\end{aligned}
\tag{7.5}
$$

where the dot and the prime denote derivatives with respect to t and w respectively. As such, the embedding associated with Eq. (7.3) does not in general yield a maximally 5-symmetric bulk, since the 6 components $C_{\mu\nu\mu\nu}$ ($\mu \neq \nu$) of the 5-dimensional Weyl tensor and the 4 components $C_{\mu 5 \mu 5}$ are all found to be proportional to C_{0505} where

$$
\begin{aligned}
C_{0505} = \frac{1}{8ef^3} \, [&4e^3 f f'' - 6e^3 f'^2 + 4e^3 f k + 6e^2 f e' f' \\
&-4e^2 f^2 e'' - 2ef^2 \ddot{f} + ef\dot{f}^2 + 2f^2 \dot{e}\dot{f}] \ .
\end{aligned}
\tag{7.6}
$$

Consequently, it is only for special values of $e(w,t)$ and $f(w,t)$ that an RW brane (either static or non-static) can embed in a conformally flat bulk such as AdS_5.

7.2 Static RW brane in an empty bulk

Since our immediate interest is in treating the static case, we shall discuss that case here and return to the non-static case below in Chapter 9.[2] Thus, for the embedding of a static RW brane in, first, an empty bulk, the only matter field will be a uniform, constant perfect fluid on the brane itself, with the Einstein equations being given by $G^M{}_N = -\kappa_5^2 T^M{}_N$ where

$$
T^M{}_N = \mathrm{diag}(-\rho, p, p, p, 0)\delta(w) \ , \tag{7.7}
$$

i.e. given by

$$
\begin{aligned}
G^0{}_0 &= -\frac{3f''}{2f} + \frac{3k}{f} = \kappa_5^2 \rho\delta(w) \ , \\
G^1{}_1 = G^2{}_2 = G^3{}_3 &= -\frac{f''}{2f} - \frac{e''}{e} + \frac{k}{f} = -\kappa_5^2 p\delta(w) \ , \\
G^5{}_5 &= -\frac{3f'e'}{2fe} + \frac{3k}{f} = 0 \ .
\end{aligned}
\tag{7.8}
$$

[2]The results presented in this chapter are based primarily on [Mannheim (2001a); Mannheim (2001b); Mannheim (2002)].

In the $k = 0$ case these equations are readily integrated, with the $G^0{}_0$ and $G^5{}_5$ equations respectively yielding[3]

$$f(w) = 1 - \frac{1}{3}\kappa_5^2\rho|w| \quad , \quad e(w) = 1 \quad . \tag{7.9}$$

However, consistency with the $G^1{}_1$ equation is not automatic, with it being found to require

$$p = -\frac{\rho}{3} \quad . \tag{7.10}$$

Thus just like the situation found for the static Einstein universe, an embedded RW brane can only be static when a very special equation of state is imposed, viz. one in which the fluid pressure is once again expressly negative.

Such a situation is found to persist if we take the 3-curvature on the brane to be non-zero, with the integration of Eq. (7.8) then readily yielding metric coefficients of the form

$$\begin{aligned} f(w) &= 1 - \frac{1}{3}\kappa_5^2\rho|w| + k|w|^2 \quad , \\ e(w) &= -\frac{1}{3}\kappa_5^2\rho + 2k|w| \quad , \end{aligned} \tag{7.11}$$

and a consistency relation

$$p = -\frac{\rho}{3} - \frac{12k}{\kappa_5^4\rho} \quad . \tag{7.12}$$

The sign of $p+\rho/3$ will thus fix the sign of k, and just like in the $k > 0$ static Einstein universe, the pressure needed to support an embedded static $k > 0$ universe would need be negative.

7.3 Static RW brane in a bulk with a cosmological constant

The need for constraints in the static case is found to persist even if we introduce a bulk cosmological constant into the model according to

$$T^M{}_N = \mathrm{diag}(-\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5) + \mathrm{diag}(-\rho, p, p, p, 0)\delta(w) \quad , \tag{7.13}$$

a situation in which the Einstein equations then take the form

$$\begin{aligned} G^0{}_0 &= -\frac{3f''}{2f} + \frac{3k}{f} = \kappa_5^2\rho\delta(w) + \kappa_5^2\Lambda_5 \quad , \\ G^1{}_1 &= G^2{}_2 = G^3{}_3 = -\frac{f''}{2f} - \frac{e''}{e} + \frac{k}{f} = -\kappa_5^2 p\delta(w) + \kappa_5^2\Lambda_5 \quad , \\ G^5{}_5 &= -\frac{3f'e'}{2fe} + \frac{3k}{f} = \kappa_5^2\Lambda_5 \quad . \end{aligned} \tag{7.14}$$

[3] In the static case a rescaling of the coordinates allows us to conveniently set $f(0) = 1$, $e(0) = 1$.

For this case the most general solution is given by

$$f(w) = \alpha e^{2b|w|} + \beta e^{-2b|w|} - \frac{k}{2b^2} \ , \quad e(w) = \alpha e^{2b|w|} - \beta e^{-2b|w|} \ ,$$
$$\frac{e^2(w)}{f(w)} = f(w) + \frac{k}{b^2} - \frac{4}{f(w)}\left(\alpha\beta - \frac{k^2}{16b^4}\right) \ , \tag{7.15}$$

where

$$b = +\left(\frac{-\Lambda_5\kappa_5^2}{6}\right)^{1/2} \tag{7.16}$$

is the AdS_5 scale parameter introduced in Eq. (2.5); with the enforcement of the junction conditions at the $\delta(w)$ singularity requiring the numerical coefficients α and β to obey

$$6b(\beta - \alpha) = \left(\alpha + \beta - \frac{k}{2b^2}\right)\kappa_5^2\rho \ ,$$
$$12b(\alpha + \beta) = (\alpha - \beta)\kappa_5^2(\rho + 3p) \ . \tag{7.17}$$

On rescaling the radial coordinate and redefining k as $k/f(0)$, we can then conveniently set $f(0) = 1$ (viz. $\alpha + \beta - k/2b^2 = 1$), with the consistency of the junction conditions then being found to impose the constraint

$$\kappa_5^4\rho(\rho + 3p) = 12\kappa_5^2\Lambda_5 - 36k \tag{7.18}$$

on the matter fields. For negative Λ_5 and non-negative k the quantity $\rho(\rho + 3p)$ thus has to be negative definite, so that for positive ρ we find for instance that the pressure on a static $k = 0$ brane embedded in a $\Lambda_5 < 0$ bulk has to be even more negative than that on the same static $k = 0$ brane when embedded in an empty bulk. The need for negative pressure to enforce a static solution is thus seen to be quite ubiquitous.

As we see, to enforce a static solution in the RW case, then just as in the analogous Randall-Sundrum M_4^+ brane-world solution, we in general need some highly specific fine-tuning relations between brane and bulk fields, with Eq. (7.18) requiring

$$\kappa_5^2\rho(\rho + 3p) = 12\Lambda_5 \tag{7.19}$$

when $k = 0$, a relation which precisely recovers the Randall-Sundrum fine-tuning condition of Eq. (2.15) when we require the fluid on the brane to obey $\rho = -p = \lambda$ (yet another situation in which the brane pressure has to be negative). However, unlike the M_4^+ case, the solution of Eq. (7.15) does not consist purely of converging exponentials in $|w|$ unless α and k just both happen to vanish. Since the imposition of $\alpha = 0$, $k = 0$ on Eq. (7.17) leads us right back to $\rho = -p$ and the M_4^+ brane-world case, we see that in cases more general than M_4^+ obtaining only purely convergent warp factors is not at all guaranteed.

Moreover, unlike the M_4^+ case, in the general $\Lambda_5 < 0$ RW case the bulk is not necessarily an AdS_5 one since the bulk Weyl tensor need not necessarily vanish.[4] However, while not being conformal to flat in general, it is nonetheless possible to find specific values for the parameters in Eq. (7.15) for which the bulk then would be, with the Weyl tensor of Eq. (7.6) being found to vanish in the solution of Eq. (7.15) when the coefficients in it obey

$$\alpha\beta - \frac{k^2}{16b^4} = 0 \ , \tag{7.20}$$

a condition under which the matter fields would then have to obey

$$\kappa_5^2\rho(2\rho + 3p) = 6\Lambda_5 \ . \tag{7.21}$$

Under such a constraint a static $k \neq 0$ RW brane then does embed in an AdS_5 bulk.[5] Even with such a constraint on the matter fields, this solution is still of interest because it represents an exact brane-world solution with an AdS_5 bulk, one which, according to Eq. (7.20), necessarily has both convergent and divergent warp factors when $k \neq 0$.[6] While we will find below that we will be able to avoid fine-tuning relations when the solutions are taken to be time dependent, as we shall also see however, the appearance of just convergent warp factors alone will prove to be the exception rather than the rule. However, as we shall also see, in certain cases it will still be possible to obtain gravitational fluctuations which are localized to the brane, and in order to find such cases we proceed first to the construction of some other exact brane-world solutions.

---

[4]When the brane geometry just happens to be maximally 4-symmetric (such as in the M_4^+ brane-world set-up where the most general metric allowed by the maximal 4-symmetry takes the form $ds^2 = dw^2 + e^{2A(w)}(dx^2 + dy^2 + dz^2 - dt^2)$), the bulk Weyl tensor then automatically vanishes (the metric being conformal to flat for any choice of $A(w)$).

[5]Once the bulk Weyl tensor is set to zero, the imposition of bulk Einstein equations with a $\Lambda_5 < 0$ source then does force the bulk to be AdS_5.

[6]With the imposition of Eq. (7.20) entailing that $\alpha^{1/2} = [(1 + k/b^2)^{1/2} + 1]/2$, $\beta^{1/2} = [(1 + k/b^2)^{1/2} - 1]/2$, we find that the metric coefficients associated with a static RW brane embedded in an AdS_5 bulk are then given as $f(w) = [\alpha^{1/2}e^{b|w|} - \beta^{1/2}e^{-b|w|}]^2 = [\cosh(b|w|) + (1 + k/b^2)^{1/2}\sinh(b|w|)]^2$, $e^2(w)/f(w) = [\alpha^{1/2}e^{b|w|} + \beta^{1/2}e^{-b|w|}]^2 = (1 + k/b^2)[\cosh(b|w|) + (1 + k/b^2)^{-1/2}\sinh(b|w|)]^2$, to thus give a metric which, following a trivial time rescaling, then recovers the $a(t) = 1$ limit of Eq. (3.18).

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Chapter 8

Embedding of de Sitter and Anti-de Sitter Branes in AdS_5

8.1 Embedding of a de Sitter brane in AdS_5

As well as the Randall-Sundrum M_4^+ and M_4^- brane-world models themselves there are two other possible maximally 4-symmetric brane-world set-ups, viz. those associated with a dS_4 or an AdS_4 geometry on the brane, with both of these branes being embeddable in AdS_5. For the dS_4 brane case first, the most general 5-dimensional metric compatible with a dS_4 symmetry can be taken to be given as

$$ds^2 = dw^2 + e^{2A(w)}[e^{2Ht}(dx^2 + dy^2 + dz^2) - dt^2] \ , \tag{8.1}$$

where $A(w)$ is for the moment an arbitrary function of w. For any arbitrary such $A(w)$ the 5-dimensional Weyl tensor associated with this metric is found to vanish identically, while the non-trivial components of the 5-dimensional Einstein tensor are given by

$$\begin{aligned} &G^0{}_0 = G^1{}_1 = G^2{}_2 = G^3{}_3 = 3e^{-2A}H^2 - 6A'^2 - 3A'' \ , \\ &G^5{}_5 = 6e^{-2A}H^2 - 6A'^2 \ . \end{aligned} \tag{8.2}$$

To support such an Einstein tensor the energy-momentum tensor has to consist of cosmological constants in both brane and bulk, viz.

$$T^M{}_N = \mathrm{diag}(-\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5) + \mathrm{diag}(-\lambda, -\lambda, -\lambda, -\lambda, 0)\delta(w) \ , \tag{8.3}$$

with the Einstein equations thus yielding

$$\begin{aligned} &3e^{-2A}H^2 - 6A'^2 - 3A'' = \kappa_5^2\lambda\delta(w) + \kappa_5^2\Lambda_5 \ , \\ &6e^{-2A}H^2 - 6A'^2 = \kappa_5^2\Lambda_5 \ . \end{aligned} \tag{8.4}$$

The general solution to these equations can readily be found [DeWolfe, Freedman, Gubser and Karch (2000); Kim and Kim (2000)] to be given by

$$e^A = \frac{H}{b}\sinh(\sigma - \epsilon(\lambda)b|w|) \ , \tag{8.5}$$

where $\sinh\sigma = b/H$, where $\epsilon(\lambda)$ is the sign of the brane tension λ, and where the dS_4 parameter H is given by

$$H = \frac{\kappa_5}{6}\left(\kappa_5^2\lambda^2 + 6\Lambda_5\right)^{1/2} \quad . \tag{8.6}$$

Comparing with the Randall-Sundrum M_4^+ fine-tuning condition of Eq. (2.15), viz. $\kappa_5^2\lambda^2 + 6\Lambda_5 = 0$, we thus see that when the quantity $\kappa_5^2\lambda^2 + 6\Lambda_5$ is positive rather than zero the geometry on the brane becomes de Sitter with no fine-tuning of matter fields then being needed.

With de Sitter brane solutions being possible for either sign of the brane tension (just like the M_4^+ and M_4^- brane-world cases), we shall refer to these two solutions as $dS_4^{\pm}$. Of these two de Sitter brane solutions we note that since the quantity e^A can never be negative for real A,[1] in the positive tension dS_4^+ case the coordinate $|w|$ is not allowed to be greater than σ/b, with $|w| = \sigma/b$ thus being a boundary for the dS_4^+ spacetime in the warp factor coordinate system, an issue we shall examine in more detail in Chapter 10 when we consider the embeddings of brane worlds in $M(4,2)$. With the dS_4^+ (w,t) plane null geodesics which are at the brane at time $t = 0$ being given by $e^{-Ht}\tanh(\sigma/2) = \tanh(\sigma/2 - b|w|/2)$, we see that it would take an infinite amount of time for a null signal to travel from the brane to $|w| = \sigma/b$, with $|w| = \sigma/b$ thus being a horizon for observers on the brane. In contrast, we note that for dS_4^- there is no constraint on the allowed range of $|w|$ ($\sinh(\sigma + b|w|)$ is never negative), with $|w|$ thus being able to range all the way from $|w| = 0$ to $|w| = \infty$ in this case. And since the null geodesics are now given by $e^{Ht}\tanh(\sigma/2) = \tanh(\sigma/2 + b|w|/2)$, a null signal can traverse dS_4^- in a finite time, with there being no horizon for brane observers. The dS_4^+ brane world is thus a globally hyperbolic spacetime with a horizon, while the dS_4^- brane world is a globally non-hyperbolic spacetime which is horizon-free.

8.2 Embedding of an anti-de Sitter brane in AdS_5

The treatment of the AdS_4 brane case is analogous to the dS_4 brane case. The most general 5-dimensional metric compatible with an AdS_4 symmetry can be taken to be given as[2]

$$ds^2 = dw^2 + e^{2A(w)}[dx^2 + e^{2Hx}(dy^2 + dz^2 - dt^2)] \quad , \tag{8.7}$$

a metric for which the 5-dimensional Weyl tensor again vanishes identically for arbitrary function $A(w)$. For this metric the non-trivial components of the 5-

[1]While e^{2A} is of course always positive, it is the sign of A itself which is fixed by Eq. (8.4) since the A'' term (viz. the one which generates the $\kappa_5^2\lambda\delta(w)$ term) in it is linear in A.

[2]The AdS_4 parameter H is related to the dS_4 one by $H(AdS) = -iH(dS)$ since $x(AdS)$ is equivalent to $it(dS)$.

dimensional Einstein tensor are given by

$$G^0{}_0 = G^1{}_1 = G^2{}_2 = G^3{}_3 = -3e^{-2A}H^2 - 6A'^2 - 3A'' \ ,$$
$$G^5{}_5 = -6e^{-2A}H^2 - 6A'^2 \ , \tag{8.8}$$

and they can also be supported by brane and bulk cosmological constants, with the Einstein equations then yielding

$$-3e^{-2A}H^2 - 6A'^2 - 3A'' = \kappa_5^2\lambda\delta(w) + \kappa_5^2\Lambda_5 \ ,$$
$$-6e^{-2A}H^2 - 6A'^2 = \kappa_5^2\Lambda_5 \ . \tag{8.9}$$

The general solution to these equations is readily found to be given by [DeWolfe, Freedman, Gubser and Karch (2000)]

$$e^A = \frac{H}{b}\cosh(\epsilon(\lambda)b|w| - \sigma) \ , \tag{8.10}$$

where $\cosh\sigma = b/H$ and where the AdS_4 parameter H is given by

$$H = \frac{\kappa_5}{6}\left(-6\Lambda_5 - \kappa_5^2\lambda^2\right)^{1/2} \ . \tag{8.11}$$

Comparing with the Randall-Sundrum M_4^+ fine-tuning condition, this time we see that when the quantity $\kappa_5^2\lambda^2 + 6\Lambda_5$ is negative the geometry on the brane becomes anti-de Sitter with no fine-tuning of matter fields being needed once again.

With anti-de Sitter brane solutions being possible for either sign of the brane tension, we shall refer to these two solutions as $AdS_4^{\pm}$, and note that unlike the $dS_4^{\pm}$ situation, for neither of the AdS_4^+ and AdS_4^- brane worlds is there any bound on the allowed values of $|w|$ ($\cosh(\epsilon(\lambda)b|w| - \sigma)$ is never negative), and neither of the spacetimes is globally hyperbolic since both can be traversed in a finite time (the null geodesics are of the form $Ht = \arcsin[\tanh(\epsilon(\lambda)b|w| - \sigma)] + \arcsin[\tanh\sigma]$). For any given assignment of values for λ and Λ_5 then, the geometry on the brane will be M_4, dS_4 or AdS_4 dependent on whether the quantity $\kappa_5^2\lambda^2 + 6\Lambda_5$ is zero, positive or negative, with there thus being some maximally 4-symmetric brane embeddable in AdS_5 no matter what values the brane and bulk cosmological constants might take.[3]

With the $M_4^{\pm}$ set-up metrics being given by

$$ds^2 = dw^2 + e^{-2\epsilon(\lambda)b|w|}[dx^2 + dy^2 + dz^2 - dt^2] \ , \tag{8.12}$$

we see that all six of the maximally 4-symmetric brane-world metrics ($M_4^{\pm}$, $dS_4^{\pm}$, $AdS_4^{\pm}$, each with its own $A(w)$) can be written in the generic separable form

$$ds^2 = dw^2 + e^{2A(w)}q_{\mu\nu}(x^\lambda)dx^\mu dx^\nu \ , \tag{8.13}$$

[3]The metrics associated with the solutions of Eqs. (8.5) and (8.10) are respectively the Z_2-symmetric generalizations of the metrics of Eqs. (3.16) and (3.17).

where each $q_{\mu\nu}(x^\lambda)$ is the induced metric at $w = 0$ ($i_{\mu\nu}(w = 0) = e^{2A(w=0)}q_{\mu\nu} = q_{\mu\nu}$). Evaluating the induced Einstein tensor $\tilde{G}_{\mu\nu}$ on the brane in each such $q_{\mu\nu}$ is found to yield

$$\tilde{G}_{\mu\nu}(M_4) = 0 \quad , \quad \tilde{G}_{\mu\nu}(dS_4) = 3H^2 q_{\mu\nu} \quad , \quad \tilde{G}_{\mu\nu}(AdS_4) = -3H^2 q_{\mu\nu} \quad . \tag{8.14}$$

Thus in terms of the effective Einstein equation

$$\tilde{G}_{\mu\nu} = \Lambda_4 q_{\mu\nu} \tag{8.15}$$

on the brane, we see that the $M_4^{\pm}$, $dS_4^{\pm}$ and $AdS_4^{\pm}$ set-ups correspond to brane worlds in which an observer on the brane sees an effective cosmological constant given by $\Lambda_4 = 0$, $\Lambda_4 = 3H^2 > 0$ and $\Lambda_4 = -3H^2 < 0$ in the respective cases, viz. one given in all cases by

$$\Lambda_4 = \frac{\kappa_5^2}{12}(\kappa_5^2\lambda^2 + 6\Lambda_5) = \frac{1}{12}(\kappa_5^4\lambda^2 - 36b^2) \quad . \tag{8.16}$$

In the brane world it is thus the residual Λ_4 which serves as the cosmological constant which is to control cosmology on the brane, and not λ, the cosmological constant carried by the brane matter fields themselves. Thus while λ would control cosmology in the non-embedded case, it is Λ_4 which is to do so in the brane case, to thus allow for a Λ_4 which could differ (and possibly even quite markedly) from the λ used in standard (non-brane-world) cosmology.

Of the six possible maximally 4-symmetric brane-world set-ups only one, viz. M_4^+, has a geometry for which the e^{2A} warp factor is converging at $|w| = \infty$. However, of the remaining five, for one of them, viz. dS_4^+, there is a horizon at a finite value of w, viz. $|w| = \sigma/b$. Since the dS_4^+ warp factor $(H^2/b^2)\sinh^2(\sigma - b|w|)$ falls monotonically all the way from the brane to $|w| = \sigma/b$, within the causally relevant region the dS_4^+ set-up thus possesses a strictly convergent warp factor [Garriga and Sasaki (2000)]. As we shall see below, the structure of gravitational fluctuations in the M_4^+ and dS_4^+ cases will thus be somewhat different from the structure associated with the other four maximally 4-symmetric brane cases. Having studied the geometries associated with the embeddings de Sitter and anti-de Sitter branes in an AdS_5 bulk, we turn next to the embedding of time dependent RW branes in the same bulk.

Chapter 9

Embedding of Non-Static Robertson-Walker Branes in AdS_5

9.1 Embedding an RW brane in a bulk with a cosmological constant

In the presence of the source of Eq. (7.13) with a now time-dependent $\rho(t)$ and $p(t)$ the Einstein equations for the embedding of a non-static maximally 3-symmetric geometry in a general (viz. not necessarily AdS_5) 5-dimensional bulk with a non-vanishing bulk cosmological constant Λ_5 follow from Eq. (7.5) and take the form

$$\begin{aligned}
G^0{}_0 &= -\frac{3f''}{2f} + \frac{3k}{f} + \frac{3\dot{f}^2}{4e^2 f} = \kappa_5^2 \rho\delta(w) + \kappa_5^2\Lambda_5 \ , \\
G^1{}_1 &= G^2{}_2 = G^3{}_3 = -\frac{f''}{2f} - \frac{e''}{e} + \frac{k}{f} + \frac{\ddot{f}}{e^2} + \frac{\dot{f}^2}{4e^2 f} - \frac{\dot{e}\dot{f}}{e^3} = -\kappa_5^2 p\delta(w) + \kappa_5^2\Lambda_5 \ , \\
G^5{}_5 &= -\frac{3f'e'}{2fe} + \frac{3k}{f} + \frac{3\ddot{f}}{2e^2} + \frac{3\dot{f}^2}{4e^2 f} - \frac{3\dot{e}\dot{f}}{2e^3} = \kappa_5^2\Lambda_5 \ , \\
G^5{}_0 &= \frac{3e'\dot{f}}{2e^3} - \frac{3\dot{f}'}{2e^2} = 0 \ .
\end{aligned} \tag{9.1}$$

To solve these equations it is most straightforward to solve them first in the bulk and to then impose the junction conditions at the brane afterwards, with general solutions having been given by a variety of authors [Csaki, Graesser, Kolda and Terning (1999); Cline, Grojean and Servant (1999); Binetruy, Deffayet and Langlois (2000); Binetruy, Deffayet, Ellwanger and Langlois (2000)]. The non-trivial ($\dot{f} \neq 0$) vanishing of $G^5{}_0$ entails that

$$e = A(t)\dot{f} \ , \tag{9.2}$$

where $A(t)$ is an arbitrary function of t. Setting $G^0{}_0$ equal to $-6b^2$ in the bulk entails that

$$f'' = \frac{1}{2A^2} + 2k + 4b^2 f \ , \tag{9.3}$$

an equation whose integration then yields

$$f = -\frac{1}{8A^2b^2} - \frac{k}{2b^2} + \alpha e^{2b|w|} + \beta e^{-2b|w|},$$
$$e = \frac{\dot{A}}{4A^2b^2} + A\dot{\alpha} e^{2b|w|} + A\dot{\beta} e^{-2b|w|}, \tag{9.4}$$

where α and β are arbitrary t-dependent integration functions. Similarly, setting ${G^5}_5$ equal to $-6b^2$ in the bulk additionally requires that $f(w,t)$ obey

$$f'^2 - 4kf - 4b^2f^2 - \frac{f}{A^2} = B(|w|) \ , \tag{9.5}$$

where $B(|w|)$ is for the moment an arbitrary integration function of $|w|$ which must be continuous at the brane;[1] with compatibility of Eq. (9.5) with Eq. (9.3) then entailing that B must actually be constant. The general bulk solution[2] is thus one in which the time dependent functions $\alpha(t)$ and $\beta(t)$ are related according to

$$\frac{(1+4A^2k)^2}{16A^4} - 16b^4\alpha\beta = b^2B. \tag{9.6}$$

Imposition of the Israel junction conditions at the brane requires satisfying the $\delta(w)$ constraints in the ${G^0}_0$ and ${G^1}_1$ equations associated with the solution of Eq. (9.4), and yields

$$6b(\beta - \alpha) = f(0,t)\kappa_5^2\rho, \quad 12b(\dot{\alpha} - \dot{\beta}) = \dot{f}(0,t)\kappa_5^2(\rho + 3p) \ . \tag{9.7}$$

With manipulation of Eq. (9.7) yielding

$$2f(0,t)\dot{\rho} + 3\dot{f}(0,t)(\rho + p) = 0 \ , \tag{9.8}$$

we see that the junction conditions entail the standard cosmological conservation conditions for the fluid on the brane.[3] With regard to the general solution, we find that though there is no longer any fine-tuning condition on the bulk and brane matter fields, nonetheless, just as in the static RW case, there is, in general (i.e. for completely arbitrary $\rho(t)$, $p(t)$), no asymptotic suppression of the metric coefficients, with the presence of both converging and diverging warp factors persisting in the non-static case.

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[1] With $B(|w|)$ being a general integration function, it could in principle take different values on the two sides of the brane, but because it is related to f'^2 it in fact does not.

[2] With the kinematic constraint ${G^5}_5 - {G^0}_0/2 - 3{G^3}_3/2 = 3(ef'' + fe'' - e'f')/2ef = 6b^2 - (1/2)\kappa_5^2(\rho - 3p)\delta(w)$ obeyed by the various components of the Einstein tensor being found to be satisfied identically in the bulk by the solution of Eqs. (9.4) and (9.6), in the bulk the ${G^3}_3$ Einstein equation is found to contain no additional information.

[3] For an embedded RW brane the covariant conservation of the 5-dimensional energy-momentum tensor, viz. ${T^{MN}}_{;N} = 0$, thus reduces to Eq. (9.8).

9.2 Embedding an RW brane in an AdS_5 bulk

While the above analysis describes the treatment of an RW brane embedded in a general 5-dimensional bulk with a cosmological constant, of more interest is the case where the bulk actually is AdS_5 even in the presence of the embedding. Specifically, requiring the bulk Weyl tensor of Eq. (7.6) to vanish obliges the metric coefficients in the solution to have to additionally obey [Mannheim (2001b)]

$$2ff'' - f'^2 - 4b^2f^2 = 0 \tag{9.9}$$

in the bulk, to thus require that the parameter B actually be zero, with α and β then having to be related to each other according to

$$\frac{(1+4A^2k)^2}{16A^4} = 16b^4\alpha\beta \ , \tag{9.10}$$

a situation in which the metric coefficient $f(w,t)$ is then found to simplify to

$$f(w,t) = (\alpha^{1/2}e^{b|w|} - \beta^{1/2}e^{-b|w|})^2 \ . \tag{9.11}$$

From Eq. (9.10) we see that the product $\alpha\beta$ must thus be non-negative (and even necessarily greater than zero in the $k > 0$ case). Hence even in the time dependent case, we see that there is still no exponential suppression of the geometry for an RW brane embedded in an AdS_5 bulk, with the situation for RW branes embedded in AdS_5 thus being quite similar to that found earlier for the embedded dS_4 and AdS_4 branes.

With the introduction of two new time dependent functions $a(t)$ and $G(t)$ defined via

$$\alpha^{1/2} = \frac{a(G-1)}{2} \ , \ \beta^{1/2} = \frac{a(G+1)}{2} \ , \tag{9.12}$$

for the case of an RW embedding in AdS_5 the metric coefficients $f(w,t)$ and $e^2(w,t)/f(w,t)$ may be brought to the form

$$\begin{aligned} f(w,t) &= a^2\left[\cosh(b|w|) - G\sinh(b|w|)\right]^2 \ , \\ \frac{e^2(w,t)}{f(w,t)} &= 4A^2\dot{a}^2\left[\cosh(b|w|) - F\sinh(b|w|)\right]^2 \ , \end{aligned} \tag{9.13}$$

where

$$F = G + \frac{a\dot{G}}{\dot{a}} \ . \tag{9.14}$$

With Eq. (9.10) entailing that $A(t)$ can be re-expressed as

$$A = \frac{1}{2[b^2a^2(G^2-1)-k]^{1/2}} \ , \tag{9.15}$$

we see that with a resetting of the time according to

$$dt' = \frac{\dot{a}dt}{[b^2a^2(G^2-1)-k]^{1/2}} = \frac{da}{[b^2a^2(G^2-1)-k]^{1/2}} \quad , \tag{9.16}$$

the metric can then be brought [Mannheim (2002)] to the convenient form[4]

$$\begin{aligned} ds^2 = &\ dw^2 - \left[\cosh(b|w|) - F\sinh(b|w|)\right]^2 dt'^2 \\ &+ a^2\left[\cosh(b|w|) - G\sinh(b|w|)\right]^2 \left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega\right] \quad , \end{aligned} \tag{9.17}$$

where

$$G = \frac{1}{ba}\left[\left(\frac{da}{dt'}\right)^2 + k + b^2a^2\right]^{1/2} \quad , \tag{9.18}$$

$$F = G + a\frac{dG}{da} = G + a\frac{dG}{dt'}\frac{dt'}{da} = \frac{1}{b^2aG}\left[\frac{d^2a}{dt'^2} + b^2a\right] \quad . \tag{9.19}$$

The redefinition $t' = t$ then recovers the metric given earlier as Eq. (3.18). With the junction conditions taking the form

$$6bG = \kappa_5^2\rho \quad , \quad 6bF = -\kappa_5^2(2\rho + 3p) \quad , \tag{9.20}$$

given an assignment of an appropriate $\rho(t)$ and $p(t)$ the dynamics of an RW brane embedded in AdS_5 is then completely specified.

The metric of Eq. (9.17) possesses two noteworthy features. First, we see that unlike the previously derived $dS_4^{\pm}$ and $AdS_4^{\pm}$ brane-world metrics, the RW brane-world metric coefficients are not separable functions of $|w|$. With these coefficients depending on both $|w|$ and t, determining the bulk null geodesics is not at all as straightforward as it was for metrics such as the dS_4^{+} one given in Chapter 8. However, it turns out that it is still possible to determine them analytically, and we shall do so below in Chapter 11 following the construction we present in Chapter 10 of the RW brane-world metric of Eq. (9.17) via an embedding of the entire 5-dimensional brane plus bulk system into the flat $M(4,2)$ into which AdS_5 itself can embed. With the RW brane-world metric coefficients not being separable, we note also that at the present time there is no treatment of gravitational fluctuations around RW branes which is as complete as the treatment of fluctuations around maximally 4-symmetric branes which is to be presented later in this monograph.

The second significant feature of the metric of Eq. (9.17) is that the specifying of $G(t)$ by the junction conditions leads to an evolution equation for the expansion radius of the form

$$\dot{a}^2 + k = \frac{a^2\kappa_5^4\rho^2}{36} - b^2a^2 \quad , \tag{9.21}$$

[4]In this form the induced metric on a brane at $w = 0$ becomes none other than a standard 4-dimensional cosmological RW metric with expansion radius $a(t')$.

an evolution equation which actually differs quite markedly from the Friedmann evolution equation of standard 4-dimensional cosmology, and in particular in the appearance of terms quadratic in the energy density ρ. The presence of such quadratic terms was first noted in [Binetruy, Deffayet and Langlois (2000)], and was subsequently shown [Shiromizu, Maeda and Sasaki (2000)] to actually be generic to brane physics, with, as we shall show in detail in Chapter 12, the Einstein equations on an embedded brane always being different from the unembedded ones, to thus provide a possible window on extra dimensions.

9.3 Difference between embedding RW and Schwarzschild branes

As such, the metrics we have presented so far ($M_4^{\pm}$, $dS_4^{\pm}$, $AdS_4^{\pm}$ and static and non-static RW) appear to constitute the entire set of known exact brane-world solutions in which a brane containing matter fields is embedded in a bulk containing a cosmological constant alone. While there are other known exact brane-world solutions, they typically require additional matter fields in the bulk. Thus while the metric [Brecher and Perry (2000)]

$$ds^2 = dw^2 + e^{-2b|w|}\left[\frac{dr^2}{(1-\beta/r)} + r^2 d\Omega_2 - \left(1 - \frac{\beta}{r}\right) dt^2\right] \tag{9.22}$$

is an exact solution to the 5-dimensional Einstein equations with bulk and brane cosmological constants which obey the fine-tuning condition $\kappa_5^2\lambda^2 + 6\Lambda_5 = 0$, just like the familiar 4-dimensional Schwarzschild solution, this particular metric actually only applies exterior to the mass sources which are needed to generate it. However, unlike the 4-dimensional Schwarzschild solution which is supported by only one $\delta^3(r)$ mass source at $\bar{r} = 0$, the solution of the 5-dimensional Eq. (9.22) has to be supported by an entire distribution of such point mass sources, one on each and every w slice, i.e. by point mass sources both on the brane and in the bulk. A similar matter field requirement is found for the analogous de Sitter and anti-de Sitter generalizations of the metric of Eq. (9.22), viz.

$$ds^2 = dw^2 + e^{2A}\left[\frac{dr^2}{(1-\beta/r \mp H^2r^2)} + r^2 d\Omega_2 - \left(1 - \frac{\beta}{r} \mp H^2 r^2\right) dt^2\right] \quad , \tag{9.23}$$

where the warp factors are as given earlier in Chapter 8 in the respective $\kappa_5^2\lambda^2 + 6\Lambda_5 > 0$ and $\kappa_5^2\lambda^2 + 6\Lambda_5 < 0$ cases. As regards a brane-world set-up in which a single point mass source is to be placed on a brane which is embedded in a bulk in which no point mass sources are to be present, no exact solution is currently known, and at the present time problems such as these (viz. the ones of most relevance for us if we are on a brane) are only amenable to the perturbative treatment which will be presented later on in this monograph.

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Chapter 10

Embedding Robertson-Walker Brane Worlds in $M(4,2)$

10.1 Embedding a $k > 0$ RW brane in $M(4,2)$

An AdS_5 spacetime can be described by the flat $M(4,2)$ metric

$$ds^2 = dW^2 + dX^2 + dY^2 + dZ^2 - dU^2 - dV^2 \tag{10.1}$$

as subject to the constraint

$$U^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = \ell^2 \ , \tag{10.2}$$

where in terms of our previous notation the radius ℓ is given by $\ell = 1/b$. For the typical case of a $k > 0$ RW brane, viz. a brane with topology S^3, we may introduce coordinates

$$\begin{aligned} X &= k^{1/2}Rr\ \sin\theta\ \cos\phi \ , \ Y = k^{1/2}Rr\ \sin\theta\ \sin\phi \ , \\ Z &= k^{1/2}Rr\ \cos\theta \ , \ W = R(1 - kr^2)^{1/2} \ , \end{aligned} \tag{10.3}$$

where $R = (W^2 + X^2 + Y^2 + Z^2)^{1/2}$, so that the metric and the constraint then take the form

$$ds^2 = dR^2 + kR^2\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - dU^2 - dV^2 \ , \tag{10.4}$$

$$U^2 + V^2 - R^2 = \ell^2 \ . \tag{10.5}$$

To embed first just the brane itself, on introducing[1] the 3-vector

$$\eta_1 = \left(\frac{k\ell^2 + a^2}{k}\right)^{1/2} \cos\beta(t) \ , \ \eta_2 = \left(\frac{k\ell^2 + a^2}{k}\right)^{1/2} \sin\beta(t) \ , \ \eta_3 = \frac{a}{k^{1/2}} \ , \tag{10.6}$$

[1] The work presented in this chapter and the next is drawn primarily from [Guth, Kaiser, Mannheim and Nayeri (2004a)].

where $a(t)$ is an arbitrary function of t and where

$$\beta(t) = k^{1/2} \int^t dt \frac{[\ell^2 \dot{a}^2 + k\ell^2 + a^2]^{1/2}}{[k\ell^2 + a^2]} \quad , \tag{10.7}$$

we directly find, upon making the identification

$$U = \eta_1 \quad , \quad V = \eta_2 \quad , \quad R = \eta_3 \quad , \tag{10.8}$$

that the metric of Eq. (10.4) then takes the form

$$ds^2 = a^2 \left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] - dt^2 \quad . \tag{10.9}$$

We recognize the metric of Eq. (10.9) as being that of a conventional closed RW cosmology with general scale factor $a(t)$.

10.2 Embedding a $k > 0$ RW brane and AdS_5 bulk in $M(4,2)$

To embed both the brane and an AdS_5 bulk, we introduce

$$\xi_1 = \frac{d\eta_1}{dt} = \frac{a\dot{a}\eta_1}{(k\ell^2 + a^2)} - \eta_2\dot{\beta} \ , \ \xi_2 = \frac{d\eta_2}{dt} = \frac{a\dot{a}\eta_2}{(k\ell^2 + a^2)} + \eta_1\dot{\beta} \ , \ \xi_3 = \frac{d\eta_3}{dt} = \frac{\dot{a}}{k^{1/2}} \quad , \tag{10.10}$$

and

$$\begin{aligned}
\chi_1 &= \eta_2\xi_3 - \eta_3\xi_2 = \frac{k^{1/2}\ell^2\dot{a}\eta_2}{(k\ell^2 + a^2)} - \frac{a\eta_1\dot{\beta}}{k^{1/2}} \quad , \\
\chi_2 &= \eta_3\xi_1 - \eta_1\xi_3 = -\frac{k^{1/2}\ell^2\dot{a}\eta_1}{(k\ell^2 + a^2)} - \frac{a\eta_2\dot{\beta}}{k^{1/2}} \quad , \\
\chi_3 &= -(\eta_1\xi_2 - \eta_2\xi_1) = -\frac{(k\ell^2 + a^2)\dot{\beta}}{k} \quad ,
\end{aligned} \tag{10.11}$$

quantities which are kinematically found to obey

$$\begin{aligned}
&\eta_1^2 + \eta_2^2 - \eta_3^2 = \ell^2 \quad , \quad \xi_1^2 + \xi_2^2 - \xi_3^2 = 1 \quad , \quad \chi_1^2 + \chi_2^2 - \chi_3^2 = -\ell^2 \quad , \\
&\eta_1\xi_1 + \eta_2\xi_2 - \eta_3\xi_3 = 0 \quad , \quad \eta_1\chi_1 + \eta_2\chi_2 - \eta_3\chi_3 = 0 \quad , \quad \xi_1\chi_1 + \xi_2\chi_2 - \xi_3\chi_3 = 0 \quad , \\
&\chi_1^2 - \ell^2\xi_1^2 - \eta_1^2 = -\ell^2 \quad , \quad \chi_2^2 - \ell^2\xi_2^2 - \eta_2^2 = -\ell^2 \quad , \quad \chi_3^2 - \ell^2\xi_3^2 - \eta_3^2 = \ell^2 \quad , \\
&\chi_1\chi_2 - \eta_1\eta_2 - \ell^2\xi_1\xi_2 = 0 \quad , \quad \dot{\beta} = \frac{k^{1/2}aG}{(k\ell^2 + a^2)} \quad , \\
&\frac{d\chi_1}{dt} = -F\xi_1 \quad , \quad \frac{d\chi_2}{dt} = -F\xi_2, \quad \frac{d\chi_3}{dt} = -F\xi_3 \quad ,
\end{aligned} \tag{10.12}$$

where

$$G = \frac{(\ell^2\dot{a}^2 + k\ell^2 + a^2)^{1/2}}{a} \quad , \quad F = \frac{(\ell^2\ddot{a} + a)}{(\ell^2\dot{a}^2 + k\ell^2 + a^2)^{1/2}} \quad . \tag{10.13}$$

Then, if we replace Eq. (10.8) by[2]

$$
\begin{aligned}
U &= \eta_1\cosh\left(\frac{w}{\ell}\right) + \chi_1\sinh\left(\frac{w}{\ell}\right) \quad , \\
V &= \eta_2\cosh\left(\frac{w}{\ell}\right) + \chi_2\sinh\left(\frac{w}{\ell}\right) \quad , \\
R &= \eta_3\cosh\left(\frac{w}{\ell}\right) + \chi_3\sinh\left(\frac{w}{\ell}\right) \quad ,
\end{aligned}
\tag{10.14}
$$

straightforward calculation then shows that because of the kinematic constraints of Eq. (10.12), the constraint of Eq. (10.5) is satisfied identically, with the metric of Eq. (10.4) being found to take the form

$$
\begin{aligned}
ds^2 = &\ dw^2 - \left[\cosh\left(\frac{w}{\ell}\right) - F(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 dt^2 \\
&+a^2\left[\cosh\left(\frac{w}{\ell}\right) - G(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 \left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega_2\right] \quad .
\end{aligned}
\tag{10.15}
$$

We recognize Eq. (10.15) as being precisely of the form of Eq. (9.17) which we derived earlier by solving the Einstein equations, with the beautiful new result of Eq. (10.15) being that it is possible to derive this self-same metric (and by extension its $k=0$ and $k<0$ analogs) without needing to make any appeal to dynamics at all, and without even needing to specify the explicit dependence of $a(t)$ on t. In addition, we note that with the form of Eq. (10.15) being fixed entirely by the RW parameters $a(t)$ and k, the specification of a cosmological dynamics on the brane alone (something we address in Chapter 12 below) will then completely determine the geometry throughout the entire AdS_5 bulk.

10.3 Embedding a $k>0$ de Sitter brane and AdS_5 bulk in $M(4,2)$

With the above development applying for any $a(t)$, it in particular applies for the special value

$$
a(t) = k^{1/2}H^{-1}\cosh(Ht)
\tag{10.16}
$$

associated with the spatially-closed ($k>0$) form of the dS_4 metric. For this choice of $a(t)$ we find that functions $\beta(t)$, $F(t)$ and $G(t)$ take the form

$$
\beta(t) = -\arctan\left(\frac{(\ell^2H^2+1)^{1/2}}{\sinh(Ht)}\right) \quad , \quad F(t) = G(t) = (\ell^2H^2+1)^{1/2} \quad ,
\tag{10.17}
$$

that the components of η_i are given by

$$
\eta_1 = -H^{-1}\sinh(Ht) \quad , \quad \eta_2 = H^{-1}(\ell^2H^2+1)^{1/2} \quad , \quad \eta_3 = H^{-1}\cosh(Ht) \quad ,
\tag{10.18}
$$

[2]With the brane being at $w=0$, and with the vector (U,V,R) reducing to (η_1,η_2,η_3) at $w=0$, while being viewable as being static in the AdS_5 space (cf. Eq. (10.15)), the brane can also be thought of as moving on the trajectory $\eta_i(t)$ with velocity $\xi_i(t)$ in the $M(4,2)$ space, with the general w transformations of Eq. (10.14) acting as Lorentz boosts in the (U,V,R) space.

and that the transformations of Eq. (10.14) simplify to

$$\begin{aligned} U &= -\frac{\sinh(Ht)}{H}\left[\cosh\left(\frac{w}{\ell}\right) - \sinh\left(\frac{w}{\ell}\right)(\ell^2H^2+1)^{1/2}\right] , \\ V &= \frac{1}{H}\left[\cosh\left(\frac{w}{\ell}\right)(\ell^2H^2+1)^{1/2} - \sinh\left(\frac{w}{\ell}\right)\right] , \\ R &= \frac{\cosh(Ht)}{H}\left[\cosh\left(\frac{w}{\ell}\right) - \sinh\left(\frac{w}{\ell}\right)(\ell^2H^2+1)^{1/2}\right] . \end{aligned} \tag{10.19}$$

With such a choice the metric of Eq. (10.15) then takes the form

$$\begin{aligned} ds^2 = &\ dw^2 + \left[\cosh\left(\frac{w}{\ell}\right) - (\ell^2H^2+1)^{1/2}\sinh\left(\frac{w}{\ell}\right)\right]^2 \\ &\times\left[\frac{k\cosh^2(Ht)}{H^2}\left(\frac{dr^2}{(1-kr^2)} + r^2d\Omega_2\right) - dt^2\right] , \end{aligned} \tag{10.20}$$

to thus describe the embedding of a spatially closed dS_4 geometry in AdS_5.

For this metric the null (w,t) geodesic solutions to[3]

$$\frac{dw}{dt} = \cosh\left(\frac{w}{\ell}\right) - (\ell^2H^2+1)^{1/2}\sinh\left(\frac{w}{\ell}\right) \tag{10.21}$$

are readily determined to be of the form

$$\sinh\left(\frac{w}{\ell} - \frac{w_0}{\ell}\right) = -\frac{1}{\sinh(Ht)} , \tag{10.22}$$

where w_0 is defined by

$$\tanh\left(\frac{w_0}{\ell}\right) = \frac{1}{(\ell^2H^2+1)^{1/2}} . \tag{10.23}$$

With the zero of the g_{00} metric coefficient occurring at

$$\tanh\left(\frac{w}{\ell}\right) = \frac{1}{(\ell^2H^2+1)^{1/2}} , \tag{10.24}$$

we thus see that this zero is located at none other than $w = w_0$, with Eq. (10.22) then entailing that it will take an infinite amount of time for a null geodesic to travel from the brane to $w = w_0$. Thus just as in our earlier discussion in Chapter 8 of the embedding of a spatially flat sectioned dS_4 metric in AdS_5, we again find that there is a horizon, this one at $w = w_0$. Now in our above construction of the $M(4,2)$ embedding we first introduced the coordinate R in Eq. (10.3) as the radius of a 3-sphere S^3. Since this is a quantity which cannot go negative, we see from the transformation of Eq. (10.19) that the spacetime of Eq. (10.20) is such that w_0 is actually the maximum allowed value for w. Consequently, when sectioned in spatially closed coordinates (viz. in coordinates which, unlike the spatially flat

[3]With the relevant Christoffel symbols associated with the metric of Eq. (10.20) being given by $\Gamma^5_{00} = -(1/2)dg_{00}/dw$ and $\Gamma^0_{05} = (1/2)d(\ln g_{00})/dw$, the geodesic equations $d^2t/d\tau^2 + (dt/d\tau)(dw/d\tau)d(\ln g_{00})/dw = 0$, $d^2w/d\tau^2 - (1/2)(dt/d\tau)^2dg_{00}/dw = 0$ admit of the exact first integrals $dt/d\tau = (g_{00})^{-1}$, $dw/d\tau = (-g_{00})^{-1/2}$, from which Eq. (10.21) then follows.

sectioned ones of Eq. (8.1), actually do cover the entire dS_4 space), we find that the horizon for the brane coincides with the boundary, w_0, of the allowed domain for the warp factor coordinate w, with the metric coefficients for a spatially closed dS_4 brane embedded in AdS_5 falling monotonically all the way from the brane to the boundary.

10.4 Analog to the Rindler horizon

From the perspective of $M(4,2)$ the above emergence of a horizon is initially somewhat puzzling since as a spacetime $M(4,2)$ does not itself possess any horizon structure at all, so that the horizon cannot be associated with any geometric properties of $M(4,2)$. However, the brane which is embedded in $M(4,2)$ is not at rest in $M(4,2)$. Rather, it has a non-zero velocity given by $\xi_i = d\eta_i/dt$, and thus an acceleration $\alpha_i = d^2\eta_i/dt^2$ given by

$$
\begin{aligned}
\alpha_1 &= \ddot{\eta}_1 = \frac{\eta_1 a\ddot{a}}{(k\ell^2+a^2)} + \frac{\eta_1 k\ell^2\dot{a}^2}{(k\ell^2+a^2)^2} - \eta_1\dot{\beta}^2 - \eta_2\ddot{\beta} - \frac{2\eta_2 a\dot{a}\dot{\beta}}{(k\ell^2+a^2)} \ , \\
\alpha_2 &= \ddot{\eta}_2 = \frac{\eta_2 a\ddot{a}}{(k\ell^2+a^2)} + \frac{\eta_2 k\ell^2\dot{a}^2}{(k\ell^2+a^2)^2} - \eta_2\dot{\beta}^2 + \eta_1\ddot{\beta} + \frac{2\eta_1 a\dot{a}\dot{\beta}}{(k\ell^2+a^2)} \ , \\
\alpha_3 &= \ddot{\eta}_3 = \frac{\ddot{a}}{k^{1/2}} \ ,
\end{aligned}
\tag{10.25}
$$

with an acceleration of the cosmology on the brane (viz. $\ddot{a} \neq 0$) thus entailing an acceleration of the brane in $M(4,2)$. However, observers who accelerate in a spacetime, even one which is flat, experience a horizon due to the very fact that they are accelerating [Rindler (1966)], with there then being some null geodesics which they are never able to encounter. Thus, for instance, in a 2-dimensional flat spacetime with metric $ds^2 = dt^2 - dx^2$, a uniformly (and thus perpetually) accelerating observer (viz. one whose covariant acceleration $w^i = d^2x^i/ds^2$ obeys $w^iw_i = 1$) moves on

$$
w^iw_i = \frac{1}{(1-v^2)^{3/2}}\frac{dv}{dt} = 1 \ , \tag{10.26}
$$

with a velocity

$$
v = \frac{t}{(1+t^2)^{1/2}} \tag{10.27}
$$

which asymptotes to the velocity of light at late times, and trajectory

$$
x^2 - t^2 = 1 \ . \tag{10.28}
$$

The observer's trajectory thus asymptotes to the null geodesic $t = x$ at late times, and, as shown in Fig. (10.1), is never able to encounter any null geodesic of the form $x = t - \alpha$ with $\alpha > 0$, with such geodesics thus being outside the observer's

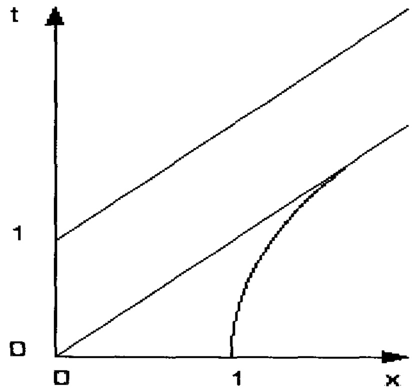

Fig. 10.1 The horizon seen by an accelerating observer in flat spacetime. The trajectory $x^2 - t^2 = 1$ begins at $x = 1$ at $t = 0$, asymptotes to the null geodesic $x = t$, and never encounters null geodesics such as the $x = t - 1$ one which originates at $x = 0$ at $t = 1$.

horizon. Comparing now with the dS_4 brane trajectory associated with Eq. (10.19), we see that a brane at $w = 0$ in AdS_5 corresponds to a brane with fixed $V = (\ell^2 H^2 + 1)^{1/2}/H$ and time dependent $U = -\sinh(Ht)/H$ and $R = -\cosh(Ht)/H$ in $M(4,2)$, with the brane thus moving on the fixed V $M(4,2)$ trajectory

$$R^2 - U^2 = \frac{1}{H^2} \ . \tag{10.29}$$

With this trajectory being of the same generic form as Eq. (10.28), and with R and U respectively being spacelike and timelike coordinates, the dS_4 brane thus has velocity

$$v = \frac{dR}{dU} = \frac{U}{R} = \frac{HU}{(1 + H^2U^2)^{1/2}} \ , \tag{10.30}$$

and constant covariant acceleration of magnitude

$$\frac{1}{(1 - v^2)^{3/2}} \frac{dv}{dU} = H \tag{10.31}$$

on a fixed V slice of the (U, V, R) space. The horizon experienced by an observer located on a dS_4 brane embedded in AdS_5 thus originates in the acceleration of the brane in $M(4,2)$.

10.5 Embedding a $k < 0$ RW brane and AdS_5 bulk in $M(4,2)$

As well as embed the $k > 0$ RW brane-world set-up in $M(4,2)$, it is also possible to embed its $k = 0$ and $k < 0$ counterparts as well. The $k < 0$ embedding can be obtained directly from the $k>0$ embedding by making the substitutions $U \to -iW$, $W \to iU$, $R \to iR$, with Eqs. (10.3) – (10.5) then taking the form

$$\begin{aligned} X &= (-k)^{1/2} Rr\ \sin\theta\ \cos\phi\ \ ,\ Y = (-k)^{1/2} Rr\ \sin\theta\ \sin\phi\ \ , \\ Z &= (-k)^{1/2} Rr\ \cos\theta\ \ ,\ \ U = R(1-kr^2)^{1/2}\ \ , \end{aligned} \tag{10.32}$$

$$ds^2 = -dR^2 - kR^2\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] + dW^2 - dV^2\ \ , \tag{10.33}$$

$$V^2 - W^2 + R^2 = \ell^2\ \ , \tag{10.34}$$

where now $R = (U^2 - X^2 - Y^2 - Z^2)^{1/2}$. Then, on defining

$$\eta_1 = \left(\frac{k\ell^2 + a^2}{k}\right)^{1/2} \cosh\beta(t)\ \ ,\ \ \eta_2 = \left(\frac{k\ell^2 + a^2}{k}\right)^{1/2} \sinh\beta(t)\ \ ,\ \ \eta_3 = \frac{a}{(-k)^{1/2}}\ \ , \tag{10.35}$$

where

$$\beta(t) = -(-k)^{1/2}\int^t dt \frac{[\ell^2\dot{a}^2 + k\ell^2 + a^2]^{1/2}}{[k\ell^2 + a^2]}\ \ , \tag{10.36}$$

we find directly that the identification

$$V = \eta_1\ \ ,\ \ W = \eta_2\ \ ,\ \ R = \eta_3\ \ , \tag{10.37}$$

brings the metric of Eq. (10.33) to the $k<0$ variant of Eq. (10.9), while the identification

$$\begin{aligned} V &= \eta_1 \cosh\left(\frac{w}{\ell}\right) - \left(\eta_2\frac{d\eta_3}{dt} - \eta_3\frac{d\eta_2}{dt}\right)\sinh\left(\frac{w}{\ell}\right)\ \ , \\ W &= \eta_2 \cosh\left(\frac{w}{\ell}\right) + \left(\eta_3\frac{d\eta_1}{dt} - \eta_1\frac{d\eta_3}{dt}\right)\sinh\left(\frac{w}{\ell}\right)\ \ , \\ R &= \eta_3 \cosh\left(\frac{w}{\ell}\right) - \left(\eta_1\frac{d\eta_2}{dt} - \eta_2\frac{d\eta_1}{dt}\right)\sinh\left(\frac{w}{\ell}\right) \end{aligned} \tag{10.38}$$

brings the metric of Eq. (10.33) to the $k<0$ variant of Eq. (10.15) just as desired.[4]

10.6 Embedding a $k=0$ RW brane and AdS_5 bulk in $M(4,2)$

For the $k=0$ embedding we introduce

$$X = R\ \sin\theta\ \cos\phi\ \ ,\ Y = R\ \sin\theta\ \sin\phi\ \ ,\ \ Z = R\ \cos\theta\ \ , \tag{10.39}$$

[4] As a check, we note that for a dS_4 brane written in $k<0$ RW coordinates where $a(t) = (-k)^{1/2}\sinh(Ht)/H$, we obtain $\beta = \text{arctanh}[\cosh(Ht)/(1+\ell^2H^2)^{1/2}]$, $\eta_1 = (1+\ell^2H^2)^{1/2}/H$, $\eta_2 = \cosh(Ht)/H$, $\eta_3 = \sinh(Ht)/H$, $\eta_2\dot{\eta}_3 - \eta_3\dot{\eta}_2 = 1/H$, $\eta_3\dot{\eta}_1 - \eta_1\dot{\eta}_3 = -(1+\ell^2H^2)^{1/2}\cosh(Ht)/H$, $\eta_1\dot{\eta}_2 - \eta_2\dot{\eta}_1 = (1+\ell^2H^2)^{1/2}\sinh(Ht)/H$, with the $k<0$ variant of Eq. (10.20) then following.

so that the $M(4,2)$ metric and constraint are then given as

$$ds^2 = dR^2 + R^2[d\theta^2 + \sin^2\theta d\phi^2] - dU^2 - d(V-W)d(V+W) \tag{10.40}$$

and

$$U^2 + (V-W)(V+W) - R^2 = \ell^2 \ . \tag{10.41}$$

On introducing

$$\eta_1 = a\beta \ , \quad \eta_+ = \ell a \ , \quad \eta_- = \frac{\ell}{a} + \frac{ar^2}{\ell} - \frac{a\beta^2}{\ell} \ , \quad \eta_4 = ar \ , \tag{10.42}$$

where

$$\beta = \int^t dt \frac{[\ell^2\dot{a}^2 + a^2]^{1/2}}{a^2} \ , \tag{10.43}$$

we directly find, upon making the identification

$$U = \eta_1 \ , \quad V + W = \eta_+ \ , \quad V - W = \eta_- \ , \quad R = \eta_4 \ , \tag{10.44}$$

that the metric of Eq. (10.40) then takes the form

$$ds^2 = a^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] - dt^2 \ , \tag{10.45}$$

to thus embed a $k = 0$ RW brane in $M(4,2)$. To embed both the brane and an AdS_5 bulk, we further introduce

$$\begin{aligned} \chi_1 &= -a^2\beta\dot{\beta} - \frac{\dot{a}\ell^2}{a} \ , \quad \chi_+ = -\ell a^2\dot{\beta} \ , \\ \chi_- &= \frac{2\ell\dot{a}\beta}{a} + \ell\dot{\beta} + \frac{a^2\beta^2\dot{\beta}}{\ell} - \frac{a^2r^2\dot{\beta}}{\ell} \ , \quad \chi_4 = -a^2r\dot{\beta} \ . \end{aligned} \tag{10.46}$$

Then on setting

$$\begin{aligned} U &= \eta_1\cosh\left(\frac{w}{\ell}\right) + \chi_1\sinh\left(\frac{w}{\ell}\right) \ , \ V + W = \eta_+\cosh\left(\frac{w}{\ell}\right) + \chi_+\sinh\left(\frac{w}{\ell}\right) \ , \\ V - W &= \eta_-\cosh\left(\frac{w}{\ell}\right) + \chi_-\sinh\left(\frac{w}{\ell}\right) \ , \ R = \eta_4\cosh\left(\frac{w}{\ell}\right) + \chi_4\sinh\left(\frac{w}{\ell}\right) \end{aligned} \tag{10.47}$$

and

$$G(t) = \frac{(\ell^2\dot{a}^2 + a^2)^{1/2}}{a} \ , \quad F(t) = \frac{(\ell^2\ddot{a} + a)}{(\ell^2\dot{a}^2 + a^2)^{1/2}} \ , \tag{10.48}$$

we find, following some algebra, that the constraint of Eq. (10.41) is satisfied identically, with the metric of Eq. (10.40) taking the form

$$\begin{aligned} ds^2 = \ & dw^2 - \left[\cosh\left(\frac{w}{\ell}\right) - F(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 dt^2 \\ & + a^2\left[\cosh\left(\frac{w}{\ell}\right) - G(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \end{aligned} \tag{10.49}$$

just as desired.

As a check on this metric we note that when we set $a(t) = 1$, the metric reduces to the $ds^2 = dw^2 + e^{-2w/\ell}[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$ metric associated with an M_4 sectioning of AdS_5, just as it should. Additionally, in this case $\beta(t)$ readily evaluates to $\beta(t) = t$, to thus yield embedding relations of the form

$$\begin{aligned} U &= te^{-w/\ell} \quad , \quad V + W = \ell e^{-w/\ell} \quad , \\ R &= re^{-w/\ell} \quad , \quad V - W = \ell e^{w/\ell} + \ell^{-1}(r^2 - t^2)e^{-w/\ell} \quad . \end{aligned} \tag{10.50}$$

For such an embedding we see that if t, r and w are allowed to range over $-\infty \leq t \leq \infty$, $0 \leq r \leq \infty$, $-\infty \leq w \leq \infty$, the embedding will generate all possible positive and negative values for both U and $V - W$ and all possible positive values for the positive definite $R = (X^2 + Y^2 + Z^2)^{1/2}$, but will only generate positive values for $V + W$. Consequently, this set of warp factor coordinates only covers half of AdS_5. However, on noting that Eq. (3.9) can be rewritten as

$$\begin{aligned} V &= te^{-w/\ell} \quad , \quad U + X = \ell e^{-w/\ell} \quad , \\ Y &= xe^{-w/\ell} \quad , \quad U - X = \ell e^{w/\ell} + \ell^{-1}(x^2 - t^2)e^{-w/\ell} \quad , \end{aligned} \tag{10.51}$$

we see that on appropriately relabeling the coordinates, the M_4-sectioned sectors of AdS_5 with $V + W > 0$ and $V + W < 0$ thus correspond to the two Poincaré patches associated with a Minkowski sectioning of an AdS space.

Additional insight into this situation can be obtained by taking the $k \to 0$ limit of Eqs. (10.3) and (10.14), with the $k > 0$ $M(4,2)$ coordinates precisely being found to limit to the $k = 0$ $M(4,2)$ coordinates of Eq. (10.47) according to

$$\begin{aligned} &\ell k^{1/2}[U(k > 0) \pm W(k > 0)] \to 2[V(k = 0) \pm W(k = 0)] \quad , \\ &V(k > 0) \to U(k = 0) \quad , \quad k^{1/2}R(k > 0)r(k > 0) \to R(k = 0)r(k = 0) \quad . \end{aligned} \tag{10.52}$$

Following a redefinition of the coordinates, the $k > 0$ embedding thus transits into the $k = 0$ one. In Eq. (10.52) we have made a point of distinguishing between $r(k > 0)$ and $r(k = 0)$. Specifically, if for the $k > 0$ embedding we define $r(k > 0) = k^{-1/2}\sin(k^{1/2}\psi)$, ψ will then serve as the third angle (along with θ and ϕ of Eq. (10.3)) to describe the S_3 3-sphere $X^2 + Y^2 + Z^2 + W^2 = R^2$. For such a sphere ψ must range from 0 to $\pi/k^{1/2}$, θ from 0 to π and ϕ from 0 to 2π in order to cover S_3. The coordinate $r(k > 0)$ thus ranges from 0 to $1/k^{1/2}$ as ψ ranges from 0 to $\pi/2k^{1/2}$ and then from $1/k^{1/2}$ back to 0 as ψ ranges from $\pi/2k^{1/2}$ to $\pi/k^{1/2}$. Hence in the $k \to 0$ limit $r(k \to 0)$ ranges from 0 to ∞ and then back to 0. S_3 consists of two hemispheres and each one of them becomes a patch with a coordinate $r(k \to 0)$ which ranges from 0 to ∞ as the S_3 curvature goes to zero. The coordinate $r(k = 0)$ of the $k = 0$ RW metric of Eq. (10.49) is thus associated with the limit of only one of the two hemispheres of S_3. With the embedding of Eq. (10.47) only covering a part of the range of allowed values for the $M(4,2)$ coordinates, the missing values (such as the negative $V + W$ in the $a(t) = 1$ case) are thus associated with a doubling of Eq. (10.47) due to the second S_3 hemisphere.

As regards the $k > 0$ and $k = 0$ embedded RW metrics given above, we additionally note that in both Eq. (10.3) and Eq. (10.39) the coordinates $R(k > 0)$ and $R(k = 0)$ serve as a radii, and thus have to be non-negative.[5] Thus not only are the spatial metric coefficients g_{rr}, $g_{\theta\theta}$ and $g_{\phi\phi}$ themselves non-negative (each is proportional to $[\cosh(w/\ell) - G(t)\sinh(w/\ell)]^2$), the quantity $\cosh(w/\ell) - G(t)\sinh(w/\ell)$ to which R itself is proportional will itself be non-negative too. Now from its definition as $G(t) = (\ell^2\dot{a}^2 + k\ell^2 + a^2)^{1/2}/a$, it follows in both the $k > 0$ and $k = 0$ cases that $G(t)$ can never be less than one, with the domain of the coordinate w thus necessarily being bounded from above in those cases, a bound which will however change with t.[6] In addition, this upper limit will itself become infinite at late times in cosmologies which are late time expanding, since in them $G(t)$ will then asymptote to one. The domain of the warp factor coordinate w will thus be bounded from above in any embedded $k > 0$ or $k = 0$ RW brane world, with the allowed values of w then having to obey $\tanh(w/\ell) \leq 1/G(t)$ at any given time.[7]

From the properties of the $M(4,2)$ embedding we thus discover that in the $k > 0$ and $k = 0$ cases the relevant domain of the coordinate w does not necessarily have to extend to $+\infty$ at any given time, even though it does do so for the embedding of an M_4 brane, to thus underscore the utility of making the $M(4,2)$ embedding. A further interesting generic feature of brane cosmologies for which $G(t)$ is greater than one at all times is that the quantity $[\cosh(w/\ell) - G(t)\sinh(w/\ell)]$ is then a monotonic function of w at any given t.[8] Consequently, for Z_2-symmetric $k > 0$ and $k = 0$ embedded cosmologies where $G(t)$ always is greater than one, at any given time the spatial metric coefficients will fall all the way from the brane to the edge of the allowed $|w|$ domain at $|w| = \text{arctanh}(1/G(t))$. While a similar argument could be made for the g_{00} metric coefficient in cases where $F(t) > 1$, unfortunately nothing of a similar generic nature can be said about the behavior of the function $F(t) = (1 + \ell^2\ddot{a}/a)/G(t)$ until dynamical information on $a(t)$ is provided so that one could then ascertain whether $\ell^2\ddot{a}/a > G(t) - 1$, a condition which $G(t) > 1$ cosmologies would however only be able to satisfy if they are accelerating ones.[9]

[5]The $k < 0$ $R(k < 0)$ also only needs to be positive, since even though positive $R(k < 0)$ only gives rise to positive $U(k < 0)$ in Eq. (10.32) (even as both signs of $X(k < 0)$, $Y(k < 0)$, and $Z(k < 0)$ are generated), the quantity $\cos(k^{1/2}\psi)$ is both positive and negative in the range $0 \leq k^{1/2}\psi \leq \pi$, with the $k < 0$ continuation of $W(k > 0) = R(k > 0)\cos(k^{1/2}\psi)$ and substitutions $W(k > 0) \to iU(k < 0)$, $R(k > 0) \to iR(k < 0)$ thus yielding both signs of $U(k < 0) = \pm R(k < 0)\cosh[(-k)^{1/2}\psi]$ from positive $R(k < 0)$ alone. The presence of two patches is to be expected since within the $k < 0$ RW geometries, the one with $a(t) = (-k)^{1/2}t$ happens to be the flat M_4.

[6]For the M_4 limiting case of the $k = 0$ RW embedding where $a(t)$ is equal to one at all times, $G(t)$ is then identically equal to one at all times, with the upper bound on w then being infinite.

[7]In and of itself the condition $\tanh(w/\ell) \leq 1/G$ with $G > 1$ allows w to be negative as well as positive, with w initially having to lie in the range $-\infty \leq w \leq \text{arctanh}(1/G)$. However, in the Z_2-symmetric extension of the $k > 0$ and $k = 0$ cases (Eqs. (10.15) and (10.49) only describe the sectioning of AdS_5 in $k > 0$ or $k = 0$ RW coordinates without imposition of any Z_2 symmetry), one can retain the $0 \leq |w| \leq \text{arctanh}(1/G)$ region alone, with $|w| = \text{arctanh}(1/G)$ then being a (time dependent) bound on the allowed values of $|w|$ in the warp factor coordinate system.

[8]Its derivative with respect to w, viz. $[\sinh(w/\ell) - G\cosh(w/\ell)]/\ell$, cannot vanish if $G > 1$.

[9]For, e.g., the permanently accelerating dS_4 cosmology, $F = (1 + \ell^2 H^2)^{1/2}$ is greater than one.

Chapter 11

Null Geodesics of AdS_5 Bulks Containing Robertson-Walker Branes

11.1 Null geodesics of the $k > 0$ RW brane world

While we were able to integrate the embedded dS_4 brane null geodesic equation of Eq. (10.21) directly because the quantity $F(t)$ which appears in the metric of Eq. (10.15) was then time independent, in the more general case of the embedding of a general RW space with arbitrary 3-curvature and arbitrary $a(t)$ into AdS_5, the integration of the non-separable

$$\frac{dw}{dt} = \pm\left[\cosh\left(\frac{w}{\ell}\right) - F(t)\sinh\left(\frac{w}{\ell}\right)\right] \tag{11.1}$$

with the relevant $F(t)$ is not as immediate.[1] Nonetheless, through use of the $M(4,2)$ embedding, it turns out [Guth, Kaiser, Mannheim and Nayeri (2004a)] that it is in fact possible to solve for the null geodesics analytically in such cases, and to even be able to do so without actually needing to have to specify any particular form for $a(t)$. To this end we note that transformations of the form

$$U = \ell\ \sec\gamma\ \cos\delta\ \ ,\ \ V = \ell\ \sec\gamma\ \sin\delta\ \ ,\ \ R = \ell\ \tan\gamma\ \ , \tag{11.2}$$

satisfy the constraint of Eq. (10.5) identically, while simultaneously bringing the $k > 0$ metric of Eq. (10.4) to the form

$$ds^2 = \frac{\ell^2}{\cos^2\gamma}\left[d\gamma^2 + k\sin^2\gamma\left(\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) - d\delta^2\right]\ \ . \tag{11.3}$$

With this rewriting of the $M(4,2)$ metric in such a simple form, the null geodesics can then be instantly given as

$$\delta + \delta_0 = \pm\gamma\ \ , \tag{11.4}$$

[1] With dw/dt being given by $+1$ or -1 at $w = 0$, the two choices of sign in Eq. (11.1) correspond to null signals which either departing from or arriving at the brane at a given time.

i.e. given by

$$\frac{R}{\ell} = \pm\frac{\hat{V}}{\hat{U}} \ , \tag{11.5}$$

where δ_0 is a constant and where

$$\hat{V} = \cos\delta_0 \ V + \sin\delta_0 \ U \quad , \quad \hat{U} = \cos\delta_0 \ U - \sin\delta_0 \ V \ . \tag{11.6}$$

Now in the $k > 0$ case every point in the $M(4,2)$ space obeys the constraint $\hat{U}^2 + \hat{V}^2 = R^2 + \ell^2$, and thus points which are null geodesics obey

$$\hat{U}^2 + \hat{V}^2 = \ell^2\frac{\hat{V}^2}{\hat{U}^2} + \ell^2 \ , \tag{11.7}$$

i.e. they obey the extremely simple

$$\hat{U} = \pm\ell \ . \tag{11.8}$$

With use of the transformations of Eq. (10.14) we may rewrite Eq. (11.8) as

$$\cosh\left(\frac{w}{\ell}\right)[\cos\delta_0 \ \eta_1 - \sin\delta_0 \ \eta_2] + \sinh\left(\frac{w}{\ell}\right)[\cos\delta_0 \ \chi_1 - \sin\delta_0 \ \chi_2] = \pm\ell \ , \tag{11.9}$$

with Eq. (11.9) thus providing us with a closed form expression for the $k > 0$ (w, t) space null geodesics.

While the use of properties of the $M(4,2)$ embedding has thus given us the exact $k > 0$ null geodesics of Eq. (11.9), it is not immediately apparent that these null geodesics do in fact satisfy the null geodesic equation of Eq. (11.1). To establish that this in fact the case it suffices to look at the null geodesics with $\delta_0 = 0$, viz. the ones which are to obey

$$\cosh\left(\frac{w}{\ell}\right)\eta_1 + \sinh\left(\frac{w}{\ell}\right)\chi_1 = \pm\ell \ . \tag{11.10}$$

On recalling the kinematic relation $\chi_1^2 - \eta_1^2 = \ell^2\xi_1^2 - \ell^2$ of Eq. (10.12), we may solve Eq. (11.10) to obtain[2]

$$\cosh\left(\frac{w}{\ell}\right) = \pm\left[\frac{\chi_1\xi_1 - \eta_1}{\ell(\xi_1^2 - 1)}\right] , \ \sinh\left(\frac{w}{\ell}\right) = \pm\left[\frac{\chi_1 - \eta_1\xi_1}{\ell(\xi_1^2 - 1)}\right] , \ \tanh\left(\frac{w}{\ell}\right) = \frac{\chi_1 - \eta_1\xi_1}{\chi_1\xi_1 - \eta_1} \ . \tag{11.11}$$

On noting also that $\ell^2\dot{\xi}_1 = -F\chi_1 - \eta_1$, $\dot{\chi}_1 = -F\xi_1$, differentiation and algebraic manipulation of the expression for $\tanh(w/\ell)$ in Eq. (11.11) then gives

$$\frac{dw}{dt} = \frac{\eta_1 - \chi_1\xi_1}{\ell(\xi_1^2 - 1)} - \frac{F(\eta_1\xi_1 - \chi_1)}{\ell(\xi_1^2 - 1)} = \mp\left[\cosh\left(\frac{w}{\ell}\right) - F\sinh\left(\frac{w}{\ell}\right)\right] \ , \tag{11.12}$$

[2]The general solution to the equation $A\cosh(w/\ell) + B\sinh(w/\ell) = \pm\ell$ is of the form $\cosh(w/\ell) = \pm[B(B^2 - A^2 + \ell^2)^{1/2} - A\ell]/(B^2 - A^2)$, $\sinh(w/\ell) = \pm[B\ell - A(B^2 - A^2 + \ell^2)^{1/2}]/(B^2 - A^2)$.

to precisely recover Eq. (11.1) just as desired. With the extension to arbitrary δ_0 following directly, we thus establish that all the null geodesics in the $k > 0$ RW embedding case are given by $\hat{U} = \pm\ell$, i.e. by

$$\cosh\left(\frac{w}{\ell}\right) = \pm\left[\frac{\hat{\chi}_1\hat{\xi}_1 - \hat{\eta}_1}{\ell(\hat{\xi}_1^2 - 1)}\right] \quad , \quad \sinh\left(\frac{w}{\ell}\right) = \pm\left[\frac{\hat{\chi}_1 - \hat{\eta}_1\hat{\xi}_1}{\ell(\hat{\xi}_1^2 - 1)}\right] \tag{11.13}$$

in an obvious notation. As a quick check on these results we note that the null geodesic for the embedding of dS_4 in AdS_5 as given by Eqs. (10.22) and (10.23) can also be written in the convenient form

$$\begin{aligned} \sinh\left(\frac{w}{\ell}\right) &= \frac{\cosh(Ht) - (\ell^2H^2 + 1)^{1/2}}{\ell H \sinh(Ht)} \quad , \\ \cosh\left(\frac{w}{\ell}\right) &= \frac{(\ell^2H^2 + 1)^{1/2}\cosh(Ht) - 1}{\ell H \sinh(Ht)} \quad , \end{aligned} \tag{11.14}$$

a relation which can then immediately be recovered when Eq. (11.11) is evaluated for $\eta_1 = -\sinh(Ht)/H$, $\xi_1 = -\cosh(Ht)$, $\chi_1 = \sinh(Ht)(1 + \ell^2H^2)^{1/2}/H$, viz. the values associated with Eq. (10.18).

11.2 Null geodesics of the $k = 0$ and $k < 0$ RW brane worlds

While our derivation of Eq. (11.11) was based on the use of properties of the embedding of a $k > 0$ RW brane plus AdS_5 bulk in $M(4,2)$, once we have the null geodesics of Eq. (11.11), we can check directly that they obey Eq. (11.1) without needing to make any further reference to the $M(4,2)$ embedding. Moreover, since the null geodesics that we have found this way turn out to be of a form in which a specific hyperbolic function of w/ℓ is to be equal to a specific closed form function of $a(t)$ and k, and since the (w,t) space null geodesic equation of Eq. (11.1) actually applies for all k (the form of the metric of Eq. (10.15) being generic to all k), it follows that the form of the null geodesic of Eq. (11.11) must thus be generic to all k. Equation (11.11) thus gives us the exact (w,t) space null geodesic solutions to Eq. (11.1) for the embedding into AdS_5 of RW branes of any of the three ($k > 0$, $k = 0$, $k < 0$) possible spatial curvatures. While the $k < 0$ null geodesics can be obtained from the $k > 0$ ones by direct replacement of the $k > 0$ η_i, ζ_i and χ_i by their $k < 0$ counterparts,[3] to get the $k = 0$ null geodesics we need to take the $k \to 0$ limit of Eq. (11.11). Noting from Eq. (10.7) that the $k \to 0$ limit of $\beta(t)/k^{1/2}$ is

[3] Alternatively, with the substitution $V = \ell\cosh\delta/\cosh\gamma$, $W = \ell\sinh\delta/\cosh\gamma$, $R = \ell\sinh\gamma/\cosh\gamma$ bringing the $k < 0$ metric of Eq. (10.33) to the form $ds^2 = (\ell^2/\cosh^2\gamma)\left[d\delta^2 - d\gamma^2 - k\sinh^2\gamma[dr^2/(1-kr^2) + r^2d\Omega_2]\right]$, the null geodesics are given by $\gamma = \delta + \delta_0$, i.e. by $R/\ell = \tilde{W}/\tilde{V}$ where $\tilde{W} = W\cosh\delta_0 + V\sinh\delta_0$, $\tilde{V} = V\cosh\delta_0 + W\sinh\delta_0$, and thus by solutions to $(R^2 - \ell^2)(\tilde{V}^2 - \ell^2) = 0$.

well-behaved, on taking this limit we obtain for the $k = 0$ null geodesics

$$\cosh\left(\frac{w}{\ell}\right) = \mp\frac{(\ell^2\dot{a}^2 + a^2)^{1/2}}{\ell\dot{a}} \quad , \quad \sinh\left(\frac{w}{\ell}\right) = \mp\frac{a}{\ell\dot{a}} \quad , \tag{11.15}$$

and it can be directly checked that these null geodesics do indeed obey the $k = 0$ null geodesic equation

$$\frac{dw}{dt} = \mp\left[\cosh\left(\frac{w}{\ell}\right) - \frac{(\ell^2\ddot{a} + a)}{(\ell^2\dot{a}^2 + a^2)^{1/2}}\sinh\left(\frac{w}{\ell}\right)\right] \quad . \tag{11.16}$$

With Eq. (11.15) entailing that

$$\coth\left(\frac{w}{\ell}\right) = \frac{(\ell^2\dot{a}^2 + a^2)^{1/2}}{a} \quad , \tag{11.17}$$

on comparing with the form for the g_{rr} metric coefficient given in Eq. (10.49), we additionally note that the edge of the warp factor coordinate system associated with a $k = 0$ RW embedding (viz. the place where $g_{rr} = 0$) is a null geodesic of AdS_5.

11.3 Alternate derivation of the $k = 0$ null geodesics

Further insight into the $k = 0$ null geodesics can be obtained by instead constructing them by an analog of the approach used in Eq. (11.3). Thus, with the transformation

$$\begin{aligned} U &= \ell\sec\gamma\,\cosh\alpha\,\cos\delta \quad , \quad V = \ell\sec\gamma\,\cosh\alpha\,\sin\delta \quad , \\ W &= \ell\sec\gamma\,\sinh\alpha \quad , \quad R = \ell\tan\gamma \quad , \end{aligned} \tag{11.18}$$

bringing the $M(4,2)$ metric of Eq. (10.40) to the form

$$ds^2 = \ell^2\sec^2\gamma[d\gamma^2 + d\alpha^2 + \sin^2\gamma\, d\Omega_2 - \cosh^2\alpha\, d\delta^2] \quad , \tag{11.19}$$

the $k = 0$ null geodesics are given by $d\alpha = \pm\cosh\alpha\, d\delta$, i.e. by

$$\sinh\alpha = \pm\frac{\sin(\delta + \delta_0)}{\cos(\delta + \delta_0)} \quad , \quad \cosh\alpha = \left|\frac{1}{\cos(\delta + \delta_0)}\right| \quad , \tag{11.20}$$

where δ_0 is a constant. In terms of the $M(4,2)$ parameters the null geodesics are thus given by the very simple relation

$$W = \pm(\sin\delta_0\, U + \cos\delta_0\, V)\theta[\cos(\delta + \delta_0)] \mp (\sin\delta_0\, U + \cos\delta_0\, V)\theta[-\cos(\delta + \delta_0)] \quad , \tag{11.21}$$

so that on setting $\delta_0 = \pi$ when $\cos(\delta + \delta_0) > 0$ (or on setting $\delta_0 = 0$ when $\cos(\delta + \delta_0) < 0$), use of Eq. (10.47) then yields

$$W + V = \ell a\cosh\left(\frac{w}{\ell}\right) - \ell(\ell\dot{a}^2 + a^2)^{1/2}\sinh\left(\frac{w}{\ell}\right) = 0 \quad , \tag{11.22}$$

with the null geodesic equation of Eq. (11.17) immediately being recovered.

As a check on these results we consider the case where the $k = 0$ RW metric is actually dS_4, viz. a case where the expansion radius is given by $a(t) = e^{Ht}$ and where the embedding in $M(4,2)$ is given by Eq. (10.47) as evaluated with parameters

$$
\begin{aligned}
\beta &= -\frac{e^{-Ht}(\ell^2H^2+1)^{1/2}}{H} \quad , \quad \eta_1 = -\frac{(\ell^2H^2+1)^{1/2}}{H} \quad , \quad \eta_+ = \ell e^{Ht} \quad , \\
\eta_- &= -\frac{e^{-Ht}}{\ell H^2} + \frac{r^2e^{Ht}}{\ell} \quad , \quad \eta_4 = re^{Ht} \quad , \quad \chi_1 = \frac{1}{H} \quad , \quad \chi_+ = -\ell e^{Ht}(\ell^2H^2+1)^{1/2} \quad , \\
\chi_- &= \frac{(\ell^2H^2+1)^{1/2}}{\ell H^2}\left[e^{-Ht} - r^2e^{Ht}H^2\right] \quad , \quad \chi_4 = -re^{Ht}(\ell^2H^2+1)^{1/2} \quad . \qquad (11.23)
\end{aligned}
$$

For this case the parameters $G(t)$ and $F(t)$ given in Eq. (10.48) both evaluate to $(\ell^2H^2+1)^{1/2}$. Since these values are precisely the same ones as obtained for the $k > 0$ form for dS_4 as given in Eq. (10.17), the null geodesic equation of Eq. (11.14) that was obtained for the $k > 0$ dS_4 case must thus hold as a $k = 0$ dS_4 null geodesic relation as well. Noting that in the particular null geodesic of Eq. (11.14) the $k = 0$ dS_4 embedding parameters of Eq. (10.47) evaluate to

$$
U = -\ell\coth(Ht) \quad , \quad V - W = \frac{1}{H}[1-\coth(Ht)] \quad , \quad V + W = \ell^2H[1+\coth(Ht)] \qquad (11.24)
$$

when we set $r = 0$, the relation

$$
-W = U\left(\frac{2\ell H}{1+\ell^2H^2}\right) + V\left(\frac{1-\ell^2H^2}{1+\ell^2H^2}\right) \qquad (11.25)
$$

will then hold on the $k = 0$ null geodesic. On setting $-2\ell H/(1+\ell^2H^2) = \sin\delta_0$, we recognize Eq. (11.25) as being of the form of Eq. (11.21), just as required.

11.4 Variation of the metric coefficients along a bulk null geodesic

Having now obtained the bulk null geodesics in closed form, we can use them to determine whether communication between the AdS_5 bulk and an observer on the brane is or is not suppressed. Specifically, we need to follow the metric coefficients back from the observer along those geodesics, to see how they vary away from the brane. The signals that can reach an observer at $w = 0$ at some time t_0 from points in the bulk with positive w are those for which dw/dt is negative at the brane, with the behavior of the g_{00} and g_{rr} metric coefficients of Eq. (10.15) and their other k counterparts thus being determined by following the quantities

$$
\begin{aligned}
-g_{00}(\text{null}) &= \left[\cosh\left(\frac{w}{\ell}\right) - F(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 \quad , \\
g_{rr}(\text{null}) &= \left[\cosh\left(\frac{w}{\ell}\right) - G(t)\sinh\left(\frac{w}{\ell}\right)\right]^2 \qquad (11.26)
\end{aligned}
$$

back along those particular bulk null geodesics. (We ignore an irrelevant $a^2(t)/(1-kr^2)^{1/2}$ factor in g_{rr} here). For the $k>0$ case we thus follow

$$\begin{aligned}
\cosh\left(\frac{w}{\ell}\right) - F(t)\sinh\left(\frac{w}{\ell}\right) &= -\left[\frac{(\hat{\eta}_1-\hat{\chi}_1\hat{\xi}_1)}{\ell(\hat{\xi}_1^2-1)} - F(t)\left(\frac{\hat{\eta}_1\hat{\xi}_1-\hat{\chi}_1}{\ell(\hat{\xi}_1^2-1)}\right)\right] \ , \\
\cosh\left(\frac{w}{\ell}\right) - G(t)\sinh\left(\frac{w}{\ell}\right) &= -\left[\frac{(\hat{\eta}_1-\hat{\chi}_1\hat{\xi}_1)}{\ell(\hat{\xi}_1^2-1)} - G(t)\left(\frac{\hat{\eta}_1\hat{\xi}_1-\hat{\chi}_1}{\ell(\hat{\xi}_1^2-1)}\right)\right] \ , \quad (11.27)
\end{aligned}$$

with analogous expressions holding in the other k cases. In Eq. (11.27) the parameter δ_0 is fixed by specifying the time t_0 at which the null geodesic is to be at the brane. Given Eq. (11.27), localization of gravity around an RW brane will thus be achievable if in the region of the bulk which is visible to the brane observer the quantities $-g_{00}(\text{null})$ and $g_{rr}(\text{null})$ are maximal at the brane.

To get a sense of how this might actually be achieved, we recall that in some brane worlds (such as the closed 3-sphere based $k>0$ RW universe) the coordinate $R(w,t)$ serves as a radius, with its positive semi-definiteness imposing a time-dependent bound $w_0(t)$ on the allowed values of the warp factor system coordinate w. This bound is given by the surface of points $w_0(t)$ which satisfy $g_{rr}=0$, viz. $\coth(w_0(t)/\ell)=G(t)$. For such brane worlds then, the domain of relevance for the brane observer at time t_0 is only out to $w_0(t_i)$, where t_i is the unique time at which a null geodesic would need to leave the $w_0(t)$ surface in order to arrive at the brane at the specified time t_0 (see e.g. Fig. (12.1) below). At such $w_0(t_i)$ the quantity $g_{rr}(\text{null})$ vanishes and is thus automatically less than the value it takes on the brane. However, $-g_{00}(\text{null})$ does not vanish at $w_0(t_i)$, instead taking the value

$$-g_{00}(\text{null}, w_0(t_i)) = \left.\frac{(G-F)^2}{(G^2-1)}\right|_{t=t_i} \quad (11.28)$$

there. As we shall see below, in some cosmologies this quantity can actually be small.

Along the null geodesics the variation of the metric coefficients is given via Eq. (11.1) as $d[(-g_{00}(\text{null}))^{1/2}]/dt = -[\sinh(w/\ell) - F\cosh(w/\ell)](-g_{00})^{1/2}/\ell - \dot{F}\sinh(w/\ell)$, and $d[(g_{rr}(\text{null}))^{1/2}]/dt = -[\sinh(w/\ell) - G\cosh(w/\ell)](-g_{00})^{1/2}/\ell - \dot{G}\sinh(w/\ell)$. At the brane we therefore obtain

$$\frac{d}{dt}\left[(-g_{00}(\text{null}))^{1/2}\right]\bigg|_{w=0} = \frac{F(t)}{\ell} \ , \quad \frac{d}{dt}\left[(g_{rr}(\text{null}))^{1/2}\right]\bigg|_{w=0} = \frac{G(t)}{\ell} \ , \quad (11.29)$$

with the strengths of the metric coefficients beginning to fall away from the brane in the event that $F(t)$ and $G(t)$ are positive. As we shall see below, in some cosmologies this will actually be the case. And in fact, we shall actually construct some cosmologies in which, in the entire region between $w_0(t_i)$ and the brane, the quantities $-g_{00}(\text{null})$ and $g_{rr}(\text{null})$ will actually be maximal at the brane itself.

Chapter 12

Generalized Einstein Equations on the Brane

12.1 Derivation of the generalized Einstein equations on the brane

In our discussion of embedded RW branes we had found that the cosmological evolution equation on an embedded brane (viz. Eq. (9.21)) departed quite substantially from the standard Friedmann evolution equation of an unembedded 4-dimensional cosmology. Such departures turn out to be generic to the brane world, and can be most conveniently be discussed [Shiromizu, Maeda and Sasaki (2000)] through use of Gaussian embedding techniques. Specifically, for the simplified case where the embedded brane set-up can be described by a Gaussian normal metric

$$ds^2 = dw^2 + i_{\mu\nu}dx^\mu dx^\nu \tag{12.1}$$

with normal $n_A = (0,0,0,0,1)$ to the brane, the induced Einstein tensor ${}^{(4)}G_{MN}$ on the brane (viz. the one calculated with the induced metric $i_{\mu\nu}$) only has non-vanishing $(\mu\nu)$ components, components which according to Eq. (4.5) are then given by

$$\begin{aligned}{}^{(4)}G_{\mu\nu} = {} & \frac{2}{3}G_{\alpha\beta}i^\alpha{}_\mu i^\beta{}_\nu + \frac{2}{3}G_{55}i_{\mu\nu} - \frac{1}{6}G^A{}_A i_{\mu\nu} \\ & -K^\alpha{}_\alpha K_{\mu\nu} + K^\alpha{}_\mu K_{\alpha\nu} + \frac{1}{2}(K^\alpha{}_\alpha)^2 i_{\mu\nu} - \frac{1}{2}K_{\alpha\beta}K^{\alpha\beta}i_{\mu\nu} - E_{\mu\nu} \quad ,\end{aligned} \tag{12.2}$$

where $E_{\mu\nu}$ and $K_{\mu\nu}$ (which now also only have non-vanishing $(\mu\nu)$ components) are respectively given by

$$E_{\mu\nu} = i^\beta{}_\mu i^\delta{}_\nu C^5{}_{\beta 5\delta} \tag{12.3}$$

and

$$K_{\mu\nu} = i^\alpha{}_\mu i^\beta{}_\nu n_{\beta;\alpha} = -i^\alpha{}_\mu i^\beta{}_\nu \Gamma^5_{\alpha\beta} \quad . \tag{12.4}$$

With Eq. (12.2) being purely kinematic, its utility in the brane world stems from the fact that the G_{MN} and $K_{\mu\nu}$ tensors which appear in it can be completely determined by the dynamics. Thus, for a Z_2 symmetric set-up in which the only matter field in the bulk is a bulk cosmological constant, and where the only other matter field present is some general energy-momentum tensor $\tau_{\mu\nu}$ on the brane, the 5-dimensional Einstein equations fix G_{MN} both on the brane and in the bulk according to

$$G_{MN} = -\kappa_5^2[-\Lambda_5 g_{MN} + \tau_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(w)] \ , \tag{12.5}$$

while the Israel junction conditions fix the extrinsic curvature at the brane according to the Z_2 symmetric extension of Eq. (5.23), viz.

$$K_{\mu\nu}(w=0^+) = -K_{\mu\nu}(w=0^-) = -\frac{\kappa_5^2}{2}\left[\tau_{\mu\nu} - \frac{1}{3}i_{\mu\nu}(w=0)\tau^{\alpha}{}_{\alpha}\right] \ . \tag{12.6}$$

To utilize Eqs. (12.5) and (12.6) we note that even though $K_{\mu\nu}$ itself has a step discontinuity at the brane, its contribution to Eq. (12.2) is actually continuous at the brane as $K_{\mu\nu}$ only contributes via terms which are quadratic in it. However, according to Eq. (12.5) G_{MN} contains both a continuous piece at the brane and a discontinuous delta function term. Now since ${}^{(4)}G_{\mu\nu}$ is calculated with the induced metric, it itself must possess no $\delta(w)$ type terms. Consequently, the contribution to Eq. (12.2) of the $-\kappa_5^2\tau_{\mu\nu}\delta(w)$ term present in $G_{\mu\nu}$ must be cancelled by the other terms which are present in Eq. (12.2). However, with all the extrinsic curvature dependent terms in Eq. (12.2) being continuous at the brane, the cancellation must thus be effected by the $E_{\mu\nu}$ term alone. The Weyl tensor term dependent must thus contain both a continuous piece at the brane which we designate by

$$\bar{E}_{\mu\nu}(w=0) = \frac{1}{2}[E_{\mu\nu}(w=0^+) + E_{\mu\nu}(w=0^-)] \ , \tag{12.7}$$

and a discontinuous delta function piece which we designate by $E^{\rm disc}_{\mu\nu}$.[1] With the singular part of the 5-dimensional Einstein tensor terms making a net delta function contribution to Eq. (12.2) of the form

$$\frac{2}{3}G_{\alpha\beta}i^{\alpha}{}_{\mu}i^{\beta}{}_{\nu} - \frac{1}{6}G^{\alpha}{}_{\alpha}i_{\mu\nu} = -\kappa_5^2\left[\frac{2}{3}\tau_{\alpha\beta}i^{\alpha}{}_{\mu}i^{\beta}{}_{\nu} - \frac{1}{6}\tau^{\alpha}{}_{\alpha}i_{\mu\nu}\right]\delta(w) \ , \tag{12.8}$$

$E^{\rm disc}_{\mu\nu}$ immediately evaluates to [Mannheim (2001c)]

$$E^{disc}_{\mu\nu} = -\kappa_5^2\left[\frac{2}{3}\tau_{\alpha\beta}i^{\alpha}{}_{\mu}i^{\beta}{}_{\nu} - \frac{1}{6}\tau^{\alpha}{}_{\alpha}i_{\mu\nu}\right]\delta(w) \ . \tag{12.9}$$

After having now taken care of the delta function contribution, the extraction of the continuous piece of Eq. (12.2) is then direct, and is found to yield [Shiromizu, Maeda and Sasaki (2000)]

$${}^{(4)}G_{\mu\nu} = \frac{1}{2}\kappa_5^2\Lambda_5 i_{\mu\nu}(w=0) - \kappa_5^4\Pi_{\mu\nu} - \bar{E}_{\mu\nu}(w=0) \ , \tag{12.10}$$

[1]This discontinuity in the Weyl tensor at the brane is the analog of the delta function discontinuity found in the Riemann tensor of an embedded 2-dimensional sheet given in Eq. (6.14).

where we have introduced

$$\Pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau_{\nu}{}^{\alpha} + \frac{1}{12}\tau^{\alpha}{}_{\alpha}\tau_{\mu\nu} + \frac{1}{8}i_{\mu\nu}(w=0)\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}i_{\mu\nu}(w=0)(\tau^{\alpha}{}_{\alpha})^2 \ . \tag{12.11}$$

Now as written Eq. (12.10) does not initially look very much like the standard (unembedded) 4-dimensional Einstein equations, not even appearing to possess a conventional Einstein source term. However this can actually be readily rectified by introducing an explicit brane cosmological constant term, with the full brane $\tau_{\mu\nu}$ then being decomposed into

$$\tau_{\mu\nu} = -\lambda i_{\mu\nu}(w=0) + S_{\mu\nu} \ , \tag{12.12}$$

where $S_{\mu\nu}$ represents all other brane matter field sources. With this decomposition, Eq. (12.10) can then be rewritten as

$${}^{(4)}G_{\mu\nu} = \Lambda_4 i_{\mu\nu}(w=0) - \kappa_4^2 S_{\mu\nu} - \kappa_5^4 \pi_{\mu\nu} - \bar{E}_{\mu\nu}(w=0) \ , \tag{12.13}$$

where we have introduced

$$\begin{aligned} \pi_{\mu\nu} &= -\frac{1}{4}S_{\mu\alpha}S_{\nu}{}^{\alpha} + \frac{1}{12}S^{\alpha}{}_{\alpha}S_{\mu\nu} + \frac{1}{8}i_{\mu\nu}(w=0)S_{\alpha\beta}S^{\alpha\beta} - \frac{1}{24}i_{\mu\nu}(w=0)(S^{\alpha}{}_{\alpha})^2 \ , \\ \Lambda_4 &= \frac{\kappa_5^2}{12}(\kappa_5^2\lambda^2 + 6\Lambda_5) \ , \quad \kappa_4^2 = \frac{\kappa_5^4\lambda}{6} \ . \end{aligned} \tag{12.14}$$

We recognize the definitions of κ_4^2 and Λ_4 as being precisely those given earlier in Eqs. (2.24) and (8.16), and see now the generic role played by the brane cosmological constant in determining the effective 4-dimensional Newtonian and cosmological constants in the brane world, with the sign of the effective κ_4^2 being fixed directly by the sign of the brane tension λ. As such, Eq. (12.13) represents the generalized Einstein equations which an observer is to experience on the brane, with the $\bar{E}_{\mu\nu}(w=0)$ and $\pi_{\mu\nu}$ terms thus signaling an explicit departure from standard 4-dimensional Einstein equation dynamics.

To put the quadratic $\pi_{\mu\nu}$ term in a slightly more tractable form, it is convenient to descend to the special case where $S_{\mu\nu}$ is taken to be the perfect fluid

$$S_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + p i_{\mu\nu}(w=0) \ , \tag{12.15}$$

since in this case $\pi_{\mu\nu}$ then also takes the form of the perfect fluid

$$\pi_{\mu\nu} = (R + P)U_{\mu}U_{\nu} + P i_{\mu\nu}(w=0) \tag{12.16}$$

whose energy density and pressure are given by

$$R = \frac{\rho^2}{12} \ , \quad P = \frac{1}{12}(\rho^2 + 2\rho p) \ . \tag{12.17}$$

Restricting now to the case where the induced metric at the brane, viz. $i_{\mu\nu}(w=0)$, is taken to be a standard RW metric, the (00) component of Eq. (12.13) then yields

$$-\frac{3(\dot{a}^2+k)}{a^2} = -\Lambda_4 - \kappa_4^2\rho - \frac{\kappa_5^4\rho^2}{12} - \bar{E}_{00}(w=0) \qquad (12.18)$$

as the relevant equation which is to describe cosmic evolution equation on the brane, to be compared with the standard (unembedded) Friedmann evolution equation

$$-\frac{3(\dot{a}^2+k)}{a^2} = -\Lambda_4 - \kappa_4^2\rho \qquad (12.19)$$

in which the $-\kappa_5^2\rho^2/12$ and $-\bar{E}_{00}(w=0)$ terms are of course absent. With regard to Eq. (12.18), we note further on that on setting $\lambda=0$ and taking the bulk to be exact AdS_5 (so that $\bar{E}_{00}(w=0)$ then vanishes identically), and on recalling that $6b^2 = -\kappa_5^2\Lambda_5$, Eq. (12.18) is then found to precisely recover Eq. (9.21) just as it must then do; though in general on the brane it is the full Eq. (12.18) which is to serve as the modified Friedmann cosmological evolution equation.[2]

12.2 Some exact solutions to the modified Friedmann equation

While cosmological evolution is explicitly modified by terms quadratic in the energy density, in an expanding cosmology in which the energy density is to redshift continually, such terms would only be competitive with terms linear in the energy density in the early universe. Consequently, for branes embedded in bulks for which the ${C^5}_{\mu 5\nu}$ components of the Weyl tensor just happen to vanish, late time cosmology on the brane would actually be completely standard. To illustrate this point explicitly, we note that when we restrict to the special case in which $k=0$, ${C^5}_{\mu 5\nu}=0$ and $\Lambda_4=0$ (so that κ_4^2 then equals $\kappa_5^2 b$), then in the radiation era where $\rho = A/a^4$, Eq. (12.18) is found to admit of the exact solution

$$a(t) = \left[\frac{2\kappa_5^2 A}{3}(2bt^2+t)\right]^{1/4} = \left[\frac{4\kappa_5^2 A}{3\ell}\right]^{1/4} t^{1/2}\left[1+\frac{\ell}{2t}\right]^{1/4} . \qquad (12.20)$$

Thus while $a(t)$ will behave as the standard $t^{1/2}$ at late times, at early times $t \ll 1/2b$ $a(t)$ will instead behave as $t^{1/4}$. The AdS_5 curvature scale $1/b$ thus sets the time scale at which departures from standard cosmology might typically be expected to occur.[3] Additionally, when Λ_4 is taken to be the non-zero $\Lambda_4 = 3H^2$ (so that κ_4^2 then equals $\kappa_5^4(H^2+b^2)^{1/2}$), the otherwise same radiation era cosmology admits of

[2] The presence of the $-\kappa_5^2\rho^2/12$ term in the brane cosmological evolution equations was first noted by [Binetruy, Deffayet and Langlois (2000)], with the generation of the $-\kappa_4^2\rho$ term having first been given by [Csaki, Graesser, Randall and Terning (2000)].

[3] When we discuss gravitational fluctuations below, we will see that it will be this same $1/b$ which will set the scale at which the embedding of a brane leads to departures from Newton's inverse square law of gravity.

the exact solution

$$a^4(t) = \frac{\kappa_5^2 Ab}{6H^2}\cosh(4Ht) - \frac{\kappa_5^2 A(H^2+b^2)^{1/2}}{6H^2} \quad , \tag{12.21}$$

a solution which, on setting $e^{4Ht_0} = b/(b^2+H^2)^{1/2}$, also agrees with the equivalent unembedded standard cosmology solution $a^4(t) = \kappa_5^2 A(b^2+H^2)^{1/2}\sinh^2[2H(t+t_0)]/3H^2$ at late times.

For $\Lambda_4 = -3H^2 < 0$ (viz. $\kappa_4^2 = \kappa_5^4(b^2-H^2)^{1/2}$) the above model is also soluble exactly, admitting of solution

$$a^4(t) = -\frac{\kappa_5^2 Ab}{6H^2}\cos(4Ht) + \frac{\kappa_5^2 A(b^2-H^2)^{1/2}}{6H^2} \quad . \tag{12.22}$$

However, unlike the $\Lambda_4 = 0$ and $\Lambda_4 > 0$ cases, this model is not late time expanding. Rather $a(t)$ only expands to a bounded maximum value given by

$$a^4_{\text{max}}(t) = \frac{\kappa_5^2 A}{6H^2}\left[b + (b^2-H^2)^{1/2}\right] \quad . \tag{12.23}$$

With the ratio of the terms in Eq. (12.18) which are quadratic and linear in ρ being given by

$$\frac{\kappa_5^4\rho^2}{12\kappa_4^2\rho} = \frac{H^2}{2[b^2-H^2+b(b^2-H^2)^{1/2}]} \tag{12.24}$$

at this maximum, we see that for H of order b, the right-hand side of Eq. (12.24) is of order one. It is thus of interest to note that there exist brane-world cosmological models in which the modified Friedmann equation on the brane never approximates the standard unembedded one at all.

Apart from radiation era perfect fluids with $p = \rho/3$, Eq. (12.18) also admits of some other interesting exact solutions when $\bar{E}_{00}(w=0) = 0$. Thus for a quintessence fluid with $p = -\rho/3$ and $\rho = A/a^2$,[4] we obtain, in the typical case where $\Lambda_4 = 3H^2 > 0$, the solution

$$a^2(t) = \frac{1}{6H^2}\left[3k - \kappa_4^2 A + [(\kappa_4^2 A - 3k)^2 - \kappa_5^4 H^2 A^2]^{1/2}\cosh(2Ht)\right] \quad , \tag{12.25}$$

with $H \to 0$ limit

$$a^2(t) = \frac{\kappa_5^4 A^2}{12(3k-\kappa_4^2 A)} + \frac{1}{3}(\kappa_4^2 A - 3k)t^2 \quad . \tag{12.26}$$

Similarly, for a quintessence fluid with $p = -2\rho/3$ and $\rho = A/a$, we obtain, in a typical case where $k=0$ and $\Lambda_4 = 0$ (i.e. $\kappa_4^2 = \kappa_5^2 b = \kappa_5^2/\ell$), the solution

$$a(t) = \gamma(t^2 - \ell^2) \quad , \tag{12.27}$$

[4]For a fluid with equation of state $p = \omega\rho$ where ω is a pure constant, the covariant conservation condition $a\dot{\rho} + 3\dot{a}(\rho + p) = 0$ (viz. Eq. (9.8) as written with $f(0,t) = a^2(t)$) entails that $\rho = A/a^{(3\omega+3)}$ where A is a pure constant.

where $\gamma = \kappa_5^2 A/12\ell$, a solution which at late times also agrees with the equivalent unembedded standard cosmology solution $a(t) = \gamma t^2$, just as it should.

The particular solution of Eq. (12.27) is actually of interest for a separate reason. For it the parameters $F(t)$ and $G(t)$ of the metric of Eq. (10.49) evaluate to $F(t) = 1$, $G(t) = (t^2 + \ell^2)/(t^2 - \ell^2)$, so that when a $p = -2\rho/3$, $k = 0$, $\Lambda_4 = 0$ RW brane with positive tension is embedded in AdS_5, the embedded metric then takes the form

$$\begin{aligned} ds^2 = & \; dw^2 - e^{-2b|w|}dt^2 \\ & + \gamma^2 t^4 e^{-2b|w|}\left[1 - \frac{\ell^2 e^{2b|w|}}{t^2}\right]^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad . \end{aligned} \tag{12.28}$$

As we see, the diverging warp factor term $e^{2b|w|}$ which is ordinarily present in g_{00} for RW embeddings just happens to be absent in this particular case since $F(t) = 1$. Consequently, in this particular case the metric coefficient g_{00} falls off indefinitely away from the brane, a situation which we had previously only encountered for maximally 4-symmetric branes. Now in our construction in Chapter 10 of the embedding of a $k = 0$ RW brane world in $M(4,2)$, we had noted that for it there was an upper bound on the allowed values of $|w|$, with the spatial metric coefficients g_{rr}, $g_{\theta\theta}$ and $g_{\phi\phi}$ at any given time falling all the way to it. A Z_2-symmetric embedded $k = 0$ cosmology with $p = -2\rho/3$ thus emerges as an example of an RW brane world in which all warp factor coordinate system metric coefficients, spatial and temporal, fall away from the brane at any given time without ever rising anywhere at all.[5]

12.3 Localization of gravity around a Robertson-Walker brane

The above found falling structure for warp factor coordinate system metric coefficients is not restricted to the permanently expanding $p = -2\rho/3$ cosmological model, with it having a late time analog in the $k = 0$ radiation era cosmological model with $\Lambda_4 = 0$, another permanently expanding cosmology whose explicit form for $a(t)$ as given in Eq. (12.20) happens to be simple enough to actually render the model completely tractable. Specifically, in this model the functions $G(t)$ and $F(t)$ which appear in the embedded metric of Eq. (10.49) are readily found to evaluate to

$$F(t) = 1 - \frac{3\ell^2}{8t^2}\left(1 + \frac{\ell}{2t}\right)^{-1} \quad , \quad G(t) = 1 + \frac{\ell^2}{8t^2}\left(1 + \frac{\ell}{2t}\right)^{-1} \quad , \tag{12.29}$$

[5]An additional feature of this particular cosmology is that for it the integral in Eq. (10.43) can be performed analytically to yield $\beta(t) = -t/\gamma(t^2 - \ell^2)$. In this case the $k = 0$ embedding functions introduced in Eqs. (10.42) and (10.46) evaluate to $\eta_1 = -t$, $\eta_+ = \gamma\ell(t^2 - \ell^2)$, $\eta_- = -1/\gamma\ell + \gamma r^2(t^2 - \ell^2)/\ell$, $\eta_4 = \gamma r(t^2 - \ell^2)$, $\chi_1 = t$, $\chi_+ = -\gamma\ell(t^2 + \ell^2)$, $\chi_- = 1/\gamma\ell - \gamma r^2(t^2 + \ell^2)/\ell$, $\chi_4 = -\gamma r(t^2 + \ell^2)$, so that the orbit of the brane in $M(4,2)$ (viz. $w = 0$) can be determined in a closed form in this case, being simply given as $V + W = \gamma\ell(U^2 - \ell^2)$.

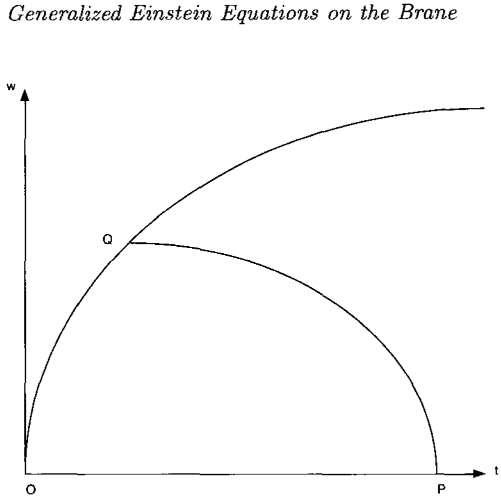

Fig. 12.1 Spacetime structure of an AdS_5 space containing an embedded radiation era $k = 0$ RW brane. The curve OQ denotes the curve $w_0(t) = \ell\text{arctanh}[1/G(t)]$, and the curve QP is the unique null geodesic which originates on the $w_0(t)$ curve at a given Q and reaches the brane at a specifically chosen P.

while the integration required for the function $\beta(t)$ in Eq. (10.43) yields

$$\beta(t) = \int^t dt \frac{\sqrt{\ell^2 \dot{a}^2 + a^2}}{a^2} = \int^t dt \frac{[\ell^2 + 4\ell t + 8t^2]}{\alpha 2^{7/4}(2t^2 + \ell t)^{5/4}}$$
$$= \frac{2^{1/4}}{\alpha(2t^2 + \ell t)^{1/4}} \left[-4t - \ell + 4t \left(\frac{\ell + 2t}{\ell} \right)^{1/4} F[3/4, 1/4, 7/4, -2t/\ell] \right] + c \,, \tag{12.30}$$

where $F[3/4, 1/4, 7/4, -2t/\ell]$ is a hypergeometric function, c is an arbitrary integration constant, and $\alpha = (4\kappa_5^2 A/3\ell)^{1/4}$. Since the function $G(t)$ of Eq. (12.29) is greater than one at all times, in the $k = 0$ RW warp factor coordinate system there will, as discussed in Chapter 10, be a bound $w_0(t)$ on $|w|$ given by $w_0(t) = \ell\text{arctanh}[1/G(t)]$, with the dependence of $w_0(t)$ on t being shown schematically in Fig. (12.1) for this specific $G(t)$. Moreover, once given a closed form for $\beta(t)$, the (w, t) null geodesics with $r = 0$ and general δ_0 can now be determined from Eqs. (11.21) and (10.47), viz.

$$\tanh\left(\frac{|w(t)|}{\ell}\right) = \frac{[\eta_+(t) - \eta_-(t) + [\eta_+(t) + \eta_-(t)]\cos\delta_0 - 2\eta_1(t)\sin\delta_0]}{[\chi_-(t) - \chi_+(t) - [\chi_+(t) + \chi_-(t)]\cos\delta_0 + 2\chi_1(t)\sin\delta_0]} \,, \tag{12.31}$$

where for convenience we have replaced δ_0 by $\pi - \delta_0$ in Eq. (11.21).

With the brane being at $w = 0$, the null geodesic which will arrive at the brane at a time t_0 will be the one for which

$$\sin\delta_0 = \frac{[2\eta_1(t_0)[\eta_+(t_0) - \eta_-(t_0)] - 2[\eta_+(t_0) + \eta_-(t_0)][\eta_1^2(t_0) + \eta_+(t_0)\eta_-(t_0)]^{1/2}]}{[4\eta_1^2(t_0) + [\eta_+(t_0) + \eta_-(t_0)]^2]} \quad ; \tag{12.32}$$

and with $\beta(t)$ of Eq. (12.30) behaving as $\beta(t) \to 2t^{1/2}/\alpha$ at times $t \gg \ell$, the null geodesic which will be at the brane at a time $t_0 \gg \ell$ (point P in Fig. (12.1)) is readily determined to be the one for which the leading order contribution to $\sin\delta_0$ is given by the very small

$$\sin\delta_0 = \frac{\ell\alpha}{t_0^{1/2}} \quad . \tag{12.33}$$

Given Eq. (12.33), we now note that if we make the identification $\sin\delta_0 = \lambda\ell\alpha/t^{1/2}$ in Eq. (12.31),[6] the large t limit of Eq. (12.31) will then take the form

$$\tanh\left(\frac{|w(t)|}{\ell}\right) = 1 - \frac{\ell^2}{8t^2}\frac{(3\lambda-1)^2}{(1-\lambda)^2} = 1 - \frac{\ell^2}{8t^2}\frac{(3t^{1/2}\sin\delta_0 - \ell\alpha)^2}{(\ell\alpha - t^{1/2}\sin\delta_0)^2} \quad . \tag{12.34}$$

With Eq. (12.34) giving the late time dependence of the null geodesic, we note that at $w = 0$ Eq. (12.34) requires that $\sin\delta_0 = (\ell\alpha/t_0^{1/2})(1 - \ell/t_0\sqrt{2})$, with compatibility with Eq. (12.33) being achieved though a non-leading term. Additionally we note that at the time t_i at which $\lambda = 1/2$, the associated $w(t_i)$ is then given by $w(t_i) = \tanh[|w(t_i)|/\ell] = 1 - \ell^2/8t_i^2$. Then, with the large t behavior of $1/G(t)$ being given by $1/G(t) = 1 - \ell^2/8t^2$, we see that the null geodesic which is at the brane at a late time is a geodesic which is also at $|w| = |w_0(t)|$ at a late time too, with the specific time t_i at which it is at the limiting $w_0(t) = \ell\mathrm{arctanh}[1/G(t)]$ curve being given by

$$4t_i = t_0 \quad . \tag{12.35}$$

With the (w,t) plane null geodesic through a given point such as the point P of Fig. (12.1) being unique, we see that the only null geodesics which can reach the brane at late times are ones which set out from the $w_0(t)$ curve at late times also. Along such null geodesics Eq. (12.34) thus holds at every point between the $w_0(t)$ curve and the brane, with the metric coefficients g_{rr} and g_{00} then readily evaluating along such null geodesics as

$$g_{rr}(\mathrm{null}) = \frac{[1 - G(t)\tanh(|w|(t)/\ell)]^2}{[1 - \tanh^2(|w(t)|/\ell)]} = \frac{(\gamma - \beta + \beta\gamma)^2}{(2\gamma - \gamma^2)} \quad ,$$
$$-g_{00}(\mathrm{null}) = \frac{[1 - F(t)\tanh(|w|(t)/\ell)]^2}{[1 - \tanh^2(|w(t)|/\ell)]} = \frac{(\gamma + 3\beta - 3\beta\gamma)^2}{(2\gamma - \gamma^2)} \quad , \tag{12.36}$$

[6] We use this particular parameterization for $\sin\delta_0$, i.e. we define a general late time as $t = \lambda^2 t_0$ where λ is of order one, since the quantities $1/t^2$, $\sin\delta_0/t^{3/2}$, $\sin^2\delta_0/t$, $\sin^3\delta_0/t^{1/2}$ and $\sin^4\delta_0$ are all of the same level of smallness for any t of order t_0.

where

$$\gamma = \frac{\sin^4\delta_0}{8\alpha^4\ell^2\lambda^4}\frac{(3\lambda-1)^2}{(1-\lambda)^2} \quad , \quad \beta = \frac{\sin^4\delta_0}{8\alpha^4\ell^2\lambda^4} \quad . \tag{12.37}$$

Other than in the region very near to the brane where $(1-\lambda)^2 \sim \sin^4\delta_0/8\alpha^4\ell^2$, $-g_{00}(\mathrm{null})$ and $g_{rr}(\mathrm{null})$ are well approximated by

$$\begin{aligned} g_{rr}(\mathrm{null}) &= \frac{\sin^4\delta_0}{\alpha^4\ell^2}\left(\frac{2\lambda-1}{\lambda(3\lambda-1)(1-\lambda)}\right)^2 \quad , \\ -g_{00}(\mathrm{null}) &= \frac{\sin^4\delta_0}{\alpha^4\ell^2}\left(\frac{3\lambda^2-3\lambda+1}{\lambda^2(3\lambda-1)(1-\lambda)}\right)^2 \quad , \end{aligned} \tag{12.38}$$

and are thus very small near $|w| = w_0(t)$. At late times, both $-g_{00}(\mathrm{null})$ and $g_{rr}(\mathrm{null})$ are found to have a very rapid initial fall off away from the brane. $g_{rr}(\mathrm{null})$ is found to fall monotonically all the way to the $w_0(t)$ curve; and while $-g_{00}(\mathrm{null})$ is found to have a minimum at $\lambda = 2/3$, the subsequent rise in $-g_{00}(\mathrm{null})$ from there to the $w_0(t)$ curve at $\lambda = 1/2$ occurs entirely in the region where it is totally suppressed by the $\sin^4\delta_0$ factor in Eq. (12.38).[7] Thus $-g_{00}(\mathrm{null})$ and $g_{rr}(\mathrm{null})$ are only non-negligible in the region very close to the brane, and both fall off very rapidly there. Additional analysis reveals that $-g_{00}(\mathrm{null})$ would actually rise, and even rise quite substantially, if evaluated via Eq. (12.31) at times prior to t_i. But since such $t < t_i$ times would take us outside the allowed $|w| \leq |w_0(t_i)|$ warp factor coordinate system region, within this region a late time observer on a $p = \rho/3$, $k = 0$ RW brane embedded in AdS_5 would look back on a $-g_{00}(\mathrm{null})$ and a $g_{rr}(\mathrm{null})$ which would both be maximal at the brane itself.

As a further example of this effect, we consider the closed 3-sphere, $k > 0$ RW brane-world model, viz. a model in which the topology of the closed 3-sphere limits $|w|$ to the range $|w| \leq |w_0(t)|$. We shall take the matter content on the brane to consist solely of a quintessence fluid with $p = -\rho/3$ (so that $\Lambda_4 = 0$ and $\rho = A/a^2$). We choose this particular model since for the special case with $\kappa_5^2 A = 6k\ell$, the brane world turns out to be soluble in closed form. Specifically, in this case we find that the RW scale factor is given by the permanently expanding $a^2(t) = k(t^2 + 2\ell t)$, with the parameter $\beta(t)$ of Eq. (10.7) then evaluating to $\beta(t) = \mathrm{arcsinh}[(t^2 + 2\ell t)^{1/2}/\ell]$. With such a time dependence, the parameters $G(t)$ and $F(t)$ of Eq. (10.13) evaluate to $G(t) = 1 + 1/\sinh^2\beta$ and $F(t) = 1 - 1/\sinh^2\beta$. Similarly, the $k > 0$ embedding parameters of Eqs. (10.6) and (10.11) evaluate to $\eta_1 = \ell\cos\beta\cosh\beta$, $\eta_2 = \ell\sin\beta\cosh\beta$, $\eta_3 = \ell\sinh\beta$, $\chi_1 = \ell\sin\beta/\sinh\beta - \eta_1$, $\chi_2 = -\ell\cos\beta/\sinh\beta - \eta_2$, $\chi_3 = -\ell\sinh\beta - \ell/\sinh\beta$. Additionally, the geodesic parameters of Eq. (11.9) evaluate to $A = \eta_1\cos\delta_0 - \eta_2\sin\delta_0 = \ell\cos(\delta_0 + \beta)\cosh\beta$,

[7] In general for a brane model with a Λ_4 and a perfect ρ, p fluid, the junction conditions at the brane (e.g. Eqs. (9.20) and (12.6)) entail that $G(t) = (1 + \ell^2\Lambda_4/3)^{1/2} + (\kappa_5^2\ell/6)\rho$, $G(t) - F(t) = (\kappa_5^2\ell/2)(\rho + p)$. Thus, in any $\Lambda_4 \geq 0$ cosmology which is late time expanding, the quantity $-g_{00}(\mathrm{null}, w_0(t_i)) = [G(t_i) - F(t_i)]^2/[G^2(t_i) - 1]$ will be small at any t_i which is itself late.

$B = \chi_1\cos\delta_0 - \chi_2\sin\delta_0 = \ell\sin(\delta_0+\beta)/\sinh\beta - \ell\cos(\delta_0+\beta)\cosh\beta$, with it being the negative $(B^2-A^2-\ell^2)^{1/2} = -\ell[\cos(\delta_0+\beta)-\sin(\delta_0+\beta)\cosh\beta/\sinh\beta]$ which is relevant here. The null geodesic of Eq. (11.9) will be at the brane at the time t_0 which obeys $\cos(\delta_0+\beta_0) = 1/\cosh\beta_0$, viz. at $\delta_0 = -\beta_0 + \arccos(1/\cosh\beta_0)$. Moreover, this same null geodesic is found to be at the limiting $w_0(t)$ curve at a time t_i which obeys $\tan(\delta_0+\beta_i) = \tanh\beta_i$. If we now require t_0 to be a late time, t_i is then given by $\beta_i = \beta_0 - \pi/4$, viz. by $\text{arcsinh}[(t_i^2+2\ell t_i)^{1/2}/\ell] = \text{arcsinh}[(t_0^2+2\ell t_0)^{1/2}/\ell] - \pi/4$, with t_i thus being late too. Consequently, a late time observer can only receive signals which leave the $w_0(t)$ curve at late times. Then with $g_{00}(\text{null}, w_0(t_i))$ being negligible at such late times, and with $g_{00}(\text{null})$ and $g_{rr}(\text{null})$ (as evaluated along the null geodesic) being found to be maximal at the brane, gravity thus localizes around a late time observer on a $p=-\rho/3$, $k>0$ RW brane embedded in AdS_5.

While the development by [Shiromizu, Maeda and Sasaki (2000)] of the generalized Einstein equations of Eq. (12.13) enabled us to characterize the differences between embedded and unembedded gravity in a very general and straightforward way, it will turn out to also be of great value in analyzing gravitational fluctuations around specific brane-world backgrounds, cases which are not in general soluble exactly. Thus for the original M_4^+ model for instance, the background itself consists of an AdS_5 invariant bulk in which a brane with cosmological constant $\lambda = (-6\Lambda_5/\kappa_5^2)^{1/2}$ is embedded, viz. a background in which $i_{\mu\nu}(w=0) = \eta_{\mu\nu}$, $\Lambda_4 = 0$, $\bar{E}_{\mu\nu}(w=0) = 0$, $S_{\mu\nu} = 0$ and $\pi_{\mu\nu} = 0$. In such a background Eq. (12.13) then requires that ${}^{(4)}G_{\mu\nu}$ be equal to zero, viz. precisely the value the 4-dimensional Einstein tensor actually does take when evaluated with an induced metric which is flat. If we now add a small perturbative source $S_{\mu\nu}$ to this brane, the addition of such a source will induce first order changes $\delta G_{\mu\nu}$ and $\delta\bar{E}_{\mu\nu}$ in the Einstein and Weyl tensor terms. Then, since Eq. (12.13) must still continue to hold, with the second order $\pi_{\mu\nu}$ now being negligible, these first order changes must thus obey

$$\delta G_{\mu\nu} = -\kappa_4^2 S_{\mu\nu} - \delta\bar{E}_{\mu\nu}(w=0) \tag{12.39}$$

on the brane. With the gravitational fluctuations around an unembedded flat M_4 background geometry obeying $\delta G_{\mu\nu} = -\kappa_4^2 S_{\mu\nu}$, we thus see that the fluctuations in the embedded case are remarkably Einstein-like, with $\kappa_4^2 = \kappa_5^4\lambda/6$ indeed serving as the relevant 4-dimensional gravitational coupling on the brane just as noted in Chapter 2, but with the fluctuations being able to actually depart from unembedded gravity through the presence of the Weyl tensor term. Since this Weyl tensor term is a projection of the bulk Weyl tensor, it involves derivatives of the fluctuation metric with respect to w as then calculated at the brane, and it is thus through the Weyl tensor term that bulk information is communicated to an observer on the brane. We thus see that brane-world gravitational fluctuations are capable of both recovering the standard features of unembedded gravity and of departing from them in monitorable ways, and so we turn now to a detailed exploration of these fluctuations to determine what does in fact occur.

Chapter 13

General Structure of Brane-World Gravitational Fluctuations

13.1 Structure of the brane-world background geometries

To deal with brane-world set-ups more complicated than the ones presented so far it is currently necessary to resort to perturbative techniques in which sources are added perturbatively to backgrounds for which exact solutions have already been found. To lowest order in these additional sources the problem then becomes that of gravitational fluctuation theory, and of the known exact background metrics, so far the only cases for which the first order tensor fluctuation equations have themselves been solved exactly are those in which the background induced metric is of the separable form $i_{\mu\nu}(w,x^\lambda)=e^{2A(w)}q_{\mu\nu}(x^\lambda)$. This class includes all brane-world set-ups in which the geometry on the brane is maximally 4-symmetric, viz. the $M_4^{\pm}$, $dS_4^{\pm}$ and $AdS_4^{\pm}$ cases, but does not include the time dependent RW branes where the warp factor of Eq. (9.17) is a non-separable function of w and t. For embedded maximally 4-symmetric brane worlds the relevant background metrics were given in Chapter 8, viz.

$$ds^2(M_4^{\pm})=dw^2+e^{-2\epsilon(\lambda)b|w|}[dx^2+dy^2+dz^2-dt^2] \ , \tag{13.1}$$

$$ds^2(dS_4^{\pm})=dw^2+\frac{H^2}{b^2}\sinh^2(\epsilon(\lambda)b|w|-\sigma)\left[e^{2Ht}(dx^2+dy^2+dz^2)-dt^2\right] \tag{13.2}$$

(where $\sinh\sigma=b/H$), and

$$ds^2(AdS_4^{\pm})=dw^2+\frac{H^2}{b^2}\cosh^2(\epsilon(\lambda)b|w|-\sigma)\left[dx^2+e^{2Hx}(dy^2+dz^2-dt^2)\right] \tag{13.3}$$

(where $\cosh\sigma=b/H$); and with all of these cases being describable by the generic metric

$$ds^2=dw^2+e^{2A(|w|)}q_{\mu\nu}(x^\lambda)dx^\mu dx^\nu \ , \tag{13.4}$$

gravitational fluctuations around these backgrounds admit of a unified treatment which we now present.[1]

13.2 General formulation of the gravitational fluctuation equations

In gravitational fluctuation theory around a given background with general equation of motion

$$G_{MN} = -\kappa_5^2 T_{MN} \tag{13.5}$$

and general background metric g_{MN}, the introduction of a perturbative source $\delta\tau_{MN}$ will induce a first order change

$$\delta g_{MN} = h_{MN} \quad , \quad \delta g^{MN} = -h^{MN} \tag{13.6}$$

in g_{MN}, a first order change δR_{MN} in the Ricci tensor, and a first order change δT_{MN} in the background T_{MN}, to thus lead to a first order fluctuation equation of the form

$$\begin{aligned}\Delta G_{MN} &= \delta G_{MN} + \kappa_5^2 \delta T_{MN} \\ &= \delta R_{MN} - \frac{1}{2} h_{MN} R^L{}_L - \frac{1}{2} g_{MN} g^{LS} \delta R_{LS} + \frac{1}{2} g_{MN} h^{LS} R_{LS} + \kappa_5^2 \delta T_{MN} \\ &= -\kappa_5^2 \delta\tau_{MN} \ .\end{aligned} \tag{13.7}$$

In Eq. (13.7) we have introduced the quantity ΔG_{MN} since in the presence of a non-vanishing background energy-momentum tensor, it is ΔG_{MN} rather than δG_{MN} which embodies the overall response of the system to $\delta\tau_{MN}$. In Eq. (13.7) the quantity δR_{MN} is given by

$$\delta R_{MN} = \frac{1}{2}[h_{;M;N} - h^L{}_{N;M;L} - h^L{}_{M;N;L} + \Box_T h_{MN}] \ , \tag{13.8}$$

where all covariant derivatives are to be calculated with respect to the background metric g_{MN}, with $\Box_T h_{MN}$ being the tensor box operator

$$\Box_T h_{MN} = g^{RL} h_{MN;R;L} \tag{13.9}$$

in which $h_{MN;R;L}$ is given as

$$\begin{aligned}h_{MN;R;L} = {} & [\partial_L \partial_R - \Gamma^S_{LR} \partial_S] h_{MN} + [\Gamma^S_{LM} \Gamma^K_{RN} + \Gamma^S_{LN} \Gamma^K_{RM}] h_{KS} \\ & + [\Gamma^S_{LN} \Gamma^K_{RS} + \Gamma^S_{LR} \Gamma^K_{SN} - \partial_L \Gamma^K_{RN} - \Gamma^K_{RN} \partial_L - \Gamma^K_{LN} \partial_R] h_{KM} \\ & + [\Gamma^S_{LM} \Gamma^K_{RS} + \Gamma^S_{LR} \Gamma^K_{SM} - \partial_L \Gamma^K_{RM} - \Gamma^K_{RM} \partial_L - \Gamma^K_{LM} \partial_R] h_{KN}\end{aligned} \tag{13.10}$$

[1]The work on fluctuations to be presented in this monograph is drawn in part from [Guth, Kaiser, Mannheim and Nayeri (2004b)].

when explicitly written in terms of the background Christoffel symbols. Use of the covariant derivative interchange relation

$$T_{LM;N;K} - T_{LM;K;N} = T^{S}{}_{M}R_{LSNK} - T_{L}{}^{S}R_{SMNK} \tag{13.11}$$

allows us to rewrite Eq. (13.8) as

$$\begin{aligned}\delta R_{MN} = \frac{1}{2}\ [&h_{;M;N} - h^{L}{}_{N;L;M} - h^{L}{}_{M;L;N} + h^{S}{}_{N}R_{SM} + h^{S}{}_{M}R_{SN} \\ &-2h^{LS}R_{MLNS} + \Box_{T}h_{MN}]\ ,\end{aligned} \tag{13.12}$$

a form which is of help with the actual evaluation of δR_{MN} once an explicit background Riemann tensor is specified.[2]

13.3 Gauge invariance considerations

Treatment of an equation such as Eq. (13.7) is greatly facilitated by taking into account the underlying gauge invariance of the theory. In general, under an infinitesimal coordinate transformation $\bar{x}^M = x^M + \epsilon^M$ where the gauge function ϵ^M is some general function of the coordinates, any general metric g_{MN} will transform in first order as $\bar{g}_{MN} = g_{MN} + \epsilon_{M;N} + \epsilon_{N;M}$. For the case where this metric can be written in the particular form of a background metric plus a fluctuation which is of the same order as ϵ^M, to lowest order in ϵ^M the metric will thus transform as

$$\bar{g}_{MN} + \bar{h}_{MN} = g_{MN} + h_{MN} + \epsilon_{M;N} + \epsilon_{N;M}\ , \tag{13.13}$$

so that

$$\bar{h}_{MN} = h_{MN} + \epsilon_{M;N} + \epsilon_{N;M}\ . \tag{13.14}$$

Consequently, either of h_{MN} and $\bar{h}_{MN}$ can be considered to be the fluctuation, with both of them describing the same physical situation; and indeed, for any given h_{MN} which obeys the fluctuation equation $\Delta G_{MN} = -\kappa_5^2\delta\tau_{MN}$, there will exist a whole family of $\bar{h}_{MN} = h_{MN} + \epsilon_{M;N} + \epsilon_{N;M}$ fluctuations which will obey it also (e.g. [Weinberg (1972)]).[3] Consequently, of the 15 components of h_{MN} (in the 5-dimensional case) five are redundant, so that one can thus reduce the problem to having to deal with only 10 independent fluctuation components; and the more judicious a choice of gauge one makes, the more tractable the fluctuation equation can then be rendered.

In order to take advantage of this gauge freedom, we need to evaluate all the background tensors and covariant derivatives which are required for Eq. (13.7).

[2]The form of Eq. (13.12) is particularly well suited to the harmonic gauge and transverse-traceless fluctuations which are to be encountered in the following.

[3]We provide a proof of this result in Appendix A.

In terms of the generic metric of Eq. (13.4) the non-vanishing components of the background Riemann tensor are found to take the form

$$R^5{}_{\mu 5\kappa} = [A''(w) + A'^2(w)]g_{\mu\kappa} \quad , \quad R^\lambda{}_{\mu\nu\kappa} = A'^2(w)[\delta^\lambda_\nu g_{\mu\kappa} - \delta^\lambda_\kappa g_{\mu\nu}] + \tilde{R}^\lambda{}_{\mu\nu\kappa} \tag{13.15}$$

while the non-vanishing elements of the background Ricci tensor and Christoffel symbols are respectively given as

$$R_{\mu\kappa} = [A''(w) + 4A'^2(w)]g_{\mu\kappa} + \tilde{R}_{\mu\kappa} \quad , \quad R_{55} = 4[A''(w) + A'^2(w)] \tag{13.16}$$

and

$$\Gamma^5_{\mu\kappa} = -A'(w)g_{\mu\kappa} \quad , \quad \Gamma^\lambda_{5\kappa} = A'(w)\delta^\lambda_\kappa \quad , \quad \Gamma^\lambda_{\mu\kappa} = \tilde{\Gamma}^\lambda_{\mu\kappa} \quad . \tag{13.17}$$

In Eqs. (13.15) – (13.17) we have used primes to denote differentiation with respect to w, and have introduced the tilde notation to indicate that a tilded quantity is to be calculated in the 4-dimensional theory associated with the induced metric $q_{\mu\nu}$. With $\tilde{R}^\lambda{}_{\mu\nu\kappa}$ being given by

$$\tilde{R}^\lambda{}_{\mu\nu\kappa} = kH^2(\delta^\lambda_\kappa q_{\mu\nu} - \delta^\lambda_\nu q_{\mu\kappa}) \tag{13.18}$$

in the maximally 4-symmetric brane backgrounds of interest [here $k = (1, 0, -1)$ for $(dS_4,\ M_4,\ AdS_4)$], and with the background energy-momentum tensor being given by

$$T_{MN} = -\Lambda_5 g_{MN} - \delta^\mu_M \delta^\nu_N \lambda q_{\mu\nu}\delta(w) \quad , \tag{13.19}$$

in each of the six $dS_4^\pm$, $M_4^\pm$ and $AdS_4^\pm$ background cases of interest (for the varying choices of the sign and magnitude of λ), the background is fully specified.

13.4 Imposition of the axial gauge in the brane world

Armed with the above information we can now choose a gauge. As first noted by Randall and Sundrum themselves [Randall and Sundrum (1999a)], in the brane world it is extremely convenient to work in the 5-dimensional axial gauge in which

$$h_{5M} = 0 \quad , \tag{13.20}$$

a gauge which is immediately suggested given the analogous $g_{5\mu} = 0$ property of the background metric. For metrics of the form of Eq. (13.4) the gauge transformation of Eq. (13.14) takes the form

$$\begin{aligned} \bar{h}_{55} &= h_{55} + 2\partial_w \xi^5 \quad , \quad \bar{h}_{5\mu} = h_{5\mu} + \partial_\mu \xi^5 + e^{2A} q_{\mu\nu}\partial_w \xi^\nu \quad , \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} + e^{2A} q_{\mu\lambda}\partial_\nu \xi^\lambda + e^{2A} q_{\nu\lambda}\partial_\mu \xi^\lambda + 2e^{2A} A' q_{\mu\nu}\xi^5 \quad , \end{aligned} \tag{13.21}$$

where A' as given by

$$A' = \frac{dA}{dw} = \frac{dA}{d|w|}\frac{d|w|}{dw} = \epsilon(w)\frac{dA}{d|w|} \tag{13.22}$$

has a step discontinuity structure in w since the relevant e^A warp factors depend on $|w|$ rather than on w itself.[4] Thus in order to implement the axial gauge while simultaneously preserving the Z_2 symmetry structure of h_{MN},[5] we set

$$\xi^5 = \epsilon(w)[f(|w|,x) - f(0,x)] \quad , \quad \xi^\nu = \xi^\nu(|w|,x) \quad , \tag{13.23}$$

and choose these gauge functions according to

$$\begin{aligned} & h_{55}(|w|,x) + 2\frac{df(|w|,x)}{d|w|} = 0 \quad , \\ & \epsilon(w)\left[h_{5\mu}(|w|,x) + \partial_\mu f(|w|,x) - \partial_\mu f(0,x) + e^{2A} q_{\mu\nu}\frac{d\xi^\nu}{d|w|}\right] = 0 \quad , \end{aligned} \tag{13.24}$$

i.e. according to

$$\begin{aligned} & f(|w|,x) - f(0,x) = -\frac{1}{2}\int_0^{|w|} d|w| h_{55}(|w|,x) \quad , \\ & q_{\mu\nu}\xi^\nu = -\int d|w| e^{-2A}\left[h_{5\mu}(|w|,x) - \frac{1}{2}\partial_\mu \int_0^{|w|} d|w| h_{55}(|w|,x)\right] \quad . \end{aligned} \tag{13.25}$$

With these particular gauge transformations we not only implement the axial gauge in the bulk, we do so on the brane as well. With $\xi^5(0,x)$ vanishing identically in Eq. (13.23), it follows that $\bar{x}^5(0,x) = x^5(0) = 0$, with Eq. (13.21) thus not leading to any reparameterization of the position of the brane.[6]

13.5 Gravitational fluctuation equations in the axial gauge

Once the axial gauge is implemented, explicit evaluation of the components of δR_{MN} in it is then fairly straightforward. In the axial gauge the non-vanishing covariant

---

[4]Similarly A'' as given by $A'' = d^2A/d|w|^2 + 2\delta(w)dA/d|w|$ has a delta function singularity, with use of the generic A'' in the following allowing us to readily track the delta function terms associated with the junction conditions at the brane.

[5]As follows directly from the form for δR_{MN} given in Eq. (13.8), $h_{55}(w,x) = h_{55}(|w|,x)$ and $h_{\mu\nu}(w,x) = h_{\mu\nu}(|w|,x)$ have to be even functions of w while $h_{5\mu}(w,x) = \epsilon(w)h_{5\mu}(|w|,x)$ has to be odd.

[6]Gauge transformations which do reparameterize the position of the brane are known in the literature as "brane bending" transformations [Garriga and Tanaka (2000); Giddings, Katz and Randall (2000)].

derivatives associated with the metric of Eq. (13.4) are given as

$$
\begin{aligned}
h_{\mu\nu;\lambda} &= \tilde{h}_{\mu\nu;\lambda} \ , \\
h_{\mu 5;\lambda} &= -A' h_{\mu\lambda} \ , \\
h_{\mu\nu;5} &= (\partial_w - 2A') h_{\mu\nu} \ , \\
h_{\mu\nu;\lambda;\sigma} &= \tilde{h}_{\mu\nu;\lambda;\sigma} + A' g_{\lambda\sigma}\partial_w h_{\mu\nu} - A'^2(2g_{\lambda\sigma}h_{\mu\nu} + g_{\mu\sigma}h_{\nu\lambda} + g_{\nu\sigma}h_{\mu\lambda}) \ , \\
h_{\mu\nu;\lambda;5} &= \partial_w \tilde{h}_{\mu\nu;\lambda} - 3A'\tilde{h}_{\mu\nu;\lambda} \ , \\
h_{\mu\nu;5;\lambda} &= \partial_w \tilde{h}_{\mu\nu;\lambda} - 3A'\tilde{h}_{\mu\nu;\lambda} \ , \\
h_{\mu 5;\lambda;\sigma} &= -A'(\tilde{h}_{\mu\lambda;\sigma} + \tilde{h}_{\mu\sigma;\lambda}) \ , \\
h_{\mu\nu;5;5} &= \partial_w^2 h_{\mu\nu} - 4A'\partial_w h_{\mu\nu} + (4A'^2 - 2A'')h_{\mu\nu} \ , \\
h_{55;\lambda;\sigma} &= 2A'^2 h_{\lambda\sigma} \ , \\
h_{\mu 5;\lambda;5} &= -A'\partial_w h_{\mu\lambda} + (2A'^2 - A'')h_{\mu\lambda} \ , \\
h_{\mu 5;5;\sigma} &= -A'\partial_w h_{\mu\sigma} + 3A'^2 h_{\mu\sigma} \ ;
\end{aligned}
\tag{13.26}
$$

with calculation of the components of δR_{MN} then directly yielding

$$
\begin{aligned}
\delta R_{55} &= \frac{1}{2}h'' + A'h' \ , \quad \delta R_{5\mu} = \frac{1}{2}\partial_w[h_{;\mu} - h^{\nu}{}_{\mu;\nu}] \ , \\
\delta R_{\mu\nu} &= \frac{1}{2}[h''_{\mu\nu} + 4A'^2 h_{\mu\nu} + A' g_{\mu\nu} h'] + e^{-2A}\widetilde{\delta R}_{\mu\nu} \ ,
\end{aligned}
\tag{13.27}
$$

where $h = g^{\mu\nu}h_{\mu\nu}$ is the trace of the fluctuation, and where $\widetilde{\delta R}_{\mu\nu}$ is given as[7]

$$
\begin{aligned}
\widetilde{\delta R}_{\mu\nu} &= \frac{1}{2}[\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{h} - \tilde{\nabla}_\lambda\tilde{\nabla}_\mu\tilde{h}^{\lambda}{}_{\nu} - \tilde{\nabla}_\lambda\tilde{\nabla}_\nu\tilde{h}^{\lambda}{}_{\mu} + \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h}_{\mu\nu}] \\
&= \frac{1}{2}[\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{h} - \tilde{\nabla}_\mu\tilde{\nabla}_\lambda\tilde{h}^{\lambda}{}_{\nu} - \tilde{\nabla}_\nu\tilde{\nabla}_\lambda\tilde{h}^{\lambda}{}_{\mu} \\
&\quad -8kH^2\tilde{h}_{\mu\nu} + 2kH^2\tilde{h}q_{\mu\nu} + \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h}_{\mu\nu}] \ .
\end{aligned}
\tag{13.28}
$$

From Eq. (13.27) we obtain the convenient relation

$$
g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2}h'' + 4A'h' + 4A'^2 h + A''h + e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu}
\tag{13.29}
$$

on bringing the warp factor dependence of $g^{\mu\nu}$ inside the w derivative terms. From its definition in Eq. (13.7) the components of ΔG_{MN} readily also follow, viz.

$$
\Delta G_{55} = -\frac{3}{2}A'h' + \frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu} - \frac{1}{2}e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} \ ,
\tag{13.30}
$$

$$
\Delta G_{5\mu} = \frac{1}{2}\partial_w[h_{;\mu} - h^{\nu}{}_{\mu;\nu}] \ ,
\tag{13.31}
$$

[7] To clarify our notation we note that even though tilded operations are to be performed with the induced metric $q_{\mu\nu}(x^\lambda)$, the fluctuation $\tilde{h}_{\mu\nu}(x^\lambda, w)$ still depends on w; and even though the covariant $h_{\mu\nu}(x^\lambda, w)$ is the same as $\tilde{h}_{\mu\nu}(x^\lambda, w)$, contravariants are given as $h^{\mu\nu} = g^{\mu A}g^{\nu B}h_{AB}$, $\tilde{h}^{\mu\nu} = q^{\mu\alpha}q^{\nu\beta}\tilde{h}_{\alpha\beta}$, with the $\tilde{\nabla}_\mu$ symbol denoting the evaluation of a covariant derivative in a theory whose metric is $q_{\mu\nu}(x^\lambda)$.

$$\begin{aligned}
\Delta G_{\mu\nu} = & \frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 8A'' - 16A'^2 - g^{\sigma\tau}\tilde{R}_{\sigma\tau} + 12b^2 - 2\kappa_5^2\lambda\delta(w)\right] h_{\mu\nu} \\
& -\frac{1}{2}\left[h'' + 4A'h' + e^{-2A}g^{\sigma\tau}\widetilde{\delta R}_{\sigma\tau} - h^{\sigma\tau}\tilde{R}_{\sigma\tau}\right] g_{\mu\nu} + e^{-2A}\widetilde{\delta R}_{\mu\nu} \\
= & \frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 8A'' - 4A'^2 - 2\kappa_5^2\lambda\delta(w)\right] h_{\mu\nu} \\
& -\frac{1}{2}\left[h'' + 4A'h' + e^{-2A}g^{\sigma\tau}\widetilde{\delta R}_{\sigma\tau} - h^{\sigma\tau}\tilde{R}_{\sigma\tau}\right] g_{\mu\nu} + e^{-2A}\widetilde{\delta R}_{\mu\nu} \quad , \qquad (13.32)
\end{aligned}$$

$$\begin{aligned}
g^{\mu\nu}\Delta G_{\mu\nu} = & -\frac{3}{2}h'' - 6A'h' - 3A''h - 6A'^2 h + 6b^2 h - \kappa_5^2\lambda h\delta(w) - \frac{1}{2}g^{\mu\nu}\tilde{R}_{\mu\nu}h \\
& +2h^{\mu\nu}\tilde{R}_{\mu\nu} - e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} \\
= & -\frac{3}{2}h'' - 6A'h' - 3A''h - \kappa_5^2\lambda h\delta(w) + 2h^{\mu\nu}\tilde{R}_{\mu\nu} \\
& -e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} \quad , \qquad (13.33)
\end{aligned}$$

with the gravitational modes of interest then being calculated as the solutions to $\Delta G_{MN} = -\kappa_5^2\delta\tau_{MN}$.[8] While we have now used the gauge freedom to reduce the number of independent components of the 15-component h_{MN} to the 10-component $h_{\mu\nu}$, and while we appear to have used up the freedom associated with the existence of a total of only five $\bar{x}^M = x^M + \epsilon^M$ coordinate invariances, it turns out that the equations of motion of Eqs. (13.30) – (13.33) also admit of some additional residual gauge symmetry which still preserves the axial gauge. We shall now exploit this additional freedom to decompose the 10-component $h_{\mu\nu}$ into two 5-component sectors, in one of which $h_{\mu\nu}$ is both transverse and traceless and in the other not.

[8]In simplifying Eqs. (13.32) and (13.33) we have used the relation $12b^2 - 12A'^2 - g^{\mu\nu}\tilde{R}_{\mu\nu} = 0$ which follows from the background Einstein equation $R_{55} - (1/2)g_{55}R^A{}_A = \kappa_5^2\Lambda_5 = -6b^2$ as applied to the metric of Eq. (13.4).

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Chapter 14

Transverse-Traceless M_4 Fluctuation Modes

14.1 NT sector fluctuations

To decompose the 10-component axial gauge fluctuation $h_{\mu\nu}$ into five transverse-traceless (TT) and five non-transverse-traceless (NT) components, we need to explicitly solve the $\Delta G_{MN} = -\kappa_5^2 \delta\tau_{MN}$ fluctuation equation in the axial gauge. To determine the gravitational propagator between two points on the brane or between one point on the brane and one in the bulk (the two cases of most physical interest for an observer on the brane), we need only consider the specific case where $\delta\tau_{MN}$ is taken to be a source on the brane. We thus set $\delta\tau_{MN} = \delta(w)\delta^\mu_M \delta^\nu_N S_{\mu\nu}$, with the source then having to obey the same covariant conservation condition[1]

$$S^{\mu\nu}{}_{;\mu} = 0 \tag{14.1}$$

as it would obey in the unembedded case, with the brane-world set-up thus being such that the brane source does not exchange energy or momentum with the bulk.

To see how the decomposition procedure works in practice, we first consider the case of a background Minkowski brane, and since the discussion is identical for either sign of the brane tension, we present the analysis for the M_4^+ case. In the presence of a perturbative source on a positive tension Minkowski brane the brane-world equations of motion of Eqs. (13.30) – (13.33) take the form

$$\Delta G_{55} = \frac{1}{2}[3b\epsilon(w)h' - e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h + \partial_\mu\partial_\nu h^{\mu\nu}] = 0 \ , \tag{14.2}$$

$$\Delta G_{5\mu} = \frac{1}{2}\partial_w(\partial_\mu h - \partial_\nu h^\nu{}_\mu) = 0 \ , \tag{14.3}$$

[1] In the brane-world set-up associated with the background metric of Eq. (13.4) and the background T_{MN} of Eq. (13.19), the standard first order covariant conservation condition $\delta\tau^{MN}{}_{;M} + \delta T^{MN}{}_{;M} + T^{NL}\delta\Gamma^M_{ML} + T^{ML}\delta\Gamma^N_{ML} = 0$ for the perturbation directly entails that $S^{\mu\nu}{}_{;\mu} = 0$, a condition which can also be written as $\tilde{S}^{\mu\nu}{}_{;\mu} = \tilde{\nabla}_\mu \tilde{S}^{\mu\nu} = 0$ if the warp factor of Eq. (13.4) is normalized to $\exp(A(w=0)) = 1$.

$$\Delta G_{\mu\nu} = \frac{1}{2}[\partial_w^2 h_{\mu\nu} - 4b^2 h_{\mu\nu} + 4bh_{\mu\nu}\delta(w) + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\sigma h^\sigma{}_\nu$$
$$-\partial_\nu\partial_\sigma h^\sigma{}_\mu - g_{\mu\nu}(h'' - 4b\epsilon(w)h' + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h - \partial_\mu\partial_\nu h^{\mu\nu})]$$
$$= -\kappa_5^2 S_{\mu\nu}\delta(w) \ , \tag{14.4}$$

$$g^{\mu\nu}\Delta G_{\mu\nu} = -\frac{3}{2}h'' + 6b\epsilon(w)h' - e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h + \partial_\mu\partial_\nu h^{\mu\nu} = -\kappa_5^2\delta(w)S \tag{14.5}$$

where $S = \eta^{\mu\nu}S_{\mu\nu}$ is the trace of $S_{\mu\nu}$. Inspection of the above equations reveals that Eqs. (14.2), (14.3) and (14.5) are purely NT, involving only the trace $h = g_{\mu\nu}h^{\mu\nu}$ and divergence $\partial_\mu h^{\mu\nu}$ of the fluctuation; and with there being as many equations as there are unknowns in the NT sector, we are actually able to solve for h and $\partial_\mu h^{\mu\nu}$ completely. Thus, on combining the ΔG_{55} and $g^{\mu\nu}\Delta G_{\mu\nu}$ equations, we obtain

$$h'' - 2b\epsilon(w)h' = \frac{2}{3}\kappa_5^2\delta(w)S \ , \tag{14.6}$$

an equation whose most general solution can be written as

$$h = e^{2b|w|}\alpha(x^\lambda) + \beta(x^\lambda) \ , \tag{14.7}$$

where $\alpha(x^\lambda)$ and $\beta(x^\lambda)$ are two w-independent Lorentz scalar functions, with the $\delta(w)$ discontinuity junction condition requiring $\alpha(x^\lambda)$ to be given as

$$\alpha(x^\lambda) = \frac{\kappa_5^2}{6b}S \ . \tag{14.8}$$

Then, on inserting the above solution for h into Eq. (14.3) we obtain

$$\partial_\nu h^\nu{}_\mu = e^{2b|w|}\partial_\mu\alpha(x^\lambda) + \gamma_\mu(x^\lambda) \tag{14.9}$$

as the most general allowed form for $\partial_\nu h^\nu{}_\mu$, where apart from not depending on w, the Lorentz 4-vector function $\gamma_\mu(x^\lambda)$ is otherwise arbitrary. Further, inserting Eqs. (14.7) and (14.9) into Eq. (14.2) gives

$$\begin{aligned}\partial_\mu\partial_\nu h^{\mu\nu} &= e^{4b|w|}\eta^{\mu\nu}\partial_\nu\partial_\mu\alpha(x^\lambda) + e^{2b|w|}\eta^{\mu\nu}\partial_\mu\gamma_\nu(x^\lambda) \\ &= e^{4b|w|}\eta^{\mu\nu}\partial_\nu\partial_\mu\alpha(x^\lambda) + e^{2b|w|}[\eta^{\mu\nu}\partial_\nu\partial_\mu\beta(x^\lambda) - 6b^2\alpha(x^\lambda)] \ ,\end{aligned} \tag{14.10}$$

to thereby yield

$$\eta^{\mu\nu}\partial_\nu\partial_\mu\beta(x^\lambda) = \eta^{\mu\nu}\partial_\nu\gamma_\mu(x^\lambda) + 6b^2\alpha(x^\lambda) \ , \tag{14.11}$$

with the function $\beta(x^\lambda)$ thus not being independent of $\alpha(x^\lambda)$ and $\gamma_\mu(x^\lambda)$. From Eqs. (14.7) and (14.9) we thus see that the dependences of h and $\partial_\nu h^\nu{}_\mu$ on w are completely determined, and that their ordinary spacetime dependences are given completely by five independent functions, the Lorentz scalar $\alpha(x^\lambda)$ and the 4-component Lorentz vector $\gamma_\mu(x^\lambda)$. With the Lorentz scalar being fixed by the junction conditions, but with Lorentz 4-vector being arbitrary, we can anticipate, and will momentarily find, that the 4-vector is a pure gauge artifact.

Since we have expressed the five components of the NT piece $h^{NT}_{\mu\nu}$ of $h_{\mu\nu}$ in terms of the five functions $\alpha(x^\lambda)$ and $\gamma_\mu(x^\lambda)$, the use of $\alpha(x^\lambda)$ and $\gamma_\mu(x^\lambda)$ thus provides us with a full accounting of the degrees of freedom in the NT sector. However, rather than use these particular degrees of freedom it is more convenient to define another Lorentz scalar and 4-vector according to

$$\beta = 8b\hat{\zeta}^5 - 2\partial_\mu\hat{\zeta}^\mu \quad , \quad \gamma_\mu(x^\lambda) = 2b\partial_\mu\hat{\zeta}^5 - \eta^{\alpha\beta}\partial_\alpha\partial_\beta\eta_{\mu\nu}\hat{\zeta}^\nu - \partial_\mu\partial_\rho\hat{\zeta}^\rho \ . \tag{14.12}$$

In terms of these new functions $\alpha(x^\lambda)$ is then given as

$$\alpha(x^\lambda) = \frac{1}{b}\eta^{\mu\nu}\partial_\mu\partial_\nu\hat{\zeta}^5 \ , \tag{14.13}$$

where the Lorentz scalar $\hat{\zeta}^5$ then obeys

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\hat{\zeta}^5 = \frac{\kappa_5^2}{6}S \ , \tag{14.14}$$

an equation whose solution is immediately given as

$$\hat{\zeta}^5 = \frac{\kappa_5^2}{6}\int d^4x' D(x-x')S(x') \ , \tag{14.15}$$

where $D(x-x')$ is the standard flat 4-dimensional spacetime massless scalar propagator. Finally, in terms of $\hat{\zeta}^5$ and $\hat{\zeta}^\mu$ we can write the most general $h^{NT}_{\mu\nu}$ as

$$h^{NT}_{\mu\nu} = \frac{1}{b}\partial_\mu\partial_\nu\hat{\zeta}^5 + e^{-2b|w|}[2b\eta_{\mu\nu}\hat{\zeta}^5 - \eta_{\mu\rho}\partial_\nu\hat{\zeta}^\rho - \eta_{\nu\rho}\partial_\mu\hat{\zeta}^\rho] \ , \tag{14.16}$$

as its trace and divergence directly recover Eqs. (14.7) and (14.9).

14.2 TT sector fluctuations

Of the five NT functions described above, one, and in fact only one of them, viz. $\hat{\zeta}^5$, participates in the junction condition at the brane. Thus of the five degrees of freedom in $h^{NT}_{\mu\nu}$, we recognize one of them as necessarily being a physical degree of freedom. To see its explicit physical implications we introduce a quantity $h^{TT}_{\mu\nu} = h_{\mu\nu} - h^{NT}_{\mu\nu}$, viz.

$$h^{TT}_{\mu\nu} = h_{\mu\nu} - \frac{1}{b}\partial_\mu\partial_\nu\hat{\zeta}^5 - e^{-2b|w|}[2b\eta_{\mu\nu}\hat{\zeta}^5 - \eta_{\mu\rho}\partial_\nu\hat{\zeta}^\rho - \eta_{\nu\rho}\partial_\mu\hat{\zeta}^\rho] \ , \tag{14.17}$$

a quantity defined to automatically obey $g^{\mu\nu}h^{TT}_{\mu\nu} = 0$, $\partial_\mu h^{TT\,\mu\nu} = 0$, and to thus necessarily be TT. As defined, the quantity $h^{TT}_{\mu\nu}$ does not contribute to the purely NT sector associated with Eqs. (14.2), (14.3) and (14.5). However, the converse is not quite true of $h^{NT}_{\mu\nu}$, as $h^{NT}_{\mu\nu}$ actually does make an explicit contribution in Eq. (14.4), the only fluctuation equation in which $h^{TT}_{\mu\nu}$ can contribute. Nonetheless, it turns out that $h^{NT}_{\mu\nu}$ actually only contributes a $\delta(w)$ discontinuity term in Eq.

(14.4), with insertion of the full $h_{\mu\nu} = h_{\mu\nu}^{TT} + h_{\mu\nu}^{NT}$ as defined by Eq. (14.17) into Eq. (14.4) yielding

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 4b^2 + 4b\delta(w) + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT}$$
$$= -\kappa_5^2 S_{\mu\nu}\delta(w) - \delta(w)\left[\frac{\partial}{\partial|w|}h_{\mu\nu}^{NT} + 2bh_{\mu\nu}^{NT} - g_{\mu\nu}\frac{\partial h}{\partial|w|}\right]$$
$$= -\kappa_5^2 S_{\mu\nu}\delta(w) - 2\delta(w)[\partial_\mu\partial_\nu\hat{\zeta}^5 - \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\hat{\zeta}^5] \ . \tag{14.18}$$

On defining

$$S_{\mu\nu}^{NT} = \frac{2}{\kappa_5^2}\left[\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\hat{\zeta}^5 - \partial_\mu\partial_\nu\hat{\zeta}^5\right] = \frac{1}{3}\left[\eta_{\mu\nu}S - \partial_\mu\partial_\nu\int d^4x' D(x-x')S(x')\right] \ , \tag{14.19}$$

and introducing $S_{\mu\nu}^{TT} = S_{\mu\nu} - S_{\mu\nu}^{NT}$, we find that as defined $S_{\mu\nu}^{TT}$ is indeed TT, and that Eq. (14.18) can be written as the pure TT

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 4b^2 + 4b\delta(w) + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = -\kappa_5^2\delta(w)S_{\mu\nu}^{TT}$$
$$= -\kappa_5^2\delta(w)\left[S_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}S + \frac{1}{3}\partial_\mu\partial_\nu\int d^4x' D(x-x')S(x')\right] \ , \tag{14.20}$$

an equation which is of key importance in brane-world studies.[2] From Eq. (14.20) we see that while $h_{\mu\nu}^{NT}$ makes no contribution to the propagation equation for $h_{\mu\nu}^{TT}$ in the bulk, the NT modes are still explicitly needed on the brane where they serve to ensure that the source of the wave operator acting on the TT modes is indeed TT. With the w dependence of the NT sector having already been unambiguously fixed above, the great utility of Eq. (14.20) is that it enables us to now extract the w dependence of $h_{\mu\nu}^{TT}$, to enable us to then determine whether the fluctuation modes fall or rise away from the brane, and thus whether or not gravity actually localizes around the brane. Moreover, recalling that in the brane world $h_{\mu\nu}^{TT}$ is taken to be a function of $|w|$ rather than of w itself, so that it then obeys

$$\frac{\partial^2}{\partial w^2}h_{\mu\nu}^{TT}(|w|) = \frac{\partial^2}{\partial|w|^2}h_{\mu\nu}^{TT}(|w|) + 2\delta(w)\frac{\partial}{\partial|w|}h_{\mu\nu}^{TT}(|w|) \ , \tag{14.21}$$

[2] Apart from the derivation given here, the TT equation of motion of Eq. (14.20) with explicit TT source was also derived by [Garriga and Tanaka (2000)] and by [Giddings, Katz and Randall (2000)] using brane bending gauge transformations, and by [DeWolfe, Freedman, Gubser and Karch (2000)] using a TT projector technique (a procedure we shall discuss further in Appendix C); with the source-free variant of Eq. (14.20) having originally been given by Randall and Sundrum themselves [Randall and Sundrum (1999a); Randall and Sundrum (1999b)].

it follows that the TT wave equation of Eq. (14.20) then breaks up into separate continuous and discontinuous pieces

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h^{TT}_{\mu\nu}(|w|) = 0 \ , \tag{14.22}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} + 2b\right] h^{TT}_{\mu\nu}(|w|) = -\kappa_5^2\delta(w)S^{TT}_{\mu\nu} \ , \tag{14.23}$$

to thus take the form of a second order differential wave equation whose solutions are required to obey a $\delta(w)$ junction constraint. For an arbitrary source on the brane solution of Eqs. (14.22) and (14.23) is generally achieved by propagator techniques (though in cases where the source is particularly simple, these equations can even admit of direct solution, some examples of which we shall present in Chapter 17 below), with the TT propagator itself being constructed out of the mode solutions to the source-free version of Eqs. (14.22) and (14.23). While we shall examine such source-free mode solutions in detail below, we note that the mode

$$h^{TT}_{\mu\nu} = e_{\mu\nu}e^{-2b|w|}e^{i\bar{p}\cdot\bar{x}-i|\bar{p}|t} \tag{14.24}$$

with a polarization tensor $e_{\mu\nu}$ which obeys $\eta_{\mu\nu}e^{\mu\nu}=0$, $p_\mu e^{\mu\nu}=0$ can immediately be seen to actually be one. As such, this particular mode acts on the brane as a massless 4-dimensional graviton, with its associated $e^{-2b|w|}$ dependence in the bulk then ensuring that the M_4^+ massless graviton is indeed localized to the brane.

14.3 Physical status of the NT and TT sector fluctuations

To establish the gauge/physical status of all of the various NT and TT degrees of freedom both on the brane and off, we consider the particular Z_2 symmetry preserving infinitesimal coordinate transformation $\bar{x}^A = x^A + \xi^A$ of the form

$$\xi^5 = \epsilon(w)\hat{\xi}^5(x^\lambda) \ , \quad \xi^\mu = -\frac{1}{2b}e^{2b|w|}\eta^{\mu\nu}\partial_\nu\hat{\xi}^5(x^\lambda) + \hat{\xi}^\mu(x^\lambda) \ , \tag{14.25}$$

to find that under it the components of h_{MN} transform as

$$\begin{aligned}\bar{h}_{\mu\nu}(|w|,x^\lambda) &= h_{\mu\nu}(|w|,x^\lambda) - \frac{1}{b}\partial_\mu\partial_\nu\hat{\xi}^5 - e^{-2b|w|}[2b\eta_{\mu\nu}\hat{\xi}^5 - \eta_{\mu\rho}\partial_\nu\hat{\xi}^\rho - \eta_{\nu\rho}\partial_\mu\hat{\xi}^\rho] \ , \\ \bar{h}_{55} &= h_{55} + 4\delta(w)\hat{\xi}^5(x^\lambda) \ , \quad \bar{h}_{5\mu} = h_{5\mu} \ .\end{aligned} \tag{14.26}$$

Other than in the generation of a delta function term in $\bar{h}_{55}$ on the brane, we see that these particular gauge transformations otherwise preserve the axial gauge everywhere else.[3] The transformations of Eq. (14.26) thus preserve the axial gauge in the bulk but not on the brane. Finally, comparing Eq. (14.26) with Eq. (14.17)

[3] For pure AdS_5 these axial gauge preserving transformations were given in [Garriga and Tanaka (2000)], with the incorporation of the $\epsilon(w)$ factor in ξ^5 allowing their extension to AdS_5/Z_2.

we see that the entire $h^{NT}_{\mu\nu}$ is pure gauge in the bulk (where the gauge is then known as the RS gauge following Randall and Sundrum's original use of it), with it thus being only the 5-component TT $h^{TT}_{\mu\nu}$ which is then physically relevant there.

Now while we could also remove the entire $h^{NT}_{\mu\nu}$ on the brane also, according to Eq. (14.26) to do so would then take $\bar{h}_{55}$ out of the axial gauge.[4] Rather than do this however, we instead prefer to stay in the axial gauge, and thus shall only avail ourselves of the four $\hat{\xi}^\mu$ transformations on the brane. Thus on the brane we see that we can reduce the 10-component axial gauge $h_{\mu\nu}$ to six (not five) degrees of freedom, viz. five TT modes and one residual NT mode. To deal with this one residual NT brane mode it is more convenient not to simply set $\hat{\xi}^\mu = \hat{\zeta}^\mu$ but rather to pick $\hat{\xi}^\mu$ according to

$$\hat{\xi}^\mu = \hat{\zeta}^\mu + \eta^{\mu\nu}\partial_\nu \left[2b\int d^4x' D(x-x')\hat{\zeta}^5(x') - \frac{1}{2}\hat{\zeta}^5\right] \quad ,$$
$$[\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu][\hat{\xi}^\rho - \hat{\zeta}^\rho] + \frac{1}{b}\partial_\mu\partial_\nu\hat{\zeta}^5 = 4b\partial_\mu\partial_\nu\int d^4x' D(x-x')\hat{\zeta}^5(x') \quad , \quad (14.27)$$

a choice under which $\bar{h}^{NT}_{\mu\nu}(w=0)$ is then given by

$$\bar{h}^{NT}_{\mu\nu}(w=0) = 2b\eta_{\mu\nu}\hat{\zeta}^5 + 4b\partial_\mu\partial_\nu\int d^4x' D(x-x')\hat{\zeta}^5(x') \quad . \quad (14.28)$$

From Eq. (14.28) we find that on the brane $\bar{h}^{NT}_{\mu\nu}(w=0)$ obeys

$$\partial^\nu \bar{h}^{NT}_{\mu\nu}(w=0) = \frac{1}{2}\partial_\mu \bar{h}(w=0) \quad , \quad (14.29)$$

with it thus having been brought to the harmonic gauge there. With $\hat{\zeta}^5$ obeying Eq. (14.15), from Eq. (14.28) we find that on the brane $\bar{h}^{NT}_{\mu\nu}(w=0)$ obeys

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta \bar{h}^{NT}_{\mu\nu}(w=0) = -2b\kappa_5^2\left[S^{NT}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right] \quad . \quad (14.30)$$

The advantage of the particular choice of gauge we have made here is that by virtue of being TT, all 5 components of $h^{TT}_{\mu\nu}$ already satisfy the harmonic gauge condition, so that with the gauge transformation of Eq. (14.27) all six components of $\bar{h}_{\mu\nu}(w=0)$ then become harmonic on the brane.

14.4 Comparison with unembedded M_4 fluctuations

The above emergence of a harmonic gauge structure for $h_{\mu\nu}$ on the brane is particularly welcome because of the role that the harmonic gauge plays in standard

[4] Since the right-hand side of Eq. (14.5) depends on a physical quantity, viz. the trace of the source energy-momentum tensor on the brane, no gauge transformation could make its left-hand side vanish. Thus if we were to go into a gauge in which the 4-trace $g^{\mu\nu}h^{NT}_{\mu\nu}$ were to vanish on the brane, this would then force the 5-trace $h_{55} + g^{\mu\nu}h^{NT}_{\mu\nu}$ to become non-zero there.

unembedded gravity where an initial 10-component $h_{\mu\nu}$ can also be reduced to six harmonic gauge degrees of freedom. Specifically, for fluctuations around a standard unembedded flat M_4 background, the fluctuation wave equation takes the familiar form

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = -2\kappa_4^2\left[S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right] \tag{14.31}$$

in the harmonic gauge, where here κ_4^2 is the standard 4-dimensional Einstein gravitational constant. If in the standard 4-dimensional theory we define a quantity $h_{\mu\nu}^{NT}$ according to

$$\begin{aligned} h_{\mu\nu}^{NT} &= \frac{\kappa_4^2}{3}\eta_{\mu\nu}\int d^4x' D(x-x')S(x') \\ &+\frac{2\kappa_4^2}{3}\partial_\mu\partial_\nu\int d^4x' D(x-x')\int d^4x'' D(x'-x'')S(x'') \ , \end{aligned} \tag{14.32}$$

we will find that this particular $h_{\mu\nu}^{NT}$ obeys both the harmonic gauge condition and the wave equation

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{NT} = -2\kappa_4^2\left[S_{\mu\nu}^{NT} - \frac{1}{2}\eta_{\mu\nu}S\right] \ , \tag{14.33}$$

where $S_{\mu\nu}^{NT}$ is as defined in Eq. (14.19). Similarly, if we define a quantity $h_{\mu\nu}^{TT} = h_{\mu\nu} - h_{\mu\nu}^{NT}$, we will find that it too will obey the harmonic gauge condition while also being TT, with the insertion of the full $h_{\mu\nu}$ into Eq. (14.31) yielding

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{TT} = -2\kappa_4^2(S_{\mu\nu} - S_{\mu\nu}^{NT}) = -2\kappa_4^2 S_{\mu\nu}^{TT} \ . \tag{14.34}$$

As we thus see, in the unembedded standard harmonic gauge case it is possible to decompose Eq. (14.31) into two separate TT and NT pieces both of which propagate as massless modes, one with spin-two and the other with spin-zero. Since full standard 4-dimensional gravity is associated with a gravitational source $S_{\mu\nu}$ which in general is neither TT or NT, standard long-range gravity only emerges in four dimensions because both of the NT and TT sectors are physical; with Eqs. (14.33) and (14.34) only being combinable into Eq. (14.31) because these equations of motion not only both have the explicit structure that they do, but because both of them also depend on only one and the same coupling constant κ_4^2.

To emphasize the role played by the NT modes, we note that there is a difference between treating Eq. (14.31) with a source and treating it without one. Specifically, in the absence of a source, the wave equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = 0$ admits of familiar massless plane wave solutions. For the typical one of these with 4-momentum $p^\mu = (p, 0, 0, p)$, the associated six degree of freedom harmonic gauge polarization tensor which satisfies $p_\mu e^{\mu\nu} = p^\nu e^\alpha{}_\alpha/2$ can be decomposed into the form

$$\begin{pmatrix} e_{00} & -e_{13} & -e_{23} & -(e_{00}+e_{33})/2 \\ -e_{13} & e_{11} & e_{12} & e_{13} \\ -e_{23} & e_{12} & -e_{11} & e_{23} \\ -(e_{00}+e_{33})/2 & e_{13} & e_{23} & e_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (e_{00}+e_{33})/2 & -e_{13} & -e_{23} & -(e_{00}+e_{33})/2 \\ -e_{13} & e_{11} & e_{12} & e_{13} \\ -e_{23} & e_{12} & -e_{11} & e_{23} \\ -(e_{00}+e_{33})/2 & e_{13} & e_{23} & (e_{00}+e_{33})/2 \end{pmatrix} + \begin{pmatrix} (e_{00}-e_{33})/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (e_{33}-e_{00})/2 \end{pmatrix}$$

in which its five specific TT components and its one NT component are then explicitly exhibited. As an equation, the source-free wave equation admits of four further $\bar{x}_\mu = x_\mu + \epsilon_\mu$ gauge transformations which preserve the harmonic gauge, viz. the four which obey $\partial_\lambda \partial^\lambda \epsilon^\mu = 0$. Of these particular four gauge transformations three of them can be chosen to obey $\partial_\mu \epsilon^\mu = 0$ with $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ then being TT,[5] with the fourth gauge transformation acting in the NT sector. Consequently, we have just the right number and just the right type of residual gauges needed to reduce the source-free region TT modes to two independent observable TT components while eliminating the NT mode altogether,[6] with it thus being only the TT modes which propagate as (on-shell) gravitational waves in the source-free region. However when it comes to determining the (off-shell) Newtonian potential, here the NT modes are also needed since, with a static source possessing the explicitly non TT energy-momentum tensor $S_{\mu\nu} = -\delta^0_\mu \delta^0_\nu \eta_{00} M \delta^3(x)$, it is from Eq. (14.31) and not Eq. (14.34) that the potential is to then be obtained.

Comparing now unembedded gravity with embedded gravity on the brane, we see that as far as the NT sector is concerned, Eqs. (14.30) and (14.33) are identical in form, with the identification $\kappa_4^2 = b\kappa_5^2$ introduced in Chapters 2 and 12 then obliging $\bar{h}^{NT}_{\mu\nu}(w=0)$ to have none other than precisely the form required of it for standard gravity. That the NT modes will make their needed contribution toward standard gravity on the brane is thus secured. However, whether the TT mode solutions to Eq. (14.20) will also make the contribution required by Eq. (14.34) is not at all as automatic, with a determination requiring the explicit construction of the TT mode solutions to Eq. (14.20) and the extraction of their coupling strengths (viz. their normalizations) to the source. While the NT sector is thus nicely in place, for the full brane-localized gravity program to succeed we will need the TT mode fluctuations to meet the twin requirements of not only having wave functions which fall monotonically away from the brane, but of also possessing a massless TT

[5]For the 4-vector $p^\mu = (1,0,0,1)$, the 4-vectors $(1,0,0,1)$, $(0,1,0,0,)$ and $(0,0,1,0)$ are transverse while $(-1,0,0,1)$ is not.

[6]The choice $\epsilon_\mu = (e_{00}/2p, -e_{13}/p, -e_{23}/p, -e_{33}/2p)$ effects the transformation to the pure TT $\bar{e}_{11} = e_{11}$, $\bar{e}_{12} = e_{12}$, $\bar{e}_{13} = 0$, $\bar{e}_{23} = 0$, $\bar{e}_{00} = 0$, $\bar{e}_{33} = 0$.

mode whose gravitational coupling is precisely the same as that of the massless NT mode. While we will show below that this will precisely turn out to be the case for fluctuations around an M_4^+ brane, in some of the other cases we shall encounter below it will prove possible to meet one or other of the two requirements without necessarily needing to satisfy them both.

While we shall explore specific TT mode solutions below, we note now that some additional insight into their contribution on the brane can be drawn from the properties of the generalized perturbative Einstein equation on the brane as given earlier (for the M_4^+ brane-world case) as Eq. (12.39), viz.

$$\delta G_{\mu\nu} = -b\kappa_5^2 S_{\mu\nu} - \delta\bar{E}_{\mu\nu}(w=0) \ . \tag{14.35}$$

As we show in Appendix B, in the M_4^+ case where $A = -b|w|$ the quantity $\delta\bar{E}_{\mu\nu}(|w|)$ at arbitrary $|w|$ explicitly evaluates to

$$\delta\bar{E}_{\mu\nu}(|w|) = \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} + 2b\frac{\partial}{\partial|w|}\right] h_{\mu\nu}(|w|) \ . \tag{14.36}$$

With the $h_{\mu\nu}^{NT}$ of Eq. (14.16) having the explicit generic form of a sum of two terms, one independent of $|w|$ and the other behaving as the warp factor $e^{-2b|w|}$, it thus follows that the NT modes make no contribution to $\delta\bar{E}_{\mu\nu}(|w|)$ at all, with $\delta\bar{E}_{\mu\nu}(|w|)$ thus being entirely TT, as is of course to be expected since the Weyl tensor is the traceless part of the Riemann tensor. Additionally, since it is to be evaluated with respect to the background induced metric, in the harmonic gauge $\delta G_{\mu\nu}$ evaluates to

$$\delta G_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h \tag{14.37}$$

on the brane. With $\delta\bar{E}_{\mu\nu}(w=0)$ being traceless, it follows from Eq. (14.35) that

$$\eta^{\mu\nu}\delta G_{\mu\nu} = -\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h = -b\kappa_5^2 S \ . \tag{14.38}$$

Thus, on now making a TT, NT decomposition of $h_{\mu\nu}$, we find from Eq. (14.35) that

$$\delta G_{\mu\nu}^{TT} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{TT} = -b\kappa_5^2 S_{\mu\nu}^{TT} - \delta\bar{E}_{\mu\nu}(w=0) \ , \tag{14.39}$$

and that

$$\begin{aligned}\delta G_{\mu\nu}^{NT} &= \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{NT} - \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h \\ &= \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{NT} - \frac{1}{2}b\kappa_5^2\eta_{\mu\nu}S = -b\kappa_5^2 S_{\mu\nu}^{NT} \ . \end{aligned} \tag{14.40}$$

We recognize Eq. (14.40) as being none other than the NT Eq. (14.30) given earlier, and while we had already established in Chapter 12 that it is the presence of the Weyl tensor term which is to herald any possible departures from standard gravity on the brane, from Eq. (14.40) we see now that any such departures are to be confined to the TT sector alone.[7] The coupling of the NT modes on the brane thus has to be canonical no matter what, with the coupling of the TT modes only being able to also be canonical when the effect of the Weyl tensor term is negligible, an issue we will examine below. Having now identified the separate roles played by the NT and TT modes in the M_4 brane case, we turn next to the dS_4 and AdS_4 cases where we will find a completely analogous outcome.

[7]With the junction condition of Eq. (14.23) entailing that on the brane $\delta\bar{E}_{\mu\nu}(w=0)$ is given by $\delta\bar{E}_{\mu\nu}(w=0) = (\partial^2_{|w|} - 4b^2)h^{TT}_{\mu\nu}/2 - b\kappa_5^2 S^{TT}_{\mu\nu}$, we note in passing that the TT sector Eq. (14.39) then emerges as none other than the $w=0$ limit of the TT wave equation given in Eq. (14.22), to thus provide an internal check on our calculations.

Chapter 15

Transverse-Traceless dS_4/AdS_4 Fluctuation Modes

15.1 Gauge invariance in the dS_4 and AdS_4 brane worlds

In order to analyze the NT/TT decomposition in the $dS_4^{\pm}$ and $AdS_4^{\pm}$ brane-world cases, rather than solve the fluctuation wave equations just as we did for $M_4^{\pm}$, guided by that analysis, this time we shall instead first construct the most general axial gauge preserving form for $h_{\mu\nu}$, and then show that it is an exact solution to the relevant fluctuation equations. Since the sign of the brane tension will play no role in effecting the NT/TT decomposition in the $dS_4^{\pm}$ and $AdS_4^{\pm}$ cases, and since the $dS_4^{\pm}$ results and $AdS_4^{\pm}$ results are obtainable from each other simply by replacing H^2 by $-H^2$, we need only treat one of the four cases, and so for definitiveness we shall present the NT/TT decomposition analysis for AdS_4^{+}, a case where the background induced Riemann tensor is given by

$$\tilde{R}_{\mu\nu\sigma\tau} = -H^2(q_{\mu\tau}q_{\nu\sigma} - q_{\mu\sigma}q_{\nu\tau}) \quad . \tag{15.1}$$

For a source on the brane, the AdS_4^{+} fluctuation equation $\Delta G_{MN} = -\kappa_5^2\delta^{\mu}_{M}\delta^{\nu}_{N}S_{\mu\nu}\delta(w)$ takes the form

$$\Delta G_{55} = -\frac{3}{2}A'h' + \frac{3}{2}H^2e^{-2A}h - \frac{1}{2}e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} = 0 \quad , \tag{15.2}$$

$$\Delta G_{5\mu} = \frac{1}{2}\partial_w[\nabla_\mu h - \nabla_\nu h^{\nu}{}_{\mu}] = 0 \quad , \tag{15.3}$$

$$\begin{aligned}\Delta G_{\mu\nu} = &\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 8A'' - 4A'^2 - 12(b^2 - H^2)^{1/2}\delta(w)\right]h_{\mu\nu} \\ &-\frac{1}{2}\left[h'' + 4A'h' + e^{-2A}g^{\sigma\tau}\widetilde{\delta R}_{\sigma\tau} - 3H^2e^{-2A}h\right]g_{\mu\nu} + e^{-2A}\widetilde{\delta R}_{\mu\nu} \\ = &-\kappa_5^2\delta(w)S_{\mu\nu} \quad ,\end{aligned} \tag{15.4}$$

$$g^{\mu\nu}\Delta G_{\mu\nu} = -\frac{3}{2}h'' - 6A'h' - 3A''h + 6H^2e^{-2A}h - 6(b^2 - H^2)^{1/2}h\delta(w)$$
$$-e^{-2A}g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} = -\kappa_5^2\delta(w)S \ , \tag{15.5}$$

where $e^A = H\cosh(b|w| - \sigma)/b$, $\cosh\sigma = b/H$, $S = q^{\mu\nu}S_{\mu\nu}$,

$$\widetilde{\delta R}_{\mu\nu} = \frac{1}{2}[\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{h} - \tilde{\nabla}_\lambda\tilde{\nabla}_\mu\tilde{h}^\lambda{}_\nu - \tilde{\nabla}_\lambda\tilde{\nabla}_\nu\tilde{h}^\lambda{}_\mu + \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h}_{\mu\nu}]$$
$$= \frac{1}{2}[\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{h} - \tilde{\nabla}_\mu\tilde{\nabla}_\lambda\tilde{h}^\lambda{}_\nu - \tilde{\nabla}_\nu\tilde{\nabla}_\lambda\tilde{h}^\lambda{}_\mu + 8H^2\tilde{h}_{\mu\nu} - 2H^2\tilde{h}q_{\mu\nu}$$
$$+\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h}_{\mu\nu}] \ ,$$
$$g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} = e^{-2A}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h} - \tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{h}^{\mu\nu}] \ , \tag{15.6}$$

and where $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{h}_{\mu\nu}$ denotes the induced metric tensor box operation.

To identify the most general axial gauge preserving form for $h_{\mu\nu}$, we note that for the generic metric of Eq. (13.4), the general gauge transformations

$$\bar{h}_{MN} = h_{MN} + \xi_{N;M} + \xi_{M;N} = h_{MN} + \partial_M\xi_N + \partial_N\xi_M - 2\Gamma^T_{MN}\xi_T \tag{15.7}$$

take the form

$$\bar{h}_{55} = h_{55} + 2\partial_w\xi_5 \ , \tag{15.8}$$

$$\bar{h}_{5\mu} = h_{5\mu} + \partial_\mu\xi_5 + \partial_w\xi_\mu - 2A'\xi_\mu \ , \tag{15.9}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + 2A'e^{2A}q_{\mu\nu}\xi_5 - 2\tilde{\Gamma}^\tau_{\mu\nu}\xi_\tau \ . \tag{15.10}$$

For these transformations to preserve the Z_2 symmetry (viz. $\bar{h}_{55}$ and $\bar{h}_{\mu\nu}$ even, $\bar{h}_{5\mu}$ odd), we need ξ_5 to be an odd function of w and ξ_μ to be an even function of w, and thus set $\xi_5 = \epsilon(w)\hat{\xi}_5$ where $\hat{\xi}_5$ is even. For these transformations to preserve the axial gauge for h_{55} and $h_{5\mu}$, Eqs. (15.8) and (15.9) require us to set

$$\xi_5 = \epsilon(w)\hat{\xi}_5(x^\lambda) \ , \quad \xi_\mu = -\frac{e^{2A}}{H^2}\frac{dA}{d|w|}\partial_\mu\hat{\xi}_5(x^\lambda) + e^{2A}\hat{\xi}_\mu(x^\lambda) \ , \tag{15.11}$$

where $\hat{\xi}_5$ and $\hat{\xi}_\mu$ are independent of w. With this choice for ξ_5 and ξ_μ we find that the components of h_{MN} then transform as

$$\bar{h}_{55} = h_{55} + 4\delta(w)\hat{\xi}^5(x^\lambda) \ , \quad \bar{h}_{5\mu} = h_{5\mu} \ ,$$
$$\bar{h}_{\mu\nu} = h_{\mu\nu} + e^{2A}[\partial_\mu\hat{\xi}_\nu + \partial_\nu\hat{\xi}_\mu - 2\tilde{\Gamma}^\tau_{\mu\nu}\hat{\xi}_\tau] + 2e^{2A}\frac{dA}{d|w|}q_{\mu\nu}\hat{\xi}_5$$
$$-\frac{2e^{2A}}{H^2}\frac{dA}{d|w|}[\partial_\mu\partial_\nu\hat{\xi}_5 - \tilde{\Gamma}^\tau_{\mu\nu}\partial_\tau\hat{\xi}_5]$$
$$= h_{\mu\nu} + e^{2A}[\tilde{\nabla}_\mu\hat{\xi}_\nu + \tilde{\nabla}_\nu\hat{\xi}_\mu] + 2e^{2A}\frac{dA}{d|w|}q_{\mu\nu}\hat{\xi}_5 - \frac{2e^{2A}}{H^2}\frac{dA}{d|w|}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\hat{\xi}_5 \ , \tag{15.12}$$

to thereby give us the most general axial gauge preserving form for $h_{\mu\nu}$.[1]

15.2 NT/TT decomposition of the fluctuation equations

Given the form for Eq. (15.12) found above, we shall thus look for a candidate solution to Eqs. (15.2) – (15.5) of the form $h_{\mu\nu} = h^{TT}_{\mu\nu} + h^{NT}_{\mu\nu}$, as written in terms of an NT component defined via

$$h^{NT}_{\mu\nu} = -e^{2A}[\tilde{\nabla}_\mu\hat{\zeta}_\nu + \tilde{\nabla}_\nu\hat{\zeta}_\mu] + \frac{2e^{2A}}{H^2}\frac{dA}{d|w|}[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\hat{\zeta}_5 - q_{\mu\nu}H^2\hat{\zeta}_5] \ , \tag{15.13}$$

and a yet to be specified component $h^{TT}_{\mu\nu}$ which is for the moment required only to be TT. For this candidate solution the trace $h = e^{-2A}\tilde{h} = g^{\mu\nu}h^{NT}_{\mu\nu}$ of $h_{\mu\nu}$ is given as

$$h = -2\tilde{\nabla}_\mu\tilde{\hat{\zeta}}^\mu + \frac{2}{H^2}\frac{dA}{d|w|}[\tilde{\nabla}_\mu\tilde{\nabla}^\mu\hat{\zeta}_5 - 4H^2\hat{\zeta}_5] \ , \tag{15.14}$$

and the divergence $\nabla_\nu h^\nu{}_\mu = e^{-2A}\tilde{\nabla}_\nu\tilde{h}^\nu{}_\mu = \nabla_\nu h^{NT\,\nu}{}_\mu$ can be written as

$$\begin{aligned}\nabla_\nu h^\nu{}_\mu &= -\tilde{\nabla}_\nu[\tilde{\nabla}_\mu\hat{\zeta}^\nu + \tilde{\nabla}^\nu\hat{\zeta}_\mu] + \frac{2}{H^2}\frac{dA}{d|w|}[\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{\nabla}^\nu\hat{\zeta}_5 - \tilde{\nabla}_\mu H^2\hat{\zeta}_5] \\ &= -\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\hat{\zeta}}^\nu - \tilde{\nabla}_\nu\tilde{\nabla}^\nu\hat{\zeta}_\mu + 3H^2\hat{\zeta}_\mu + \frac{2}{H^2}\frac{dA}{d|w|}\tilde{\nabla}_\mu[\tilde{\nabla}_\nu\tilde{\nabla}^\nu\hat{\zeta}_5 - 4H^2\hat{\zeta}_5] \ ,\end{aligned} \tag{15.15}$$

where in deriving the second form of Eq. (15.15) use has been made of the differential geometric relation

$$\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{V}^\nu - \tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{V}^\nu = \tilde{V}^\sigma\tilde{R}^\nu{}_{\sigma\mu\nu} = -3H^2\tilde{V}_\mu \tag{15.16}$$

obeyed by a general 4-vector $\tilde{V}^\mu$ in an AdS_4 geometry.

With $g^{\mu\nu}\widetilde{\delta R}_{\mu\nu}$ having the form exhibited in Eq. (15.6), inspection of Eqs. (15.2), (15.3) and (15.5) reveals that these three equations depend on h and $\nabla_\nu h^\nu{}_\mu$ alone, and so we first check to see whether our candidate solution satisfies these particular three equations. Thus on inserting Eqs. (15.14) and (15.15) into Eq. (15.3), we immediately confirm that the $\Delta G_{5\mu}$ equation is satisfied identically for arbitrary x^λ dependences of the functions $\hat{\zeta}_\mu$ and $\hat{\zeta}_5$. To check Eq. (15.2) we evaluate $g^{\mu\nu}\widetilde{\delta R}_{\mu\nu}$

[1] With this choice we find that $\bar{h}_{55} = h_{55} + 4\delta(w)\hat{\xi}_5$, to thus actually take us out of the axial gauge on the brane. Since we do not wish to do this we shall thus never actually ever make the $\xi_5 = \epsilon(w)\hat{\xi}_5$ transformation in the following. Nonetheless, we can still use this transformation here to determine the needed form for $h_{\mu\nu}$, and can then check that the found form does indeed obey the fluctuation equations without ever needing to make any reference to the $\xi_5 = \epsilon(w)\hat{\xi}_5$ transformation at all.

in our candidate solution, to obtain

$$
\begin{aligned}
g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} = & -2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{\nabla}_\mu\tilde{\hat{\zeta}}^{\mu} + \frac{2}{H^2}\frac{dA}{d|w|}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha[\tilde{\nabla}_\beta\tilde{\nabla}^\beta\hat{\zeta}_5 - 4H^2\hat{\zeta}_5] \\
& +\tilde{\nabla}_\mu\tilde{\nabla}_\nu[\tilde{\nabla}^\mu\tilde{\hat{\zeta}}^{\nu} + \tilde{\nabla}^\nu\tilde{\hat{\zeta}}^{\mu}] - \frac{2}{H^2}\frac{dA}{d|w|}\tilde{\nabla}_\mu\tilde{\nabla}^\mu[\tilde{\nabla}_\nu\tilde{\nabla}^\nu\hat{\zeta}_5 - 4H^2\hat{\zeta}_5] \\
= & -\tilde{\nabla}_\mu\tilde{\nabla}^\mu\tilde{\nabla}_\nu\tilde{\hat{\zeta}}^{\nu} - 3H^2\tilde{\nabla}_\mu\tilde{\hat{\zeta}}^{\mu} + \tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\nabla}^\nu\tilde{\hat{\zeta}}^{\mu} \quad ,
\end{aligned}
\tag{15.17}
$$

an expression in which $\hat{\zeta}_5$ has conveniently dropped out identically. Use of the additional geometric relation

$$
\begin{aligned}
\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{\nabla}_\mu\tilde{V}_\nu = & \tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}_\mu\tilde{V}_\nu + \tilde{R}_{\nu\sigma\alpha\beta}\tilde{\nabla}_\mu\tilde{V}^\sigma - \tilde{R}_{\sigma\mu\alpha\beta}\tilde{\nabla}^\sigma\tilde{V}_\nu \\
= & \tilde{\nabla}_\alpha(\tilde{\nabla}_\mu\tilde{\nabla}_\beta\tilde{V}_\nu + \tilde{R}_{\nu\sigma\mu\beta}\tilde{V}^\sigma) + \tilde{R}_{\nu\sigma\alpha\beta}\tilde{\nabla}_\mu\tilde{V}^\sigma - \tilde{R}_{\sigma\mu\alpha\beta}\tilde{\nabla}^\sigma\tilde{V}_\nu \\
= & \tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{V}_\nu + \tilde{R}_{\nu\sigma\mu\alpha}\tilde{\nabla}_\beta\tilde{V}^\sigma - \tilde{R}_{\sigma\beta\mu\alpha}\tilde{\nabla}^\sigma\tilde{V}_\nu \\
& +\tilde{\nabla}_\alpha(\tilde{R}_{\nu\sigma\mu\beta}\tilde{V}^\sigma) + \tilde{R}_{\nu\sigma\alpha\beta}\tilde{\nabla}_\mu\tilde{V}^\sigma - \tilde{R}_{\sigma\mu\alpha\beta}\tilde{\nabla}^\sigma\tilde{V}_\nu
\end{aligned}
\tag{15.18}
$$

allows us to infer that in an AdS_4 geometry we have

$$
\begin{aligned}
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{\nabla}_\mu\tilde{V}_\nu - \tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{V}_\nu = & 2H^2 q_{\mu\nu}\tilde{\nabla}_\alpha\tilde{V}^\alpha - 2H^2\tilde{\nabla}_\nu\tilde{V}_\mu - 3H^2\tilde{\nabla}_\mu\tilde{V}_\nu \quad , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{\nabla}_\nu\tilde{V}^\nu - \tilde{\nabla}_\nu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{V}^\nu = & 3H^2\tilde{\nabla}_\alpha\tilde{V}^\alpha \quad ,
\end{aligned}
\tag{15.19}
$$

so that from Eq. (15.19) we then find that in our candidate solution $g^{\mu\nu}\widetilde{\delta R}_{\mu\nu}$ can be written very compactly as

$$
g^{\mu\nu}\widetilde{\delta R}_{\mu\nu} = -6H^2\tilde{\nabla}_\mu\tilde{\hat{\zeta}}^{\mu} \quad . \tag{15.20}
$$

Finally, with use of the relation

$$
\frac{d^2A}{d|w|^2} = H^2 e^{-2A} \tag{15.21}
$$

obeyed by the AdS_4 warp factor[2] entailing that in our candidate solution the first two terms in ΔG_{55} are given by

$$
-\frac{3}{2}A'h' + \frac{3}{2}H^2 e^{-2A}h = -3e^{-2A}H^2\tilde{\nabla}_\mu\tilde{\hat{\zeta}}^{\mu} \quad , \tag{15.22}
$$

(an expression in which $\hat{\zeta}_5$ again drops out), we thus conclude that in our candidate solution ΔG_{55} also vanishes identically, just as desired.

To check Eq. (15.5), we note that as well as possessing a continuous piece, Eq. (15.5) also contains a discontinuous delta function piece since $g^{\mu\nu}\Delta G_{\mu\nu} = -\kappa_5^2\delta(w)S$. Through use of Eqs. (15.20) and (15.21) the continuous piece is then directly found to vanish identically without need for any constraint on $\hat{\zeta}_5$ or $\hat{\zeta}_\mu$. At the same time, with $dA(w=0)/d|w|$ being given by $dA(w=0)/d|w| = -(b^2 -$

[2]Analogously, the dS_4 warp factor obeys $d^2A/d|w|^2 = -H^2e^{-2A}$.

$H^2)^{1/2}$,[3] the delta function junction constraint in Eq. (15.5) is found to fix $\hat{\zeta}_5$ according to

$$\tilde{\nabla}_\mu \tilde{\nabla}^\mu \hat{\zeta}_5 - 4H^2 \hat{\zeta}_5 = \frac{\kappa_5^2}{6} S \quad . \tag{15.23}$$

Since our candidate solution for $h^{NT}_{\mu\nu}$ contains five independent degrees of freedom, viz. $\hat{\zeta}_5$ and $\hat{\zeta}_\mu$, Eq. (15.13) thus gives the general NT sector solution we seek, with both h and $\nabla_\nu h^\nu{}_\mu$ being completely specified, and with their w dependences being completely fixed.

Following our previous discussion of the NT/TT decomposition in the M_4^+ case, we next need to show that $h^{NT}_{\mu\nu}$ makes no contribution to the continuous piece of $\Delta G_{\mu\nu}$ given in Eq. (15.4). To this end we note that from Eqs. (15.16) and (15.19) we can obtain for any AdS_4 scalar such as $\hat{\zeta}_5$

$$\begin{aligned} &\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 - 2H^2 q_{\mu\nu} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \hat{\zeta}_5 + 5H^2 \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 \\ &\quad = \tilde{\nabla}_\mu \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{\nabla}_\nu \hat{\zeta}_5 = \tilde{\nabla}_\mu \tilde{\nabla}_\alpha \tilde{\nabla}_\nu \tilde{\nabla}^\alpha \hat{\zeta}_5 = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \hat{\zeta}_5 - 3H^2 \tilde{\nabla}_\nu \hat{\zeta}_5) \quad , \end{aligned} \tag{15.24}$$

viz.

$$\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \hat{\zeta}_5 = 2H^2 q_{\mu\nu} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \hat{\zeta}_5 - 8H^2 \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 \quad . \tag{15.25}$$

Using this relation and Eq. (15.19) we can then show that the contribution of $h^{NT}_{\mu\nu}$ to $e^{-2A}\widetilde{\delta R}_{\mu\nu}$ is given by

$$e^{-2A}\widetilde{\delta R}^{NT}_{\mu\nu} = -3H^2(\tilde{\nabla}_\mu \tilde{\hat{\zeta}}_\nu + \tilde{\nabla}_\nu \tilde{\hat{\zeta}}_\mu) + \frac{dA}{d|w|}(4\tilde{\nabla}_\mu \tilde{\nabla}_\nu - q_{\mu\nu}\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha)\hat{\zeta}_5 \quad , \tag{15.26}$$

from which it can then readily be shown that $h^{NT}_{\mu\nu}$ indeed does not contribute to the continuous piece of $\Delta G_{\mu\nu}$. Despite this however, $h^{NT}_{\mu\nu}$ does contribute to the discontinuity in $\Delta G_{\mu\nu}$, yielding

$$\Delta G^{NT}_{\mu\nu} = -2\delta(w)[q_{\mu\nu}\tilde{\nabla}_\sigma \tilde{\nabla}^\sigma \hat{\zeta}_5 - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 - 3q_{\mu\nu}H^2\hat{\zeta}_5] \quad , \tag{15.27}$$

a discontinuity to which only $\hat{\zeta}_5$ but not $\tilde{\hat{\zeta}}_\nu$ contributes. Finally, on setting $S_{\mu\nu} = S^{TT}_{\mu\nu} + S^{NT}_{\mu\nu}$ where $S^{NT}_{\mu\nu}$ is defined as

$$S^{NT}_{\mu\nu} = \frac{2}{\kappa_5^2}[q_{\mu\nu}\tilde{\nabla}_\sigma \tilde{\nabla}^\sigma \hat{\zeta}_5 - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \hat{\zeta}_5 - 3q_{\mu\nu}H^2\hat{\zeta}_5] \quad , \tag{15.28}$$

from Eq. (15.4) we then find that the TT $h^{TT}_{\mu\nu}$ obeys

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 4b^2 + 4(b^2 - H^2)^{1/2}\delta(w) + e^{-2A}[\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + 4H^2]\right] h^{TT}_{\mu\nu} = -\kappa_5^2 \delta(w) S^{TT}_{\mu\nu} \quad , \tag{15.29}$$

[3]For dS_4^+ $dA(w=0)/d|w| = -(b^2 + H^2)^{1/2}$.

viz. (on using Eq. (B.13))

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 4\frac{dA}{d|w|}\delta(w) + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h^{TT}_{\mu\nu} = -\kappa_5^2\delta(w)S^{TT}_{\mu\nu} \ . \quad (15.30)$$

Equation (15.30) is thus our desired AdS_4^+ TT wave equation,[4] and we can check explicitly that as defined $S^{TT}_{\mu\nu}$ is indeed both transverse and traceless with respect to the AdS_4 background, just as it should be. As we see then, just as in the M_4^+ case, the NT modes play an explicit physical role on the brane; with the general solution to the fluctuation equations of Eqs. (15.2) – (15.5) thus being writable in the form $h_{\mu\nu} = h^{TT}_{\mu\nu} + h^{NT}_{\mu\nu}$ where $h^{NT}_{\mu\nu}$ is given in Eq. (15.13) and $h^{TT}_{\mu\nu}$ is determined from Eq. (15.30), explicit solutions to which we will provide below. As well as provide a wave equation for the TT modes, for completeness we note that we can also write one for the NT modes, with application of the operator $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2$ to Eq. (15.13) and use of Eqs. (15.19) and (15.25) yielding (at any w)

$$\begin{aligned}
&\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h^{NT}_{\mu\nu} - H^2 q_{\mu\nu}h \\
&= \frac{1}{2}e^{2A}[\tilde{\nabla}_\mu(3H^2 - \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha)\tilde{\hat{\zeta}}_\nu + \tilde{\nabla}_\nu(3H^2 - \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha)\tilde{\hat{\zeta}}_\mu] - 2H^2[e^{2A} - 1]q_{\mu\nu}\tilde{\nabla}_\alpha\tilde{\hat{\zeta}}^\alpha \\
&\quad + \frac{1}{H^2}e^{2A}\frac{dA}{d|w|}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 6H^2\tilde{\nabla}_\mu\tilde{\nabla}_\nu - q_{\mu\nu}H^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 6H^4 q_{\mu\nu}\right]\hat{\zeta}_5 \\
&\quad + 2[e^{2A} - 1]\frac{dA}{d|w|}q_{\mu\nu}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 4H^2]\hat{\zeta}_5 \ , \qquad (15.31)
\end{aligned}$$

where $\hat{\zeta}_5$ can be related to the source via Eq. (15.23). As a wave equation, Eq. (15.31) contains no derivatives with respect to w.

It is also of interest to write $S^{NT}_{\mu\nu}$ and $S^{TT}_{\mu\nu}$ in the form of explicit projection operations on the full $S_{\mu\nu}$ in the AdS_4 case. We thus introduce an AdS_4 space scalar propagator according to

$$[q^{-1/2}\partial_\mu q^{1/2}\partial^\mu - 4H^2]E(x,x') = q^{-1/2}\delta^4(x-x') \ , \quad (15.32)$$

so that the solution to Eq. (15.23) can then be written as

$$\hat{\zeta}_5 = \frac{\kappa_5^2}{6}\int d^4x' q^{1/2}(x')E(x,x')S(x') \ . \quad (15.33)$$

In terms of this propagator $S^{NT}_{\mu\nu}$ and $S^{TT}_{\mu\nu}$ then be written as closed form projections

[4]By being written purely as a function of $A(w)$, Eq. (15.30) is thus generic, holding for all the six maximally 4-symmetric brane-world cases of interest to us in this monograph.

of $S_{\mu\nu}$, viz.

$$
\begin{aligned}
S^{NT}_{\mu\nu} &= \frac{1}{3}\left[q_{\mu\nu}S - \left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - H^2 q_{\mu\nu}\right)\int d^4x' q^{1/2}(x')E(x,x')S(x')\right] \quad , \\
S^{TT}_{\mu\nu} &= S_{\mu\nu} - \frac{1}{3}\left[q_{\mu\nu}S - \left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - H^2 q_{\mu\nu}\right)\int d^4x' q^{1/2}(x')E(x,x')S(x')\right] \quad . \quad (15.34)
\end{aligned}
$$

As such Eq. (15.34) thus generalizes the M_4 TT projection operation of Eq. (14.19) to AdS_4.

To determine the gravity which is to emerge on the brane, it is most convenient to first put our brane-embedded solution into the AdS_4 harmonic gauge where it is to then obey $\nabla_\nu h^\nu{}_\mu(w=0) - \nabla_\mu h(w=0)/2 = 0$ (i.e. $e^{-2A}[\tilde{\nabla}_\nu \tilde{h}^\nu{}_\mu(w=0) - \tilde{\nabla}_\mu \tilde{h}(w=0)/2] = 0$) on the brane. With $\nabla_\nu h^\nu{}_\mu - \nabla_\mu h/2$ (a quantity to which $h^{TT}_{\mu\nu}$ does not contribute) evaluating at any w (as per Eqs. (15.14) and (15.15)) to

$$
\nabla_\nu h^\nu{}_\mu - \frac{1}{2}\nabla_\mu h = [3H^2 - \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha]\tilde{\hat{\zeta}}_\mu + \frac{1}{H^2}\frac{dA}{d|w|}\tilde{\nabla}_\mu[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 4H^2]\hat{\zeta}_5 \quad , \quad (15.35)
$$

we thus use the gauge freedom on $\hat{\zeta}_\mu$ to fix $\tilde{\hat{\zeta}}_\mu$ according to

$$
[3H^2 - \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha]\tilde{\hat{\zeta}}_\mu - \frac{(b^2-H^2)^{1/2}}{H^2}\tilde{\nabla}_\mu[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 4H^2]\hat{\zeta}_5 = 0 \quad . \quad (15.36)
$$

(Choosing the gauge functions $\hat{\xi}_\mu$ and $\hat{\xi}_5$ of Eq. (15.12) so as to separately cancel the $\tilde{\hat{\zeta}}_\mu$-dependent and $\hat{\zeta}_5$-dependent terms in Eq. (15.35) would take us out of the axial gauge on the brane.) With the general $h^{NT}_{\mu\nu}$ obeying Eq. (15.31), on making the gauge choice of Eq. (15.36) the brane NT modes are then found to obey

$$
\begin{aligned}
&\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h^{NT}_{\mu\nu}(w=0) - H^2 q_{\mu\nu}h(w=0) \\
&\quad = (b^2-H^2)^{1/2}\left[2\tilde{\nabla}_\mu\tilde{\nabla}_\nu - 6H^2 q_{\mu\nu} + q_{\mu\nu}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]\hat{\zeta}_5 \\
&\quad = -(b^2-H^2)^{1/2}\kappa_5^2\left[S^{NT}_{\mu\nu} - \frac{1}{2}q_{\mu\nu}S\right] \quad , \\
&\left[\frac{1}{2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 3H^2\right]\tilde{h}(w=0) = (b^2-H^2)^{1/2}\kappa_5^2 S \quad . \quad (15.37)
\end{aligned}
$$

We recognize Eq. (15.37) as the curved space generalization of Eq. (14.30), and with the TT modes automatically being harmonic, once we impose Eq. (15.36), on the brane the entire $h_{\mu\nu}(w=0)$ solution will then obey the AdS_4 harmonic gauge condition.

15.3 Comparison with unembedded AdS_4 fluctuations

To determine whether standard gravity might emerge on the brane in the AdS_4^+ brane-world case, we need to compare Eqs. (15.30) and (15.37) with the structure of standard 4-dimensional non-brane AdS_4 gravity in this same harmonic gauge. For pure AdS_4 where $R_{\mu\nu} = 3H^2 q_{\mu\nu}$, in the harmonic gauge the AdS_4 fluctuation tensors straightforwardly evaluate to

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} + 4H^2 h_{\mu\nu} - q_{\mu\nu}H^2 h \ , \\ \delta G_{\mu\nu} &= \frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - 2H^2 h_{\mu\nu} - \frac{1}{4}q_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \frac{1}{2}q_{\mu\nu}H^2 h \ . \end{aligned} \tag{15.38}$$

With the background Einstein tensor being given by $G_{\mu\nu} = -3H^2 q_{\mu\nu}$, the addition of a perturbation $S_{\mu\nu}$ generates a first order response

$$\begin{aligned}\Delta G_{\mu\nu} &= \delta G_{\mu\nu} + 3H^2 h_{\mu\nu} \\ &= \frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} + H^2 h_{\mu\nu} - \frac{1}{4}q_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \frac{1}{2}q_{\mu\nu}H^2 h = -\kappa_4^2 S_{\mu\nu} \ , \end{aligned} \tag{15.39}$$

so that

$$\frac{1}{2}\nabla_\alpha\nabla^\alpha h - 3H^2 h = \kappa_4^2 S \ , \tag{15.40}$$

to thereby yield

$$\frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} + H^2 h_{\mu\nu} - H^2 q_{\mu\nu} h = -\kappa_4^2 [S_{\mu\nu} - \frac{1}{2}q_{\mu\nu} S] \tag{15.41}$$

as the requisite harmonic gauge AdS_4 fluctuation equation.[5] In terms of the scalar propagator $E(x,x')$ given in Eq. (15.32) and an as yet to be defined scalar $\hat{S}$, it is convenient to introduce a quantity

$$\begin{aligned} h^{NT}_{\mu\nu} = & \frac{\kappa_4^2}{3}q_{\mu\nu}\int d^4x' q^{1/2}(x')E(x,x')\hat{S}(x') + \frac{2\kappa_4^2}{3}(\nabla_\mu\nabla_\nu - q_{\mu\nu}H^2) \\ & \times \int d^4x' q^{1/2}(x')E(x,x')\int d^4x'' q^{1/2}(x'')E(x',x'')\hat{S}(x'') \end{aligned} \tag{15.42}$$

with trace

$$h^{NT} = 2\kappa_4^2 \int d^4x' q^{1/2}(x')E(x,x')\hat{S}(x') \ . \tag{15.43}$$

With use of Eqs. (15.16) and (15.32) it can then readily be shown that this $h^{NT}_{\mu\nu}$ identically obeys the AdS_4 harmonic gauge condition no matter what the choice

[5] In passing we note that the combination $\hat{S}_{\mu\nu} = S_{\mu\nu} - q_{\mu\nu}S/2$ actually satisfies the harmonic gauge condition $\nabla_\mu \hat{S}^{\mu\nu} - \nabla^\nu \hat{S}/2 = 0$ in any background.

of $\hat{S}$. On now defining $\hat{S}$ to be a quantity which is related to the trace of the energy-momentum tensor according to

$$S = \hat{S} - 2H^2 \int d^4x' q^{1/2}(x') E(x,x') \hat{S}(x') \ , \tag{15.44}$$

(a choice which enables h^{NT} to then obey the harmonic gauge Eq. (15.40) so that we can identify it with the harmonic gauge h), use of Eq. (15.25) then enables us to show that $h_{\mu\nu}^{NT}$ obeys

$$\frac{1}{2}[\nabla_\alpha \nabla^\alpha + 2H^2] h_{\mu\nu}^{NT} - H^2 q_{\mu\nu} h^{NT} = -\kappa_4^2 \left[S_{\mu\nu}^{NT} - \frac{1}{2} q_{\mu\nu} S \right] \ , \tag{15.45}$$

where $S_{\mu\nu}^{NT}$ is defined in Eq. (15.34). Additionally, the quantity $h_{\mu\nu}^{TT} = h_{\mu\nu} - h_{\mu\nu}^{NT}$ (a quantity which is also harmonic once $h_{\mu\nu}$ is, and thus TT as well once $h^{NT} = h$) is found to obey

$$\frac{1}{2}[\nabla_\alpha \nabla^\alpha + 2H^2] h_{\mu\nu}^{TT} = -\kappa_4^2 S_{\mu\nu}^{TT} \ , \tag{15.46}$$

where $S_{\mu\nu}^{TT}$ is also defined in Eq. (15.34). Comparing (15.45) with Eq. (15.37) we thus see that with the identification $\kappa_4^2 = (b^2 - H^2)^{1/2} \kappa_5^2$ (viz. precisely the value $\kappa_5^4 \lambda/6$ required by the generalized brane Einstein equation given in Eq. (12.13)), the NT mode Eq. (15.37) precisely becomes Eq. (15.45). On the brane then the AdS_4^+ NT modes propagate as spin-zero modes which move on the AdS_4 lightcone. Whether or not standard gravity will be recovered on an AdS_4^+ brane then depends on whether the TT wave equation of Eq. (15.30) contains a massless spin-two mode which recovers the TT Eq. (15.46) on the brane,[6] an issue which we will address in detail below for AdS_4^+ and its analogs on a case by case basis.

However, before actually doing so, we note that it is possible to address some general aspects of the issue using the analysis of Chapter 12. Thus, to first order in a brane perturbation $\delta\tau_{\mu\nu} = S_{\mu\nu}\delta(w)$ around a background with an induced $\Lambda_4 = -3H^2$, the general Eq. (12.13) entails that on the brane the first order change in the brane $\widetilde{G}_{\mu\nu}$ due to the perturbation is given by

$$\begin{aligned} &\widetilde{\delta G}_{\mu\nu}(w=0) + 3H^2 \tilde{h}_{\mu\nu}(w=0) \\ &= \frac{1}{2} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{h}_{\mu\nu} + H^2 \tilde{h}_{\mu\nu} - \frac{1}{4} q_{\mu\nu} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{h} + \frac{1}{2} q_{\mu\nu} H^2 \tilde{h} \\ &= -(b^2 - H^2)^{1/2} \kappa_5^2 S_{\mu\nu} - \delta \bar{E}_{\mu\nu}(w=0) \ . \end{aligned} \tag{15.47}$$

Now as we show in Appendix B, for fluctuations around the AdS_4^+ brane world,

[6] Since the pure AdS_4 TT modes obey $\Delta G_{\mu\nu}^{TT} = (1/2)[\nabla_\alpha \nabla^\alpha + 2H^2] h_{\mu\nu}^{TT} = -\kappa_4^2 S_{\mu\nu}^{TT}$, the equation of motion for a massless spin two graviton mode propagating on the AdS_4 lightcone is thus given by $[\nabla_\alpha \nabla^\alpha + 2H^2] h_{\mu\nu}^{TT} = 0$.

$\delta\bar{E}_{\mu\nu}$ is given at arbitrary w by

$$\delta\bar{E}_{\mu\nu} = \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 2\frac{dA}{d|w|}\frac{\partial}{\partial|w|} - 2\frac{d^2A}{d|w|^2}\right]h_{\mu\nu} \ . \tag{15.48}$$

With the contribution of the NT sector $h^{NT}_{\mu\nu}$ of Eq. (15.13) to $\delta\bar{E}_{\mu\nu}$ being found to vanish identically, and with $h^{NT}_{\mu\nu}(w=0)$ obeying Eq. (15.37) on the brane, the TT sector contribution of $h^{TT}_{\mu\nu}(w=0)$ to Eq. (15.47) is then given as[7]

$$\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h^{TT}_{\mu\nu}(w=0) + (b^2-H^2)^{1/2}\kappa_5^2 S^{TT}_{\mu\nu} = -\delta\bar{E}^{TT}_{\mu\nu}(w=0) \ . \tag{15.49}$$

Comparing with the TT Eq. (15.46), we thus see that as long as there is a massless spin two graviton mode present in the TT sector, standard AdS_4 gravity will be recovered on the brane in any kinematic region in which the Weyl tensor term contribution may be ignored. With explicit calculation showing that $\delta E_{\mu\nu}$ actually vanishes identically for an $h^{TT}_{\mu\nu}$ whose w dependence is precisely that of the warp factor e^{2A} itself, we thus see that whenever there is a massless TT graviton in the spectrum (one whose wave function always is the warp factor itself in separable metric brane worlds), its contribution alone would lead to the vanishing of both sides of Eq. (15.49), so that in any long-distance limit on the brane in which a TT massless mode would dominate, the standard gravity Eq. (15.46) would then be recovered. The vanishing or non-vanishing of the Weyl tensor projection $\delta E_{\mu\nu}$ at large distances on the brane is thus the measure of whether or not standard long-distance gravity on the brane ultimately emerges. Having now determined the relevant TT wave equation, we turn next to a discussion of an appropriate normalization condition for the associated TT modes which are needed in order to construct the TT wave equation propagator.

[7] Just as in the analogue M_4^+ case, Eq. (15.49) is the $w=0$ limit of the TT wave equation of Eq. (15.30).

Chapter 16

Normalization of Fluctuation Modes and Curved Space Gauss's Law

16.1 Orthogonality measure for TT modes

For all of the six brane-world set-ups with maximally 4-symmetric branes the TT mode wave equation is given generically by Eq. (15.30), viz. by

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2}-4\left(\frac{dA}{d|w|}\right)^2-4\frac{dA}{d|w|}\delta(w)+e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h^{TT}_{\mu\nu}=-\kappa_5^2\delta(w)S^{TT}_{\mu\nu}\ . \quad (16.1)$$

With a Z_2-symmetric $h^{TT}_{\mu\nu}$ being a function of $|w|$, the TT mode wave equation will typically break up into two separate pieces, one of which is continuous at the brane and the other of which involves the discontinuous delta function junction term, viz.

$$\left[\frac{\partial^2}{\partial|w|^2}-4\left(\frac{dA}{d|w|}\right)^2+e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h^{TT}_{\mu\nu}=0\ , \quad (16.2)$$

$$\delta(w)\left[\frac{\partial}{\partial|w|}-2\frac{dA}{d|w|}\right]h^{TT}_{\mu\nu}=-\kappa_5^2\delta(w)S^{TT}_{\mu\nu}\ , \quad (16.3)$$

with the latter condition implementing the junction condition at the brane. The standard way to construct a propagator for Eq. (16.1) is to build it out of modes which satisfy the source-free limit of Eqs. (16.2) and (16.3), viz. which satisfy

$$\left[\frac{\partial^2}{\partial|w|^2}-4\left(\frac{dA}{d|w|}\right)^2+e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h^{TT}_{\mu\nu}=0\ , \quad (16.4)$$

as subject to the constraint

$$\delta(w)\left[\frac{\partial}{\partial|w|}-2\frac{dA}{d|w|}\right]h^{TT}_{\mu\nu}=0\ , \quad (16.5)$$

and in order to determine the particular set or subset of these modes which will suffice for such a construction, the key requirement is that the selected modes form a complete basis, complete in the sense that any localized fluctuation can be expanded in terms of them. Now while the most convenient type of basis to use is of course one which is orthonormal with respect to some appropriate measure, in general modes which make up a complete basis need not be normalizable at all.[1] And as we shall see in the following, while it will indeed prove possible to construct complete orthonormal bases for both convergent and divergent warp factor brane worlds, for the latter, precisely because they involve geometries in which information can come in from infinity in a finite time, there will also exist a second family of basis modes which do not in fact vanish asymptotically and which are consequently not normalizable at all. The distinction between these various kinds of bases lies only in their behavior at infinity (an infinity reachable by a null signal in a finite time in the divergent warp factor case), with the existence of two rather than one possible complete sets of basis modes in divergent warp factor brane worlds being a quite novel and unusual aspect of such theories, something we will have to take into consideration when we construct and assess such brane-world propagators below.

To determine an appropriate measure for orthonormalizability, we note that if we introduce a generic separation constant for Eq. (16.2) of the form[2]

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2kH^2]h^{TT}_{\mu\nu} = m^2 h^{TT}_{\mu\nu} \quad , \tag{16.6}$$

defined here so that modes with $m^2 = 0$ will propagate on the appropriate ($k = 1, 0, -1$) maximally 4-symmetric lightcone, Eq. (16.4) will then take the form

$$\left[\frac{\partial^2}{\partial|w|^2} - 4\left(\frac{dA}{d|w|}\right)^2 + 2ke^{-2A}H^2 + e^{-2A}m^2\right] h^{TT}_{\mu\nu} = 0 \ . \tag{16.7}$$

Then, with separable solutions to Eqs. (16.6) and (16.7) being of the generic form $h^{TT}_{\mu\nu} = f_m(|w|)e_{\mu\nu}(x^\lambda, m)$, any pair of them with different m^2 values will identically obey

$$\begin{aligned} &e^{-2A}(m_1^2 - m_2^2)f_{m_1}f_{m_2} \\ &= \frac{d}{d|w|}\left[f_{m_1}\left(\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right)f_{m_2} - f_{m_2}\left(\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right)f_{m_1}\right] \ ; \end{aligned} \tag{16.8}$$

and given the junction condition of Eq. (16.5), will consequently obey

$$\begin{aligned} &(m_1^2 - m_2^2)\int_0^\infty d|w|e^{-2A}f_{m_1}f_{m_2} \\ &= \lim_{|w|\to\infty}\left[f_{m_1}\left(\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right)f_{m_2} - f_{m_2}\left(\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right)f_{m_1}\right] \ . \end{aligned} \tag{16.9}$$

Thus whenever the modes are well enough behaved at $|w| = \infty$ so as to cause the right-hand side of Eq. (16.9) to vanish, modes with unequal m^2 would then be orthogonal with respect to an e^{-2A} measure.

[1] A case in point is flat space plane waves which do not have a finite norm but which still serve as a basis for functions which do.

[2] In the dS_4 and AdS_4 cases where $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}$ is not (as we explicitly show below) diagonal in its $(\mu\nu)$ indices, the allowed m^2 would be the eigenvalues of the $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2kH^2$ operator.

16.2 Energy of gravitational fluctuations around flat spacetime

While the establishment of an orthogonality criterion for pairs of modes only requires use of the wave equation itself (viz. an equation which is first order in the fluctuation even though Eq. (16.9) itself is second order), to establish a normalization criterion for an individual mode will require a condition which is bilinear in a single mode. Of such possible bilinear quantities the most natural one to consider would be one based on the time independent energy carried by a gravitational wave, since the finiteness of such an energy would serve as a bilinear normalization condition on the modes which would then hold at all times. Now in trying to construct a conserved total energy for gravitational waves it is customary to work not with a $T^{\mu\nu}$ which obeys a true covariant conservation condition $\nabla_\mu T^{\mu\nu} = 0$, but rather with some non-covariant pseudo-tensor which obeys an ordinary conservation condition $\partial_\mu T^{\mu\nu} = 0$ instead, since it is an ordinary conservation condition which can translate into a time independent total energy. And while we will present a formalism below which will allow us to actually work with a covariant conservation condition, in order to see how things work we shall first consider the one case where it is valid to use an an ordinary conservation condition, namely fluctuations around a flat background.

Thus, for the familiar case of fluctuations around a flat 4-dimensional spacetime[3] where the term in the Ricci tensor which is linear in a general fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is given by[4]

$$R^{(1)}_{\mu\kappa} = \frac{1}{2}\left(\partial_\mu\partial_\kappa h^\lambda{}_\lambda - \partial_\lambda\partial_\kappa h^\lambda{}_\mu - \partial_\lambda\partial_\mu h^\lambda{}_\kappa + \partial_\lambda\partial^\lambda h_{\mu\kappa}\right) \quad , \tag{16.10}$$

the exact Einstein equations in the presence of a perturbative source $\tau_{\mu\kappa}$ can be written as

$$G^{(1)}_{\mu\kappa} = R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}R^{(1)\lambda}{}_\lambda = -\kappa_4^2(\tau_{\mu\kappa} + t_{\mu\kappa}) \quad , \tag{16.11}$$

where

$$\kappa_4^2 t_{\mu\kappa} = R_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}R^\lambda{}_\lambda - R^{(1)}_{\mu\kappa} + \frac{1}{2}\eta_{\mu\kappa}R^{(1)\lambda}{}_\lambda \quad . \tag{16.12}$$

With use of the form of $R^{(1)}_{\mu\kappa}$ given in Eq. (16.10), the left-hand side of Eq. (16.11) is readily found to obey the linearized Bianchi identity

$$\frac{\partial}{\partial x^\mu}\left(R^{(1)\mu\kappa} - \frac{1}{2}\eta^{\mu\kappa}R^{(1)\lambda}{}_\lambda\right) = 0 \quad . \tag{16.13}$$

With the perturbative source itself also being conserved according to $\partial_\mu \tau^{\mu\kappa} = 0$, it thus follows that the purely gravitational $t^{\mu\kappa}$ identically obeys the ordinary

[3]The discussion presented here is based on [Weinberg (1972)].

[4]All raising and lowering of indices is to be done here with the flat background metric $\eta_{\mu\nu}$.

conservation condition

$$\frac{\partial}{\partial x^\mu} t^{\mu\kappa} = 0 \tag{16.14}$$

for modes which are solutions to the equation of motion. With Eq. (16.14) being an ordinary conservation condition rather than a covariant one, we see that if the system is contained in a volume L^3 bounded by a surface S with outward normal n_i, the integration of Eq. (16.14) would then entail that

$$\frac{\partial}{\partial t} \int d^3x t^{0\kappa} = -\int dS t^{i\kappa} n_i \quad , \tag{16.15}$$

with the asymptotic vanishing of the surface term in Eq. (16.15) in its turn entailing the time independence of the 4-component

$$P^\kappa = \int d^3x t^{0\kappa} \quad . \tag{16.16}$$

With the purely gravitational $t_{\mu\kappa}$ appearing as a source term on the right-hand side of Eq. (16.11), $E = P^0$ can be thought of as being the energy carried by gravity itself.

As introduced, the quantity $t_{\mu\kappa}$ represents the difference between the full Einstein tensor and its first order piece, to thus begin in second order, with the substitution $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in the expression for the general Riemann tensor

$$\begin{aligned} R_{\lambda\mu\nu\kappa} = &\frac{1}{2}\left(\partial_\kappa\partial_\mu g_{\lambda\nu} - \partial_\kappa\partial_\lambda g_{\mu\nu} - \partial_\nu\partial_\mu g_{\lambda\kappa} + \partial_\nu\partial_\lambda g_{\mu\kappa}\right) \\ &+ g_{\eta\sigma}(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}) \end{aligned} \tag{16.17}$$

leading to a second order in $h_{\mu\nu}$ contribution $t^{(2)}_{\mu\kappa}$ to $t_{\mu\kappa}$ of the form

$$\kappa_4^2 t^{(2)}_{\mu\kappa} = R^{(2)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}\eta^{\alpha\beta}R^{(2)}_{\alpha\beta} - \frac{1}{2}h_{\mu\kappa}\eta^{\alpha\beta}R^{(1)}_{\alpha\beta} + \frac{1}{2}\eta_{\mu\kappa}h^{\alpha\beta}R^{(1)}_{\alpha\beta} \quad , \tag{16.18}$$

where $R^{(2)}_{\mu\kappa}$ is given by[5]

$$\begin{aligned} R^{(2)}_{\mu\kappa} &= -\frac{1}{2}h^{\lambda\nu}\left(\partial_\kappa\partial_\mu h_{\lambda\nu} - \partial_\kappa\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h_{\lambda\kappa} + \partial_\nu\partial_\lambda h_{\mu\kappa}\right) \\ &\quad + \frac{1}{4}\left(2\partial_\nu h^\nu{}_\sigma - \partial_\sigma h^\nu{}_\nu\right)\left(\partial_\kappa h^\sigma{}_\mu + \partial_\mu h^\sigma{}_\kappa - \partial^\sigma h_{\mu\kappa}\right) \\ &\quad - \frac{1}{4}\left(2\partial_\lambda h_{\sigma\mu}\partial^\lambda h^\sigma{}_\kappa - 2\partial_\lambda h^\sigma{}_\mu \partial_\sigma h^\lambda{}_\kappa + \partial_\mu h_{\sigma\lambda}\partial_\kappa h^{\sigma\lambda}\right) \quad . \end{aligned} \tag{16.19}$$

While it is tempting to identify the above $t^{(2)}_{\mu\kappa}$ as the energy-momentum tensor of a gravitational wave, such an identification cannot initially be made, since for

[5]While a first order term in $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ would require a second order term in its $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\sigma h^{\sigma\nu}$ inverse in order to maintain $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu$ through second order, the quadratic term in $g^{\mu\nu}$ does not generate any quadratic terms in the second order Ricci tensor.

fluctuations which obey the first order wave equation

$$R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}R^{(1)\lambda}{}_{\lambda} = -\kappa_4^2\tau_{\mu\kappa} \quad , \tag{16.20}$$

the satisfying of Eq. (16.11) order by order in $h_{\mu\nu}$ would then entail that $t^{(2)}_{\mu\kappa}$ would have to vanish identically. In order to avoid such an outcome there needs to be a further second order term, to be labelled $\hat{t}^{(2)}_{\mu\kappa}$, present in $t_{\mu\kappa}$, though with its presence one would then have to initially anticipate that it would only be the sum $t^{(2)}_{\mu\kappa} + \hat{t}^{(2)}_{\mu\kappa}$ which would be conserved, rather than $t^{(2)}_{\mu\kappa}$ alone. To identify the needed additional $\hat{t}^{(2)}_{\mu\kappa}$, we note that up to second order rather than just to first, the metric itself would be given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \hat{g}_{\mu\nu} \quad , \tag{16.21}$$

where $\hat{g}_{\mu\nu}$ is an explicit second order change in the metric, with there being an accompanying second order change

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\sigma}h^{\sigma\nu} - \hat{g}^{\mu\nu} \tag{16.22}$$

in its inverse. From the form of the general Riemann tensor given in Eq. (16.17) we find that $\hat{g}_{\mu\nu}$ induces a second order change in $t_{\mu\kappa}$ of the form

$$\begin{aligned} \hat{t}^{(2)}_{\mu\kappa} = {} & \frac{1}{2}\left(\partial_\mu\partial_\kappa\hat{g}^{\lambda}{}_{\lambda} - \partial_\lambda\partial_\kappa\hat{g}^{\lambda}{}_{\mu} - \partial_\lambda\partial_\mu\hat{g}^{\lambda}{}_{\kappa} + \partial_\lambda\partial^\lambda\hat{g}_{\mu\kappa}\right) \\ & -\frac{1}{2}\eta_{\mu\kappa}\left(\partial_\alpha\partial^\alpha\hat{g}^{\beta}{}_{\beta} - \partial_\alpha\partial_\beta\hat{g}^{\alpha\beta}\right) \quad , \end{aligned} \tag{16.23}$$

with it then being $t^{(2)}_{\mu\kappa} + \hat{t}^{(2)}_{\mu\kappa}$ which would vanish in modes which obey Eq. (16.20), with the condition

$$\hat{t}^{(2)}_{\mu\kappa} = -t^{(2)}_{\mu\kappa} \tag{16.24}$$

enabling us to determine $\hat{g}_{\mu\nu}$ in terms of the $h_{\mu\nu}$ found from the first order Eq. (16.20). As such, the expression given for $\hat{t}^{(2)}_{\mu\kappa}$ in Eq. (16.23) is quite remarkable, since a replacement of $\hat{g}_{\mu\nu}$ by $h_{\mu\nu}$ in this expression would cause $\hat{t}^{(2)}_{\mu\kappa}$ to become identical in form to the quantity $R^{(1)}_{\mu\kappa} - (1/2)\eta_{\mu\kappa}R^{(1)\lambda}{}_{\lambda}$ which appears on the left-hand side of Eq. (16.11). Thus not only can Eq. (16.24) be thought of as an Einstein equation for the second order $\hat{g}_{\mu\nu}$ in which $\hat{g}_{\mu\nu}$ is determinable in terms of a source $t^{(2)}_{\mu\kappa}$ which depends quadratically on the first order $h_{\mu\nu}$, moreover, precisely because $R^{(1)}_{\mu\kappa} - (1/2)\eta_{\mu\kappa}R^{(1)\lambda}{}_{\lambda}$ kinematically obeys a Bianchi identity, it follows that $\hat{t}^{(2)}_{\mu\kappa}$, and thus $t^{(2)}_{\mu\kappa}$, must do so also, with both $\hat{t}^{(2)}_{\mu\kappa}$ and $t^{(2)}_{\mu\kappa}$ then necessarily having to obey

$$\frac{\partial}{\partial x^\mu}\hat{t}^{(2)\mu\kappa} = 0 \quad , \quad \frac{\partial}{\partial x^\mu}t^{(2)\mu\kappa} = 0 \tag{16.25}$$

in fluctuations which obey the first order Eq. (16.20). Thus after the fact we discover that $t^{(2)}_{\mu\kappa}$, viz. that second order part of $t_{\mu\kappa}$ which arises purely from setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, is conserved on its own after all, just as we had initially wanted.

To assess the relative importance of the $t^{(2)}_{\mu\kappa}$ and $\hat{t}^{(2)}_{\mu\kappa}$ terms, we note that for an oscillating, spherically symmetric source of mass M for instance where the $h_{\mu\nu}$ solutions to Eq. (16.20) typically behave as $\kappa_4^2 M e^{i\omega t}/r$, it follows from Eq. (16.24) that $\hat{g}_{\mu\nu}$ will behave as $\kappa_4^4 M^2 e^{2i\omega t}/r^2$. Thus when an oscillating source emits a gravitational wave, it is only its $h_{\mu\nu}$ component which will affect a distant receiver in any substantial way, with the energy imparted to that receiver then being well approximated by the energy carried by the $t^{(2)}_{\mu\kappa}$ piece of the full $t_{\mu\kappa}$ alone. With this $t^{(2)}_{\mu\kappa}$ piece being conserved, in solutions to the source-free region equation $G^{(1)}_{\mu\kappa} = 0$ for which the asymptotic surface term on the right-hand side of Eq. (16.15) vanishes we may thus identify the fluctuation bilinear

$$E^{(2)} = \int d^3x t^{(2)00} = \frac{1}{\kappa_4^2} \int d^3x \left(R^{(2)00} - \frac{1}{2}\eta^{00} R^{(2)\lambda}{}_{\lambda} \right) \tag{16.26}$$

as the lowest order energy deliverable to a detector by a gravitational wave,[6] with the finiteness of this energy then being able to serve as a normalization criterion for fluctuation configurations in which $E^{(2)}$ is time independent.[7] Then, finally, for a source-free region 3-box normalized massless harmonic gauge plane wave mode $h_{\mu\nu} = 2\kappa_4 e_{\mu\nu} e^{ip\cdot x}/(2p^0)^{1/2} L^{3/2} + \text{c.c.}$ with $p_\mu p^\mu = 0$, $p_\mu e^\mu{}_\nu = (1/2) p_\nu e^\alpha{}_\alpha$, $E^{(2)}$ is readily found to evaluate to the time-independent

$$E^{(2)} = p^0 \left(e^{\mu\nu} e_{\mu\nu} - \frac{1}{2} (e^\alpha{}_\alpha)^2 \right) \tag{16.27}$$

just as one would want of an energy.

16.3 First order total gravitational energy as a surface term

As well as construct the energy in a gravitational wave (a second order quantity) it is also possible to recover Gauss's law (the first order weak gravity limit of the theory). Specifically, it is possible to write the linearized Einstein tensor as a total derivative, since on introducing

$$Q^{\rho\nu\lambda} = \frac{1}{2} \left(\frac{\partial h^\mu{}_\mu}{\partial x_\nu} \eta^{\rho\lambda} - \frac{\partial h^\mu{}_\mu}{\partial x_\rho} \eta^{\nu\lambda} - \frac{\partial h^{\mu\nu}}{\partial x^\mu} \eta^{\rho\lambda} + \frac{\partial h^{\mu\rho}}{\partial x^\mu} \eta^{\nu\lambda} + \frac{\partial h^{\nu\lambda}}{\partial x_\rho} - \frac{\partial h^{\rho\lambda}}{\partial x_\nu} \right) , \tag{16.28}$$

[6]For the moment though such an identification must be regarded as only being provisional, pending an analysis of the behavior of $E^{(2)}$ under gauge transformations which we provide below.

[7]For the purposes of constructing the $h_{\mu\nu}$ propagator needed to integrate Eq. (16.20), we only need information about the normalization of $h_{\mu\nu}$, and require no knowledge of $\hat{g}_{\mu\nu}$.

a quantity which obeys

$$Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda} \ , \tag{16.29}$$

we find directly that

$$R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}R^{(1)\lambda}{}_{\lambda} = \frac{\partial}{\partial x^{\rho}}Q^{\rho}{}_{\mu\kappa} \ . \tag{16.30}$$

With the lowest order Einstein equation thus taking the form

$$\frac{\partial}{\partial x^{\rho}}Q^{\rho}{}_{\mu\kappa} = -\kappa_4^2\tau_{\mu\kappa} \ , \tag{16.31}$$

on using the antisymmetry of Eq. (16.29) the total energy injected into the system by the perturbative source is then found to be given by

$$\begin{aligned} E^{(1)} &= \int d^3x\tau^{00} = -\frac{1}{\kappa_4^2}\int d^3x\frac{\partial}{\partial x^{\rho}}Q^{\rho 00} = -\frac{1}{\kappa_4^2}\int d^3x\frac{\partial}{\partial x^{i}}Q^{i00} \\ &= -\frac{1}{\kappa_4^2}\int dSn_iQ^{i00} = -\frac{1}{2\kappa_4^2}\int dSn_i\left(\frac{\partial h^{j}{}_{j}}{\partial x_i} - \frac{\partial h^{ij}}{\partial x^{j}}\right) \ , \end{aligned} \tag{16.32}$$

a purely first order relation. With Q^{i00} being given by $(1/2)(\partial^i h^j{}_j - \partial_j h^{ij})$, we see that $\partial_i Q^{i00}$ remains unchanged under the replacement $\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, so that Eq. (16.32) is manifestly gauge invariant just as it has to be since the injected τ^{00} is a physical observable. For a static point source $\tau_{\mu\nu} = \delta^0_\mu\delta^0_\nu M\delta^3(x)$, the non-vanishing components of $\tau_{\mu\nu} - \eta_{\mu\nu}\tau/2$ are given by $\tau_{00} - \eta_{00}\tau/2 = \tau_{11} - \eta_{11}\tau/2 = \tau_{22} - \eta_{22}\tau/2 = \tau_{33} - \eta_{33}\tau/2 = M\delta^3(x)/2$, so that in the convenient harmonic gauge the non-vanishing components of $h_{\mu\nu}$ then obey $\nabla^2 h_{00} = \nabla^2 h_{11} = \nabla^2 h_{22} = \nabla^2 h_{33} = -\kappa_4^2 M\delta^3(x)$. On defining $h_{00} = -2\phi$ (and thus $h_{11} = h_{22} = h_{33} = -2\phi$ in this case), the Newtonian potential is then found to obey $\nabla^2\phi = \kappa_4^2 M\delta^3(x)/2 = 4\pi MG\delta^3(x)$ (so that $\phi = -MG/r$), so that Eq. (16.32) then yields

$$\frac{1}{2}\kappa_4^2 M = -\frac{1}{2}\int dSn_i\frac{\partial h^{00}}{\partial x_i} = -\frac{1}{2}\int d^3x\nabla^2 h^{00} = \int d^3x\nabla^2\phi \ . \tag{16.33}$$

We recognize Eq. (16.33) as Gauss's law of gravity.

16.4 Second order total gravitational energy as a surface term

As well as being able to write the first order $E^{(1)} = \int d^3x\tau^{00}$ in the form of a surface term, using the above analysis it is also possible to write the second order $E^{(2)}$ of Eq. (16.26) in such a form also. Specifically, we had noted earlier that the second order $\hat{t}^{(2)}_{\mu\kappa}$ of Eq. (16.23) had a form identical to that of the first order Einstein tensor given in Eq. (16.11). Thus if we introduce a quantity

$$\hat{Q}^{\rho\nu\lambda} = \frac{1}{2}\left(\frac{\partial \hat{g}^{\mu}{}_{\mu}}{\partial x_{\nu}}\eta^{\rho\lambda} - \frac{\partial \hat{g}^{\mu}{}_{\mu}}{\partial x_{\rho}}\eta^{\nu\lambda} - \frac{\partial \hat{g}^{\mu\nu}}{\partial x^{\mu}}\eta^{\rho\lambda} + \frac{\partial \hat{g}^{\mu\rho}}{\partial x^{\mu}}\eta^{\nu\lambda} + \frac{\partial \hat{g}^{\nu\lambda}}{\partial x_{\rho}} - \frac{\partial \hat{g}^{\rho\lambda}}{\partial x_{\nu}}\right) \ , \tag{16.34}$$

the same reasoning that leads to Eq. (16.30) coupled with use of Eq. (16.24) enables us to obtain

$$\frac{\partial}{\partial x^{\rho}}\hat{Q}^{\rho}{}_{\mu\kappa} = \hat{t}_{\mu\kappa} = -t^{(2)}_{\mu\kappa} \ , \tag{16.35}$$

to thereby yield for the energy

$$\begin{aligned} E^{(2)} &= \int d^3x t^{(2)00} = -\int d^3x \frac{\partial}{\partial x^i}\hat{Q}^{i00} \\ &= -\frac{1}{2}\int d^3x \frac{\partial}{\partial x^i}\left(\frac{\partial \hat{g}^{j}{}_{j}}{\partial x_i} - \frac{\partial \hat{g}^{ij}}{\partial x^j}\right) = -\frac{1}{2}\int dS n_i \left(\frac{\partial \hat{g}^{j}{}_{j}}{\partial x_i} - \frac{\partial \hat{g}^{ij}}{\partial x^j}\right) \ , \end{aligned} \tag{16.36}$$

a remarkably compact relation.

16.5 Second order gauge transformation properties of the energy

Once given a relation such as Eq. (16.36), it is relatively simple to determine how $E^{(2)}$ behaves under an infinitesimal gauge transformation $\bar{x}^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$, though to do so we first need to determine how $\hat{g}_{\mu\nu}$ behaves under such a transformation. Since $(\partial \bar{x}^{\mu}/\partial x^{\sigma})(\partial x^{\sigma}/\partial \bar{x}^{\nu}) = \delta^{\mu}_{\nu}$, it follows that under such an infinitesimal gauge transformation the relations

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{\sigma}} = \delta^{\mu}_{\sigma} - \frac{\partial \epsilon^{\mu}}{\partial x^{\sigma}} \ , \quad \frac{\partial x^{\sigma}}{\partial \bar{x}^{\nu}} = \delta^{\sigma}_{\nu} + \frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \frac{\partial \epsilon^{\alpha}}{\partial x^{\nu}}\frac{\partial \epsilon^{\sigma}}{\partial x^{\alpha}} \ , \quad \frac{\partial}{\partial \bar{x}^{\mu}} = \frac{\partial}{\partial x^{\mu}} + \frac{\partial \epsilon^{\sigma}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\sigma}} \tag{16.37}$$

hold through second order. Under the $\bar{x}^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$ gauge transformation the change in any general rank two tensor $A_{\mu\nu}$ is given by

$$\bar{A}_{\mu\nu}(\bar{x}) = \bar{A}_{\mu\nu}(x-\epsilon) = \frac{\partial x^{\sigma}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\tau}}{\partial \bar{x}^{\nu}}A_{\sigma\tau}(x) \ , \tag{16.38}$$

to yield

$$\begin{aligned} &\bar{A}_{\mu\nu}(x) - \epsilon^{\lambda}\frac{\partial \bar{A}_{\mu\nu}(x)}{\partial x^{\lambda}} + \frac{1}{2}\epsilon^{\lambda}\epsilon^{\rho}\frac{\partial^2 \bar{A}_{\mu\nu}(x)}{\partial x^{\lambda}\partial x^{\rho}} = A_{\mu\nu}(x) + A_{\mu\tau}(x)\frac{\partial \epsilon^{\tau}}{\partial x^{\nu}} \\ &+ A_{\nu\sigma}(x)\frac{\partial \epsilon^{\sigma}}{\partial x^{\mu}} + A_{\mu\tau}(x)\frac{\partial \epsilon^{\alpha}}{\partial x^{\nu}}\frac{\partial \epsilon^{\tau}}{\partial x^{\alpha}} + A_{\nu\sigma}(x)\frac{\partial \epsilon^{\alpha}}{\partial x^{\mu}}\frac{\partial \epsilon^{\sigma}}{\partial x^{\alpha}} + A_{\sigma\tau}(x)\frac{\partial \epsilon^{\sigma}}{\partial x^{\mu}}\frac{\partial \epsilon^{\tau}}{\partial x^{\nu}} \end{aligned} \tag{16.39}$$

through second order. Using Eq. (16.39) itself to expand the $\bar{A}_{\mu\nu}(x)$ derivative terms on its left-hand side, we can then write $\bar{A}_{\mu\nu}(x)$ as

$$\begin{aligned}\bar{A}_{\mu\nu}(x) = &\ A_{\mu\nu}(x) + A_{\mu\tau}(x)\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} + A_{\nu\sigma}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}} + \epsilon^{\lambda}\frac{\partial A_{\mu\nu}(x)}{\partial x^{\lambda}} \\ &+\epsilon^{\lambda}\frac{\partial}{\partial x^{\lambda}}\left(A_{\mu\tau}(x)\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} + A_{\nu\sigma}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}}\right) - \frac{1}{2}\epsilon^{\lambda}\epsilon^{\rho}\frac{\partial^2 A_{\mu\nu}(x)}{\partial x^{\lambda}\partial x^{\rho}} \\ &+A_{\mu\tau}(x)\frac{\partial\epsilon^{\alpha}}{\partial x^{\nu}}\frac{\partial\epsilon^{\tau}}{\partial x^{\alpha}} + A_{\nu\sigma}(x)\frac{\partial\epsilon^{\alpha}}{\partial x^{\mu}}\frac{\partial\epsilon^{\sigma}}{\partial x^{\alpha}} + A_{\sigma\tau}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}}\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} \end{aligned} \tag{16.40}$$

through second order. If we now set $A_{\mu\nu}(x) = A^{(0)}_{\mu\nu}(x)+A^{(1)}_{\mu\nu}(x)+A^{(2)}_{\mu\nu}(x)$, $\bar{A}_{\mu\nu}(x) = \bar{A}^{(0)}_{\mu\nu}(x) + \bar{A}^{(1)}_{\mu\nu}(x) + \bar{A}^{(2)}_{\mu\nu}(x)$, we find that

$$\begin{aligned}\bar{A}^{(0)}_{\mu\nu}(x) = &\ A^{(0)}_{\mu\nu}(x)\ , \\ \bar{A}^{(1)}_{\mu\nu}(x) = &\ A^{(1)}_{\mu\nu}(x) + A^{(0)}_{\mu\tau}(x)\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} + A^{(0)}_{\nu\sigma}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}} + \epsilon^{\lambda}\frac{\partial A^{(0)}_{\mu\nu}(x)}{\partial x^{\lambda}}\ , \\ \bar{A}^{(2)}_{\mu\nu}(x) = &\ A^{(2)}_{\mu\nu}(x) + A^{(1)}_{\mu\tau}(x)\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} + A^{(1)}_{\nu\sigma}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}} + \epsilon^{\lambda}\frac{\partial A^{(1)}_{\mu\nu}(x)}{\partial x^{\lambda}} \\ &+\epsilon^{\lambda}\frac{\partial}{\partial x^{\lambda}}\left(A^{(0)}_{\mu\tau}(x)\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}} + A^{(0)}_{\nu\sigma}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}}\right) - \frac{1}{2}\epsilon^{\lambda}\epsilon^{\rho}\frac{\partial^2 A^{(0)}_{\mu\nu}(x)}{\partial x^{\lambda}\partial x^{\rho}} \\ &+A^{(0)}_{\mu\tau}(x)\frac{\partial\epsilon^{\alpha}}{\partial x^{\nu}}\frac{\partial\epsilon^{\tau}}{\partial x^{\alpha}} + A^{(0)}_{\nu\sigma}(x)\frac{\partial\epsilon^{\alpha}}{\partial x^{\mu}}\frac{\partial\epsilon^{\sigma}}{\partial x^{\alpha}} + A^{(0)}_{\sigma\tau}(x)\frac{\partial\epsilon^{\sigma}}{\partial x^{\mu}}\frac{\partial\epsilon^{\tau}}{\partial x^{\nu}}\ . \end{aligned} \tag{16.41}$$

With Eq. (16.41) holding for any rank two tensor, it applies to the metric itself, so that in the flat background case where $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, we see that under a gauge transformation $\bar{x}^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$ the second order term in the metric transforms as

$$\begin{aligned}\bar{\hat{g}}_{\mu\nu}(x) = &\ \hat{g}_{\mu\nu}(x) + h_{\mu\lambda}(x)\partial_{\nu}\epsilon^{\lambda} + h_{\nu\lambda}(x)\partial_{\mu}\epsilon^{\lambda} + \epsilon^{\lambda}\partial_{\lambda}h_{\mu\nu}(x) \\ &+\partial_{\mu}(\epsilon^{\lambda}\partial_{\lambda}\epsilon_{\nu}) + \partial_{\nu}(\epsilon^{\lambda}\partial_{\lambda}\epsilon_{\mu}) + \partial_{\mu}\epsilon_{\lambda}\partial_{\nu}\epsilon^{\lambda}\ . \end{aligned} \tag{16.42}$$

Given Eq. (16.42), we find that under a gauge transformation the quantity $\partial_i\hat{Q}^{i00} = (1/2)(\partial_i\partial^i\hat{g}^j{}_j - \partial_i\partial_j\hat{g}^{ij})$ does not remain unchanged, with the term quadratic in ϵ_{μ} for instance inducing a change

$$\partial_i\bar{\hat{Q}}^{i00} - \partial_i\hat{Q}^{i00} = \frac{1}{2}\left(\partial_i\partial_j\epsilon_{\lambda}\partial^i\partial^j\epsilon^{\lambda} - \partial_i\partial^i\epsilon_{\lambda}\partial_j\partial^j\epsilon^{\lambda}\right)\ . \tag{16.43}$$

From Eq. (16.43) we thus establish that $\hat{t}^{(2)}_{\mu\kappa}$ is not at all gauge invariant. However, with Eq. (16.41) also holding for the Einstein tensor, we see that when evaluated in fluctuations around flat space (viz. around $G^{(0)}_{\mu\nu} = 0$) which obey the source-free region wave equation $G^{(1)}_{\mu\nu} = 0$, the second order term in the Einstein tensor will then obey $\bar{G}^{(2)}_{\mu\nu} = G^{(2)}_{\mu\nu}$, with the sum $t^{(2)}_{\mu\kappa} + \hat{t}^{(2)}_{\mu\kappa}$ thus being gauge invariant in source-free region solutions to the theory.[8] Gauge invariance thus requires the presence

[8]The gauge invariance of $t^{(2)}_{\mu\kappa} + \hat{t}^{(2)}_{\mu\kappa}$ cannot immediately be inferred from Eq. (16.11), since while the the first order $G^{(1)}_{\mu\kappa}$ term on its left-hand side is gauge invariant, it is only gauge invariant

of both $t^{(2)}_{\mu\kappa}$ and $\hat{t}^{(2)}_{\mu\kappa}$ in second order, with neither of them being separately gauge invariant.

16.6 Gauge invariant conservation condition for the total energy

Despite the above interplay between $t^{(2)}_{\mu\kappa}$ and $\hat{t}^{(2)}_{\mu\kappa}$, something quite remarkable happens when we consider not the functions $t^{(2)}_{\mu\kappa}$ and $\hat{t}^{(2)}_{\mu\kappa}$ themselves but rather their derivatives. We had noted earlier that from the very form for $\hat{t}^{(2)}_{\mu\kappa}$ given in Eq. (16.23) we could deduce that $\partial_\mu \hat{t}^{(2)\mu\kappa}$ vanished identically for arbitrary $\hat{g}_{\mu\nu}$. Consequently, following a gauge transformation the transformed $\partial_\mu \bar{\hat{t}}^{(2)\mu\kappa}$ would still vanish when evaluated with $\bar{\hat{g}}_{\mu\nu}$, with the condition $\partial_\mu \hat{t}^{(2)\mu\kappa} = 0$ thus being a gauge invariant one. Then with the sum $t^{(2)}_{\mu\kappa} + \hat{t}^{(2)}_{\mu\kappa}$ being gauge invariant in source-free region solutions to the theory, it follows that the condition $\partial_\mu t^{(2)\mu\kappa} = 0$ is gauge invariant too. Hence even though $t^{(2)\mu\kappa}$ is not itself gauge invariant, nonetheless its derivative is. Moreover, on now integrating over all space we not only obtain the second order version of Eq. (16.15) viz.

$$\frac{\partial E^{(2)}}{\partial t} = \frac{\partial}{\partial t}\int d^3x t^{(2)00} = -\int dS t^{(2)i0} n_i \ , \tag{16.44}$$

we also find that with a change in the spatial coordinates $\bar{x}_i = x_i - \epsilon_i$ only inducing a higher order change in the integration measure, Eq. (16.44) is also completely gauge invariant through second order. Thus we can actually get as far as the integral relation of Eq. (16.44) while retaining full gauge invariance through second order. In fact the only place where gauge invariance would be lost would be in the subsequent setting of the asymptotic surface term in Eq. (16.44) to zero, since even if this term were in fact to vanish in some given gauge, one could always make a gauge transformation with a badly enough behaved gauge function to some other gauge in which it would no longer do so. It is thus the dropping of the surface term which is the only step in the analysis which is not gauge invariant. Despite this however, a further remarkable thing happens, namely, as we now show, within the broad class of gauges for which the surface term does in fact vanish, $E^{(2)}$ is not only time independent, but also it takes the very same time-independent value in all of those gauges.

To establish this result we note that with the surface term actually vanishing in harmonic gauge massless plane wave solutions of the form $h_{\mu\nu} = 2\kappa_4 e_{\mu\nu} e^{ip\cdot x}/(2p^0)^{1/2}L^{3/2}$ + c.c. where $p_\mu p^\mu = 0$, we can thus consider the class of non-harmonic fluctuations of the form $\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)$ where the

through first order. Specifically, under a gauge transformation the change in $\partial/\partial x^\mu$ exhibited in Eq. (16.37) causes the gauge transform of $G^{(1)}_{\mu\kappa}$ to acquire a second order piece, with $\bar{G}^{(2)}_{\mu\kappa}$ thus being dependent on $G^{(1)}_{\mu\kappa}$ just as indicated in Eq. (16.41). However, such terms in $\bar{G}^{(2)}_{\mu\kappa}$ vanish in modes which obey $G^{(1)}_{\mu\kappa} = 0$.

arbitrary gauge function $\epsilon_\lambda(x)$ takes us out of the harmonic gauge (i.e. does not obey $\partial_\alpha \partial^\alpha \epsilon_\lambda = 0$) but does so in a way which admits of a convergent plane wave expansion, viz. $\epsilon_\lambda(x) = \sum_q a_\lambda(q)(e^{iq\cdot x} + e^{-iq\cdot x})$ where $q_0^2 - \bar{q}^2 \neq 0$. This set of ϵ_λ thus gives the full set of $\bar{h}_{\mu\nu}$ for which the surface term still vanishes. For this set the change in the $\partial_i \hat{Q}^{i00}$ term in Eq. (16.36) due to the term quadratic in ϵ_λ is then writable as

$$\partial_i \bar{\hat{Q}}^{i00} - \partial_i \hat{Q}^{i00} = \frac{1}{2} \sum_{q,q'} a_\lambda(q) a^\lambda(q') [q_i q^{i'} q_j q^{j'} - q_i q^i q'_j q^{j'}] \times \left[e^{i(q+q')\cdot x} + e^{i(q-q')\cdot x} + e^{-i(q-q')\cdot x} + e^{-i(q+q')\cdot x} \right] \quad , (16.45)$$

while that due to the $h_{\mu\nu}$, ϵ_μ type cross-terms of Eq. (16.42) is given as

$$\begin{aligned} \partial_i \bar{\hat{Q}}^{i00} - \partial_i \hat{Q}^{i00} &= \frac{\kappa_4}{i(2p^0)^{1/2}(L)^{3/2}} \sum_q a^\lambda(q)' \left[e^{i(p+q)\cdot x} - e^{-i(p+q)\cdot x} \right] \\ &\times (p_i + q_i) \left[(p^i + q^i)(p_\lambda e^k{}_k - 2p_k e^k{}_\lambda) - (p_k + q_k)(p_\lambda e^{ik} - 2p^i e^k{}_\lambda) \right] \\ &+ \frac{\kappa_4}{i(2p^0)^{1/2}(L)^{3/2}} \sum_q a^\lambda(q) \left[e^{i(p-q)\cdot x} - e^{-i(p-q)\cdot x} \right] \\ &\times (p_i - q_i) \left[(p^i - q^i)(p_\lambda e^k{}_k - 2p_k e^k{}_\lambda) - (p_k - q_k)(p_\lambda e^{ik} - 2p^i e^k{}_\lambda) \right] \quad . (16.46) \end{aligned}$$

On now integrating these two expressions with $\int d^3x$, the 3-dimensional delta functions which are generated then force the changes exhibited in both Eq. (16.45) and Eq. (16.46) to vanish identically (despite the time dependence which is not integrated over), to thereby yield $\int d^3x \partial_i \bar{\hat{Q}}^{i00} = \int d^3x \partial_i \hat{Q}^{i00}$ at all times, and thus the gauge invariance of $E^{(2)}$ for this entire class of gauges.[9] With $E^{(2)}$ being gauge invariant for this broad class of gauges,[10] we now can conclude that it really can serve as the energy deposited in a detector by a localized gravitational wave, since if such an energy is to lead to a physically observable effect such as the excitation of a gravitational antenna, it had better be a gauge invariant one on which observers in different gauges can all agree.[11] The finiteness of this $E^{(2)}$ then is the fluctuation

[9]Gauge choices not in this class such as, for instance, the non-localized $\epsilon_\lambda = a_\lambda(x_1^2 + x_2^2 + x_3^2)$ lead to a momentum-independent contribution to Eq. (16.43) of the form $(1/2)(\partial_i \partial_j \epsilon_\lambda \partial^i \partial^j \epsilon^\lambda - \partial_i \partial^i \epsilon_\lambda \partial_j \partial^j \epsilon^\lambda) = -12 a_\lambda a^\lambda$, and thus to a non-vanishing, infinite change in $\partial_i \hat{Q}^{i00}$ on spatially integrating.

[10]Included in this broad class of gauges are those which preserve the harmonic gauge, so that if $E^{(2)}$ is to be evaluated in any particular gauge in which $h_{\mu\nu}$ is localized, the resulting expression for $E^{(2)}$ should then exhibit any residual gauge invariance possessed by that gauge; with the harmonic gauge expression for $E^{(2)}$ given in Eq. (16.27) for instance nicely doing so since under $e_{\mu\nu} \to e_{\mu\nu} + p_\mu e_\nu + p_\nu e_\mu$ the harmonic nature of this gauge is maintained by $p_\mu p^\mu = 0$, with the quantity $e^{\mu\nu} e_{\mu\nu} - (1/2)(e^\alpha{}_\alpha)^2$ then remaining unchanged.

[11]The essential point here is that in gauges in which the asymptotic surface term in Eq. (16.44) does vanish, one can calculate the response of an antenna purely from the energy $E^{(2)}$ deposited in it. However in gauges for which this is not the case the response of the antenna would have to include the deposition of both a now time-dependent energy and a momentum flux coming from the asymptotic surface term. Since the conserved energy-momentum tensor $\hat{\tau}_{\mu\nu}$ of the antenna couples to the fluctuation with the gauge invariant coupling $I = \int d^4x g^{1/2} h^{\mu\nu} \hat{\tau}_{\mu\nu}$, its response

mode normalization criterion we need for constructing the propagator for localized fluctuations.

16.7 Origin of the gauge invariant energy conservation condition

While it is very nice to be able to establish the gauge invariance of the conservation condition given in Eq. (16.44), the emergence of such a gauge invariance is somewhat puzzling, since Eq. (16.44) involves the integral of a function which is not itself gauge invariant. To understand this point we note that rather than obtain the first order Eq. (16.20) from the Einstein equations, we could instead obtain (the source-free region limit of) it by variation of the second order action

$$I_{EH}^{(2)} = \frac{1}{4\kappa_4^2}\int d^4x h^{\mu\nu}G_{\mu\nu}^{(1)} = \frac{1}{4\kappa_4^2}\int d^4x h^{\mu\nu}\left(R_{\mu\nu}^{(1)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}^{(1)}\right) \qquad (16.47)$$

with respect to the fluctuation $h^{\mu\nu}$, since this particular variation precisely leads right back to $G_{\mu\nu}^{(1)} = 0$. As we show in Appendix D, the action given in Eq. (16.47) is that action which is generated in an expansion around a flat background of the full Einstein-Hilbert action through second order in $h_{\mu\nu}$. Moreover, with the first order $G_{\mu\nu}^{(1)}$ being both conserved and gauge invariant under $\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, it follows that the action of Eq. (16.47) is gauge invariant through second order,[12] as indeed it must be if its variation with respect to $h^{\mu\nu}$ is to lead to an equation of motion which is gauge invariant in first order. Now the action of Eq. (16.47) is not the full second order term in the Einstein-Hilbert action, but rather only that part of it which depends on $h_{\mu\nu}$, i.e. that part which does not depend on $\hat{g}_{\mu\nu}$. Thus despite the lack of gauge invariance of $t_{\mu\kappa}^{(2)}$, with the action of Eq. (16.47) we see that even on its own $h_{\mu\nu}$ is nonetheless still capable of producing a second order action which is gauge invariant (as must indeed be the case since the resulting gauge invariant first order equation $G_{\mu\nu}^{(1)} = 0$ is completely independent of $\hat{g}_{\mu\nu}$). Moreover, from the action of Eq. (16.47) we can also construct an energy-momentum tensor $2g^{-1/2}\delta I_{EH}/\delta g_{\mu\nu}$ via a variation with respect to a general background metric $g_{\mu\nu}$ of its covariant generalization $I_{EH}^{(2)} = (1/4\kappa_4^2)\int d^4x g^{1/2}h^{\mu\nu}\Delta G_{\mu\nu}$ (viz. a general coordinate scalar action since the $\Delta G_{\mu\nu}$ of Eq. (13.7) transforms as a true tensor with respect to the background). The energy-momentum tensor so obtained would not itself be gauge invariant, since we could just as easily have varied the action with respect to $\bar{g}_{\mu\nu} = g_{\mu\nu} + \epsilon_{\nu;\mu} + \epsilon_{\mu;\nu}$ instead. However, all such obtained energy-momentum tensors while not equal to each other would still be gauge equivalent to

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to the fluctuation is gauge invariant no matter which gauge we use to do the calculation, with the conservation condition of Eq. (16.44) indeed being gauge invariant. It is just that it is much simpler to do the calculation in gauges for which the surface term does in fact vanish.

[12] Under a gauge transformation the action of Eq. (16.47) changes by $(1/4\kappa_4^2)\int d^4x(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu)G_{\mu\nu}^{(1)} \equiv -(1/4\kappa_4^2)\int d^4x(\epsilon^\mu\partial^\nu + \epsilon^\nu\partial^\mu)G_{\mu\nu}^{(1)}$ through second order, with the kinematic conservation of $G_{\mu\nu}^{(1)}$ then securing gauge invariance in every variational path to this order.

each other. Moreover, all of them would automatically be covariantly conserved,[13] and so the covariant conservation condition $(2g^{-1/2}\delta I_{EH}/\delta g_{\mu\nu})_{;\mu} = 0$ would therefore hold in every gauge, to thus provide us with the gauge invariant condition we seek. Below we shall construct this energy-momentum tensor explicitly and show that it leads us right back to Eq. (16.27) just as we would want it to do. A further advantage of an approach based on Eq. (16.47) is that it readily generalizes to curved space, and while the expressions given in Eqs. (16.18) and (16.23) for $t^{(2)}_{\mu\kappa}$ and $\hat{t}^{(2)}_{\mu\kappa}$ could readily be covariantized, it will turn out to be simpler to use a generalization of Eq. (16.47) in order to discuss the curved space case. However, whatever prescription we might wish to choose in the curved space case, we still need to deal with the fact that we would now be using a covariant conservation condition rather than an ordinary one, and so it is to this issue that we now turn.

16.8 Abbott and Deser treatment of the curved space case

While the above analysis gives us the derivation of both an energy for gravitational waves and a Gauss's law, both derivations were based on a gravitational energy-momentum tensor which obeyed an ordinary conservation condition. To try to continue to use an ordinary conservation condition in cases where the background is not flat would oblige us to have to construct a so-called energy-momentum pseudo-tensor which obeys an ordinary conservation condition, to then lead us to an energy-momentum tensor with inappropriate general coordinate transformation properties and a consequently non-gauge invariant generalization of the energy conservation condition of Eq. (16.44). To remedy this shortcoming Abbott and Deser [Abbott and Deser (1982)] developed a totally different approach to the problem, one in which coordinate invariance can be retained throughout, one which allows for the construction of a gauge invariant energy conservation condition for a gravitational wave propagating in an arbitrary background in an arbitrary dimension, and for the construction of a covariant Gauss's law in the restricted case of maximally symmetric backgrounds, viz. backgrounds which include just the ones of interest to us in the AdS_5 brane world. The essence of the Abbott and Deser analysis is to note that since a Killing vector K_M in some general D-dimensional spacetime obeys the antisymmetric condition $K_{M;N} = -K_{N;M}$, from any symmetric true rank two tensor T^{MN} which obeys the covariant conservation condition $T^{MN}{}_{;M} = 0$, one can construct a true vector $J^M = T^{MN}K_N$ which will then obey

$$J^M{}_{;M} = g^{-1/2}\partial_M(g^{1/2}J^M) = T^{MN}{}_{;M}K_N + T^{MN}K_{N;M} = 0 \ , \qquad (16.48)$$

to thereby allow the construction of a fully covariant energy conservation condition once an appropriate T^{MN} has been found. With most spaces of interest usually having some Killing vectors, and with AdS_5 having no less than 15, the construction

[13]It is generally true that the rank two tensor formed by variation of a general coordinate scalar action with respect to the metric is always covariantly conserved for fields which are stationary.

of Abbott and Deser is thus immediately of use in the brane world, and will be applied below.

Using this same Killing vector technique Abbott and Deser also constructed a covariant generalization of Gauss's law in the particular case of fluctuations around maximally symmetric backgrounds. The construction involves the introduction of three general quantities K^{MANB}, X^{MN} and F^{AM} which are defined via

$$\begin{aligned}
K^{MANB} &= \frac{1}{2}[g^{MB}h^{NA} + g^{NA}h^{MB} + g^{MN}g^{AB}h] \\
&\quad -\frac{1}{2}[g^{MN}h^{AB} + g^{AB}h^{MN} + g^{MB}g^{NA}h] \ , \\
X^{MN} &= R^{N}{}_{BLA}K^{MALB} = \frac{1}{2}\left(R^{MN}h - R^{N}{}_{L}h^{ML} - R^{N}{}_{B}{}^{M}{}_{A}h^{AB}\right) \ , \\
F^{AM} &= K^{MBNA}\nabla_B K_N - K_N\nabla_B K^{MANB} \ ,
\end{aligned} \tag{16.49}$$

where g_{MN}, R_{MANB}, K_N (and ∇_M below) refer to the background. Without any assumption at all as to the form of the background, it can directly be shown using only Eq. (13.11) and the general relation $\nabla_A\nabla_B K_N = -R^{L}{}_{ABN}K_L$ obeyed by Killing vectors, that the lowest order quantity δG_{MN} introduced in Eq. (13.7) can be rewritten as

$$\begin{aligned}
\delta G_{MN} &= \delta R_{MN} - \frac{1}{2}h_{MN}R^{L}{}_{L} - \frac{1}{2}g_{MN}g^{LS}\delta R_{LS} + \frac{1}{2}g_{MN}h^{LS}R_{LS} \\
&= X_{MN} - \nabla_A\nabla_B K^{MANB} \\
&\quad -\frac{1}{2}h_{MN}R^{L}{}_{L} - \frac{1}{2}R_{MN}h + h_{MS}R^{S}{}_{N} + \frac{1}{2}g_{MN}h_{LS}R^{LS} \ ,
\end{aligned} \tag{16.50}$$

and that F^{AM} kinematically obeys

$$\nabla_A F^{AM} = (X_{MN} - \nabla_A\nabla_B K^{MANB})K_N \ . \tag{16.51}$$

With use of the antisymmetry properties of Killing vector derivatives and the interchange properties of K^{MANB},[14] it can also be shown that F^{AM} obeys $F^{AM} = -F^{MA}$. With the covariant derivative of any antisymmetric F^{AM} being given by $\nabla_A F^{AM} = g^{-1/2}\partial_A(g^{1/2}F^{AM})$, we can thus conclude that

$$\nabla_M\nabla_A F^{AM} = g^{-1/2}\partial_M(g^{1/2}\nabla_A F^{AM}) = g^{-1/2}\partial_M\partial_A(g^{1/2}F^{AM}) = 0 \ , \tag{16.52}$$

a purely kinematic relation which involves ordinary derivatives alone.

With the perturbative Einstein equations taking the form $\Delta G_{MN} = \delta G_{MN} + \kappa_5^2\delta T_{MN} = -\kappa_5^2\delta\tau_{MN}$ for first order perturbations around a background which obeys $G_{MN} + \kappa_5^2 T_{MN} = 0$, the vector quantity $J^M = \Delta G^{MN}K_N$ (a quantity which is automatically conserved according to

$$\nabla_M J^M = 0 \tag{16.53}$$

[14] As defined K^{MANB} obeys $K^{MANB} = -K^{AMNB} = -K^{MABN} = K^{NBMA}$.

since ΔG_{MN} is symmetric) can thus be written as a total derivative

$$J^M = \Delta G^{MN} K_N = \nabla_A F^{AM} \quad , \tag{16.54}$$

whenever the condition

$$\kappa_5^2 \delta T_{MN} = \frac{1}{2} h_{MN} R^L{}_L + \frac{1}{2} R_{MN} h - h_{MS} R^S{}_N - \frac{1}{2} g_{MN} h_{LS} R^{LS} \tag{16.55}$$

holds. As noted by Abbott and Deser, this condition does in fact hold for maximally symmetric backgrounds such as pure AdS_5 (i.e. without any Z_2 symmetry) where $R_{MLNS} = -b^2(g_{MS}g_{NL} - g_{MN}g_{NS})$ and $\kappa_5^2 \delta T_{MN} = 6b^2 h_{MN}$. Inspection shows that the condition also holds in the axial gauge TT sectors of the six maximally 4-symmetric brane worlds of interest to us where R_{MLNS} is given by Eq. (13.15) and $\kappa_5^2 \delta T_{MN} = 6b^2 h_{MN} + 6(dA/d|w|)\delta(w)\delta^\mu_M \delta^\nu_N h_{\mu\nu}$, with Eq. (16.55) thus being the criterion we want.

On recalling the perturbative equation of motion $\Delta G^{MN} = -\kappa_5^2 \delta\tau^{MN}$, we thus find in cases in which Eq. (16.55) holds that

$$-\kappa_5^2 \delta\tau^{MN} K_N = g^{-1/2} \partial_A (g^{1/2} F^{AM}) \quad ; \tag{16.56}$$

and since $F^{00} = 0$, conclude that the total injected perturbative energy then obeys

$$\begin{aligned} E &= \int d^3x \int_{-\infty}^{\infty} dw g^{1/2} \delta\tau^{0N} K_N = -\frac{1}{\kappa_5^2} \int d^3x \int_{-\infty}^{\infty} dw [\partial_w (g^{1/2} F^{50}) + \partial_i (g^{1/2} F^{i0})] \\ &= -\frac{1}{\kappa_5^2} \int d^3x \left[g^{1/2} F^{50} \right] \Bigg|_{w=-\infty}^{w=\infty} - \frac{1}{\kappa_5^2} \int_{-\infty}^{\infty} dw \int dS n_i g^{1/2} F^{i0} \quad . \end{aligned} \tag{16.57}$$

We thus recognize Eq. (16.57) as a covariant generalization of Gauss's law, a special case of which had been given earlier as Eq. (2.19).[15]

16.9 The needed curved space Killing vectors

In order to apply the construction of Abbott and Deser to the brane world, we need to find both an appropriate Killing vector and an appropriate T^{MN}. As regards first the Killing vector we recall that the various AdS_5 brane-world metrics can be written as $ds^2 = dw^2 + e^{2A} q_{\mu\nu} dx^\mu dx^\nu$, with the non-zero Christoffel symbols being given in Eq. (13.17) as $\Gamma^5_{\mu\kappa} = -A'(w) g_{\mu\kappa}$, $\Gamma^\lambda_{5\kappa} = A'(w)\delta^\lambda_\kappa$, $\Gamma^\lambda_{\mu\kappa} = \tilde{\Gamma}^\lambda_{\mu\kappa}$.[16] With

[15] Specifically, in the M_4^+ case with $K^N = (-1,0,0,0,0)$ (see below), for Z_2-symmetric axial gauge TT modes the quantities $g^{1/2}F^{50}$ and $g^{1/2}F^{i0}$ evaluate to $g^{1/2}F^{50} = (1/2)e^{6A}[\partial_w + 2A']h^{TT\,00} = (1/2)e^{2A}[\partial_w - 2A']h^{TT}_{00} = (1/2)\epsilon(w)e^{-2b|w|}[\partial_{|w|} + 2b]h^{TT}_{00}$ and $g^{1/2}F^{i0} = (1/2)e^{2A}g^{ij}(\partial_j h^{TT}_{00} - \partial_0 h^{TT}_{j0})$, just as required to recover Eq. (2.19) for a static $h^{TT}_{\mu\nu}$ in the presence of a source $\delta\tau_{00} = \delta(w)S^{TT}_{00}$.

[16] For Cartesian M_4 we have $\tilde{\Gamma}^\lambda_{\mu\kappa} = 0$, for the dS_4 induced metric of Eq. (13.2) the non-vanishing Christoffel symbols are given by $\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = H$, $\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = He^{2Ht}$, while for the AdS_4 induced metric of Eq. (13.3) the non-vanishing Christoffel symbols are given by $\Gamma^0_{10} = \Gamma^2_{12} = \Gamma^3_{13} = H$, $\Gamma^1_{22} = \Gamma^1_{33} = -\Gamma^1_{00} = -He^{2Hx}$.

the Killing vectors obeying $\partial_M K_N + \partial_N K_M - 2\Gamma^A_{MN}K_A = 0$, it is straightforward to check that for both an M_4 and an AdS_4 sectioning of AdS_5 the vector $K^M = (-1,0,0,0,0)$ is a timelike Killing vector; while for a dS_4 sectioning of AdS_5 the vector $K^M = (-1, Hx, Hy, Hz, 0)$ is a timelike Killing vector for events which obey $p^2 - x^2 - y^2 - z^2 > 0$ where $p = \int dt/e^{Ht} = -e^{-Ht}/H$ is the conformal time.

16.10 The needed brane-world energy-momentum tensor

To find an appropriate brane world T^{MN} from which we can construct a TT mode normalization, it is very convenient to first recast the 4-dimensional TT mode conditions $h^{\mu\nu}{}_{;\nu} = 0$, $g_{\mu\nu}h^{\mu\nu} = 0$ in the form of 5-dimensional ones. Specifically, with the general 5-dimensional covariant derivative taking the form

$$h^{MN}{}_{;N} = \partial_N h^{MN} + \Gamma^M_{NK}h^{KN} + \Gamma^N_{NK}h^{MK} \quad , \tag{16.58}$$

through use of Eq. (13.17) we find in the $h_{5M} = 0$ axial gauge that the $M = 5$ component of Eq. (16.58) is given by

$$h^{5N}{}_{;N} = \Gamma^5_{\nu\kappa}h^{\kappa\nu} = -A' g_{\nu\kappa}h^{\nu\kappa} \quad , \tag{16.59}$$

while an $M = \mu$ component is given by

$$h^{\mu N}{}_{;N} = \partial_\nu h^{\mu\nu} + \Gamma^\mu_{\nu\kappa}h^{\kappa\nu} + \Gamma^\nu_{\nu\kappa}h^{\mu\kappa} = h^{\mu\nu}{}_{;\nu} \quad , \tag{16.60}$$

with the dependence on the fifth index having completely dropped out. Then with the 5-trace $g_{MN}h^{MN}$ being equal to the 4-trace $g_{\mu\nu}h^{\mu\nu}$ in the axial gauge, we see that axial gauge modes which are 4-dimensional TT are 5-dimensional TT also.

With the source-free region AdS_5 bulk mode wave equation being given by $\Delta G_{MN} = 0$, evaluating ΔG_{MN} of Eq. (13.7) for the above 5-dimensional TT modes then yields

$$\begin{aligned}\Delta G_{MN} = {} & \frac{1}{2}(\overset{TT}{h}_{SN}R^S{}_M + \overset{TT}{h}_{SM}R^S{}_N - 2\overset{TT}{h}_{LS}R_M{}^L{}_N{}^S + \nabla_A\nabla^A\overset{TT}{h}_{MN}) \\ & -\frac{1}{2}\overset{TT}{h}_{MN}R^L{}_L + 6b^2\overset{TT}{h}_{MN} + 6\frac{dA}{d|w|}\delta(w)\delta^\mu_M\delta^\nu_N\overset{TT}{h}_{\mu\nu} = 0 \quad ,\end{aligned} \tag{16.61}$$

an expression which evaluates to the very compact

$$\frac{1}{2}\left(\nabla_A\nabla^A\overset{TT}{h}_{MN} + 2b^2\overset{TT}{h}_{MN}\right) = 0 \tag{16.62}$$

in all of the six maximally 4-symmetric brane worlds of interest to us. Through use of the explicit form for $\nabla_A\nabla^A h_{MN}$ given in Eq. (13.10) it can be shown that this wave equation explicitly reduces to that of a sourceless Eq. (16.1) just as it should. Since we can view the 5-dimensional TT mode wave equation of Eq. (16.62) as that of a tensor field $\overset{TT}{h}_{MN}$ propagating in a background g_{MN}, we can consider the theory to be that of an independent field $\overset{TT}{h}_{MN}$ coupled to a background g_{MN}. In such a theory we can introduce an action for the h^{TT}_{MN} field which would then generate this very same wave equation under stationary variation with respect to h^{TT}_{MN}, viz.

$$S = \frac{1}{8\kappa_5^2}\int d^5x g^{1/2}\left(h^{TT\,MN;S}{}_{;S}h^{TT}_{MN} + 2b^2 h^{TT\,MN}h^{TT}_{MN}\right) \quad , \tag{16.63}$$

an action which we note is bilinear in the h^{TT}_{MN} fluctuation, with the h^{TT}_{MN} kinetic energy term having the correct (viz non-ghost) signature just as it should.[17] Given such an action, its variation with respect to g_{MN} then allows us to covariantly construct a true rank two tensor $T^{MN} = 2g^{-1/2}\delta S/\delta g_{MN}$ with respect to the background g_{MN}, one which would automatically be conserved in solutions to the h^{TT}_{MN} wave equation. Calculation of such a T^{MN} is straightforward though lengthy (see Appendix D) and is found to yield

$$\begin{aligned} T^{MN} = \frac{1}{8\kappa_5^2}\Big(&2h^{TT\,AB;M}h^{TT}{}_{AB}{}^{;N} + 2b^2 g^{MN}h^{TT\,AB}h^{TT}_{AB} - g^{MN}h^{TT\,AB;S}h^{TT}_{AB;S} \\ &-2h^{TT\,AB;M}h^{TT\,N}{}_{A;B} - 2h^{TT\,AB;N}h^{TT\,M}{}_{A;B} \\ &+2h^{TT\,M}{}_{A}h^{TT\,AB}R_B{}^N - 2h^{TT\,M}{}_{A}h^{TT\,BC}R^A{}_C{}^N{}_B \\ &+2h^{TT\,N}{}_{A}h^{TT\,AB}R_B{}^M - 2h^{TT\,N}{}_{A}h^{TT\,BC}R^A{}_C{}^M{}_B \\ &+2h^{TT\,B}{}_{A}h^{TT\,MA;N}{}_{;B} + 2h^{TT\,B}{}_{A}h^{TT\,NA;M}{}_{;B}\Big) \end{aligned} \tag{16.64}$$

(R_{ACNB} is given in Eq. (13.15)), viz. a T^{MN} which is manifestly a rank two tensor with respect to the AdS_5 background, with its covariant conservation with respect to this background being explicitly checked in a typical case in Appendix D. For this T^{MN} the total energy[18]

$$E = \int_{-\infty}^{\infty} d^3x \int_{-\infty}^{\infty} dw g^{1/2}T^{0N}K_N \tag{16.65}$$

[17] In Appendix D we show that this particular action, with its particular $1/8\kappa_5^2$ coefficient, is the specific second order term associated with the perturbative expansion of the Einstein-Hilbert action $S = -(1/2\kappa_5^2)\int d^5x g^{1/2}g^{MN}R_{MN}$ around an AdS_5 background. The structure found for the action of Eq. (16.63) is not totally surprising because in such a perturbative expansion there is present a bilinear term $(1/2\kappa_5^2)\int d^5x g^{1/2}h^{MN}\delta R_{MN}$. With δR_{MN} being given by Eq. (13.12), the TT contribution to this bilinear term is found to evaluate to $S = (1/4\kappa_5^2)\int d^5x g^{1/2}(h^{TT\,MN}\nabla_A\nabla^A h^{TT}_{MN} + 10b^2 h^{TT\,MN}h^{TT}_{MN})$. Moreover, in constructing a general coordinate scalar action which is quadratic in h^{TT}_{MN}, the only possible second derivative terms are of the form $\nabla^A h^{TT\,MN}\nabla_A h^{TT}_{MN}$ and $\nabla^A h^{TT\,MN}\nabla_M h^{TT}_{AN}$. With this latter term being writable in a form which contains a total derivative, viz. $\nabla_M(\nabla^A h^{TT\,MN}h^{TT}_{AN}) + (h^{TT\,SN}R^A{}_S + h^{TT}_{MS}R^{SNAM})h^{TT}_{AN}$, in an AdS_5 background it only generates an $h^{TT\,MN}h^{TT}_{MN}$ type term in the action. With the relative weights of the two terms in the action of Eq. (16.63) being fixed so as to give the wave equation of Eq. (16.62), for TT fluctuations around AdS_5, up to an overall coefficient the action of Eq. (16.63) is thus the most general bilinear TT action there is.

[18] With K_N transforming as $\bar{K}_N = K_N + K^L\epsilon_{L;N} + K_{N;L}\epsilon^L$ under a gauge transformation $\bar{x}^M = x^M - \epsilon^M$, gauge transformations on K_N only begin to effect $T^{MN}K_N$ in third order.

will obey

$$\frac{\partial E}{\partial t} = -\int_{-\infty}^{\infty} d^3x \int_{-\infty}^{\infty} dw \left[\frac{\partial}{\partial w}\left(g^{1/2}T^{5N}K_N\right) + \frac{\partial}{\partial x^i}\left(g^{1/2}T^{iN}K_N\right)\right] \quad , \qquad (16.66)$$

and will be time independent whenever the momentum fluxes $g^{1/2}T^{5N}K_N$ and $g^{1/2}T^{iN}K_N$ vanish asymptotically, with the time independence of the resulting E then providing us with the normalization criterion we want.[19] As a quick check on our result, we note that if we evaluate the 4-dimensional flat spacetime analog of Eq. (16.64), viz.

$$\begin{aligned} T^{\mu\nu} = \; & \frac{1}{8\kappa_4^2}\Big(2\partial^\mu h^{TT\,\alpha\beta}\partial^\nu h^{TT}_{\alpha\beta} - \eta^{\mu\nu}\partial^\sigma h^{TT\,\alpha\beta}\partial_\sigma h^{TT}_{\alpha\beta} - 2\partial^\mu h^{TT\,\alpha\beta}\partial_\beta h^{TT\,\nu}{}_\alpha \\ & -2\partial^\nu h^{TT\,\alpha\beta}\partial_\beta h^{TT\,\mu}{}_\alpha + 2h^{TT\,\beta}{}_\alpha\partial_\beta\partial^\mu h^{TT\,\nu\alpha} + 2h^{TT\,\beta}{}_\alpha\partial_\beta\partial^\nu h^{TT\,\mu\alpha}\Big) \quad , (16.67) \end{aligned}$$

for a massless TT plane wave $h^{TT}_{\mu\nu} = 2\kappa_4 e^{ip\cdot x}e_{\mu\nu}(p^\lambda)/(2p^0)^{1/2}L^{3/2}$+c.c., we find that with $K^\mu = (-1,0,0,0)$ the energy is then given by $E = \int d^3x T^{00}K_0 = p^0 e_{\mu\nu}e^{\mu\nu}$, in complete accord with the answer we obtained previously in Eq. (16.27) since a TT mode does obey the harmonic gauge condition, only with $p_\mu e^{\mu\nu} = 0$, $e^\mu{}_\mu = 0$.[20]

16.11 M_4^+ total energy, normalization and completeness relations

While the explicit evaluation of E in the brane-world case will require specifying the particular mode basis associated with Eq. (16.4) in each individual case (something we will address in the upcoming chapters), to quickly exhibit the implications of Eq. (16.65) we consider its evaluation in the illustrative M_4^+ brane-world case where $A = -b|w|$. Thus for a typical M_4^+ TT plane wave with separable wave function $h^{TT}_{\mu\nu} = 2\kappa_5 f_m(|w|)e_{\mu\nu}(p^\lambda, m)e^{ip\cdot x}/(2p^0)^{1/2}L^{3/2}$ + c.c. where $f_m(|w|)$ is associated with Eqs. (16.6) and (16.7) and m^2 is given by $(p^0)^2 - \bar{p}^2 = m^2$, the associated

[19] In fact, once we have a T^{MN}, and regardless of how we actually obtained it, its covariant conservation is then all that is needed to construct a time-independent fluctuation norm.

[20] In passing we should note that while we are using a negative timelike contravariant Killing vector here and throughout, for the purposes of the Abbott-Deser procedure we could just as readily have used a positive timelike one instead, a sign change which would, however, alter the overall sign of E. The significance of the Abbott-Deser procedure then lies in the fact that as long as there is a unique choice of the overall sign of the Killing vector for which the energy (and thus the norm which carries the same overall sign) of every possible field configuration is positive, the background will then be stable against small fluctuations. In the event of there being some configurations with positive energy and some others with negative energy, for no choice of the sign of K^M would it then be possible to make the energies of all such configurations be positive.

energy-momentum $T_{\mu\nu}$ as given in Appendix D readily evaluates to

$$\begin{aligned} T_{\mu\nu} = &\frac{\eta^{\alpha\delta}\eta^{\beta\gamma}e_{\alpha\beta}e_{\delta\gamma}}{4p^0L^3}[2e^{-4A}(\eta_{\mu\nu}\eta^{\sigma\tau}p_\sigma p_\tau - p_\mu p_\nu)(e^{2ip\cdot x}+e^{-2ip\cdot x})f_m^2 \\ &+4e^{-4A}p_\mu p_\nu f_m^2 - e^{-2A}\eta_{\mu\nu}(e^{2ip\cdot x}+e^{-2ip\cdot x}+2)\partial_w\left(f_m(\partial_w-2A')f_m\right)] \\ &+\frac{\eta^{\alpha\beta}e_{\alpha\mu}e_{\beta\nu}}{p^0L^3}e^{-2A}\left(e^{2ip\cdot x}+e^{-2ip\cdot x}+2\right)\partial_w(A'f_m^2) \end{aligned} \tag{16.68}$$

($A=-b|w|$), while $T_{5\mu}$ evaluates to

$$T_{5\mu} = \frac{e^{-4A}}{2p^0L^3}\eta^{\alpha\delta}\eta^{\beta\gamma}\left(e^{2ip\cdot x}-e^{-2ip\cdot x}\right)ip_\mu e_{\delta\gamma}e_{\alpha\beta}f_m(\partial_w-2A')f_m \quad . \tag{16.69}$$

On setting $K^N=(-1,0,0,0,0)$ the energy E is then given by $E=\int dwd^3xe^{2A}T_{00}$, with all the terms in E which could make it time-dependent (through the time dependence of $e^{ip\cdot x}=e^{-ip^0t+i\bar{p}\cdot\bar{x}}$) only contributing in the asymptotic w surface terms which result on doing the w integration.[21] Thus on requiring the modes to vanish asymptotically according to

$$f_m^2\to 0 \quad , \quad f_m(\partial_w-2A')f_m\to 0 \quad , \tag{16.70}$$

(conditions which can be met by the asymptotic vanishing of f_m itself),[22] the energy is then given by the time-independent

$$E = p^0\eta^{\alpha\delta}\eta^{\beta\gamma}e_{\alpha\beta}e_{\delta\gamma}\int_{-\infty}^{\infty}dwe^{-2A}f_m^2 = 2p^0\eta^{\alpha\delta}\eta^{\beta\gamma}e_{\alpha\beta}e_{\delta\gamma}\int_0^{\infty}d|w|e^{-2A}f_m^2 \quad . \tag{16.71}$$

Under these same asymptotic conditions the momentum fluxes $g^{1/2}T^{5N}K_N = -e^{4A}T_{50}$, $g^{1/2}T^{iN}K_N=-e^{2A}T_{i0}$ are both found to vanish just as the must in any configuration in which the energy is time-independent. We thus identify the finiteness of $\int d|w|e^{-2A}f_m^2$ as the time-independent normalization condition we seek, and note that the normalization measure we obtain this way is none other than precisely the one we found earlier in our discussion of an orthogonality criterion.[23] With the asymptotic $T_{5\mu}$ momentum flux behaving as in Eq. (16.69), we thus see

[21]The time dependent $(-\eta^{\sigma\tau}p_\sigma p_\tau - p_0p_0)(e^{2ip\cdot x}+e^{-2ip\cdot x}) = (-\bar{p}^2)(e^{2ip\cdot x}+e^{-2ip\cdot x})$ term is canceled identically via the $\delta^3(\bar{p})$ term which is generated by the $\int d^3x$ integration.

[22]For the massless TT graviton mode the quantity $\eta^{\alpha\beta}e_{\alpha 0}e_{\beta 0}$ happens to vanish identically for any p^μ on the M_4 lightcone. (For the 4-vector $p^\mu=((p^2+m^2)^{1/2},0,0,p)$ for instance the five TT polarization states are given by $e_{00}=-e_{33}=-p/(p^2+m^2)^{1/2}e_{30}=-p/(p^2+m^2)^{1/2}e_{03}$; $e_{01}=e_{10}=-p/(p^2+m^2)^{1/2}e_{31}=-p/(p^2+m^2)^{1/2}e_{13}$; $e_{02}=e_{20}=-p/(p^2+m^2)^{1/2}e_{32}=-p/(p^2+m^2)^{1/2}e_{23}$; $e_{11}=-e_{22}$; $e_{12}=e_{21}$; with $\eta^{\alpha\beta}e_{\alpha 0}e_{\beta 0}$ vanishing in all of them when $m=0$.) Consequently, for the $m=0$ modes it is the asymptotic vanishing of $f_m(\partial_w-2A')f_m$ which secures the time independence of the energy.

[23]The use of the Abbott-Deser technique in the brane world was first considered by [DeWolfe, Freedman, Gubser and Karch (2000)], in an approach which also led to the Eq. (16.71) normalization condition, though through the use of a fluctuation energy-momentum tensor which differs from the one being considered here.

that the asymptotic vanishing of the $g^{1/2}T^{5N}K_N$ momentum flux entails the time independence of the total energy in gravitational waves, and thus its finiteness at all times for a gravitational wave which is generated by the injection of a finite amount of energy at some initial time. Moreover, recalling our previous discussion of an orthogonality condition, we recognize the right hand side of Eq. (16.9) as possessing the same w-dependence as this momentum flux. Asymptotic vanishing of the $g^{1/2}T^{5N}K_N$ flux thus simultaneously imposes time independence of the energy and orthogonality of the modes.

For finite energy modes whose asymptotic momentum flux does vanish we can choose their normalization so that the $f_m(w)$ modes then form an orthonormal basis according to

$$\int_{-\infty}^{\infty} dw e^{-2A} f_m(|w|) f_{m'}(|w|) = \delta_{m,m'} \quad , \tag{16.72}$$

and with this normalization the completeness relation for the modes will then take the form

$$\sum_m f_m(|w|) f_m(|w'|) = e^{2A}\delta(w - w') \quad . \tag{16.73}$$

16.12 Construction of the M_4^+ propagator via completeness

Once given a completeness relation such as that of Eq. (16.73) (and regardless in fact of whether or not the underlying basis modes actually are normalizable), we can then construct a TT brane-world propagator.[24] Specifically, with the (4-space) retarded propagator for a free massive scalar field in ordinary M_4 spacetime obeying

$$[\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2]D(x - x', m) = \delta^4(x - x') \tag{16.74}$$

with solution

$$D(x, m) = -\frac{\delta(x^0 - |\bar{x}|)}{4\pi|\bar{x}|} + \frac{\theta(x^0 - |\bar{x}|) m J_1[m((x^0)^2 - \bar{x}^2)^{1/2}]}{4\pi[(x^0)^2 - \bar{x}^2]^{1/2}} \quad , \tag{16.75}$$

we can, for modes which obey Eq. (16.73), write the M_4^+ (or M_4^-) TT basis mode propagator in the generic form[25]

$$G^{TT}(x, x', w, w') = \sum_m f_m(|w|) f_m(|w'|) D(x - x', m) \quad . \tag{16.76}$$

[24] Even non-normalizable modes can still obey a relation such as Eq. (16.73), as is of course the case for plane waves where $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-x')} = \delta(x - x')$. However, in the divergent warp factor brane worlds to be studied below, we will find complete non-normalizable mode bases for which there is no analog of Eq. (16.73) at all.

[25] In Appendix E we give the retarded propagators for pure AdS_4 and dS_4 spacetimes which are needed for construction of the analog propagators of the $AdS_4^\pm$ and $dS_4^\pm$ brane worlds, and for completeness of the presentation also give a derivation of the pure M_4 spacetime Eq. (16.75).

Then, through use of Eqs. (16.4), (16.5) and (16.73) (and without any reference to Eq. (16.72)), we find that this propagator then obeys

$$
\begin{aligned}
&\left[\frac{\partial^2}{\partial w^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 4\frac{dA}{d|w|}\delta(w) + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] G^{TT}(x,x',w,w') \\
&= \sum_m \left[\frac{\partial^2}{\partial |w|^2} + 2\delta(w)\frac{\partial}{\partial |w|} - 4\left(\frac{dA}{d|w|}\right)^2 - 4\frac{dA}{d|w|}\delta(w) + e^{-2A}m^2\right] \\
&\quad \times f_m(|w|)f_m(|w'|)D(x-x',m) \\
&\quad + e^{-2A}\sum_m f_m(|w|)f_m(|w'|)\delta^4(x-x') \\
&= \delta(w-w')\delta^4(x-x') \ , \qquad (16.77)
\end{aligned}
$$

just as it should, with the solution to Eq. (16.1) thus being given by

$$h^{TT}_{\mu\nu}(x,w) = -2\kappa_5^2 \int d^4x' G^{TT}(x,x',w,0)S^{TT}_{\mu\nu}(x') \ . \qquad (16.78)$$

While Eq. (16.78) is by construction an exact solution to Eq. (16.1), in passing it should be pointed out that it does not in fact also satisfy Eqs. (16.2) and (16.3). Rather, as can be seen from its derivation above, the crucial $\delta(w-w')\delta^4(x-x')$ term present on the right-hand side of Eq. (16.77) is generated not because any of the $f_m(|w|)$ modes individually obey Eq. (16.3) (in fact each one obeys the source-free Eq. (16.5)), but because of the action of the $e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha$ term on the $D(x-x',m)$ propagator in Eq. (16.76) as then combined with the completeness relation of Eq. (16.73). Thus rather than break up Eq. (16.1) as exhibited in Eqs. (16.2) and (16.3), the solution of Eq. (16.78) will instead break up Eq. (16.1) into two equations in which it will be the analog of the derivative operator in Eq. (16.2) which will be associated with the $\delta(w)S^{TT}_{\mu\nu}$ source term, viz. a break-up of the form

$$\frac{1}{2}\left[\frac{\partial^2}{\partial |w|^2} - 4\left(\frac{dA}{d|w|}\right)^2 + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h^{TT}_{\mu\nu} = -\kappa_5^2\delta(w)S^{TT}_{\mu\nu} \ , \qquad (16.79)$$

$$\delta(w)\left[\frac{\partial}{\partial |w|} - 2\frac{dA}{d|w|}\right] h^{TT}_{\mu\nu} = 0 \ . \qquad (16.80)$$

With it being the break-up according to Eqs. (16.2) and (16.3) which leads to the perturbative generalized Einstein equation of Eq. (14.35) on the brane (Eqs. (16.2) and (16.3) entail that

$$\left[\frac{\partial^2}{\partial |w|^2} + \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h^{TT}_{\mu\nu} = 4\left(\frac{dA}{d|w|}\right)^2 h^{TT}_{\mu\nu} = 2\frac{dA}{d|w|}\left[\frac{\partial h^{TT}_{\mu\nu}}{\partial |w|} + \kappa_5^2 S^{TT}_{\mu\nu}\right] \qquad (16.81)$$

holds at $w = 0$, from which

$$\frac{1}{2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu} = \frac{dA}{d|w|}\kappa_5^2 S^{TT}_{\mu\nu} - \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 2\frac{dA}{d|w|}\frac{\partial}{\partial|w|}\right] h^{TT}_{\mu\nu} \tag{16.82}$$

follows), we see that the solution in the form given in Eq. (16.78) does not satisfy the generalized Einstein equation on the brane. Rather, it obeys

$$\frac{1}{2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu} = -\kappa_5^2\delta(w) S^{TT}_{\mu\nu} - \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 2\frac{dA}{d|w|}\frac{\partial}{\partial|w|}\right] h^{TT}_{\mu\nu} \tag{16.83}$$

instead. In the following we shall therefore construct an alternate form for the brane-world propagator (cf. Eq. (17.30) below), one which then will break up according to Eqs. (16.2) and (16.3). And indeed, one of our tasks in the following will be to compare and contrast these various candidate forms for the brane-world propagator.

To conclude this chapter, we note that given the completeness relation of Eq. (16.73), it is actually possible to analytically perform the summation on m needed to determine the propagator given in Eq. (16.76). Specifically, with a $J_1(y)$ Bessel function being expandable as

$$yJ_1(y) = \sum_{k=0}^{k=\infty} a_k y^{2(k+1)} \quad , \tag{16.84}$$

where

$$a_k = \frac{(-1)^k}{2^{2k+1}k!(k+1)!} \quad , \tag{16.85}$$

and with the M_4^+ $f_m(|w|)$ modes obeying

$$\mathcal{D} f_m = m^2 f_m \quad , \tag{16.86}$$

where

$$\mathcal{D} = e^{-2b|w|}\left[4b^2 - \frac{d^2}{d|w|^2}\right] \quad , \tag{16.87}$$

the insertion of Eq. (16.75) into Eq. (16.76) yields [with $-x^2 = (x^0)^2 - \bar{x}^2$]

$$\begin{aligned} G^{TT}(x, 0, w, w') &= -\frac{\delta(x^0 - |\bar{x}|)e^{-2b|w|}\delta(w - w')}{4\pi|\bar{x}|} \\ &+ \frac{\theta(x^0 - |\bar{x}|)}{4\pi(-x^2)}\left[\sum_{k=0}^{k=\infty} a_k(-x^2)^{k+1}\mathcal{D}^{k+1}\right][e^{-2b|w|}\delta(w - w')] \quad , \end{aligned} \tag{16.88}$$

i.e.

$$G^{TT}(x,0,w,w') = -\frac{\delta(x^0 - |\bar{x}|)e^{-2b|w|}\delta(w-w')}{4\pi|\bar{x}|}$$
$$+ \frac{\theta(x^0 - |\bar{x}|)\mathcal{D}^{1/2}}{4\pi(-x^2)^{1/2}} J_1[(-x^2\mathcal{D})^{1/2}][e^{-2b|w|}\delta(w-w')] \ , \tag{16.89}$$

to thus exhibit the singularity structure of the propagator.

To study the implications of Eq. (16.89) it is convenient to set $z = e^{2b|w|}/2b$, $e^{-2b|w|}\delta(w) = \delta(z)$, so that

$$\mathcal{D} = -2b\left[z\frac{d^2}{dz^2} + \frac{d}{dz} - \frac{1}{z}\right] \ . \tag{16.90}$$

Then, on integrating over a test function $g(x,w)$ with the $e^{2b|w|}$ measure of Eq. (16.72) (equivalent, incidentally, to trying to solve Eq. (13.7) with a source $\delta\tau_{MN}$ which is not necessarily localized to the brane) we obtain first

$$\int dw e^{2b|w|} g(x,w)\mathcal{D}[e^{-2b|w|}\delta(w-w')]$$
$$= \int dz g(x,z)\mathcal{D}\delta(z-z')$$
$$= -2b\int dz\left[z\frac{d^2g}{dz^2} + 2\frac{dg}{dz} - \frac{dg}{dz} - \frac{g}{z}\right]\delta(z-z')$$
$$= \int dz\delta(z-z')\mathcal{D}g(x,z) = \mathcal{D}g(x,z') \ , \tag{16.91}$$

and then

$$\int dw e^{2b|w|} g(x,w)\mathcal{D}^n[e^{-2b|w|}\delta(w-w')] = \mathcal{D}^n g(x,z') \ , \tag{16.92}$$

to yield finally

$$\int dw' e^{2b|w'|} G^{TT}(x,0,w,w')g(x,w')$$
$$= -\frac{\delta(x^0 - |\bar{x}|)g(x,z)}{4\pi|\bar{x}|} + \frac{\theta(x^0 - |\bar{x}|)\mathcal{D}^{1/2}}{4\pi(-x^2)^{1/2}} J_1[(-x^2\mathcal{D})^{1/2}]g(x,z) \ . \tag{16.93}$$

Rather than use the operator $\mathcal{D}$ given in Eq. (16.90), we note that if we set $z = y^2$ we can rewrite $\mathcal{D}$ in a more familiar form, viz.

$$\mathcal{D} = -\frac{b}{2}\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{4}{y^2}\right] \ , \tag{16.94}$$

which we recognize as being in the same form as the operator which appears in the Bessel equation

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{\nu^2}{y^2}\right] Z_\nu(y) = -Z_\nu(y) \ . \tag{16.95}$$

Thus in passing we note that if we choose the test function to be given by $g(x,z) = J_2(z^{1/2}) = J_2(e^{b|w|}/(2b)^{1/2})$ (so that $\mathcal{D}J_2(z^{1/2}) = bJ_2(z^{1/2})/2$), for such a case Eq. (16.93) reduces to

$$\int dw' e^{2b|w'|} G^{TT}(x,0,w,w') J_2\left(\frac{e^{b|w'|}}{(2b)^{1/2}}\right)$$
$$= -\frac{1}{4\pi}\left[\frac{\delta(x^0-|\bar{x}|)}{|\bar{x}|} - \frac{\theta(x^0-|\bar{x}|)b^{1/2}}{2^{1/2}(-x^2)^{1/2}} J_1\left(\left(\frac{-bx^2}{2}\right)^{1/2}\right)\right] J_2\left(\frac{e^{b|w|}}{(2b)^{1/2}}\right), \quad (16.96)$$

a compact, completely exact relation.

To round out the discussion of the propagator we turn now to a determination of the explicit mode bases to be used in all the specific brane-world cases of interest to us, exploring first the implications of having vanishing asymptotic momentum flux, and then, in divergent warp factor cases, additionally analyzing what is to happen if this is not to be the case. While propagators of the generic form given in Eq. (16.76) (viz. propagators which are explicitly built out of modes which obey the completeness relation of Eq. (16.73) and which are based on 4-space retarded propagators such as $D(x-x',m)$) are necessarily (by construction) 4-space retarded, it does not automatically follow that they are 5-space retarded or 5-space causal as well, and thus in the following we shall also explicitly explore the degree to which they, and also propagators which are built out of non-normalizable modes, might be retarded and causal as well.

Chapter 17

Fluctuations around an Embedded Positive-Tension Minkowski Brane

17.1 M_4^+ mode basis

To determine the TT modes associated with the positive-tension M_4^+ brane world we set $A(|w|) = -b|w|$ in Eqs. (16.4) and (16.5), to obtain

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = 0 \quad , \tag{17.1}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} + 2b\right] h_{\mu\nu}^{TT} = 0 \quad . \tag{17.2}$$

Separable TT mode solutions to these equations with separation constant m^2 take the form $h_{\mu\nu}^{TT} = 2\kappa_5 f_m(|w|)e_{\mu\nu}(p^\lambda, m)e^{ip\cdot x}/(2p^0)^{1/2}L^{3/2}$ + c.c. where $(p^0)^2 - \bar{p}^2 = m^2$, $\eta^{\mu\nu}e_{\mu\nu} = 0$, $p_\mu e^{\mu\nu} = 0$, with $f_m(|w|)$ having to obey

$$\left[\frac{d^2}{d|w|^2} - 4b^2 + e^{2b|w|}m^2\right] f_m(|w|) = 0 \quad . \tag{17.3}$$

Under the change of variable $y = me^{b|w|}/b$ (viz. $d/d|w| = byd/dy$) Eq. (17.3) can be brought to the Bessel equation form

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + 1 - \frac{4}{y^2}\right] f_m(y) = 0 \quad . \tag{17.4}$$

With m^2 only having been introduced as a separation constant, it could in principle be positive, zero or negative. For $m^2 > 0$ the solutions to Eq. (17.3) can be written in terms of Bessel functions, viz.

$$f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y) \quad , \tag{17.5}$$

where α_m and β_m are y-independent constants, while for $m^2 = 0$ the solutions can be read off from Eq. (17.3) directly, viz.

$$f_0(y) = \alpha_0 e^{-2b|w|} + \beta_0 e^{2b|w|} \quad . \tag{17.6}$$

For $m^2 = -\mu^2 < 0$ the change of variable $\hat{y} = \mu e^{b|w|}/b$ brings Eq. (17.3) to the form

$$\left[\frac{d^2}{d\hat{y}^2} + \frac{1}{\hat{y}}\frac{d}{d\hat{y}} - 1 - \frac{4}{\hat{y}^2}\right] f_\mu(\hat{y}) = 0 \ , \tag{17.7}$$

with the solutions being modified Bessel functions, viz.

$$f_\mu(\hat{y}) = \alpha_\mu I_2(\hat{y}) + \beta_\mu K_2(\hat{y}) \ . \tag{17.8}$$

With the Bessel functions obeying the recurrence relations[1]

$$\begin{aligned} \frac{dJ_2(y)}{dy} &= J_1(y) - \frac{2}{y}J_2(y) \ , \quad \frac{dY_2(y)}{dy} = Y_1(y) - \frac{2}{y}Y_2(y) \ , \\ \frac{dI_2(\hat{y})}{d\hat{y}} &= I_1(\hat{y}) - \frac{2}{\hat{y}}I_2(\hat{y}) \ , \quad \frac{dK_2(\hat{y})}{d\hat{y}} = -K_1(\hat{y}) - \frac{2}{\hat{y}}K_2(\hat{y}) \ , \end{aligned} \tag{17.9}$$

the various solutions thus obey

$$\left[\frac{d}{d|w|} + 2b\right] f_m = m e^{b|w|}[\alpha_m J_1(y) + \beta_m Y_1(y)] \ , \tag{17.10}$$

$$\left[\frac{d}{d|w|} + 2b\right] f_0 = 4b\beta_0 e^{2b|w|} \ , \tag{17.11}$$

$$\left[\frac{d}{d|w|} + 2b\right] f_\mu = \mu e^{b|w|}[\alpha_\mu I_1(\hat{y}) - \beta_\mu K_1(\hat{y})] \ . \tag{17.12}$$

Satisfying the junction condition of Eq. (17.2) at the brane thus requires

$$\alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0 \ , \tag{17.13}$$

$$\beta_0 = 0 \ , \tag{17.14}$$

$$\alpha_\mu I_1(\mu/b) - \beta_\mu K_1(\mu/b) = 0 \tag{17.15}$$

in the various cases. It thus possible to find solutions to Eqs. (17.1) and (17.2) for all values of m^2 between $-\infty$ to $+\infty$ including the massless (from the point of view of the 4-dimensional $(p^0)^2 - p^2 = m^2$) graviton with wave function $f_0 = \alpha_0 e^{-2b|w|}$.

[1] The various Bessel and associated Legendre function relations used in this monograph can be found in [Erdelyi, Magnus, Oberhettinger, and Tricomi (1953)], [Abramowitz and Stegun (1965)], and [Gradshteyn and Rhyzik (1980)].

With these various solutions behaving asymptotically as

$$f_m \to \left(\frac{2}{\pi y}\right)^{1/2} [\alpha_m \cos(y - 5\pi/4) + \beta_m \sin(y - 5\pi/4)] \quad ,$$
$$f_0 \to \alpha_0 e^{-2b|w|} \quad ,$$
$$f_\mu = \left(\frac{1}{2\pi\hat{y}}\right)^{1/2} \alpha_\mu e^{\hat{y}} + \left(\frac{\pi}{2\hat{y}}\right)^{1/2} \beta_\mu e^{-\hat{y}} \tag{17.16}$$

for large $|w|$ (and thus large y or $\hat{y}$), and with the momentum flux of Eq. (16.69) being given by

$$T_{5\mu} = \frac{e^{4b|w|}}{2p^0 L^3} \eta^{\alpha\delta} \eta^{\beta\gamma} \left(e^{2ip\cdot x} - e^{-2ip\cdot x}\right) ip_\mu e_{\delta\gamma} e_{\alpha\beta}$$
$$\times \epsilon(w) f_m(|w|) \left(\frac{\partial}{\partial |w|} + 2b\right) f_m(|w|) \quad , \tag{17.17}$$

we see that momentum flux $g^{1/2} T^{5N} K_N = -e^{-4b|w|} T_{50}$ associated with the TT fluctuation mode energy-momentum tensor introduced in Chapter 16 will vanish asymptotically for the $m^2 > 0$ and $m^2 = 0$ solutions, but not for the $m^2 < 0$ ones. The spectrum of states with a time-independent energy $E = \int dwd^3x e^{-2b|w|} T_{00}$ thus consists of a massless graviton mode together with an infinite KK continuum of $m^2 > 0$ modes which begins at $m^2 = 0$, with the tachyonic $m^2 < 0$ modes being excluded by this criterion. With the energy of the time-independent states being given by

$$E = 2p^0 \eta^{\alpha\delta} \eta^{\beta\gamma} e_{\alpha\beta} e_{\delta\gamma} \int_0^\infty d|w| e^{2b|w|} f_m^2(|w|) \tag{17.18}$$

according to Eq. (16.71), the allowed states not only have a time-independent energy, from Eq. (17.18) it follows that the energy in each allowed solution is explicitly non-negative. Consequently, all of these states can be (bound state or continuum) normalized according to Eq. (16.72), viz. according to

$$2\int_0^\infty d|w| e^{2b|w|} f_m(|w|) f_{m'}(|w|) = \delta_{m,m'} \quad . \tag{17.19}$$

With the vanishing of the asymptotic momentum flux in each and every one of the allowed $m^2 \geq 0$ modes, this set of modes is thus both a normalizable set ($0 < E < \infty$) and an orthogonal one (Eq. (16.9)), to thus provide a complete orthonormal basis for constructing the M_4^+ TT propagator, with associated completeness relation

$$\sum_m f_m(|w|) f_m(|w'|) = e^{-2b|w|} \delta(w - w') \quad . \tag{17.20}$$

(As we shall see below, in constructing the propagator, we will not need to include the $m^2 < 0$ solutions as well.) Then finally, with all of the allowed modes in the basis having wave functions which fall off exponentially fast away from the brane (the graviton as $e^{-2b|w|}$ and the $m^2>0$ KK modes as $e^{-b|w|/2}$)), it follows that the gravitational TT propagator localizes around an M_4^+ brane, the key result of the Randall-Sundrum program [Randall and Sundrum (1999b)].

From Eq. (17.19) it follows that the normalized $m^2=0$ graviton mode wave function is explicitly given as

$$h^{TT}_{\mu\nu}(m=0)=e^{-2b|w|}\frac{2\kappa_5 b^{1/2}}{(2p^0)^{1/2}L^{3/2}}e_{\mu\nu}(p^\lambda,m=0)e^{i\bar{p}\cdot\bar{x}-ip^0t}+\text{c.c.} \qquad (17.21)$$

With the $m^2>0$ KK modes forming a continuum, we shall give them the same orthonormality conditions ($2\int_1^L dx x f_{m_i}(m_i x/b)f_{m_j}(m_j x/b)=\delta_{ij}L/\pi$) as box normalized flat space Bessel functions, to yield the wave function given by Garriga and Tanaka [Garriga and Tanaka (2000)] in their analogous construction of the gravitational propagator, viz.[2]

$$\begin{aligned} h^{TT}_{\mu\nu}(m)= &\ \kappa_5 m^{1/2}e_{\mu\nu}(p^\lambda,m)e^{i\bar{p}\cdot\bar{x}-ip^0t} \\ &\times\frac{[Y_1(m/b)J_2(me^{b|w|}/b)-J_1(m/b)Y_2(me^{b|w|}/b)]}{(p^0)^{1/2}L^{3/2}[J_1^2(m/b)+Y_1^2(m/b)]^{1/2}}+\text{c.c.} \end{aligned} \qquad (17.22)$$

While the massive TT KK modes have five polarization states, we note that the $\hat{\xi}^\mu$-dependent piece of the axial gauge preserving gauge transformation given earlier as Eq. (14.26), viz. $\bar{h}_{\mu\nu}(|w|,x^\lambda)=h_{\mu\nu}(|w|,x^\lambda)+e^{-2b|w|}[\eta_{\mu\rho}\partial_\nu\hat{\xi}^\rho+\eta_{\nu\rho}\partial_\mu\hat{\xi}^\rho]$, has precisely the same dependence on w as does $h^{TT}_{\mu\nu}(m=0)$. Since $\eta_{\mu\rho}\partial_\nu\hat{\xi}^\rho+\eta_{\nu\rho}\partial_\mu\hat{\xi}^\rho$ will be TT if $\hat{\xi}^\mu$ obeys $\partial_\mu\hat{\xi}^\mu=0$, $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\hat{\xi}^\mu=0$, we see that in total there are thus three TT preserving residual gauge transformations which we are free to impose on the massless TT graviton in the axial gauge.[3] Consequently, the massless $h^{TT}_{\mu\nu}(m=0)$ only possesses two observable propagating degrees of freedom, both on the brane and in the bulk, just as it should.

17.2 Mode propagation in M_4^+ and long distance gravity

Once we have obtained the explicit normalized wave functions, the requisite M_4^+ TT basis mode propagator can now be constructed according to Eq. (16.76), to yield

$$G^{TT}(x,x',w,w')=\sum_m f_m(|w|)f_m(|w'|)D(x-x',m)\ , \qquad (17.23)$$

[2]Use of the indefinite integrals $\int dx x J_2(ax)J_2(bx)=x[bJ_2(ax)J_1(bx)-aJ_1(ax)J_2(bx)]/(a^2-b^2)$, $\int dx x J_2(ax)Y_2(bx)=x[bJ_2(ax)Y_1(bx)-aJ_1(ax)Y_2(bx)]/(a^2-b^2)$, $\int dx x Y_2(ax)Y_2(bx)=x[bY_2(ax)Y_1(bx)-aY_1(ax)Y_2(bx)]/(a^2-b^2)$ confirms the orthogonality of f_m and $f_{m'}$ when $m'\neq m$, with use of $\int dx x J_2^2(ax)=(x^2/2)[J_2^2(ax)-J_1(ax)J_3(ax)]$, $\int dx x Y_2^2(ax)=(x^2/2)[Y_2^2(ax)-Y_1(ax)Y_3(ax)]$ and the asymptotic limits $J_\nu(x)\to(2/\pi x)^{1/2}\cos(x-\nu\pi/2-\pi/4)$, $Y_\nu(x)\to(2/\pi x)^{1/2}\sin(x-\nu\pi/2-\pi/4)$ fixing the coefficient of the divergent part of the norm of f_m.

[3]The ξ_5-dependent piece of the gauge transformation of Eq. (14.26) affects the brane.

i.e.

$$
\begin{aligned}
G^{TT}(x,0,w,0) = \; & be^{-2b|w|}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{[(p^0)^2-\bar{p}^2+i\epsilon\epsilon(p^0)]} \\
& +\sum_m \frac{b[Y_1(m/b)J_2(me^{b|w|}/b)-J_1(m/b)Y_2(me^{b|w|}/b)]}{\pi[J_1^2(m/b)+Y_1^2(m/b)]} \\
& \times \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{[(p^0)^2-\bar{p}^2-m^2+i\epsilon\epsilon(p^0)]} \quad ; \qquad (17.24)
\end{aligned}
$$

with the presence of the massless graviton and the massive KK modes with energy $E_p = (\bar{p}^2+m^2)^{1/2}$ immediately being revealed, since in addition to a circle at infinity contribution, a 4-space retarded contour integration in the complex p^0 plane[4] generates an explicit singular pole term contribution to Eq. (17.24) of the form

$$
\begin{aligned}
G^{TT}(x,0,w,0;\mathrm{SING}) = \; & -ibe^{-2b|w|}\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|}\left[e^{-i|p|t}-e^{i|p|t}\right] \\
& -i\sum_m \frac{b[Y_1(m/b)J_2(me^{b|w|}/b)-J_1(m/b)Y_2(me^{b|w|}/b)]}{\pi[J_1^2(m/b)+Y_1^2(m/b)]} \\
& \times \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt}-e^{iE_pt}\right] \quad . \qquad (17.25)
\end{aligned}
$$

In terms of the propagator of Eq. (17.24) the solution to Eq. (16.1) is given by

$$
h_{\mu\nu}^{TT}(x,w) = -2\kappa_5^2\int d^4x' G^{TT}(x,x',w,0)S_{\mu\nu}^{TT}(x') \quad , \qquad (17.26)
$$

with the specific massless graviton contribution to it being given by

$$
h_{\mu\nu}^{TT}(x,w) = -2\kappa_5^2 be^{-2b|w|}\int d^4x' D(x-x',m=0)S_{\mu\nu}^{TT}(x') \quad , \qquad (17.27)
$$

a contribution which thus obeys

$$
\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{TT}(x,w) = -\kappa_5^2 be^{-2b|w|}S_{\mu\nu}^{TT}(x) \quad . \qquad (17.28)
$$

Since our analysis of the M_4^+ NT modes in Chapter 14 established that the NT modes propagate on the brane as massless particles with an effective 4-dimensional gravitational constant given by $\kappa_4^2 = b\kappa_5^2$, we see now that on the brane the massless TT modes couple to matter with precisely the same strength as the NT ones. Thus, following our discussion of Chapter 14, we see that on the brane the net contribution of the M_4^+ NT modes and the massless TT graviton is precisely the same as that associated with a 4-dimensional massless graviton in standard, non-brane gravity. In the M_4^+ brane world then the massless graviton sector is both localized to the brane and completely canonical on it.

[4]The $i\epsilon\epsilon(p^0)$ prescription puts both of the $p^0 = \pm(\bar{p}^2+m^2-i\epsilon\epsilon(p^0))^{1/2} = \pm(\bar{p}^2+m^2)^{1/2} \mp i\epsilon\epsilon(p^0)/2(\bar{p}^2+m^2)^{1/2}$ poles below the real axis according to $p^0 = \pm(\bar{p}+m^2)^{1/2} - i\epsilon/2(\bar{p}^2+m^2)^{1/2}$.

Since long distance gravity on an M_4^+ brane is canonical, it must be the case that the KK modes make a non-leading contribution at large distances on the brane. However, even though our analysis of Chapter 14 guarantees us that this must be so, such an outcome is not immediately apparent from Eq. (17.26) since the KK mode continuum begins at $m^2 = 0$. However, as noted in [Randall and Sundrum (1999b)], there is a suppression in the KK mode couplings to matter which makes them non-leading at large r after all. Specifically, for a static source $S_{\mu\nu} = -M\delta^0_\mu\delta^0_\nu\eta_{00}\delta^3(x)$ (viz. $S^{TT}_{00} = 2M\delta^3(x)/3$), evaluation of the net KK contribution to Eq. (17.26) is straightforward, and on the brane is found to yield (at any r) the Yukawa type potential[5]

$$h^{TT}_{00}(r, w=0) = \kappa_5^2 M \int_0^\infty \frac{dm}{b} f^2_m(w=0)\frac{e^{-mr}}{3\pi r} \quad . \tag{17.29}$$

In the large r limit the continuum integral in Eq. (17.29) will be dominated by the small-m region, and with the normalized $f_m(w=0)$ behaving as $f_m(w=0) \sim m^{1/2}$ in this limit,[6] it follows that the continuum integral ($\sim (\kappa_5^2 M/br)\int dm\, m e^{-mr}$) then only generates a non-leading contribution of order $\kappa_5^2 M/br^3$ to the $h^{TT}_{00}(r, w=0)$ potential. It is thus because the massive $f_m(w=0)$ vanish as $m^{1/2}$ for small m that the very light continuum modes do not compete with the massless M_4^+ graviton at large distances on the brane. Thus despite the absence of any gap between the massless graviton and the lowest-lying KK modes (as would be the case in models with a compactified fifth dimension), the KK mode contribution to long-range gravity on the brane is still non-leading; with this particular aspect of the brane-localized gravity program being just as vital to its viability as the very existence of the localized massless graviton itself. Finally, with the massless graviton generating a long-range potential $h^{TT}_{00}(r, w=0) = \kappa_5^2 bM/3\pi r$, we see that the non-leading large distance net KK contribution on the brane differs from it by a factor of order $1/b^2r^2$. With long distance gravity being known to be canonical down to millimeter region distances, the length scale $1/b$ of AdS_5 would thus need to be no bigger than 10^{-3} meters in order for M_4^+ gravity to be phenomenologically viable at large distances on the brane. Phenomenological viability of brane-localized gravity thus requires a large rather than a small value for the AdS_5 input parameter $-\kappa_5^2\Lambda_5/6 = b^2$ of the theory.

Apart from recovering Newton's law of gravity, as a first order correction to the metric $h_{\mu\nu}$ also embodies the gravitational bending of light and the gravitational redshift, the lowest order manifestations of general relativistic corrections to Newtonian gravity. Specifically, for a test particle moving with velocity $\bar{v} = d\bar{r}/dt$ in background metric of the generic form $ds^2 = [1 + b(r)]dt^2 - [1 + a(r)][dr^2 + r^2 d\Omega]$,

[5] With each f_m being a function of the dimensionless $y = me^{b|w|}/b$, we use the dimensionless m/b as the integration measure here.

[6] For small values of their arguments $J_\nu(y)$ and $Y_\nu(y)$ behave as $y^\nu/2^\nu\Gamma(\nu+1)$ and $-\Gamma(\nu)2^\nu/\pi y^\nu$ respectively, while for equal values of their arguments their Wronskian is given by $Y_\nu(y)J_{\nu+1}(y) - J_\nu(y)Y_{\nu+1}(y) = 2/\pi y$.

the motion of the particle is symbolically describable as being associated with a metric $ds^2 = dt^2[1 + b(r) - v^2(1 + a(r)]$. For a slowly moving particle in a weak gravitational background, v^2 will be of the same order as the first order gravitational contribution, $b_1(r)$, to $b(r)$, so that Newtonian gravity is associated with $ds^2 = dt^2[1 + b_1(r) - v^2]$. For light however where $v = c$, the gravitational redshift is associated with $ds^2 = dt^2[1+b_1(r)]$ (viz. time dilation), while gravitational bending is associated with $ds^2 = dt^2[1 + b_1(r) - c^2(1 + a_1(r)]$ (time dilation plus length contraction), with bending thus involving the lowest order correction, $a_1(r)$, to $a(r)$ as well. For slowly moving particles the first relativistic correction to Newtonian planetary orbits (viz. perihelion precession) is associated with the second order correction to the background metric, viz. $ds^2 = dt^2[1+b_1(r)+b_2(r)-v^2(1+a_1(r)]$, and is thus not contained in the first order Eq. (17.27). With the TT graviton coupling in Eq. (17.27) being the same as the NT coupling in Eq. (14.30), the full first order graviton coupling on the brane is given by Eq. (14.31), with gravitational bending of light on a positive tension M_4 brane thus being identical to the bending which occurs in unembedded gravity. A similar situation is found in second order as well, with a second order calculation [Giannakis and Ren (2001)] recovering the standard precession of perihelia. As far as gravitational effects due to the exchange of the massless graviton is concerned then, through second order, gravity on an embedded positive tension M_4 brane is completely indistinguishable from standard unembedded gravity.[7]

17.3 Causality issues in the M_4^+ brane world

While we have now obtained the gravitational propagator exhibited in Eqs. (17.23) – (17.25), its consisting of a sum over terms of the form of functions of $|w|$ multiplied by $D(x - x', m)$ raises an apparent puzzle, namely how can this propagator be 5-space causal (i.e. take support only on and within the 5-dimensional AdS_5-space lightcone) when the w-independent $D(x - x', m)$ itself takes support on and within the 4-dimensional M_4-space lightcone alone. Thus, with the non-spacelike (w, t) geodesics in the background metric $ds^2 = dw^2 + e^{-2b|w|}\eta_{\mu\nu}dx^\mu dx^\nu$ which are at the brane at $t = 0$ being of the form $|bt| \geq |e^{b|w|} - 1|$, we need to show that the $G^{TT}(x, 0, w, 0)$ propagator does not take support in $|e^{b|w|} - 1| > |bt|$.

To this end it is convenient to use a very different approach for constructing the propagator, one developed by [Giddings, Katz and Randall (2000)] who noted that in the M_4^+ case a general solution to Eqs. (16.2) and (16.3) can be written in the

[7]There is as yet no brane gravity calculation of the next leading order effect, viz. the gravitational radiation reaction associated with the decay of the orbit of a binary pulsar which is to be accompanied by the emission of gravitational radiation. This is in a sense unfortunate, since unlike the static background gravitational field treatment needed for the three standard tests of general relativity (redshift, bending and precession), with the binary pulsar one tests directly that gravitational information is indeed communicated with finite velocity, the true hallmark of a covariant theory.

generic form

$$\begin{aligned} h^{TT}_{\mu\nu} &= -\frac{\kappa_5^2}{(2\pi)^4}\int d^4x'd^4p e^{ip\cdot(x-x')}\frac{[\alpha_q J_2(qe^{b|w|}/b)+\beta_q Y_2(qe^{b|w|}/b)]}{q[\alpha_q J_1(q/b)+\beta_q Y_1(q/b)]}S^{TT}_{\mu\nu}(x') \\ &= -2\kappa_5^2\int d^4x'\hat{G}^{TT}(x,x',w,0)S^{TT}_{\mu\nu}(x') \ , \end{aligned} \tag{17.30}$$

where $q^2=(p^0)^2-\bar{p}^2$ and where α_q and β_q are arbitrary functions of q. Equation (17.30) serves to define a propagator $\hat{G}^{TT}(x,x',w,0)$, and in the following we shall study the degree to which such $\hat{G}^{TT}(x,x',w,0)$ type propagators and the previously introduced $G^{TT}(x,x',w,0)$ type propagators of Eq. (17.24) are related, finding some quite substantive differences in some cases. While Eq. (17.30) is written in terms of the J_2 and Y_2 Bessel functions alone, because p^0 and $\bar{p}$ range over all values from $-\infty$ to $+\infty$ (as is needed to generate the $\delta^4(x-x')$ term in Eq. (17.31) below), Eq. (17.30) encompasses both $q^2>0$ and $q^2<0$ regions, and thus also the I_2 and K_2 Bessel functions to which the J_2 and Y_2 Bessel functions with pure imaginary argument are related.[8] With $J_\nu(z)$ being single valued for integer ν and with $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ having a semi-infinite branch cut from $z=0$ to $|z|=\infty$ for any ν,[9] whenever the parameter β_q is non-zero, the integrand in Eq. (17.30) will possess a branch point at $q=0$ together with a pole at $q^2=0$ coming from the singular behavior of Y_2 at zero value of its argument. Because of this multiple-valuedness, we shall once and for all define the parameter q to be of the same sign as p^0 (a requirement which gives not only their real parts a common sign, but their imaginary parts a common sign also), so that q is taken to be positive in the $0<p^0<\infty$ region of the p^0 integration in Eq. (17.30). Similarly, q will be taken to be negative in the $-\infty<p^0<0$ region, with the $-\infty<p^0<0$ region integrand in Eq. (17.30) then being determined by analytic continuation. To establish that Eq. (17.30) is indeed a solution to Eq. (16.1), we note that no matter how α_q and β_q depend on q, use of Eqs. (17.3) and (17.10) yields

$$\begin{aligned} &\frac{1}{2}\left[\frac{\partial^2}{\partial w^2}-4b^2+4b\delta(w)+e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right]h^{TT}_{\mu\nu}(x,|w|) \\ &= -\delta(w)\frac{\kappa_5^2}{(2\pi)^4}\int d^4x'd^4p e^{ip\cdot(x-x')}S^{TT}_{\mu\nu}(x') \\ &= -\delta(w)\kappa_5^2\int d^4x'\delta^4(x-x')S^{TT}_{\mu\nu}(x') = -\delta(w)\kappa_5^2 S^{TT}_{\mu\nu}(x) \ , \end{aligned} \tag{17.31}$$

just as required.

---

[8] $J_2(iz)=-I_2(z)$, $J_1(iz)=iI_1(z)$, $iY_2(iz)=I_2(z)+2iK_2(z)/\pi$, $iY_1(iz)=-iI_1(z)-2K_1(z)/\pi$.

[9] While the integer $J_n(z)$ can be written as power series (see e.g. Eq. (16.84)), the integer $Y_n(z)$ are given by $(2/\pi)J_n(z)\log z$ plus a power series. The multiple-valuedness of the Bessel functions can also be exhibited directly in the $\nu=1/2$ case where they are known in closed form as: $J_{1/2}(z)=(2/\pi z)^{1/2}\sin z$, $Y_{1/2}(z)=-(2/\pi z)^{1/2}\cos z$, $J_{1/2}(z)\pm iY_{1/2}(z)=\mp i(2/\pi z)^{1/2}e^{\pm iz}$.

17.4 Causality in flat spacetime

While the propagator given in Eq. (17.30) does not initially look all that much like the one given in Eq. (17.24), we shall now show that on performing the p^0 contour integration required to give a meaning to Eq. (17.30), none other than $G^{TT}(x, 0, w, 0, \mathrm{SING})$ of Eq. (17.25) will then emerge as its singular part. However, before doing this explicitly, we first need to make some general introductory remarks regarding the use of contour integration for propagators. For the illustrative case of the free massless 4-dimensional flat spacetime retarded scalar propagator

$$D_{\mathrm{RET}}(x, m=0) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4p \frac{e^{ip\cdot x}}{[(p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2+i\epsilon\epsilon(p^0)]} \quad , \tag{17.32}$$

there are two straightforward ways to evaluate it as a contour integral. With all of the singularities of the integrand lying below the real p^0 axis, on closing the contour below the real p^0 axis we can evaluate $D_{\mathrm{RET}}(x, m=0)$ (a p^0 integration along the real p^0 axis from $p^0 = -\infty$ to $p^0 = +\infty$) according to

$$D_{\mathrm{RET}}(x, m=0) + D_{\mathrm{LHP}} = D_{\mathrm{SING}} \quad , \tag{17.33}$$

where D_{LHP} denotes the contribution along a half circle of large radius P in the lower half p^0 plane along which p^0 is given by $p^0 = Pe^{i\theta}$ where θ goes clockwise from $\theta = 2\pi$ to $\theta = \pi$, and where D_{SING} denotes the contribution from the singularities of integrand of Eq. (17.32) as evaluated using Cauchy's theorem. Alternatively, we can close the contour in the upper half p^0 plane, and since there are no singularities there, then obtain

$$D_{\mathrm{RET}}(x, m=0) + D_{\mathrm{UHP}} = 0 \quad , \tag{17.34}$$

where D_{UHP} denotes the contribution along a half circle of large radius P in the upper half p^0 plane along which p^0 is again given by $p^0 = Pe^{i\theta}$ but where θ now goes counter-clockwise from $\theta = 0$ to $\theta = \pi$.

Since the choice of contour is arbitrary, the two above prescriptions must necessarily yield the same $D_{\mathrm{RET}}(x, m=0)$. However, since the entire time dependence in Eq. (17.32) is in the $\exp(-ip^0t) = \exp(-iPt\cos\theta + Pt\sin\theta)$ factor, this would seem to suggest that when t is positive D_{LHP} (where $\sin\theta$ is negative) would vanish in the $P \to \infty$ limit, so that $D_{\mathrm{RET}}(x, m=0)$ would then be given by D_{SING}. In contrast, the same argument would also seem to suggest that for the same positive t D_{UHP} (where $\sin\theta$ is positive) would then be infinite in this limit; in consequence of which Eq. (17.34) would imply that $D_{\mathrm{RET}}(x, m=0)$ itself would then be infinite, and thus not equal to the specifically finite value $[\delta(t+|\bar{x}|) - \delta(t-|\bar{x}|)]/4\pi|\bar{x}|$ that D_{SING} explicitly evaluates to. Study of the behavior of $\exp(ip^0t)$ in the complex p^0 plane thus cannot be a good guide as to the behavior of D_{LHP} or D_{UHP}, with this failure actually being due to the fact that it fails to take into account the $dp^0 = iPd\theta e^{i\theta}$

integration (or the d^3p one for that matter). For example, along the half circle in the lower half p^0 plane a typical integral such as $\int dp^0 \exp(-ip^0 t)$ is given by

$$\int dp^0 e^{-ip^0 t} = iP \int_{2\pi}^{\pi} d\theta e^{i\theta} e^{-iPe^{i\theta}t} = ie^{-iPe^{i\theta}t}\Big|_{2\pi}^{\pi} = -2\sin(Pt) \ , \qquad (17.35)$$

a function which despite the presence of the converging $\exp(Pt\sin\theta)$ factor, does not in fact vanish in the limit of large P (the important contributions to the integral come from the region near the end points where the vanishing of the $\sin\theta$ factor in $Pt\sin\theta$ removes the exponential damping). Similarly, the upper half plane analog of the same integral is given by $-2\sin(Pt)$, a quantity which does not grow exponentially in P (the oscillations of the $e^{i\theta}\exp(-iPt\cos\theta)$ factor that multiplies $\exp(Pt\sin\theta)$ removes the exponential growth). The contributions of the lower and upper half circle terms thus have to be evaluated completely before any assessment as to their potential importance may be made.

To explicitly evaluate the massless scalar field $D_{\rm UHP}$, it is convenient to introduce $\mu = -ip^0 = -ip^0_R + p^0_I$, quantity whose real part is positive in the upper half p^0 plane, with $D_{\rm UHP}$ then being given by (r denotes $|\bar{x}|$)

$$\begin{aligned} D_{\rm UHP} &= -D_{\rm RET}(x, m=0) = \frac{1}{(2\pi)^4}\int_0^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}t} \int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{(-\mu^2 - \bar{p}^2)} \\ &= -\frac{1}{8\pi^2 r}\int_0^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}t} e^{-\mu r} = -\frac{1}{8\pi^2 r}\int_0^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}(t-r)} \\ &= \frac{1}{4\pi^2 r}\frac{\sin[P(t-r)]}{(t-r)} \rightarrow \frac{1}{4\pi r}\delta(t-r) \ , \end{aligned} \qquad (17.36)$$

with the $P \to \infty$ limit having been taken. Not only is $D_{\rm UHP}$ seen to be finite, with $r = |\bar{x}|$ being non-negative, $\delta(t-r)$ can only take support in $t \geq 0$, with $D_{\rm RET}(x, m=0)$ therefore vanishing at negative times. The prescription of locating all complex p^0 plane singularities below the real p^0 axis thus automatically leads to a $D_{\rm RET}(x, m=0)$ which is indeed retarded (one which additionally cannot take support outside the $t = r$ flat spacetime lightcone, to thus be causal). Comparing with the massless propagator term given by the $m \to 0$ limit of Eq. (16.75), we see that even without ever needing to evaluate $D_{\rm SING}$ explicitly, the closing of the p^0 contour in the upper half plane can nonetheless lead us to a complete determination of $D_{\rm RET}(x, m=0)$. (That this $D_{\rm RET}(x, m=0) = -\delta(t-r)/4\pi r$ does obey $\eta^{\alpha\beta}\partial_\alpha\partial_\beta D_{\rm RET}(x, m=0) = \delta^4(x)$ is checked directly since $\nabla^2(1/r) = -4\pi\delta^3(x)$.)

To explicitly evaluate the lower half plane $D_{\rm LHP}$, it is convenient to introduce $\mu = ip^0 = ip^0_R - p^0_I$, a quantity whose real part is positive in the lower half p^0 plane, with $D_{\rm LHP}$ then being given by

$$
\begin{aligned}
D_{\rm LHP} &= -D_{\rm RET}(x, m=0) + D_{\rm SING} \\
&= \frac{1}{(2\pi)^4}\int_{2\pi}^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}t}\int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{(-\mu^2-\bar{p}^2)} \\
&= -\frac{1}{8\pi^2 r}\int_{2\pi}^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}t} e^{-\mu r} = -\frac{1}{8\pi^2 r}\int_{2\pi}^{\pi} iPd\theta e^{i\theta} e^{-iPe^{i\theta}(t+r)} \\
&= \frac{1}{4\pi^2 r}\frac{\sin[P(t+r)]}{(t+r)} \to \frac{1}{4\pi r}\delta(t+r) \ .
\end{aligned}
\tag{17.37}
$$

Then since $\delta(t+r)$ has no support when t is positive (r being positive), we see that $D_{\rm LHP}(t>0)$ vanishes, so that in this particular case (but not in general as we shall see below) the retarded $D_{\rm RET}(x, m=0)$ is indeed given as $D_{\rm SING}$ after all.[10]

17.5 Upper half plane determination of the M_4^+ propagator

Turning now to the evaluation of Eq. (17.30), we note that there are essentially four characteristic cases to consider, viz. the two Hankel function based combinations in which $(\alpha_q = 1, \beta_q = i)$, $(\alpha_q = 1, \beta_q = -i)$, together with the pure J_2 and Y_2 based combinations $(\alpha_q = 1, \beta_q = 0)$, $(\alpha_q = 0, \beta_q = 1)$ which will be of some interest in the following. For all of these four combinations, even though the right-hand side of Eq. (17.30) is not in general integrable exactly for an arbitrary source $S^{TT}_{\mu\nu}(x')$, to explore the causal properties of Eq. (17.30) in the (w, t) space, it suffices to consider sources which depend only on t', with the simplest choice, viz. $S^{TT}_{\mu\nu}(x') = A_{\mu\nu}\delta(t')$ where $A_{\mu\nu}$ is a pure constant TT tensor, being one which is rich enough for detailed exploration of the causality structure, while being simple enough to actually be tractable. For such a source Eq. (17.30) reduces to

$$
h^{TT}_{\mu\nu} = -\frac{\kappa_5^2 A_{\mu\nu}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t}\frac{[\alpha_q J_2(p^0 e^{b|w|}/b) + \beta_q Y_2(p^0 e^{b|w|}/b)]}{p^0[\alpha_q J_1(p^0/b) + \beta_q Y_1(p^0/b)]} \ , \tag{17.38}
$$

where now $q = p^0$. With the propagator of Eq. (17.38) being required to only possess singularities below the real p^0 axis,[11] to evaluate Eq. (17.38) we only need to evaluate the contribution over a large half circle in the upper half p^0 plane. When

[10] Since the evaluation of $D_{\rm LHP}$ and $D_{\rm UHP}$ did not depend on where the singularities were located, $D_{\rm LHP}$ and $D_{\rm UHP}$ evaluate the same way even when the singularities are all located in the upper half p^0 plane. The advanced $D_{\rm ADV}(x, m=0)$ is thus given as $D_{\rm ADV}(x, m=0) = -D_{\rm LHP} = -\delta(t+r)/4\pi r$, with $\delta(t+r)$ only taking support in $t<0$. Whether the propagator of Eq. (17.32) is to represent a retarded or an advanced propagator is thus fixed entirely by whether the singularities are located in the lower or the upper complex p^0 plane.

[11] Our strategy here is to once and for all define Eq. (17.38) as being an integral whose singularities are all located below the real p^0 axis, and to then check for which particular (α_q, β_q) combinations such a propagator actually is in fact retarded and causal. Our use below of the notation $h^{TT}_{\mu\nu}$(RET) is to imply only that all singularities are in the lower half p^0 plane (the retarded contour prescription), and not that any particular (α_q, β_q) combination being considered is necessarily retarded.

$|\arg z| < \pi$ (which is on the upper half circle for $p^0 = Pe^{i\theta}$ with $0 < \theta < \pi$) the relevant Bessel functions have asymptotic behavior as $|z|$ goes to infinity of the form

$$\begin{aligned}
J_\nu(z) \to\ & \left(\frac{1}{2\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\left[1-\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \\
&+\left(\frac{1}{2\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\left[1+\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \ , \\
iY_\nu(z) \to\ & \left(\frac{1}{2\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\left[1-\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \\
&-\left(\frac{1}{2\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\left[1+\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \ , \\
J_\nu(z)-iY_\nu(z) \to\ & \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)}\left[1+\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \ , \\
J_\nu(z)+iY_\nu(z) \to\ & \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)}\left[1-\frac{(4\nu^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \ . \quad (17.39)
\end{aligned}$$

Thus, along the upper half circle we obtain the leading behavior

$$\begin{aligned}
\frac{[J_2(p^0e^{b|w|}/b)+iY_2(p^0e^{b|w|}/b)]}{p^0[J_1(p^0/b)+iY_1(p^0/b)]} &\to -\frac{ie^{ip^0(e^{b|w|}-1)/b}}{p^0e^{b|w|/2}}\left[1-\frac{15b}{8ip^0e^{b|w|}}+\frac{3b}{8ip^0}\right]+\dots \ , \\
\frac{[J_2(p^0e^{b|w|}/b)-iY_2(p^0e^{b|w|}/b)]}{p^0[J_1(p^0/b)-iY_1(p^0/b)]} &\to \frac{ie^{-ip^0(e^{b|w|}-1)/b}}{p^0e^{b|w|/2}}\left[1+\frac{15b}{8ip^0e^{b|w|}}-\frac{3b}{8ip^0}\right]+\dots \ , \\
\frac{J_2(p^0e^{b|w|}/b)}{p^0J_1(p^0/b)} &\to \frac{[e^{ip^0(e^{b|w|}+1)/b}+ie^{-ip^0(e^{b|w|}-1)/b}]}{p^0e^{b|w|/2}[1+ie^{2ip^0/b}]}+\dots \ , \\
\frac{Y_2(p^0e^{b|w|}/b)}{p^0Y_1(p^0/b)} &\to -\frac{[e^{ip^0(e^{b|w|}+1)/b}-ie^{-ip^0(e^{b|w|}-1)/b}]}{p^0e^{b|w|/2}[1-ie^{2ip^0/b}]}+\dots \ .
\end{aligned} \quad (17.40)$$

However, despite the suppression of the non-leading terms by inverse powers of P, on performing the actual integration along the upper half circle in the p^0 plane, we will nonetheless find that these terms still make a contribution comparable to that found for the leading term itself.

For the two Hankel function cases first, we see that with $e^{b|w|}$ always being greater than one when $|w| > 0$, along the half circle in the upper half p^0 plane, the leading behavior associated with the $J_2 - iY_2$ based combination diverges as $P \to \infty$, while that associated with the $J_2 + iY_2$ based one converges. For the $J_2 + iY_2$ based combination the integration of Eq. (17.38) yields a leading upper half circle contribution of the form

$$h^{TT}_{\mu\nu}(\mathrm{UHP};\mathrm{J_2+iY_2}) = -\frac{\kappa_5^2A_{\mu\nu}}{2\pi e^{b|w|/2}}\int_0^\pi id\theta e^{-iPe^{i\theta}\alpha}(-i)\left[1-\frac{15b}{8iPe^{i\theta}e^{b|w|}}+\frac{3b}{8iPe^{i\theta}}\right], \quad (17.41)$$

where

$$\alpha = \frac{1}{b}(bt - e^{b|w|} + 1) \quad . \tag{17.42}$$

The evaluation of the integrals needed for Eq. (17.41) is given in Appendix E, to yield, on taking the $P \to \infty$ limit,

$$\begin{aligned} -h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 + \mathrm{iY}_2) &= h^{TT}_{\mu\nu}(\mathrm{UHP}; \mathrm{J}_2 + \mathrm{iY}_2) \\ &= -\frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}\theta(\alpha)\left[1 + \frac{15b\alpha}{8e^{b|w|}} - \frac{3b\alpha}{8} + O(\alpha^2)\right] \quad . \end{aligned} \tag{17.43}$$

While we would need to sum the power series in α in Eq. (17.43) in order to determine an explicit value for $h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 + \mathrm{iY}_2)$,[12] as we see, the leading and the non-leading terms in Eq. (17.40) all give rise to a common dependence on the step function $\theta(\alpha)$.[13] Consequently, with the AdS_5 lightcone and its interior being given by $\alpha \geq 0$, the propagator of Eq. (17.30) as evaluated with $\alpha_q = 1$, $\beta_q = i$ is seen to be both and causal and retarded (with b being positive, the positivity of α required by $\theta(\alpha)$ entails the positivity of t), just as desired.

Similarly, in terms of the quantity

$$\beta = \frac{1}{b}(bt + e^{b|w|} - 1) = \alpha + \frac{2e^{b|w|}}{b} - \frac{2}{b} \ , \tag{17.44}$$

the analog expression for the $J_2 - iY_2$ based combination given in Eq. (17.40) takes the form

$$\begin{aligned} -h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 - \mathrm{iY}_2) &= h^{TT}_{\mu\nu}(\mathrm{UHP}; \mathrm{J}_2 - \mathrm{iY}_2) \\ &= \frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}\theta(\beta)\left[1 - \frac{15b\beta}{8e^{b|w|}} + \frac{3b\beta}{8} + O(\beta^2)\right] \ , \end{aligned} \tag{17.45}$$

an expression which, as we thus see, takes support outside the AdS_5 lightcone (β can be positive even when α is negative). Of the two Hankel function combinations then, only the $J_2 + iY_2$ based propagator is retarded and causal, and it is thus the one we need for causal propagation in the M_4^+ brane world.[14] As such, basing a propagator on $J_2 + iY_2$ is of course the natural thing to do (the case expressly considered in [Giddings, Katz and Randall (2000)] in fact), since $J_2 + iY_2$ is of the form associated with an outgoing bulk travelling wave Hankel function $H_2^{(1)}$. Indeed, with it taking an infinite amount of time for an M_4^+ null signal $bt = e^{b|w|} - 1$

[12] With application of the wave operator $(1/2)[\partial_w^2 - 4b^2 + 4b\delta(w) - e^{2b|w|}\partial_t^2]$ to the quantity $\kappa_5^2 A_{\mu\nu}e^{-b|w|/2}\theta(\alpha)$ giving $-(15/32)b^2\kappa_5^2 A_{\mu\nu}e^{-b|w|/2}\theta(\alpha) + (3/2)b\kappa_5^2 A_{\mu\nu}\delta(w)\theta(t) - \kappa_5^2 A_{\mu\nu}\delta(w)\delta(t)$, we see that the solution of Eq. (17.43) has the correct generic form to be a solution to the wave equation in the presence of the source, with the rest of the series in Eq. (17.43) being needed in order for the solution to satisfy the wave equation exactly.

[13] In Appendix F we study some instructive lower dimensional brane-world models which are exactly soluble without need for recourse to the asymptotic expansions of Eq. (17.39), to find this same overall $\theta(\alpha)$ dependence in the analogous M_3^+ brane-world propagator.

[14] With β also being able to be positive when t is negative, the $J_2 - iY_2$ based combination serves as the advanced propagator when the singularities are located in the upper half p^0 plane.

to travel from $w = 0$ to $w = \infty$, and then another infinite amount for it to return to the brane, causality only permits a brane source to generate outgoing signals and never incoming ones, thus making it impossible to build a causal propagator based on an incoming $H_2^{(2)} = J_2 - iY_2$ travelling wave.[15]

Unlike the $J_2 + iY_2$ and $J_2 - iY_2$ based combinations which are both found to lead to values for $h^{TT}_{\mu\nu}(\text{UHP})$ which are real (a non-trivial outcome since for such combinations the integrand in Eq. (17.38) is not itself real), an analogous evaluation of the J_2 and Y_2 based combinations given in Eq. (17.40) leads to expressions for $h^{TT}_{\mu\nu}(\text{UHP})$ which are not in fact real.[16] However, since Eq. (16.1) is a real equation with a real source, the real and imaginary parts of $h^{TT}_{\mu\nu}(\text{UHP})$ must independently satisfy Eq. (16.1). But with the source being real, the imaginary part of $h^{TT}_{\mu\nu}(\text{UHP})$ must thus satisfy a homogeneous, source-free wave equation. Consequently, in the end it will only be the real part of $h^{TT}_{\mu\nu}(\text{UHP})$ which will be of interest to us. To determine the step function domains of relevance to $h^{TT}_{\mu\nu}(\text{UHP})$ in the J_2 and Y_2 based cases [to be denoted jointly by $J_2(Y_2)$] we power series expand the denominator of Eq. (17.40) as $1/(1 \pm ie^{2ip^0/b}) \sim 1 \mp ie^{2ip^0/b} - e^{4ip^0/b} + \ldots$ (we are able to do this along a circle in the upper half p^0 plane since the modulus of $\exp[ip^0/b] = \exp[i(P/b)\cos\theta - (P/b)\sin\theta]$ is never greater than one when $0 \leq \theta \leq \pi$). On making this expansion, the J_2 and Y_2 based propagators are then given as

$$\begin{aligned}
-h^{TT}_{\mu\nu}(\text{RET}; \text{J}_2(\text{Y}_2)) &= h^{TT}_{\mu\nu}(\text{UHP}; \text{J}_2(\text{Y}_2)) \\
&= \frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}[\mp i\theta(bt - e^{b|w|} - 1) \ + \theta(bt + e^{b|w|} - 1) \\
&\qquad - \theta(bt - e^{b|w|} - 3) \ \mp i\theta(bt + e^{b|w|} - 3) \\
&\qquad \pm i\theta(bt - e^{b|w|} - 5) \ - \theta(bt + e^{b|w|} - 5) \\
&\qquad + \theta(bt - e^{b|w|} - 7) \ \pm i\theta(bt + e^{b|w|} - 7) + \ldots] \ .
\end{aligned} \tag{17.46}$$

With the non-leading terms in the asymptotic expansion of Eq. (17.40) generating analogous step function type terms, Eq. (17.46) thus characterizes the step function domains appropriate to the J_2 and Y_2 based propagators. With inspection of these

[15]Despite the naturalness of basing the propagator on the outgoing $J_2 + iY_2$, in our study of the M_4^- brane world to be given in Chapter 18 (a globally non-hyperbolic brane world which can be traversed in a finite time) we shall see a switch in the roles of $J_2 + iY_2$ and $J_2 - iY_2$ based propagators, with the $J_2 - iY_2$ based one serving as the retarded propagator and the $J_2 + iY_2$ based one serving as the advanced.

[16]With $J_2(p^0e^{b|w|}/b)/p^0J_1(p^0/b)$ appearing to transform into itself under $p^0 \to -p^0$, one might initially expect the integral $\int dp^0 e^{-ip^0(t-t')}J_2(p^0e^{b|w|}/b)/p^0J_1(p^0/b)$ to actually be real. However, because the integrand is singular, we have defined the integral using the retarded contour prescription in which singularities in both $p^0 > 0$ and $p^0 < 0$ are to lie below the real p^0 axis. Consequently, under $p^0 \to -p^0$ the $p^0 > 0$ singularities below the real axis actually map into $p^0 < 0$ singularities above the real axis rather than into $p^0 < 0$ singularities below the real axis. (While putting singularities both above and below the real axis (e.g. the Feynman contour) could avoid this problem, the associated propagators would then take support outside the light cone.) As defined then the J_2 and Y_2 based propagators are not in fact obliged to necessarily be real.

step function domains revealing that the real parts of both of these two propagators take support outside the M_4^+ lightcone, we see that it is $J_2 + iY_2$ based propagator alone which is causal in M_4^+. This of course is just as one would expect since a theory can only contain one causal propagator (the solution to the Cauchy initial value problem is unique within the Cauchy development of the initial configuration), and for the M_4^+ brane world the outgoing travelling wave $J_2{+}iY_2$ based propagator is it. (As we shall see in Chapter 18 however, for the globally non-hyperbolic negative tension M_4^- brane world, it will not be the $J_2 + iY_2$ based propagator which will be causal, but the $J_2 - iY_2$ based one instead.)

17.6 Lower half plane determination of the M_4^+ propagator

The evaluation of Eq. (17.38) when closed in the lower half p^0 plane proceeds analogously,[17] and for the $J_2 + iY_2$ based combination yields, on making the transformation $\theta \to \theta - \pi$,

$$h^{TT}_{\mu\nu}(\mathrm{LHP;J_2+iY_2}) = -\frac{\kappa_5^2 A_{\mu\nu}}{2\pi e^{b|w|/2}}\int_\pi^0 id\theta e^{+iPe^{i\theta}\alpha}(-1)i\left[1+\frac{15b}{8iPe^{i\theta}e^{b|w|}}-\frac{3b}{8iPe^{i\theta}}\right] \tag{17.47}$$

with Eq. (17.47) immediately evaluating to

$$\begin{aligned} -h^{TT}_{\mu\nu}(\mathrm{RET;J_2+iY_2}) + h^{TT}_{\mu\nu}(\mathrm{SING;J_2+iY_2}) &= h^{TT}_{\mu\nu}(\mathrm{LHP;J_2+iY_2}) \\ &= \frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}\theta(-\alpha)\left[1+\frac{15b\alpha}{8e^{b|w|}}-\frac{3b\alpha}{8}+O(\alpha^2)\right] \ . \end{aligned} \tag{17.48}$$

Since $\theta(-\alpha) = 1 - \theta(\alpha)$, combining Eqs. (17.43) and (17.48) yields

$$h^{TT}_{\mu\nu}(\mathrm{SING;J_2+iY_2}) = \frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}\left[1+\frac{15b\alpha}{8e^{b|w|}}-\frac{3b\alpha}{8}+O(\alpha^2)\right] \ , \tag{17.49}$$

$$h^{TT}_{\mu\nu}(\mathrm{RET;J_2+iY_2}) = \theta(\alpha)h^{TT}_{\mu\nu}(\mathrm{SING;J_2+iY_2}) \ . \tag{17.50}$$

While Eq. (17.50) thus expresses the retarded causal M_4^+ brane-world propagator entirely in terms of its singularities, we note that according to Eq. (17.49) $h^{TT}_{\mu\nu}(\mathrm{SING;J_2+iY_2})$ itself actually does in fact take support outside the AdS_5 lightcone – nonetheless though, it is only its value on and within the AdS_5 lightcone which is of relevance for $h^{TT}_{\mu\nu}(\mathrm{RET;J_2+iY_2})$.

[17]To avail ourselves of the use of Eq. (17.39) for functions on the lower half circle where $|\arg z| > \pi$, we first make the transformation $\theta \to \theta - \pi$, viz. $Pe^{i\theta} \to -Pe^{i\theta}$, and then use the analytic continuation rules $J_1(-z) = -J_1(z)$, $J_2(-z) = J_2(z)$, $Y_1(-z) = -Y_1(z) - 2iJ_1(z)$, $Y_2(-z) = Y_2(z) + 2iJ_2(z)$, $H_1^{(1)}(-z) = H_1^{(2)}(z)$, $H_2^{(1)}(-z) = -H_2^{(2)}(z)$, $H_1^{(2)}(-z) = H_1^{(1)}(z)$, $H_2^{(2)}(-z) = -H_2^{(1)}(z)$.

For the $J_2 - iY_2$ based combination we similarly obtain

$$
\begin{aligned}
-h^{TT}_{\mu\nu}&(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2-\mathrm{iY}_2)\\
&=-\frac{\kappa_5^2 A_{\mu\nu}}{2\pi e^{b|w|/2}}\int_\pi^0 id\theta e^{+iPe^{i\theta}\beta}(-1)(-i)\left[1-\frac{15b}{8iPe^{i\theta}e^{b|w|}}+\frac{3b}{8iPe^{i\theta}}\right]\\
&=-\frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}\theta(-\beta)\left[1-\frac{15b\beta}{8e^{b|w|}}+\frac{3b\beta}{8}+O(\beta^2)\right] \quad , \qquad (17.51)
\end{aligned}
$$

where β is given in Eq. (17.44). In this case while we obtain an analogous relation of the form $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)=\theta(\beta)h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)$, this particular relation also holds outside the AdS_5 lightcone.

For completeness, we also evaluate the J_2 and Y_2 based propagators in the lower half p^0 plane, to obtain the leading behavior

$$
\begin{aligned}
-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)\ &=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2)\\
=\frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}[-i\theta(-bt-e^{b|w|}-1)\ &+\ \theta(-bt+e^{b|w|}-1)\\
-\ \theta(-bt-e^{b|w|}-3)\ &-i\theta(-bt+e^{b|w|}-3)\\
+i\theta(-bt-e^{b|w|}-5)\ &-\ \theta(-bt+e^{b|w|}-5)\\
+\ \theta(-bt-e^{b|w|}-7)\ &+i\theta(-bt+e^{b|w|}-7)+....] \quad , \qquad (17.52)
\end{aligned}
$$

$$
\begin{aligned}
-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2)\ &+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2)=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{Y}_2)\\
=\frac{\kappa_5^2 A_{\mu\nu}}{e^{b|w|/2}}[\ &-\frac{i}{3}\theta(-bt-e^{b|w|}-1)+\ \theta(-bt+e^{b|w|}-1)\\
&-\ \frac{1}{9}\theta(-bt-e^{b|w|}-3)-\frac{i}{3}\theta(-bt+e^{b|w|}-3)\\
&+\frac{i}{27}\theta(-bt-e^{b|w|}-5)-\ \frac{1}{9}\theta(-bt+e^{b|w|}-5)\\
&+\ \frac{1}{81}\theta(-bt-e^{b|w|}-7)+\frac{i}{27}\theta(-bt+e^{b|w|}-7)+....] \quad , \ (17.53)
\end{aligned}
$$

with $h^{TT}_{\mu\nu}(\mathrm{SING};J_2)$ and $h^{TT}_{\mu\nu}(\mathrm{SING};Y_2)$ both taking support outside the AdS_5 lightcone,[18] and with there being no straightforward relation between $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2(\mathrm{Y}_2))$ and $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2(\mathrm{Y}_2))$ of the type exhibited for the J_2+iY_2 based propagator in Eq. (17.50).

---

[18]The difference between the forms of Eqs. (17.52) and (17.53) originates in the markedly different behaviors of the $J_1(z)$, $J_2(z)$ and $Y_1(z)$, $Y_2(z)$ Bessel functions under analytic continuation to negative argument, and will be reflected below in the fact that, unlike the J_2 based propagator, the singular part of the Y_2 based propagator contains cuts.

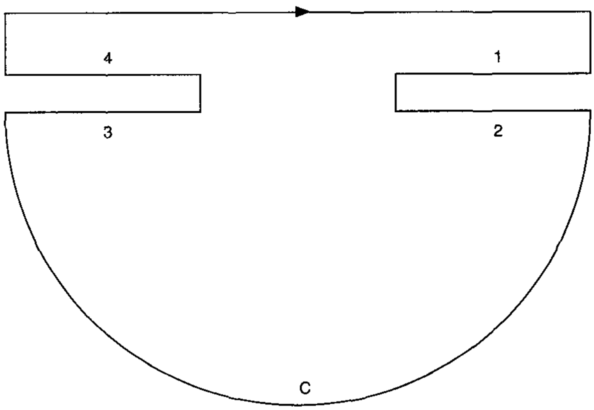

Fig. 17.1 The p^0 contour.

17.7 Evaluation of the cut discontinuities of the M_4^+ propagator

Having now analyzed the causal structure of the M_4^+ brane-world propagator given in Eq. (17.30), we turn next to an explicit evaluation of $h_{\mu\nu}^{TT}(\mathrm{SING}; \mathrm{J}_2 + \mathrm{iY}_2)$ in the causal $J_2 + iY_2$ based case, a calculation which can actually be carried through for an arbitrary source without restriction. In explicitly evaluating the singular part of the $J_2 + iY_2$ based propagator we have to deal with the branch point at $q = [(p^0)^2 - \bar{p}^2]^{1/2} = 0$ which both the Y_1 and Y_2 Bessel functions possess. Thus for the $J_2 + iY_2$ based TT retarded propagator associated with Eq. (17.30), viz.

$$\hat{G}^{TT}(x, x', w, 0) = \frac{1}{2(2\pi)^4}\int d^4p e^{ip\cdot(x-x')}\frac{[J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)]}{q[J_1(q/b) + iY_1(q/b)]}\ , \tag{17.54}$$

we see that in the complex p^0 plane, as well as containing poles at $p^0 = -|p| - i\epsilon$ and $p^0 = |p| - i\epsilon$, the propagator also contains branch points at $p^0 = -|p|$ and $p^0 = |p|$. For the retarded propagator we take the two associated branch cuts to lie below the real p^0 axis, taking them to respectively extend to $p^0 = -\infty$ and $p^0 = +\infty$.[19]

To evaluate the singular contribution $\hat{G}^{TT}(x, x', w, 0)(\mathrm{SING})$ to the retarded propagator, we introduce the clockwise closed contour of Fig. (17.1) which goes from $p^0 = -\infty$ to $p^0 = +\infty$, then back along the segment labelled (1) from $p^0 = +\infty$ to the branch point at $p^0 = |p|$, around the branch point and then along segment (2) to $p^0 = +\infty$. Then around a large half circle (C) below the real axis to $p^0 = -\infty$, then along segment (3) to the branch point at $p^0 = -p$, around that branch point and then finally along segment (4) to return to the starting point at $p^0 = -\infty$.

[19]With the branch points not being connected by a branch cut going from $p^0 = -|p|$ to $p^0 = |p|$, there is no discontinuity in the $q^2 < 0$ region. The evaluation of the p^0 contour integration in Eq. (17.54) will thus only involve the ordinary Bessel function mode solutions of Eq. (17.5) and not the modified Bessel function ones of Eq. (17.8).

With the poles being located below the real p^0 axis, we can thus symbolically set

$$\int_{-\infty}^{+\infty} + \int_{+\infty}^{|p|} (1) + \int_{|p|}^{+\infty} (2) + \int_{+\infty}^{-\infty} (C) + \int_{-\infty}^{-|p|} (3) + \int_{-|p|}^{-\infty} (4) = \text{ POLE TERM } , \tag{17.55}$$

with the singular part of the retarded propagator being given as

$$\int_{-\infty}^{+\infty} (\text{SING}) = \int_{|p|}^{+\infty} (1) - \int_{|p|}^{+\infty} (2) + \int_{-|p|}^{-\infty} (3) - \int_{-|p|}^{-\infty} (4) + \text{ POLE TERM } . \tag{17.56}$$

On symbolically defining the relevant part of the integrand in Eq. (17.54) as $e^{-ip^0t}f^{(i)}(p^0_R, p^0_I, q_R, q_I)$ on each segment (i), and on recalling that the relevant p^0 near the real p^0 axis is to be taken to be the retarded $p^0 + i\epsilon\epsilon(p^0)$, we thus obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dp^0 e^{-ip^0t} f(p^0)(\text{SING}) - \text{ POLE TERM}\\
&= \int_{-\infty}^{+\infty} dp^0 e^{-ip^0t} f(p^0)(\text{CUT})\\
&= \int_{|p|}^{+\infty} dp^0 e^{-ip^0t} f^{(1)}(+,+,+,+) - \int_{|p|}^{+\infty} dp^0 e^{-ip^0t} f^{(2)}(+,+,+,+)\\
&\quad + \int_{-|p|}^{-\infty} dp^0 e^{-ip^0t} f^{(3)}(-,-,-,-) - \int_{-|p|}^{-\infty} dp^0 e^{-ip^0t} f^{(4)}(-,-,-,-)\\
&= \int_{|p|}^{+\infty} dp^0 e^{-ip^0t} f^{(1)}(+,+,+,+) - \int_{|p|}^{+\infty} dp^0 e^{-ip^0t} f^{(2)}(+,+,+,+)\\
&\quad - \int_{|p|}^{\infty} dp^0 e^{ip^0t} f^{(3)}(+,+,+,+) + \int_{|p|}^{\infty} dp^0 e^{ip^0t} f^{(4)}(+,+,+,+) \ ,
\end{aligned} \tag{17.57}$$

where we have used the transformation $p^0 + i\epsilon\epsilon(p^0) \rightarrow -[p^0 + i\epsilon\epsilon(p^0)]$ in the last two of the integrals. For the pole term, on recalling that $J_1(y)$, $J_2(y)$, $Y_1(y)$ and $Y_2(y)$ respectively behave as $y/2$, $y^2/8$, $-2/\pi y + O(y)$ and $-4/\pi y^2 - 1/\pi$ near $y = 0$ (so that $[J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)]/q[J_1(q/b) + iY_1(q/b)] \sim 2be^{-2b|w|}/q^2$), we see that the singular behavior of the irregular $Y_2(y)$ Bessel function at small argument generates a pole term contribution to Eq. (17.54) of the form

$$\hat{G}^{TT}(x, x', w, 0)(\text{POLE}) = \frac{1}{2(2\pi)^4} \int d^4p e^{ip\cdot(x-x')} \frac{2be^{-2b|w|}}{[(p^0)^2 - \bar{p}^2 + i\epsilon\epsilon(p^0)]} \ , \tag{17.58}$$

viz. of the form

$$\hat{G}^{TT}(x, x', w, 0)(\text{POLE}) = -ibe^{-2b|w|} \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|} \left[e^{-i|p|t} - e^{i|p|t}\right] \ . \tag{17.59}$$

The pole term in Eq. (17.54) thus precisely recovers the massless graviton pole contribution given in Eq. (17.25).

To determine the cut contributions, we recall that under analytic continuation the Hankel functions are given for general ν and integer m by

$$\begin{aligned}\sin(\nu\pi)[J_\nu(ze^{im\pi})+iY_\nu(ze^{im\pi})] &= -\sin[(m-1)\nu\pi][J_\nu(z)+iY_\nu(z)] \\ &\quad +(-1)^{(\nu+1)}\sin(m\nu\pi)[J_\nu(z)-iY_\nu(z)] \ , \end{aligned} \tag{17.60}$$

so that for an $m=1$, $e^{i\pi}$ continuation we obtain

$$\begin{aligned} J_2(-z)+iY_2(-z) &= -[J_2(z)-iY_2(z)] \ , \\ J_1(-z)+iY_1(-z) &= J_1(z)-iY_1(z) \ . \end{aligned} \tag{17.61}$$

Noting also that at $w=0$ we have

$$\begin{aligned} &\left[\frac{d}{dw}+2b\right]\left[\frac{J_2(qe^{b|w|}/b)+iY_2(qe^{b|w|}/b)}{q[J_1(q/b)+iY_1(qb)]}\right]\Bigg|_{w=0} = 1 \ , \\ &\left[\frac{d}{dw}+2b\right]\left[\frac{J_2(-qe^{b|w|}/b)+iY_2(-qe^{b|w|}/b)]}{(-q)[J_1(-q/b)+iY_1(-qb)]}\right]\Bigg|_{w=0} \\ &= \left[\frac{d}{dw}+2b\right]\left[\frac{J_2(qe^{b|w|}/b)-iY_2(qe^{b|w|}/b)]}{q[J_1(q/b)-iY_1(qb)]}\right]\Bigg|_{w=0} = 1 \ , \end{aligned} \tag{17.62}$$

we see that all of the combinations which appear in Eq. (17.62) are capable of generating the $\delta^4(x-x')$ (as opposed to $-\delta^4(x-x')$) term in Eq. (17.31) which is needed to implement the junction condition at the brane, with all of these combinations thus having just the right weight for Eq. (17.57). With the navigation of the branch points in Fig. (17.1) being counterclockwise both in going from (1) to (2) and in going from (3) to (4), we shall determine the value of the integrand on (1) and (3) to be given by the $J_2(y)+iY_2(y)$ combinations, with the determinations on (2) and (4) thus being given by the $J_2(y)-iY_2(y)$ combinations, as they respectively differ from (3) and (1) by an $e^{i\pi}$ phase. Thus, from Eq. (17.57) we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} dp^0 e^{-ip^0t} f(p^0)(\mathrm{CUT}) \\ &= \int_{|p|}^{\infty} dp^0 e^{-ip^0t}\left[\frac{J_2(qe^{b|w|}/b)+iY_2(qe^{b|w|}/b)}{q[J_1(q/b)+iY_1(qb)]}\right] \\ &\quad - \int_{|p|}^{\infty} dp^0 e^{-ip^0t}\left[\frac{J_2(qe^{b|w|}/b)-iY_2(qe^{b|w|}/b)}{q[J_1(q/b)-iY_1(qb)]}\right] \\ &\quad - \int_{|p|}^{\infty} dp^0 e^{ip^0t}\left[\frac{J_2(qe^{b|w|}/b)+iY_2(qe^{b|w|}/b)}{q[J_1(q/b)+iY_1(qb)]}\right] \\ &\quad + \int_{|p|}^{\infty} dp^0 e^{ip^0t}\left[\frac{J_2(qe^{b|w|}/b)-iY_2(qe^{b|w|}/b)}{q[J_1(q/b)-iY_1(qb)]}\right] \ , \end{aligned} \tag{17.63}$$

where q is now everywhere positive. With the change of variable $m=+((p^0)^2-$

$\bar{p}^2)^{1/2}$, the continuum cut contribution may be written as

$$\hat{G}^{^{TT}}(x,0,w,0)(\mathrm{CUT}) = \frac{1}{(2\pi)^4}\int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times \int dm \left[\frac{[J_2(me^{b|w|}/b) + iY_2(me^{b|w|}/b)]}{[J_1(m/b) + iY_1(mb)]} - \frac{[J_2(me^{b|w|}/b) - iY_2(me^{b|w|}/b)]}{[J_1(m/b) - iY_1(mb)]}\right]$$
$$= -\frac{i}{(2\pi)^3}\int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times \int dm \left[\frac{[Y_1(m/b)J_2(me^{b|w|}/b) - J_1(m/b)Y_2(me^{b|w|}/b)]}{\pi[J_1^2(m/b) + Y_1^2(mb)]}\right] \quad , \tag{17.64}$$

where $E_p = (\bar{p}^2 + m^2)^{1/2}$. On setting $\sum_m = \int dm/b$, we then recognize Eq. (17.64) as being precisely none other than the continuum contribution to Eq. (17.25) given earlier, just as we would want; and for an arbitrary source can thus identify the general $h_{\mu\nu}^{^{TT}}(\mathrm{SING}; \mathrm{J}_2 + \mathrm{iY}_2)$ with the contribution $h_{\mu\nu}^{^{TT}}(x,w;\mathrm{SING}) = -2\kappa_5^2\int d^4x' G^{^{TT}}(x,x',w,0;\mathrm{SING})S_{\mu\nu}^{^{TT}}(x')$ of the singular $G^{^{TT}}(x,x',w,0;\mathrm{SING})$ of Eq. (17.25) to the general $h_{\mu\nu}^{^{TT}}(x,w)$ of Eq. (17.26). Finally, in light of Eq. (17.50), we see that we may write the complete retarded causal propagator of the M_4^+ brane world not actually as the singular Eq. (17.25) itself, but as $\theta(\hat{\alpha})$ times it, viz.

$$\hat{G}^{^{TT}}(x,0,w,0)(\mathrm{J}_2 + \mathrm{iY}_2) = \theta(\hat{\alpha})[\hat{G}^{^{TT}}(x,0,w,0)(\mathrm{POLE}) + \hat{G}^{^{TT}}(x,0,w,0)(\mathrm{CUT})] \quad , \tag{17.65}$$

with the M_4 invariance of the M_4^+ brane world having allowed us to generalize the α term of Eq. (17.42) to the general t, $\bar{x}$, $|w|$ case $\hat{\alpha}$ given by

$$\hat{\alpha} = (t^2 - |\bar{x}|^2)^{1/2} - \frac{1}{b}(e^{b|w|} - 1) \quad . \tag{17.66}$$

The fact that $h_{\mu\nu}^{^{TT}}(\mathrm{SING}; \mathrm{J}_2 + \mathrm{iY}_2)$ as evaluated in Eq. (17.49) takes support outside the AdS_5 lightcone has a serious consequence for the normalized mode basis propagator $G^{^{TT}}(x,0,w,0)$ of Eq. (17.24), a propagator whose evaluation as a lower half p^0 plane contour results in the $G^{^{TT}}(x,0,w,0;\mathrm{SING})$ contribution of Eq. (17.25) together with a $G^{^{TT}}(x,0,w,0;\mathrm{LHP})$ contribution from the lower half p^0 plane circle at infinity. Specifically, according to Eq. (17.23) this circle contribution is built as a mode sum over the lower half p^0 plane circle at infinity contributions $D(x,m;\mathrm{LHP})$ to the 4-space retarded propagators $D(x,m)$ as given in Eq. (E.61). Since such contributions vanish identically when t is positive ($D(x,m;\mathrm{LHP}) = -D(x,m;\mathrm{ADV})$), for points with $t > 0$ we are lead to the conclusion that the full $G^{^{TT}}(x,0,w,0)$ is given as $G^{^{TT}}(x,0,w,0;\mathrm{SING})$. With $h^{^{TT}}(x,0,w,0;\mathrm{SING})$ having been shown above to have the same causal properties as $h_{\mu\nu}^{^{TT}}(\mathrm{SING}; \mathrm{J}_2 + \mathrm{iY}_2)$, we have to conclude that the full normalized mode basis $G^{^{TT}}(x,0,w,0)$ propagator must take support outside the 5-space AdS_5 lightcone (where $e^{b|w|} - 1 > b|t|$) even though it never takes support outside the 4-space M_4 lightcone (where $|\bar{x}| > |t|$). The normalized mode basis propagator of Eq. (17.24) is thus not causal, leaving Eq. (17.65) as the unique causal propagator for the M_4^+ brane world.

Despite the fact that Eq. (17.26) is not causal, for static sources it will, nonetheless, suffice to use it. Specifically, for static sources, causality is not at issue since the source is always on, with an observer at any given instant being able to receive all signals which set off from the static source at all prior times which are within the backward lightcone of the observer. On treating a static source as a set of $\delta(t)$ sources which are uniformly distributed over all times t, the integration in Eq. (17.30) over such a set of sources makes the $\theta(\hat{\alpha})$ factor in Eq. (17.65) redundant, with $\hat{G}^{TT}(x,0,w,0)(\mathrm{J}_2+\mathrm{iY}_2)$ of Eq. (17.65) then yielding the same fluctuation as does $G^{TT}(x,0,w,0)$ of Eq. (17.26) in the static case. The discussion given earlier in this chapter of M_4^+ brane-world long range static gravity as based on Eq. (17.26) thus remains intact.[20]

The connection between the J_2+iY_2 based $h_{\mu\nu}^{TT}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)$ and the normalized mode basis $h_{\mu\nu}^{TT}(x,w;\mathrm{SING})$ that we have found has an interesting implication for the way the general TT fluctuation wave equation of Eq. (16.1) is to be implemented. Specifically, it is $h_{\mu\nu}^{TT}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)$ which obeys the implementation of Eq. (16.1) given as Eqs. (16.2) and (16.3), while it is $h_{\mu\nu}^{TT}(x,w)$ of Eq. (17.26) which obeys the implementation given as Eqs. (16.79) and (16.80). Now the normalized mode basis $h_{\mu\nu}^{TT}(x,w)$ is defined via the propagator of Eq. (16.76), a propagator which is itself built out the retarded flat spacetime propagator $D(x-x',m)$ given in Eq. (16.75). As such, this latter propagator obeys the Green's function equation of Eq. (16.74), viz. $[\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2]D(x-x',m) = \delta^4(x-x')$, and indeed it is because of the inhomogeneity of this particular equation that the $\delta(w)$ term in Eq. (16.79) is actually generated. Now in, for instance, the typical $m=0$ case, we explicitly evaluated the retarded $D(x-x',m=0)$ as a contour integral in the upper half p^0 plane, to obtain for it the value $-\delta(t-r)/4\pi r$ given in Eq. (17.36). Now we also evaluated this same retarded $D(x-x',m=0)$ as a contour integral in the lower half p^0 plane, to then obtain for it two contributions, a singular term of the form $D(x-x',m=0;\mathrm{SING}) = -\delta(t-r)/4\pi r + \delta(t+r)/4\pi r$, and a circle at infinity contribution of the form $-D(x-x',m=0;\mathrm{LHP}) = -\delta(t+r)/4\pi r$, with their addition then giving back $-\delta(t-r)/4\pi r$. It is from this $D(x-x',m=0;\mathrm{SING})$ that $G^{TT}(x,x',w,0;\mathrm{SING})$ of Eq. (17.25) is then

[20] Alternatively, we may note that with $h_{\mu\nu}^{TT}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)$ being calculable directly as a real p^0 integral when the source is time-independent (the t' integration generates a $\delta(p^0)$ term), for an observer on the brane and static source $S_{\mu\nu} = -M\delta^0_\mu\delta^0_\nu\eta_{00}\delta^3(x)$, on using the Bessel function identity $Z_2(z) = 2Z_1(z)/z - Z_0(z)$ and the relation $K_\nu(z) = (i\pi/2)e^{i\pi\nu/2}[J_\nu(iz)+iY_\nu(iz)]$, we obtain $h_{00}^{TT}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2,w=0) = (M\kappa_5^2 b/6\pi^3)\int d^3p e^{i\bar{p}\cdot\bar{x}}[1/p^2 + K_0(p/b)/2bpK_1(p/b)]$ where $p = [(p^1)^2+(p^2)^2+(p^3)^2]^{1/2}$. At large distances this brings us right back to $h_{00}^{TT}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2,w=0) = M\kappa_5^2 b/3\pi r$, together with a KK contribution which is suppressed by a factor of order $1/b^2r^2$. (At large r the KK contribution integrand is dominated by the small p limit $K_0(p/b)/pK_1(p/b) \to -(1/b)\log(p/b)$.)

built, even though $h^{TT}_{\mu\nu}(x,w)$ of Eq. (17.26) is built from the full $D(x-x',m=0)$. Now while the full $D(x-x',m=0) = -\delta(t-r)/4\pi r$ obeys the inhomogeneous $\eta^{\alpha\beta}\partial_\alpha\partial_\beta D(x-x',m=0) = \delta^4(x-x')$, its singular part obeys the homogeneous $\eta^{\alpha\beta}\partial_\alpha\partial_\beta D(x-x',m=0;\mathrm{SING}) = \eta^{\alpha\beta}\partial_\alpha\partial_\beta[-\delta(t-r)/4\pi r + \delta(t+r)/4\pi r] = 0$. In consequence of this, $h^{TT}_{\mu\nu}(x,w;\mathrm{SING})$ must obey the homogeneous (i.e. source-free) variant of Eqs. (16.79) and (16.80). However, since this particular $h^{TT}_{\mu\nu}(x,w;\mathrm{SING})$ is precisely the quantity $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)$, it quite surprisingly therefore follows that $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)$ must obey a source-free wave equation as well, and that it must do so even as $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)$ obeys a source-dependent one.[21]

In addition to actually constructing the M_4^+ causal propagator of Eq. (17.65), in recovering $G^{TT}(x,x',w,0;\mathrm{SING})$ of Eq. (17.25) as its singular part, our analysis also enables us to confirm that the completeness relation of Eq. (17.20) as built out of the normalized graviton and KK modes of Eqs. (17.21) and (17.22) is indeed the correct completeness relation for those modes. Moreover, in confirming the correctness of the completeness relation, we see that its confirmation via the use of Eq. (17.30) did not at all require the use of the orthonormality conditions of Eq. (17.19). Rather, it was the normalization of the propagator to the junction condition at the brane according to Eq. (17.31) which is what fixed the specific, explicitly finite weight with which each of the various modes is to appear in Eq. (17.65). Consequently, the steps leading to $\hat{G}^{TT}(x,0,w,0)(\mathrm{POLE})$ and $\hat{G}^{TT}(x,0,w,0)(\mathrm{CUT})$ can be carried through for any choice of α_q and β_q in Eq. (17.30), with the propagator weights which result still being able to be finite regardless of whether or not the relevant modes are normalizable at all. It is thus of interest to explore what happens when the Eq. (17.31) junction condition at the brane is satisfied by non-normalizable modes, with the most interesting cases being the modes associated with the J_2 and Y_2 based propagators which we discuss now, and those associated with the actually non-normalizable graviton mode of the $A = b|w|$ divergent warp factor negative tension M_4^- brane world which we discuss in Chapter 18.

17.8 Alternate propagators in M_4^+

For the J_2 based M_4^+ combination, Eq. (17.30) leads to a propagator of the form

$$\hat{G}^{TT}(x,x',w,0)(\mathrm{J}_2) = \frac{1}{2(2\pi)^4}\int d^4p e^{ip\cdot(x-x')}\frac{J_2(qe^{b|w|}/b)}{qJ_1(q/b)} \ . \tag{17.67}$$

While this particular propagator is free of branch points, it nonetheless possesses poles whenever $J_1(q/b)$ is zero. Its poles thus occur at the zeroes, j_i, of the J_1 Bessel

[21] The fact that $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)$ is to obey a source-free wave equation while $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)$ obeys a source-dependent one is not immediately apparent given that they are related via Eqs. (17.50) and (17.65). We shall thus demonstrate this result explicitly in the exactly soluble lower-dimensional M_3^+ brane-world model to be discussed below in Appendix F.

function, an infinite discrete set of zeroes, the first positive three of which occur at 3.832, 7.016 and 10.173, with the large ones being well-approximated by $j_n \approx (n + 1/4)\pi$. While the infinite KK continuum has been replaced by an infinite discrete set of modes, we note that this particular propagator possesses no massless mode, since the $J_2(qe^{b|w|}/b)$ term in the numerator of Eq. (17.67) vanishes quadratically at $q = 0$.[22] With a Taylor series expansion of $J_1(q/b)$ around any j_i yielding

$$J_1\left(\frac{q}{b}\right) = \left(\frac{q}{b} - j_i\right) J'(j_i) = \left(\frac{q}{b} - j_i\right)\left[\frac{J(j_i)}{j_i} - J_2(j_i)\right] = -\left(\frac{q}{b} - j_i\right) J_2(j_i) \ , \tag{17.68}$$

and with each j_i also being a zero of $J_1(-q/b)$, the relevant singular part of the p^0 contour integration in Eq. (17.67) is found to be given by the sum of pole terms

$$\begin{aligned}
&\frac{1}{2\pi}\int dp^0 e^{-ip^0(t-t')}\frac{J_2(qe^{b|w|}/b)}{qJ_1(q/b)}(\mathrm{POLE}; \mathrm{J}_2) \\
&= \frac{1}{2\pi}\sum_i \int dp^0 e^{-ip^0(t-t')}\left[-\frac{J_2(j_i e^{b|w|})}{j_i(q - bj_i)J_2(j_i)} - \frac{J_2(-j_i e^{b|w|})}{(-j_i)(q + bj_i)J_2(-j_i)}\right] \\
&= -\frac{1}{2\pi}\sum_i \int dp^0 e^{-ip^0(t-t')}\frac{J_2(j_i e^{b|w|})}{j_i J_2(j_i)}\left[\frac{1}{(q - bj_i)} - \frac{1}{(q + bj_i)}\right] \\
&= -\frac{1}{2\pi}\sum_i \int dp^0 e^{-ip^0(t-t')}\frac{J_2(j_i e^{b|w|})}{J_2(j_i)}\frac{2b}{[(p^0)^2 - E_i^2 + i\epsilon\epsilon(p^0)]} \\
&= i\sum_i \frac{2bJ_2(j_i e^{b|w|})}{J_2(j_i)}\left[\frac{e^{-iE_i(t-t')} - e^{iE_i(t-t')}}{2E_i}\right] \ ,
\end{aligned} \tag{17.69}$$

where $E_i = (\bar{p}^2 + b^2 j_i^2)^{1/2}$. With there being no cuts, the singular part of the $J_2(y)$ based propagator is thus given by

$$\hat{G}^{TT}(x, 0, w, 0)(\mathrm{SING}; \mathrm{J}_2) = i\sum_i f_i(|w|)f_i(0)\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_i}\left[e^{-iE_i t} - e^{iE_i t}\right] \ , \tag{17.70}$$

where we have introduced a basis of modes $f_i(|w|)$ defined as

$$f_i(|w|) = \frac{b^{1/2}J_2(j_i e^{b|w|})}{J_2(j_i)} \ . \tag{17.71}$$

Each one of these $f_i(|w|)$ modes is readily checked to obey the source-free Eqs. (17.1) and (17.2). That these modes obey Eq. (17.1) is automatic since the modes are $J_2(j_i e^{b|w|})$ Bessel functions. And, then, in the absence of any $\beta_m Y_1(m/b)$ type terms in Eq. (17.13), the junction condition constraint of Eq. (17.2) is instead

[22] This pole cancellation occurs only at $q = 0$ since none of the other zeroes of $J_2(q)$ coincide with any of zeroes of $J_1(q)$. (The zeroes of $J_2(q)$ and $J_1(q)$ interlace each other with the first three positive zeroes of $J_2(q)$ occurring at 5.136, 8.417 and 11.620, and the large ones at $j_n \approx (n+3/4)\pi$.)

secured by the vanishing of $J_1(m/b)$ itself at its zeroes. Since the discrete $f_i(|w|)$ modes are as convergent at infinity as the $f_m(|w|)$ KK continuum modes of Eq. (17.22), the discrete $f_i(|w|)$ modes also have vanishing asymptotic momentum flux. The discrete $f_i(|w|)$ modes thus form an orthogonal basis of modes (cf. Eq. (16.9)), with each such mode possessing a time-independent energy. However, evaluation of the energy (viz. Eq. (17.18)) that each $f_i(|w|)$ mode possesses shows it to be infinite, though nonetheless positive.[23] The discrete $f_i(|w|)$ modes while orthogonal are not orthonormal, but despite their not being normalizable, the modes still appear with finite weight in Eq. (17.70). Thus, the weights with which modes appear in a propagator are not necessarily fixed by their normalization constants.

As well as fix the overall magnitude of the weight with which each of the discrete modes appear in Eq. (17.70), the junction condition of Eq. (17.31) also fixes the overall sign. Thus, on comparing the overall signs of the pole terms in the propagators of Eqs. (17.70) and (17.59), we see that unlike the massless graviton which appears with positive signature in the J_2+iY_2 based propagator, the discrete $f_i(|w|)$ modes appear in the J_2 based propagator with negative signature. Such ghost signature is highly undesirable, making the J_2 based propagator physically unacceptable (though still mathematically interesting), and in field theory is often characteristic of acausal behavior, behavior which, as indicated earlier, the J_2 based propagator explicitly possesses. Now it is possible to avoid ghost signature by defining the basis modes not as the $f_i(|w|)$ given above, but as $\hat{f}_i(|w|) = if_i(|w|)$ instead, a change which would still leave bilinear functions of the modes such as the propagator real. However, in its turn, such a change would also reverse the sign of the energy in each mode, to thereby replace ghost signature by an equally undesirable negative energy. Either way then, the J_2 based propagator is not physically acceptable.

A similar such negative signature is found for the modes of both the Y_2 based propagator and the J_2-iY_2 based propagator, the two other acausal propagators of the M_4^+ brane world. For the Y_2 based propagator, the treatment of the Y_2 analog of Eq. (17.67), viz.

$$\hat{G}^{TT}(x,x',w,0)(\mathrm{Y}_2) = \frac{1}{2(2\pi)^4}\int d^4p e^{ip\cdot(x-x')}\frac{Y_2(qe^{b|w|}/b)}{qY_1(q/b)} \tag{17.72}$$

proceeds analogously, generating both discrete poles and a continuum cut contribution. As well as a discrete graviton pole at $q^2=0$ (near $q^2=0$ the leading singular behaviors of J_2+iY_2 and Y_2 are the same), the other poles are at the discrete zeroes of the Y_1 Bessel function, another infinite family of Bessel function zeroes, the first positive three of which are at 2.197, 5.430 and 8.596. With an analog of Eq. (17.68) also holding for the Y_1 Bessel function, the pole contribution to Eq. (17.72) is thus

[23] Such an infinite positive energy was also found for the KK continuum modes of Eq. (17.22), but for them such an infinity was acceptable since the modes were normalized in the continuum. For discrete modes however, an infinite energy is not physically acceptable.

given by

$$\hat{G}^{TT}(x,0,w,0)(\mathrm{POLE;Y_2}) = -ibe^{-2b|w|}\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|}\left[e^{-i|p|t} - e^{i|p|t}\right]$$
$$+\, i\sum_i \tilde{f}_i(|w|)\tilde{f}_i(0)\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_i}\left[e^{-iE_it} - e^{iE_it}\right] \ , \qquad (17.73)$$

where

$$\tilde{f}_i(|w|) = \frac{b^{1/2}Y_2(y_ie^{b|w|})}{Y_2(y_i)} \ , \quad E_i = (\bar{p}^2 + b^2y_i^2)^{1/2} \ . \qquad (17.74)$$

The cut contribution follows the derivation of Eq. (17.64) to yield

$$\hat{G}^{TT}(x,0,w,0)(\mathrm{CUT;Y_2}) = \frac{1}{(2\pi)^4}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times\int dm\left[\frac{Y_2(me^{b|w|}/b)}{Y_1(mb)} - \frac{[Y_2(me^{b|w|}/b) + 2iJ_2(me^{b|w|}/b)]}{[Y_1(m/b) + 2iJ_1(mb)]}\right]$$
$$= -\frac{i}{(2\pi)^3}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times\int dm\left[\frac{[Y_1(m/b)J_2(me^{b|w|}/b) - J_1(m/b)Y_2(me^{b|w|}/b)]}{\pi[4J_1^2(m/b) + Y_1^2(mb)]}\right]\left[1 - 2i\frac{J_2(me^{b|w|}/b)}{Y_1(m/b)}\right] \ ,$$
$$(17.75)$$

where use has been made of the continuation relation $Y_\nu(-z) = (-1)^\nu Y_\nu(z) + 2i\cos(\nu\pi)J_\nu(z)$ obeyed by the Y_ν Bessel functions. As we thus see, just like their J_2 sector counterparts, the discrete Y_2 based $f_i(|w|)$ modes of $\hat{G}^{TT}(x,0,w,0)(\mathrm{POLE;Y_2})$ also couple with negative signature.

As regards the $J_2 - iY_2$ based propagator, even though its graviton pole term couples with the same positive signature as the graviton pole of the $J_2 + iY_2$ based propagator (near $q^2 = 0$ the leading singular behaviors of $J_2 + iY_2$ and $J_2 - iY_2$ are the same), due to the change in integrand from outgoing $(J_2 + iY_2)$ to incoming $(J_2 - iY_2)$ Hankel functions, the cut contribution differs from that given in Eq. (17.64) by an overall change in sign, viz.

$$\hat{G}^{TT}(x,0,w,0)(\mathrm{CUT;J_2 - iY_2}) = \frac{1}{(2\pi)^4}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times\int dm\left[\frac{[J_2(me^{b|w|}/b) - iY_2(me^{b|w|}/b)]}{[J_1(m/b) - iY_1(mb)]} - \frac{[J_2(me^{b|w|}/b) + iY_2(me^{b|w|}/b)]}{[J_1(m/b) + iY_1(mb)]}\right]$$
$$= \frac{i}{(2\pi)^3}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt} - e^{iE_pt}\right]$$
$$\times\int dm\left[\frac{[Y_1(m/b)J_2(me^{b|w|}/b) - J_1(m/b)Y_2(me^{b|w|}/b)]}{\pi[J_1^2(m/b) + Y_1^2(mb)]}\right] \ ,$$
$$(17.76)$$

to thus make the continuum of the $J_2 - iY_2$ based propagator ghostlike. Consequently, in the M_4^+ brane world only the $J_2 + iY_2$ based propagator is ghost free, and thus it uniquely is the only physically acceptable propagator for the M_4^+ brane world.

Despite not being acceptable bases physically, it is still of interest to ask whether either of the above two sets of discrete bases might nonetheless still be complete,[24] and whether the discrete $f_i(|w|) = b^{1/2}J_2(j_i e^{b|w|})/J_2(j_i)$ basis modes or the discrete $\tilde{f}_i(|w|) = b^{1/2}Y_2(y_i e^{b|w|})/Y_2(y_i)$ plus $\tilde{f}_0(|w|) = b^{1/2}e^{-2b|w|}$ graviton basis modes might respectively obey completeness relations of candidate form

$$\sum_i \lambda_i f_i(|w|) f_i(0) = \delta(w) \quad , \tag{17.77}$$

$$\sum_i \lambda_i f_i(|w|) \tilde{f}_i(0) + \lambda_0 \tilde{f}_0(|w|) \tilde{f}_0(0) = \delta(w) \quad , \tag{17.78}$$

where the λ_i are weight factors which can accommodate any minus signs that might be needed because of ghost signature. To test for the validity of Eq. (17.77) we apply $\int_0^\infty d|w| e^{2b|w|} e^{-2b|w|}$ to it, i.e. we integrate it with respect to the M_4^+ $e^{2b|w|}$ measure of interest as multiplied by the $e^{-2b|w|}$ graviton wave function. However, with both the graviton and all of the $f_i(|w|)$ modes being solutions to the equations of motion which possess vanishing asymptotic flux, these modes are all orthogonal to each other, to thus cause the integration of the left-hand side of Eq. (17.77) to vanish. Since the integral of the right-hand side is simply equal to 1/2, Eq. (17.77) cannot be valid. For Eq. (17.78) we apply $\int_0^\infty d|w| e^{2b|w|} J_2(j_1 e^{b|w|})$, where j_1 is any one of the zeroes of the J_1 Bessel function. With the $J_2(j_1 e^{b|w|})$ mode also being an exact solution to the equations of motion with vanishing asymptotic flux, and with none of the zeroes of J_1 and Y_1 coinciding (the J_1 and Y_1 zeroes interlace each other), it follows that any $J_2(j_1 e^{b|w|})$ mode is orthogonal to every mode in the basis of Eq. (17.78), including the graviton. Since applying $\int_0^\infty d|w| e^{2b|w|} J_2(j_1 e^{b|w|})$ to the left-hand side of Eq. (17.78) thus gives zero, and since the integral of the right-hand side gives the non-zero $J_2(j_1)/2$ (none of the zeroes of J_1 coincides with any of the zeroes of J_2, with these two sets of zeroes also interlacing each other), we see that Eq. (17.78) cannot be valid either. The reason for the failure of Eqs. (17.77) and (17.78) lies in the fact that for the J_2 and Y_2 based propagators, neither is given solely by its singular term alone. Thus unlike the causal $J_2 + iY_2$ based propagator for which Eq. (17.50) (and consequently the completeness relation of Eq. (17.20)) hold, for the J_2 and Y_2 based sectors there is no straightforward relation between the propagator and its singular part due to the presence of the contributions of the upper and lower half p^0 plane circles at infinity in Eqs. (17.52) and (17.53).

[24]Like the discrete f_i basis of the J_2 propagator, the $\tilde{f}_i$ basis modes plus massless graviton of the Y_2 propagator also form an orthogonal set; though, just like the f_i basis modes, other than the graviton itself, none of the other $\tilde{f}_i$ modes has a finite norm.

Now in addition we could also ask whether the modes might still be complete but not obey relations such as Eq. (17.77) or (17.78), i.e. we can also ask whether we could still expand a localized square step (of which $\delta(w)$ is a limiting case) in terms of the above bases, but with coefficients other than the ones suggested by the structure of the pole terms in $\hat{G}^{TT}(x,0,w,0)(\mathrm{POLE};\mathrm{J}_2)$ and $\hat{G}^{TT}(x,0,w,0)(\mathrm{POLE};\mathrm{Y}_2)$, since satisfying this more general criterion would still be sufficient to establish completeness of a basis. Thus we need to determine whether we can make the expansions

$$V_J(|w|) = \sum_i V_i J_2(j_i e^{b|w|}) \quad , \tag{17.79}$$

and

$$V_Y(|w|) = \sum_i V_i Y_2(y_i e^{b|w|}) + V_0 e^{-2b|w|} \quad , \tag{17.80}$$

where $V_J(|w|)$ and $V_Y(|w|)$ are arbitrary square steps ($V_{J(Y)} = \hat{V}$, $\alpha \leq e^{b|w|} \leq \beta$, $V_{J(Y)} = 0$ otherwise, $\hat{V}$, α and β arbitrary constants), and the various V_i and V_0 are coefficients to be determined. However, on projecting Eq. (17.79) with $\int_0^\infty d|w|e^{2b|w|}e^{-2b|w|}$, we see that we would have to satisfy $(\hat{V}/b)\log(\beta/\alpha) = 0$ which cannot of course be true; while projecting Eq. (17.80) with $\int_0^\infty d|w|e^{2b|w|}J_2(j_1 e^{b|w|})$ would require the validity of $(\hat{V}/bj_1^2)[j_1\alpha J_1(j_1\alpha) + 2J_0(j_1\alpha) - j_1\beta J_1(j_1\beta) - 2J_0(j_1\beta)] = 0$ which could not be true either. Hence neither of the two discrete bases is complete, with the M_4^+ brane world thus only possessing one complete basis, the continuum orthonormal $J_2 + iY_2$ based basis with its bilinear completeness relation of the form given in Eq. (17.20).[25] While we have now determined the completeness properties of bases in the M_4^+ case, in our discussion in Chapter 18 of the negative tension M_4^- brane world, we shall see the emergence of two independent types of complete bases, as well as an interchange of roles played by the various kinds of bases. Specifically, it will be the M_4^- analog of the discrete J_2 type modes which will then prove to be complete and positive signatured, while the Y_2 based graviton plus KK continuum basis, even though complete this time, will instead possess a graviton with ghost signature. Moreover, both the M_4^- $J_2 + iY_2$ and $J_2 - iY_2$ based bases will be found to possess a ghost-signatured graviton as well, but despite this, the latter will nonetheless turn out to be causal.

17.9 Some exact solutions to the M_4^+ fluctuation equations

Further insight into the various M_4^+ propagators we have encountered, as well as some independent testing of their validity, can be provided by their reconciliation

[25]Since the $J_2 + iY_2$ based KK mode plus massless graviton basis is complete, the expansion of the $J_2 - iY_2$ based propagator in Eq. (17.76) is thus an expansion in terms of a complete basis with coefficients other than the ones associated with their bilinear completeness relation (viz. the positive signatured Eq. (17.20)) since the $J_2 - iY_2$ based continuum cut contribution is ghostlike.

with some particular exact solutions that the two different realizations of Eq. (16.1) we have been considering, viz.

$$\frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = 0 \ , \ \delta(w)\left[\frac{\partial}{\partial|w|} + 2b\right] h_{\mu\nu}^{TT} = -\kappa_5^2\delta(w)S_{\mu\nu}^{TT} \ , \tag{17.81}$$

$$\frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = -\kappa_5^2\delta(w)S_{\mu\nu}^{TT} \ , \ \delta(w)\left[\frac{\partial}{\partial|w|} + 2b\right] h_{\mu\nu}^{TT} = 0 \ , \tag{17.82}$$

happen to possess. Specifically, exact solutions to Eqs. (17.81) and (17.82) can explicitly be found for any TT brane source which obeys $\eta^{\alpha\beta}\partial_\alpha\partial_\beta S_{\mu\nu}^{TT} = 0$. Thus, as can readily be checked, for such a source the asymptotically bounded

$$h_{\mu\nu}^{TT} = -2b\kappa_5^2 e^{-2b|w|}\int d^4x' D(x-x', m=0)S_{\mu\nu}^{TT}(x') - \frac{\kappa_5^2}{2b}S_{\mu\nu}^{TT} \ , \tag{17.83}$$

and the asymptotically unbounded

$$h_{\mu\nu}^{TT} = -\frac{\kappa_5^2}{4b}e^{2b|w|}S_{\mu\nu}^{TT} \tag{17.84}$$

both satisfy Eq. (17.81) without any approximation at all.[26] To both of these solutions we could additionally add on the fluctuation

$$\begin{aligned} h_{\mu\nu}^{TT} &= \frac{\alpha}{2b}e^{-2b|w|}S_{\mu\nu}^{TT} - \frac{\hat{\alpha}}{2b}S_{\mu\nu}^{TT} + \frac{\hat{\alpha}}{4b}e^{2b|w|}S_{\mu\nu}^{TT} \\ &\quad - 2\hat{\alpha}be^{-2b|w|}\int d^4x' D(x-x', m=0)S_{\mu\nu}^{TT}(x') \ , \end{aligned} \tag{17.85}$$

as it identically obeys the source-free (viz. $\kappa_5^2 = 0$) limit of Eq. (17.81) (and thus also the $\kappa_5^2 = 0$ limit of Eq. (17.82)) for arbitrary α and $\hat{\alpha}$, a source-free solution which will prove to be of use to us below. Analogously, for the same source, on recalling that $\delta(w)\theta(|w|) = \delta(w)/2$, $\delta(|w|) = \delta(w)$, Eq. (17.82) is found to admit of two solutions, again one with an $m = 0$ graviton term and one without, both of which turn out to be asymptotically unbounded, viz.

$$\begin{aligned} h_{\mu\nu}^{TT} &= -2b\kappa_5^2 e^{-2b|w|}\int d^4x' D(x-x', m=0)S_{\mu\nu}^{TT}(x') \\ &\quad - \frac{\kappa_5^2}{4b}(e^{2b|w|} - e^{-2b|w|})\theta(|w|)S_{\mu\nu}^{TT} - \frac{\kappa_5^2}{2b}S_{\mu\nu}^{TT} + \frac{3\kappa_5^2}{8b}e^{2b|w|}S_{\mu\nu}^{TT} \ , \end{aligned} \tag{17.86}$$

$$h_{\mu\nu}^{TT} = -\frac{\kappa_5^2}{4b}(e^{2b|w|} - e^{-2b|w|})\theta(|w|)S_{\mu\nu}^{TT} + \frac{\kappa_5^2}{8b}e^{2b|w|}S_{\mu\nu}^{TT} \ . \tag{17.87}$$

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[26] Despite the presence of the $(\kappa_5^2/2b)S_{\mu\nu}^{TT}$ term, the solution of Eq. (17.83) can still be regarded as being localized to the brane, since this term can be removed everywhere in the bulk by one of the ξ^5-dependent gauge transformations given in Eq. (14.26).

However, in this case, because of the freedom associated with the $\hat{\alpha}$-dependent solution to the source-free wave equation, these two specific solutions are not actually independent of each other, and so we shall restrict the discussion to the first of the two of them in the following. Of all of the exact solutions we have found above, only the very first one given in Eq. (17.83) actually localizes to the brane, with all of the others being ones that grow without bound at infinity, to thus show that even in a convergent warp factor theory, a source on the brane is able to generate divergent as well as convergent fluctuations.

Since the above solutions are solutions to the wave equation, each one must be reproduced when an appropriate choice of Eq. (17.30) or Eq. (17.26) is evaluated with a harmonic source, something we now show for a particularly tractable harmonic source, viz. a massless plane wave source $S^{TT}_{\mu\nu} = A_{\mu\nu}e^{i\bar{k}\cdot\bar{x}-ikt}$ where $A_{\mu\nu}$ is a pure constant. Considering first the M_4^+ $J_2 + iY_2$ based causal propagator

$$h^{TT}_{\mu\nu} = -\frac{\kappa_5^2}{(2\pi)^4}\int d^4x'd^4pe^{ip\cdot(x-x')}\frac{[J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)]}{q[J_1(q/b) + iY_1(q/b)]}S^{TT}_{\mu\nu}(x') \quad , \quad (17.88)$$

we see that in an evaluation with the massless plane wave source, because of the generation of a $\delta^4(p-k)$ term, only the $q^2 = 0$ contribution will be relevant. With the $Y_1(z)$ and $Y_2(z)$ Bessel functions behaving as the singular $Y_1(z) \to -2/\pi z$ and $Y_2(z) \to -4/\pi z^2 - 1/\pi$ at small z, the integrand in this propagator thus behaves as

$$\frac{[J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)]}{q[J_1(q/b) + iY_1(q/b)]} \to \frac{2be^{-2b|w|}}{q^2} + \frac{1}{2b} \quad (17.89)$$

at small q^2. Through the use of this limit the solution of Eq. (17.83) is then immediately recovered. As well as provide a consistency check on our work, we also establish that of all of the exact solutions, it is precisely the asymptotically bounded one which is associated with the causal propagator, just as one would want.

The second of the exact solutions is associated with the J_2 based propagator given in Eq. (17.67). Specifically, with the small argument behavior of the $J_1(z)$ and $J_2(z)$ Bessel functions being given by $J_1(z) \to z/2$, $J_2(z) \to z^2/8$, the small q^2 behavior of the integrand of Eq. (17.67) is given by $J_2(qe^{b|w|}/b)/qJ_1(q/b) \to e^{2b|w|}/4b$, with the solution of Eq. (17.84) immediately being recovered, just as desired. As we see, this time it is the ghost-signatured J_2 based propagator which is needed in order to recover the exact solution; and thus while Eq. (17.84) is an exact solution to the theory, no source on a positive tension Minkowski brane would ever be physically capable of generating it since the J_2 based propagator is not causal. As regards this particular solution, we additionally note that even though every pole term in Eq. (17.67) has a residue which is well-behaved at large $|w|$ (at the $q/b = j_i$ zeroes of $J_1(q/b)$, the residues behave as $J_2(j_ie^{2b|w|}) \to e^{-b|w|/2}$ at large $|w|$), nonetheless, the full solution still grows at infinity. Having modes which individually converge is not sufficient to ensure the convergence of their sum.

The final one of the exact solutions is associated with the normalized mode based solution of Eq. (17.26), viz.

$$h^{TT}_{\mu\nu}(x,w) = -2\kappa_5^2 \sum_m f_m(|w|) f_m(|w'|) \int d^4x' D(x-x',m) S^{TT}_{\mu\nu}(x') \quad , \tag{17.90}$$

as summed over the normalized massless graviton and KK continuum basis modes of Eq. (17.24). And indeed, the graviton term given in Eq. (17.86) is already precisely in the form of the graviton contribution as given in Eq. (17.27) (for any source in fact),[27] so that we need to show that the KK contribution can generate the other terms according to

$$\begin{aligned} h^{TT}_{\mu\nu}(KK) &= -2\kappa_5^2 \sum_{m\neq 0} f_m(|w|) f_m(0) \int d^4x' D(x-x',m) S^{TT}_{\mu\nu}(x') \\ &= -\frac{\kappa_5^2}{4b}(e^{2b|w|} - e^{-2b|w|})\theta(|w|) S^{TT}_{\mu\nu} - \frac{\kappa_5^2}{2b} S^{TT}_{\mu\nu} + \frac{3\kappa_5^2}{8b} e^{2b|w|} S^{TT}_{\mu\nu} \ . \end{aligned} \tag{17.91}$$

To perform the required integration, we note that the integration of $e^{ik\cdot x}$ with respect to a massive flat spacetime retarded propagator yields

$$\begin{aligned} \int d^4x' D(x-x',m) e^{ik\cdot x'} &= \frac{1}{2\pi} \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dp^0 \frac{e^{i\bar{k}\cdot\bar{x}} e^{-ip^0(t-t')} e^{-ikt'}}{[(p^0)^2 - \bar{k}^2 - m^2 + i\epsilon\epsilon(p^0)]} \\ = -ie^{i\bar{k}\cdot\bar{x}} \int_{-\infty}^{t} dt' e^{-ikt'} &\left[\frac{e^{-iE_k(t-t')}}{2E_k} - \frac{e^{iE_k(t-t')}}{2E_k} \right] = -\frac{e^{ik\cdot x}}{m^2} \quad , \end{aligned} \tag{17.92}$$

where $E_k = (\bar{k}^2 + m^2)^{1/2}$. For the $A_{\mu\nu}e^{ik\cdot x}$ source then we obtain $h^{TT}_{\mu\nu}(KK) = 2\kappa_5^2 S^{TT}_{\mu\nu} F(|w|)$ where $F(|w|)$ is given in terms of the basis of Eq. (17.22) as

$$F(|w|) = \sum_{m\neq 0} \frac{f_m(|w|) f_m(0)}{m^2} \quad , \quad F(0) = \frac{2b}{\pi^2} \int_0^{\infty} dm \frac{1}{m^3 [J_1^2(m/b) + Y_1^2(m/b)]} \ . \tag{17.93}$$

Use now of the wave equation obeyed by the $f_m(|w|)$ modes and the completeness relation of Eq. (17.20) which they obey allow us to infer that $F(|w|)$ obeys

$$\begin{aligned} \left(\frac{d^2}{d|w|^2} - 4b^2 \right) F(|w|) &= -e^{2b|w|} \sum_{m\neq 0} f_m(|w|) f_m(0) \\ &= -e^{2b|w|}[e^{-2b|w|}\delta(w) - be^{-2b|w|}] = -\delta(w) + b \quad , \\ \delta(w) \left(\frac{d}{d|w|} + 2b \right) F(|w|) &= 0 \quad , \end{aligned} \tag{17.94}$$

to yield a solution of the form

$$F(|w|) = -\frac{1}{8b}(e^{2b|w|} - e^{-2b|w|})\theta(|w|) - \frac{1}{4b} + \frac{3}{16b} e^{2b|w|} \ . \tag{17.95}$$

[27]For the $S^{TT}_{\mu\nu} = A_{\mu\nu} e^{i\bar{k}\cdot\bar{x} - ikt}$ source, the integral $\int d^4x' D(x-x', m=0)$ diverges, with the α-dependent term in Eq. (17.85) being available to cancel its infinite piece.

Given this form for $F(|w|)$, the solution of Eq. (17.86) is then recovered, just as required. The completeness relation for the modes is thus checked in a completely soluble model, with the critical role played in the completeness relation by the positive-signatured massless graviton in generating the $f_{m=0}(|w|)f_{m=0}(0) = be^{-2b|w|} > 0$ term in Eq. (17.94) being revealed. As we see then, the various propagators we have considered are fully consistent with the exact solutions we have found, with it being just one of them, viz. the $J_2 + iY_2$ based one, which is both causal and capable of generating gravitational fluctuations which are localized to the brane; with our exact solution analysis confirming that it is the $J_2 + iY_2$ based propagator rather than the normalized mode based one which is the appropriate one for the brane world.

As well as being an exact solution to the theory, we note that the particular localizing solution given in Eq. (17.83) is also of interest for an additional reason, namely that since it holds for an arbitrary harmonic source, it consequently also holds for a brane source familiar from Robertson-Walker cosmology, viz. a maximally 3-symmetric, traceless perfect fluid source. Specifically, for the perturbative source $S^{TT}_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m\eta_{\mu\nu}$ with spatially independent ρ_m and p_m and equation of state $p_m = \rho_m/3$, covariant conservation with respect to an M_4^+ background according to $\partial_\mu S^{TT\,\mu\nu} = 0$ would then require ρ_m and p_m to be time-independent, to thereby yield a completely space and time independent source, a source which would then automatically obey $\eta^{\alpha\beta}\partial_\alpha\partial_\beta S^{TT}_{\mu\nu} = 0$. Given this feature of the solution of Eq. (17.83), we turn now to a more detailed analysis of RW perturbations on positive tension M_4 brane.

17.10 Connection to RW perturbations on the brane

Given the above analysis, it is of interest to explore the connection between the constant $S^{TT}_{\mu\nu}$ case and radiation era RW perturbations in a little more detail. Specifically, as can directly be checked, for a perturbative RW $S^{TT}_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m\eta_{\mu\nu}$ source with $p_m = \rho_m/3$, Eq. (17.81) admits of an exact solution of the form

$$\begin{aligned}
h^{TT}_{00} &= -\kappa_5^2 b p_m e^{-2b|w|}[x^2+y^2+z^2] - \frac{3}{2b}\kappa_5^2 p_m \ , \quad h^{TT}_{12} = \frac{\kappa_5^2 b}{3} p_m e^{-2b|w|} xy \ , \\
h^{TT}_{23} &= \frac{\kappa_5^2 b}{3} p_m e^{-2b|w|} yz \ , \quad h^{TT}_{31} = \frac{\kappa_5^2 b}{3} p_m e^{-2b|w|} zx \ , \\
h^{TT}_{11} &= h^{TT}_{22} = h^{TT}_{33} = -\frac{\kappa_5^2 b}{3} p_m e^{-2b|w|}[x^2+y^2+z^2] - \frac{\kappa_5^2}{2b} p_m \ ,
\end{aligned} \tag{17.96}$$

with the presence of the non-diagonal components h^{TT}_{12}, h^{TT}_{23} and h^{TT}_{31} being needed in order to enforce the TT condition $\partial_\mu h^{TT\,\mu\nu} = 0$.[28] To check this solution against

[28]All of these non-diagonal terms are solutions to the source-free wave equation and do not participate in the junction conditions since the $S^{TT}_{\mu\nu}$ being considered here is diagonal in its indices.

Eq. (17.83) we note that with cut-off Λ we obtain

$$\int d^4x' D(x-x', m=0) = -\frac{1}{4\pi}\int \frac{d^3x'}{|\bar{x}-\bar{x}'|}\int_{-\infty}^{t} dt'\delta(t-t'-|\bar{x}-\bar{x}'|)$$
$$= -\frac{1}{2r}\int_0^r dr' 2r'^2 - \frac{1}{2r}\int_r^{\Lambda} dr' 2r' r = -\frac{\Lambda^2}{2} + \frac{(x^2+y^2+z^2)}{6} \ , \qquad (17.97)$$

which is just of the form needed for Eq. (17.96).

Now in Chapter 9 we obtained the exact solution of Eq. (9.17) which described the geometry associated with the (non-perturbative) embedding of a general RW brane in AdS_5, viz.

$$ds^2 = dw^2 - [\cosh(b|w|) - F(t)\sinh(b|w|)]^2 dt^2$$
$$+a^2(t)[\cosh(b|w|) - G(t)\sinh(b|w|)]^2\left[\frac{dr^2}{(1-kr^2)} + r^2 d\Omega\right] \ , \qquad (17.98)$$

where

$$G(t) = \frac{[\dot{a}^2 + k + b^2a^2]^{1/2}}{ba} \ , \quad F(t) = \frac{[\ddot{a} + b^2 a]}{b[\dot{a}^2 + k + b^2a^2]^{1/2}} \ , \qquad (17.99)$$

and where the junction conditions give

$$1 + \frac{k}{a^2b^2} + \frac{\dot{a}^2}{a^2b^2} = -\frac{\kappa_5^2}{6\Lambda_5}(\lambda + \rho_m)^2 \ , \quad a\dot{\rho}_m + 3\dot{a}(\rho_m + p_m) = 0 \qquad (17.100)$$

for a brane with tension λ and perfect fluid with energy density ρ_m. Since Eq. (17.98) is an exact solution, it must also hold in the limit in which ρ_m is small and in which $\kappa_5^2\lambda^2 + 6\Lambda_5 = 0$, i.e. in which a perturbative RW cosmology is added to a Minkowski brane. For ρ_m and p_m being of order some small parameter ϵ, it will then be the case that $G-1$, $F-1$, k, and $a-1$ will all begin in order ϵ also. Both ρ_m and p_m ($=\rho_m/3$ in the radiative era) must thus be static to lowest order, and to get the time dependence of $a_1(t)$ we will need to calculate through second order. With the exact radiation era ρ_m being given by $\rho_m = A/a^4$, on setting $A = A_1 + A_2$ and $a = 1 + a_1$, through second order ρ_m will thus be given by $\rho_m = A_1 - 4a_1A_1 + A_2$. On setting $k = k_1 + k_2$, the junction conditions then give

$$k_1\lambda - 2b^2A_1 = 0 \ , \quad \lambda^2k_2 + \lambda^2\dot{a}_1^2 + 4\lambda b^2A_1a_1 - 2\lambda b^2A_2 - b^2A_1^2 = 0 \qquad (17.101)$$

through second order, with the time dependence of $a_1(t)$ then being given as

$$a_1(t) = -\frac{k_1t^2}{2} + \frac{k_1}{8b^2} + \frac{A_2}{2A_1} - \frac{k_2}{2k_1} \ . \qquad (17.102)$$

Consequently, to first order $G(t)$ and $F(t)$ are given by

$$G(t) = 1 + G_1 = 1 + \frac{k_1}{2b^2} \ , \quad F(t) = 1 + F_1 = 1 - \frac{3k_1}{2b^2} \ , \qquad (17.103)$$

so that to first order the perturbative metric can be written as $ds^2 = dw^2 + [e^{-2b|w|}\eta_{\mu\nu} + h_{\mu\nu}]dx^\mu dx^\nu$ where

$$\begin{aligned} h_{00} &= F_1 - F_1 e^{-2b|w|} \quad , \quad h_{rr} = -G_1 + (G_1 + 2a_1 + k_1 r^2)e^{-2b|w|} \quad , \\ h_{\theta\theta} &= r^2[-G_1 + (G_1 + 2a_1)e^{-2b|w|}] \quad , \quad h_{\phi\phi} = \sin^2\theta h_{\theta\theta} \end{aligned} \tag{17.104}$$

in polar coordinates; with the perturbative metric thus being writable as

$$\begin{aligned} h_{00} &= F_1 - F_1 e^{-2b|w|} \quad , \quad h_{01} = h_{02} = h_{03} = 0 \quad , \\ h_{11} &= -G_1 + (G_1 + k_1 x^2 + 2a_1)e^{-2b|w|} \quad , \\ h_{22} &= -G_1 + (G_1 + k_1 y^2 + 2a_1)e^{-2b|w|} \quad , \\ h_{33} &= -G_1 + (G_1 + k_1 z^2 + 2a_1)e^{-2b|w|} \quad , \\ h_{12} &= k_1 xye^{-2b|w|} \quad , \quad h_{23} = k_1 yze^{-2b|w|} \quad , \quad h_{31} = k_1 xze^{-2b|w|} \end{aligned} \tag{17.105}$$

in Cartesian coordinates. Interestingly, the perturbative metric is found to automatically be in the axial gauge, and even though the full RW metric of Eq. (17.98) possesses both converging ($e^{-2b|w|}$) and diverging ($e^{2b|w|}$) terms, the perturbative metric of Eq. (17.105) possesses the converging, localizing $e^{-2b|w|}$ alone.

Under a gauge transformation it is possible to bring this metric to a more familiar form. Specifically, on picking the various constants so that $a_1(t)$ can be written for simplicity as $a_1(t) = -k_1 t^2/2$, under the axial gauge preserving transformation $h_{\mu\nu} \to h_{\mu\nu} + e^{-2b|w|}(\eta_{\mu\rho}\partial_\nu\hat{\xi}^\rho + \eta_{\nu\rho}\partial_\mu\hat{\xi}^\rho)$, where

$$\begin{aligned} \hat{\xi}^1 &= \frac{k_1}{2}xt^2 - \frac{2k_1}{9}x^3 - \frac{k_1}{6}x(y^2 + z^2) - \frac{k_1}{4b^2}x \quad , \\ \hat{\xi}^2 &= \frac{k_1}{2}yt^2 - \frac{2k_1}{9}y^3 - \frac{k_1}{6}y(x^2 + z^2) - \frac{k_1}{4b^2}y \quad , \\ \hat{\xi}^3 &= \frac{k_1}{2}zt^2 - \frac{2k_1}{9}z^3 - \frac{k_1}{6}z(x^2 + y^2) - \frac{k_1}{4b^2}z \quad , \\ \hat{\xi}^0 &= \frac{k_1}{2}t(x^2 + y^2 + z^2) + \frac{3k_1}{4b^2}t \quad , \end{aligned} \tag{17.106}$$

the perturbative metric is brought to the now transverse-traceless form

$$\begin{aligned} h_{00} &= -\frac{3k_1}{2b^2} - k_1(x^2 + y^2 + z^2)e^{-2b|w|} \quad , \\ h_{11} &= h_{22} = h_{33} = -\frac{k_1}{2b^2} - \frac{k_1}{3}(x^2 + y^2 + z^2)e^{-2b|w|} \quad , \\ h_{12} &= \frac{k_1}{3}xye^{-2b|w|} \quad , \quad h_{23} = \frac{k_1}{3}yze^{-2b|w|} \quad , \quad h_{31} = \frac{k_1}{3}xze^{-2b|w|} \quad , \\ h_{01} &= h_{02} = h_{03} = 0 \quad . \end{aligned} \tag{17.107}$$

With k_1 being given by $k_1 = 2b^2 A_1/\lambda = \kappa_5^2 bp_m$ in lowest order, we recognize Eq. (17.107) as being none other than the perturbative solution given earlier as Eq. (17.96). With the deriving of this solution, we see that as well as providing us with a nice consistency check on our entire treatment of axial gauge, transverse-traceless fluctuations, we also learn that the metric of Eq. (17.96) is one in which, despite the presence of the perturbation, the bulk is still exact AdS_5. Thus despite the fact that an RW brane geometry need not in general embed in an exact AdS_5 invariant bulk, we find that for perturbative RW fluctuations around an M_4^+ brane world, the AdS_5 symmetry of the background AdS_5 is not in fact broken. Now in the unbounded solution of Eq. (17.84), it can readily be shown that the AdS_5 invariance of the bulk actually is broken.[29] However, as we noted earlier, this non-AdS_5 invariant solution is forbidden since a non-causal propagator is required for its production. Hence, and quite remarkably, putting a perturbative RW cosmology on an M_4^+ brane does not lower the symmetry of the bulk.

With the M_4^+ bulk being AdS_5 both before and after the introduction of the RW brane perturbation, it must be impossible for a bulk observer to ascertain that the perturbation had ever been introduced. Hence the solution of Eq. (17.96) must be pure gauge in the bulk. Thus on defining

$$\begin{aligned}
\hat{\xi}^0 &= -\frac{3k_1}{4}t(x^2+y^2+z^2) - \frac{k_1}{4}t^3 \ , \quad \hat{\xi}^1 = -\frac{k_1}{36}x(x^2+3y^2+3z^2+27t^2) \ , \\
\hat{\xi}^2 &= -\frac{k_1}{36}y(3x^2+y^2+3z^2+27t^2) \ , \quad \hat{\xi}^3 = -\frac{k_1}{36}z(3x^2+3y^2+z^2+27t^2) \ , \\
\hat{\xi}^5 &= \frac{k_1}{4b}(x^2+y^2+z^2+3t^2) \ ,
\end{aligned} \tag{17.108}$$

we indeed find that Eq. (17.96) can explicitly be written in the form of gauge transformation of Eq. (14.26), viz.

$$h_{\mu\nu}(|w|,x^\lambda) = -\frac{1}{b}\partial_\mu\partial_\nu\hat{\xi}^5 - 2be^{-2b|w|}\eta_{\mu\nu}\hat{\xi}^5 - e^{-2b|w|}[\eta_{\mu\rho}\partial_\nu\hat{\xi}^\rho + \eta_{\nu\rho}\partial_\mu\hat{\xi}^\rho] \ , \tag{17.109}$$

so that a bulk observer is indeed unable to detect the presence of the RW perturbation on the brane. However, since the introduction of the RW brane puts physical energy density and pressure on the brane, there has to be some observable effect. Now as we noted in Chapter 14, while the gauge transformation $\xi^5 = \epsilon(w)\hat{\xi}^5(x^\lambda)$, $\xi^\mu = -(1/2b)e^{2b|w|}\eta^{\mu\nu}\partial_\nu\hat{\xi}^5(x^\lambda) + \hat{\xi}^\mu(x^\lambda)$ does implement the transformation of Eq. (17.108), it also causes h_{55} to transform into $\bar{h}_{55} = h_{55} + 4\delta(w)\hat{\xi}^5(x^\lambda)$. The gauge transformation of Eq. (17.108) will thus take the geometry on the brane out of the axial gauge, with a brane observer being able to detect $\bar{h}_{55} = 4\delta(w)\hat{\xi}^5(x^\lambda)$. This though is the only effect of an RW brane perturbation, so that in and of itself, an RW brane perturbation does not undo the localized gravity already present in the M_4^+ brane-world background. Having now completed our analysis of the M_4^+ brane world, we turn next to the M_4^- brane world where we shall find complete bases of varying signature, but with it being the graviton which will now emerge as a ghost.

[29]Because of the $e^{2b|w|}$ factor in the solution of Eq. (17.84), the perturbative Weyl tensor projection $\delta\bar{E}_{\mu\nu} = (1/2)[\partial^2/\partial|w|^2 + 2b\partial/\partial|w|]h_{\mu\nu}$ of Eq. (14.36) is found not to vanish.

Chapter 18

Fluctuations around an Embedded Negative-Tension Minkowski Brane

18.1 M_4^- mode basis

While the determination of the TT mode spectrum associated with a negative-tension M_4^- brane world follows the discussion given in the M_4^+ case, because it possesses a divergent warp factor there will be some substantive conceptual differences. For M_4^- we set $A(|w|) = +b|w|$ in Eqs. (16.4) and (16.5), to obtain

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{-2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = 0 \quad , \tag{18.1}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2b\right] h_{\mu\nu}^{TT} = 0 \quad . \tag{18.2}$$

Separable TT mode solutions to these equations with separation constant m^2 take the form $h_{\mu\nu}^{TT} = 2\kappa_5 f_m(|w|)e_{\mu\nu}(p^\lambda, m)e^{ip\cdot x}/(2p^0)^{1/2}L^{3/2}$ + c.c. where $(p^0)^2 - \bar{p}^2 = m^2$, $\eta^{\mu\nu}e_{\mu\nu} = 0$, $p_\mu e^{\mu\nu} = 0$, with $f_m(|w|)$ having to obey

$$\left[\frac{d^2}{d|w|^2} - 4b^2 + e^{-2b|w|}m^2\right] f_m(|w|) = 0 \quad . \tag{18.3}$$

Under the change of variable $y = me^{-b|w|}/b$ Eq. (18.3) can be brought to the Bessel equation form

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + 1 - \frac{4}{y^2}\right] f_m(y) = 0 \quad . \tag{18.4}$$

For positive separation constant m^2 the solutions to Eq. (18.3) can thus be written in terms of Bessel functions, viz.

$$f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y) \quad , \tag{18.5}$$

where α_m and β_m are y-independent constants, while for $m^2 = 0$ the solution is given directly as

$$f_0(y) = \alpha_0 e^{-2b|w|} + \beta_0 e^{2b|w|} \quad . \tag{18.6}$$

For $m^2 = -\mu^2 < 0$ the change of variable $\hat{y} = \mu e^{-b|w|}/b$ brings Eq. (18.3) to the form

$$\left[\frac{d^2}{d\hat{y}^2} + \frac{1}{\hat{y}}\frac{d}{d\hat{y}} - 1 - \frac{4}{\hat{y}^2}\right] f_\mu(\hat{y}) = 0 \quad , \tag{18.7}$$

with solution

$$f_\mu(\hat{y}) = \alpha_\mu I_2(\hat{y}) + \beta_\mu K_2(\hat{y}) \quad . \tag{18.8}$$

With the Bessel functions obeying the recurrence relations

$$\begin{aligned} \frac{dJ_2(y)}{dy} &= J_1(y) - \frac{2}{y}J_2(y) \ , \quad \frac{dY_2(y)}{dy} = Y_1(y) - \frac{2}{y}Y_2(y) \ , \\ \frac{dI_2(\hat{y})}{d\hat{y}} &= I_1(\hat{y}) - \frac{2}{\hat{y}}I_2(\hat{y}) \ , \quad \frac{dK_2(\hat{y})}{d\hat{y}} = -K_1(\hat{y}) - \frac{2}{\hat{y}}K_2(\hat{y}) \ , \end{aligned} \tag{18.9}$$

the various solutions thus obey

$$\left[\frac{d}{d|w|} - 2b\right] f_m = -me^{-b|w|}[\alpha_m J_1(y) + \beta_m Y_1(y)] \quad , \tag{18.10}$$

$$\left[\frac{d}{d|w|} - 2b\right] f_0 = -4b\alpha_0 e^{-2b|w|} \quad , \tag{18.11}$$

$$\left[\frac{d}{d|w|} - 2b\right] f_\mu = -\mu e^{-b|w|}[\alpha_\mu I_1(\hat{y}) - \beta_\mu K_1(\hat{y})] \quad . \tag{18.12}$$

Satisfying the junction condition of Eq. (18.2) at the brane thus requires

$$\alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0 \quad , \tag{18.13}$$

$$\alpha_0 = 0 \quad , \tag{18.14}$$

$$\alpha_\mu I_1(\mu/b) - \beta_\mu K_1(\mu/b) = 0 \tag{18.15}$$

in the various possible cases.

Unlike the M_4^+ case where the argument of the Bessel functions (viz. $y = me^{b|w|}/b$) becomes large when $|w|$ is large, in the M_4^- case the $y = me^{-b|w|}/b$ argument becomes very small at large w, a region where the $Y_2(y)$, $e^{2b|w|}$ and $K_2(\hat{y})$ modes then diverge. Consequently, the various M_4^- solutions behave asymptotically not in analog to Eq. (17.16), but rather as

$$\begin{aligned} f_m &\to \frac{\alpha_m y^2}{8} - \frac{4\beta_m}{\pi y^2} \ , \\ f_0 &\to \beta_0 e^{2b|w|} \ , \\ f_\mu &= \frac{\alpha_\mu y^2}{8} + \frac{2\beta_\mu}{y^2} \end{aligned} \tag{18.16}$$

at large $|w|$. With the momentum flux of Eq. (16.69) being given by

$$\begin{aligned} T_{5\mu} &= \frac{e^{-4b|w|}}{2p^0L^3}\eta^{\alpha\delta}\eta^{\beta\gamma}\left(e^{2ip\cdot x}-e^{-2ip\cdot x}\right)ip_\mu e_{\delta\gamma}e_{\alpha\beta} \\ &\quad\times \epsilon(w)f_m(|w|)\left(\frac{\partial}{\partial|w|}-2b\right)f_m(|w|) \end{aligned} \tag{18.17}$$

in an M_4^- mode, we see that momentum flux $g^{1/2}T^{5N}K_N = -e^{4b|w|}T_{50}$ associated with the M_4^- TT fluctuation mode energy-momentum tensor will only vanish asymptotically for the convergent $J_2(y)$ and $I_2(\hat{y})$ modes (modes which behave asymptotically as $e^{-2b|w|}$), while additionally actually doing so for the $e^{2b|w|}$ graviton mode as well, since even though this mode is highly divergent, it nonetheless has a wave function for which $(\partial_{|w|}-2b)f_0(|w|)$ just happens to vanish identically. However, since $I_1(\mu/b)$ has no zeroes on the real axis, the only allowed M_4^- mode solutions which satisfy the junction condition at the brane and have vanishing asymptotic momentum flux are therefore given by modes which obey

$$J_1(m_i/b=j_i)=0 \quad , \tag{18.18}$$

(a thus discrete spectrum of modes with masses m_i/b given by the zeroes, j_i, of the J_1 Bessel function), together with the massless graviton mode $\beta_0 e^{2b|w|}$ which satisfies the junction condition for arbitrary β_0. All of these particular modes thus have a time-independent energy given by

$$E = 2p^0\eta^{\alpha\delta}\eta^{\beta\gamma}e_{\alpha\beta}e_{\delta\gamma}\int_0^\infty d|w|e^{-2b|w|}f_m^2(|w|) \quad . \tag{18.19}$$

Of them, the $J_2(m_ie^{-b|w|}/b)$ modes are found to have an energy which is finite. The massless M_4^- graviton, however, is found to have an energy which is infinite, to thus exclude it from the normalizable mode spectrum. Normalizability thus excludes the massless graviton in the M_4^- brane world (a point we return to below), leaving as normalizable a discrete spectrum of $J_2(m_ie^{-b|w|}/b)$ modes alone, with each such mode having a wave function which, by behaving asymptotically as $e^{-2b|w|}$, is not only localized to the brane, but which is as localized to the brane as the most convergent of the modes of the M_4^+ brane world.

For the $J_2(me^{-b|w|}/b)$ modes with masses which obey $J_1(m/b)=0$ we have (on setting $x=e^{-b|w|}/b$)

$$\begin{aligned} &\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)J_2(m'e^{-b|w|}/b) = b\int_0^{1/b}dx xJ_2(mx)J_2(m'x) \\ &= bx\left[\frac{mJ_1(mx)J_2(m'x)-m'J_1(m'x)J_2(mx)}{(m'^2-m^2)}\right]\Bigg|_0^{1/b} \\ &= b\left[\frac{mJ_1(m/b)J_2(m'/b)-m'J_1(m'/b)J_2(m/b)}{(m'^2-m^2)}\right]=0 \end{aligned} \tag{18.20}$$

when m is not equal to m', while for a single m we have

$$\int_0^\infty d|w|e^{-2b|w|}J_2^2(me^{-b|w|}/b) = b\int_0^{1/b} dxxJ_2^2(mx)$$
$$= b\frac{x^2}{2}\left[J_2^2(mx) - J_1(mx)J_3(mx)\right]\Big|_0^{1/b} = \frac{J_2^2(m/b)}{2b} \quad . \tag{18.21}$$

With normalization

$$f_i(|w|) = \frac{b^{1/2}J_2(j_ie^{-b|w|})}{J_2(j_i)} \quad , \tag{18.22}$$

the $f_i(|w|)$ modes thus form an orthonormal basis which obeys

$$2\int_0^\infty d|w|e^{-2b|w|}f_i(|w|)f_{i'}(|w|) = \delta_{i,i'} \quad , \tag{18.23}$$

with the associated completeness relation thus taking the form

$$\sum_i f_i(|w|)f_i(|w'|) = e^{2b|w|}\delta(w - w') \quad . \tag{18.24}$$

Then, with the four-dimensional Minkowski spacetime retarded propagator of Eq. (16.75) being unconnected to the sign of the brane tension, the normalized mode TT propagator for the M_4^- brane world is given by the $G^{TT}(x, x', w, w')$ propagator of Eq. (16.76) as reckoned with the propagator of Eq. (16.75) and the modes of Eq. (18.22).

18.2 Completeness test for normalizable M_4^- basis modes

While the above discrete basis of modes is by construction the complete set of normalizable modes which obey the junction and vanishing asymptotic momentum flux conditions, it is nonetheless instructive (and reassuring) to test for this completeness directly. We thus need to determine whether it is possible to expand the localized square step $V_J = \hat{V}$, $\alpha \le e^{-b|w|}/b \le \beta$, $V_J = 0$ otherwise in terms of the modes of this basis, viz. we seek to find a set of V_m from which we can reconstruct the square step according to

$$V_J(|w|) = \sum_m V_m J_2(me^{-b|w|}/b) \quad . \tag{18.25}$$

To achieve this objective we apply $\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)$ to Eq. (18.25), and find through the use of Eqs. (18.20) and (18.21) that $V_J(|w|)$ is to be given by

$$V_J(|w|) = \sum_m \frac{2bB_m}{J_2^2(m/b)}J_2(me^{-b|w|}/b) \quad , \tag{18.26}$$

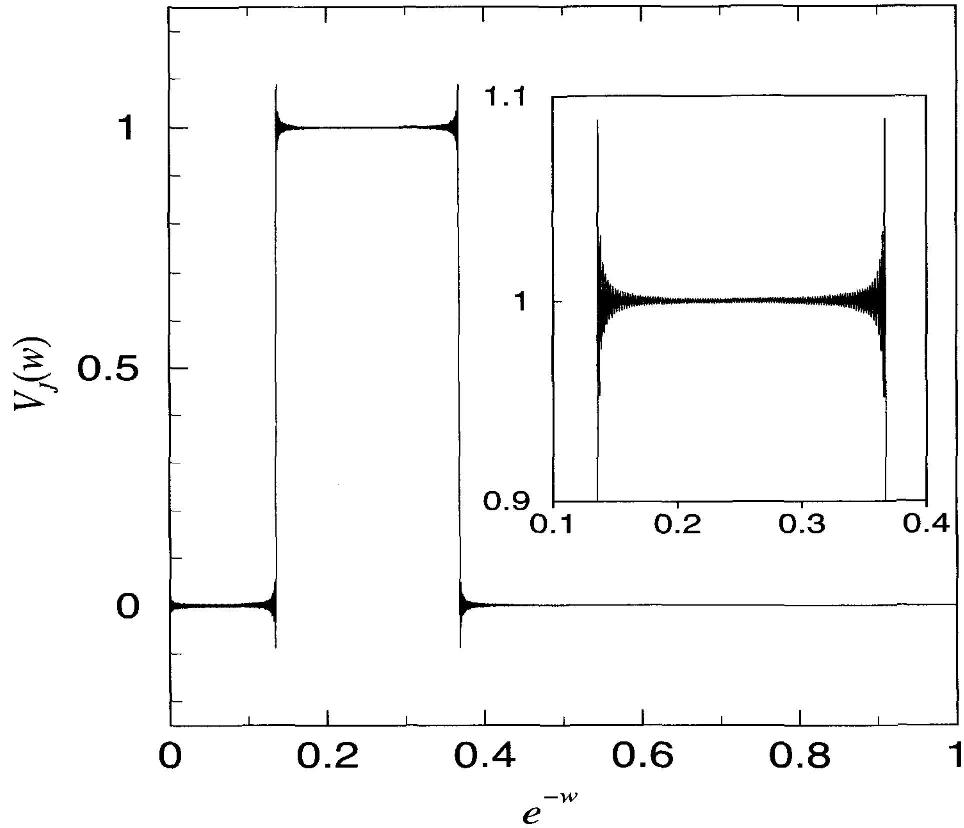

Fig. 18.1 Test of completeness of the discrete $J_2(j_i e^{-b|w|})$ TT mode basis by reconstructing the square step $V_J(|w|) = 1$, $1 < |w| < 2$, $V_J = 0$ otherwise. The parameter b is set equal to one.

where

$$\begin{aligned} B_m &= \int_0^\infty d|w| e^{-2b|w|} V_J(|w|) J_2(me^{-b|w|}/b) = -b\hat{V} \int_\alpha^\beta x dx J_2(mx) \\ &= -\frac{b\hat{V}}{m^2} \int_{m\alpha}^{m\beta} [2J_1(x) - xJ_0(x)] = \frac{b\hat{V}}{m^2} [2J_0(x) + xJ_1(x)] \Big|_{m\alpha}^{m\beta} \\ &= \frac{b\hat{V}}{m^2} [2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2} [2J_0(m\alpha) + m\beta J_1(m\alpha)] \quad . \quad (18.27) \end{aligned}$$

The function given in Eq. (18.26) can readily be plotted and is displayed in Fig. (18.1) through use of the first 1000 modes in the sum.[1] As we see, the basis is indeed capable of generating the square step to very high accuracy, with its completeness thus being confirmed.

[1]The various completeness tests presented in this chapter are taken from [Guth, Kaiser, Mannheim and Nayeri (2004b)] and [Mannheim and Simbotin (2004)].

Moreover, as we see from the inset in the figure, the mode sum explicitly displays the Gibbs phenomenon associated with trying to fit a discontinuity with a complete basis, with there being an overshoot (to near $V_J = 1.1$ in the figure) at the top of the discontinuity and an accompanying undershoot at the bottom, an overshoot and undershoot which were found to get narrower (in $|w|$) as the number of modes in the sum was increased, but to never change in height, always reaching close to $V_J = 1.1$ in the figure, just as the Gibbs phenomenon requires.

18.3 Completeness test for non-normalizable M_4^- basis modes

Even though the above basis is mathematically interesting since it is composed of modes all of which are localized to the brane, such a basis is not of much physical interest since it does not contain any massless (or even near-massless for that matter) 4-dimensional graviton (the lowest lying mode of the $J_2(me^{-b|w|}/b)$ basis lies at a mass $m_1 = 3.832b$ and is actually as close to the second mode at mass $m_2 = 7.016b$ as it is to $m = 0$), and would thus lead to a totally unviable gravity on the brane.[2] It is thus of interest to explore whether it might be possible to relax the vanishing asymptotic momentum flux condition to see if the massless graviton could then be restored to the spectrum (though if it were to be restored it would be with a wave function which behaves like $\beta_0 e^{2b|w|}$, to thus not lead to localization of gravity to the brane at all.) Now as had been noted in Chapter 16, it is not actually normalizability which is the essential criterion for completeness. Rather, for a basis to be complete, the requisite condition is that its modes, even if not normalizable themselves, nonetheless serve as an expansion basis for configurations which are. Thus we shall now look to see whether we can expand the same square step given above in terms of an entirely different basis, one built out of modes which are not normalizable at all. By analog with the discrete $J_2(me^{-b|w|}/b)$ basis, we instead take a basis built out of the divergent massless graviton mode $\beta_0 e^{2b|w|}$ and the equally divergent $Y_2(ne^{-b|w|}/b)$ modes.[3] With the massless graviton already satisfying the junction condition, the $Y_2(ne^{-b|w|}/b)$ modes will do so also provided their masses

[2] As we had discussed in Chapter 14, without a massless TT mode such a gravity would be dominated by the massless NT modes at large distances ($r \gg 1/3.832b$) on the brane, leading to a long-distance gravity which would effectively be scalar rather than tensor. Specifically, while standard (unembedded) long-distance 4-dimensional gravity, and also long distance gravity on an embedded positive tension M_4 brane, are controlled by $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}(x, w=0) = -2b\kappa_5^2[S_{\mu\nu} - \eta_{\mu\nu}S/2]$, long-distance gravity on an embedded negative tension M_4 brane would instead be controlled by the NT $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}^{NT}(x, w=0) = -2b\kappa_5^2[S_{\mu\nu}^{NT} - \eta_{\mu\nu}S/2]$. While for a static brane source $S_{\mu\nu} = -M\delta_\mu^0\delta_\nu^0\eta_{00}\delta^3(x)$ this would lead to $\nabla^2 h_{00} = -b\kappa_5^2 M\delta^3(x)$, $\nabla^2 h_{ij} = -b\kappa_5^2 M\eta_{ij}\delta^3(x)$ in the M_4^+ brane-world case, it would instead lead to $\nabla^2 h_{00}^{NT} = (b\kappa_5^2 M/3)\delta^3(x)$, $\nabla^2 h_{ij}^{NT} = -(b\kappa_5^2 M/3)\eta_{ij}\delta^3(x) + (b\kappa_5^2 M/6\pi)(3x_i x_j/r^5 - \eta_{ij}/r^3)$ in the M_4^- case. Consequently, in contrast to standard gravity, in a normalizable M_4^- brane world the long-distance Newtonian gravity due to h_{00}^{NT} and the gravitational bending contribution generated by h_{ij}^{NT} would have opposite signs.

[3] We lump these particular modes together since $Y_2(ne^{-b|w|}/b)$ behaves as $e^{2b|w|}$ at small n, and for clarity use m_i to denote the zeroes of $J_1(m/b)$ and n_i to denote those of $Y_1(n/b)$.

obey

$$Y_1(n_i/b = y_i) = 0 \ , \tag{18.28}$$

where this time the y_i are the zeroes of Y_1, an equally infinite set of zeroes which interlace both the zeroes of J_1 and the zeroes of Y_2, with the first three positive ones being at 2.197, 5.430 and 8.596 and with the large ones being well-approximated by $y_n \approx (n - 1/4)\pi$.

To test for completeness of this basis we try to reconstruct the square step via the expansion

$$V_Y(|w|) = \sum_n V_n Y_2(ne^{-b|w|}/b) + V_0 e^{2b|w|} \ . \tag{18.29}$$

Since $Y_2(y)$ behaves as $-4/\pi y^2 - 1/\pi$ at small argument, the various coefficients in Eq. (18.29) will first have to be chosen so as to cancel the divergent piece of $V_Y(|w|)$, since at large $|w|$ we have

$$V_Y(|w|) \to e^{2b|w|}\left[V_0 - \frac{4b^2}{\pi}\sum_n \frac{V_n}{n^2}\right] - \frac{1}{\pi}\sum_n V_n \ ; \tag{18.30}$$

and so we constrain the coefficients to obey the two conditions

$$\frac{4b^2}{\pi}\sum_n \frac{V_n}{n^2} = V_0 \ , \quad \sum_n V_n = 0 \ . \tag{18.31}$$

In order to explicitly extract the coefficients in the expansion of Eq. (18.29) we have found it very convenient to apply $\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)$ to it, where the m/b are the zeroes of J_1 and not those of Y_1. With none of the $J_1(m/b)$ zeroes coinciding with any of the zeroes of $Y_1(n/b)$, the needed overlap integrals are given (on setting $x = e^{-b|w|}/b$) by

$$\begin{aligned}
&\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)Y_2(ne^{-b|w|}/b) = b\int_0^{1/b} dx x J_2(mx)Y_2(nx) \\
&= bx\left[\frac{nY_1(nx)J_2(mx) - mJ_1(mx)Y_2(nx)}{(m^2 - n^2)}\right]\Bigg|_0^{1/b} = \frac{2bm^2}{\pi n^2(n^2 - m^2)} \ ,
\end{aligned} \tag{18.32}$$

and

$$\begin{aligned}
&\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)e^{2b|w|} = \frac{1}{b}\int_0^{1/b}\frac{dx}{x}J_2(mx) \\
&= -\frac{1}{b}\int_0^{1/b} dx \frac{d}{dx}\left(\frac{J_1(mx)}{mx}\right) = \frac{1}{2b} \ ,
\end{aligned} \tag{18.33}$$

and are thus nicely finite. For the square step $V_Y(|w|) = \hat{V}$, $\alpha \le e^{-b|w|}/b \le \beta$,

$V_Y(|w|) = 0$ otherwise, the expansion coefficients must thus obey

$$\begin{aligned}\frac{V_0}{2b} + \frac{2b}{\pi}\sum_n V_n \frac{m^2}{n^2(n^2-m^2)} &= \frac{V_0}{2b} + \frac{2b}{\pi}\sum_n V_n \left[\frac{1}{(n^2-m^2)} - \frac{1}{n^2}\right] \\ &= \frac{2b}{\pi}\sum_n \frac{V_n}{(n^2-m^2)} = B_m \end{aligned} \tag{18.34}$$

for all m, where

$$\begin{aligned} B_m &= -b\hat{V}\int_\alpha^\beta dx x J_2(mx) = \frac{b\hat{V}}{m^2}[2J_0(mx) + mxJ_1(mx)]\bigg|_\alpha^\beta \\ &= \frac{b\hat{V}}{m^2}[2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2}[2J_0(m\alpha) + m\alpha J_1(m\alpha)] \ . \end{aligned} \tag{18.35}$$

With the B_m being given in closed form, Eq. (18.34) is thus a set of N equations for N unknowns and can be viewed as an eigenvalue equation for the V_n. The V_n coefficients can thus be found numerically, and lead, for the case of the first 1000 modes in the basis to the plot displayed in Fig. (18.2). As we thus see, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one, and is thus every bit as complete.[4] Non-normalizable as the massless M_4^- graviton might be, it is just as capable of belonging to a complete basis as the normalizable graviton mode of M_4^+.

Beyond the two discrete bases given above, we note that in analogy with the KK continuum plus massless graviton mode basis of the M_4^+ brane world (as used for the $J_2 + iY_2$ based and $J_2 - iY_2$ based propagators), in the M_4^- brane world we can thus also construct a basis associated with the massless graviton plus a KK continuum of modes with $Y_1(m/b)J_2(me^{-b|w|}/b) - J_1(m/b)Y_2(me^{-b|w|}/b)$ type wave functions, wave functions which in the M_4^- brane world are non-normalizable. Now within such a basis there will be a subset whose masses obey $Y_1(n_i/b) = 0$, with the subset consisting of the massless graviton together with modes $Y_2(n_i e^{-b|w|}/b)$ where $n_i/b = y_i$ are the y_i zeroes of the Y_1 Bessel function.[5] Since we have just shown that we can build a localized square step out of this subset, it thus follows that despite its lack of normalizability, the M_4^- graviton plus KK continuum mode type basis is complete also, and not just complete but possibly even overcomplete in addition. Moreover, given the existence of both $J_2 + iY_2$ and $J_2 - iY_2$ based propagators in the M_4^+ brane world, there will thus be two KK continuum plus massless graviton mode type propagators in M_4^- as well, to thus augment those which can be built on the discrete J_2 and Y_2 type bases introduced above. Given the existence then of no less than four candidate types of propagator, in order to determine which, if

[4]The reconstruction of the square step using the divergent mode basis is so good that the only perceptible difference between Figs. (18.1) and (18.2) is that in the region close to $e^{-w} = 0$ the $J_2(me^{-b|w|}/b)$ contribution is ever so slightly thicker. (The constraints of Eq. (18.31) force a more rapid convergence on the $Y_2(ne^{-b|w|}/b)$ mode sum.)

[5]At masses $m = by_i$, $[Y_1(m/b)J_2(me^{-b|w|}/b) - J_1(m/b)Y_2(me^{-b|w|}/b)]/[J_1^2(m/b) + Y_1^2(m/b)]$ reduces to $-Y_2(y_i e^{-b|w|})/J_1(y_i)$.

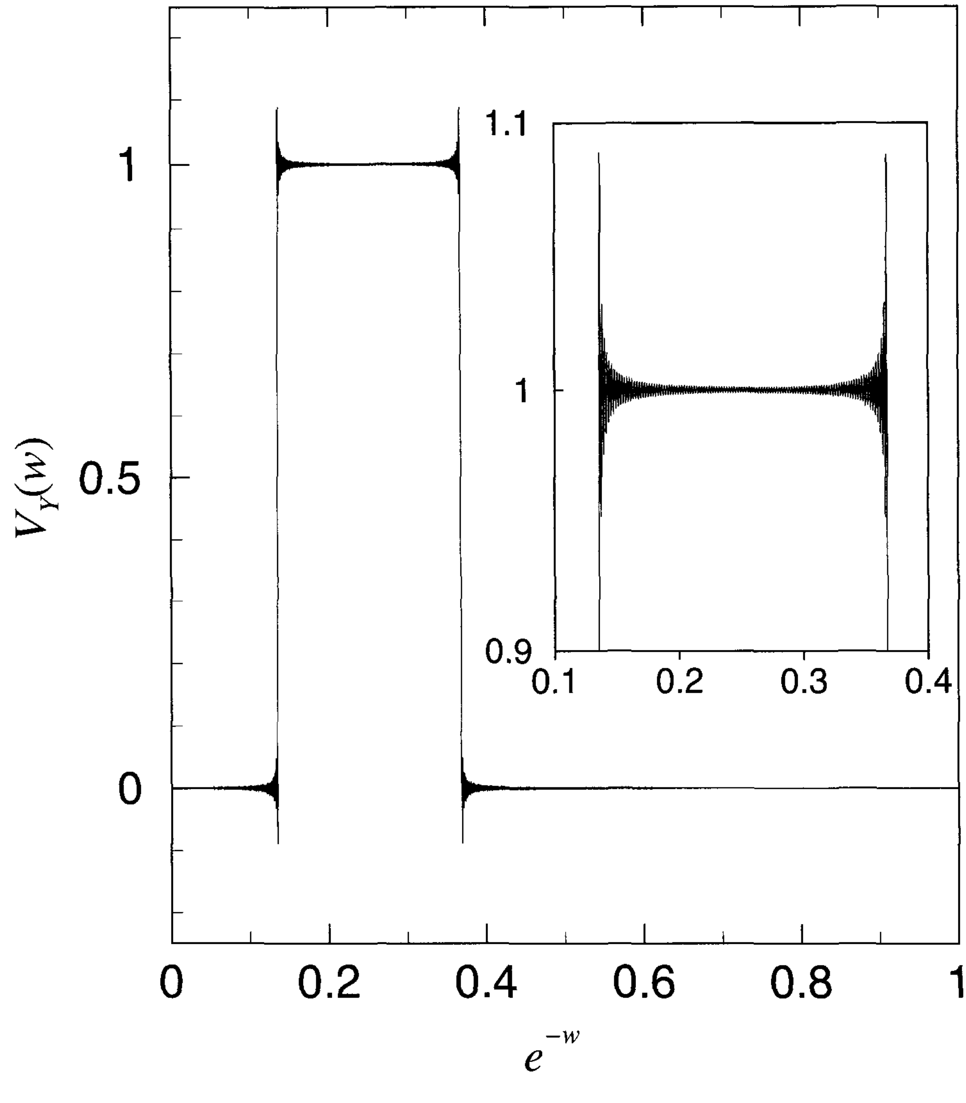

Fig. 18.2 Test of completeness of the discrete $Y_2(y_i e^{-b|w|})$ plus $e^{2b|w|}$ TT mode basis by reconstructing the square step $V_Y(|w|) = 1$, $1 < |w| < 2$, $V_Y = 0$ otherwise. The parameter b is set equal to one.

any, of them is to be physically preferred, we need to explore their causality and retardation properties. And as we shall now see, despite the lack of normalizability of its basis modes, the $J_2 - iY_2$ type propagator will prove to be causal, with its causality owing precisely to the above apparent overcompleteness of its basis.

18.4 Upper half plane determination of the M_4^- propagator

To explicitly explore causality in the M_4^- brane world we construct the M_4^- analog of Eq. (17.30), viz.

$$\begin{aligned} h_{\mu\nu}^{TT} &= \frac{\kappa_5^2}{(2\pi)^4}\int d^4x' d^4p e^{ip\cdot(x-x')}\frac{[\alpha_q J_2(qe^{-b|w|}/b)+\beta_q Y_2(qe^{-b|w|}/b)]}{q[\alpha_q J_1(q/b)+\beta_q Y_1(q/b)]}S_{\mu\nu}^{TT}(x') \\ &= -2\kappa_5^2\int d^4x'\hat{G}^{TT}(x,x',w,0)S_{\mu\nu}^{TT}(x') \ , \end{aligned} \tag{18.36}$$

with Eq. (18.36) readily being checked to be an exact solution to Eq. (16.1) for arbitrary α_q and β_q. Central to our discussion in the following will be the change in overall sign between Eqs. (17.30) and (18.36) due to the change in sign between the right-hand sides of Eqs. (17.10) and (18.10) occasioned by the change in the sign of the tension of the brane.[6] Paralleling the discussion leading to Eq. (17.40), along a half circle of radius $Pe^{i\theta}$ in the upper half p^0 plane we find the leading behavior

$$\begin{aligned} \frac{[J_2(p^0e^{-b|w|}/b)+iY_2(p^0e^{-b|w|}/b)]}{p^0[J_1(p^0/b)+iY_1(p^0/b)]} &\to -\frac{ie^{ip^0(e^{-b|w|}-1)/b}}{p^0e^{-b|w|/2}}\left[1-\frac{15b}{8ip^0e^{-b|w|}}+\frac{3b}{8ip^0}\right] , \\ \frac{[J_2(p^0e^{-b|w|}/b)-iY_2(p^0e^{-b|w|}/b)]}{p^0[J_1(p^0/b)-iY_1(p^0/b)]} &\to \frac{ie^{-ip^0(e^{-b|w|}-1)/b}}{p^0e^{-b|w|/2}}\left[1+\frac{15b}{8ip^0e^{-b|w|}}-\frac{3b}{8ip^0}\right] , \\ \frac{J_2(p^0e^{-b|w|}/b)}{p^0J_1(p^0/b)} &\to \frac{[e^{ip^0(e^{-b|w|}+1)/b}+ie^{-ip^0(e^{-b|w|}-1)/b}]}{p^0e^{-b|w|/2}[1+ie^{2ip^0/b}]}+\dots , \\ \frac{Y_2(p^0e^{-b|w|}/b)}{p^0Y_1(p^0/b)} &\to -\frac{[e^{ip^0(e^{-b|w|}+1)/b}-ie^{-ip^0(e^{-b|w|}-1)/b}]}{p^0e^{-b|w|/2}[1-ie^{2ip^0/b}]}+\dots \end{aligned} \tag{18.37}$$

for the four combinations of interest to us. For the J_2+iY_2 based combination the integration of Eq. (18.36) for the same illustrative source $S_{\mu\nu}^{TT}(x')=A_{\mu\nu}\delta(t')$ as used in Chapter 17 yields a leading upper half circle contribution of the form

$$\begin{aligned} & h_{\mu\nu}^{TT}(\mathrm{UHP};\mathrm{J_2+iY_2}) \\ &= \frac{\kappa_5^2A_{\mu\nu}e^{b|w|/2}}{2\pi}\int_0^{\pi} id\theta e^{-iPe^{i\theta}\alpha}(-i)\left[1-\frac{15be^{b|w|}}{8iPe^{i\theta}}+\frac{3b}{8iPe^{i\theta}}\right] , \end{aligned} \tag{18.38}$$

---

[6] As introduced in Eq. (18.36) the $\hat{G}^{TT}(x,x',w,0)$ propagator obeys the Green's function equation $[\partial_w^2-4b^2-4b\delta(w)+e^{-2b|w|}\eta^{\mu\nu}\partial_\mu\partial_\nu]\hat{G}^{TT}(x,x',w,0)=\delta(w)\delta^4(x-x')$, with the coefficient of the delta function term being the conventional plus one and not minus one. However, with this definition, $\hat{G}^{TT}(x,x',w,0)$ can be written as $\hat{G}^{TT}(x,x',w,0)=-(1/2(2\pi)^4)\int d^4pe^{ip\cdot(x-x')}[J_2(qe^{-b|w|}/b)+iY_2(qe^{-b|w|}/b)]/q[J_1(q/b)+iY_1(q/b)]$, a relation which, due to the change in sign of the tension of the brane, differs from Eq. (17.54) by an overall sign. In consequence of this, in the analysis presented below M_4^- basis modes will be found to have opposite signature to their M_4^+ counterparts.

where

$$\alpha = \frac{1}{b}(bt - e^{-b|w|} + 1) \ . \tag{18.39}$$

Then, on taking the $P \to \infty$ limit and using the integrals given in Appendix E, we obtain

$$\begin{aligned} -h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 + \mathrm{iY}_2) &= h^{TT}_{\mu\nu}(\mathrm{UHP}; \mathrm{J}_2 + \mathrm{iY}_2) \\ &= \kappa_5^2 A_{\mu\nu} e^{b|w|/2}\theta(\alpha)\left[1 + \frac{15be^{b|w|}\alpha}{8} - \frac{3b\alpha}{8} + O(\alpha^2)\right] \ . \end{aligned} \tag{18.40}$$

Similarly, in terms of the quantity

$$\beta = \frac{1}{b}(bt + e^{-b|w|} - 1) = \alpha + \frac{2e^{-b|w|}}{b} - \frac{2}{b} \ , \tag{18.41}$$

the analog expression for the $J_2 - iY_2$ based combination takes the form

$$\begin{aligned} -h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 - \mathrm{iY}_2) &= h^{TT}_{\mu\nu}(\mathrm{UHP}; \mathrm{J}_2 - \mathrm{iY}_2) \\ &= -\kappa_5^2 A_{\mu\nu} e^{b|w|/2}\theta(\beta)\left[1 - \frac{15be^{b|w|}\beta}{8} + \frac{3b\beta}{8} + O(\beta^2)\right] \ . \end{aligned} \tag{18.42}$$

With the M_4^- brane-world lightcone and its interior being given by $|t| \geq |1/b - e^{-b|w|}/b|$, i.e. by $\beta > 0$ for $t > 0$, in the M_4^- brane world, and in sharp contrast to the M_4^+ brane world, this time it is the incoming $J_2 - iY_2$ based propagator which is causal and retarded,[7] and the outgoing $J_2 + iY_2$ based one which is not (since $e^{-b|w|} - 1$ is never positive, α can be positive even when β is negative). This switch over between the roles of the outgoing and incoming travelling waves is a reflection of the fact that the M_4^- brane world is a globally non-hyperbolic spacetime in which information can come in from infinity in a finite time.

Despite the fact that the $J_2 - iY_2$ based propagator is causal, we see that with its overall $e^{b|w|/2}$ prefactor, this particular propagator does not lead to an $h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 - \mathrm{iY}_2)$ which is localized to the brane. This is of course to be expected since the M_4^- massless graviton and KK continuum modes associated with $h^{TT}_{\mu\nu}(\mathrm{SING}; \mathrm{J}_2 - \mathrm{iY}_2)$ all blow up at spatial infinity with none of them being normalizable. Beyond the phenomenological fact that the fluctuations in the $J_2 - iY_2$ sector are not localized to the brane, of altogether more serious concern is the fact that there is a growth of $h^{TT}_{\mu\nu}(\mathrm{RET}; \mathrm{J}_2 - \mathrm{iY}_2)$ with $|w|$ at all, as it entails that the $h^{TT}_{\mu\nu}$ fluctuations grow without bound. Consequently, this sector of the M_4^- brane world is not stable against small perturbations. However, before concluding that M_4^- is unstable, we must first look at the J_2 based propagator since it is built out of normalizable modes, modes which are much better behaved at infinity.

[7]For points with $e^{-b|w|} < 1$, the positivity of β required by $\theta(\beta)$ entails the positivity of t.

For the J_2 (and Y_2) based combinations we obtain leading p^0 plane upper half circle contributions of a form analogous to those given in Eq. (17.46), viz.

$$\begin{aligned}
-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2(\mathrm{Y}_2)) &= h^{TT}_{\mu\nu}(\mathrm{UHP};\mathrm{J}_2(\mathrm{Y}_2)) \\
&= -\kappa_5^2 A_{\mu\nu} e^{b|w|/2}[\mp i\theta(bt-e^{-b|w|}-1) \;+\theta(bt+e^{-b|w|}-1) \\
&\qquad -\theta(bt-e^{-b|w|}-3) \;\mp i\theta(bt+e^{-b|w|}-3) \\
&\qquad \pm i\theta(bt-e^{-b|w|}-5) \;-\theta(bt+e^{-b|w|}-5) \\
&\qquad +\theta(bt-e^{-b|w|}-7) \;\pm i\theta(bt+e^{-b|w|}-7)+\ldots.] \ . \qquad (18.43)
\end{aligned}$$

As with the M_4^+ case, the above expressions nicely display the appropriate step function domains, but, unlike the analog M_4^+ case, for M_4^- we find a totally different outcome. First of all we note that since the quantity $e^{-b|w|}$ can never be greater than one, both $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2)$ vanish identically when t is negative, with both of these propagators thus being retarded. Secondly, both propagators vanish when t is zero and $|w|$ is non-zero, so that with the M_4^- lightcone and its interior being given by $bt \geq 1-e^{-b|w|}$, at time $t=0$ neither propagator takes support outside the lightcone. Thirdly, we note that in the range $0 < bt < 1$, the only step function which contributes to the above propagators is the $\theta(bt+e^{-b|w|}-1)$ one, with both of the propagators being given by $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2(\mathrm{Y}_2)) = \kappa_5^2 A_{\mu\nu} e^{b|w|/2}\theta(bt+e^{-b|w|}-1)$ in this particular time interval, an expression which is completely causal. Moreover, not only is this expression causal, it agrees completely with the value that $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$ takes in the same time interval since $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2) = \kappa_5^2 A_{\mu\nu} e^{b|w|/2}\theta(bt+e^{-b|w|}-1)$ to leading order.[8] Now there has to be an equivalence such as this since a theory can only possess one causal propagator. However, an initial glance at the various propagators at times $bt > 1$ might suggest a possible difference between the various propagators at later times, since while $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$ continues to be given by Eq. (18.42), at times $bt > 1$ other step functions in $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2(\mathrm{Y}_2))$ begin to contribute as well. With inspection showing that these contributions are also causal, it must be the case that on their own, all of these additional terms must obey not the inhomogeneous wave equation with a source, but rather a source-free homogeneous wave equation instead. Hence any difference between $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)$ (or $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2)$) must itself be a fully retarded, fully causal solution to the source-free wave equation.[9] Consequently, with the full content of the theory being contained in the sector which couples to the source, the unique, causal and retarded response of the M_4^- brane world to a $\delta(t)$ source on the brane is given in leading order by $h^{TT}_{\mu\nu}(\mathrm{RET}) = \kappa_5^2 A_{\mu\nu} e^{b|w|/2}\theta(bt+e^{-b|w|}-1)$, and in all orders by $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$. Moreover, even without causality considerations, as

[8]In Appendix F where we determine the analog M_3^- propagators exactly, we explicitly show that the analog M_3^- results hold as exact properties of the theory.

[9]In Appendix F we demonstrate explicitly how this achieved in the M_3^- brane world.

far as $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)$ itself is concerned, as is seen from Eq. (18.43), the diverging $e^{b|w|/2}$ prefactor is present in it anyway, and so even if it were to be the only causal propagator in the theory it would still lead to instability and lack of localization of gravity.[10] Thus all causal propagators are unstable, and remain so even if we drop their homogeneous parts. The restriction to the normalizable sector alone is thus not seen to engender localization around the brane or stability of the fluctuations in the divergent warp factor case. And with it anyway being $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$ which contains the non-trivial coupling to the source, and with the massless graviton being contained in its singular part, there is no reason to leave out the massless graviton in brane worlds in which it is not normalizable.

18.5 Lower half plane determination of the M_4^- propagator

To get further insight into the structure we have just found for the M_4^- brane world, we look now at the evaluation of the various propagators in the lower half p^0 plane where the singularities of the propagators specifically reside. The evaluation of Eq. (18.36) when closed in the lower half p^0 plane proceeds just as in Chapter 17, and for the J_2+iY_2 based propagator yields

$$\begin{aligned}-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)&=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2+\mathrm{iY}_2)\\&=-\kappa_5^2A_{\mu\nu}e^{b|w|/2}\theta(-\alpha)\left[1+\frac{15b\alpha e^{b|w|}}{8}-\frac{3b\alpha}{8}+O(\alpha^2)\right]\ .\end{aligned}\tag{18.44}$$

Consequently, on combining Eqs. (18.40) and (18.44) we obtain

$$h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)=-\kappa_5^2A_{\mu\nu}e^{b|w|/2}\left[1+\frac{15b\alpha e^{b|w|}}{8}-\frac{3b\alpha}{8}+O(\alpha^2)\right]\ ,\tag{18.45}$$

and thus

$$h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2+\mathrm{iY}_2)=\theta(\alpha)h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2)\ ,\tag{18.46}$$

[10] Even while every one of the normalizable $J_2(me^{-b|w|}/b)$ modes of Eq. (18.22) behaves as $m^2e^{-2b|w|}/b^2$ at small Bessel function argument, such a behavior is only of relevance when m is held fixed and $|w|$ is allowed to go to infinity. However, for any given value of $|w|$ there will always be some m which are of order $be^{b|w|}$ and then an infinite number of m which are a lot larger. With the sum over the entire infinite set of discrete modes thus being one for which the argument of $J_2(me^{-b|w|}/b)$ is not always small at any given large $|w|$, the infinite sum is not automatically suppressed at large $|w|$, with both the analysis above and the numerical study presented below explicitly showing that the sum actually diverges as $e^{b|w|/2}$ for a $\delta(t)$ source on the brane.

to thus allow a construction of the $J_2 + iY_2$ based propagator entirely in terms of its singularities. Similarly, for the $J_2 - iY_2$ based combination we obtain

$$
\begin{aligned}
&-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2-\mathrm{iY}_2)\\
&=\frac{\kappa_5^2A_{\mu\nu}e^{b|w|/2}}{2\pi}\int_{\pi}^{0}id\theta e^{+iPe^{i\theta}\beta}(-1)(-i)\left[1-\frac{15be^{b|w|}}{8iPe^{i\theta}}+\frac{3b}{8iPe^{i\theta}}\right]\\
&=\kappa_5^2A_{\mu\nu}e^{b|w|/2}\theta(-\beta)\left[1-\frac{15b\beta e^{b|w|}}{8}+\frac{3b\beta}{8}+O(\beta^2)\right] \ ,
\end{aligned}
\tag{18.47}
$$

to thus yield

$$
h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)=\kappa_5^2A_{\mu\nu}e^{b|w|/2}\left[1-\frac{15b\beta e^{b|w|}}{8}+\frac{3b\beta}{8}+O(\beta^2)\right] \ ,
\tag{18.48}
$$

and

$$
h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)=\theta(\beta)h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2) \ ,
\tag{18.49}
$$

to thus allow a construction of the $J_2 - iY_2$ based propagator entirely in terms of its singularities as well, with $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2-\mathrm{iY}_2)$ not taking support outside the M_4^- lightcone even while $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)$ itself does (no step function in Eq. (18.48)).

Further, in analog to Eqs. (17.52) and (17.53), for the J_2 and Y_2 based propagators we obtain the leading behavior

$$
\begin{aligned}
&-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2)\\
&=-\kappa_5^2A_{\mu\nu}e^{b|w|/2}[\ -i\theta(-bt-e^{-b|w|}-1)+\theta(-bt+e^{-b|w|}-1)\\
&\qquad-\theta(-bt-e^{-b|w|}-3)-i\theta(-bt+e^{-b|w|}-3)\\
&\qquad+i\theta(-bt-e^{-b|w|}-5)-\theta(-bt+e^{-b|w|}-5)\\
&\qquad+\theta(-bt-e^{-b|w|}-7)+i\theta(-bt+e^{-b|w|}-7)+....] \ ,
\end{aligned}
\tag{18.50}
$$

and

$$
\begin{aligned}
&-h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2)+h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2)=h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{Y}_2)\\
&=-\kappa_5^2A_{\mu\nu}e^{b|w|/2}[\ -\frac{i}{3}\theta(-bt-e^{-b|w|}-1)+\theta(-bt+e^{-b|w|}-1)\\
&\qquad-\frac{1}{9}\theta(-bt-e^{-b|w|}-3)-\frac{i}{3}\theta(-bt+e^{-b|w|}-3)\\
&\qquad+\frac{i}{27}\theta(-bt-e^{-b|w|}-5)-\frac{1}{9}\theta(-bt+e^{-b|w|}-5)\\
&\qquad+\frac{1}{81}\theta(-bt-e^{-b|w|}-7)+\frac{i}{27}\theta(-bt+e^{-b|w|}-7)+...] \ .
\end{aligned}
\tag{18.51}
$$

With $h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{Y}_2)$ both being seen to be causal in the $t<0$ region, the vanishing of the retarded $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2)$ at negative t entails that $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2)$ themselves are causal in the $t<0$ region. Additionally, with $h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{LHP};\mathrm{Y}_2)$ both being found to vanish identically in the $t>0$ region ($e^{-b|w|}$ is never greater than one), in this region then $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)$ and $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2)$ are both directly related (i.e. without the presence of any causal type step function such as the one exhibited in Eq. (18.49)) to their corresponding retarded propagators according to

$$\begin{aligned} h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2) &= \theta(t)h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2) \ , \\ h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{Y}_2) &= \theta(t)h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2) \ , \end{aligned} \tag{18.52}$$

and thus both are causal in the $t>0$ region too. Hence, unlike $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)$ which is not itself causal, neither $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)$ or $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{Y}_2)$ takes support anywhere outside the M_4^- lightcone.

18.6 Negative signature modes in the M_4^- propagator

As well as study the causal structure of the propagators, we also need to determine whether there might be any states with negative signature, and so to look for possible ghosts we need to perform the contour integrals in Eq. (18.36) in the various cases of interest. Such calculations completely parallel those performed in the analog M_4^+ brane world, and in the M_4^- brane world lead to singular contributions to the propagators of interest of the form

$$\begin{aligned} &\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2+\mathrm{iY}_2) = ibe^{2b|w|}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|}\left[e^{-i|p|t}-e^{i|p|t}\right] \\ &+\frac{i}{(2\pi)^3}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt}-e^{iE_pt}\right] \\ &\times\int dm\left[\frac{[Y_1(m/b)J_2(me^{-b|w|}/b)-J_1(m/b)Y_2(me^{-b|w|}/b)]}{\pi[J_1^2(m/b)+Y_1^2(mb)]}\right] \ , \end{aligned} \tag{18.53}$$

$$\begin{aligned} &\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2) = ibe^{2b|w|}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|}\left[e^{-i|p|t}-e^{i|p|t}\right] \\ &-\frac{i}{(2\pi)^3}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt}-e^{iE_pt}\right] \\ &\times\int dm\left[\frac{[Y_1(m/b)J_2(me^{-b|w|}/b)-J_1(m/b)Y_2(me^{-b|w|}/b)]}{\pi[J_1^2(m/b)+Y_1^2(mb)]}\right] \ , \end{aligned} \tag{18.54}$$

$$\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2) = -i\sum_i f_i(|w|)f_i(0)\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_i}\left[e^{-iE_it}-e^{iE_it}\right] \ , \tag{18.55}$$

where

$$f_i(|w|) = \frac{b^{1/2}J_2(j_ie^{-b|w|})}{J_2(j_i)} \ , \quad E_i = (\bar{p}^2+b^2j_i^2)^{1/2} \ , \tag{18.56}$$

and

$$\begin{aligned}\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{Y}_2) &= ibe^{2b|w|}\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2|p|}\left[e^{-i|p|t}-e^{i|p|t}\right]\\ &-i\sum_i \tilde{f}_i(|w|)\tilde{f}_i(0)\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_i}\left[e^{-iE_it}-e^{iE_it}\right]\\ &+\frac{i}{(2\pi)^3}\int d^3p\frac{e^{i\bar{p}\cdot\bar{x}}}{2E_p}\left[e^{-iE_pt}-e^{iE_pt}\right]\int dm\left[1-2i\frac{J_2(me^{-b|w|}/b)}{Y_1(m/b)}\right]\\ &\times\left[\frac{[Y_1(m/b)J_2(me^{-b|w|}/b)-J_1(m/b)Y_2(me^{-b|w|}/b)]}{\pi[4J_1^2(m/b)+Y_1^2(mb)]}\right] \ ,\end{aligned} \tag{18.57}$$

where

$$\tilde{f}_i(|w|) = \frac{b^{1/2}Y_2(y_ie^{-b|w|})}{Y_2(y_i)} \ , \quad E_i = (\bar{p}^2+b^2y_i^2)^{1/2} \ . \tag{18.58}$$

Comparing these expressions with their M_4^+ counterparts, we see only two changes, the minor one of the replacement of $e^{b|w|}$ by $e^{-b|w|}$ as required by the change in the warp factor from $e^{-2b|w|}$ by $e^{2b|w|}$, and the major one of an overall change in sign due to the change in overall sign between Eqs. (17.30) and (18.36) brought about by the change in sign of the tension on the brane. However, in general agreement with the situation found in the M_4^+ case, we again find that modes which are not normalizable can nonetheless appear in the M_4^- propagators with finite weights.

Because of the change in the overall sign, and because the signature of each mode is fixed by the junction conditions at the brane, modes which were positive signatured in the M_4^+ brane world become ghost signatured in M_4^- and vice versa. In particular, the massless graviton which had been a normal, positive signatured state, now becomes an undesirable negative signatured one, while the discrete modes associated with the zeroes of J_1 and Y_1 now become positive signatured. Similarly, the J_2+iY_2 continuum now becomes ghost signatured, while the J_2-iY_2 continuum becomes positive signatured. Since all three of the J_2+iY_2, J_2-iY_2 and Y_2 based propagators possess the graviton pole, in M_4^- all three of them possess ghostlike states, with the ghost state heralding the existence of an instability, either the presence of fluctuations which grow without bound in the classical theory or the potential presence of negative norm Hilbert space states in the associated quantum theory.[11] This dichotomy between the M_4^+ and M_4^- brane worlds (a spatially converging causal sector metric in one case and a spatially diverging one in the other) is a reflection of the change in topology of the brane-world geometry from globally hyperbolic to globally non-hyperbolic, showing that globally hyperbolic and globally non-hyperbolic brane worlds can differ in quite substantial ways.

18.7 Temporal evolution of localized square steps in M_4^-

Further insight into the causal structure of the M_4^- brane world can be obtained by monitoring the temporal development of the two localized square steps we constructed earlier, one built out of the normalizable $f_i(|w|)$ modes which we now know to have positive signature in $\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2)$, and the other built out of the non-normalizable $\tilde{f}(|w|)$ modes and the non-normalizable graviton (a mode we now know to have ghost signature in $\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{Y}_2)$). Of these two discrete bases, the one associated with the J_2 based propagator is a completely standard, positive signatured, orthonormal basis with completely standard completeness relation of the form given in Eq. (18.24), viz.

$$\sum_i f_i(|w|)f_i(0) = \sum_i \frac{bJ_2(j_i e^{-b|w|})}{J_2(j_i)} = \delta(w) \quad , \tag{18.59}$$

and indeed Eq. (18.59) is nothing more than a limit of the square step constructed in Eq. (18.26).[12] However, even though the discrete basis associated with the poles of the Y_2 based propagator is complete also, it possesses no completeness relation of a form such as that given in Eq. (18.59). And indeed, in our construction of the square step given in Eq. (18.29), the various V_n coefficients do not at all behave like $b/Y_2(y_i)$ – rather they are determined by a complete diagonalization of Eq. (18.34).[13]

To monitor the time evolution of the steps, two types of time evolution are

[11] Apart from the overall signs of the graviton and continuum contributions to $\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)$, we note that these contributions also have opposite relative sign. Hence even the use of a propagator which would obey the unconventional $[\partial_w^2 - 4b^2 - 4b\delta(w) + e^{-2b|w|}\eta^{\mu\nu}\partial_\mu\partial_\nu]\hat{G}^{TT}(x,x',w,0) = -\delta(w)\delta^4(x-x')$ would still not yield a $\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2-\mathrm{iY}_2)$ which would be free of negative signature states.

[12] Indeed, we have even checked numerically that the sum of $bJ_2(j_i e^{-b|w|})/J_2(j_i)$ over the first 1000 positive zeroes of J_1 actually is shaped like a delta function, though one of course with a finite height $1000b$ at $w=0$.

[13] We have numerically constructed a delta function via Eq. (18.29) using the graviton and the first 1000 positive zeroes of Y_1, one again with a finite height. The resulting numerical coefficients were not found to behave anything like $b/Y_2(y_i)$. Additionally, the choice of coefficients $V_n = b/Y_2(y_i)$, $V_0 = -b$ was not found to obey the constraints of Eq. (18.31). The complete discrete Y_2 type basis is thus an example of a complete basis which does not obey a canonical completeness relation such as that exhibited in Eq. (18.59).

readily suggested, either

$$V_J(|w|,t) = \sum_m V_m J_2\left(\frac{me^{-b|w|}}{b}\right)\cos(mt) \quad , \tag{18.60}$$

and

$$V_Y(|w|,t) = \sum_n V_n Y_2\left(\frac{ne^{-b|w|}}{b}\right)\cos(nt) + V_0 e^{2b|w|} \quad , \tag{18.61}$$

or

$$V_J(|w|,t) = \theta(t)\sum_m V_m J_2\left(\frac{me^{-b|w|}}{b}\right)\frac{\sin(mt)}{m} \quad , \tag{18.62}$$

and

$$V_Y(|w|,t) = \sum_n V_n Y_2\left(\frac{ne^{-b|w|}}{b}\right)\frac{\sin(nt)}{n} + V_0 e^{2b|w|}t \quad . \tag{18.63}$$

Of these two types of evolution, both of Eqs. (18.60) and (18.61) are exact solutions to the source-free Eqs. (18.1) and (18.2) at all times, with the time dependence having been chosen so that both solutions reduce to the square steps of Eqs. (18.26) and (18.29) themselves at $t = 0$. The sinusoidal time dependence chosen in Eqs. (18.62) and (18.63) is motivated by the explicit time dependence exhibited in $\hat{G}^{TT}(x,0,w,0)(\text{SING};\text{J}_2)$ and $\hat{G}^{TT}(x,0,w,0)(\text{SING};\text{Y}_2)$,[14] while the $\theta(t)$ dependence in Eq. (18.62) is motivated by Eq. (18.52). As a consequence of this, the solution of Eq. (18.62) has a particularly significant property – when the mode sum is used to construct a delta function and yield a retarded TT fluctuation of the form

$$h^{TT}_{\mu\nu}(t,|w|;\text{RET}) = 2\kappa_5^2 b\theta(t)A_{\mu\nu}\sum_i \frac{J_2(j_i e^{-b|w|})}{J_2(j_i)}\frac{\sin(j_i bt)}{j_i b} \quad , \tag{18.64}$$

then due to the explicit $\theta(t)$ factor and the completeness relation given in Eq. (18.59), this particular solution is explicitly found to obey the wave equation of Eq. (16.1) as evaluated with the illustrative source $S_{\mu\nu}(x,t) = A_{\mu\nu}\delta(t)$ considered earlier. However, because there is no analogous completeness relation for the $\tilde{f}(|w|)$ modes, the delta function limit of $V_Y(|w|,t)$ of Eq. (18.63) would not satisfy Eq. (16.1) even if Eq. (18.63) were to be given a $\theta(t)$ factor. Without any such $\theta(t)$ factor Eq. (18.63) does however satisfy the source-free wave equation, and so we include it in our sample time evolutions anyway. And in fact, as regards the causality properties which were uncovered in a numerical study [Guth, Kaiser, Mannheim and Nayeri (2004b)], it was found that both of the two above $V_Y(|w|,t)$ yielded an identical outcome anyway. Similarly, the two $V_J(|w|,t)$ were also found to yield

[14] With reference to Eqs. (18.55) and (18.57), $i\int d^3x \int d^3p e^{i\bar{p}\cdot\bar{x}}\left[e^{-iEt} - e^{iEt}\right]/16\pi^3 E = \sin(mt)/m$.

an identical outcome, an outcome however which was, in one significant way, quite different from that found for the evolution of the two $V_Y(|w|,t)$.

With reference to Fig. (3.1), what was found for a narrow $V_J(|w|,t)$ square step located very close to $|w| = 0$ at $t = 0$ (and with either choice of time evolution) was that the step started to spread into the $|w|$ bulk, with the leading edge of the step moving along the M_4^- null geodesic $bt = 1 - e^{-b|w|}$, and with reflections back toward $|w| = 0$ occurring only after the signal had reached the edge of the AdS_5 spacetime at time $t = 1/b$. This behavior is completely causal, with information only propagating within the Cauchy development of the initial square step configuration at the initial time $t = 0$. Moreover, in the numerical work the drop in $V_J(|w|,t)$ at points only marginally outside the leading edge was found to be quite precipitous, with $V_J(|w|,t)$ immediately dropping by a factor of one thousand or so in models in which the step was built from a one thousand mode sum. As such, the numerical study confirms, and in quite dramatic fashion, the form given for $h^{TT}_{\mu\nu}(\mathrm{RET};\mathrm{J}_2)$ in Eq. (18.52), with $h^{TT}_{\mu\nu}(\mathrm{SING};\mathrm{J}_2)$ (a sum over the discrete modes given for $\hat{G}^{TT}(x,0,w,0)(\mathrm{SING};\mathrm{J}_2)$ in Eq. (18.55)) indeed being found to be causal. Moreover, in a numerical evolution of $h^{TT}_{\mu\nu}(t,|w|;\mathrm{RET})$ of Eq. (18.64), it was explicitly found that as the leading edge of the disturbance moved away from the brane along a null geodesic, the magnitude of $h^{TT}_{\mu\nu}(1/b - e^{-b|w|}/b, |w|;\mathrm{RET})$ precisely grew as the delocalizing $e^{b|w|}/2$, just as required by Eqs. (18.43) and (18.50). Additionally, the numerical study also confirmed the generic reflection properties exhibited in Eq. (F.68) which occur subsequent to the arrival of the signal at the edge of the spacetime, with the point of maximum intensity in the signal returning to the brane along a null geodesic. It is noteworthy to emphasize here that despite the fact that $V_J(|w|,t)$ is built entirely out of massive modes, its leading edge nonetheless still moved along an AdS_5 null geodesic. The reason for this is that, as such, the original wave equation of Eqs. (16.4) and (16.5) is actually an equation for one massless 5-dimensional graviton moving on the 5-dimensional AdS_5 lightcone. It is only from the perspective of a 4-dimensional decomposition that the 5-dimensional massless graviton can be thought of as being an infinite tower of massive 4-dimensional states. In the bulk this entire tower of massive modes thus acts collectively as one massless 5-dimensional graviton.[15]

However, a quite different behavior was found for the propagation of a very narrow $V_Y(|w|,t)$ square step located very close to $|w| = 0$ at time $t = 0$. While the step did indeed begin to spread into the bulk with a leading edge which moved along the same AdS_5 null geodesic $bt = 1 - e^{-b|w|}$, in addition a second signal started to come from the $|w| = \infty$ boundary of the AdS_5 spacetime, with this signal having a leading edge which also moved on an AdS_5 null geodesic, viz. the $bt = e^{-b|w|}$ one which originated at $|w| = \infty$ at $t = 0$. This signal met the signal emerging from

[15]Even without any brane at all, the closely related 5-dimensional massless scalar field propagator $D_S(x,x',m=0)$ of pure AdS_5 can still be decomposed (see Eq. (E.7) of Appendix E) as an infinite tower of (normalizable) massive 4-dimensional $J_2(me^{-bw}/b)$ modes.

the step at the point $e^{-b|w|} = 1/2$ at a time $t = 1/2b$. Consequently, this time the Cauchy development of the step was only predictable within the $0 < b|w| < \ln 2$, $0 < bt < 1/2$ region, after which time new information from infinity was able to enter the Cauchy development of the $|w| = 0$ region.[16] Thus of our two candidate discrete bases we see that the $J_2(me^{-b|w|}/b)$ based one actually is causal, while the discrete $Y_2(ne^{-b|w|}/b)$ based one is not.

Now at first the discovery that the discrete $Y_2(ne^{-b|w|}/b)$ basis is not causal appears to be at variance with Eqs. (18.49) and (18.51) which explicitly demonstrate that the $J_2 - iY_2$ and Y_2 based propagators are in fact causal. However, those particular propagators have singular parts (viz. Eqs. (18.54) and (18.57)) which contain an entire continuum of KK modes and not just the discrete approximation to it associated with the zeroes of Y_1. The causality of the $J_2 - iY_2$ and Y_2 based propagators outside the M_4^- lightcone is thus generated by destructive interference between the discrete modes with masses $m = by_i$ and the rest of the KK continuum. Hence, while the entire KK continuum might be more than complete for the purposes of constructing localized square steps, it is not overcomplete for causality purposes, with all of its modes (and the massless graviton of course) being needed for Eqs. (18.49) and (18.51). Thus to sum up, we see that the M_4^- brane-world is indeed causal, that its causal propagator is given as $\hat{G}^{TT}(x, 0, w, 0)(\mathrm{RET}; \mathrm{J}_2 - \mathrm{iY}_2)$ as evaluated with the retarded contour prescription, that this propagator does not lead to localization of gravity to the brane but rather to unstable fluctuations which grow exponentially away from the brane, and that it contains a non-normalizable massless graviton with ghost signature.

18.8 Some exact solutions to the M_4^- fluctuation equations

Some further insight into the ghost nature of the M_4^- graviton is provided by the M_4^- generalized Einstein equation on the brane which, by analog with the M_4^+ Eq. (14.35), takes the form

$$\delta G_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = b\kappa_5^2 S_{\mu\nu} - \delta\bar{E}_{\mu\nu}(w=0) \quad , \tag{18.65}$$

where

$$\delta\bar{E}_{\mu\nu}(|w|) = \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 2b\frac{\partial}{\partial|w|}\right]h_{\mu\nu}(|w|) \quad . \tag{18.66}$$

Because of the positivity of the $b\kappa_5^2$ coefficient of the $S_{\mu\nu}$ term in Eq. (18.65), when realized in solutions which possess a graviton mode, the graviton would then have to appear with ghost signature. But when realized in solutions which possess no

[16]Within this $0 < b|w| < \ln 2$, $0 < bt < 1/2$ region the propagation of $V_Y(|w|, t)$ and the propagation of $V_J(|w|, t)$ were found to be identical, as of course has to be the case since the Cauchy initial value problem has a unique solution prior to the entry of new information into the Cauchy development of the initial configuration.

graviton, the $b\kappa_5^2 S_{\mu\nu}$ term would have to be counterbalanced by a non-vanishing Weyl tensor term instead. The only graviton which could ever appear in M_4^- would thus have to have ghost signature.

These two distinct realizations to Eq. (18.65), one with a ghost graviton, the other with no massless graviton at all, are nicely exhibited in two particular exact solutions that the M_4^- TT wave equation

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{-2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right] h_{\mu\nu}^{TT} = 0 \;, \quad \delta(w)\left[\frac{\partial}{\partial|w|} - 2b\right] h_{\mu\nu}^{TT} = -\kappa_5^2\delta(w)S_{\mu\nu}^{TT} \tag{18.67}$$

happens to possess. Specifically, for a TT source which obeys $\eta^{\alpha\beta}\partial_\alpha\partial_\beta S_{\mu\nu}^{TT} = 0$, it can readily be checked that both the asymptotically unbounded

$$h_{\mu\nu}^{TT} = 2b\kappa_5^2 e^{2b|w|}\int d^4x' D(x-x', m=0)S_{\mu\nu}^{TT}(x') + \frac{\kappa_5^2}{2b}S_{\mu\nu}^{TT} \;, \tag{18.68}$$

and the asymptotically bounded

$$h_{\mu\nu}^{TT} = \frac{\kappa_5^2}{4b}e^{-2b|w|}S_{\mu\nu}^{TT} \;, \tag{18.69}$$

are exact solutions to Eq. (18.67). On the brane the former obeys

$$\delta G_{\mu\nu}^{TT} = b\kappa_5^2 S_{\mu\nu}^{TT} \;, \tag{18.70}$$

while evaluation of Eq. (18.66) shows that the latter obeys

$$b\kappa_5^2 S_{\mu\nu}^{TT} - \delta\bar{E}_{\mu\nu}(w=0) = 0 \;. \tag{18.71}$$

In analog to the analysis of the equivalent M_4^+ exact solutions presented in Chapter 17 as Eqs. (17.83) and (17.84), we find that for the typical massless plane wave source $S_{\mu\nu}^{TT} = A_{\mu\nu}e^{i\bar{k}\cdot\bar{x}-ikt}$, the solution of Eq. (18.68) is generated by the $J_2 - iY_2$ based M_4^- propagator of Eq. (18.36) as evaluated with $\alpha_q = 1, \beta_q = -i$,[17] while the solution of Eq. (18.69) is generated by the $J_2(j_i e^{-b|w|})$ based M_4^- propagator of Eq. (18.36) as evaluated with $\alpha_q = 1, \beta_q = 0$.[18] The solution given in Eq. (18.68), and in particular its overall change in sign compared with the graviton-containing M_4^+ solution of Eq. (17.83), thus reveals the ghost nature of the M_4^- massless graviton. The solution given in Eq. (18.69) is also of interest, since it shows how it is in principle possible for so Einstein-looking an equation as Eq. (18.65) to be able to admit solutions which do not possess a massless graviton at all. Having thus explored the structure of the divergent warp factor M_4^- brane world, we turn now to another divergent warp factor brane world, viz. the positive-tension AdS_4^+ one in which the geometry on the brane is taken to be AdS_4.

[17] The small q behavior of $[J_2(qe^{-b|w|}/b) - iY_2(qe^{-b|w|}/b)]/q[J_1(q/b) - iY_1(q/b)]$ of relevance in Eq. (18.36) for the $S_{\mu\nu}^{TT} = A_{\mu\nu}e^{i\bar{k}\cdot\bar{x}-ikt}$ source is given by $2be^{2b|w|}/q^2 + 1/2b$.

[18] The relevant small q behavior of $J_2(qe^{-b|w|}/b)/qJ_1(q/b)$ is given by $e^{-2b|w|}/4b$.

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Chapter 19

Fluctuations around an Embedded Positive-Tension AdS Brane

19.1 AdS_4^+ mode basis

To determine the TT modes associated with the globally non-hyperbolic positive-tension AdS_4^+ brane world where a signal can come in to the brane from infinity in a finite time, we set $e^{A(|w|)} = H\cosh(b|w|-\sigma)/b$ (where $\cosh\sigma = b/H$) in Eqs. (16.4) and (16.5), to obtain as wave equation

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2\tanh^2(b|w|-\sigma) + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h_{\mu\nu}^{TT} = 0 \ , \tag{19.1}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} + 2(b^2-H^2)^{1/2}\right] h_{\mu\nu}^{TT} = 0 \ , \tag{19.2}$$

where $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{\mu\nu}^{TT} = q^{\alpha\beta}\tilde{h}_{\mu\nu;\alpha;\beta}^{TT}$ is the tensor box operator associated with an AdS_4 metric $q_{\mu\nu}$. In this case, and unlike the previous M_4^+ and M_4^- cases, $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{\mu\nu}^{TT}$ is not diagonal in its indices. Specifically, for an induced AdS_4 metric of the form $q_{\mu\nu}dx^\mu dx^\nu = dx^2 + e^{2Hx}(-dt^2+dy^2+dz^2)$, direct evaluation of Eq. (13.10) in a TT mode yields

$$\begin{aligned}
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{11}^{TT} &= e^{-4Hx}[\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2+\partial_3^2-\partial_0^2) - 4H^2][e^{4Hx}h_{11}^{TT}] \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{12}^{TT} &= e^{-2Hx}[\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2+\partial_3^2-\partial_0^2) - 4H^2][e^{2Hx}h_{12}^{TT}] \\
&\quad +2H\partial_2 h_{11}^{TT} \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{22}^{TT} &= [\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2+\partial_3^2-\partial_0^2) - 4H^2]h_{22}^{TT} \\
&\quad +2H^2e^{2Hx}h_{11}^{TT} + 4H\partial_2 h_{12}^{TT} \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{23}^{TT} &= [\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2+\partial_3^2-\partial_0^2) - 4H^2]h_{23}^{TT} + 2H\partial_3 h_{12}^{TT} \\
&\quad +2H\partial_2 h_{13}^{TT} \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h_{00}^{TT} &= [\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2+\partial_3^2-\partial_0^2) - 4H^2]h_{00}^{TT} \\
&\quad -2H^2e^{2Hx}h_{11}^{TT} - 4H\partial_0 h_{10}^{TT}
\end{aligned} \tag{19.3}$$

on every $|w|$ slice, with the other components of $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}$ being given by symmetry. With $h^{TT}_{\mu\nu}$ being an AdS_4 TT mode, on every $|w|$ slice its components obey the TT conditions

$$\begin{aligned}
\partial_1 h^{TT}_{11} + e^{-2Hx}[\partial_2 h^{TT}_{12} + \partial_3 h^{TT}_{13} - \partial_0 h^{TT}_{10}] + 4Hh^{TT}_{11} &= 0 \ , \\
\partial_1 h^{TT}_{12} + e^{-2Hx}[\partial_2 h^{TT}_{22} + \partial_3 h^{TT}_{23} - \partial_0 h^{TT}_{20}] + 3Hh^{TT}_{12} &= 0 \ , \\
\partial_1 h^{TT}_{13} + e^{-2Hx}[\partial_2 h^{TT}_{23} + \partial_3 h^{TT}_{33} - \partial_0 h^{TT}_{30}] + 3Hh^{TT}_{13} &= 0 \ , \\
\partial_1 h^{TT}_{10} + e^{-2Hx}[\partial_2 h^{TT}_{20} + \partial_3 h^{TT}_{30} - \partial_0 h^{TT}_{00}] + 3Hh^{TT}_{10} &= 0 \ , \\
h^{TT}_{11} + e^{-2Hx}[h^{TT}_{22} + h^{TT}_{33} - h^{TT}_{00}] &= 0 \ ,
\end{aligned} \tag{19.4}$$

with Eqs. (19.3) and (19.4) thus specifying the structure of the TT sector.

With TT tensor fluctuations obeying $(\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2)h^{TT}_{\mu\nu} = 0$ on the AdS_4 light-cone, we shall thus look to separate Eq. (19.1) in mass eigenstates defined by

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h^{TT}_{\mu\nu} = m^2 h^{TT}_{\mu\nu} \ . \tag{19.5}$$

As such, Eq. (19.5) is an eigenvalue equation which in general requires a rediagonalization of the $h^{TT}_{\mu\nu}$ modes in the (μ,ν) space. However, given the structure displayed in Eq. (19.3), we see that for solutions which obey $h^{TT}_{1\mu}(|w|) = 0$,[1] the operator $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}(|w|)$ happens to then be diagonal in its indices at every $|w|$. We thus only need to solve the theory for modes which obey $h^{TT}_{1\mu}(|w|) = 0$, with the $h^{TT}_{1\mu}(|w|) \neq 0$ components of the spin two KK modes then being fixed by their five-fold degeneracy. Thus in the following we can restrict our discussion to modes which obey $h^{TT}_{1\mu}(|w|) = 0$, modes for which Eq. (19.5) then takes the very convenient diagonal form

$$[\partial_1^2 - H\partial_1 + e^{-2Hx}(\partial_2^2 + \partial_3^2 - \partial_0^2) - 2H^2]h^{TT}_{\mu\nu} = m^2 h^{TT}_{\mu\nu} \ . \tag{19.6}$$

Separable solutions to Eqs. (19.1) and (19.6) are readily constructed. As can readily be checked, they take the form

$$h^{TT}_{\mu\nu}(m) = e_{\mu\nu} f_m(|w|) e^{-ip^0 t + ip^2 y + ip^3 z} e^{Hx/2} J_{\nu+1/2}(p^1 e^{-Hx}/H) + \text{c.c.} \ , \tag{19.7}$$

where

$$p^1 = [(p^0)^2 - (p^2)^2 - (p^3)^2]^{1/2} \ , \tag{19.8}$$

[1] While this condition is analogous to an AdS_4 axial gauge condition, we make no such gauge transformation here. Nor in fact could we, since for such a gauge transformation to be of utility in Eq. (19.1), we would have to apply it at all $|w|$, and this we cannot do since the available AdS_5 axial gauge preserving Eq. (15.12) can only serve to reduce the number of degrees of freedom for modes whose $|w|$ dependence is precisely that specified in Eq. (15.12), something which is not in general the case for arbitrary solutions to Eq. (19.10) below. Bulk KK mode solutions with $h^{TT}_{1\mu}(|w|) \neq 0$ are just as physical as those for which $h^{TT}_{1\mu}(|w|) = 0$.

and where we have chosen the particular Bessel function $J_{\nu+1/2}(p^1 e^{-Hx}/H)$ since it is the only Bessel function which is well-behaved over the entire $-\infty < x < \infty$ range of allowed values for x. In Eq. (19.7) the parameter ν is defined by

$$\nu = \left(\frac{9}{4} + \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2} \ , \tag{19.9}$$

the tensor $e_{\mu\nu}$ is an appropriately chosen TT polarization tensor, and, with $y = \tanh(b|w| - \sigma)$, the $|w|$-dependent $f_m(|w|)$ obeys

$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \frac{m^2}{H^2} + 2 - \frac{4}{(1-y^2)}\right] f_m(y) = 0 \ , \tag{19.10}$$

viz.

$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \nu(\nu+1) - \frac{4}{(1-y^2)}\right] f_m(y) = 0 \ . \tag{19.11}$$

Equation (19.11) is recognized as an associated Legendre equation, an equation which possesses the associated Legendre functions $P^2_\nu(y)$ and $Q^2_\nu(y)$ of the first and second kind as solutions. For an argument y which lies between -1 and 1 (the case here where $y = \tanh(b|w| - \sigma)$ lies between $-\tanh\sigma = -(1 - H^2/b^2)^{1/2}$ and 1, with $y = -\tanh\sigma$ corresponding to $|w| = 0$ and $y = 1$ corresponding to $|w| = \infty$), these particular associated Legendre functions (and the closely allied $P^1_\nu(y)$ and $Q^1_\nu(y)$) are related to the ordinary $P_\nu(y)$ and $Q_\nu(y)$ Legendre functions according to

$$\begin{aligned} P^2_\nu(y) &= (1-y^2)\frac{d^2P_\nu(y)}{dy^2} \ , \quad Q^2_\nu(y) = (1-y^2)\frac{d^2Q_\nu(y)}{dy^2} \ , \\ P^1_\nu(y) &= -(1-y^2)^{1/2}\frac{dP_\nu(y)}{dy} \ , \quad Q^1_\nu(y) = -(1-y^2)^{1/2}\frac{dQ_\nu(y)}{dy} \ . \end{aligned} \tag{19.12}$$

Such Legendre functions exist for integer ν, non-integer ν and also complex ν, with those with $\nu = -1/2 + i\lambda$ being known as conical functions which are distinguished by the fact that, despite their complex index, the $P_{-1/2+i\lambda}(y)$ with $|y| < 1$ (but not the $Q_{-1/2+i\lambda}(y)$) are nonetheless real. The $P_\nu(y)$ (but not the $Q_\nu(y)$) obey the index symmetry $P_\nu(y) = P_{-\nu-1}(y)$, to allow us to restrict the $P_\nu(y)$ to $\mathrm{Re}[\nu] \geq -1/2$. For integer ν the $P_\nu(y)$ are the ordinary Legendre polynomials, with the integer $P^2_\nu(y)$ thus vanishing when $\nu < 2$. For integer ν both the $P_\nu(y) = P^0_\nu(y)$ and $Q_\nu(y) = Q^0_\nu(y)$ Legendre functions can be written in terms of elementary functions, with some typical ordinary and associated Legendre functions being given as

$$
\begin{aligned}
&P^0_0(y) = 1 \ , \ P^0_1(y) = y \ , \ P^0_2(y) = \frac{(3y^2-1)}{2} \ , \\
&P^1_0(y) = 0 \ , \ P^1_1(y) = -(1-y^2)^{1/2} \ , \ P^1_2(y) = -3y(1-y^2)^{1/2} \ , \\
&P^2_0(y) = 0 \ , \ P^2_1(y) = 0 \ , \ P^2_2(y) = 3(1-y^2) \ , \\
&Q^0_0(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \ , \ Q^0_1(y) = \frac{y}{2}\ln\left(\frac{1+y}{1-y}\right) - 1 \ , \\
&Q^0_2(y) = \frac{(3y^2-1)}{4}\ln\left(\frac{1+y}{1-y}\right) - \frac{3y}{2} \ , \\
&Q^1_0(y) = -\frac{1}{(1-y^2)^{1/2}} \ , \ Q^1_1(y) = -\frac{y}{(1-y^2)^{1/2}} - \frac{(1-y^2)^{1/2}}{2}\ln\left(\frac{1+y}{1-y}\right) \ , \\
&Q^1_2(y) = \frac{(2-3y^2)}{(1-y^2)^{1/2}} - \frac{3y(1-y^2)^{1/2}}{2}\ln\left(\frac{1+y}{1-y}\right) \ , \ Q^2_0(y) = \frac{2y}{(1-y^2)} \ , \\
&Q^2_1(y) = \frac{2}{(1-y^2)} \ , \ Q^2_2(y) = \frac{(5y-3y^3)}{(1-y^2)} + \frac{3(1-y^2)}{2}\ln\left(\frac{1+y}{1-y}\right) \ . \qquad (19.13)
\end{aligned}
$$

For arbitrary ν (integer or otherwise) all the $P^2_\nu(y)$ are well-behaved at $y=1$ behaving there as $P(\nu)[(1-y)-(1-y)^2(\nu^2+\nu-3)/6]$ where $P(\nu)=\nu(\nu^2-1)(\nu+2)/4$, though the non-integer ones diverge as $\hat{P}(\nu)/(1+y)$ at $y=-1$ where $\hat{P}(\nu)=2\nu(\nu^2-1)(\nu+2)/\Gamma(2-\nu)\Gamma(3+\nu)$. For both integer and non-integer ν the $Q^2_\nu(y)$ are divergent near $y=1$, behaving there as $Q(\nu)[1/(1-y)+(\nu^2+\nu-1)/2+O\left((1-y)\ln(1-y)\right)]$ where $Q(\nu)=\nu(1-\nu^2)(\nu+2)\pi/\Gamma(2-\nu)\Gamma(3+\nu)\sin\nu\pi$, while also being divergent at $y=-1$ where they diverge as $\hat{Q}(\nu)/(1+y)$ with $\hat{Q}(\nu)=\nu(\nu^2-1)(\nu+2)\pi/\Gamma(2-\nu)\Gamma(3+\nu)\tan\nu\pi$.[2] With the range of allowed values of y in the AdS_4^+ case of interest being $-\tanh\sigma \le y \le 1$, we see that the $P^2_\nu(y)$ are well-behaved over the entire allowed range, but the $Q^2_\nu(y)$ are not (they at diverge $|w|=\infty$). We can thus anticipate that it will be the $P^2_\nu(y)$ which will be normalizable, and the $Q^2_\nu(y)$ not. With the massless graviton possessing $\nu=1$, we note that with $Q^2_1(y)$ behaving as $(1-y^2)^{-1}$, viz. as the $\cosh^2(b|w|-\sigma)$ AdS_4^+ warp factor itself, we can also anticipate that in the AdS_4^+ brane world the graviton will not be normalizable.

With Eq. (19.11) being a second order differential equation, it will possess two independent solutions for each value of the parameter $\nu(\nu+1)$. In terms of the associated Legendre functions, almost of all of these solutions can be written in the

[2]The normalization we are using here for the associated Legendre functions is the conventional one in which the $P_\nu(y)$ and $Q_\nu(y)$ obey $\int_{-1}^{1} dy[P_\nu(y)]^2 = [2\pi^2 - 4\sin^2(\pi\nu)\psi'(\nu+1)]/[\pi^2(2\nu+1)]$, $\int_{-1}^{1} dy P_\nu(y)Q_\nu(y) = -\sin(2\pi\nu))\psi'(\nu+1)]/\pi(2\nu+1)$, where $\psi(x)=(d/dx)\log\Gamma(x)$ and $\Gamma(x)=\int_0^\infty dt e^{-t}t^{x-1}$, with the first integral having the familiar integer index limit $\int_{-1}^{1} dy[P_n(y)]^2 = 2/(2n+1)$. While we shall thus use this normalization to once and for all define the associated Legendre functions, in the following the normalization of AdS_4^+ brane-world wave functions will be made with respect to an integration measure in which y ranges not from -1 to 1, but from $-\tanh\sigma$ to 1 instead.

form

$$f_m(y) = \alpha_m P^2_\nu(y) + \beta_m Q^2_\nu(y) \tag{19.14}$$

with $|w|$-independent coefficients α_m and β_m, a class which includes the massless graviton with wave function

$$f_0(y) = \beta_0 Q^2_1(y) \ . \tag{19.15}$$

However, because of the kinematic vanishing of $P^2_0(y) = P^2_{-1}(y)$ and $P^2_1(y) = P^2_{-2}(y)$, the class of solutions given in Eq. (19.14) would be missing one solution in each of the $\nu = 0$ and $\nu = 1$ cases, with these cases thus being in need of separate treatment. One way to isolate these missing solutions is to take the derivative of Eq. (19.11) with respect to the index ν, so that any $P^2_\nu(y)$ which obeys Eq. (19.11) will also obey

$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \nu(\nu+1) - \frac{4}{(1-y^2)}\right]\frac{dP^2_\nu(y)}{d\nu} + (2\nu+1)P^2_\nu(y) = 0 \ . \tag{19.16}$$

Consequently, it will be the ν derivatives $dP^2_\nu(y)/d\nu$ which will provide the missing solutions to Eq. (19.11), as such ν derivatives will themselves be solutions to Eq. (19.11) for any ν for which $P^2_\nu(y)$ itself happens to vanish identically. However, while it is possible to determine the needed ν derivatives directly,[3] it is more instructive to simply return to Eq. (19.1) itself and solve it directly in the two particular cases.

Thus, for the $\nu = 0$ case first, we note that $\nu = 0$ corresponds to the tachyonic $m^2 = -2H^2$. And with such tachyonic modes obeying $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu} = -4H^2 h^{TT}_{\mu\nu}$, in the $\nu = 0$ case Eq. (19.1) reduces to the very simple

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2\right] h^{TT}_{\mu\nu} = 0 \ , \tag{19.17}$$

to yield solutions of the form $\exp(2b|w|)$ and $\exp(-2b|w|)$. Such solutions can be written as linear combinations of the $Q^2_0(y) = 2y/(1-y^2) = \sinh(2b|w| - 2\sigma) = (1/2)[\exp(2b|w| - 2\sigma) - \exp(-2b|w| + 2\sigma)]$ solution given in Eq. (19.14)[4] and an independent solution of the form $\exp(-2b|w| + 2\sigma) = (1-y)/2(1+y)$, so that for

[3]The $dP_\nu(y)/d\nu$ derivatives can be determined as the solutions to the $y \to -y$ invariant differential equation $\left[(1-y^2)d^2_y - 2yd_y + \nu(\nu+1)\right] dP_\nu(y)/d\nu + (2\nu+1)P_\nu(y) = 0$ which diverge at $y = -1$ but not at $y = 1$. (With $P_\nu(y)$ behaving as $P_\nu(y) \to 1-(\nu^2+\nu)(1-y)/2$ as $y \to 1$, it follows that in this limit $dP_\nu(y)/d\nu$ has to behave as the non-singular $dP_\nu(y)/d\nu \to -(2\nu+1)(1-y)/2$.) For $\nu = 0$ we thus find that $(dP_\nu(y)/d\nu)|_{\nu=0} = \ln[(1+y)/2]$, while for $\nu = 1$ we obtain $(dP_\nu(y)/d\nu)|_{\nu=1} = y + y\ln[(1+y)/2] - 1$. Consequently, for $\nu = 0$ we have $(dP^2_\nu(y)/d\nu)|_{\nu=0} = -(1-y)/(1+y)$, while for $\nu = 1$ we have $(dP^2_\nu(y)/d\nu)|_{\nu=1} = (1-y)(2+y)/(1+y) = 2/(1+y) - y$.

[4]In passing we note that we have actually met this particular functional dependence on $|w|$ already, albeit in a somewhat different guise. Specifically, the $|w|$ dependence of the $\hat{\xi}_5$-dependent term in the most general axial gauge preserving gauge transformations of Eq. (15.12) is given by $\exp(2A)dA/d|w|$, i.e. by precisely $\sinh(2b|w| - 2\sigma)$.

$\nu = 0$ we replace Eq. (19.14) by

$$f_{(m^2=-2H^2)}(y) = \alpha_{(m^2=-2H^2)}\frac{(1-y)}{2(1+y)} + \beta_{(m^2=-2H^2)}Q_0^2(y) \quad . \tag{19.18}$$

Similarly, for the $\nu = 1$ case, we note that $\nu = 1$ corresponds to modes with $m^2 = 0$. Since $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu} = -2H^2 h^{TT}_{\mu\nu}$ in such modes, in the $\nu = 1$ case Eq. (19.1) reduces to

$$\left[\frac{d^2}{d|w|^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 2\frac{d^2A}{d|w|^2}\right] f_0(|w|) = 0 \quad . \tag{19.19}$$

To solve Eq. (19.19) we try as solution $f_0(|w|) = \exp[2A(|w|)]g(|w|)$, so that the function $g(|w|)$ thus has to obey

$$e^{-2A}\frac{d}{d|w|}\left(e^{4A}\frac{dg}{d|w|}\right) = 0 \quad , \tag{19.20}$$

an equation for which the solution can conveniently be written as[5]

$$g(|w|) = -\frac{\alpha_0 b^2}{H^2}\left[3\tanh(b|w|-\sigma) - \tanh^3(b|w|-\sigma)\right] + \frac{2\alpha_0 b^2}{H^2} + \frac{2\beta_0 b^2}{H^2} \quad . \tag{19.21}$$

With $Q_1^2(y)$ being given by $Q_1^2(y) = 2/(1-y^2) = 2\cosh^2(b|w|-\sigma) = 2b^2\exp(2\text{A})/H^2$, for $\nu = 1$ (viz. $m^2 = 0$) Eq. (19.14) is replaced by

$$f_0(y) = \alpha_0\left(\frac{2}{(1+y)} - y\right) + \beta_0 Q_1^2(y) \quad . \tag{19.22}$$

With the help of Eq. (19.16) we thus identify all solutions to Eq. (19.11), with the separation parameter m^2 being as yet unconstrained.

To implement the junction condition of Eq. (19.2), we note that the associated Legendre function solutions of Eq. (19.14) obey the identity

$$\begin{aligned}
&\left[\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right]\left[\alpha_m P_\nu^2(|w|) + \beta_m Q_\nu^2(|w|)\right] \\
&\qquad = b\left[(1-y^2)\frac{d}{dy} - 2y\right]\left[\alpha_m P_\nu^2(y) + \beta_m Q_\nu^2(y)\right] \\
&\qquad = b(\nu-1)(\nu+2)(1-y^2)^{1/2}\left[\alpha_m P_\nu^1(y) + \beta_m Q_\nu^1(y)\right] \quad .
\end{aligned} \tag{19.23}$$

Solutions to the junction condition of the form of Eq. (19.14) are thus given by any α_m, β_m combinations which satisfy

$$\alpha_m P_\nu^1(-\tanh\sigma) + \beta_m Q_\nu^1(-\tanh\sigma) = 0 \tag{19.24}$$

[5]We have chosen the particular form $2(\alpha_0 + \beta_0)b^2/H^2$ for one of the two integration constants so that the resulting α_0 dependent term in $f_0(|w|)$ will be non-singular in the $y \to 1$ limit, a limit where the general $dP_\nu^2(y)/d\nu$ derivative behaves as $dP_\nu^2(y)/d\nu \to (2\nu^3 + 3\nu^2 - \nu - 1)(1-y)/2$.

since $y = -\tanh\sigma$ at the brane. Additionally, for the $\nu = 0$ solution of Eq. (19.18), the linear combination which obeys the junction condition is found to be given by

$$f_{(m^2=-2H^2)} = \cosh(2b|w| - \sigma) \ , \tag{19.25}$$

a solution which should not be confused with the $\cosh^2(b|w| - \sigma)$ warp factor. Finally, for $\nu = 1$, the $m^2 = 0$ solution which obeys the junction condition is found to be given by

$$f_0 = \beta_0 Q_1^2(y) \ , \tag{19.26}$$

viz. by the AdS_4^+ warp factor itself.[6] With this massless mode having the same dependence on $|w|$ as the $\hat{\xi}_\mu$-dependent term in the general axial gauge preserving gauge transformation of Eq. (15.12), the $\bar{h}_{\mu\nu} = h_{\mu\nu} + e^{2A}[\tilde{\nabla}_\mu \hat{\xi}_\nu + \tilde{\nabla}_\nu \hat{\xi}_\mu]$ gauge transformations allow us to reduce the AdS_4^+ massless TT graviton to precisely two propagating degrees of freedom, both on the brane and in the bulk.

19.2 Normalizability criterion for AdS_4^+ modes

To determine which of the above modes are normalizable, we need to evaluate the mode energy-momentum tensor given in Chapter 16. Since it is only the $|w|$-dependence which is of relevance for normalizability, it is sufficient to evaluate the energy-momentum tensor of Eq. (16.64) in a particularly convenient solution in which Eq. (19.3) greatly simplifies, viz. in a mode whose only non-zero component is an h_{23}^{TT} $(= h_{32}^{TT})$ which only depends on the coordinates $|w|$, x and t and is independent of y and z, viz. a mode which, by construction, identically satisfies the TT conditions of Eq. (19.4). As such, this particular mode is an AdS_4 analog of a polarized M_4 plane wave propagating in the x direction, and for it the entire set of equations given in Eq. (19.5) for the entire $h_{\mu\nu}^{TT}$ then reduces to just the one non-trivial equation

$$\left[\partial_x^2 - H\partial_x - e^{-2Hx}\partial_0^2 - 2H^2\right] h_{23}^{TT}(|w|, x, t) = m^2 h_{23}^{TT}(|w|, x, t) \ , \tag{19.27}$$

with immediate solution

$$h_{23}^{TT}(m) = f_m(|w|) e^{-ip^0 t} e^{Hx/2} J_{\nu+1/2}(p^0 e^{-Hx}/H) + \text{c.c.} \tag{19.28}$$

With the needed AdS_4^+ timelike Killing vector being of the form $K^M = (-1, 0, 0, 0, 0)$ (see Chapter 16), in the particular solution of Eq. (19.28) the relevant

[6]However, as we shall see below, for small H^2/b^2 the α_0-dependent term in Eq. (19.22) will also satisfy the junction condition, with $P_\nu^1(-\tanh\sigma)$ then vanishing for a ν very close to $\nu = 1$.

components of $T^{MN}K_N$ are found to evaluate to

$$\begin{aligned}
8\kappa_5^2 g^{1/2}T^{50}K_0 &= -4e^{-Hx}\partial_0 h_{23}^{TT}\left[\partial_w - 2A'\right]h_{23}^{TT} \quad , \\
8\kappa_5^2 g^{1/2}T^{10}K_0 &= -4e^{-2A}e^{-Hx}\partial_0 h_{23}^{TT}(\partial_x - 2H)h_{23}^{TT} \quad , \\
8\kappa_5^2 g^{1/2}T^{00}K_0 &= 2e^{-2A}e^{-3Hx}\partial_0 h_{23}^{TT}\partial_0 h_{23}^{TT} \\
&\quad +2e^{-2A}e^{-Hx}\left(\partial_x - 2H\right)h_{23}^{TT}\left(\partial_x - 2H\right)h_{23}^{TT} \\
&\quad +2e^{-Hx}\left(\partial_w - 2A'\right)h_{23}^{TT}\left(\partial_w - 2A'\right)h_{23}^{TT} \quad .
\end{aligned} \tag{19.29}$$

With use of the wave equation in the form

$$\left[\partial_w^2 - 4A'^2 + e^{-2A}\left(\partial_x^2 - H\partial_x - e^{-2Hx}\partial_0^2 - 4H^2\right)\right]h_{23}^{TT}(|w|, x, t) = 0 \tag{19.30}$$

allowing us to rewrite $g^{1/2}T^{00}K_0$ as

$$\begin{aligned}
8\kappa_5^2 g^{1/2}T^{00}K_0 &= 2e^{-2A}e^{-3Hx}\left[\partial_0 h_{23}^{TT}\partial_0 h_{23}^{TT} - h_{23}^{TT}\partial_0^2 h_{23}^{TT}\right] \\
&\quad +2e^{-2A}\partial_x\left[e^{-Hx}h_{23}^{TT}(\partial_x - 2H)h_{23}^{TT}\right] \\
&\quad +2e^{-Hx}\partial_w\left[h_{23}^{TT}\left(\partial_w - 2A'\right)h_{23}^{TT}\right] \quad ,
\end{aligned} \tag{19.31}$$

we see that in modes for which there is both an asymptotic vanishing of the $g^{1/2}T^{50}K_0$ flux as $w \to \pm\infty$ and an asymptotic vanishing of the $g^{1/2}T^{10}K_0$ flux as $x \to \pm\infty$, the energy

$$\begin{aligned}
E &= \frac{1}{4\kappa_5^2}\int dw dx e^{-2A}e^{-3Hx}[\partial_0 h_{23}^{TT}\partial_0 h_{23}^{TT} - h_{23}^{TT}\partial_0^2 h_{23}^{TT}] \\
&= \frac{2H}{\kappa_5^2}\int_0^\infty d|w|e^{-2A}f_m^2(|w|)\int_0^\infty dz z J_{\nu+1/2}^2(z) \\
&= \frac{2b}{H\kappa_5^2}\int_{-\tanh\sigma}^1 dy f_m^2(y)\int_0^\infty dz z J_{\nu+1/2}^2(z)
\end{aligned} \tag{19.32}$$

will then be time independent. For the massive modes with $m^2 \neq 0$, the asymptotic vanishing of the $g^{1/2}T^{10}K_0$ flux is met by the $e^{Hx/2}J_{(\nu+1/2)}$ modes (though it would not be met by $e^{Hx/2}Y_{(\nu+1/2)}$ modes), while the asymptotic vanishing of the $g^{1/2}T^{50}K_0 \sim f_m[(1-y^2)d/dy - 2y]f_m$ flux is met by the $P_\nu^2(y)$ modes alone and not by any of the $Q_\nu^2(y)$ modes or the tachyonic $f_{(m^2=-2H^2)} = \cosh(2b|w| - \sigma)$ mode given in Eq. (19.25). With the $P_\nu^2(y)$ modes being convergent within the $-\tanh\sigma \leq y \leq 1$ range, the $P_\nu^2(y)$ type modes (with integer or non-integer ν) thus make a finite, explicitly positive contribution to the energy ($E \sim \int dy(P_\nu^2(y))^2$),[7] and are thus the normalizable modes appropriate to the problem. Then with $P_\nu^2(\tanh(b|w| - \sigma))$ behaving asymptotically as $1/\cosh^2(b|w| - \sigma)$, all such modes are localized to the brane.

As regards the massless mode, a mode for which $f_0(|w|)$ is found to be the warp factor $e^{2A} = H^2\cosh^2(b|w| - \sigma)/b^2$ itself, this massless mode actually meets our

[7] The Bessel function contribution however still needs to be continuum normalized.

criterion of vanishing asymptotic $g^{1/2}T^{50}K_0$ flux (since $[d/d|w| - 2(dA/d|w|)]e^{2A}$ vanishes identically), and thus actually has a time independent energy. Moreover this mode even satisfies the junction condition of Eq. (19.2). However, while the vanishing of the asymptotic flux guarantees the time independence of the energy, this is not sufficient to make the energy finite. And with the integral $\int_0^\infty d|w|e^{-2A}(e^{2A})^2$ in fact being infinite (though positive), the AdS_4^+ massless spin-two graviton mode is non-normalizable.[8]

19.3 Discrete normalizable mode basis for AdS_4^+

In order for the junction condition at the brane to be satisfied by the normalizable modes alone, the $P_\nu^2(y)$ modes would have to satisfy Eq. (19.24) on their own. The modes would thus need to satisfy

$$P_\nu^1(-\tanh\sigma) = P_\nu^1(-(1 - H^2/b^2)^{1/2}) = 0 \quad , \tag{19.33}$$

with the allowed values of ν then being determined by the zeroes of $P_\nu^1(-\tanh\sigma)$. As such, for any given value of $\tanh\sigma$, the function $P_\nu^1(-\tanh\sigma)$ actually has an infinite number of such zeroes, and with these zeroes occurring at discrete values of ν, the normalizable AdS_4^+ spectrum is thus a discrete one.[9] While the positions of the zeroes of $P_\nu^1(-\tanh\sigma)$ would in general have to be determined numerically for any arbitrarily assigned value of $\tanh\sigma$,[10] in two particularly interesting limits the positions of the zeroes of $P_\nu^1(y)$ are actually known in a closed form, viz. $y \to -1$ and $y \to 0$.[11] Since the quantity $\tanh\sigma = +(1 - H^2/b^2)^{1/2}$ is positive for AdS_4^+ ($\tanh\sigma$ would be the negative $-(1 - H^2/b^2)^{1/2}$ for AdS_4^-), the allowed values of ν in the AdS_4^+ case are thus completely determinable in a closed form in the $H \to b$ and $H \to 0$ limits. Given the known closed form expression

$$P_\nu^\mu(0) = \frac{2^\mu \pi^{1/2}}{\Gamma(\nu/2 - \mu/2 + 1)\Gamma(-\nu/2 - \mu/2 + 1/2)} \tag{19.34}$$

for the associated Legendre functions at zero argument, we find that in the $H \to b$ small brane tension limit (a limit in which we nonetheless continue to retain the delta function constraint) the allowed values of ν are given by $\nu = 2, 4, 6, \ldots$. With

[8] While the non-normalizable graviton is the only solution to the $m^2 = 0$ wave equation which obeys the junction condition, we note that the other massless solution exhibited in Eq. (19.22), viz. the $\alpha_0(2/(1+y) - y)$ one, actually is normalizable. In the following we shall uncover a role for this mode in the small H limit of the normalizable $P_\nu^2(\tanh(b|w| - \sigma))$ sector of AdS_4^+.

[9] That the normalizable spectrum of the AdS_4^+ model is discrete was first noted by [Karch and Randall (2001)] via a numerical analysis of the wave equation, with related studies of the model being made by [Miemiec (2001)], by [Schwartz (2001)] and by [Giannakis, Liu and Ren (2003)].

[10] Since $P_\nu^1(-\tanh\sigma)$ has no zeroes with $\nu < 1$, there would however be no normalizable tachyonic modes no matter what particular value of $\tanh\sigma$ might be chosen.

[11] The positions of the zeroes are also known in a closed form in the $y \to 1$ limit, a limit of relevance to the negative tension AdS_4^- brane world.

the parameter ν obeying Eq. (19.9), it follows that the $H \to b$ mass spectrum is then given by

$$m^2 = n(n+3)b^2 \quad , \quad n = 1, 3, 5, 7, \ldots \quad , \tag{19.35}$$

with all the allowed values of n being odd.[12] In this limit then all masses are of order the AdS_5 scale b, and for non-small b, none is massless or even close to massless for that matter.

Similarly, with the small ϵ zeroes of $P^1_\nu(-1+\epsilon^2/2)$ being given by $\nu = 1 + n + (n+1)(n+2)\epsilon^2/4$ where $n = 0, 1, 2, 3, 4, \ldots\ldots$, we find in the $H \to 0$ limit where Λ_4 is small and $-(1 - H^2/b^2)^{1/2} \to -1 + H^2/2b^2$, that the mass spectrum is then given by

$$m^2 = n(n+3)H^2 + O\left(H^4/b^2\right) \quad , \quad n = 0, 1, 2, 3, 4, \ldots \quad , \tag{19.36}$$

with n being allowed to be either an even or an odd integer this time. In this limit the allowed masses are small (viz. of order H) since H is small, with the $n = 0$ mode having an even smaller mass of order H^2/b, to thus be close to massless (though still with five TT polarization states).[13] With Eqs. (19.35) and (19.36) representing the two extreme limits of the model (the brane tension $\lambda = 6(b^2 - H^2)^{1/2}/\kappa_5^2$ always has to be real), we see that the mass of the lightest normalizable AdS_4^+ mode always has to lie in the range $0 < m \leq 2b$, being close to zero for $H \ll b$.[14]

Further insight into the structure of $H \ll b$ mass spectrum can be obtained by looking at the associated wave functions. For the lightest of the modes the parameter ν is given by $\nu = 1 + H^2/2b^2$, and so the wave function can be written

---

[12] In passing we note that study of this $H \to b$ limit conveniently allows us to pinpoint the specific role played by the junction condition at the brane, as the above analysis would also apply to a treatment of the modes of a pure, non-brane-world, $H = b$ AdS_5 when sectioned in AdS_4 coordinates, a limit which is obtained from an AdS_4^+ brane world with $\cosh\sigma = H/b = 1$ (i.e. $\sigma = 0$) by dropping the Z_2 symmetry on w and the delta function constraint at the brane. In such a case $y = \tanh(bw)$ is then allowed to range over the entire $(-1, 1)$ interval as w ranges from $-\infty$ to ∞, with the normalization condition then only permitting the $P^2_\nu(y)$ modes with integer ν. In this case the allowed ν values are then given as $\nu = 2, 3, 4, 5, \ldots$ and the allowed mass values are again given as $m^2 = n(n+3)b^2$ only where now $n = 1, 2, 3, 4, \ldots$, i.e. both even and odd integers. Of these modes then, as originally noted by [Karch and Randall (2001)], only the odd n ones remain in the spectrum once the junction condition at the brane is introduced.

[13] In our discussion in Chapter 20 of an equivalent Schrödinger equation treatment of the wave equation, we discuss why this one particular state does in fact lie much lower than all the others.

[14] In passing we note that for the negative tension AdS_4^- brane world, the only significant change in the calculation is that the modes are given as the solutions to $P^1_\nu(+(1-H^2/b^2)^{1/2}) = 0$ because of the change in the sign of the brane tension. With the values of ν which make $P^1_\nu(\cos\phi)$ vanish in the small ϕ limit being given by $\nu = [j_i \mathrm{cosec}(\phi/2) - 1]/2$ where the j_i are the zeroes of the J_1 Bessel function, the small-H mass spectrum for AdS_4^- is given by $m_i^2 = b^2 j_i^2 - 9H^2/4$, a relation which we recognize as being a small-H generalization of the normalizable M_4^- mass spectrum given in Eq. (18.18). In contrast to AdS_4^+ then, in the AdS_4^- brane world there is no very low mass mode in the small H limit.

entirely in terms of elementary functions as

$$P^2_{1+H^2/2b^2}(y) = P^2_1(y) + \frac{H^2}{2b^2}\frac{dP^2_\nu}{d\nu}\bigg|_{\nu=1} = \frac{H^2}{2b^2}\left[\frac{2}{(1+y)} - y\right] \ . \tag{19.37}$$

Quite remarkably, we recognize this function as being none other than the α_0-dependent mode which we encountered earlier in Eq. (19.22).[15] To check that this mode does indeed satisfy the junction condition we note that

$$\left((1-y^2)\frac{d}{dy} - 2y\right) P^2_{1+H^2/2b^2}(y) = -\frac{3H^2}{2b^2}(1-y^2) = -\frac{3H^4e^{-2A}}{2b^4} \ , \tag{19.38}$$

a quantity which thus vanishes at the brane (where $e^{2A} = 1$) to leading order in H^2/b^2.[16] Further, with $P^2_{1+H^2/2b^2}(y)$ being well-behaved as $y \to 1$, as is described in more detail below, its normalization is given in leading order by

$$N_{1+H^2/2b^2} = \frac{2b}{H^2}\int_{-1+H^2/2b^2}^{1} dy[P^2_{1+H^2/2b^2}(y)]^2 = \frac{4}{b} \ . \tag{19.39}$$

Recalling that $\cosh\sigma = b/H$, to lowest order in H/b we can set $\sinh\sigma = b/H - H/2b$, $e^{-\sigma} = H/2b$, and

$$y = \tanh(b|w| - \sigma) = \frac{(e^{-2\sigma} - e^{-2b|w|})}{(e^{-2\sigma} + e^{-2b|w|})} \to \frac{(H^2 - 4b^2e^{-2b|w|})}{(H^2 + 4b^2e^{-2b|w|})} \ , \tag{19.40}$$

and can thus write the lowest lying wave function as

$$P^2_{1+H^2/2b^2}(y) = 2e^{-2b|w|} + \frac{4H^2e^{-2b|w|}}{(H^2 + 4b^2e^{-2b|w|})} \ , \tag{19.41}$$

with the normalized wave function then being given by

$$f_0(y) = \frac{P^2_{1+H^2/2b^2}(y)}{N^{1/2}_{1+H^2/2b^2}} = b^{1/2}e^{-2b|w|} + \frac{2b^{1/2}H^2e^{-2b|w|}}{(H^2 + 4b^2e^{-2b|w|})} \ . \tag{19.42}$$

From Eq. (19.42) we see that not only is the $H \to 0$ limit of $f_0(y)$ non-singular (just as was found in the numerical analysis of [Karch and Randall (2001)]), in the limit in which $e^{-b|w|}b/H \gg 1$ (viz. not too large $|w|$) $f_0(y)$ behaves as none other than $b^{1/2}e^{-2b|w|}$, viz. as precisely none other than the wave function of the normalized graviton of the M_4^+ brane world. However, while $f_0(y)$ does behave as $b^{1/2}e^{-2b|w|}$ when $e^{-b|w|}b/H \gg 1$, it does not behave as $b^{1/2}e^{-2b|w|}$ in the intermediate $|w|$

[15] As well as being a solution to Eq. (19.16) with $\nu = 1$, $(dP^2_\nu/d\nu)|_{\nu=1}$ is also a solution to the same equation when $\nu(\nu+1) = 2 + 3H^2/2b^2$, since the no longer vanishing $(2\nu+1)P^2_\nu(y)$ term is given to lowest order in H^2/b^2 by $3(\nu-1)(dP^2_\nu/d\nu)|_{\nu=1} = (3H^2/2b^2)(dP^2_\nu/d\nu)|_{\nu=1}$.

[16] Alternatively, we can evaluate $P^1_\nu(-\tanh\sigma)$ near $\nu = 1$, to find first that at any y we have $P^1_\nu(y) = P^1_1(y) + (\nu-1)(dP^1_\nu/d\nu)|_{\nu=1} = -(1-y^2)^{1/2}\left[1 + (\nu-1)\left[\log[(1+y)/2] - 1/(1+y) + 2\right]\right]$, from which it follows for small H that $P^1_{1+H^2/2b^2}(-\tanh\sigma) = P^1_{1+H^2/2b^2}(-1 + H^2/2b^2) = -(H/b)\left[1 + (H^2/2b^2)[2 - 2b^2/H^2 + \log(H^2/4b^2)]\right]$, a quantity which vanishes identically through order H^2/b^2.

region, and though it does behave as $e^{-2b|w|}$ in the large $|w|$ limit where $e^{-b|w|}b/H \ll 1$, it does so with a coefficient (viz. $3b^{1/2}$) which is different from that (viz. $b^{1/2}$) possessed by the M_4^+ graviton wave function in the same kinematic region. As we shall explicitly see below when we construct the AdS_4^+ normalizable mode sector propagator, because the small $|w|$ behavior of the lowest lying small H AdS_4^+ mode is so much like that of the M_4^+ graviton, its contribution to static, large distance Newtonian gravity on the brane will be very similar to that of a massless mode, though no matter how small H might be, as long as H is in fact non-zero, the lowest lying normalizable AdS_4^+ mode will still be a five-component non-zero mass mode rather than a two-component zero mass one.

The small H behavior just found for the lowest lying of the normalizable AdS_4^+ modes is not unique to it, with there being some analog to it in the behavior of the other small H modes as well, since all $P_\nu^2(y)$ have both a common dependence on y in the $y \to 1$ limit (viz. $P_\nu^2(y) \sim (1-y)$) and a common dependence on y in the $y \to -1$ limit (viz. $P_\nu^2(y) \sim (1+y)^{-1}$). Thus, for $\nu = 2$ for instance, the quantity $(dP_\nu/d\nu)|_{\nu=2}$ is given by

$$\left.\frac{dP_\nu}{d\nu}\right|_{\nu=2} = \frac{7y^2}{4} + \frac{1}{2}(3y^2-1)\log\left(\frac{1+y}{2}\right) - \frac{3y}{2} - \frac{1}{4} \tag{19.43}$$

as normalized so that as $y \to 1$, $(dP_\nu/d\nu)|_{\nu=2} \to -(2\nu+1)(1-y)/2 = -5(1-y)/2$. From Eq. (19.43) $(dP_\nu^2/d\nu)|_{\nu=2}$ is then constructed as

$$\left.\frac{dP_\nu^2}{d\nu}\right|_{\nu=2} = (1-y^2)\left(8 - \frac{3}{(1+y)} - \frac{1}{(1+y)^2} + 3\log\left(\frac{1+y}{2}\right)\right) . \tag{19.44}$$

Thus for the mode with $\nu = 2 + 3H^2/2b^2$, the wave function is given as

$$P^2_{2+3H^2/2b^2}(y) = P_2^2(y) + \frac{3H^2}{2b^2}\left.\frac{dP_\nu^2}{d\nu}\right|_{\nu=2} , \tag{19.45}$$

and can thus be written entirely in terms of elementary functions as

$$\begin{aligned} P^2_{2+3H^2/2b^2}(y) &= 3(1-y^2)\left[1 + \frac{H^2}{2b^2}\left(8 - \frac{3}{(1+y)} - \frac{1}{(1+y)^2} + 3\log\left(\frac{1+y}{2}\right)\right)\right] \\ &= \frac{12z}{(1+z)^2}\left[1 + \frac{H^2}{2b^2}\left(8 - \frac{3(1+z)}{2z} - \frac{(1+z)^2}{4z^2} + 3\log\left(\frac{z}{1+z}\right)\right)\right] , \end{aligned} \tag{19.46}$$

where $z = H^2e^{2b|w|}/4b^2$. In the small $He^{b|w|}/b$ limit in which $z \to 0$ the $P^2_{2+3H^2/2b^2}(y)$ wave function thus behaves as $-6e^{-2b|w|}$, in complete analog to the behavior found for the lowest lying of the modes in the same kinematic region. However, while the $P^2_{2+3H^2/2b^2}(y)$ wave function even behaves as $e^{-2b|w|}$ in the large $|w|$ limit in which $z \to \infty$, unlike the lowest lying of the modes, it tends to $48b^2e^{-2b|w|}/H^2$, with the coefficient of $e^{-2b|w|}$ explicitly depending on H. A further difference between this mode and the lowest lying one is in its normalization. Specifically, given the form of Eq. (19.46), the integral $\int_{-1+H^2/2b^2}^{1} dy[P^2_{2+3H^2/2b^2}(y)]^2$ can be performed analytically, with it being found to take the finite value 48/5 when $H^2/b^2 \ll 1$. However, from Eq. (19.39), we see that its norm is given by $N_{2+3H^2/2b^2} = 96b/5H^2$, a norm which is thus not finite in the $H \to 0$ limit. Moreover, this behavior repeats in all the higher lying modes as well, since with the general allowed small H ν values being given by $\nu = 1+n+(n-1)(n_2)H^2/4b^2$, the general $P^2_\nu(y)$ is very close to $P^2_{1+n}(y)$, with the leading term in $\int_{-1+H^2/2b^2}^{1} dy[P^2_v(y)]^2$ thus being given by the H-independent $\int_{-1}^{1} dy[P^2_{1+n}(y)]^2 = 2(n+3)!/(2n+3)(n-1)!$. Consequently, all these higher modes have norms which also behave as $1/H^2$ in the small H limit. The lowest lying of the small H modes is thus distinguished from all of the others by being the only one which has a finite norm in the $H \to 0$ limit.

As we have thus found, when the $H \to 0$ limit of modes which obey $P^1_\nu(-\tanh\sigma) = 0$ is taken, in order to continue maintaining the junction condition at the brane, as H goes to zero the allowed masses have to go to zero with it, with the (unnormalized) wave functions of such modes all collapsing onto the massless M_4^+ graviton wave function $e^{-b|w|}$ in the small $He^{b|w|}/b$ limit. Moreover, a similar situation is found for the non-normalizable AdS_4^+ massless graviton wave function (viz. the warp factor wave function $Q^2_1 \sim e^{2A}$) as well, as it also continues directly into the M_4^+ graviton warp factor wave function. However, as well as these particular modes, the AdS_4^+ brane world also possesses modes whose masses do not go to zero in the small H limit, viz. the infinite continuum of non-normalizable KK modes with wave functions of the form $f_m(|w|) = \alpha_m P^2_\nu(\tanh(b|w|-\sigma)) + \beta_m Q^2_\nu(\tanh(b|w|-\sigma))$ as given in Eq. (19.14). Now in Chapter 17 we noted that the massive M_4^+ brane-world modes were of the generic form $f_m(|w|) = \alpha_m J_2(me^{b|w|}/b) + \beta_m Y_2(me^{b|w|}/b)$ given in Eq. (17.5). It thus of interest to see to degree to which these particular sets of massive modes might continue into other in the small H limit. To this end we note first that there is a particular limit in which the associated Legendre functions actually become Bessel functions, viz. a $\nu \to \infty$, fixed x in the real range $-1 \leq x \leq 1$, fixed non-negative μ limit wherein $\nu^\mu P^{-\mu}_\nu(\cos(x/\nu)) \to J_\mu(x)$, $\nu^\mu Q^{-\mu}_\nu(\cos(x/\nu)) \to -(\pi/2)Y_\mu(x)$. For our purposes here we note that for masses which remain fixed as $H^2 \to 0$ (viz. which do not vanish as $H^2 \to 0$), the small H limit of the index of the Legendre functions is given as the divergent $\nu \to m/H$, while the small H limit of the argument of the Legendre functions is given by $y = \tanh(b|w| - \sigma) \to -(1 - H^2e^{2b|w|}/2b^2)$, viz. by $y \to -\cos(x/\nu)$ where $x = me^{b|w|}/b$. Consequently, to bring the associated Legendre functions to a form where we can use the large ν limit, we need to relate associated Legendre functions with positive degree and argument to those with negative degree and argument.

Thus on using the general relations

$$
\begin{aligned}
P^{\mu}_{\nu}(x) &= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[\cos(\mu\pi)P^{-\mu}_{\nu}(x)+\frac{2}{\pi}\sin(\mu\pi)Q^{-\mu}_{\nu}(x)\right] \quad , \\
Q^{\mu}_{\nu}(x) &= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[\cos(\mu\pi)Q^{-\mu}_{\nu}(x)-\frac{\pi}{2}\sin(\mu\pi)P^{-\mu}_{\nu}(x)\right] \quad , \\
P^{\mu}_{\nu}(-x) &= \cos[(\nu+\mu)\pi]P^{\mu}_{\nu}(x)-\frac{2}{\pi}\sin[(\nu+\mu)\pi]Q^{\mu}_{\nu}(x) \\
&= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[\cos(\nu\pi)P^{-\mu}_{\nu}(x)-\frac{2}{\pi}\sin(\nu\pi)Q^{-\mu}_{\nu}(x)\right] \quad , \\
Q^{\mu}_{\nu}(-x) &= -\cos[(\nu+\mu)\pi]Q^{\mu}_{\nu}(x)-\frac{\pi}{2}\sin[(\nu+\mu)\pi]P^{\mu}_{\nu}(x) \\
&= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[-\cos(\nu\pi)Q^{-\mu}_{\nu}(x)-\frac{\pi}{2}\sin(\nu\pi)P^{-\mu}_{\nu}(x)\right] \quad , \qquad (19.47)
\end{aligned}
$$

which hold for general real argument x in the range $-1 \leq x \leq 1$ and without restriction on ν or μ, in the integer $\mu = n$ cases of interest to us we obtain

$$
\begin{aligned}
&P^{n}_{\nu}(y) \to \left(\frac{m}{H}\right)^{n}\left[\ \cos\left(\frac{\pi m}{H}\right)J_n\left(\frac{me^{b|w|}}{b}\right)+\sin\left(\frac{\pi m}{H}\right)Y_n\left(\frac{me^{b|w|}}{b}\right)\right] \quad , \\
&\frac{2}{\pi}Q^{n}_{\nu}(y) \to \left(\frac{m}{H}\right)^{n}\left[-\sin\left(\frac{\pi m}{H}\right)J_n\left(\frac{me^{b|w|}}{b}\right)+\cos\left(\frac{\pi m}{H}\right)Y_n\left(\frac{me^{b|w|}}{b}\right)\right] \quad , \\
&\cos(\nu\pi)P^{n}_{\nu}(y)-\frac{2}{\pi}\sin(\nu\pi)Q^{n}_{\nu}(y) \to \left(\frac{m}{H}\right)^{n}J_n\left(\frac{me^{b|w|}}{b}\right) \quad , \\
&\sin(\nu\pi)P^{n}_{\nu}(y)+\frac{2}{\pi}\cos(\nu\pi)Q^{n}_{\nu}(y) \to \left(\frac{m}{H}\right)^{n}Y_n\left(\frac{me^{b|w|}}{b}\right) \quad , \\
&\left[\alpha_m\cos(\nu\pi)+\beta_m\sin(\nu\pi)\right]P^{n}_{\nu}(y)+\frac{2}{\pi}\left[\beta_m\cos(\nu\pi)-\alpha_m\sin(\nu\pi)\right]Q^{n}_{\nu}(y) \\
&\qquad\qquad \to \left(\frac{m}{H}\right)^{n}\left[\alpha_m J_n\left(\frac{me^{b|w|}}{b}\right)+\beta_m Y_n\left(\frac{me^{b|w|}}{b}\right)\right] \quad , \\
&P^{n}_{\nu}(y)\pm\frac{2i}{\pi}Q^{n}_{\nu}(y) \to \left(\frac{m}{H}\right)^{n}e^{\mp i\pi m/H}\left[J_n\left(\frac{me^{b|w|}}{b}\right)\pm iY_n\left(\frac{me^{b|w|}}{b}\right)\right] \quad ,
\end{aligned}
$$

$$(19.48)$$

where $y = \tanh(b|w| - \sigma)$. With Eq. (19.48) holding for both $n = 2$ and $n = 1$, we see that Eq. (19.14) expressly continues into Eq. (17.5), and, simultaneously, the AdS_4^+ junction condition of Eq. (19.24) expressly continues into the M_4^+ one given in Eq. (17.13). The non-normalizable, massive AdS_4^+ KK continuum modes thus continue directly into the massive M_4^+ KK modes, with the presence of the $1/H^2$ factor multiplying $J_2(me^{b|w|}/b)$ in Eq. (19.48) when $n = 2$ reflecting the fact that the ($H = 0$) M_4^+ continuum of KK modes has to be continuum normalized. Thus none of the discrete, massive AdS_4^+ $P^2_\nu(y)$ modes continue into any of the massive modes that we had found for M_4^+.[17] The AdS_4^+ brane world with non-zero H thus possesses a family of massive modes which do continue into massive M_4^+ modes, together with a second family of massive modes which instead continue into the massless M_4^+ graviton mode alone. While we shall not consider the limits given in Eq. (19.48) any further at the moment, we shall have occasion to return to them below when we analyze the causal structure of AdS_4^+ brane world. However, before doing so we shall first return to our exploration of the discrete normalizable sector.

As regards the issue of the explicit normalization condition which is to be used for the normalizable discrete family of AdS_4^+ $P_\nu^2(y)$ modes, we note that with these $P_\nu^2(y)$ modes both obeying the junction condition at the brane and having vanishing asymptotic flux, all of these modes will, as Eq. (16.9) indicates, thus be orthogonal to each other in the e^{-2A} measure. Since these modes also have finite energy, we can thus normalize them according to the orthonormality relation of Eq. (16.72), viz.

$$\int_{-\infty}^{\infty} dwe^{-2A} f_m(|w|) f_{m'}(|w|) = \delta_{m,m'} \quad , \tag{19.49}$$

as written with the AdS_4^+ warp factor, with the normalized basis modes thus being given by

$$f_m(|w|) = \frac{P_\nu^2(y)}{N_\nu^{1/2}} \quad , \tag{19.50}$$

where

$$N_\nu = \int_{-\infty}^{\infty} dwe^{-2A}[P_\nu^2(|w|)]^2 = 2\int_0^{\infty} d|w|e^{-2A}[P_\nu^2(|w|)]^2 = \frac{2b}{H^2}\int_{-\mathrm{tanh}\sigma}^{1} dy[P_\nu^2(y)]^2. \tag{19.51}$$

19.4 Completeness test for normalizable AdS_4^+ basis modes

Given the form of Eq. (19.49) above, we can anticipate that the normalized modes will obey a completeness relation of the form

$$\sum_m f_m(|w|) f_m(|w'|) = e^{2A}\delta(w - w') \quad . \tag{19.52}$$

[17]While we did find an M_4^+ set of discrete $J_2(m_i e^{b|w|}/b)$ modes with $m_i/b = j_i$ where the j_i are the zeroes of the J_1 Bessel function, none of these masses are in any way small as even the smallest of the j_i are of order one – thus rather than being reachable as a limit of a set of $P_\nu^2(y)$ which obey $P_\nu^1(-\mathrm{tanh}\sigma) = 0$, this particular set of modes is instead reached as the limit of the $\cos(\nu\pi)P_\nu^2(y) - (2/\pi)\sin(\nu\pi)Q_\nu^2(y)$ combinations given in Eq. (19.48) with indices $\nu = (9/4 + b^2 j_i^2/H^2)^{1/2} - 1/2$, combinations which, when H is still non-zero, thus satisfy the AdS_4^+ junction conditions by an interplay of the modes rather than by the modes individually doing so.

However, since it is not immediately obvious that this particular basis of modes is in fact complete, for consistency we check for completeness directly. We thus need to be able to reconstruct a localized square step out of the basis modes, i.e. we need to find a set of expansion coefficients V_m for which the expansion

$$V_P = \sum V_m P^2_\nu(y) \tag{19.53}$$

reproduces the square step $V_P = \hat{V}$ when $|w_1| < |w| < |w_2|$, $V_P = 0$ otherwise. (The discrete m values needed here are related to the allowed discrete ν values by $m^2/H^2 = (\nu-1)(\nu+2)$.) Given the orthogonality of the basis modes, the V_m coefficients are immediately given as $V_m = B_m/N_\nu$, where

$$\begin{aligned}
B_m &= \hat{V}\int_{|w_1|}^{|w_2|} d|w| e^{-2A} P^2_\nu(|w|) = \frac{\hat{V}b}{H^2}\int_{y_1}^{y_2} dy(1-y^2)\frac{d^2P_\nu(y)}{dy^2} \\
&= \frac{\hat{V}b}{H^2}\int_{y_1}^{y_2}\left[\frac{d}{dy}\left(\nu P_{\nu-1} - \nu y P_\nu + 2yP_\nu\right) - 2P_\nu\right] \\
&= \frac{\hat{V}b}{H^2}\left[\frac{(\nu+1)(\nu+2)P_{\nu-1} - \nu(\nu-1)P_{\nu+1}}{2\nu+1}\right]\Bigg|_{y_1}^{y_2} ,
\end{aligned} \tag{19.54}$$

and where use has been made of some convenient properties of the Legendre functions.[18] For the typical case with tanh$\sigma = 0.9$ (viz. $H/b = 0.43589$)[19] and step interval $y_1 = 0.1$, $y_2 = 0.2$, the summation over the first 1000 $P^2_\nu(y)$ modes yields the step shown in Fig. (19.1).[20] As we see, the step is reproduced remarkably well, with the inset showing the Gibbs phenomenon analogous to the one one met in Chapter 18 in the M_4^- case. We are thus assured that the discrete $P^2_\nu(y)$ mode basis is indeed complete.

19.5 Normalizable AdS_4^+ basis mode propagator

With the completeness of the discrete $P^2_\nu(y)$ mode basis and the completeness relation of Eq. (19.52) having now been confirmed, we can now use this specific basis to construct an AdS_4^+ brane-world propagator as a direct analog of Eq. (16.76). Specifically, following Appendix E we first introduce the pure AdS_4 propagator $D_S(x, x', m)$ of Eq. (E.7) which obeys

$$[\partial_x^2 + 3H\partial_x + e^{-2Hx}(\partial_y^2 + \partial_z^2 - \partial_t^2) - m^2]D_S(x,x',m) = e^{-3Hx}\delta^4(x-x') , \tag{19.55}$$

[18] The relations $(1-y^2)dP_\nu/dy = \nu(P_{\nu-1} - yP_\nu) = (\nu+1)(yP_\nu - P_{\nu+1})$ and $(\nu+1)P_{\nu+1} + \nu P_{\nu-1} = (2\nu+1)yP_\nu$ entail that $d(P_{\nu+1} - P_{\nu-1})/dy = (2\nu+1)P_\nu$.

[19] For this particular case the first few zeroes of $P^1_\nu(-0.9)$ occur at $\nu = 1.088$, 2.216 and 3.362.

[20] As with the M_4^- completeness tests, the AdS_4^+ completeness tests presented here are taken from [Guth, Kaiser, Mannheim and Nayeri (2004b)] and [Mannheim and Simbotin (2004)].

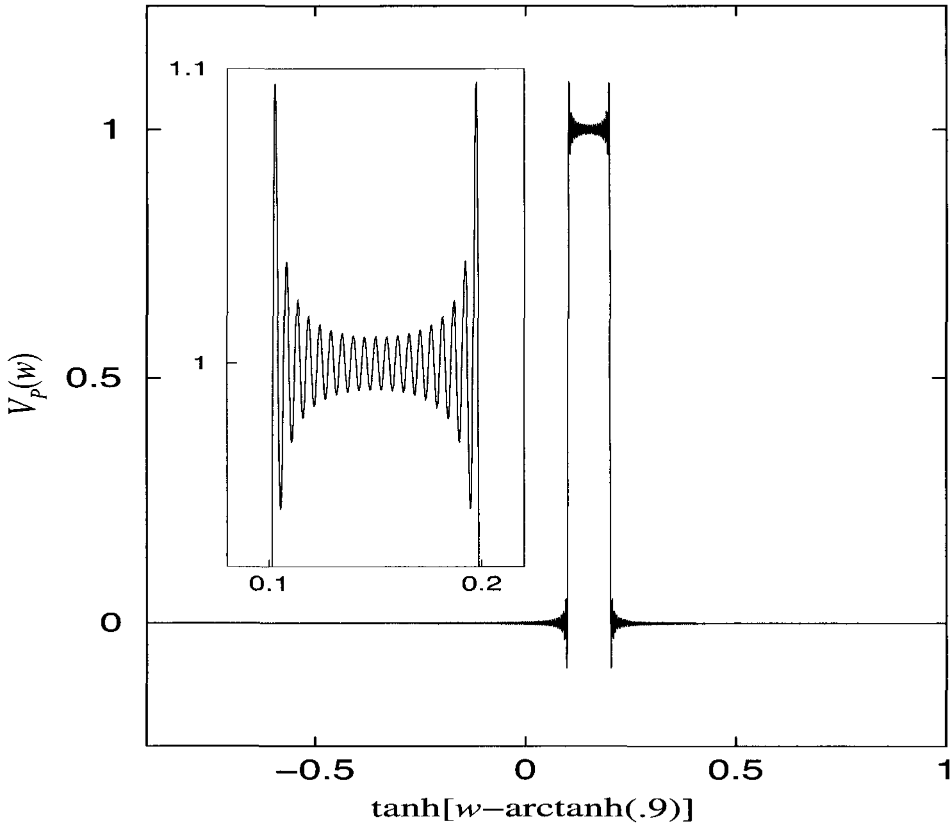

Fig. 19.1 Test of the completeness of the discrete $P^2_\nu(\tanh(b|w|-\sigma))$ TT mode basis by reconstructing the square step $V_P(w)=1$, $0.1 \leq \tanh(b|w|-\text{arctanh}(0.9)) \leq 0.2$, $V_P(w)=0$ otherwise, in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$, and $b=1$.

and in terms of it define a propagator $\hat{D}_S(x,x',m) = e^{2Hx}D_S(x,x',m)e^{2Hx'}$ which obeys

$$[\partial_x^2 - H\partial_x + e^{-2Hx}(\partial_y^2+\partial_z^2-\partial_t^2) - 2H^2 - m^2]\hat{D}_S(x,x',m) = e^{Hx}\delta^4(x-x') \ . \tag{19.56}$$

From this $\hat{D}_S(x,x',m)$ we then construct an AdS_4^+ TT propagator

$$G^{TT}(x,x',w,w') = \sum f_m(|w|)f_m(|w'|)\hat{D}_S(x,x',m) \tag{19.57}$$

which, owing to Eq. (19.52), then obeys

$$\left[\frac{\partial}{\partial|w|^2} + 2\delta(w)\frac{\partial}{\partial|w|} - 4b^2\tanh(b|w|-\sigma) + 4(b^2-H^2)^{1/2}\delta(w)\right.$$
$$\left.+e^{-2A}\left[\partial_x^2 - H\partial_x + e^{-2Hx}(\partial_y^2+\partial_z^2-\partial_t^2) - 4H^2\right]\right]G^{TT}(x,x',w,w')$$
$$= \delta(w-w')e^{Hx}\delta^4(x-x') \ . \tag{19.58}$$

Given such a propagator, the TT fluctuation

$$h_{\mu\nu}^{TT} = -2\kappa_5^2\sum_m f_m(w)f_m(0)\int d^4x' e^{-Hx'}\hat{D}_S(x,x',m)S_{\mu\nu}^{TT}(x') \tag{19.59}$$

is then an exact solution to Eq. (16.1).

Since the solution of Eq. (19.59) is built from normalizable modes alone, each of these modes has a wave function which localizes around the brane (as $|w|\to\infty$, $P_\nu^2(\tanh(b|w|-\sigma)) \to e^{-2b|w|}$). However, as noted in our discussion of the $\delta(t)$ source in the normalizable sectors of the divergent warp factor M_4^- model of Chapter 18 and the divergent warp factor M_3^- model of Appendix F, and as we shall explicitly test for below, the fact that modes may individually localize around the brane does not mean that an infinite summation over all of them necessarily does so as well. Nonetheless, in static, large $|\bar{x}-\bar{x}'|$ kinematic configurations in which only one or a few low lying mass states contribute (so that there is then no need for any infinite summation), there still could be localization, and in $H \ll b$ cases where, according to Eq. (19.36), one mode is altogether lighter than all the others, there could [Karch and Randall (2001)] even be a good approximation to standard long-distance Newtonian gravity. Specifically, even while Eq. (19.59) could not lead to strict standard long-distance gravity on the brane since all of the modes involved in Eq. (19.59) are massive,[21] nonetheless, we can still ask how close Eq. (19.59) might actually come to being able to do so. With the fluctuation of Eq. (19.59) obeying

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h_{\mu\nu}^{TT}(w=0) = -2\kappa_5^2\sum f_m^2(0)S_{\mu\nu}^{TT}(x)$$
$$-2\kappa_5^2\sum f_m^2(0)m^2\int d^4x' e^{-Hx'}\hat{D}_S(x,x',m)S_{\mu\nu}^{TT}(x') \ , \tag{19.60}$$

and with Eq. (15.46) requiring the TT sector to reduce to

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h_{\mu\nu}^{TT}(w=0) = -2\kappa_5^2(b^2-H^2)^{1/2}S_{\mu\nu}^{TT}(x) \tag{19.61}$$

[21] Our discussion of Chapter 15 shows that on the brane even the lightest AdS_4^+ TT mode cannot compete with the AdS_4^+ NT modes at large enough distances since the NT modes propagate in Eq. (15.37) as strictly massless modes on the brane. Since Newton's Law of Gravity is ordinarily applied out to astrophysical scales, the inverse mass of the lightest of the TT modes would thus need to be at least of astrophysical scale if it is to compete with the massless NT modes on such scales.

at large distances if standard gravity is to be obtained on the brane, we see that in the $H^2/b^2 \ll 1$ limit, the large distance domination in Eq. (19.59) of a very light $P^2_\nu(y)$ mode of mass $m = O\left(H^2/b\right)$ would recover standard long distance gravity on the brane if its inverse mass b/H^2 were phenomenologically to be of astrophysical scale, and if its wave function were to come close to obeying $f^2_m(0) \sim (b^2 - H^2)^{1/2}$. Now, as shown in Eq. (19.42), on the brane the wave function of the lowest lying normalized mode is given by $f_m(0) \sim b^{1/2} + H^2/2b^{3/2}$, and thus it is indeed very close to the form $f^2_m(0) \sim (b^2 - H^2)^{1/2}$ required by Eq. (19.61).[22] Now while the normalizable sector of the AdS_4^+ model thus could lead to a viable Newtonian phenomenology if $H \ll b$, this is still quite a severe restriction on the model since it would not be achievable in any other region of the (H, b) parameter space. Also even within this $H \ll b$ region, there could still be (and, as we shall shortly see, in fact are) other kinematic situations which involve non-localizing infinite summations over all of the modes. Finally, this analysis completely leaves out of consideration the non-localizing, non-normalizable sector, and so it is to this issue that we now turn.

19.6 Completeness test for non-normalizable AdS_4^+ basis modes

As with the divergent warp factor M_4^- brane-world model studied in Chapter 18, we can ask whether the non-normalizable AdS_4^+ graviton might also belong to a complete basis. To this end we note that since the AdS_4^+ warp factor wave function $f_0(y) = \beta_0 Q^2_1(y) = 2\beta_0/(1-y^2)$ is an explicit solution to Eqs. (19.10) and (19.24) with asymptotic behavior $f_0(y) \to \beta_0/(1-y)$ as $y \to 1$, the AdS_4^+ massless graviton can then readily be associated with a family of massive $Q^2_\nu(y)$ modes since all such modes also diverge in precisely this same $1/(1-y)$ fashion. All $Q^2_\nu(y)$ type modes automatically satisfy Eq. (19.10), and each will in addition also obey the Eq. (19.24) junction condition for any index ν for which the condition

$$Q^1_\nu(-\tanh\sigma) = Q^1_\nu(-(1 - H^2/b^2)^{1/2}) = 0 \tag{19.62}$$

is satisfied. Thus in precisely the same way as Eq. (19.33) of the $P^2_\nu(y)$ sector is satisfied, for any given value of $\tanh\sigma$ Eq. (19.62) is found to be satisfied in the $Q^2_\nu(y)$ sector by an infinite set of discrete ν values.[23]

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[22] To see how close it comes numerically, we note that for $H^2/b^2 = 0.002$ (viz. $\nu = 1.001$, $m/H = 0.045$, $(b^2 - H^2)^{1/2} = 0.998999b$), $f^2_m(0)$ evaluates to $0.994718b$; while for $H^2/b^2 = 0.00001$ (viz. $\nu = 1.000005$, $m/H = 0.003$, $(b^2 - H^2)^{1/2} = 0.999995b$), $f^2_m(0)$ evaluates to $0.99995b$.

[23] For instance, for the typical value of $\tanh\sigma = 0.9$ considered earlier, the first three non-negative ν values are found to be given as 0.536, 1.649 and 2.788, of which the first one actually corresponds to an $m^2 < 0$ tachyon since it has $\nu < 1$. And in fact, for any given value of $(1 - H^2/b^2)^{1/2}$ it can be shown numerically that Eq. (19.62) will always possess one solution between $\nu = -1/2$ and $\nu = 1$ ($\nu = -1/2$ is the turning point of $m^2/H^2 = (\nu - 1)(\nu + 2)$), i.e. always possess one tachyonic mode. For instance, with the $y \to -1$ limit of $Q^1_{1/2}(y)$ being given by the non-singular $-3\pi(1 + y)^{1/2}/8\sqrt{2}$, for small H the function $Q^1_\nu(-1 + H^2/b^2)$ will have a zero close to $\nu = 1/2$. Moreover, with the general $Q^1_\nu(y)$ behaving as $Q^1_\nu(y) \to -\nu(\nu + 1)\pi\cot(\nu\pi)/\Gamma(1 - \nu)\Gamma(2 + \nu)(1 +$

As with the $P^2_\nu(y)$ modes, to check for completeness of the candidate non-normalizable mode basis we need to be able to expand the localized square step $V_Q = \hat{V}$ when $|w_1| \leq |w| \leq |w_2|$, $V_Q = 0$ otherwise as

$$V_Q = \sum_n V_n Q^2_\nu(y) + \frac{V_0}{1-y^2} \quad . \tag{19.63}$$

(We use n^2 here to denote the squared masses $n^2/H^2 = (\nu - 1)(\nu + 2)$ of the $Q^2_\nu(y)$ basis modes, to distinguish from m^2 which we use for the $P^2_\nu(y)$ modes.) With the left-hand side of Eq. (19.63) being zero for $y = 1$, and with the $Q^2_\nu(y)$ having asymptotic behavior $Q^2_\nu(y) \to Q(\nu)[1/(1-y) + (\nu^2 + \nu - 1)/2]$ as $y \to 1$ (where $Q(\nu) = \nu(1-\nu^2)(\nu+2)\pi/\Gamma(2-\nu)\Gamma(3+\nu)\mathrm{sin}\nu\pi$), the coefficients on the right-hand side of Eq. (19.63) have to be constrained according to

$$\sum_n V_n Q(\nu) + \frac{V_0}{2} = 0 \quad , \quad \frac{1}{2}\sum_n V_n Q(\nu)(\nu^2 + \nu - 1) + \frac{V_0}{4} = 0 \quad . \tag{19.64}$$

We can thus rewrite the square step expansion as

$$V_Q = \sum_n V_n \left[Q^2_\nu(y) - \frac{2Q(\nu)}{1-y^2} \right] \quad , \tag{19.65}$$

as subject to the constraint

$$\sum_n V_n Q(\nu) \left[\nu^2 + \nu - 2\right] = \sum_n V_n Q(\nu) \frac{n^2}{H^2} = 0 \quad . \tag{19.66}$$

To explicitly determine the needed V_n coefficients, it is very convenient to apply $\int_0^\infty d|w| e^{-2A} P^2_{\nu'}(y)$ to both sides of Eq. (19.65) where we now use ν' to label the $P^2_{\nu'}(y)$ sector so that its squared masses are now given by $m^2/H^2 = (\nu' - 1)(\nu' + 2)$. Since both the $P^2_{\nu'}(y)$ and $Q^2_\nu(y)$ are solutions to the source-free wave equation, for any pair of them Eq. (16.9) applies, to yield[24]

$$\begin{aligned}(m^2 - n^2) &\int_0^\infty d|w| e^{-2A} P^2_{\nu'}(y) Q^2_\nu(y) \\ &= \lim_{y \to 1} \left[b P^2_{\nu'}(y) \left((1-y^2)\frac{d}{dy} - 2y \right) Q^2_\nu(y) - b Q^2_\nu(y) \left((1-y^2)\frac{d}{dy} - 2y \right) P^2_{\nu'}(y) \right] .\end{aligned} \tag{19.67}$$

From the known $y \to 1$ behavior of $P^2_{\nu'}(y)$ and $Q^2_\nu(y)$ given earlier we thus obtain

$$\int_0^\infty d|w| e^{-2A} P^2_{\nu'}(y) Q^2_\nu(y) = \frac{4bP(\nu')Q(\nu)}{(m^2 - n^2)} \quad , \tag{19.68}$$

$y)^{1/2}\sqrt{2} + O\left((1+y)^{1/2}\right)$ as $y \to -1$, every half-integer indexed $Q^1_\nu(y)$ will be non-singular in this limit, with the small H solutions to $Q^1_\nu(-\mathrm{tanh}\sigma)$ thus being close to $\nu = 1/2, 3/2, 5/2, \ldots$.

[24]For any given $\mathrm{tanh}\sigma$ Eqs. (19.33) and (19.62) have no common solution, with their zeroes interlacing each other.

where $P(\nu')=\nu'(\nu'^2-1)(\nu'+2)/4$, with the overlap of $P^2_{\nu'}(y)$ and $Q^2_{\nu}(y)$ actually being finite despite the divergence of $Q^2_{\nu}(y)$. With the warp factor graviton also obeying the wave equation, we additionally obtain

$$\int_0^\infty d|w|e^{-2A}\frac{P^2_{\nu'}(y)}{(1-y^2)}=\frac{2bP(\nu')}{m^2} \quad . \tag{19.69}$$

Given Eqs. (19.68) and (19.69), the application of $\int_0^\infty d|w|e^{-2A}P^2_{\nu'}(y)$ to Eq. (19.65) then yields

$$4b\sum_n V_n P(\nu')Q(\nu)\left[\frac{1}{(m^2-n^2)}-\frac{1}{m^2}\right]=B_m \quad , \tag{19.70}$$

where

$$B_m=\int_0^\infty d|w|e^{-2A}P^2_{\nu'}(y)V_Q \tag{19.71}$$

was already given in Eq. (19.54).

With Eq. (19.70) being an eigenvalue equation for the V_n as subject to the constraint of Eq. (19.66), it can readily be diagonalized numerically once a set of m and n are given. Thus for the typical case of $\tanh\sigma=0.9$ and step interval $y_1=0.1$, $y_2=0.2$, the use of the first 1000 $P^2_{\nu'}(y)$ mode m values and the first 1000 $Q^2_{\nu}(y)$ n values for the diagonalization then yields for the summation of Eq. (19.65) the figure shown in Fig. (19.2). As we see, the step is reproduced remarkably well,[25] with the non-normalizable discrete $Q^2_{\nu}(y)$ plus graviton mode basis thus being every bit as complete as the normalizable $P^2_{\nu'}(y)$ basis. Finally, since this discrete basis is itself a subset of the graviton plus full KK continuum basis of $\nu\geq 1$ modes $Q^1_{\nu}(-\tanh\sigma)P^2_{\nu}(y)-P^1_{\nu}(-\tanh\sigma)Q^2_{\nu}(y)$ which satisfy Eq. (19.24) by an interplay between the $P^2_{\nu}(y)$ and $Q^2_{\nu}(y)$ modes, we see that the AdS_4^+ graviton plus KK continuum mode basis is complete also.

19.7 Some exact solutions to the AdS_4^+ fluctuation equations

With it being possible to obtain complete bases both with or without a massless graviton, some additional insight into the structure of the AdS_4^+ brane world can be obtained by looking at some exact solutions to Eq. (16.1). Thus, first we look at some solutions which its realization via Eqs. (16.2) and (16.3), viz.

$$\left[\frac{\partial^2}{\partial|w|^2}-4\left(\frac{dA}{d|w|}\right)^2+e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h^{TT}_{\mu\nu}=0 \quad , \tag{19.72}$$

[25] The construction is so good that the only perceptible difference between Figs. (19.1) and (19.2) is that in the regions close to the edges of the steps the Gibbs phenomenon overshoot as shown in the Fig. (19.1) blow-up is ever so slightly closer to 1.1 than the one shown in the blow-up of Fig. (19.2).

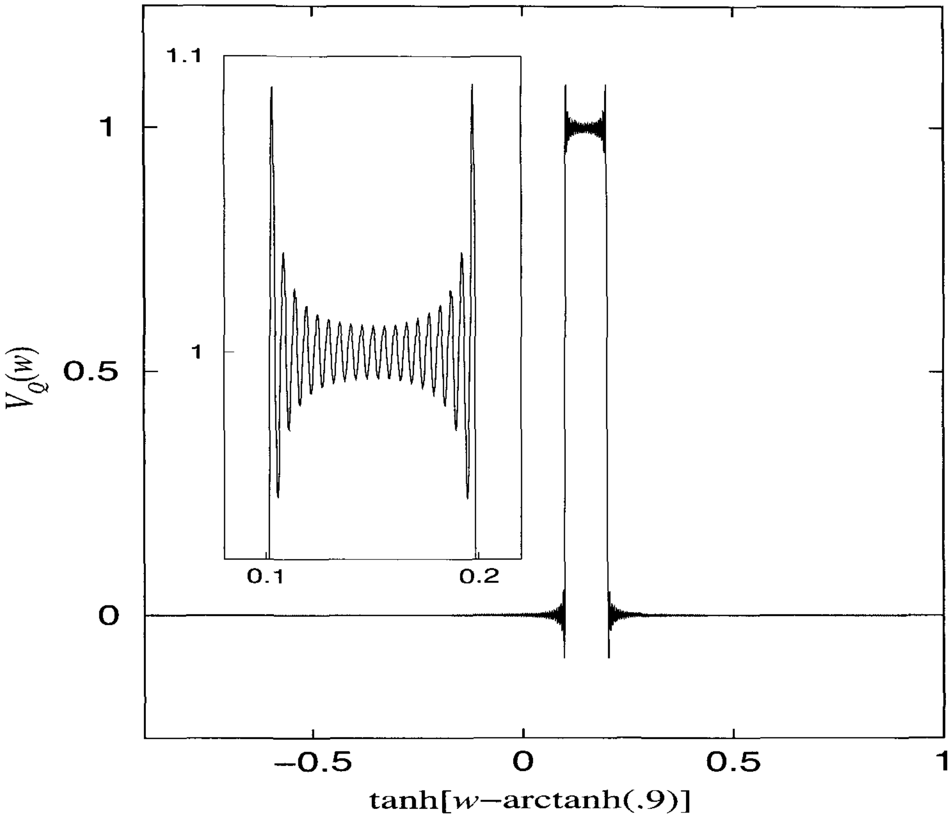

Fig. 19.2 Test of the completeness of the discrete $Q^2_\nu(\tanh(b|w|-\sigma))$ plus $\cosh^2(b|w|-\sigma)$ TT mode basis by reconstructing the square step $V_Q(w)=1$, $0.1 \leq \tanh(b|w| - \text{arctanh}(0.9)) \leq 0.2$, $V_Q(w)=0$ otherwise, in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$, and $b=1$.

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2\frac{dA}{d|w|}\right] h^{TT}_{\mu\nu} = -\kappa_5^2\delta(w)S^{TT}_{\mu\nu} \ , \tag{19.73}$$

happens to possess. Specifically, we consider the special case of a source which obeys $[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]S^{TT}_{\mu\nu} = 0$ (viz. one which, just like the source considered for analogous purposes in Chapter 17, lies on the relevant induced metric lightcone), and note that such sources will obey Eq. (19.6) as evaluated with $m^2 = 0$, and will thus in general be given by the typical form exhibited in Eq. (19.7) as evaluated with $\nu = 1$, or by the special solutions e^{2Hx} or e^{-Hx} that Eq. (19.6) also happens to possess when $m^2 = 0$. For such a source two independent solutions to Eqs. (19.72) and (19.73) can be found, one with and the other without a massless graviton. For the case which includes a graviton we consider as candidate solution

$$h_{\mu\nu}^{TT} = \beta e^{2A} \int d^4x' e^{-Hx'} \hat{D}_S(x, x', m = 0) S_{\mu\nu}^{TT}(x') + f(|w|) S_{\mu\nu}^{TT}(x) \quad , \tag{19.74}$$

where β is a constant, to find that it obeys

$$\begin{aligned} &\left[\frac{\partial^2}{\partial |w|^2} - 4 \left(\frac{dA}{d|w|} \right)^2 + e^{-2A} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \right] h_{\mu\nu}^{TT} \\ &= \left(\beta + \frac{d^2 f}{d|w|^2} - 4 \left(\frac{dA}{d|w|} \right)^2 f - 2 \frac{d^2 A}{d|w|^2} f \right) S_{\mu\nu}^{TT}(x) \quad . \end{aligned} \tag{19.75}$$

The candidate solution will thus obey Eq. (19.72) provided

$$e^{-2A} \frac{d}{d|w|} \left(e^{4A} \frac{dg}{d|w|} \right) + \beta = 0 \quad , \tag{19.76}$$

where $f(|w|) = e^{2A} g(|w|)$. The function $g(|w|)$ must thus obey

$$\begin{aligned} \frac{dg}{d|w|} &= -e^{-4A} \beta \int d|w| e^{2A} \\ &= -\frac{\beta b}{2H^2 \cosh^4(b|w| - \sigma)} \Big(b|w| - \sigma + \cosh(b|w| - \sigma) \sinh(b|w| - \sigma) + \eta \Big) \end{aligned} \tag{19.77}$$

(η is another constant), from which it follows that

$$\begin{aligned} f(|w|) &= \alpha e^{2A} - \frac{\eta \beta}{6b^2} \left(3\cosh(b|w| - \sigma) \sinh(b|w| - \sigma) - \frac{\sinh^3(b|w| - \sigma)}{\cosh(b|w| - \sigma)} \right) \\ &+ \frac{\beta}{6b^2} \left(1 - \frac{(b|w| - \sigma) \sinh(b|w| - \sigma)}{\cosh(b|w| - \sigma)} - 2(b|w| - \sigma) \cosh(b|w| - \sigma) \sinh(b|w| - \sigma) \right. \\ &\left. + 2\cosh^2(b|w| - \sigma) \log[\cosh(b|w| - \sigma)] \right) \quad , \end{aligned} \tag{19.78}$$

where α is yet another integration constant. Finally, in order that the junction condition of Eq. (19.73) also be satisfied, the various constants need to obey

$$\frac{dg(|w| = 0)}{d|w|} = \frac{\beta H^2}{2b^3} \left[\text{arccosh} \left(\frac{b}{H} \right) + \frac{b^2}{H^2} \left(1 - \frac{H^2}{b^2} \right)^{1/2} - \eta \right] = -\kappa_5^2 \quad , \tag{19.79}$$

to thus give us the requisite solution. With Eq. (19.79) thus yielding a β which is finite, we see that despite its lack of normalizability, the graviton contribution to the fluctuation can still be both of the generic form of Eq. (19.59) and possess a finite coefficient. As we see from Eqs. (19.74) and (19.78), the solution which contains the graviton is indeed non-localizing, with this solution being the $H^2 \neq 0$ analog of the M_4^+ Eq. (17.83) where the analog graviton contribution $\int d^4x' D(x - x', m = 0) S_{\mu\nu}^{TT}(x')$ is also multiplied by a warp factor (a warp factor which explicitly is the

$H \to 0$ limit of the AdS_4^+ warp factor), and where the analog α-dependent term of the source-free Eq. (17.85) equally has a coefficient which is not constrained by the fluctuation equations.

For the graviton-independent solution we just set $h_{\mu\nu}^{TT} = f(|w|)S_{\mu\nu}^{TT}(x) = e^{2A}g(|w|)S_{\mu\nu}^{TT}(x)$, to find that this solution will satisfy Eq. (19.72) on its own provided

$$e^{-2A}\frac{d}{d|w|}\left(e^{4A}\frac{dg}{d|w|}\right) = 0 \ . \tag{19.80}$$

Since we have already met this particular equation earlier as Eq. (19.20), the solution is immediately given by Eq. (19.22), which, with a relabeling of the coefficients, is of the form

$$\begin{aligned} f(|w|) &= \gamma\left(\frac{2}{[1+\tanh(b|w|-\sigma)]} - \tanh(b|w|-\sigma)\right) + \delta e^{2A} \\ &= \gamma\left(e^{2\sigma}e^{-2b|w|} + \frac{2}{(1+e^{-2\sigma}e^{2b|w|})}\right) \\ &\quad + \frac{\delta H^2}{4b^2}\left(2 + e^{-2\sigma}e^{2b|w|} + e^{2\sigma}e^{-2b|w|}\right) \quad , \end{aligned} \tag{19.81}$$

to thus involve not just one but both of the two $m^2 = 0$ solutions to the wave equation, one localizing and the other not. The solution of Eq. (19.81) will additionally satisfy the Eq. (19.73) junction condition provided

$$\frac{dg(|w|=0)}{d|w|} = -\frac{3\gamma H^2}{b} = -\kappa_5^2 \ , \tag{19.82}$$

a condition which fixes the parameter γ as $\gamma = \kappa_5^2 b/3H^2$ while leaving the coefficient δ of the non-localizing term unconstrained.

To constrain the parameter δ, we recall that in Chapter 17 we found two independent solutions to the equations of motion for an M_4^+ source which obeyed $\eta^{\alpha\beta}\partial_\alpha\partial_\beta S_{\mu\nu}^{TT} = 0$, viz. the M_4^+ graviton-containing solution of Eq. (17.83) and the M_4^+ graviton-independent solution of Eq. (17.84). Because of the continuity of the fluctuation equations and of the particular choice of sources (both on their respective lightcones) when the $H^2 \to 0$ limit is taken, in this limit the above two AdS_4^+ solutions must continue into their M_4^+ counterparts. As far as the graviton-containing solution of Eq. (19.74) is concerned, we note that according to Eq. (19.79), the parameter β which serves as the coefficient of the AdS_4^+ graviton contribution in Eq. (19.74) tends to $\beta = -2b\kappa_5^2$ in the $H^2 \to 0$ limit (we assume η to be well-behaved in the limit). The β coefficient thus tends to precisely the coefficient of the M_4^+ graviton contribution as given in Eq. (17.83).[26]

[26]In passing we recall that every solution to Eqs. (19.72) and (19.73) must obey the general AdS_4^+ brane relation $[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h_{\mu\nu}^{TT}(w=0) = -2(b^2-H^2)^{1/2}\kappa_5^2 S_{\mu\nu}^{TT} - 2\delta\bar{E}_{\mu\nu}^{TT}(w=0)$ given in Eq. (15.49). With the solution of Eq. (19.74) obeying the relation $[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]h_{\mu\nu}^{TT} = \beta e^{2A}S_{\mu\nu}^{TT}$

While the $H^2 \to 0$ limit of the coefficient β is continuous, this is not the case for the $\gamma = \kappa_5^2 b/3H^2$ coefficient. Hence to maintain continuity in the graviton-independent sector the parameter δ in Eq. (19.81) will have to cancel the singular terms generated by γ, and can therefore not be arbitrarily assigned. Since the solution of Eq. (19.81) behaves as

$$f(|w|) \to \gamma\left(\frac{4b^2e^{-2b|w|}}{H^2}+2-\frac{H^2e^{2b|w|}}{2b^2}\right) + \frac{\delta H^2}{4b^2}\left(2+\frac{H^2e^{2b|w|}}{4b^2}+\frac{4b^2e^{-2b|w|}}{H^2}\right) \tag{19.83}$$

in the small H limit, Eq. (17.84) will explicitly be recovered if the leading small H behaviors of the γ and δ parameters of Eqs. (19.81) and (19.82) are of the form

$$\gamma = \frac{\kappa_5^2 b}{3H^2}+\tau \quad , \quad \delta = -\frac{4\kappa_5^2 b^3}{3H^4}-\frac{4b^2\tau}{H^2}+\frac{\alpha\kappa_5^2}{2b} \quad , \tag{19.84}$$

where τ is arbitrary and α is the source-free equation coefficient given in Eq. (17.85). The emergence of a dependence of δ on the coupling constant κ_5^2 in its key leading $-4\kappa_5^2 b^3/3H^4$ term is at first surprising since we had previously found that the δ term in Eq. (19.81) decoupled completely from the junction condition equation of Eq. (19.73). However, while this is the case for the exact solution, it is not the case for its leading small H term alone. Specifically, with $A'(|w|=0) = -b\tanh\sigma$ behaving as $A'(|w|=0) = -b + H^2/2b$ at small H, the action of the left hand side of Eq. (19.73) on the γ-dependent term in Eq. (19.83) is found to yield

$$\delta(w)\left(\frac{d}{d|w|}+2b-\frac{H^2}{b}\right)\gamma\left(\frac{4b^2e^{-2b|w|}}{H^2}+2-\frac{H^2e^{2b|w|}}{2b^2}\right) = -\frac{4\kappa_5^2}{3}\delta(w) \quad , \tag{19.85}$$

while its action on the δ-dependent term in Eq. (19.83) is found to yield the non-zero

$$\delta(w)\left(\frac{d}{d|w|}+2b-\frac{H^2}{b}\right)\frac{\delta H^2}{4b^2}\left(2+\frac{H^2e^{2b|w|}}{4b^2}+\frac{4b^2e^{-2b|w|}}{H^2}\right) = \frac{\kappa_5^2}{3}\delta(w) \quad . \tag{19.86}$$

Thus both terms now contribute to the junction condition, and it is only their sum which produces the net $-\kappa_5^2\delta(w)$ term required for the right hand side of Eq. (19.73). Finally then, since the δ parameter cannot be set equal to zero, the solution of Eq. (19.81) thus does not localize around the AdS_4^+ brane. Hence in contrast to M_4^+ where one of the two solutions associated with our specifically chosen source is localizing, in AdS_4^+ neither of them is. Despite the nice localizing behavior of the discrete, graviton-independent $P_\nu^2(y)$ modes then, we see that in the graviton-independent sector of AdS_4^+ there are sources for which the fluctuations associated with Eqs. (19.72) and (19.73) do not localize to the brane.

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identically, we see that other than for small H, unless the parameter η is judiciously chosen, compatibility of this solution with Eq. (15.49) can only be achieved via a non-vanishing Weyl tensor contribution on the brane, a situation in which the couplings of the massless TT modes on the brane would then not correspond to those of the massless NT modes given in Eq. (15.37).

In addition to the realization of Eq. (16.1) via Eqs. (19.72) and (19.73), we must also look at its realization via Eqs. (16.79) and (16.80), a realization which is based on the propagator of Eq. (19.59) as built out of the normalizable, graviton-non-containing, $P^2_\nu(y)$ basis modes. On conveniently specializing to the particular source $S^{TT}_{\mu\nu} = A_{\mu\nu}e^{-Hx}$ which obeys $[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + 2H^2]S^{TT}_{\mu\nu} = 0$, the integration required for Eq. (19.59) can then be readily carried out. Thus, on setting $\sigma = (9/4+m^2/H^2)^{1/2}$ in Eq. (E.7) and on making the change of variable $v = e^{-Hx'}/H$, we obtain

$$\int d^4x' e^{-Hx'}\hat{D}_S(x,x',m)S^{TT}_{\mu\nu}(x')$$
$$= -\frac{A_{\mu\nu}}{H}\int_{-\infty}^{\infty} dx' \int_0^\infty \frac{d\hat{k}}{\hat{k}} e^{Hx/2}e^{-3Hx'/2}J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}e^{-Hx'}H)$$
$$= -\frac{A_{\mu\nu}e^{Hx/2}}{H^{1/2}}\int_0^\infty dv v^{1/2}\int_0^\infty \frac{d\hat{k}}{\hat{k}}J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}v)$$
$$= -\frac{A_{\mu\nu}e^{Hx/2}}{2\sigma H^{1/2}}\left(\int_0^{e^{-Hx}/H} dv H^\sigma e^{\sigma Hx}v^{1/2+\sigma} + \int_{e^{-Hx}/H}^\infty dv H^{-\sigma}e^{-\sigma Hx}v^{1/2-\sigma}\right)$$
$$= -\frac{A_{\mu\nu}e^{-Hx}}{H^2(\sigma^2-9/4)} = -\frac{A_{\mu\nu}e^{-Hx}}{m^2} \quad , \tag{19.87}$$

which we in passing note is nicely in concordance with the analogous M_4^+ expression given in Eq. (17.92). For the $S^{TT}_{\mu\nu} = A_{\mu\nu}e^{-Hx}$ source then Eq. (19.59) simplifies to

$$h^{TT}_{\mu\nu} = 2\kappa_5^2 A_{\mu\nu}e^{-Hx}\sum_m \frac{f_m(|w|)f_m(0)}{m^2} \equiv 2\kappa_5^2 S^{TT}_{\mu\nu}F(|w|) \quad . \tag{19.88}$$

With each discrete $f_m(|w|) = P^2_\nu(y)/N_\nu^{1/2}$ normalizable mode obeying Eqs. (19.1), (19.2) and (19.52), the $F(|w|)$ mode sum is then found to obey

$$\left[\frac{d^2}{d|w|^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 2H^2e^{-2A}\right]F(|w|) = -e^{-2A}\sum_m f_m(|w|)f_m(0) = -\delta(w) \quad ,$$
$$2\delta(w)\left[\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right]F(|w|) = 0 \quad , \tag{19.89}$$

with explicit, graviton-non-containing, exact solution

$$F(|w|) = \frac{(e^{2A}-e^{-2A})}{4(b^2-H^2)^{1/2}}\theta(|w|) + \frac{b}{6H^2}\left(\frac{2}{[1+\tanh(b|w|-\sigma)]} - \tanh(b|w|-\sigma)\right) \quad . \tag{19.90}$$

With the resulting $h^{TT}_{\mu\nu}$ fluctuation thus being found to actually diverge at large $|w|$, we see that despite the fact that every single mode in the complete basis of Eq. (19.52) is itself localized, their sum is not. This particular example thus shows how a summation over an infinite set of localizing modes need not necessarily lead to a sum which localizes too.

As such, the implications of the above three exact solutions are quite severe for the AdS_4^+ brane world – in none of the cases do we get localization of gravity to the brane, not even in the last case where we restrict to localized, normalizable modes alone.[27] Since gravity does not localize if we do include the massless graviton, and does not do so even if we exclude it, we have to conclude that the AdS_4^+ brane world is not of relevance to the brane-gravity localization program. However, still unresolved is the status of the graviton in the theory, and whether it is anyway legitimate to leave it out just because it is not normalizable. And indeed, our reconstruction earlier of localized square steps shows that the non-normalizable sector is every bit as complete as the normalizable one. Hence, to resolve the issue we therefore turn to an analysis of causality in the AdS_4^+ brane world.

19.8 Constructing the causal AdS_4^+ propagator

Given our previous analysis of causality in the M_4^+ and M_4^- brane worlds, we need to construct the AdS_4^+ brane-world propagator not as an Eq. (19.57) type mode sum, but rather as an analog of the M_4^+ propagator given in Eq. (17.30) as

$$\hat{G}^{TT}(x,x',w,0;M_4^+) = \frac{1}{2(2\pi)^4}\int d^4p e^{ip\cdot(x-x')}\frac{[\alpha_q J_2(qe^{b|w|}/b) + \beta_q Y_2(qe^{b|w|}/b)]}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]} , \tag{19.91}$$

where $q^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$. While we have not been able to construct an AdS_4^+ analog of Eq. (19.91) which is valid for arbitrary H, we have been able to construct one which is valid for small H; and this will in fact suffice for our purposes here as it will not only yield an AdS_4^+ propagator which explicitly continues into the M_4^+ one in the $H \to 0$ limit, it will also enable us to monitor causality in the small, but non-zero, H AdS_4^+ brane world itself, an adequate enough measure of the causality structure of the AdS_4^+ brane world. As such, the propagator of Eq. (19.91) consists of two components, a w-dependent piece and an integration over a complete set of plane wave modes associated with the M_4 wave equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h^{TT}_{\mu\nu} = q^2 h^{TT}_{\mu\nu}$. The analog of the w-dependent piece is readily given as

$$\begin{aligned} &B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu) \\ &= \frac{1}{H(\nu-1)(\nu+2)}\left[\frac{\hat{\alpha}_\nu P_\nu^2(\tanh(b|w|-\sigma)) + \hat{\beta}_\nu Q_\nu^2(\tanh(b|w|-\sigma))}{\hat{\alpha}_\nu P_\nu^1(-\tanh\sigma) + \hat{\beta}_\nu Q_\nu^1(-\tanh\sigma)}\right] \end{aligned} \tag{19.92}$$

since it obeys

$$\delta(w)\left[\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right] B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu) = \delta(w) \ . \tag{19.93}$$

[27] Failure to localize for even just one brane source is sufficient to establish lack of localization.

In Eq. (19.92) ν is given by $\nu = (9/4+q^2/H^2)^{1/2}-1/2$ and q^2 is again given by $q^2 = (p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2$. The needed analog mode basis is given as the solutions to Eq. (19.6). In terms of the variables $v = e^{-Hx}/H$ and $k = [(p^0)^2-(p^2)^2-(p^3)^2]^{1/2}$, solutions to Eq. (19.6) of the form $e^{-ip^0t+ip^2y+ip^3z}A_\tau(k,v)$ will obey

$$\left[\frac{d^2}{dv^2}+\frac{2}{v}\frac{d}{dv}+k^2-\frac{2}{v^2}-\frac{q^2}{H^2v^2}\right]A_\tau(k,v)=0 \quad , \tag{19.94}$$

to thus be of the form $A_\tau(k,v) = Z_\tau(kv)/v^{1/2}$ where $\tau = (9/4+q^2/H^2)^{1/2}$, and $Z_\tau(kv)$ is an appropriate Bessel function.

Since Eq. (19.91) involves a four-dimensional integration over four sets of eigenvalues, viz. p^0, p^2, p^3 and p^1, the analog integration in the AdS_4^+ case would have to be over the four eigenvalues associated with the solutions of Eq. (19.94), viz. p^0, p^2, p^3 and τ, and even though we can trade p^1 for τ since $\tau = [9/4+(k^2-(p^1)^2)/H^2]^{1/2}$, nonetheless, this time the need to sum over τ necessitates finding a completeness relation for Bessel functions with respect to index. To determine such a possible relation we note that manipulation of Eq. (19.94) obliges pairs of mode solutions to have to obey

$$\begin{aligned}(\tau_1^2-\tau_2^2)\int_0^\infty dvA_{\tau_1}(k_1,v)A_{\tau_2}(k_2,v)-(k_1^2-k_2^2)\int_0^\infty dvv^2A_{\tau_1}(k_1,v)A_{\tau_2}(k_2,v)\\ =\left(v^2A_{\tau_2}(k_2,v)\frac{d}{dv}A_{\tau_1}(k_1,v)-v^2A_{\tau_1}(k_1,v)\frac{d}{dv}A_{\tau_2}(k_2,v)\right)\Bigg|_0^\infty\end{aligned} \tag{19.95}$$

(as integrated over the allowed $0\le v\le\infty$ range for $v = e^{-Hx}/H$). Any two Bessel function solutions $Z_{\tau_1}(k_1v)$ and $\hat{Z}_{\tau_2}(k_2v)$ thus have to obey

$$\begin{aligned}(\tau_1^2-\tau_2^2)\int_0^\infty \frac{dv}{v}Z_{\tau_1}(k_1v)\hat{Z}_{\tau_2}(k_2v)-(k_1^2-k_2^2)\int_0^\infty dvvZ_{\tau_1}(k_1v)\hat{Z}_{\tau_2}(k_2v)\\ =\left(v\hat{Z}_{\tau_2}(k_2v)\frac{d}{dv}Z_{\tau_1}(k_1v)-vZ_{\tau_1}(k_1v)\frac{d}{dv}\hat{Z}_{\tau_2}(k_2v)\right)\Bigg|_0^\infty \quad .\end{aligned} \tag{19.96}$$

Consequently, for the two most relevant choices of Bessel function combinations (viz. the $J_\tau(kv)$ based one associated with the pure AdS_4 spacetime causal propagator as constructed in Appendix E, and, for completeness, the Hankel function based one associated with the dS_4 causal propagator as also given in Appendix E), we obtain for modes with different index but the same k

$$\begin{aligned}(\tau_1^2-\tau_2^2)\int_0^\infty\frac{dv}{v}J_{\tau_1}(kv)J_{\tau_2}(kv)&=\frac{2}{\pi}\sin\left(\frac{\pi(\tau_1-\tau_2)}{2}\right)-O(v^{\tau_1+\tau_2})\Bigg|_{v=0}\quad ,\\ (\tau_1^2-\tau_2^2)\int_0^\infty\frac{dv}{v}H^{(1)}_{\tau_1}(kv)H^{(2)}_{\tau_2}(kv)&=\frac{4i}{\pi}\cos\left(\frac{\pi(\tau_1-\tau_2)}{2}\right)+\frac{4}{\pi}\sin\left(\frac{\pi(\tau_1-\tau_2)}{2}\right)\\ -\left[\frac{(\tau_2-\tau_1)}{\sin(\pi\tau_1)\sin(\pi\tau_2)\Gamma(1-\tau_1)\Gamma(1-\tau_2)}\left(\frac{2}{kv}\right)^{\tau_1+\tau_2}\right.&\left.\left(1+O(v^2)\right)\right]\Bigg|_{v=0}\quad ,\end{aligned} \tag{19.97}$$

with the $v = 0$ lower limit contribution to the $J_\tau(kv)$ based integral vanishing when $\mathrm{Re}[\tau_1 + \tau_2] > 0$. Similarly, for modes with a common τ but different k we obtain

$$\int_0^\infty dvvJ_\tau(k_1v)J_\tau(k_2v) = \frac{\delta(k_1 - k_2)}{k_1} - \frac{\sin(\pi\tau)\delta(k_1 + k_2)}{(k_1k_2)^{1/2}}$$
$$- \left[\frac{\cos(\pi\tau)\cos(k_1v + k_2v)}{\pi(k_1k_2)^{1/2}(k_1 + k_2)}\right]\Bigg|_{v=\infty},$$
$$\int_0^\infty dvvH_\tau^{(1)}(k_1v)H_\tau^{(2)}(k_2v) = \frac{2\delta(k_1 - k_2)}{k_1} - \left[\frac{2i\cos(k_1v - k_2v)}{\pi(k_1k_2)^{1/2}(k_1 - k_2)}\right]\Bigg|_{v=\infty}$$
$$- \left[\frac{2^{2\tau-1}\tau}{\sin^2(\pi\tau)\Gamma(1 - \tau)\Gamma(2 - \tau)(k_1k_2)^\tau}v^{2-2\tau}\left(1 + O(v^2)\right)\right]\Bigg|_{v=0}, \tag{19.98}$$

with the $v = 0$ lower limit contribution to the $J_\tau(k_1v)$ based integral expression given in Eq. (19.98) being the one obtained when $\mathrm{Re}[\tau] > 0$, and that given for the Hankel function based integral being the one obtained when $\mathrm{Re}[\tau] > 1$.[28] As we see, while $J_\tau(kv)$ modes of different k_1 and k_2 are orthogonal when $k_2 \neq -k_1$,[29] unfortunately $J_\tau(kv)$ modes of different τ are not. Despite this, we have nonetheless found that it is still possible to obtain an orthogonality relation for $J_\tau(kv)$ modes of different τ in the special case where H is small, one which, despite the differences, we will even be able to extend to the Hankel function combination in our analysis of the dS_4^+ brane world which we give in Chapter 21 below.

Thus, with $J_\tau(kv)$ modes of the same τ obeying the orthogonality relation given in Eq. (19.98) when $\mathrm{Re}[\tau] > 0$, on respectively replacing the v, k_1 and k_2 variables in Eq. (19.98) by k, v_1 and v_2, for the non-negative $v_1 = e^{-Hx}/H$ and $v_2 = e^{-Hx'}/H$ we obtain

$$\int_0^\infty dkkJ_\tau(kv_1)J_\tau(kv_2) = \frac{\delta(v_1 - v_2)}{v_1} = He^{Hx_1}e^{Hx_2}\delta(x_1 - x_2) \tag{19.99}$$

when $\mathrm{Re}[\tau] > 0$. However, even though taking the $\tau_1 \to \tau_2$ limit of Eq. (19.97) allows us to determine a finite normalization integral of the form[30]

$$\int_0^\infty \frac{dv}{v}[J_\tau(kv)]^2 = \frac{1}{2\tau} \tag{19.100}$$

[28]An alternate expression for the leading contribution of the Hankel function expression lower limit term is obtained when $1 > \mathrm{Re}[\tau] > 0$, with all of the Hankel function $\mathrm{Re}[\tau] > 0$ relations being adaptable to the $\mathrm{Re}[\tau] < 0$ case since $H_{-\tau}^{(1)}(z) = e^{i\pi\tau}H_\tau^{(1)}(z)$, $H_{-\tau}^{(2)}(z) = e^{-i\pi\tau}H_\tau^{(2)}(z)$. The treatment of the $J_\tau(k_1v)$ based integral in cases where $\mathrm{Re}[\tau] < 0$ is, however, not as straightforward, since $J_{-\tau}(z) = \cos(\tau\pi)J_\tau(z) - \sin(\tau\pi)Y_\tau(z)$.

[29]We need $k_1 \neq -k_2$ in order to get the $\cos(k_1v + k_2v)$ term to oscillate away at $v = \infty$. And in passing we also note that with the $\cos(k_1v + k_2v)$ term being an even function of v, an extension of the range of v to $-\infty$ yields the additional orthogonality relation $\int_{-\infty}^\infty dvvJ_\tau(k_1v)J_\tau(k_2v) = 2\delta(k_1 - k_2)/k_1 - 2\sin(\pi\tau)\delta(k_1 + k_2)/(k_1k_2)^{1/2}$ in cases where $\mathrm{Re}[\tau] > 0$ (viz. cases where the integrand has no singularity at $v = 0$).

[30]The analog Hankel function based integral diverges.

when $\mathrm{Re}[\tau] > 0$, because of the lack of orthogonality of the $J_\tau(kv)$ with differing τ we cannot infer that there is any completeness relation of a form, for example, such as

$$
\begin{aligned}
&-\int_{(9/4+k^2/H^2)^{1/2}}^{i\infty} d\tau\tau J_\tau(kv_1)J_\tau(kv_2) \\
&= \int_0^\infty \frac{dp^1 p^1}{H^2} J_\tau(kv_1)J_\tau(kv_2)\Big|_{\tau=(9/4+k^2/H^2-(p^1)^2/H^2)^{1/2}} \\
&= v_1\delta(v_1 - v_2) = \frac{\delta(x_1 - x_2)}{H} \quad . \qquad (19.101)
\end{aligned}
$$

Now even while any Eq. (19.101) type relation might not be expected to hold in general, from Eq. (19.97) we note that in the small H limit $J_\tau(kv)$ modes with different τ do become orthogonal in first order since when both $\mathrm{Re}[\tau_1]$ and $\mathrm{Re}[\tau_2]$ are greater than zero, their overlap integrals behave as the second order

$$
\int_0^\infty \frac{dv}{v} J_{\tau_1}(kv)J_{\tau_2}(kv) = \frac{2}{\pi(\tau_1^2 - \tau_2^2)}\sin\left(\frac{\pi(\tau_1 - \tau_2)}{2}\right) \to O\left(H^2\right) \qquad (19.102)
$$

in the limit. Consequently, a relation such as Eq. (19.101) might possibly at least be valid to lowest order in H, and in fact, as we shall immediately see, despite our initial motivation, not only will such a relation prove to hold for small H, it will more generally even be found to hold for values of τ real or complex, including those with $\mathrm{Re}[\tau] < 0$.

To check for this possibility we need to appeal to the behavior of the $J_\tau(z)$ Bessel functions when both τ and z are large and complex, with consideration of four specific cases being needed. As we had noted in our study of the $\hat{G}^{TT}(x, x', w, 0; M_4^+)$ propagator in Chapter 17, our convention for defining the retarded contour integration is to give $q = (k^2 - (p^1)^2)^{1/2}$ and p^0 a common sign. Thus with τ behaving as q/H when H is small, for $p_R^0 > 0$ (and thus $p_I^0 = \epsilon\epsilon(p_R^0) > 0$ for the retarded contour prescription) we have $\tau_R = +(k^2 - (p^1)^2)^{1/2}/H, \tau_I = 0$ when $k^2 > (p^1)^2$, and have $\tau_R = 0, \tau_I = +((p^1)^2 - k^2)^{1/2}/H$ when $k^2 < (p^1)^2$. Similarly, for $p_R^0 < 0$ (and thus $p_I^0 < 0$) we have $\tau_R = -(k^2 - (p^1)^2)^{1/2}/H, \tau_I = 0$ when $k^2 > (p^1)^2$, and have $\tau_R = 0, \tau_I = -((p^1)^2 - k^2)^{1/2}/H$ when $k^2 < (p^1)^2$. For the asymptotic behavior of the Bessel functions when both a complex index and a complex argument become large, the leading behavior is given by [Watson (1962)]

$$
J_\tau\left(\frac{\tau}{\cosh\gamma}\right) \pm iY_\tau\left(\frac{\tau}{\cosh\gamma}\right) \to \frac{2^{1/2}e^{\pm(\tau\tanh\gamma - \tau\gamma - i\pi/4)}}{[-i\tau\pi\tanh\gamma]^{1/2}} \quad , \qquad (19.103)
$$

where $\gamma = \alpha + i\beta$ is in general complex. For the case first where $p_R^0 > 0$ and $k^2 > (p^1)^2$ we have $\tau = +(k^2 - (p^1)^2)^{1/2}/H$, and with $\cosh(\alpha + i\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta$ and $\tau/\cosh\gamma = ke^{Hx}/H$ thus set $\alpha = 0$. With the argument of each Bessel function now being of the form $\tau/\cos\beta$, the large τ behavior is thus given by $J_\tau(\tau/\cos\beta) \to (e^{i\psi} + e^{-i\psi})/(2\pi\tau\tan\beta)^{1/2}$ where $\psi = \tau(\tan\beta - \beta) - \pi/4$. Thus, with

$\tanh\gamma = i\tan\beta = i[(p^1)^2 - k^2 Hx]/p^1(k^2-(p^1)^2)^{1/2}$ when H is small (and analogously for the x' term), we find that the requisite β and ψ are given as $\beta = \arctan(p^1/(k^2-(p^1)^2)^{1/2}) - Hx(k^2-(p^1)^2)^{1/2}/p^1$ and $\psi = p^1/H - ((k^2-(p^1)^2)^{1/2}/H)\arctan(p^1/(k^2-(p^1)^2)^{1/2}) - \pi/4 - p^1 x$ when H is small. Consequently, on dropping the terms which oscillate away on integrating (viz. on only retaining the $e^{i(\psi_1-\psi_2)} + e^{-i(\psi_1-\psi_2)}$ terms in the $(e^{i\psi_1} + e^{-i\psi_1})(e^{i\psi_2} + e^{-i\psi_2})$ cross-product), in the small H limit we obtain

$$J_\tau\left(\frac{ke^{-Hx}}{H}\right) J_\tau\left(\frac{ke^{-Hx'}}{H}\right) \to \frac{H}{2\pi p^1}\left(e^{-ip^1(x-x')} + e^{-ip^1(x'-x)}\right) \tag{19.104}$$

when $p_R^0 > 0$ and $k^2 > (p^1)^2$. The case with $p_R^0 > 0$ and $k^2 < (p^1)^2$ (a situation in which k^2, when equal to $(p^0)^2 - (p^2)^2 - (p^3)^2$, could be positive or negative) corresponds to $\tanh\gamma = [(p^1)^2 - k^2 Hx]/p^1((p^1)^2 - k^2)^{1/2}$ for small H, and is treated completely analogously, leading again to the same limiting form as displayed on the right-hand side of Eq. (19.104). Similarly, the cases with p_R^0 negative only differ from their $p_R^0 > 0$ counterparts by a shift of β by π, and thus also recover the right-hand side of Eq. (19.104), just as required to thereby validate Eq. (19.101) (with its now justified numerical factors) in the small H limit for large complex τ.

Once we now have Eq. (19.104) for all the above combinations of signs of p_R^0 and $k^2 - (p^1)^2$, the AdS_4^+ propagator defined as

$$\begin{aligned}
&\hat{G}^{TT}(x, x', w, 0; AdS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) \\
&= \frac{1}{2H(2\pi)^3}\int_{-\infty}^{\infty} dp^0 dp^2 dp^3 \int_0^{\infty} dp^1 p^1 B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) e^{Hx/2} e^{Hx'/2} \\
&\quad \times e^{-ip^0(t-t') + ip^2(y-y') + ip^3(z-z')} J_\tau(ke^{-Hx}/H) J_\tau(ke^{-Hx'}/H)
\end{aligned} \tag{19.105}$$

(where k is still given by $k = [(p^0)^2 - (p^2)^2 - (p^3)^2]^{1/2}$ and τ by $\tau = \nu + 1/2 = [9/4 + k^2/H^2 - (p^1)^2/H^2]^{1/2}$) is then readily found to obey

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial w^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 4\frac{dA}{d|w|}\delta(w) + e^{-2A}\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha\right] \hat{G}^{TT}(x, x', w, 0; AdS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) \\
&= e^{Hx}\delta(x - x')\delta(t - t')\delta(y - y')\delta(z - z')\delta(w) \ ,
\end{aligned} \tag{19.106}$$

in the small H limit just as we want. Consequently, the fluctuation

$$h_{\mu\nu}^{TT}(x) = -2\kappa_5^2 \int d^4x' e^{-Hx'} \hat{G}^{TT}(x, x', w, 0; AdS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) S_{\mu\nu}^{TT}(x') \tag{19.107}$$

is an exact AdS_4^+ brane world small H solution to Eqs. (19.72) and (19.73) for an arbitrary source on the brane.[31]

[31]In constructing the $h_{\mu\nu}^{TT}(x)$ fluctuation from Eq. (19.107), it is understood that one should take the real part alone if necessary.

To make contact with the M_4^+ propagator of Eq. (19.91) we recall from Eq. (19.48) that the small H limit of $B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu)$ is given by

$$B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu)\to\frac{[\alpha_q J_2(qe^{b|w|}/b)+\beta_q Y_2(qe^{b|w|}/b)]}{q[\alpha_q J_1(q/b)+\beta_q Y_1(q/b)]} \tag{19.108}$$

when $\hat{\alpha}_\nu=\alpha_q\cos(\nu\pi)+\beta_q\sin(\nu\pi)$, $\hat{\beta}_\nu=(2/\pi)[-\alpha_q\sin(\nu\pi)+\beta_q\cos(\nu\pi)]$. Thus, given Eq. (19.104), we confirm that the AdS_4^+ propagator precisely continues into the M_4^+ one in the limit in which H goes to zero, just as we would want it to.

Having obtained the propagator of Eq. (19.105), we can now make contact with Eq. (19.57) when it too is evaluated for small H. Specifically, when we set $\hat{\beta}_\nu=0$ in B_ν, the $P^1_\nu(-\tanh\sigma)$ term in the denominator of Eq. (19.105) will then generate a set of poles at the discrete $\nu_n=(9/4+m_n^2/H^2)^{1/2}-1/2=\tau_n-1/2$ zeroes of $P^1_\nu(-\tanh\sigma)$, masses which, as noted previously, will themselves be small when H is small, though none will actually be zero.[32] Such poles thus provide a pole contribution to the propagator of the form

$$\begin{aligned}&\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu=1,\hat{\beta}_\nu=0,\mathrm{POLE})\\&=\sum_n\frac{1}{2(2\pi)^3}\int_{-\infty}^{\infty}dp^0dp^2dp^3\int_0^\infty dp^1p^1e^{-ip^0(t-t')+ip^2(y-y')+ip^3(z-z')}\\&\times\frac{e^{Hx/2}e^{Hx'/2}J_{\tau_n}(k_ne^{-Hx}/H)J_{\tau_n}(k_ne^{-Hx'}/H)P^2_{\nu_n}(\tanh(b|w|-\sigma))(2\nu_n+1)}{(\nu_n-1)(\nu_n+2)(q^2-m_n^2)(dP^1_\nu(-\tanh\sigma)/d\nu)|_{\nu=\nu_n}}\ .\end{aligned} \tag{19.109}$$

In Eq. (19.109) we have set $k\to k_n=((p^1)^2+m_n^2)^{1/2}$ since that is the value that k will take at each pole once the p^0 contour integration is performed. In terms of the small H, small mass pure AdS_4 propagator

$$\begin{aligned}\hat{D}_S(x,x',m)&=\frac{(e^{Hx/2}e^{Hx'/2})}{(2\pi)^3H}\int_{-\infty}^{\infty}dp^0dp^2dp^3\int_0^\infty dp^1p^1J_\tau(p^1e^{-Hx}/H)\\&\times J_\tau(p^1e^{-Hx'}/H)\left(\frac{e^{-ip^0(t-t')+ip^2(y-y')}+ip^3(z-z')}{(p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2-m^2+i\epsilon\epsilon(p^0)}\right)\end{aligned} \tag{19.110}$$

constructed in Appendix E, the pole contribution of Eq. (19.109) can thus be compactly written as

$$\begin{aligned}&\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu=1,\hat{\beta}_\nu=0,\mathrm{POLE})\\&=\sum_n\frac{HP^2_{\nu_n}(\tanh(b|w|-\sigma))(2\nu_n+1)\hat{D}_S(x,x',m_n)}{2(\nu_n-1)(\nu_n+2)(dP^1_\nu(-\tanh\sigma)/d\nu)|_{\nu=\nu_n}}\ .\end{aligned} \tag{19.111}$$

In the small H case of interest where Eq. (19.105) is valid, the relevant ν_n and m_n parameters were given earlier as $\nu_n=1+n+(n+1)(n+2)H^2/4b^2$ and $m_n^2=$

[32] In the pure $P^2_\nu(y)$ sector the $(\nu-1)$ term in the denominator of B_ν does not generate a pole at $\nu=1$ since at $\nu=1$ the $P^2_\nu(y)$ term in the numerator vanishes identically.

$n(n+3)H^2+(2n+3)(n+1)(n+2)H^4/4b^2$ where $n=0,1,2,3,...$, with the τ_n indices required for Eq. (19.109) thus being given by $\tau_n=3/2+n+(n+1)(n+2)H^2/4b^2$. Thus for small H we can set

$$\begin{aligned}&\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu=1,\hat{\beta}_\nu=0,\text{POLE})\\&=\sum_{n=0}^{\infty}\frac{HP^2_{\nu_n}(\tanh(b|w|-\sigma))(2n+3)\hat{D}_S(x,x',n(n+3)H^2)}{2[n(n+3)+(2n+3)(n+1)(n+2)H^2/4b^2](dP^1_\nu(-\tanh\sigma)/d\nu)|_{\nu=\nu_n}}\ .\end{aligned} \tag{19.112}$$

Compatibility of Eq. (19.112) with the pole structure of Eq. (19.57) thus requires that the relation

$$\begin{aligned}&\frac{P^2_{\nu_n}(-\tanh\sigma)}{N_{\nu_n}}\\&=\frac{H(2n+3)}{2[n(n+3)+(2n+3)(n+1)(n+2)H^2/4b^2](dP^1_\nu(-\tanh\sigma)/d\nu)|_{\nu=\nu_n}}\end{aligned} \tag{19.113}$$

hold where N_{ν_n} is the normalization constant which was introduced in Eq. (19.51).

To verify the consistency of Eq. (19.113), we check it directly for the representative $n=0$ case, and thus need to confirm the validity of the relation

$$\begin{aligned}&P^2_{1+H^2/2b^2}(-1+H^2/2b^2)(dP^1_\nu(-1+H^2/2b^2)/d\nu)|_{\nu=1+H^2/2b^2}\\&=\frac{2b^3}{H^3}\int_{-1+H^2/2b^2}^{1}dy[P^2_{1+H^2/2b^2}(y)]^2\end{aligned} \tag{19.114}$$

in leading order in H. To this end we note that the quantity $P^2_{1+H^2/2b^2}(y)$ is given in Eq. (19.37) as $P^2_{1+H^2/2b^2}(y)=(H^2/2b^2)[2/(1+y)-y]$; and with $(dP_\nu(y)/d\nu)|_{\nu=1}$ having been given earlier as $(dP_\nu(y)/d\nu)|_{\nu=1}=y+y\ln[(1+y)/2]-1$, the quantity $(dP^1_\nu(y)/d\nu)|_{\nu=1}$ is thus given as $-(1-y^2)^{1/2}[1+\ln[(1+y)/2]+y/(1+y)]$. To leading order then, the left hand side of Eq. (19.114) evaluates to $(H^2/2b^2)(4b^2/H^2)\times(-H/b)(-2b^2/H^2)=4b/H$, while the right hand side evaluates to $(2b^3/H^3)(2H^2/b^2)=4b/H$, just as required.

In addition, we also note that for modes with $\nu_n=1+n+(n+1)(n+2)H^2/4b^2$, the Bessel functions which appear in Eq. (19.110) have indices which are very close to $\tau=3/2,5/2,7/2,....$ For such indices, the large index expansion of Eq. (19.103) is not applicable. However, since the half-integer index Bessel functions are known in closed form, for them it can readily be shown that the combinations which appear in $\hat{D}_S(x,x',n(n+3)H^2)$ in Eq. (19.110) all have small H limit of the form

$$J_{n+3/2}\left(\frac{p^1e^{-Hx}}{H}\right)J_{n+3/2}\left(\frac{p^1e^{-Hx'}}{H}\right)\to\frac{H}{2\pi p^1}\left(e^{-ip^1(x-x')}+e^{-ip^1(x'-x)}\right)\ . \tag{19.115}$$

For such small mass modes then, in the limit in which H goes to zero, the pure AdS_4 propagator of Eq. (19.110) becomes the massless M_4 propagator of flat spacetime, just as it must.

Having now checked the consistency of our candidate small H AdS_4^+ propagator, it is of interest to discuss the graviton pole contribution to it. To this end we note that since $(\nu-1)(\nu+2) = q^2/H^2$, the graviton pole will be generated by the $(\nu-1)(\nu+2)$ term in B_ν rather than by the $Q_\nu^2(\tanh(b|w|-\sigma))/Q_\nu^2(-\tanh\sigma)$ term, since even though this latter term does possess poles, none of them is at $\nu=1$. On recalling that $P_1^2(y) = 0$, the graviton pole contribution is readily found to be of the form

$$\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu,\hat{\beta}_\nu,\text{graviton}) = \frac{HQ_1^2(\tanh(b|w|-\sigma))\hat{D}_S(x,x',m=0)}{2[(\hat{\alpha}_1/\hat{\beta}_1)P_1^1(-\tanh\sigma)+Q_1^1(-\tanh\sigma)]} \ . \tag{19.116}$$

From the forms for $P_1^1(y)$ and $Q_1^1(y)$ given in Eq. (19.13) we find that

$$P_1^1(-\tanh\sigma) = -\frac{H}{b} \ , \quad Q_1^1(-\tanh\sigma) = \frac{b}{H}\left(1-\frac{H^2}{b^2}\right)^{1/2} + \frac{H}{b}\text{arccosh}\left(\frac{b}{H}\right) \ , \tag{19.117}$$

with the graviton contribution thus being given by

$$\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu,\hat{\beta}_\nu,\text{graviton}) \\ = \frac{be^{2A}\hat{D}_S(x,x',m=0)}{[-(\hat{\alpha}_1/\hat{\beta}_1)(H^2/b^2)+(1-H^2/b^2)^{1/2}+(H^2/b^2)\text{arccosh}(b/H)]} \ , \tag{19.118}$$

to yield via Eq. (19.107) a contribution to $h_{\mu\nu}^{TT}$ of the form

$$h_{\mu\nu}^{TT}(x) = -e^{2A}\int d^4x' e^{-Hx'}\hat{D}_S(x,x',m=0)S_{\mu\nu}^{TT}(x') \\ \times\frac{2b\kappa_5^2}{[-(\hat{\alpha}_1/\hat{\beta}_1)(H^2/b^2)+(1-H^2/b^2)^{1/2}+(H^2/b^2)\text{arccosh}(b/H)]} \ . \tag{19.119}$$

Thus, despite the lack of normalizability of the graviton wave function, we see that the contribution of the graviton pole to the propagator is nonetheless finite; and not only that, in addition we see that its contribution is of precisely the same generic form as that of the exact fluctuation solution associated with an $S_{\mu\nu}^{TT}(x') = A_{\mu\nu}e^{-Hx'}$ source that we presented earlier as Eq. (19.74), a solution whose coefficient β was fixed by the junction condition in Eq. (19.79) to be given by none other than

$$\beta = -\frac{2b\kappa_5^2}{[-\eta(H^2/b^2)+(1-H^2/b^2)^{1/2}+(H^2/b^2)\text{arccosh}(b/H)]} \ . \tag{19.120}$$

Moreover, not only is Eq. (19.120) completely in accord with Eq. (19.119), in addition we are now able to identify the integration parameter η which appears in Eq. (19.77) as $\eta = \hat{\alpha}_1/\hat{\beta}_1$.

Our ability to exactly recover Eq. (19.120) is actually somewhat surprising since the solution of Eq. (19.74) was obtained as an exact solution for an $S^{TT}_{\mu\nu}(x') = A_{\mu\nu}e^{-Hx'}$ source without regard to whether H was small or large. The use of $\hat{G}^{TT}(x, x', w, 0; AdS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu)$ of Eq. (19.105) as the AdS_4^+ propagator might thus contain some ingredients which have validity beyond the small H region in which Eq. (19.105) was derived. To isolate one such possible ingredient, on recalling that $B_\nu(\tanh(b|w| - \sigma))$ obeys Eq. (19.93) for all values of H and not just for small ones, we return to Eq. (19.111), and on comparing it with Eq. (19.57), ask whether the relation

$$\begin{aligned}
&\frac{P^2_{\nu_n}(-\tanh\sigma)(\nu_n - 1)(\nu_n + 2)(dP^1_\nu(-\tanh\sigma)/d\nu)|_{\nu=\nu_n}}{(2\nu_n + 1)} \\
&= \frac{HN_{\nu_n}}{2} = \frac{b}{H}\int_{-\tanh\sigma}^{1} dy[P^2_{\nu_n}(y)]^2
\end{aligned} \tag{19.121}$$

as evaluated at the zeroes of $P^1_\nu(-\tanh\sigma)$ might possibly be valid when H is large. To this end, we note that with the zeroes of $P^1_\nu(-\tanh\sigma)$ being determinable numerically once any particular value for $\tanh\sigma$ is specified, we can test for Eq. (19.121) numerically by again considering the typical case where $\tanh\sigma = 0.9$, viz. where $b/H = 2.29416$. With the first positive zero of $P^1_{\nu_n}(-0.9)$ being at $\nu = 1.08775$ in this case, for it we evaluate numerically to obtain $P^2_{1.08775}(-0.9) = 1.83681$ and $(dP^1_\nu(-0.9)/d\nu)|_{\nu=1.08775} = 5.09207$, with the left hand side of Eq. (19.121) then evaluating to $1.83681 \times 0.08775 \times 3.08775 \times 5.09207/(2 \times 1.08775 + 1) = 0.79802$. With the right hand side of Eq. (19.121) evaluating to $2.29416 \times 0.34785 = 0.79802$, we thus confirm the validity of Eq. (19.121) when $\nu = 1.08775$. With a similar numerical check being found for the first ten zeroes of $P^1_\nu(-0.9)$, viz. a typical set of values, we can thus regard Eq. (19.121) as being valid for any H large or small. With the $G^{TT}(x, x', w, 0)$ propagator of Eq. (19.57) obeying Eq. (19.58) for any H large or small, it is thus possible that the expression for $\hat{G}^{TT}(x, x', w, 0; AdS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu)$ given in Eq. (19.105) might even have some validity beyond the small H region in which it was initially derived. However, regardless of any such possibility, the ability of the propagator of Eq. (19.105) to serve as a propagator for small H alone still suffices for testing causality, and so it is that issue to which we now turn. In the following we shall evaluate contour integrals assuming Eq. (19.105) to be exact for all H, with the results we obtain then being understood as (at least) being valid to lowest non-trivial order in H.

19.9 Causal propagation in the AdS_4^+ brane world

To explore the analytic structure of $\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu,\hat{\beta}_\nu)$ in the complex p^0 plane as is needed to test for causality, we note that as well as poles, the propagator also possesses cuts. Specifically, with the dependence of B_ν on p^0 being purely in the index $\nu=(9/4+q^2/H^2)^{1/2}-1/2$, we see that ν has a square root branch point at the tachyonic $q^2=-9H^2/4$. Since $-\nu-1=-(9/4+q^2/H^2)^{1/2}-1/2$, B_ν has a branch cut beginning at $q^2=-9H^2/4$ with the values of it on the two sides being given by associated Legendre functions with indices ν and $-\nu-1$ respectively. On noting that the associated Legendre functions obey the relations

$$
\begin{aligned}
P^{\mu}_{-\nu-1}(z) &= P^{\mu}_{\nu}(z) \ , \\
Q^{\mu}_{-\nu-1}(z) &= Q^{\mu}_{\nu}(z)+e^{i\pi\mu}\cos(\nu\pi)\Gamma[\nu+\mu+1]\Gamma[\mu-\nu]P^{-\mu}_{\nu}(z) \\
&= \frac{\sin[\pi(\nu+\mu)]}{\sin[\pi(\nu-\mu)]}Q^{\mu}_{\nu}(z)-\frac{\pi e^{i\pi\mu}\cos(\nu\pi)}{\sin[\pi(\nu-\mu)]}P^{\mu}_{\nu}(z) \ ,
\end{aligned}
\tag{19.122}
$$

we see that p^0 plane cuts in B_ν arise solely from the Q^μ_ν sector. However, the B_ν sector is not the only potential cause of multiple-valuedness in $\hat{G}^{TT}(x,x',w,0;AdS_4^+,\hat{\alpha}_\nu,\hat{\beta}_\nu)$ since the Bessel functions also depend on the very same index, with the Bessel functions with opposite square root signs for $\tau=(9/4+q^2/H^2)^{1/2}$ being related by

$$
J_{-\tau}(z)=\cos(\tau\pi)J_\tau(z)-\sin(\tau\pi)Y_\tau(z) \ . \tag{19.123}
$$

Since the retarded contour prescription is to put all singularities in the lower half p^0 plane, a contour integration determination of the retarded $\hat{G}^{TT}(x,x',w,0;AdS_4^+;\hat{\alpha}_\nu,\hat{\beta}_\nu,\mathrm{RET})$ is then most easily achieved by closing the contour above the real p^0 axis, with all multiple-valuedness issues then being avoided. The retarded $\hat{G}^{TT}(x,x',w,0;AdS_4^+;\hat{\alpha}_\nu,\hat{\beta}_\nu,\mathrm{RET})$ can thus be evaluated as (the negative of) an integral around an upper half p^0 plane half circle with large radius $p^0=Pe^{i\theta}$. For such large values of p^0 both the index and the argument of the Bessel functions in Eq. (19.105) become large, and so for them we can use the limit of Eq. (19.104), to yield

$$
\begin{aligned}
&\hat{G}^{TT}(x,x',w,0;AdS_4^+;\hat{\alpha}_\nu,\hat{\beta}_\nu,\mathrm{RET}) \\
&= -\frac{iP}{2(2\pi)^4}\int_0^\pi d\theta e^{i\theta}\int_{-\infty}^{\infty}dp^2dp^3\int_0^\infty dp^1 B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu) \\
&\quad\times e^{-iPe^{i\theta}(t-t')+ip^2(y-y')+ip^3(z-z')}e^{Hx/2}e^{Hx'/2}\left(e^{-ip^1(x-x')}+e^{-ip^1(x'-x)}\right) \\
&= -\frac{iP}{2(2\pi)^4}\int_0^\pi d\theta e^{i\theta}\int_{-\infty}^{\infty}dp^1dp^2dp^3 B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu) \\
&\quad\times e^{-iPe^{i\theta}(t-t')+ip^1(x-x')+ip^2(y-y')+ip^3(z-z')}e^{Hx/2}e^{Hx'/2} \ ,
\end{aligned}
\tag{19.124}
$$

with the last equality following since the index ν is an even function of p^1.

To evaluate B_ν on the half circle, we note that unlike the small H expansion of Eqs. (19.47) and (19.48) where the indices of the associated Legendre functions became large while their arguments became small, the large p^0 limit only involves the limit of the indices of the associated Legendre functions alone. In this particular limit the leading terms are given not by Eq. (19.47) but by the limits

$$
\begin{aligned}
&P^\mu_\nu(-\cos\phi) \to \\
&\left(\frac{2\nu+1}{2}\right)^{\mu-1/2}\left(\frac{2}{\pi\sin\phi}\right)^{1/2}\cos[(\nu+1/2)(\pi-\phi)-\pi/4+\mu\pi/2]+O\left(\frac{\nu^\mu}{\nu^{3/2}}\right), \\
&Q^\mu_\nu(-\cos\phi) \to \\
&\left(\frac{2\nu+1}{2}\right)^{\mu-1/2}\left(\frac{\pi}{2\sin\phi}\right)^{1/2}\cos[(\nu+1/2)(\pi-\phi)+\pi/4+\mu\pi/2]+O\left(\frac{\nu^\mu}{\nu^{3/2}}\right)
\end{aligned}
\tag{19.125}
$$

instead, limits which hold for any ϕ which lies in the range $\epsilon \leq \phi \leq \pi - \epsilon$ where ϵ is not too close to zero. For our purposes here it is convenient to set $\cos\phi = -\tanh(b|w|-\sigma)$ and $\cos\hat{\phi} = \tanh\sigma$, and can thus choose ϕ and $\hat{\phi}$ to lie in the required $(0,\pi)$ range at all $|w|$,[33] in consequence of which we can we can also set $\sin\phi = 1/\cosh(b|w|-\sigma) = He^{-A}/b$, $\sin\hat{\phi} = 1/\cosh\sigma = H/b$ within the relevant range. Then with $\nu+1/2$ being given by $\nu+1/2 = q/H$ when it is large, we find that B_ν of Eq. (19.92) limits to

$$
\begin{aligned}
&B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\hat{\beta}_\nu) \to \frac{1}{q}\left(\frac{\cosh(b|w|-\sigma)}{\cosh\sigma}\right)^{1/2} \\
&\times\left(\frac{\hat{\alpha}_\nu\cos[q(\pi-\phi)/H+3\pi/4]+(\pi\hat{\beta}_\nu/2)\cos[q(\pi-\phi)/H+5\pi/4]}{\hat{\alpha}_\nu\cos[q(\pi-\hat{\phi})/H+\pi/4]+(\pi\hat{\beta}_\nu/2)\cos[q(\pi-\hat{\phi})/H+3\pi/4]}\right),
\end{aligned}
\tag{19.126}
$$

i.e. to

$$
\begin{aligned}
&B_\nu(\tanh(b|w|-\sigma),\hat{\alpha}_\nu,\beta_\nu) \to \frac{e^{A(|w|)/2}}{q} \\
&\left(\frac{\hat{\alpha}_\nu(e^{iq(\pi-\phi)/H}+ie^{-iq(\pi-\phi)/H})+(\pi\hat{\beta}_\nu/2)(ie^{iq(\pi-\phi)/H}+e^{-iq(\pi-\phi)/H})}{\hat{\alpha}_\nu(-ie^{iq(\pi-\hat{\phi})/H}-e^{-iq(\pi-\hat{\phi})/H})+(\pi\hat{\beta}_\nu/2)(e^{iq(\pi-\hat{\phi})/H}+ie^{-iq(\pi-\hat{\phi})/H})}\right),
\end{aligned}
\tag{19.127}
$$

where $A(|w|)$ is the AdS_4^+ warp factor.

For the brane source $S^{TT}_{\mu\nu} = A_{\mu\nu}e^{Hx'/2}\delta(t')$ which is particularly convenient for exploration of the causal properties of the propagator, the fluctuation of Eq.

[33] When $|w|$ is zero, the angle ϕ is equal to $\hat{\phi}$, with $\hat{\phi}$ lying between 0 and $\pi/2$ since σ is positive. When $|w| = \sigma/b$, ϕ is equal to $\pi/2$; and as $|w| \to \infty$, ϕ asymptotes to π. As defined then, the angle ϕ thus increases monotonically with $|w|$, always lying in the range $\hat{\phi} \leq \phi \leq \pi$.

(19.107) is readily given as

$$h_{\mu\nu}^{TT} = \frac{i\kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2}}{2\pi} \int_0^\pi d\theta e^{-iPe^{i\theta}t}$$
$$\times \left[\frac{(\hat{\alpha}_\nu + i\pi\hat{\beta}_\nu/2) e^{iPe^{i\theta}(\pi-\phi)/H} + (i\hat{\alpha}_\nu + \pi\hat{\beta}_\nu/2) e^{-iPe^{i\theta}(\pi-\phi)/H}}{(-i\hat{\alpha}_\nu + \pi\hat{\beta}_\nu/2) e^{iPe^{i\theta}(\pi-\hat{\phi})/H} - (\hat{\alpha}_\nu - i\pi\hat{\beta}_\nu/2) e^{-iPe^{i\theta}(\pi-\hat{\phi})/H}} \right] . \tag{19.128}$$

As with our previous discussion of the M_4^+ and M_4^- brane-world cases, we evaluate Eq. (19.128) in four typical cases, to find leading contributions of the form

$$h_{\mu\nu}^{TT}(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = 2i/\pi) = \frac{\kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2}}{2\pi} \int_0^\pi d\theta e^{-iPe^{i\theta}(t-\phi/H+\hat{\phi}/H)}$$
$$= \kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2} \theta(t - \phi/H + \hat{\phi}/H) \ , \tag{19.129}$$

$$h_{\mu\nu}^{TT}(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = -2i/\pi) = -\kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2} \theta(t + \phi/H - \hat{\phi}/H) \ , \tag{19.130}$$

$$h_{\mu\nu}^{TT}(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = 0) = \kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2} \times [\theta(t - \phi/H + \hat{\phi}/H)$$
$$- i\theta(t + \phi/H + \hat{\phi}/H - 2\pi/H) - i\theta(t - \phi/H + 3\hat{\phi}/H - 2\pi/H)] \ , \tag{19.131}$$

$$h_{\mu\nu}^{TT}(\hat{\alpha}_\nu = 0, \hat{\beta}_\nu = 2/\pi) = \kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2} \times [\theta(t - \phi/H + \hat{\phi}/H)$$
$$+ i\theta(t + \phi/H + \hat{\phi}/H - 2\pi/H) + i\theta(t - \phi/H + 3\hat{\phi}/H - 2\pi/H)] \ . \tag{19.132}$$

(In the $\hat{\alpha}_\nu = 1$, $\hat{\beta}_\nu = 0$ and $\hat{\alpha}_\nu = 0$, $\hat{\beta}_\nu = 2/\pi$ cases where we need to keep both of the $\exp[-iPe^{i\theta}(\pi - \hat{\phi})/H]$ and $\exp[iPe^{i\theta}(\pi - \hat{\phi})/H]$ exponentials in the denominator, the modulus of the latter exponential never gets large in the upper half p^0 plane, to thus permit a power series expansion of the denominator.) As a quick check on the validity of these relations, we note from Eqs. (19.48) and (19.108) that the $P_\nu^2(y) \pm (2i/\pi)Q_\nu^2(y)$ based AdS_4^+ propagators should carry over into their $J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)$ based M_4^+ counterparts in the small H limit. Then, with ϕ and $\hat{\phi}$ respectively behaving as the small quantities $He^{b|w|}/b$ and H/b in this limit, we see that Eqs. (19.129) and (19.130) precisely limit to the leading terms of Eqs. (17.43) and (17.45), just as they should. As with our analysis of the $\delta(t)$ source in the divergent warp factor M_4^- and M_3^- brane worlds, we see that all four of the fluctuations in Eqs. (19.129) – (19.132) diverge as $|w| \to \infty$, even including the pure $P_\nu^2[\tanh(b|w| - \sigma)]$ based one of Eq. (19.131). Thus no matter which one of the propagators we might use, none will lead to localization of gravity around the brane, with the pure $P_\nu^2[\tanh(b|w| - \sigma)]$ based one providing an example of how an integration over an integrand which is convergent at large $|w|$ can generate a fluctuation which does not itself converge.

To determine the retardation properties of Eqs. (19.129) – (19.132), on recalling that ϕ always has to lie in the $\hat{\phi} \leq \phi \leq \pi$ range, the $\theta(t - \phi/H + \hat{\phi}/H)$ step function thus takes support in $Ht > \phi - \hat{\phi}$ where the quantity $\phi - \hat{\phi}$ has to lie in the always positive range $0 \leq \phi - \hat{\phi} \leq \pi - \hat{\phi}$. In contrast, the $\theta(t + \phi/H - \hat{\phi}/H)$ step function takes support in $Ht > \hat{\phi} - \phi$, a region which includes negative t. The $\theta(t + \phi/H + \hat{\phi}/H - 2\pi/H)$ step function takes support in $Ht > 2\pi - \phi - \hat{\phi}$ with $2\pi - \phi - \hat{\phi}$ having to lie in the always positive range $\pi - \hat{\phi} \leq 2\pi - \phi - \hat{\phi} \leq 2\pi - 2\hat{\phi}$. Finally, the $\theta(t - \phi/H + 3\hat{\phi}/H - 2\pi/H)$ step function takes support in $Ht > 2\pi + \phi - 3\hat{\phi}$ with $2\pi + \phi - 3\hat{\phi}$ having to lie in the always positive range $2\pi - 2\hat{\phi} \leq 2\pi + \phi - 3\hat{\phi} \leq 3\pi - 3\hat{\phi}$. With this same pattern repeating for the non-leading terms as well, we see that of the various propagators listed in Eqs. (19.129) – (19.132) then, all but the $(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = -2i/\pi)$ one of Eq. (19.130) are retarded.

To determine the causality properties of Eqs. (19.129) – (19.132) we need to determine the $(|w|, t)$ plane null geodesics of AdS_4^+. The null geodesics for which the coordinate x is conveniently taken to be zero are given by $dt = bdw/H\cosh(b|w| - \sigma)$, i.e. by

$$\sin(Ht - Ht_0) = \tanh(b|w| - \sigma) \quad . \tag{19.133}$$

The choice $\sin(Ht_0) = \tanh\sigma$ then yields a null geodesic which is at the brane at $t = 0$, with such geodesics also being writable as

$$\sin(Ht) = \frac{[\sinh(b|w| - \sigma) + \sinh(\sigma)]}{\cosh(b|w| - \sigma)\cosh\sigma} \quad , \quad \cos(Ht) = \frac{[1 - \sinh(b|w| - \sigma)\sinh(\sigma)]}{\cosh(b|w| - \sigma)\cosh\sigma} \quad . \tag{19.134}$$

In terms of the parameters ϕ and $\hat{\phi}$ these relations can also be written as

$$\sin(Ht) = \sin(\phi - \hat{\phi}) \quad , \quad \cos(Ht) = \cos(\phi - \hat{\phi}) \quad . \tag{19.135}$$

Now while the three of the above propagators which are retarded do differ in form from each other, we see that in the time interval $0 < Ht < \pi - \hat{\phi}$ only the $\theta(t - \phi/H + \hat{\phi}/H)$ step function is operative, with all the three propagators thus being identical in this particular region. Since according to Eq. (19.135) a null signal which leaves the brane at $t = 0$ will just reach the $|w| = \infty$ boundary of the spacetime at a time $t = (\pi - \hat{\phi})/H$, we see that in the time interval $0 < Ht < \pi - \hat{\phi}$ the region of support of the $\theta(t - \phi/H + \hat{\phi}/H)$ step function is precisely on and within the interior of the AdS_4^+ lightcone. All three of the retarded propagators (or at the least their small H limits) are thus causal. Now in the time interval $\pi - \hat{\phi} < Ht < 3\pi - 3\hat{\phi}$ (viz. the time it takes for the null signal to return to the brane and then go back out to $|w| = \infty$ a second time) both of the $\theta(t + \phi/H + \hat{\phi}/H - 2\pi/H)$ and $i\theta(t - \phi/H + 3\hat{\phi}/H - 2\pi/H)$ step functions will also become operative, causing the $(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = 0)$ and $(\hat{\alpha}_\nu = 0, \hat{\beta}_\nu = 2/\pi)$ based propagators to not only depart from each other but also from the $(\hat{\alpha}_\nu = 1, \hat{\beta}_\nu = 2i/\pi)$ propagator as well. However, as we noted in our discussion of the analogous situation in the M_4^- and M_3^- brane worlds (brane worlds which also possess divergent warp factors), these three propagators can only differ from each other by terms which satisfy the source-free fluctuation equation; with the full content of the theory being contained in the common $h^{TT}_{\mu\nu} = \kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2} \theta(t - \phi/H + \hat{\phi}/H)$ term that they all possess, since this is the term which in the Eq. (19.73) junction condition recovers the $\delta(t)$ source term on the brane.[34] To conclude then, we thus see that causality directly requires us to use the $P^2_\nu[\tanh(b|w| - \sigma)] + (2i/\pi)Q^2_\nu[\tanh(b|w| - \sigma)]$ based propagator in the AdS_4^+ brane world, just as we would expect since it is precisely the $P^2_\nu[\tanh(b|w| - \sigma)] + (2i/\pi)Q^2_\nu[\tanh(b|w| - \sigma)]$ based propagator which transits in the small H limit to the $J_2(qe^{b|w|}/b) + iY_2(qe^{b|w|}/b)$ based causal propagator of M_4^+. With the $P^2_\nu[\tanh(b|w| - \sigma)] + (2i/\pi)Q^2_\nu[\tanh(b|w| - \sigma)]$ based propagator necessarily possessing a massless graviton pole, we see, just like in our study of the M_4^- and M_3^- brane worlds, that despite its lack of normalizability, the massless graviton has to be included in the propagator, with gravity thus not localizing around an AdS_4^+ brane. Finally, since the Cauchy development into the bulk of an initial disturbance on the brane is uniquely prescribed (until such time as new information from the boundary could reach the leading edge of the disturbance – cf. Fig. (3.1)), there can only be one causal propagator. Thus once we have shown that the non-localizing $P^2_\nu[\tanh(b|w| - \sigma)] + (2i/\pi)Q^2_\nu[\tanh(b|w| - \sigma)]$ based propagator is causal, the $P^2_\nu[\tanh(b|w| - \sigma)]$ based propagator would then either have to not be causal, or if causal would, despite containing a convergent integrand, have to be non-localizing too. Consequently, localization of gravity cannot be achieved in the divergent warp factor AdS_4^+ brane world.

[34]Since $\delta(w)(d\phi/d|w|) = H\delta(w)$, in the junction conditions the $\theta(t + \phi/H + \hat{\phi}/H - 2\pi/H)$ and $\theta(t - \phi/H + 3\hat{\phi}/H - 2\pi/H)$ step function terms respectively generate spurious $-i\kappa_5^2\delta(w)A_{\mu\nu}e^{Hx/2}\delta(t + 2\hat{\phi}/H - 2\pi/H)$ and $i\kappa_5^2\delta(w)A_{\mu\nu}e^{Hx/2}\delta(t + 2\hat{\phi}/H - 2\pi/H)$ terms in $\delta(w)(d/d|w|)h^{TT}_{\mu\nu}(\hat{\alpha}_\nu = 1, \beta_\nu = 0)$ (and analogously for $\delta(w)(d/d|w|)h^{TT}_{\mu\nu}(\hat{\alpha}_\nu = 0, \beta_\nu = 2/\pi)$, terms which then precisely pairwise cancel in Eq. (19.73); while the delta function generated by the action of $\delta(w)d/d|w|$ on $\kappa_5^2 A_{\mu\nu} e^{Hx/2} e^{A(|w|)/2}\theta(t - \phi/H + \hat{\phi}/H)$ is precisely of the form $-\kappa_5^2\delta(w)A_{\mu\nu}e^{Hx/2}\delta(t) = -\kappa_5^2\delta(w)S^{TT}_{\mu\nu}$ that the junction conditions require.

Chapter 20

Equivalent Schrödinger Equation Treatment of the Fluctuation Equation

20.1 Equivalent brane-world Schrödinger equation

Rather than work directly with the source-free fluctuation equation

$$\frac{1}{2}\left[\frac{\partial^2}{\partial w^2}-4\left(\frac{dA}{d|w|}\right)^2-4\frac{dA}{d|w|}\delta(w)+e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h^{TT}_{\mu\nu}=0\ , \tag{20.1}$$

Randall and Sundrum [Randall and Sundrum (1999b)] noted that under a sequence of coordinate transformations and field rescalings Eq. (20.1) can be brought to the form of an equivalent one-dimensional Schrödinger equation potential problem. Specifically, on defining the fluctuations to be of the form

$$ds^2=e^{2A(z)}[dz^2+(q_{\mu\nu}+e^{-3A(z)/2}\psi_{\mu\nu})dx^\mu dx^\nu]\ , \tag{20.2}$$

they found that for any $h^{TT}_{\mu\nu}$ whose components obeyed

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha-2kH^2]h^{TT}_{\mu\nu}=m^2h^{TT}_{\mu\nu}\ , \tag{20.3}$$

where $k=1,0,-1$ in the maximally 4-symmetric dS_4, M_4 and AdS_4 brane worlds of interest to us, each component of the fluctuation $\psi_{\mu\nu}$ would then obey the Schrödinger type equation

$$\left[-\frac{\partial^2}{\partial z^2}+\frac{9A'^2}{4}+\frac{3A''}{2}-m^2\right]\psi(z)=0\ , \tag{20.4}$$

where the primes here denote differentiation with respect to z. In the specific M_4^+ brane-world case that they studied, the Schrödinger equation took the form

$$\left[-\frac{\partial^2}{\partial z^2}+\frac{15b^2}{4(1+b|z|)^2}-3b\delta(z)-m^2\right]\psi(z)=0\ , \tag{20.5}$$

where in terms of the original coordinate w, the coordinate z is given by $z=\epsilon(w)(e^{b|w|}-1)/b$. With the resulting potential having a shape somewhat reminiscent of a volcano, potentials associated with equations such as Eqs. (20.4) and (20.5) are generically known as volcano potentials in the literature. Given our familiarity with Schrödinger problems, study of equations such as these can be very helpful for developing insight into the structure of the TT fluctuation spectra in a readily accessible way. In fact this is all the more so since on transcribing the orthonormality condition $\int_{-\infty}^{\infty} d|w|e^{-2A}f_1(|w|)f_2(|w|) = \delta_{12}$ of Eqs. (16.72), (17.19) and (19.49) to the metric fluctuations of Eq. (20.2), we find that the orthonormality condition becomes none other than

$$\int_{-\infty}^{\infty} dz\psi_1(z)\psi_2(z) = \delta_{12} \quad , \tag{20.6}$$

i.e. none other than the standard non-relativistic flat spacetime Schrödinger equation orthonormality condition.[1] Study of the TT modes in the maximally 4-symmetric dS_4, M_4 and AdS_4 brane worlds thus reduces to a study of Eqs. (20.4) and (20.6), and can thus provide insight into the mode structure in the various cases of interest to us.

Thus for the M_4^+ case itself, Eq. (20.5) is recognized as being a standard Schrödinger equation with an attractive delta function potential, to therefore immediately admit [Randall and Sundrum (1999b)] of just one normalizable bound state, viz. a massless mode with a wave function $\psi(z) = b^{1/2}/(1+b|z|)^{1/2}$ which is localized around $z = 0$, to wit, a massless graviton which is localized to the brane. All the other normalizable solutions to Eq. (20.5) are oscillating modes which form the M_4^+ KK continuum of continuum-normalizable modes with non-tachyonic masses $m^2 \geq 0$ and wave functions $\psi_m(z) = \alpha_m(1+b|z|)^{1/2}J_2\left(m(1+b|z|)/b\right) + \beta_m(1+b|z|)^{1/2}Y_2\left(m(1+b|z|)/b\right)$ with coefficients which are fixed by the delta function in Eq. (20.5) to obey the junction constraint $\delta(z)[xd/dx+2]\psi(x) = 0$ where $x = m(1+b|z|)/b$, a condition which brings us right back to Eq. (17.13).

For the AdS_4^+ case which was treated in Chapter 19, we find that the equivalent Schrödinger equation problem takes the form [Karch and Randall (2001)]

$$\left[-\frac{\partial^2}{\partial z^2} - \frac{9H^2}{4} + \frac{15H^2}{4\sin^2(Hz_0 - H|z|)} - 3H\cot(Hz_0)\delta(z) - m^2\right]\psi(z) = 0 \quad , \tag{20.7}$$

where $Hz = \epsilon(w)(\arcsin[\operatorname{sech}(b|w| - \sigma)] - Hz_0)$ and $\sin(Hz_0) = \operatorname{sech}\sigma = H/b$. On taking ψ to be a function of $|z|$, Eq. (20.7) then yields

$$\left[-\frac{\partial^2}{\partial |z|^2} - \frac{9H^2}{4} + \frac{15H^2}{4\sin^2(Hz_0 - H|z|)} - m^2\right]\psi(|z|) = 0 \quad , \tag{20.8}$$

$$[2\psi'(z) + 3H\psi(z)\cot(Hz_0)]\delta(z) = 0 \quad , \tag{20.9}$$

the solutions to which can readily be found. On now setting $y = \cos(H|z| + Hz_0) =$

[1] This simplification was first noted by [DeWolfe, Freedman, Gubser and Karch (2000)] within the context of the M_4^+ brane world.

$\tanh(b|w| - \sigma)$ and $\psi(y) = (1-y^2)^{1/4}\chi(y)$ we find that Eq. (20.8) takes the form

$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \frac{m^2}{H^2} + 2 - \frac{4}{(1-y^2)}\right]\chi(y) = 0 \ , \tag{20.10}$$

with its general solution then being the $\chi(y) = \alpha P_\nu^2(y) + \beta Q_\nu^2(y)$ associated Legendre function solution we found earlier. Similarly, use of Eq. (20.9) leads us right back to the requirement that $\alpha P_\nu^1(y) + \beta Q_\nu^1(y)$ vanish at the brane. Then finally, with the $Q_\nu^2(y)$ modes not being normalizable under the norm of Eq. (20.6), we again see that it is the discrete $P_\nu^2(y)$ modes which are the normalizable modes of the AdS_4^+ brane world.

20.2 Instructive approximate brane-world Schrödinger equation

Even though we have now obtained the exact solution, and even though we now see why the normalizable spectrum is discrete, nonetheless it still turns out to be instructive [Schwartz (2001)] to replace the exact Schrödinger problem by a much simpler approximate one. Specifically, with the potential of Eq. (20.7) being a singular one with infinitely high walls at $z = (-\pi/H + z_0)$ and $z = (\pi/H - z_0)$ and with a delta function which drops down to $-\infty$ at $z = 0$ (viz. a potential which is shaped like the letter Y), we can approximate the potential by a simple attractive delta function together with two confining walls on which the potential is infinite. We thus consider the Schrödinger problem

$$\left[-\frac{\partial^2}{\partial z^2} - 2\lambda\delta(z)\right]\psi(z) = E\psi(z) \tag{20.11}$$

with walls at $z = \pm a$ such that $\psi(\pm a) = 0$. Such a Schrödinger problem admits of the bound state

$$\psi(|z|) = \sinh(\alpha a - \alpha|z|) \tag{20.12}$$

with energy $E = -\alpha^2$, provided the equation

$$\alpha = \lambda \tanh(\alpha a) \tag{20.13}$$

has a solution, i.e. provided $\lambda a > 1$.[2] This Schrödinger problem also has $E = \nu^2$ positive energy solutions of the form

$$\psi(|z|) = \nu\cos(\nu|z|) - \lambda\sin(\nu|z|) \ , \tag{20.14}$$

with the vanishing of the wave functions at the walls forcing the allowed energy eigenvalues to obey

$$\nu = \lambda \tan(\nu a) \ . \tag{20.15}$$

[2] As x increases the function $f(x) = x/\tanh x$ increases from an initial value of $f(0) = 1$.

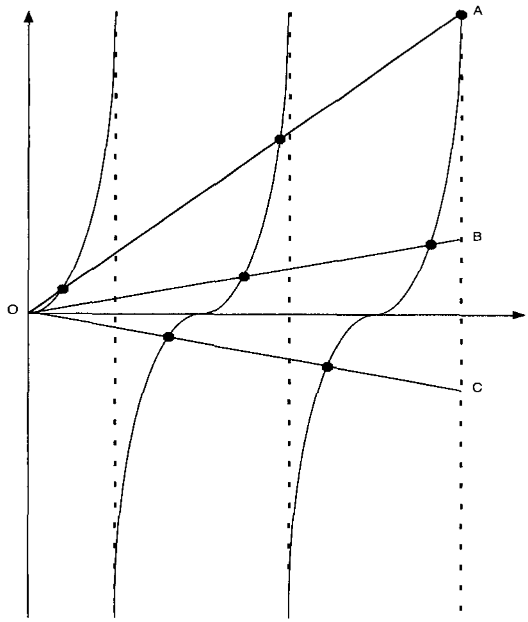

Fig. 20.1 Plot of $\lambda a \tan(\nu a)$ versus νa used for determining the positive energy eigenstates for delta function potentials in the $\lambda a > 1$, $0 < \lambda a < 1$ and $\lambda a < 0$ cases (straight lines OA, OB and OC).

Solutions to Eq. (20.15) yield a discrete spectrum, with there always being one solution between any pair of adjacent asymptotes of the tangent no matter what the value of λ, i.e. there will always be one solution between $\nu a = \pi/2$ and $\nu a = 3\pi/2$, always one solution between $\nu a = 3\pi/2$ and $\nu a = 5\pi/2$, and so on (c.f. the intercepts of lines OA and OB in Fig. (20.1)). However, there will only be a positive energy solution between $\nu a = 0$ and $\nu a = \pi/2$ if $\lambda a < 1$ (intercept of line OA),[3] with this solution being replaced by the negative energy solution of Eq. (20.12) if $\lambda a > 1$. Thus no matter what the magnitude of λ, there will always be one solution which lies below $\nu a = \pi/2$, a solution which would be oscillating for small λ but bound for large λ. We thus see why the full AdS_4^+ problem possesses a light mode when the brane tension $\lambda = 6(b^2 - H^2)^{1/2}/\kappa_5^2$ is big (cf. the small H limit given in Eq. (19.36)) and why it does not do so when the brane tension is small (cf. the large H limit given in Eq. (19.35)). Similarly we also see why the rest of the modes in the spectrum have to be discrete.

[3]Between $x = 0$ and $x = \pi/2$ the function $f(x) = x/\tan x$ decreases from $f(0) = 1$ to $f(\pi/2) = 0$.

The role that the boundary condition at the walls at $z = \pm a$ plays in producing bound and oscillating states has an immediate parallel in the M_4^- brane world, since it also possesses a warp factor which diverges at infinity. Specifically, for M_4^- we find that the M_4^+ potential of Eq. (20.5) is replaced by the singular

$$\left[-\frac{\partial^2}{\partial z^2} + \frac{15b^2}{4(1-b|z|)^2} + 3b\delta(z) - m^2\right]\psi(z) = 0 \ , \tag{20.16}$$

where now $z = \epsilon(w)(1-e^{-b|w|})/b$, with the range of z now being the finite interval $(-1/b, 1/b)$. The potential of Eq. (20.16) has infinitely high walls at $z = -1/b$ and $z = +1/b$ together with a delta function which rises up to $+\infty$ at $z = 0$ (i.e. the potential is shaped like the letter W). Consequently, we can approximate this potential by a simple repulsive delta function, viz.

$$\left[-\frac{\partial^2}{\partial z^2} + 2\lambda\delta(z)\right]\psi(z) = E\psi(z) \ , \tag{20.17}$$

where $\lambda > 0$, together with two confining walls at $z = \pm a$ at which the potential is infinite. Such a potential has no negative energy solutions. However there are $E = \nu^2$ positive energy solutions which are given by

$$\psi(|z|) = \nu\cos(\nu|z|) + \lambda\sin(\nu|z|) \ , \tag{20.18}$$

with the allowed energy eigenvalues being given by

$$\nu = -\lambda\tan(\nu a) \ . \tag{20.19}$$

Since we have changed the sign of coefficient of the delta function, this time the lowest-lying mode has to lie between $\nu a = \pi/2$ and $\nu a = 3\pi/2$, with the next mode lying between $\nu a = 3\pi/2$ and $\nu a = 5\pi/2$, and so on (intercepts of line OC in Fig. (20.1)). The normalizable spectrum for the repulsive delta function potential in a box is thus very similar to that of the attractive one, with, as we see in Fig. (20.1), the only significant difference being that now there is no very light mode. Rather, just as we found for the exact normalizable solutions to the M_4^- brane world given in Eq. (18.18) (and equally for the negative tension AdS_4^- brane world referred to in Chapter 19), there is a finite lower bound on the mass of the lowest allowed normalizable mode. As we see then, the pattern of normalizable mode masses that we found in the divergent warp factor AdS_4^+, M_4^- and AdS_4^- brane worlds is readily understood from the perspective of the equivalent Schrödinger equation approach. The only drawback to such an analysis is that while the non-normalizable modes would be excluded in a treatment of a true quantum-mechanical Schrödinger equation (such modes would not lie in the quantum-mechanical Hilbert space), in the classical gravity case of interest to us here, we have seen in Chapters 18 and 19 that non-normalizable solutions to Eq. (20.1) are just as physically relevant as the non-normalizable ones.

As a final comment on the brane-world mass spectrum, it is of interest to note that there is a remarkable degree of similarity between the equivalent Schrödinger equation analysis given above and the problem (see e.g. [Mannheim (1968)] and references therein) of the phonon spectrum associated with crystals which contain point defects (viz. defect atoms with masses which differ from those of the atoms of the otherwise pure harmonic crystal hosts into which they are substitutionally inserted). Specifically, when the inserted defect is heavier than the host atom which it replaces all the phonon modes are shifted down from the phonon band maximum of the pure crystal, but when the defect is lighter all the modes are shifted upwards, with one of them actually being shifted beyond the band maximum when the defect is light enough. While all the shifted band modes remain oscillatory for both lighter and heavier defects, modes which are shifted outside the band (the so-called localized modes) are found to be localized to the defect site and to fall off exponentially from it, with some pure crystal oscillating phonon modes thus being converted into bound state ones when a sufficiently light defect is inserted into the pure crystal. In this sense then the bound state gravitons of convergent warp factor brane worlds are very much akin to the localized modes of the crystal impurity problem. Having now analyzed brane gravity from the perspective of an equivalent Schrödinger equation approach, we turn next to the one remaining maximally 4-symmetric brane world of interest to us, viz. the dS_4^+ one, and as we shall see, because of the horizon which it possesses, it will prove to be much closer in spirit to the convergent warp factor M_4^+ brane world than to the divergent warp factor AdS_4^+ and M_4^- brane worlds we have just studied.

Chapter 21

Fluctuations around an Embedded Positive-Tension dS Brane

21.1 dS_4^+ mode basis

The treatment of the dS_4^+ brane world parallels that of the AdS_4^+ one, and differs from it primarily in the fact that, as noted in Chapters 8 and 10, the dS_4^+ brane world has both a limit on the range of allowed $|w|$ and a horizon at this limit, with a null signal needing an infinite amount of time to travel from the brane to the $|w| = \sigma/b$ limit. To determine the TT modes of dS_4^+, we set $e^{A(|w|)} = H\sinh(\sigma - b|w|)/b$ (where $\sinh\sigma = b/H$) in Eqs. (16.4) and (16.5), to obtain as wave equation

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2\coth^2(\sigma - b|w|) + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h^{TT}_{\mu\nu} = 0 \ , \tag{21.1}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} + 2(b^2 + H^2)^{1/2}\right] h^{TT}_{\mu\nu} = 0 \ , \tag{21.2}$$

where $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu} = q^{\alpha\beta}\tilde{h}^{TT}_{\mu\nu;\alpha;\beta}$ is the tensor box operator associated with a dS_4 metric $q_{\mu\nu}$. As with the AdS_4 induced metric, the operator $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}$ is not diagonal in its indices. Specifically, for an induced dS_4 metric of the form $q_{\mu\nu}dx^\mu dx^\nu = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2)$, direct evaluation of Eq. (13.10) in a TT mode, or straightforward substitution of $H \to -iH$, $t \to -ix$, $x \to it$ in Eq. (19.3), yields

$$\begin{aligned}
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{00} &= e^{-4Ht}[-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_1^2 + \partial_2^2 + \partial_3^2) + 4H^2][e^{4Ht}h^{TT}_{00}] \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{01} &= e^{-2Ht}[-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_1^2 + \partial_2^2 + \partial_3^2) + 4H^2][e^{2Ht}h^{TT}_{01}] \\
&\quad -2H\partial_1 h^{TT}_{00} \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{11} &= [-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_1^2 + \partial_2^2 + \partial_3^2) + 4H^2]h^{TT}_{11} \\
&\quad +2H^2e^{2Ht}h^{TT}_{00} - 4H\partial_1 h^{TT}_{01} \ , \\
\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{12} &= [-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_1^2 + \partial_2^2 + \partial_3^2) + 4H^2]h^{TT}_{12} - 2H\partial_1 h^{TT}_{02} \\
&\quad -2H\partial_2 h^{TT}_{01} \ ,
\end{aligned} \tag{21.3}$$

on every $|w|$ slice, with the other components of $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}$ being given by symmetry. With $h^{TT}_{\mu\nu}$ being an AdS_4 TT mode, on every $|w|$ slice its components obey the TT conditions

$$\begin{aligned}
\partial_0 h^{TT}_{10} - e^{-2Ht}[\partial_1 h^{TT}_{11} + \partial_2 h^{TT}_{12} + \partial_3 h^{TT}_{13}] + 3Hh^{TT}_{10} &= 0 \ , \\
\partial_0 h^{TT}_{20} - e^{-2Ht}[\partial_1 h^{TT}_{21} + \partial_2 h^{TT}_{22} + \partial_3 h^{TT}_{23}] + 3Hh^{TT}_{20} &= 0 \ , \\
\partial_0 h^{TT}_{30} - e^{-2Ht}[\partial_1 h^{TT}_{31} + \partial_2 h^{TT}_{32} + \partial_3 h^{TT}_{33}] + 3Hh^{TT}_{30} &= 0 \ , \\
\partial_0 h^{TT}_{00} - e^{-2Ht}[\partial_1 h^{TT}_{01} + \partial_2 h^{TT}_{02} + \partial_3 h^{TT}_{03}] + 4Hh^{TT}_{00} &= 0 \ , \\
h^{TT}_{00} - e^{-2Ht}[h^{TT}_{11} + h^{TT}_{22} + h^{TT}_{33}] &= 0 \ ,
\end{aligned} \tag{21.4}$$

with Eqs. (21.3) and (21.4) thus specifying the structure of the TT sector.

With TT tensor fluctuations obeying $(\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2H^2)h^{TT}_{\mu\nu} = 0$ on the dS_4 light-cone, we shall thus look to separate Eq. (21.1) in mass eigenstates defined by

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2H^2]h^{TT}_{\mu\nu} = m^2 h^{TT}_{\mu\nu} \ . \tag{21.5}$$

As such, Eq. (21.5) is an eigenvalue equation which in general requires a rediagonalization of the $h^{TT}_{\mu\nu}$ modes in the (μ,ν) space. However, given the structure displayed in Eq. (21.3), we see that for solutions which obey $h^{TT}_{0\mu}(|w|) = 0$,[1] the operator $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha h^{TT}_{\mu\nu}(|w|)$ happens to then be diagonal in its indices at every $|w|$. We thus only need to solve the theory for modes which obey $h^{TT}_{0\mu}(|w|) = 0$, with the $h^{TT}_{0\mu}(|w|) \neq 0$ components of the spin two KK modes then being fixed by their five-fold degeneracy. Thus in the following we can restrict our discussion to modes which obey $h^{TT}_{0\mu}(|w|) = 0$, modes for which Eq. (21.5) then takes the very convenient diagonal form

$$[-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_1^2 + \partial_2^2 + \partial_3^2) + 2H^2]h^{TT}_{\mu\nu} = m^2 h^{TT}_{\mu\nu} \ . \tag{21.6}$$

Separable solutions to Eqs. (21.1) and (21.6) are readily constructed. As can readily be checked, they take the form

$$h^{TT}_{\mu\nu}(m) = e_{\mu\nu} f_m(|w|) e^{ip^1x + ip^2y + ip^3z} e^{Ht/2} Z_{\nu+1/2}(p^0 e^{-Ht}/H) + \text{c.c.} \ , \tag{21.7}$$

where

$$p^0 = [(p^1)^2 + (p^2)^2 + (p^3)^2]^{1/2} \ , \tag{21.8}$$

[1] While this condition is analogous to a dS_4 synchronous gauge condition, as in the discussion of the use of the condition $h^{TT}_{1\mu}(|w|) = 0$ in the AdS_4^+ case, we make no such gauge transformation here, with bulk KK mode solutions with $h^{TT}_{0\mu}(|w|) \neq 0$ being just as physical as those with $h^{TT}_{0\mu}(|w|) = 0$.

and where $Z_{\nu+1/2}(p^0 e^{-Ht}/H)$ is a Bessel function. In Eq. (21.7) the parameter ν is defined by

$$\nu = \left(\frac{9}{4} - \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2} \ , \tag{21.9}$$

the tensor $e_{\mu\nu}$ is an appropriately chosen TT polarization tensor, and, with the change of variable $y = \coth(\sigma - b|w|)$, the $|w|$-dependent $f_m(|w|)$ is then found to obey

$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} - \frac{m^2}{H^2} + 2 - \frac{4}{(1-y^2)}\right] f_m(y) =$$
$$\left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \nu(\nu+1) - \frac{4}{(1-y^2)}\right] f_m(y) = 0 \ . \tag{21.10}$$

With Eq. (21.10) being the same associated Legendre equation which was encountered in the AdS_4^+ brane world, its solutions are immediately given as the associated Legendre functions $P_\nu^2(y)$ and $Q_\nu^2(y)$; and in analog to the AdS_4^+ case, the junction condition of Eq. (21.2) will be satisfied by any linear combinations $\alpha_m P_\nu^2(y) + \beta_m Q_\nu^2(y)$ of the modes which obey

$$\alpha_m P_\nu^1(\coth\sigma) + \beta_m Q_\nu^1(\coth\sigma) = 0 \ . \tag{21.11}$$

While the AdS_4^+ and dS_4^+ basis modes are quite similar to each other in their generic structure, in one crucial regard they differ significantly. Specifically, because of the horizon in the dS_4^+ case, the range of allowed $|w|$ values is from $|w| = 0$ to $|w| = \sigma/b$ only, with the allowed values of y then lying in the range $\coth\sigma = (1 + H^2/b^2)^{1/2} \le y \le \infty$, a range which is to be contrasted with the $-\tanh\sigma = -(1-H^2/b^2)^{1/2} \le y \le 1$ range encountered in the AdS_4^+ case.[2] Since the associated Legendre functions behave at large $y \gg 1$ as $P_\nu^2(y) \to O(y^\nu) + O(y^{-\nu-1})$, $Q_\nu^2(y) \to O(y^{-\nu-1})$, the only modes which are well-behaved at the horizon are the $\nu = 1$ massless graviton with warp factor wave function $Q_1^2(y) = 2/(1-y^2)$ (we recall that $P_1^2(y)$ is identically zero) and all $P_\nu^2(y)$ and $Q_\nu^2(y)$ modes with a complex $\nu = -1/2 \pm i(m^2/H^2 - 9/4)^{1/2}$ whose real part is equal to $-1/2$ and whose masses obey $m^2/H^2 \ge 9/4$. With its wave function being the dS_4^+ warp factor, the massless graviton is able to satisfy the junction condition at the brane by itself, while the modes of the KK continuum which begins at $m = 3H/2$ satisfy the junction condition by an interplay between the modes of the form

[2] In the complex z plane, the $P_\nu^2(z)$ and $Q_\nu^2(z)$ Legendre functions are both multiple-valued functions of z which possess a branch cut which extends from $z = -\infty$ to $z = 1$. With the dS_4 horizon restricting the relevant y to lie a range in which y is always greater than one, we see that the dS_4 Legendre functions can be defined on the real axis in a z plane which is cut from $z = 1$ to $z = -\infty$. (For the AdS_4 Legendre functions whose $-(1-H^2/b^2)^{1/2} \le y \le 1$ argument actually lies on the cut, the ones with argument in the real range $-(1-H^2/b^2)^{1/2} \le y \le 1$ which were used in Chapter 19 are defined as the linear combinations $P_\nu^\mu(y) = [e^{i\pi\mu/2} P_\nu^\mu(y+i\epsilon) + e^{-i\pi\mu/2} P_\nu^\mu(y-i\epsilon)]/2$, $Q_\nu^\mu(y) = e^{-i\pi\mu}[e^{-i\pi\mu/2} Q_\nu^\mu(y+i\epsilon) + e^{i\pi\mu/2} Q_\nu^\mu(y-i\epsilon)]/2$.)

$\alpha_m = Q^1_\nu(\coth\sigma)$, $\beta_m = -P^1_\nu(\coth\sigma)$. For all of these particular modes the flux $f_{m_1}(d/d|w| - 2dA/d|w|)f_{m_2} - f_{m_2}(d/d|w| - 2dA/d|w|)f_{m_1}$ which appears in Eq. (16.9) vanishes both at the brane and at the horizon, so that within the horizon the modes are thus orthogonal with respect to the e^{-2A} measure. Consequently, we can normalize them according to

$$\int_{-\sigma/b}^{\sigma/b} dwe^{-2A} f_m(|w|) f_{m'}(|w|) = \delta_{m,m'} \ , \tag{21.12}$$

since each of these modes has a norm which is finite.[3] In particular, for the graviton with wave function $f_0(|w|) = e^{2A}/N_0^{1/2}$, the norm immediately evaluates to

$$N_0 = \frac{2H^2}{b^2}\int_0^{\sigma/b} d|w|\sinh^2(\sigma - b|w|) = \frac{1}{b}\left(1+\frac{H^2}{b^2}\right)^{1/2} - \frac{H^2}{b^3}\text{arcsinh}\left(\frac{b}{H}\right) \ , \tag{21.13}$$

a norm which is thus expressly finite.[4] When normalized within the horizon then, and as originally noted by [Garriga and Sasaki (2000)], the normalizable dS_4^+ TT spectrum consists of a warp factor wave function massless spin-two graviton together with a continuum of KK modes which begins at $m = 3H/2$. Thus unlike the normalizable TT spectrum found in the M_4^+ brane world, in the dS_4^+ brane world there is an explicit mass gap between the graviton and the KK continuum. With its normalizable mode spectrum possessing both a massless graviton and a KK continuum, the dS_4^+ brane world is much closer in spirit to the M_4^+ brane world than is the AdS_4^+ one. And indeed, with all of the dS_4^+ normalizable modes having wave functions which fall monotonically all the way from the brane to the horizon (where all of them actually vanish), then just as with their analogous M_4^+ counterparts, the gravity associated with their propagation will localize to the brane.

21.2 Causality in the dS_4^+ brane world

With the dS_4^+ normalizable graviton and KK continuum basis modes $f_m(|w|)$ obeying a completeness relation of the form given in Eq. (16.73), we are able to construct a propagator analogous to that given in Eq. (16.76). Specifically, in terms of the pure dS_4 spacetime scalar propagator $D_S(x, x', m)$ given in Eq. (E.47) we introduce

[3]While the e^{-2A} measure actually diverges at the $|w| = \sigma/b$ horizon, for the normalizable modes the quantity $f_m^2(|w|)$ vanishes faster there.

[4]For general z we replace $Q_1^2(y)$ and $Q_1^1(y)$ of Eq. (19.13) by $Q_1^2(z) = 2/(z^2-1)$ and $Q_1^1(z) = (z^2-1)^{1/2}[(1/2)\log[(z+1)/(z-1)] - z/(z^2-1)]$, so that $Q_1^2(\coth(\sigma - b|w|)) = 2\sinh^2(\sigma - b|w|)$, $Q_1^1(\coth\sigma) = (H/b)\text{arcsinh}(b/H) - (b^2+H^2)^{1/2}/H$. We can thus identify $Q_1^2(\coth(\sigma - b|w|)) = 2b^2e^{2A}/H^2$, $Q_1^1(\coth\sigma) = -b^2N_0/H$, and in passing note the parallel to the structure of Eq. (19.116). Additionally, we also note that in the limit $H \to 0$, the dS_4^+ graviton normalization behaves as $N_0^{1/2} \to 1/b^{1/2}$, i.e. just as the normalization of the M_4^+ graviton given in Eq. (17.21).

a propagator $\hat{D}_S(x,x',m) = e^{2Ht}D_S(x,x',m)e^{2Ht'}$ which then obeys

$$[-\partial_t^2 + H\partial_t + e^{-2Ht}(\partial_x^2 + \partial_y^2 + \partial_z^2) + 2H^2 - m^2]\hat{D}_S(x,x',m) = e^{Ht}\delta^4(x-x') \quad , \tag{21.14}$$

with the requisite normalized mode basis propagator which obeys Eq. (16.77) with an $e^{Ht}\delta(w-w')\delta^4(x-x')$ source then being given by

$$G^{TT}(x,x',w,w') = \sum f_m(|w|)f_m(|w'|)\hat{D}_S(x,x',m) \quad . \tag{21.15}$$

However, as we had noted in our analog study of the convergent warp factor M_4^+ brane world given in Chapter 17, this particular form for the propagator is not the correct one for causality purposes. Rather, to implement causality in the dS_4^+ brane world we instead need to construct a dS_4^+ propagator as an analog of the M_4^+ and AdS_4^+ propagators given in Eqs. (17.30) and (19.105). Given our construction in Appendix E of the retarded propagator of a pure dS_4 spacetime, we see that the dS_4^+ analog of the AdS_4^+ brane-world propagator of Eq. (19.105) will have to be based not on $J_\tau J_\tau$ type Bessel function combinations but on $H_\tau^{(1)}H_\tau^{(2)}$ type Hankel function combinations instead. For such Hankel function combinations we need to find an integral over index relation for small H analogous to the one given in Eq. (19.101). On setting $\tau = (9/4 + k^2/H^2 - (p^0)^2/H^2)^{1/2}$, we thus try as a candidate such relation

$$\frac{1}{4}\int_{-\infty}^{\infty}\frac{dp^0p^0}{H^2}H_\tau^{(1)}\left(\frac{ke^{-Ht}}{H}\right)H_\tau^{(2)}\left(\frac{ke^{-Ht'}}{H}\right)\Bigg|_{\tau=(9/4+k^2/H^2-(p^0)^2/H^2)^{1/2}}$$
$$= \frac{e^{-Ht}}{H}\delta\left(\frac{e^{-Ht}}{H} - \frac{e^{-Ht'}}{H}\right) = \frac{\delta(t-t')}{H} \quad , \tag{21.16}$$

where k here is any parameter which is independent of p^0. With the large τ behavior of $H_\tau^{(i)}$ Bessel functions with argument $\tau/\cos\beta$ being given by

$$H_\tau^{1)}\left(\frac{\tau}{\cos\beta}\right) \to \left(\frac{2}{\pi\tau\tan\beta}\right)^{1/2}e^{i\psi}\left[1 - \frac{i}{\tau}\left(\frac{1}{8\tan\beta} + \frac{5}{24\tan^3\beta}\right) + O\left(\frac{1}{\tau^2}\right)\right] \quad ,$$
$$H_\tau^{2)}\left(\frac{\tau}{\cos\beta}\right) \to \left(\frac{2}{\pi\tau\tan\beta}\right)^{1/2}e^{-i\psi}\left[1 + \frac{i}{\tau}\left(\frac{1}{8\tan\beta} + \frac{5}{24\tan^3\beta}\right) + O\left(\frac{1}{\tau^2}\right)\right] \quad , \tag{21.17}$$

where $\psi = \tau(\tan\beta - \beta) - \pi/4$, on setting $\cos\beta = Qe^{Ht}/k$ where $Q = (k^2 - (p^0)^2)^{1/2}$,[5] we find that the requisite small H β and ψ are given as $\beta = \arctan(p^0/Q) - HtQ/p^0$

[5]Here we explicitly treat the case where k^2 is greater than $(p^0)^2$, and p^0 and k are both real and positive, a case where the quantity Q is conveniently then both real and positive, and the quantity Q/k (viz. the small H limit of Qe^{Ht}/k) is real and less than one. Since Eq. (19.103) holds for Hankel functions of complex index and argument, Eq. (21.18) to be obtained below will hold for the other values of k and p^0 which are needed for Eq. (21.16) as well.

and $\psi = p^0/H - (Q/H)\arctan(p^0/Q) - p^0 t - \pi/4$. Consequently, in the small H limit we obtain

$$H_\tau^{(1)}\left(\frac{ke^{-Ht}}{H}\right)H_\tau^{(2)}\left(\frac{ke^{-Ht'}}{H}\right) \to \frac{2H}{\pi p^0}e^{-ip^0(t-t')}\left[1+O\left(\frac{iH}{(k^2-(p^0)^2)^{1/2}}\right)\right] \ , \tag{21.18}$$

just as required to validate Eq. (21.16) to lowest order in H.

Given Eq. (21.16), to obtain the requisite dS_4^+ brane-world propagator, we simply need to make the substitutions $H \to -iH$, $t \to -ix$, $x \to it$ in Eq. (19.105), to find the dS_4^+ propagator to be of the form

$$\begin{aligned}
&\hat{G}^{TT}(x, x', w, 0; dS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) \\
&= \frac{1}{4H(2\pi)^3}\int_{-\infty}^{\infty} dp^1 dp^2 dp^3 \int_{-\infty}^{\infty} dp^0 p^0 B_\nu(\coth(\sigma - b|w|), \hat{\alpha}_\nu, \hat{\beta}_\nu) e^{Ht/2} e^{Ht'/2} \\
&\times e^{ip^1(x-x')+ip^2(y-y')+ip^3(z-z')} H_\tau^{(1)}\left(\frac{ke^{-Ht}}{H}\right) H_\tau^{(2)}\left(\frac{ke^{-Ht'}}{H}\right) \ ,
\end{aligned} \tag{21.19}$$

where now $k = [(p^1)^2 + (p^2)^2 + (p^3)^2]^{1/2}$, $\tau = \nu + 1/2 = [9/4 - q^2/H^2]^{1/2}$, and $q^2 = (p^0)^2 - k^2$. With B_ν as defined by

$$\begin{aligned}
&B_\nu(\coth(\sigma - b|w|), \hat{\alpha}_\nu, \hat{\beta}_\nu) \\
&= \frac{1}{H(\nu-1)(\nu+2)}\left[\frac{\hat{\alpha}_\nu P_\nu^2(\coth(\sigma - b|w|)) + \hat{\beta}_\nu Q_\nu^2(\coth(\sigma - b|w|))}{\hat{\alpha}_\nu P_\nu^1(\coth\sigma) + \hat{\beta}_\nu Q_\nu^1(\coth\sigma)}\right]
\end{aligned} \tag{21.20}$$

obeying

$$\delta(w)\left[\frac{d}{d|w|} - 2\frac{dA}{d|w|}\right] B_\nu(\coth(\sigma - b|w|), \hat{\alpha}_\nu, \hat{\beta}_\nu) = \delta(w) \ , \tag{21.21}$$

it can readily be checked that for $e^{A(|w|)} = H\sinh(\sigma - b|w|)/b$, the propagator of Eq. (21.19) does indeed obey

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial w^2} - 4\left(\frac{dA}{d|w|}\right)^2 - 4\frac{dA}{d|w|}\delta(w) + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]\hat{G}^{TT}(x, x', w, 0; dS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) \\
&= e^{Ht}\delta(x - x')\delta(t - t')\delta(y - y')\delta(z - z')\delta(w)
\end{aligned} \tag{21.22}$$

for small H, just as required. Finally then, the fluctuation

$$h_{\mu\nu}^{TT}(x) = -2\kappa_5^2 \int d^4x' e^{-Ht'} \hat{G}^{TT}(x, x', w, 0; dS_4^+, \hat{\alpha}_\nu, \hat{\beta}_\nu) S_{\mu\nu}^{TT}(x') \tag{21.23}$$

is an exact dS_4^+ brane world small H solution to

$$\left[\frac{\partial^2}{\partial |w|^2} - 4\left(\frac{dA}{d|w|}\right)^2 + e^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right] h_{\mu\nu}^{TT} = 0 \ , \tag{21.24}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|}-2\frac{dA}{d|w|}\right]h^{TT}_{\mu\nu}=-\kappa_5^2\delta(w)S^{TT}_{\mu\nu}\ , \tag{21.25}$$

for an arbitrary source on the brane.

Before exploring the implications of Eq. (21.23), it is instructive to confirm that the dS_4^+ propagator of Eq. (21.19) does indeed go into the M_4^+ one of Eq. (17.30) in the $H \to 0$ limit. To this end we note that in the large ν limit, associated Legendre functions with degree μ and argument $z > 1$ held fixed behave as

$$\begin{aligned}\nu^{\mu}P_{\nu}^{-\mu}\left(\cosh\left(\frac{z}{\nu}\right)\right) &\to I_{\mu}(z)=e^{-i\pi\mu/2}J_{\mu}(iz)\ ,\\ \nu^{-\mu}e^{-i\pi\mu}Q_{\nu}^{\mu}\left(\cosh\left(\frac{z}{\nu}\right)\right) &\to K_{\mu}(z)=\frac{i\pi}{2}e^{i\pi\mu/2}[J_{\mu}(iz)+iY_{\mu}(iz)]\ .\end{aligned} \tag{21.26}$$

In consequence of this, the small H limit of $B_{\nu}(\coth(\sigma-b|w|),\hat{\alpha}_{\nu},\hat{\beta}_{\nu})$ [viz. the limit $\nu \to (k^2-(p^0)^2)^{1/2}/H$, $\coth(\sigma-b|w|)\to 1+H^2e^{2b|w|}/2b^2\equiv\cosh(He^{b|w|}/b)=\cosh((k^2-(p^0)^2)^{1/2}e^{b|w|}/b\nu)$] is thus given by[6]

$$B_{\nu}(\coth(\sigma-b|w|),\hat{\alpha}_{\nu},\hat{\beta}_{\nu})\to \frac{1}{i(k^2-(p^0)^2)^{1/2}}\left[\frac{\hat{\alpha}_{\nu}J_2(iz)+(i\pi\hat{\beta}_{\nu}/2)[J_2(iz)+iY_2(iz)]}{\hat{\alpha}_{\nu}J_1(iz_0)+(i\pi\hat{\beta}_{\nu}/2)[J_1(iz_0)+iY_1(iz_0)]}\right]\ , \tag{21.27}$$

where $z=(k^2-(p^0)^2)^{1/2}e^{b|w|}/b$, $z_0=(k^2-(p^0)^2)^{1/2}/b$. Given the limit of Eq. (21.18), on defining the parameter $q=((p^0)^2-k^2)^{1/2}$ of Eq. (17.30) as $q=i(k^2-(p^0)^2)^{1/2}$ (so that then $\tau=\nu+1/2=-i[(p^0)^2/H^2-k^2/H^2-9/4]^{1/2}$), the dS_4^+ propagator of Eq. (21.19) is thus seen to limit to the analogous M_4^+ one, just as it should. In fact given the structure of Eq. (21.26) and our prior demonstration in Chapter 17 that it is the $J_2(z)+iY_2(z)$ based propagator which is the causal one in the M_4^+ brane world, we can immediately anticipate that it will be the pure $Q_{\nu}^2(\coth(\sigma-b|w|))$ based propagator which will be the causal one in the dS_4^+ brane world.[7]

To check for causality explicitly we introduce the convenient source $S^{TT}_{\mu\nu}(x^{\lambda})=A_{\mu\nu}e^{i\epsilon x}\delta(t)$ in Eq. (21.23) where ϵ is a small parameter which will be needed for the small argument divergence of the Hankel functions, where x is the first of the three dS_4 spatial coordinates, and where the choice $A_{\mu\nu}\sim\delta_{\mu}^2\delta_{\nu}^3$ then enables $A_{\mu\nu}e^{i\epsilon x}\delta(t)$ to obey the TT relations of Eq. (21.4). With this choice of source the spatial

[6]For integer degree m, the $P_{\nu}^{m}(z)$ with positive m can be determined from the $P_{\nu}^{-m}(z)$ via $P_{\nu}^{-m}(z)=P_{\nu}^{m}(z)\Gamma(\nu-m+1)/\Gamma(\nu+m+1)$.

[7]In this regard we note a difference between the dS_4^+ and AdS_4^+ brane worlds – in the dS_4^+ brane world it is the $Q_{\nu}^2(\coth(\sigma-b|w|))$ based propagator which continues into the $J_2(z)+iY_2(z)$ based propagator of M_4^+, while in the AdS_4^+ brane world it is the $P_{\nu}^2(-\tanh(\sigma-b|w|))+iQ_{\nu}^2(-\tanh(\sigma-b|w|))$ based propagator which continues into it. This difference stems from the fact that the large ν limit of the linear combinations $P_{\nu}^{\mu}(y)=[e^{i\pi\mu/2}P_{\nu}^{\mu}(y+i\epsilon)+e^{-i\pi\mu/2}P_{\nu}^{\mu}(y-i\epsilon)]/2$, $Q_{\nu}^{\mu}(y)=e^{-i\pi\mu}[e^{-i\pi\mu/2}Q_{\nu}^{\mu}(y+i\epsilon)+e^{i\pi\mu/2}Q_{\nu}^{\mu}(y-i\epsilon)]/2$ with argument less than one of relevance to the AdS_4^+ brane world is given not by Eq. (21.26), but by $\nu^{\mu}P_{\nu}^{-\mu}(\cos(x/\nu))\to J_{\mu}(x)$, $\nu^{\mu}Q_{\nu}^{-\mu}(\cos(x/\nu))\to -(\pi/2)Y_{\mu}(x)$ instead.

integrals can then readily be performed in Eq. (21.23), to yield

$$h^{TT}_{\mu\nu}(x) = -\frac{A_{\mu\nu}\kappa_5^2 e^{Ht/2}e^{i\epsilon x}}{2H}\int_{-\infty}^{\infty} dp^0 p^0 B_\nu H^{(1)}_\tau\left(\frac{\epsilon e^{-Ht}}{H}\right) H^{(2)}_\tau\left(\frac{\epsilon}{H}\right) \quad , \tag{21.28}$$

where now $\tau = \nu + 1/2 = [9/4 + \epsilon^2/H^2 - (p^0)^2/H^2]^{1/2}$. With the retarded contour prescription putting all singularities in the lower half complex p^0 plane, we can thus evaluate Eq. (21.28) via an integration along a closed half circle of large radius P in the upper half p^0 plane, to obtain

$$h^{TT}_{\mu\nu}(x) = \frac{iA_{\mu\nu}\kappa_5^2 e^{Ht/2}e^{i\epsilon x}}{2H}\int_0^{\pi} d\theta P^2 e^{2i\theta} B_\nu H^{(1)}_\tau\left(\frac{\epsilon e^{-Ht}}{H}\right) H^{(2)}_\tau\left(\frac{\epsilon}{H}\right) \quad . \tag{21.29}$$

Since we are only interested in the using the above expression for small H, use of Eq. (21.18) allows us to rewrite Eq. (21.29) in the convenient form

$$h^{TT}_{\mu\nu}(x) = \frac{iA_{\mu\nu}\kappa_5^2 e^{Ht/2}e^{i\epsilon x}}{\pi}\int_0^{\pi} d\theta P e^{i\theta} B_\nu e^{-iPe^{i\theta}t}\left[1 + O\left(\frac{H}{Pe^{i\theta}}\right)\right] \tag{21.30}$$

when H is small.

As regards the behavior of B_ν along the large circle in the upper half p^0 plane where $\nu + 1/2 \to -iPe^{i\theta}/H$,[8] we note that in the $|\nu| \to \infty$ limit with μ and z held fixed, the associated Legendre functions have asymptotic expansions of the form

$$\begin{aligned} P^\mu_\nu(z) &\to \frac{1}{(2\pi)^{1/2}}\frac{1}{(z^2-1)^{1/4}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} \\ &\times \left[[z + (z^2-1)^{1/2}]^{(\nu+1/2)} + ie^{-i\pi\mu}[z - (z^2-1)^{1/2}]^{(\nu+1/2)}\right]\left[1 + O\left(\frac{1}{\nu}\right)\right] \quad , \\ Q^\mu_\nu(z) &\to \frac{\pi^{1/2}}{2^{1/2}}\frac{e^{i\pi\mu}}{(z^2-1)^{1/4}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}[z - (z^2-1)^{1/2}]^{(\nu+1/2)}\left[1 + O\left(\frac{1}{\nu}\right)\right] \quad . \end{aligned} \tag{21.31}$$

Consequently, along the large circle in the upper half p^0 plane, B_ν behaves as

$$\begin{aligned} B_\nu(\coth(\sigma - b|w|), \hat{\alpha}_\nu, \hat{\beta}_\nu) &\to \frac{1}{-iPe^{i\theta}}\left(\frac{\sinh\hat{\sigma}}{\sinh\sigma}\right)^{1/2} \\ &\times \left[\frac{\alpha_\nu[\coth(\hat{\sigma}/2)]^{(-iPe^{i\theta}/H)} + (i\alpha_\nu + \pi\beta_\nu)[\tanh(\hat{\sigma}/2)]^{(-iPe^{i\theta}/H)}}{\alpha_\nu[\coth(\sigma/2)]^{(-iPe^{i\theta}/H)} - (i\alpha_\nu + \pi\beta_\nu)[\tanh(\sigma/2)]^{(-iPe^{i\theta}/H)}}\right] \quad , \end{aligned} \tag{21.32}$$

where $\hat{\sigma} = \sigma - b|w|$. With the insertion of Eq. (21.32) into Eq. (21.30) and the use of Eq. (E.25) we thus obtain for the leading contribution to the fluctuation

$$h^{TT}_{\mu\nu}(x) \to A_{\mu\nu}\kappa_5^2 e^{Ht/2}e^{i\epsilon x}\left(\frac{\sinh\hat{\sigma}}{\sinh\sigma}\right)^{1/2}\theta\left[t - \frac{1}{H}\log\left(\frac{\coth(\hat{\sigma}/2)}{\coth(\sigma/2)}\right)\right] \tag{21.33}$$

[8] Our identification above of $\nu+1/2$ as $\nu+1/2 = -i((p^0)^2 - k^2)^{1/2}/H$ yields $\nu+1/2 = -ip^0/H$ when k is zero, since, as per its introduction in Eq. (17.30), the quantity $((p^0)^2 - k^2)^{1/2}$ is always taken to have the same sign as p^0.

when we set $\alpha_\nu = 0$, and obtain

$$h^{TT}_{\mu\nu}(x) \to -A_{\mu\nu}\kappa_5^2 e^{Ht/2} e^{i\epsilon x}\left(\frac{\sinh\hat{\sigma}}{\sinh\sigma}\right)^{1/2}\theta\left[t+\frac{1}{H}\log\left(\frac{\coth(\hat{\sigma}/2)}{\coth(\sigma/2)}\right)\right] \tag{21.34}$$

when we set $i\hat{\alpha}_\nu + \pi\hat{\beta}_\nu = 0$. With the dS_4^+ metric (w,t) plane null geodesic which passes through $w=0, t=0$ and its interior being given in closed form as

$$e^{Ht} \geq \frac{\coth(\sigma/2 - b|w|/2)}{\coth(\sigma/2)} \quad , \tag{21.35}$$

we thus see that it is precisely, and in fact only, the pure $\hat{\beta}_\nu$ based dS_4^+ propagator which is causal, just as we had anticipated.[9] As a final check on Eq. (21.33), we note that if we let H go to zero, then since $\sinh\hat{\sigma} \to be^{-b|w|}/H$ and $\log(\coth(\hat{\sigma}/2)) \to He^{b|w|}/b$ in the limit, the dS_4^+ brane-world leading term as given in Eq. (21.33) precisely transits into the M_4^+ brane-world leading term as given in Eq. (17.43), just as it should. The $Q_\nu^2(\coth(\sigma - b|w|))$ based propagator, viz.

$$\begin{aligned}
&\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)\\
&= \frac{1}{4H(2\pi)^3}\int_{-\infty}^{\infty} dp^1dp^2dp^3\int_{-\infty}^{\infty} dp^0 p^0\frac{e^{Ht/2}e^{Ht'/2}}{H(\nu-1)(\nu+2)}\frac{Q_\nu^2(\coth(\sigma-b|w|))}{Q_\nu^1(\coth\sigma)}\\
&\times e^{ip^1(x-x')+ip^2(y-y')+ip^3(z-z')}H_\tau^{(1)}\left(\frac{ke^{-Ht}}{H}\right)H_\tau^{(2)}\left(\frac{ke^{-Ht'}}{H}\right) \quad ,
\end{aligned} \tag{21.36}$$

is thus indeed the small H dS_4^+ brane-world causal propagator we seek; and with it leading in Eq. (21.33) to fluctuations which behave as $\sinh^{1/2}(\sigma - b|w|)$, it thus leads to a gravity which localizes to the brane.

It is also of interest to explore the complex p^0 plane singularity structure of the causal $\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)$ propagator. With the $Q_\nu^1(z)$ associated Legendre function possessing no zeroes when its argument is greater than one,[10] the only pole present in $\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)$ is the one at $\nu=1$ associated with the massless graviton. With the ν index in $Q_\nu^2(\coth(\sigma-b|w|)/Q_\nu^1(\coth\sigma)$ being a multiple-valued function of p^0, the $\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)$ propagator also possesses a branch point at $\nu=-1/2$.

As regards the massless graviton contribution, we note that with $Q_1^1(\coth\sigma)$ being given by

$$Q_1^1(\coth\sigma) = \frac{\sigma}{\sinh\sigma} - \cosh\sigma = \frac{H}{b}\text{arcsinh}\left(\frac{b}{H}\right) - \frac{b}{H}\left(1+\frac{H^2}{b^2}\right)^{1/2} \quad , \tag{21.37}$$

[9] With the $i\hat{\alpha}_\nu + \pi\hat{\beta}_\nu = 0$ based propagator being acausal, any B_ν combination of associated Legendre functions which involves a non-zero $\hat{\alpha}_\nu$ will then be acausal as well.

[10] The zeroes that $Q_\nu^1(z)$ does possess occur when its argument is between -1 and 1, and as described in Chapter 19, lead then to the presence of an infinite set of isolated discrete massive poles in the analog AdS_4^+ brane-world propagator.

the graviton pole contribution to Eq. (21.36) readily evaluates to

$$\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1,\text{graviton})$$
$$=\frac{be^{2A}\hat{D}_S(x,x',m=0)}{[(1+H^2/b^2)^{1/2}-(H^2/b^2)\text{arcsinh}(b/H)]}\ , \tag{21.38}$$

with its contribution to $h_{\mu\nu}^{TT}$ accordingly being of the form

$$\begin{aligned}h_{\mu\nu}^{TT}(w,x,\text{graviton}) &= -\frac{2b\kappa_5^2e^{2A}}{[(1+H^2/b^2)^{1/2}-(H^2/b^2)\text{arcsinh}(b/H)]}\\ &\quad\times\int d^4x'e^{-Ht'}\hat{D}_S(x,x',m=0)S_{\mu\nu}^{TT}(x')\ ,\end{aligned} \tag{21.39}$$

where the massless $\hat{D}_S(x,x',m=0)$ propagator is given by

$$\begin{aligned}\hat{D}_S(x,x',m=0) &= \frac{(e^{Ht}e^{Ht'})^{1/2}}{4(2\pi)^3H}\int_{-\infty}^{\infty}d^4pp_0H_T^{(1)}\left(\frac{p^0e^{-Ht}}{H}\right)H_T^{(2)}\left(\frac{p^0e^{-Ht'}}{H}\right)\\ &\quad\times\frac{e^{ip^1(x-x')+ip^2(y-y')+ip^3(z-z')}}{[(p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2+i\epsilon\epsilon(p^0)]}\ .\end{aligned} \tag{21.40}$$

Comparing with Eq. (21.13), we see that the graviton pole contribution to the causal $\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)$ propagator of Eq. (21.36) is identical in form to its contribution to the normalized basis mode $G^{TT}(x,x',w,w')$ propagator given in Eq. (21.15).

As well as obtain a massless graviton wave function which peaks at the brane, in order to secure a phenomenologically viable brane world, we additionally need the graviton to couple canonically on the brane, i.e. we need the graviton to behave the same way on the brane as it would in an unembedded dS_4 spacetime where the TT and NT modes behave as (cf. Eqs. (15.45) and (15.46) as adapted to dS_4^+)

$$\frac{1}{2}[\nabla_\alpha\nabla^\alpha-2H^2]h_{\mu\nu}^{NT}+H^2q_{\mu\nu}h^{NT}=-\kappa_4^2\left[S_{\mu\nu}^{NT}-\frac{1}{2}q_{\mu\nu}S\right]\ , \tag{21.41}$$

$$\frac{1}{2}[\nabla_\alpha\nabla^\alpha-2H^2]h_{\mu\nu}^{TT}=-\kappa_4^2S_{\mu\nu}^{TT} \tag{21.42}$$

in the harmonic gauge. (Here κ_4^2 is the 4-dimensional gravitational constant which appears in the ordinary (unembedded) 4-dimensional Einstein equations.) In this same gauge in the dS_4^+ brane world the NT modes obey (cf. Eq. (15.37))

$$\begin{aligned}&\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha-2H^2]h_{\mu\nu}^{NT}(w=0)+H^2q_{\mu\nu}h^{NT}(w=0)\\ &=-(b^2+H^2)^{1/2}\kappa_5^2\left[S_{\mu\nu}^{NT}-\frac{1}{2}q_{\mu\nu}S\right]\end{aligned} \tag{21.43}$$

on the brane, while the full TT fluctuation obeys the generalized brane Einstein equation (cf. Eq. (15.49))

$$\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2H^2]h^{TT}_{\mu\nu}(w=0) = -b\left(1+\frac{H^2}{b^2}\right)^{1/2}\kappa_5^2 S^{TT}_{\mu\nu} - \delta\bar{E}^{TT}_{\mu\nu}(w=0) \ , \quad (21.44)$$

with $\delta\bar{E}^{TT}_{\mu\nu}(w=0)$ being as given in Eq. (15.48).

Comparing first the NT Eqs. (21.41) and (21.43), we see that we need to make the identification $\kappa_4^2 = \kappa_5^2(b^2+H^2)^{1/2}$. Next, we note that in consequence of Eq. (21.14), the dS_4^+ TT graviton contribution as given in Eq. (21.39) obeys

$$\frac{1}{2}[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2H^2]h^{TT}_{\mu\nu}(w=0,x,\text{graviton})$$
$$= -\frac{b\kappa_5^2 S^{TT}_{\mu\nu}(x)}{[(1+H^2/b^2)^{1/2} - (H^2/b^2)\text{arcsinh}(b/H)]} \ , \quad (21.45)$$

on the brane. And while the coefficients of the $S^{TT}_{\mu\nu}(x)$ terms in Eqs. (21.44) and (21.45) look somewhat different from each other, we recall that even while Eq. (21.44) was derived in Chapter 15 without restriction on H, our derivation of Eq. (21.45) is only valid to lowest non-trivial order in H. Then, with the denominator in Eq. (21.45) tending to $1 + H^2/2b^2 + (H^2/b^2)\log(H/2b)$ when H is small, we see that to first order in H the graviton couples to $S^{TT}_{\mu\nu}(x)$ in Eq. (21.45) with just the strength needed to match Eq. (21.42). Gravity in a small H dS_4^+ brane world is thus canonical on the brane.[11]

As regards the discontinuity in the causal $\hat{G}^{TT}(x,x',w,0;dS_4^+,\hat{\alpha}_\nu=0,\hat{\beta}_\nu=1)$ propagator, we note that with the index ν being given by $\nu = (9/4 - q^2/H^2)^{1/2} - 1/2$ where $q^2 = (p^0)^2 - \bar{p}^2$, the index thus has a square root branch point at $q^2 = 9H^2/4$. The quantity B_ν thus has a branch cut beginning at $q^2 = 9H^2/4$ also, i.e. beginning precisely at the point where the dS_4^+ normalizable KK continuum mode basis begins, with the values of B_ν on the two sides of the branch cut being given by associated Legendre functions with indices ν and $-\nu-1$ respectively. Recalling the relation

$$Q^\mu_{-\nu-1}(z) = \frac{\sin[\pi(\nu+\mu)]}{\sin[\pi(\nu-\mu)]}Q^\mu_\nu(z) - \frac{\pi e^{i\pi\mu}\cos(\nu\pi)}{\sin[\pi(\nu-\mu)]}P^\mu_\nu(z) \ , \quad (21.46)$$

we thus see that the branch cut discontinuity in the causal propagator is determined

[11]The situation which is to obtain for large H remains to be explored.

by the difference

$$\frac{Q^2_{-\nu-1}(\coth(\sigma-b|w|))}{Q^1_{-\nu-1}(\coth\sigma)}-\frac{Q^2_\nu(\coth(\sigma-b|w|))}{Q^1_\nu(\coth\sigma)}$$
$$=\frac{[Q^2_\nu(\coth(\sigma-b|w|))-\pi\cot(\pi\nu)P^2_\nu(\coth(\sigma-b|w|))]}{[Q^1_\nu(\coth\sigma)-\pi\cot(\pi\nu)P^1_\nu(\coth\sigma)]}-\frac{Q^2_\nu(\coth(\sigma-b|w|))}{Q^1_\nu(\coth\sigma)}$$
$$=\pi\cot(\pi\nu)\left(\frac{P^1_\nu(\coth\sigma)Q^2_\nu(\coth(\sigma-b|w|))-Q^1_\nu(\coth\sigma)P^2_\nu(\coth(\sigma-b|w|))}{Q^1_\nu(\coth\sigma)[Q^1_\nu(\coth\sigma)-\pi\cot(\pi\nu)P^1_\nu(\coth\sigma)]}\right)\ . \tag{21.47}$$

With the above discontinuity formula applying for all q^2 which are greater than $9H^2/4$, and with the relevant ν then, as had been noted earlier, being given by $\nu=-1/2-i(q^2/H^2-9/4)^{1/2}$, we recognize the form of the numerator of Eq. (21.47) as being none other than that associated with the KK continuum basis mode combinations which satisfy Eq. (21.11). Additionally, when $\nu=-1/2-i(q^2/H^2-9/4)^{1/2}$, the denominator of Eq. (21.47) gets to be compactly rewritten as $Q^1_\nu(\coth\sigma)[Q^1_\nu(\coth\sigma)-i\pi\tanh((q^2/H^2-9/4)^{1/2})P^1_\nu(\coth\sigma)]$.[12] Finally, then, and in complete analog to the situation found for the M_4^+ brane world, we see that the singular part of the causal dS_4^+ brane world propagator of Eq. (21.36) is precisely given by that of the normalized basis mode propagator of Eq. (21.15), though given the parallel with M_4^+, it follows that in the dS_4^+ brane world it is only the propagator of Eq. (21.36) which is causal, and not the one given in Eq. (21.15).

As well as the positive tension dS_4^+, it is also of interest to explore the globally non-hyperbolic negative tension dS_4^- brane world. Here the warp factor is given by $e^A=(H/b)\sinh(\sigma+b|w|)$ where $\sinh\sigma=b/H$, and the basis modes are again the associated Legendre functions, only with an argument $y=\coth(\sigma+b|w|)$ which now lies in the range $1\leq y\leq\coth\sigma$. Because of this change in range, in dS_4^- neither the graviton nor the KK continuum modes will be normalizable (all the $Q^2_\nu(y)$ diverge at $y=1$) or have wave functions which localize to the brane. The only allowed modes which will be normalizable and localized are those $P^2_{\nu_i}(y)$ for which $P^1_{\nu_i}(\coth\sigma)=0$, an infinite, discrete set of allowed such ν_i. Given the limit for $P^2_\nu(y)$ exhibited in Eq. (21.26), when $H\to 0$, the masses of those $P^1_{\nu_i}(\coth\sigma)=0$ modes which do not themselves vanish with H will transit into the discrete normalizable mode masses of M_4^- (viz. those associated with the $m_i=bj_i$ zeroes of $J_1(m_i/b)$ discussed in Chapter 18). However it will be the linear combination with $\hat{\beta}_\nu=-\hat{\alpha}_\nu/i\pi$ in the dS_4^- analog of Eq. (21.27) which will transit into the causal J_2-iY_2 based propagator of M_4^-, with the thus causal $\hat{\beta}_\nu=-\hat{\alpha}_\nu/i\pi$ based dS_4^- propagator possessing a graviton whose $e^{2A}=(H^2/b^2)\sinh^2(\sigma+b|w|)$ wave function does not localize to the brane. Of the positive and negative tension dS_4 based brane worlds then, only the positive tension one can lead to a phenomenologically viable gravity.

[12]We note that via Eq. (21.26) this particular denominator has a small H limit which recovers the $J_1^2(q/b)+Y_1^2(q/b)$ factor associated with the denominator of the continuum contribution to the analog M_4^+ causal propagator as given in Eq. (17.64).

Finally then, to summarize, we see that just like the M_4^+ brane world, the small H dS_4^+ brane-world causal propagator also possesses a localized, massless graviton,[13] with the fluctuations which are generated by this propagator via Eq. (21.23) being localized to the brane, and with the small H graviton being canonical on it. Thus even while a small H AdS_4^+ brane world is not phenomenologically viable (as we saw in Chapter 19, its gravity does not in general localize to the brane), the small H dS_4^+ brane world is as viable a brane world as the original M_4^+ brane world which Randall and Sundrum found.

[13]Even though we did not construct a large H causal propagator explicitly, we anticipate that whatever its form, it would continue into the small H one in the small H limit, with it not to be expected that the causal structure of the dS_4^+ brane world would suddenly change discontinuously at some particular value of H.

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Chapter 22

Dynamically Generated Domain Walls and the Brane World

22.1 Kink generated M_4 domain wall

While we have seen the great value to the brane-localized gravity program of embedding the brane in a bulk which is AdS_5 invariant, left unaddressed in this program are two issues which are quite central to it. Thus, on the one hand we need to ask just exactly where such an AdS_5 bulk might actually come from, and on the other hand we also need to ask what is to happen if the bulk were to be endowed with more fields than just a 5-dimensional cosmological constant, a situation in which the bulk would then typically have a symmetry which would be lower than AdS_5.

With regard to the first of these two issues, it is of interest to note that in the 10-dimensional string theory program to which we referred in Chapter 1, an AdS_5 geometry might actually play a special role. Specifically, in keeping with the general idea of Kaluza-Klein compactification, in the original formulation of string theory it was presupposed that the six dimensions beyond the four ordinary ones would be compactified into a closed 6-dimensional sphere S_6 of microscopic, Planck length, size, with the resulting geometry then being $M(3,1) \times S_6$. However it was noticed by Maldecena [Maldacena1998] that a very particular alternate compactification chain, viz. to $AdS_5 \times S_5$, may also be favored, one which would enable string theory to make contact with conformal field theories, the so called AdS/CFT correspondence, in which properties of string theories could be used to learn about properties of ordinary conformal field theories (in flat spacetime) and vice versa. In this way then a non-compactified AdS_5 spacetime might emerge as a favored outcome of string theory, to thereby provide a potential rationale for why a spacetime such as AdS_5 might occupy a somewhat privileged position in it.

The issue of having additional field structure in some general, non AdS_5 invariant, 5-dimensional bulk has been pursued by various authors [DeWolfe, Freedman, Gubser and Karch (2000); Csaki, Erlich, Hollowood and Shirman (2000); Gremm 2000a; Gremm 2000b; Behrndt, 2000; Davidson and Mannheim (2000)] both in and of itself, and with a view to recovering the Randall-Sundrum braneworld picture in some appropriate limit. Typically, in such studies the brane is considered not to be infinitesimally thin (from the point of view of the fifth dimension), but rather to be an analog of a ferromagnetic domain wall which actually extends into the fifth dimension. Such so-called thick branes might then transit into infinitesimally thin ones in some appropriate limit. Now in many-body theory a domain structure is typically maintained by the existence of some long order parameter (such as the spontaneous magnetization of a ferromagnet) which is not a pure constant everywhere, but which takes differing values in differing domains, and which changes gradually from the value it possesses in one domain to the different value that it possesses in an adjacent one. In such situations the thickness of the domain wall is given as the size of the region over which the order parameter effects this change. An analog of a domain wall order parameter can be realized in field theory by a scalar field whose potential energy possesses more than one minimum. The scalar field can then be in different minima of the potential in differing domains, and interpolate between them as it traverses the domain wall which then separates them. Such a scalar field would then be spatially varying, and thus necessarily have a non-zero kinetic energy. In consequence of this, in addition to its potential energy, the kinetic energy of the scalar field would also contribute to its energy-momentum tensor, and thus provide a source for gravity which would not be a spacetime-independent constant, and which would thus not act as the pure cosmological constant which is to be obtained in the single domain situation where a non spatially varying scalar field is always in just one of the potential energy minima. In cases were the scalar field is spatially varying then, the geometry generated in the Einstein equations by the source would not be the maximally 5-symmetric AdS_5 spacetime that the source would generate if the scalar field were to be a pure constant, but would instead be some other spacetime with lower symmetry. To monitor what is to then occur in such a situation, and to see whether an AdS_5 bulk might be recovered when we take the thin-brane limit, we turn to some exactly soluble models.

We consider first the case where the domain wall is to have M_4 symmetry, and thus take as metric

$$ds^2 = dw^2 + e^{2A}(dx^2 + dy^2 + dz^2 - dt^2) \quad , \tag{22.1}$$

as defined here in terms of an as yet undetermined warp factor A which only depends on the fifth coordinate w. For such a metric we note that it is only when A is linear in w that the metric will describe an AdS_5 invariant spacetime. For a scalar field ϕ with 5-dimensional energy-momentum tensor $T_{MN} = \partial_M\phi\partial_N\phi - g_{MN}[(1/2)\partial_K\phi\partial^K\phi + V(\phi)]$ with some general potential energy $V(\phi)$, whenever g_{MN} and ϕ depend only on w, the 5-dimensional gravitational and scalar field equations take the form (primes denote derivatives with respect to w)

$$\begin{aligned} 3A'' + 6A'^2 &= -\kappa_5^2 e^{-2A} T_{00} = -\kappa_5^2\left[\frac{1}{2}\phi'^2 + V(\phi)\right] \quad , \\ 6A'^2 &= \kappa_5^2 T_{55} = \kappa_5^2\left[\frac{1}{2}\phi'^2 - V(\phi)\right] \quad , \end{aligned} \tag{22.2}$$

$$\phi'' + 4A'\phi' = \frac{dV(\phi)}{d\phi} \ . \tag{22.3}$$

Given that the double well potential $V(\phi) = \lambda(\phi^2 - f^2)^2$ (where λ and f are constants) supports the so-called kink solution

$$\phi(w) = \alpha\tanh(\beta w) \tag{22.4}$$

in the flat spacetime case,[1] it is immediately suggested to see what potential might possibly support the configuration of Eq. (22.4) in the curved space case. And for a potential of the form

$$V(\phi) = \frac{\alpha^2\beta^2}{2}\left[\left(1 - \frac{\phi^2}{\alpha^2}\right)^2 - \frac{4\kappa_5^2}{27}\phi^2\left(3 - \frac{\phi^2}{\alpha^2}\right)^2\right] \ , \tag{22.5}$$

it was thus found [DeWolfe, Freedman, Gubser and Karch (2000)] that the above kink solution then actually was an exact solution to Eqs. (22.2) and (22.3), with the warp factor being given by

$$\begin{aligned} A &= \frac{\kappa_5^2\alpha^2}{18}\left[\frac{1}{\cosh^2(\beta w)} - \ln\cosh^4(\beta w)\right] \ , \\ A' &= -\frac{\kappa_5^2\alpha^2\beta}{9}\left[\frac{\sinh(\beta w)}{\cosh^3(\beta w)} + 2\tanh(\beta w)\right] \ , \\ e^{2A} &= \exp\left(\frac{\kappa_5^2\alpha^2}{9\cosh^2(\beta w)}\right)\left(\cosh(\beta w)\right)^{\frac{-4\kappa_5^2\alpha^2}{9}} \ . \end{aligned} \tag{22.6}$$

Thus rather than a quartic potential, this time it is a sextic one which is needed. While not a particularly nice potential since it is unbounded below ($V \to -\infty$ as $\phi \to \pm\infty$), use of this potential is still very instructive. Specifically, the potential has two degenerate local minima at $\phi = \pm\alpha$ where it takes the negative value

$$V(\pm\alpha) = -\frac{8\kappa_5^2\alpha^4\beta^2}{27} \ , \tag{22.7}$$

with the kink solution of Eq. (22.4) thus interpolating between these two minima according to $\phi(-\infty) = -\alpha$, $\phi(\infty) = \alpha$. Because of this interpolation $\phi(w)$ must be an odd function of w, with the form of the equations of motion then obliging A to be an even function of w. Because of the kink structure then, A naturally becomes a function of $|w|$, to thus give the warp factor the Z_2 symmetry required of the brane world. When gravity is coupled to a domain wall scalar field then, we see that the Z_2 symmetry of the brane-world metric can be generated dynamically. Moreover, the warp factor is not only found to be a function of $|w|$, the particular form found

[1] For the flat spacetime solution the α and β coefficients are given by $\alpha = f$, $\beta = (2\lambda)^{1/2}f$, with $\phi(w)$ taking the values $\phi(-\infty) = -f$, $\phi(\infty) = f$, to thus interpolate between the two minima of the double well potential at $\phi = -f$, $\phi = f$.

for it in Eq. (22.6) is one which peaks at $w = 0$ and then falls monotonically to zero as $|w| \to \infty$. The geometry is thus seen to localize around the domain wall.

It is also possible to construct a thin-brane limit for this solution by taking a particular limit of it, one in which α goes to zero and β goes to infinity while the product $\alpha^2\beta$ is held fixed. On noting that in the $\beta \to \infty$ limit the quantity $[\cosh(\beta w)]^{-1/\beta}$ limits to $e^{-|w|}$, we see that $A(w)$ and $\phi(w)$ limit to

$$A \to -\frac{2\kappa_5^2\alpha^2\beta|w|}{9} \quad , \quad \phi \to \alpha\epsilon(w) \quad , \tag{22.8}$$

with $\phi(w)$ becoming a function which changes discontinuously at $w = 0$, and with the form of $A(w)$ becoming none other than that of the warp factor associated with the M_4^+ brane world. The Randall-Sundrum thin brane thus emerges as a particular limit of a domain wall thick brane. Additionally, on defining $V(\pm\alpha) = \Lambda_5$ in Eq. (22.7) and recalling Eq. (2.5), we see that can rewrite the limit as

$$A \to -\left(\frac{-\kappa_5^2\Lambda_5}{6}\right)^{1/2}|w| = -b|w| \ . \tag{22.9}$$

In such a limit we find that $A' \to -b\epsilon(w)$, $A'' \to -2b\delta(w)$, to thus generate jump and delta function singularities at the brane. In order to determine just exactly where such a delta function singularity might come from in our model, we note from Eq. (22.2) that the simultaneous presence of a delta function singularity in A'' and the absence of one in A' entail that $\phi'^2/2$ and $V(\phi)$ must both generate the same delta function in the limit. And indeed from Eq. (22.4) we see immediately that since

$$\phi'^2 = \frac{\alpha^2\beta^2}{\cosh^4(\beta w)} = \frac{\alpha^2\beta}{3}\frac{d}{dw}\left(2\tanh(\beta w) + \frac{\tanh(\beta w)}{\cosh^2(\beta w)}\right) \quad , \tag{22.10}$$

in the limit we therefore obtain

$$\phi'^2 \to \frac{2\alpha^2\beta}{3}\frac{d}{dw}\epsilon(w) = \frac{4\alpha^2\beta}{3}\delta(w) \quad , \tag{22.11}$$

to thus generate a term $e^{2A}[\phi'^2/2+V(\phi)] \to (4\alpha^2\beta/3)\delta(w)$ in T_{00}, an effective brane tension term $\lambda\delta(w) = (4\alpha^2\beta/3)\delta(w)$ which then automatically obeys the condition $\Lambda_5 + \kappa_5^2\lambda^2/6 = 0$, i.e. which obeys none other than the Randall-Sundrum fine-tuning condition introduced in Eq. (2.15). Thus what must be thought of as a fine tuning condition in a Randall-Sundrum theory with an a priori brane possessing a tension λ which is introduced by hand is now seen as being a dynamical output to a kink-induced brane theory, with λ being due to the energy density of the bulk kink when the kink is squeezed to the selfsame brane which it itself generates. Not only is the Randall-Sundrum fine-tuning condition generated dynamically, we see that, if desired, it can even be avoided by not letting the brane become too thin.

Some additional insight into the structure of the above solution can be obtained by noting that without needing to specify any particular form for the potential, manipulation of Eq. (22.2) allows us to write

$$3A'' = -\kappa_5^2(e^{-2A}T_{00} + T_{55}) = -\kappa_5^2\phi'^2 \quad , \tag{22.12}$$

to yield the integral relation

$$A'(\infty) - A'(-\infty) = -\frac{\kappa_5^2}{3}\int_{-\infty}^{\infty} dw(e^{-2A}T_{00} + T_{55}) = -\frac{\kappa_5^2}{3}\int_{-\infty}^{\infty} dw\phi'^2 \quad . \tag{22.13}$$

Now as such, Eq. (22.13) holds both before and after the thin-brane limit is taken. However, after the limit is taken, the only contribution to an integration of the energy-momentum tensor across the brane will be the singular one due to the localizing of the scalar field kinetic and potential energy to it. With such a localization being equivalent to the trapping of a $T_{\mu\nu} = -\lambda\eta_{\mu\nu}\delta(w)$ source on the brane, Eq. (22.13) thus yields the limit

$$A'(\infty) - A'(-\infty) \to -\frac{\kappa_5^2}{3}\lambda \quad , \tag{22.14}$$

a generic relation which must hold whenever a kink type solution supports a domain wall with a well-behaved thin-brane limit. Now we recall that any $T_{\mu\nu} = -\lambda\eta_{\mu\nu}\delta(w)$ discontinuity at $w = 0$ always obeys the Israel junction condition

$$K_{\mu\nu}(0^+) - K_{\mu\nu}(0^-) = -\kappa_5^2\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}\right) = -\frac{\kappa_5^2}{3}\eta_{\mu\nu}\lambda \quad . \tag{22.15}$$

Consequently, with the extrinsic curvature of an M_4 surface embedded according to Eq. (22.1) being given by $K_{\mu\nu}(w) = e^{2A}A'\eta_{\mu\nu}$ (cf. Eq. (4.8)), we obtain

$$A'(0^+) - A'(0^-) = -\frac{\kappa_5^2}{3}\lambda \quad . \tag{22.16}$$

Comparing with Eq. (22.14), we see that while Eq. (22.16) will hold whenever there is a brane (either introduced by hand or generated dynamically), in addition Eq. (22.14) will also hold if the brane is actually generated dynamically as a limit of a domain wall. While the discussion leading to Eq. (22.14) is potential-independent, we note that in the sextic potential case the quantity ϕ'^2 is expressed in Eq. (22.10) as a total derivative. The insertion of Eq. (22.10) into Eq. (22.13) therefore yields

$$A'(\infty) - A'(-\infty) = -\frac{\kappa_5^2\alpha^2\beta}{9}\left(2\tanh(\beta w) + \frac{\tanh(\beta w)}{\cosh^2(\beta w)}\right)\Big|_{-\infty}^{\infty} = -\frac{4\kappa_5^2\alpha^2\beta}{9} \quad , \tag{22.17}$$

to lead us right back to the relation $\lambda = 4\alpha^2\beta/3$ found above, since A' as given in Eq. (22.6) behaves as $A' \to -(2\kappa_5^2\alpha_2\beta/9)\tanh(\beta w) \equiv -(2\kappa_5^2\alpha^2\beta/9)\epsilon(w)$ in the limit of large w, while also behaving as $A' \to -(2\kappa_5^2\alpha^2\beta/9)\epsilon(w)$ in the limit of large β. The limiting behavior of the thin brane can thus be monitored not only by looking at the junction discontinuity at the brane, but also by looking at the total change in the quantity $A'(\infty) - A'(-\infty)$ across the entire bulk.

22.2 Soliton generated M_4 domain wall

While the above analysis shows how it is possible to recover the brane world starting from a thick domain wall, because the needed potential given in Eq. (22.5) is both ad hoc and unbounded below,[2] it would be nice if one could obtain this same result in a model with a fully bounded, less ad hoc, potential.

To motivate a possible choice for such a potential, we note that there is an interesting geometric connection between maximally symmetric spaces and the sine-Gordon equation, an equation which also supports solutions which interpolate between potential minima, viz. the so-called solitons. Specifically, the general (Euclidean or Minkowski) signatured 2-dimensional metric

$$ds^2 = dx^2[1 + \cos\theta(x, y)] \pm dy^2[1 - \cos\theta(x, y)] \tag{22.18}$$

will describe a space of constant 2-curvature K provided (see e.g. [Coleman (1985)]) the metric coefficient $\theta(x, y)$ obeys the condition

$$\frac{\partial^2\theta}{\partial x^2} \mp \frac{\partial^2\theta}{\partial y^2} = -2K\sin\theta \ , \tag{22.19}$$

a condition which we recognize as being of the form of none other than the sine-Gordon equation. As such, Eq. (22.19) will support the y-independent soliton solution

$$\tan\left(\frac{\theta}{4}\right) = e^{(-2K)^{1/2}x} \tag{22.20}$$

if K is negative, with θ varying over the bounded range from $\theta = 0$ to $\theta = 2\pi$ as x ranges from $-\infty$ to ∞. On recalling now that our sought AdS_5 brane-world spacetime is itself a space of constant negative curvature, it is thus suggested to try as potential the sine-Gordon potential

$$V(\phi) = \frac{\alpha^2\beta^2}{8} - \frac{\alpha^2\beta^2}{8}\left(1 + \frac{\kappa_5^2\alpha^2}{3}\right)\sin^2\left(\frac{2\phi}{\alpha}\right) \ , \tag{22.21}$$

a potential which happens to actually be bounded below. On setting

$$\tan\left(\frac{\phi}{\alpha}\right) = \tanh\left(\frac{\beta w}{2}\right) \ , \ \phi' = \frac{\alpha\beta}{2}\text{sech}(\beta w) = \frac{\alpha\beta}{2}\cos\left(\frac{2\phi}{\alpha}\right) \ , \tag{22.22}$$

viz. on introducing a solitonic domain wall, we find [Gremm 2000a; Behrndt, 2000; Davidson and Mannheim (2000)] that for such a domain wall an exact solution to

[2]Apart from arbitrarily being taken to be sextic in the first place, technically, the potential of Eq. (22.5) is even somewhat fine-tuned, since the most general sextic potential should possess four independent parameters rather than just the two displayed in Eq. (22.5). Moreover, since this sextic potential is unbounded below, even while the above static kink solution is actually a bounded one in which $\phi(w)$ (and thus $V(\phi)$) never become large, the model could nonetheless still possess time dependent solutions which are unbounded.

the entire set of field equations of Eqs. (22.2) and (22.3) is then obtained with the metric coefficient being given by

$$A' = -\frac{\kappa_5^2\alpha^2\beta}{12}\tanh(\beta w) \quad , \quad e^A = \left(\cosh(\beta w)\right)^{\frac{-\kappa_5^2\alpha^2}{12}} \quad . \tag{22.23}$$

We recognize this solution as being precisely of the thick-brane form, with the warp factor being an even function of w which is maximal at $w = 0$ and which falls off on either side of it.

Like the kink-based solution, this soliton-based solution also admits of a thin-brane limit. Specifically, we again consider the limit in which α goes to zero and β goes to infinity while the product $\alpha^2\beta$ is held fixed, to find that $A(w)$ and $\phi(w)$ then limit to

$$A \to -\frac{\kappa_5^2\alpha^2\beta|w|}{12} \quad , \quad \phi \to \frac{\pi\alpha}{4}\epsilon(w) \quad , \tag{22.24}$$

with the Randall-Sundrum M_4^+ brane-world warp factor form thus emerging once again. Additionally, on defining the value that the sine-Gordon potential takes at its degenerate minima as $V(\pm\pi\alpha/4) = \Lambda_5$, we obtain $\Lambda_5 = -\kappa_5^2\alpha^4\beta^2/24$, to thereby yield

$$A \to -\left(\frac{-\kappa_5^2\Lambda_5}{6}\right)^{1/2}|w| = -b|w| \tag{22.25}$$

in the thin-brane limit. With the thick brane A' as given in Eq. (22.23) behaving as $A' \to -(\kappa_5^2\alpha^2\beta/12)\epsilon(w)$ in the limit of large w, and then continuing to behave this way in any limit in which $\alpha^2\beta$ is held constant, from the generic analysis of Eq. (22.14) we see that we can define the thin-brane limit λ as $\lambda = \alpha^2\beta/2$. Consequently, λ and Λ_5 thus obey the Randall-Sundrum fine tuning condition $\Lambda_5 + \kappa_5^2\lambda^2/6 = 0$ just as desired, with the thin-brane limit of the solitonic sine-Gordon thick-brane theory leading us right back to the Randall-Sundrum M_4^+ brane world.

In both the kink and the soliton-based thick-brane models then, we see how we can start with a single patch of spacetime containing a spatially-varying scalar field, and take a limit of it in which the domain wall becomes an infinitesimally thin one which then divides the original spacetime into two separate patches. The two patch Randall-Sundrum brane world can thus be generated dynamically. Now in our general analysis of fluctuation modes given earlier we had noted that for generic metrics of the form given in Eq. (13.4), a massless TT graviton mode with wave function e^{2A} would be an exact solution to the source-free TT wave equations given as Eqs. (16.4) and (16.5). Thus whether we consider the thick (Eq. (22.23)) or thin (Eq. (22.24)) brane soliton-based models, in either case (and likewise for kink-based models) we would obtain a massless graviton which would be localized around the $w = 0$ region. The most significant feature of the Randall-Sundrum brane world, viz. a localized massless graviton, is thus seen to be obtainable even without needing to actually take the thin-brane limit at all. Similarly, even before taking the thin-brane limit, the fluctuation spectrum would also include a continuum of KK modes. The only possible departure from the Randall-Sundrum picture would be in the possible existence of fluctuations which would be odd functions of w rather than even ones. Specifically, while such fluctuations are forbidden in the Randall-Sundrum picture by identifying points at infinity (cf. Chapter 2), no such identification occurs in the domain wall situation, as it generates the brane wall as a limit which acts on a single decompactified patch of spacetime. It would thus be of interest to ascertain whether or not there actually are any stable, w-odd fluctuations in the domain wall case, as the possible absence of any such fluctuations would then allow us to generate a brane world in which one could dispense with the need to identify points at infinity altogether.

22.3 dS_4 domain wall

It is also possible to explore the domain wall picture in the case where the geometry on the wall is taken to be dS_4. In such a situation the metric is given by

$$ds^2 = dw^2 + e^{2A}[e^{2Ht}(dx^2 + dy^2 + dz^2) - dt^2] \ , \qquad (22.26)$$

with general warp factor A which only depends on w. In the presence of a 5-dimensional scalar field ϕ Eq. (22.2) is replaced by

$$\begin{aligned} 3A'' + 6A'^2 - 3H^2e^{-2A} &= -\kappa_5^2 e^{-2A}T_{00} = -\kappa_5^2\left[\frac{1}{2}\phi'^2 + V(\phi)\right] \ , \\ 6A'^2 - 6H^2e^{-2A} &= \kappa_5^2 T_{55} = \kappa_5^2\left[\frac{1}{2}\phi'^2 - V(\phi)\right] \ , \end{aligned} \qquad (22.27)$$

while Eq. (22.3) remains unchanged. As we recall from Chapters 8 and 10, for the choice of energy-momentum tensor of the form $T_{00} = e^{2A}\Lambda_5 + \lambda\delta(w)$, $T_{55} = -\Lambda_5$, Eq. (22.27) admits of a Randall-Sundrum type thin-brane model with warp factor

$$e^A = \frac{\sinh(bw_0 - b|w|)}{\sinh(bw_0)} \ , \quad A' = -b\epsilon(w)\coth(bw_0 - b|w|) \ , \qquad (22.28)$$

where

$$H\sinh(bw_0) = \left(\frac{-\kappa_5^2\Lambda_5}{6}\right)^{1/2} = b \ , \quad \coth(bw_0) = \frac{\kappa_5^2\lambda}{6b} \ , \quad \Lambda_5 + \frac{\kappa_5^2\lambda^2}{6} = \frac{6H^2}{\kappa_5^2} > 0 \ , \qquad (22.29)$$

and where the associated AdS_5 bulk possesses a horizon at $|w| = w_0$, with $\sinh(bw_0)$ varying inversely with H, the residual cosmological constant on the brane.

In order to determine what is needed to construct such a dS_4 brane as a limit of a domain wall, we again look for a soliton or kink-like solution in which the scalar field interpolates between two degenerate minima of the potential.[3] However, this time we shall require the scalar field to be in the minima not at $w = \pm\infty$ but at $w = \pm w_0$ instead, with ϕ' then having to vanish at $w = \pm w_0$. Given the singularity structure of the thin-brane warp factor at $w = \pm w_0$ which is exhibited in Eq. (22.28), we need to determine how a singular behavior such as this (assuming it to be present prior to taking the thin-brane limit) would affect the behavior of the scalar field at the singular points. Thus, taking the scalar field to behave as $\phi(w) = \phi(w_0) + E(w_0 - w)^n$ near $w = w_0$, from Eq. (22.3) we see that $dV(\phi)/d\phi$ needs to behave as $\phi'' + 4A'\phi' = En(n+3)(w_0 - w)^{n-2}$ near $w = w_0$. If $V(\phi)$ is to be a well-behaved, Taylor series expandable potential with smooth minima, then it must behave near any minimum $\phi_{\rm min}$ as $dV/d\phi \sim (\phi - \phi_{\rm min})d^2V(\phi_{\rm min})/d\phi^2$, and thus near $w = w_0$ as $dV/d\phi \sim E[w_0 - w]^n d^2V(\phi_{\rm min})/d\phi^2$, a form which we see is simply not compatible with $\phi'' + 4A'\phi'$ no matter what the value of n. Hence, near its minima $V(\phi)$ cannot in fact be as well-behaved as a standard particle physics Higgs potential. Rather, $dV(\phi)/d\phi$ must also behave as $(w_0 - w)^{n-2}$, i.e. as $[\phi(w_0) - \phi]^{(n-2)/n}$, so that the (necessarily real) potential itself must behave as the non-analytic $(|\phi - \phi(w_0)|)^{2(n-1)/n}$, and thus have a cusp at $w = w_0$. With the same behavior having to occur at $w = -w_0$ as well, rather than being a smooth potential, $V(\phi)$ would have to be a double (or multiple) well or periodic ($\simeq \sin|\phi|$) cusp potential. The presence of a horizon at $w = \pm w_0$ is thus heralded by the loss of analyticity at the two cusp minima of the potential, with a potential which has such cusps then being able to generate the dS_4^+ brane-world horizon while also providing a mechanism with which to disconnect the $|w| < w_0$ and $|w| > w_0$ regions from each other.

While the presence of a cusp potential is thus seen to be the key ingredient needed to produce a thin dS_4 brane as a limit of a thick brane, no specific exact thick-brane solution for such a set-up appears to be known. However, an exact thin dS_4 brane solution which involves a cusp potential is known to exist, one in which the scalar field itself is found to diverge at $w = \pm w_0$. And despite the fact that this is an at first somewhat disquieting requirement, we now present an explicit closed form solution to the field equations which allows such a divergent ϕ to support a thin dS_4 brane without there being any divergence in the energy density. In this solution the warp factor is again given by the thin-brane Eq. (22.28), the field configuration (viz. a configuration for which $\phi(w = \pm w_0) = \pm\infty$) is given by

$$\frac{\phi(w)}{\nu} = \frac{\epsilon(w)}{b}\text{logtanh}\left(\frac{bw_0 - b|w|}{2}\right) - \epsilon(w)\frac{\mu}{\nu} \quad , \quad \frac{\mu}{\nu} = \frac{1}{b}\text{logtanh}\left(\frac{bw_0}{2}\right) \quad , \tag{22.30}$$

[3]The work of this section is taken from [Davidson and Mannheim (2000)], with some additional discussion of thick dS_4 branes being given in [Gremm 2000b].

the potential is taken to have the "shine-Gordon" form[4]

$$V = V_0 - \frac{3\nu^2}{2}\sinh^2\left(\frac{-b|\phi| + b\mu}{\nu}\right) \quad , \tag{22.31}$$

and the energy-momentum tensor is taken to have components

$$\begin{aligned} e^{-2A}T_{00} &= \frac{\phi'^2}{2} + V(\phi) + \lambda\delta(w) \quad , \\ T_{55} &= \frac{\phi'^2}{2} - V(\phi) \quad , \end{aligned} \tag{22.32}$$

with Eq. (22.27) thus being replaced by

$$\begin{aligned} 3A'' + 6A'^2 - 3H^2e^{-2A} &= -\kappa_5^2 e^{-2A}T_{00} = -\kappa_5^2\left[\frac{1}{2}\phi'^2 + V(\phi) + \lambda\delta(w)\right] \quad , \\ 6A'^2 - 6H^2e^{-2A} &= \kappa_5^2 T_{55} = \kappa_5^2\left[\frac{1}{2}\phi'^2 - V(\phi)\right] \quad . \end{aligned} \tag{22.33}$$

In this solution the various parameters are found to obey[5]

$$\begin{aligned} -\kappa_5^2 V_0 &= 6b^2 \quad , \\ 3H^2\sinh^2(bw_0) - 3b^2 &= -\kappa_5^2\nu^2 \quad , \\ \kappa_5^2\lambda &= 6b\coth(bw_0) \quad , \\ \frac{\lambda^2\kappa_5^2}{6} + V_0 &= \frac{18b^2H^2}{\kappa_5^2(3b^2 - \kappa_5^2\nu^2)} \quad , \end{aligned} \tag{22.34}$$

to yield a structure quite similar to that exhibited in Eq. (22.29). Despite the fact that the potential of Eq. (22.31) is unbounded from below, the quantity $e^{2A}[\phi'^2/2 + V(\phi) + \lambda\delta(w)]$ remains bounded because of compensating zeroes in the metric. Thus while a potential such as that of Eq. (22.31) could not be considered in flat space physics, in the event of curvature it is possible for the energy density in the gravitational field to compensate and render the theory well-behaved. With such a diverging of ϕ at $w = \pm w_0$ we thus uncover a mechanism which can actually serve to disconnect the $|w| < w_0$ and $|w| > w_0$ regions and thereby force the dS_4^+ brane-world horizon upon us in the first place. Moreover, we can even consider the points $\phi = \pm\infty$ as being effective "minima" of the theory since they are the points at which $V(\phi)$ takes its lowest values within the $|w| \leq w_0$ region (with the horizon

[4]The general (Minkowski or Euclidean) signatured 2-dimensional metrics $ds^2 = dx^2[1 + \cosh\theta(x,y)] \pm dy^2[1 - \cosh\theta(x,y)]$ will be spaces of constant 2-curvature K provided θ obeys the conditions $\partial^2\theta/\partial x^2 \mp \partial^2\theta/\partial y^2 = -2K\sinh\theta$, and will support the y-independent mode $\tanh(\theta/2) = \sin[(-2K)^{1/2}x]$ if K is negative, a mode in which θ becomes infinite at finite x.

[5]This particular solution is one of a generic family of solutions in which $e^A = (\phi')^{-B}$ and $A'\phi'/\phi'' = -B$ where B is constant, solutions in which Eq. (22.3) admits of an exact integral $V(\phi) = V_0 + \phi'^2(1-4B)/2$, and in which the second of Eqs. (22.33) can thus be written as the first order $6A'^2 - 6H^2e^{-2A} = \kappa_5^2(2B\phi'^2 - V_0) = \kappa_5^2(2Be^{-2A/B} - V_0)$. Additionally, in the solution given in Eq. (22.30) the scalar field $\phi(w)$ is negative when w is positive, and is positive when w is negative, so that in the solution we can replace the $\epsilon(\phi)$ term in $dV/d\phi$ by $-\epsilon(w)$.

thus cutting off the region where the shine-Gordon potential becomes unbounded below); with the point $\phi(w = 0) = 0$ at which the brane is located being a local maximum, one around which the geometry is then localized. In this way then a soliton-like scalar field configuration is able to support a dS_4 brane.

Appendix A

Gauge Invariance of the Fluctuation Equation

To establish the gauge invariance of the fluctuation equation $\Delta G_{MN} = -\kappa_5^2 \delta\tau_{MN}$ of Eq. (13.7), we need to first determine the dependence of the general δR_{MN} of Eq. (13.8) on the background Ricci tensor in the case where the fluctuation h_{MN} is taken to be the pure gauge function $\epsilon_{M;N} + \epsilon_{N;M}$ itself. To this end, we note that in any fluctuation the first order change in the Ricci tensor is in general given by the Palatini identity

$$\delta R_{MN} = (\delta\Gamma^L_{ML})_{;N} - (\delta\Gamma^L_{MN})_{;L} \quad , \tag{A.1}$$

where

$$2\delta\Gamma^L_{MN} = g^{LR}[h_{RM;N} + h_{RN;M} - h_{MN;R}] \quad . \tag{A.2}$$

Thus evaluating $\delta\Gamma^L_{MN}$ in the special case where $h_{MN} = \epsilon_{M;N} + \epsilon_{N;M}$ we obtain

$$\begin{aligned} 2\delta\Gamma^L_{MN} &= g^{LR}[\epsilon_{R;M;N} + \epsilon_{R;N;M} + \epsilon_{M;R;N} - \epsilon_{M;N;R} + \epsilon_{N;R;M} - \epsilon_{N;M;R}] \\ &= \epsilon^L{}_{;M;N} + \epsilon^L{}_{;N;M} - \epsilon_S R^S{}_M{}^L{}_N - \epsilon_S R^S{}_N{}^L{}_M \quad , \end{aligned} \tag{A.3}$$

where use has been made of differential geometry identity

$$\epsilon_{M;R;N} - \epsilon_{M;N;R} = -\epsilon_S R^S{}_{MRN} \quad , \tag{A.4}$$

as defined for the interchange of covariant derivatives with respect to the background metric g_{MN}. On contracting Eq. (A.3) we obtain

$$2\delta\Gamma^L_{ML} = \epsilon^L{}_{;M;L} + \epsilon^L{}_{;L;M} + \epsilon_S R^S{}_M \quad , \tag{A.5}$$

which we can rewrite in the two forms

$$\delta\Gamma^L_{ML} = \epsilon^L{}_{;L;M} \quad , \quad \delta\Gamma^L_{ML} = \epsilon^L{}_{;M;L} + \epsilon_S R^S{}_M \quad , \tag{A.6}$$

since contracting Eq. (A.4) yields

$$\epsilon^L{}_{;L;M} = \epsilon^L{}_{;M;L} + \epsilon_S R^S{}_M \quad . \tag{A.7}$$

On taking the covariant derivative of the first form of Eq. (A.6), we obtain for the contracted Christoffel symbol term

$$(\delta\Gamma^L_{ML})_{;N} = \epsilon^L{}_{;L;M;N} = \epsilon^L{}_{;L;N;M} = (\delta\Gamma^L_{NL})_{;M} \tag{A.8}$$

since the quantity $\epsilon^L{}_{;L}$ transforms as a scalar with respect to the background. Similarly, on taking the covariant derivative of the second form of Eq. (A.6) we obtain

$$\begin{aligned}(\delta\Gamma^L_{ML})_{;N} &= \epsilon^L{}_{;M;L;N} + \epsilon_{S;N}R^S{}_M + \epsilon_S R^S{}_{M;N} \\ &= \epsilon^L{}_{;N;L;M} + \epsilon_{S;M}R^S{}_N + \epsilon_S R^S{}_{N;M} \quad ,\end{aligned} \tag{A.9}$$

to yield

$$\begin{aligned}2(\delta\Gamma^L_{ML})_{;N} = \; &\epsilon^L{}_{;M;L;N} + \epsilon^L{}_{;N;L;M} + \epsilon_{S;N}R^S{}_M \\ &+\epsilon_{S;M}R^S{}_N + \epsilon_S R^S{}_{M;N} + \epsilon_S R^S{}_{N;M} \quad .\end{aligned} \tag{A.10}$$

On using the identity

$$\epsilon^L{}_{;M;N;K} - \epsilon^L{}_{;M;K;N} = \epsilon^S{}_{;M}R^L{}_{SNK} - \epsilon^L{}_{;S}R^S{}_{MNK} \quad , \tag{A.11}$$

viz.

$$\epsilon^L{}_{;M;L;K} - \epsilon^L{}_{;M;K;L} = \epsilon^S{}_{;M}R_{SK} - \epsilon_{L;S}R^S{}_M{}^L{}_K \quad , \tag{A.12}$$

we then obtain

$$\begin{aligned}2(\delta\Gamma^L_{ML})_{;N} = \; &\epsilon^L{}_{;M;N;L} + \epsilon^L{}_{;N;M;L} - \epsilon_{L;S}R^S{}_M{}^L{}_N - \epsilon_{L;S}R^S{}_N{}^L{}_M \\ &+2\epsilon^S{}_{;N}R_{SM} + 2\epsilon^S{}_{;M}R_{SN} + \epsilon_S R^S{}_{M;N} + \epsilon_S R^S{}_{N;M} \quad .\end{aligned} \tag{A.13}$$

Similarly for the uncontracted Christoffel symbol term we obtain

$$\begin{aligned}2(\delta\Gamma^L_{MN})_{;L} = \; &\epsilon^L{}_{;M;N;L} + \epsilon^L{}_{;N;M;L} - \epsilon_{S;L}R^S{}_M{}^L{}_N \\ &-\epsilon_S R^S{}_M{}^L{}_{N;L} - \epsilon_{S;L}R^S{}_N{}^L{}_M - \epsilon_S R^S{}_N{}^L{}_{M;L} \quad .\end{aligned} \tag{A.14}$$

Recalling that

$$R^L{}_{MKE;L} = R_{KE}{}^L{}_{M;L} = R_{ME;K} - R_{MK;E} \quad , \tag{A.15}$$

and relabeling some indices we can thus write

$$\begin{aligned}2(\delta\Gamma^L_{MN})_{;L} = \; &\epsilon^L{}_{;M;N;L} + \epsilon^L{}_{;N;M;L} - \epsilon_{L;S}R^L{}_M{}^S{}_N - \epsilon_{L;S}R^L{}_N{}^S{}_M \\ &-\epsilon^S[R_{NM;S} - R_{NS;M}] - \epsilon^S[R_{MN;S} - R_{MS;N}] \quad .\end{aligned} \tag{A.16}$$

Finally, subtracting Eq. (A.16) from Eq. (A.13) we obtain

$$\delta R_{MN} = \epsilon_{S;M}R^S{}_N + \epsilon_{S;N}R^S{}_M + \epsilon^S R_{MN;S} \quad , \tag{A.17}$$

to thus give us our desired expression for δR_{MN} when evaluated in the fluctuation $h_{MN} = \epsilon_{M;N} + \epsilon_{N;M}$.

Given Eq. (A.17), for the pure gauge fluctuation $h_{MN} = \epsilon_{M;N} + \epsilon_{N;M}$ the quantity δG_{MN} of Eq. (13.7) then readily evaluates to $\delta G_{MN} = \delta G_{MN}(\epsilon_S)$ where

$$\begin{aligned}\delta G_{MN}(\epsilon_S) = &\ \epsilon_{S;M}R^S{}_N + \epsilon_{S;N}R^S{}_M + \epsilon^S R_{MN;S} \\ &-\frac{1}{2}[\epsilon_{M;N}R + \epsilon_{N;M}R + g_{MN}\epsilon^S R_{;S}] \ .\end{aligned} \tag{A.18}$$

Thus suppose we have some given fluctuation h_{MN} which is a solution to the fluctuation equation $\Delta G_{MN} = \delta G_{MN} + \kappa_5^2 \delta T_{MN} = -\kappa_5^2 \delta\tau_{MN}$, and we now make a gauge transformation to bring h_{MN} to the form $\bar{h}_{MN} = h_{MN} + \epsilon_{M;N} + \epsilon_{N;M}$. Under this transformation δG_{MN} will transform into $\bar{\delta G}_{MN} = \delta G_{MN} + \delta G_{MN}(\epsilon_S)$ since in first order δG_{MN} is linear in the fluctuation. Since T_{MN} is a tensor with respect to the background, under a gauge transformation it will transform into $\bar{T}_{MN} = T_{MN} + \delta T_{MN}(\epsilon_S)$ where $\delta T_{MN}(\epsilon_S)$ is the Lie derivative of T_{MN} with respect to ϵ_S, viz.

$$\delta T_{MN}(\epsilon_S) = \epsilon_{S;M}T^S{}_N + \epsilon_{S;N}T^S{}_M + \epsilon^S T_{MN;S} \ . \tag{A.19}$$

Since the background equation of motion is $R_{MN} - (1/2)g_{MN}R = -\kappa_5^2 T_{MN}$, we thus see that $-\kappa_5^2 \delta T_{MN}(\epsilon_S)$ is given by

$$\begin{aligned}-\kappa_5^2 \delta T_{MN}(\epsilon_S) = &\ \epsilon_{S;M}R^S{}_N + \epsilon_{S;N}R^S{}_M + \epsilon^S R_{MN;S} \\ &-\frac{1}{2}[\epsilon_{M;N}R + \epsilon_{N;M}R + g_{MN}\epsilon^S R_{;S}] \ ,\end{aligned} \tag{A.20}$$

from which it follows that $\delta G_{MN}(\epsilon_S) + \kappa_5^2 \delta T_{MN}(\epsilon_S)$ vanishes identically. The fluctuations h_{MN} and $\bar{h}_{MN}$ thus both obey the same fluctuation equation.

Appendix B

Perturbative Weyl Tensor

From its definition in Eq. (2.2) the first order change in the general 5-dimensional Weyl tensor is given as

$$\begin{aligned}
\delta C_{LMNK} = {} & \delta R_{LMNK} + \frac{1}{12}[g^{AB}\delta R_{AB} - h^{AB}R_{AB}][g_{LN}g_{MK} - g_{LK}g_{MN}] \\
& + \frac{1}{12}R^A{}_A[g_{LN}h_{MK} - g_{LK}h_{MN} + h_{LN}g_{MK} - h_{LK}g_{MN}] \\
& - \frac{1}{3}[g_{LN}\delta R_{MK} - g_{LK}\delta R_{MN} - g_{MN}\delta R_{LK} + g_{MK}\delta R_{LN}] \\
& - \frac{1}{3}[h_{LN}R_{MK} - h_{LK}R_{MN} - h_{MN}R_{LK} + h_{MK}R_{LN}] \ . \qquad \text{(B.1)}
\end{aligned}$$

With the components of δR_{MN} being given in Eq. (A.1), to evaluate δC_{LMNK} we additionally need to determine δR_{LMNK}. To this end we note that from the definition of $R^L{}_{MNK}$ as

$$R^L{}_{MNK} = \frac{\partial\Gamma^L_{MN}}{\partial x^K} + \Gamma^L_{KS}\Gamma^S_{MN} - \frac{\partial\Gamma^L_{MK}}{\partial x^N} - \Gamma^L_{NS}\Gamma^S_{MK} \ , \qquad \text{(B.2)}$$

the first order change in the Riemann tensor is given as

$$\begin{aligned}
\delta R^L{}_{MNK} = {} & \frac{\partial\delta\Gamma^L_{MN}}{\partial x^K} + \Gamma^L_{KS}\delta\Gamma^S_{MN} - \Gamma^S_{MK}\delta\Gamma^L_{NS} \\
& - \frac{\delta\partial\Gamma^L_{MK}}{\partial x^N} - \Gamma^L_{NS}\delta\Gamma^S_{MK} + \Gamma^S_{MN}\delta\Gamma^L_{KS} \ , \qquad \text{(B.3)}
\end{aligned}$$

a form which, on adding and subtracting $\Gamma^S_{KN}\delta\Gamma^L_{MS}$, can be written very compactly as

$$\delta R^L{}_{MNK} = \delta\Gamma^L_{MN;K} - \delta\Gamma^L_{MK;N} \ . \qquad \text{(B.4)}$$

We recognize Eq. (B.4) as the generalization of Eq. (A.1), and with $\delta\Gamma^L_{MN}$ being given covariantly in Eq. (A.2) as a true tensor with respect to the background metric, confirm that $\delta R^L{}_{MNK}$ does indeed transform as a true tensor with respect to the background metric just as it should.

From Eq. (B.3) we find for a background metric of the generic form of Eq. (13.4), that $\delta R^5{}_{\mu 5\kappa}$ is given as

$$\delta R^5{}_{\mu 5\kappa} = \partial_\kappa \delta\Gamma^5_{5\mu} - A' g_{\kappa\sigma}\delta\Gamma^\sigma_{5\mu} + A' g_{\mu\kappa}\delta\Gamma^5_{55} - \partial_w \delta\Gamma^5_{\mu\kappa} + A'\delta\Gamma^5_{\mu\kappa} \ . \tag{B.5}$$

With Eq. (13.26) entailing that in the axial gauge the requisite perturbative Christoffel symbols are given as

$$\delta\Gamma^5_{5\mu} = 0 \ , \quad \delta\Gamma^5_{55} = 0 \ , \quad \delta\Gamma^5_{\mu\kappa} = -\frac{1}{2}h'_{\mu\kappa} \ , \quad \delta\Gamma^\sigma_{5\mu} = \frac{1}{2}g^{\sigma\rho}(\partial_w - 2A')h_{\mu\rho} \ , \tag{B.6}$$

in the axial gauge $\delta R^5{}_{\mu 5\kappa}$ then takes the form

$$\delta R^5{}_{\mu 5\kappa} = \frac{1}{2}[\partial_w^2 - 2A'\partial_w + 2A'^2]h_{\mu\kappa} \ . \tag{B.7}$$

Then finally, with the axial gauge components of R_{MN} and δR_{MN} being given in Eqs. (13.16) and (13.27), direct evaluation of the $\delta C^5{}_{\mu 5\kappa}$ component of the axial gauge Weyl tensor itself is found to yield

$$\begin{aligned}
\delta C^5{}_{\mu 5\kappa} =&\ \delta R^5{}_{\mu 5\kappa} + \frac{1}{12}g_{\mu\kappa}[\delta R_{55} + g^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta}] + \frac{1}{12}h_{\mu\kappa}[R_{55} + g^{\alpha\beta}R_{\alpha\beta}] \\
&-\frac{1}{3}[\delta R_{\mu\kappa} + g_{\mu\kappa}\delta R_{55}] - \frac{1}{3}h_{\mu\kappa}R_{55} \\
=&\ \frac{1}{2}[\partial_w^2 - 2A'\partial_w + 2A'^2]h_{\mu\kappa} - \frac{1}{4}g_{\mu\kappa}\delta R_{55} + \frac{1}{12}g_{\mu\kappa}g^{\alpha\beta}\delta R_{\alpha\beta} - \frac{1}{3}\delta R_{\mu\kappa} \\
&+\frac{1}{12}h_{\mu\kappa}[4A'^2 - 8A'' + g^{\alpha\beta}\tilde{R}_{\alpha\beta}] - \frac{1}{12}g_{\mu\kappa}[4A'^2h + A''h + h^{\alpha\beta}\tilde{R}_{\alpha\beta}] \\
=&\ \frac{1}{3}[\partial_w^2 - 3A'\partial_w + 2A'^2 - 2A'']h_{\mu\kappa} + \frac{1}{12}h_{\mu\kappa}g^{\alpha\beta}\tilde{R}_{\alpha\beta} - \frac{1}{12}g_{\mu\kappa}h^{\alpha\beta}\tilde{R}_{\alpha\beta} \\
&+\frac{1}{12}e^{-2A}[g_{\mu\kappa}g^{\alpha\beta}\widetilde{\delta R}_{\alpha\beta} - 4\widetilde{\delta R}_{\mu\kappa}] - \frac{1}{12}g_{\mu\kappa}[\partial_w^2 + A'\partial_w]h \ ,
\end{aligned} \tag{B.8}$$

with the third form of Eq. (B.8) explicitly exhibiting the way in which $\delta C^5{}_{\mu 5\kappa}$ depends on w. As a check on our calculation, we note that since

$$\begin{aligned}
g^{\mu\nu}\partial_w h_{\mu\nu} =&\ \partial_w h - h_{\mu\nu}\partial_w g^{\mu\nu} = h' + 2A'h \\
g^{\mu\nu}\partial_w^2 h_{\mu\nu} =&\ (\partial_w + 2A')(g^{\mu\nu}\partial_w h_{\mu\nu}) = h'' + 4A'h' + 2A''h + 4A'^2h \ ,
\end{aligned} \tag{B.9}$$

and since $\tilde{R}_{\alpha\beta}$ is given by

$$\tilde{R}_{\alpha\beta} = -3kH^2q_{\alpha\beta} = -3kH^2g_{\alpha\beta}e^{-2A} = 3\frac{d^2A}{d|w|^2}g_{\alpha\beta} \tag{B.10}$$

in each of the $dS_4^\pm$, $M_4^\pm$ and $AdS_4^\pm$ background brane geometries of interest to us, it readily follows that in all of these cases $g^{\mu\kappa}\delta C^5{}_{\mu 5\kappa} = 0$, as should of course be the case since the Weyl tensor is traceless.

As such, Eq. (B.8) gives the kinematic structure of $\delta C^5{}_{\mu 5\kappa}$. To determine its specific dynamical structure when the brane-world equations of motion are imposed, we note that as well as being writable in the form $\Delta G_{MN} = -\kappa_5^2 \delta\tau_{MN}$ given in Eq. (13.7), the brane-world equations of motion can also be written in the form

$$\begin{aligned}\Delta R_{MN} &= \delta R_{MN} - 4b^2 h_{MN} + \frac{4}{3}\kappa_5^2 \lambda h_{MN}\delta(w) - \kappa_5^2 \lambda \delta^\mu_M \delta^\nu_N h_{\mu\nu}\delta(w) \\ &= -\kappa_5^2(\delta\tau_{MN} - \frac{1}{3}g_{MN}g^{LR}\delta\tau_{LR}) \ . \qquad \text{(B.11)}\end{aligned}$$

In the case where $\delta\tau_{MN}$ is taken to be a delta function source $\delta\tau_{MN} = \delta^\mu_M \delta^\nu_N \delta\tau_{\mu\nu}\delta(w)$ on the brane, the piece of δR_{MN} which is continuous at the brane is then given very simply by $\delta R_{MN} = 4b^2 h_{MN}$. Insertion of this piece into the second form of Eq. (B.8) then enables us to determine the explicit change in continuous piece of the Weyl tensor due to the introduction of a perturbative source on the brane, with a Z_2-symmetric axial gauge $\delta\bar{E}_{\mu\nu}(|w|)$ then evaluating (at any $|w|$) to

$$\begin{aligned}\delta\bar{E}_{\mu\nu}(|w|) = & \left[\frac{1}{2}\frac{\partial^2}{\partial|w|^2} - \frac{dA}{d|w|}\frac{\partial}{\partial|w|} + \frac{4}{3}\left(\frac{dA}{d|w|}\right)^2 - \frac{4}{3}b^2 + \frac{1}{3}\frac{d^2A}{d|w|^2}\right] h_{\mu\nu}(|w|) \\ & + \frac{1}{3}\left[b^2 - \left(\frac{dA}{d|w|}\right)^2 - \frac{d^2A}{d|w|^2}\right] g_{\mu\nu}h(|w|) \ . \qquad \text{(B.12)}\end{aligned}$$

With all six of the maximally 4-symmetric brane worlds of interest to us having warp factors which happen to obey

$$b^2 - \left(\frac{dA}{d|w|}\right)^2 - \frac{d^2A}{d|w|^2} = 0 \ , \qquad \text{(B.13)}$$

$\delta\bar{E}_{\mu\nu}(|w|)$ can finally be written very compactly as

$$\delta\bar{E}_{\mu\nu}(|w|) = \frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - 2\frac{dA}{d|w|}\frac{\partial}{\partial|w|} - 2\frac{d^2A}{d|w|^2}\right] h_{\mu\nu}(|w|) \ , \qquad \text{(B.14)}$$

an expression in which the dependence on the trace of the fluctuation has dropped out identically. With the $\delta\bar{E}_{\mu\nu}(|w|)$ of interest now being expressed entirely in terms of the warp factor, we immediately see that $\delta\bar{E}_{\mu\nu}(|w|)$ vanishes identically for an $h_{\mu\nu}(|w|)$ whose $|w|$ dependence is the warp factor e^{2A} itself. Moreover, for warp factors which expressly obey Eq. (B.13), $\delta\bar{E}_{\mu\nu}(|w|)$ is also found to vanish for an $h_{\mu\nu}(|w|)$ whose $|w|$ dependence is $e^{2A}dA/d|w|$; while for a warp factor for which $d^2A/d|w|^2$ just happens to be zero (viz. $M_4^{\pm}$), $\delta\bar{E}_{\mu\nu}(|w|)$ also vanishes for any piece of $h_{\mu\nu}(|w|)$ which is independent of $|w|$ altogether. Consequently, none of the NT sector modes of any of the six maximally 4-symmetric brane worlds of interest to us in this monograph actually contribute to $\delta\bar{E}_{\mu\nu}(|w|)$ at all, with $\delta\bar{E}_{\mu\nu}(|w|)$ thus being purely TT in all such cases.

As regards the discontinuous piece $E^{\rm disc}_{\mu\kappa}$ of $\delta C^5{}_{\mu 5\kappa}$, we note that for an $h_{\mu\kappa}$ which only depends on $|w|$, Eq. (B.11) entails for the delta function source $\delta\tau_{MN} =$

$\delta^\mu_M \delta^\nu_N \delta\tau_{\mu\nu}\delta(w)$, that the fluctuation obeys the junction conditions

$$\begin{aligned}\frac{dh_{\mu\kappa}}{d|w|}\delta(w) &= -\left[\frac{1}{3}\kappa_5^2\lambda h_{\mu\kappa} + \kappa_5^2\delta\tau_{\mu\kappa} - \frac{1}{3}\kappa_5^2 q_{\mu\kappa}\delta\tau\right]\delta(w) \ , \\ \frac{dh}{d|w|}\delta(w) &= \frac{1}{3}\kappa_5^2\delta\tau\delta(w) \end{aligned} \tag{B.15}$$

at the brane. With the third form of Eq. (B.8) entailing that $E^{\rm disc}_{\mu\kappa}$ is given by

$$E^{\rm disc}_{\mu\kappa} = \left[\frac{2}{3}\frac{dh_{\mu\kappa}}{d|w|} - \frac{4}{3}\frac{dA}{d|w|}h_{\mu\kappa} - \frac{1}{6}g_{\mu\kappa}\frac{dh}{d|w|}\right]\delta(w) \ , \tag{B.16}$$

use of the kinematic relations

$$\kappa_5^2\lambda = 6(b^2 + kH^2)^{1/2} \ , \quad \frac{dA(w=0)}{d|w|} = -(b^2 + kH^2)^{1/2} \ , \tag{B.17}$$

which hold in all of the maximally 4-symmetric brane cases of interest to us, then enables us to obtain

$$E^{\rm disc}_{\mu\kappa} = -\kappa_5^2\left[\frac{2}{3}\delta\tau_{\mu\kappa} - \frac{1}{6}q_{\mu\kappa}\delta\tau\right]\delta(w) \ , \tag{B.18}$$

which we recognize as the perturbative form of Eq. (12.9).

Appendix C

Transverse and Transverse-Traceless Projection

C.1 Transverse projection for vector fields

In order to see how to construct a transverse-traceless projection operator for tensor fields, it is instructive to first see how the transverse projection technique works in the Maxwell case. We thus start with a general Maxwell field A_μ, and wish to decompose it into its transverse and longitudinal parts. While the longitudinal part of a vector can always be written as the divergence $\partial_\mu\phi$ of a scalar ϕ, we note that the arbitrary $\partial_\mu\phi$ is not necessarily longitudinal since the quantity $\partial_\mu[\partial^\mu\phi]$ would vanish for a ϕ which is harmonic, with $\partial_\mu\phi$ then being transverse. We must thus seek an alternate definition of the longitudinal part, one which can never be transverse. To determine such an object we first introduce the retarded flat spacetime massless scalar propagator $D(x-x')$ which obeys

$$\partial_\nu\partial^\nu D(x-x') = \delta^4(x-x') \quad , \tag{C.1}$$

and note the kinematic identity

$$\begin{aligned}&\phi(x')\partial_\nu\partial^\nu D(x-x')\\&= D(x-x')\partial_\nu\partial^\nu\phi(x') + \partial_\nu[\phi(x')\partial^\nu D(x-x') - D(x-x')\partial^\nu\phi(x')] \quad ,\end{aligned} \tag{C.2}$$

which holds for a general scalar ϕ (the ∂_ν derivatives here are with respect to x'). On integrating Eq. (C.2) we obtain

$$\begin{aligned}\phi(x) = &\int d^4x' D(x-x')\partial_\nu\partial^\nu\phi(x')\\&+\int dS_\nu[\phi(x')\partial^\nu D(x-x') - D(x-x')\partial^\nu\phi(x')] \quad ,\end{aligned} \tag{C.3}$$

a relation whose validity we check by noting that on applying $\partial_\mu\partial^\mu$ to it we obtain

$$\partial_\mu\partial^\mu\phi(x) = \partial_\nu\partial^\nu\phi(x) + \int dS_\nu[\phi(x')\partial^\nu\delta^4(x-x') - \delta^4(x-x')\partial^\nu\phi(x')] \quad , \tag{C.4}$$

with the asymptotic surface term in Eq. (C.4) thus having to vanish for arbitrary ϕ, which it actually does since the delta function and its derivatives only take support at the origin. The decomposition on the right-hand side of Eq. (C.3) thus breaks up a general ϕ into two parts, the first part of which (to be labelled ϕ^L) the box operator $\partial_\mu\partial^\mu$ does not annihilate and the second part of which (ϕ^T) it does. The quantity

$$\partial_\mu\phi^L = \partial_\mu \int d^4x' D(x-x')\partial_\nu\partial^\nu\phi(x') \tag{C.5}$$

which obeys

$$\partial_\mu\partial^\mu\phi^L = \partial_\nu\partial^\nu\phi \ , \tag{C.6}$$

and consequently

$$\partial_\mu\phi^L = \partial_\mu \int d^4x' D(x-x')\partial_\nu\partial^\nu\phi^L(x') \ , \tag{C.7}$$

is thus the longitudinal function we seek since $\partial_\mu\phi^L$ is a function which can never be transverse.[1]

Given the above ϕ^L we can now decompose a general Maxwell field into its transverse and longitudinal components as

$$A_\mu = A_\mu^T + \partial_\mu \int d^4x' D(x-x')\partial_\nu\partial^\nu\phi(x') \ , \tag{C.8}$$

where the transverse and longitudinal pieces obey

$$\partial_\mu A^{T\mu} = 0 \ , \quad \partial_\mu A^\mu = \partial_\nu\partial^\nu\phi \ . \tag{C.9}$$

Finally, on inserting Eq. (C.9) back into Eq. (C.8) we find that the general decomposition of A_μ can be written as

$$A_\mu = A_\mu^T + \partial_\mu \int d^4x' D(x-x')\partial_\nu A^\nu(x') = A_\mu^T + A_\mu^L \ , \tag{C.10}$$

to give us our sought-after result.

However, before we can use Eq. (C.10), we note that with it initially being defined for arbitrary A_μ, there will be some A_μ for which the integral in Eq. (C.10) will not be finite. Noting that this integral only involves the longitudinal part of A_μ (its transverse part does not contribute to $\partial_\nu A^\nu$), we shall be able rectify this problem by a gauge transformation. Thus, if we make some gauge transformation

$$\bar{A}_\mu = A_\mu + \partial_\mu\chi \ , \tag{C.11}$$

we will then generate an $\bar{A}_\mu$ which will admit of its own transverse/longitudinal decomposition

$$\bar{A}_\mu = \bar{A}_\mu^T + \partial_\mu \int d^4x' D(x-x')\partial_\nu\bar{A}^\nu(x') \ , \tag{C.12}$$

[1]The only $\partial_\nu\partial^\nu\phi$ which could vanish would be the one for which ϕ^L itself vanishes.

where

$$\partial_\nu \bar{A}^\nu = \partial_\nu A^\nu + \partial_\nu \partial^\nu \chi = \partial_\nu A^\nu + \partial_\nu \partial^\nu \chi^{L} \quad . \tag{C.13}$$

Since we are allowed to make gauge transformations with arbitrary χ, for any given $\partial_\nu A^\nu$ we can thus always find a χ for which the integral in Eq. (C.12) associated with the gauge transformed $\bar{A}_\mu$ then will be finite; and thus from now on in using equations such as Eqs. (C.10) and (C.12) we shall assume that both equations involve only finite integrals with any necessary gauge transformations already having been made.

To determine how the transverse and longitudinal parts of A_μ behave under a general gauge transformation, we find from Eqs. (C.10) and (C.12) that under a general $\bar{A}_\mu = A_\mu + \partial_\mu \chi$, the difference between the transverse parts is given by

$$\bar{A}^{T}_\mu - A^{T}_\mu = \partial_\mu \chi + \partial_\mu \int d^4x' D(x-x') \partial_\nu \partial^\nu \chi(x') \quad . \tag{C.14}$$

On using Eq. (C.3) we can rewrite Eq. (C.14) as

$$\bar{A}^{T}_\mu = A^{T}_\mu + \partial_\mu \chi^{T} \quad , \tag{C.15}$$

where we have introduced

$$\chi^{T} = \int dS_\nu [\chi(x') \partial^\nu D(x-x') - D(x-x') \partial^\nu \chi(x')] \quad . \tag{C.16}$$

With Eq. (C.15) giving us the change in the transverse A^{T}_μ, the change in the longitudinal A^{L}_μ is then given as

$$\begin{aligned} \bar{A}^{L}_\mu - A^{L}_\mu &= \bar{A}_\mu - A_\mu - \bar{A}^{T}_\mu + A^{T}_\mu = \partial_\mu \chi - \partial_\mu \chi^{T} \\ &= \partial_\mu \chi^{L} = \partial_\mu \int d^4x' D(x-x') \partial_\nu \partial^\nu \chi(x') \quad , \end{aligned} \tag{C.17}$$

with the longitudinal and transverse gauge transformations thus acting independently in their separate sectors. With χ^{T} obeying $\partial_\nu \partial^\nu \chi^{T} = 0$ (this being a generic feature of the surface term in Eq. (C.2)), from Eq. (C.15) we see that not only is transversality preserved under a general gauge transformation (viz. $\partial_\mu \bar{A}^{T\mu} = \partial_\mu A^{T\mu}$), but with $\partial_\nu \partial^\nu \chi^{T}$ vanishing, we additionally see that the quantity $\partial_\nu \partial^\nu A^{T}_\mu$ transforms as

$$\partial_\nu \partial^\nu \bar{A}^{T}_\mu = \partial_\nu \partial^\nu A^{T}_\mu \tag{C.18}$$

and is thus gauge invariant under transverse (and also longitudinal for that matter) gauge transformations.

With A^{T}_μ being given by Eq. (C.10), $\partial_\nu \partial^\nu A^{T\mu}$ is given by

$$\partial_\nu \partial^\nu A^{T\mu} = \partial_\nu [\partial^\nu A^\mu - \partial^\mu A^\nu] = \partial_\nu F^{\nu\mu} \quad , \tag{C.19}$$

with the Maxwell equations $\partial_\nu F^{\nu\mu} = J^\mu$ thus being writable purely in terms of A^T_μ as

$$\partial_\nu \partial^\nu A^{T\mu} = J^\mu \quad , \tag{C.20}$$

something to be anticipated since the source J^μ is transverse.[2] Moreover, with $\partial_\nu \partial^\nu A^T_\mu$ being gauge invariant, we thus confirm the gauge invariance of the Maxwell equations in our formalism. As a final comment on Eq. (C.20), we note that even though Eq. (C.20) is of precisely the same generic form as the standard Lorentz gauge Maxwell equations (where $\partial_\nu A^\nu = 0$), our derivation of Eq. (C.20) involved no need to make any such choice of gauge.[3]

It is also possible to recast the transverse/longitudinal decomposition in the language of projectors. Thus in light of Eq. (C.10) we introduce the non-local projector $\Pi_{\mu\nu}$ defined as[4]

$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial}{\partial x^\mu} \int d^4x' D(x - x') \frac{\partial}{\partial x'^\nu} \quad , \tag{C.21}$$

and rewrite Eq. (C.10) as

$$A^T_\mu = \Pi_{\mu\nu} A^\nu \quad . \tag{C.22}$$

Given Eq. (C.10) we can confirm that $\Pi_{\mu\nu}$ is indeed a projection operator by noting that

$$\begin{aligned}
\Pi_{\mu\nu}\Pi^\nu{}_\sigma &= \Pi_{\mu\sigma} \quad , \\
\Pi_{\mu\nu} A^{T\nu} &= A^T_\mu - \partial_\mu \int d^4x' D(x-x') \partial_\nu A^{T\nu}(x') = A^T_\mu \quad , \\
\Pi_{\mu\nu} A^{L\nu} &= \partial_\mu \int d^4x' D(x-x') \partial_\nu A^\nu(x') \\
&\quad - \partial_\mu \int d^4x' D(x-x') \partial_\nu \partial^\nu \int d^4x'' D(x'-x'') \partial_\sigma A^\sigma(x'') = 0 \ ,
\end{aligned} \tag{C.23}$$

with $\Pi_{\mu\nu}$ returning a transverse vector when it acts on one.[5] The projector behaves

[2] Similarly, with $F_{\mu\nu}$ itself being writable as $F_{\mu\nu} = \partial_\mu A^T_\nu - \partial_\nu A^T_\mu$ (viz. an expression which does not involve A^L_μ), the Maxwell action can be written purely in terms of A^T_μ, to give a still fully gauge invariant starting action for quantizing electrodynamics in which the number of independent degrees of freedom has already been reduced.

[3] That a projection is different from a gauge transformation is to be anticipated, since unlike a gauge transformation, subsequent to a projection there is no operation which will take us back to the state on which the projection had been made.

[4] We do not need to symmetrize $\Pi_{\mu\nu}$ since it is understood to always act on an object with a contravariant index ν.

[5] It is the specific sequencing of derivatives in Eq. (C.21) which enables $\Pi_{\mu\nu}$ to obey the projector relation $\Pi_{\mu\nu}\Pi^\nu{}_\sigma = \Pi_{\mu\sigma}$.

exactly the same way on a gauge function where it acts as

$$\Pi_{\mu\nu}\partial^\nu\chi = \partial_\mu\chi - \partial_\mu \int d^4x' D(x-x')\partial_\nu\partial^\nu\chi \ , \tag{C.24}$$

to yield, with use of Eq. (C.7),

$$\Pi_{\mu\nu}\partial^\nu\chi^{T} = \partial_\mu\chi^{T} \ , \ \Pi_{\mu\nu}\partial^\nu\chi^{L} = 0 \ , \tag{C.25}$$

with the transverse gauge function being one of its eigenstates.

As a final comment on the Maxwell field we discuss the generalization of the above analysis to a general curved spacetime with some general metric $g_{\mu\nu}$. Since the longitudinal part of A_μ will still be the (covariant) derivative of a scalar even in the curved space case, we only need to use the curved space massless scalar propagator, which obeys

$$\nabla_\mu(x)\nabla^\mu(x)D(x,x') = g^{-1/2}\delta^4(x-x') \ , \tag{C.26}$$

to construct the curved space Maxwell field projector. We thus define

$$\Pi_{\mu\nu} = g_{\mu\nu}(x) - \nabla_\mu(x)\int d^4x' g^{1/2}(x')D(x,x')\nabla_\nu(x') \ , \tag{C.27}$$

to find that its action on A_μ yields

$$\begin{aligned}\Pi_{\mu\nu}A^\nu &= A_\mu(x) - \nabla_\mu(x)\int d^4x' g^{1/2}(x')D(x,x')\nabla_\nu(x')A^\nu(x') \\ &= A_\mu(x) - A^{L}_\mu(x) = A^{T}_\mu(x) \ ,\end{aligned} \tag{C.28}$$

with A^{T}_μ being found to obey the transversality condition $\nabla_\mu A^{T\mu} = 0$ just as required, and with $A^{L\mu}$ indeed being found to both be the covariant derivative of a scalar and to obey $\nabla_\mu A^{L\mu} = \nabla_\mu A^\mu$ just as it should. Applying the covariant box operator to A^{T}_μ of Eq. (C.28) yields

$$\begin{aligned}&\nabla_\nu\nabla^\nu A^{T\mu} - \nabla_\nu\nabla^\nu A^\mu \\ &= -\nabla_\nu\nabla^\nu\nabla^\mu \int d^4x' g^{1/2}(x')D(x,x')\nabla_\sigma(x')A^\sigma(x') \\ &= -\nabla_\nu\nabla^\mu\nabla^\nu \int d^4x' g^{1/2}(x')D(x,x')\nabla_\sigma(x')A^\sigma(x') \\ &= -\nabla^\mu\nabla_\sigma A^\sigma + R^\mu{}_\nu\nabla^\nu \int d^4x' g^{1/2}(x')D(x,x')\nabla_\sigma(x')A^\sigma(x') \\ &= -\nabla^\mu\nabla_\sigma A^\sigma + R^\mu{}_\nu(A^\nu - A^{T\nu}) \ ,\end{aligned} \tag{C.29}$$

i.e.

$$\nabla_\nu\nabla^\nu A^{T\mu} + R^\mu{}_\nu A^{T\nu} = \nabla_\nu\nabla^\nu A^\mu - \nabla^\mu\nabla_\sigma A^\sigma + R^\mu{}_\nu A^\nu \ . \tag{C.30}$$

Then with the curved space Maxwell equations taking the form

$$\nabla_\nu \nabla^\nu A^\mu - \nabla_\nu \nabla^\mu A^\nu = \nabla_\nu \nabla^\nu A^\mu - \nabla^\mu \nabla_\nu A^\nu + R^\mu{}_\nu A^\nu = J^\mu \quad , \tag{C.31}$$

we obtain

$$\nabla_\nu \nabla^\nu A^{T\mu} + R^\mu{}_\nu A^{T\mu} = J^\mu \quad . \tag{C.32}$$

Even in the curved space case then the Maxwell equations can be written purely in terms of A^T_μ. And moreover, Eq. (C.32) is of precisely the same generic form as the standard covariant Lorentz gauge Maxwell equations (where $\nabla_\nu A^\nu = 0$). Transverse projection thus yields a familiar result.

C.2 Transverse projection for flat space tensor fields

Having now completed the discussion of the Maxwell case, we turn next to a discussion of the tensor case. Again we would like to construct a longitudinal component, and while we might expect it to be of the form $\partial_\mu V_\nu + \partial_\nu V_\mu$, with the derivative of this function being given by

$$\partial_\nu[\partial^\mu V^\nu + \partial^\nu V^\mu] = \partial^\mu \partial_\nu V^\nu + \partial_\nu \partial^\nu V^\mu \quad , \tag{C.33}$$

we see that this form will not suffice since there are some V_μ (such as harmonic, transverse ones) for which this derivative could vanish. Thus we need to construct a two index object which could never be transverse. Proceeding by analog with the Maxwell case we have found that for some general vector field W_μ, the requisite object is given (in flat space) by

$$\begin{aligned} W^L_{\mu\nu} = {} & \partial_\mu \int d^4x' D(x-x') \partial_\sigma \partial^\sigma W_\nu(x') + \partial_\nu \int d^4x' D(x-x') \partial_\sigma \partial^\sigma W_\mu(x') \\ & - \partial_\mu \partial_\nu \int d^4x' D(x-x') \partial_\sigma \int d^4x'' D(x'-x'') \partial_\tau \partial^\tau W^\sigma(x'') \quad , \end{aligned} \tag{C.34}$$

since its derivative

$$\partial_\mu W^{L\mu\nu} = \partial_\sigma \partial^\sigma W^\nu \tag{C.35}$$

could never vanish without $W^{L\mu\nu}$ itself vanishing. Comparing Eq. (C.33) with Eq. (C.35), we thus see that if we set $\partial_\sigma \partial^\sigma W_\nu = \partial_\sigma \partial^\sigma V_\nu + \partial_\nu \partial_\sigma V^\sigma$ we can then decompose a general $V_{\mu\nu} = \partial_\mu V_\nu + \partial_\nu V_\mu$ into

$$\partial_\mu V_\nu + \partial_\nu V_\mu = V^T_{\mu\nu} + V^L_{\mu\nu} \quad , \tag{C.36}$$

where

$$V^{L}_{\mu\nu} = \partial_\mu \int d^4x' D(x-x')[\partial_\sigma\partial^\sigma V_\nu(x') + \partial_\nu\partial_\sigma V^\sigma(x')]$$
$$+\partial_\nu \int d^4x' D(x-x')[\partial_\sigma\partial^\sigma V_\mu(x') + \partial_\mu\partial_\sigma V^\sigma(x')]$$
$$-\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')[\partial_\tau\partial^\tau V^\sigma(x'') + \partial^\sigma\partial_\tau V^\tau(x'')], \quad (C.37)$$

and

$$V^{T}_{\mu\nu} = \partial_\mu V_\nu + \partial_\nu V_\mu$$
$$-\partial_\mu \int d^4x' D(x-x')[\partial_\sigma\partial^\sigma V_\nu(x') + \partial_\nu\partial_\sigma V^\sigma(x')]$$
$$-\partial_\nu \int d^4x' D(x-x')[\partial_\sigma\partial^\sigma V_\mu(x') + \partial_\mu\partial_\sigma V^\sigma(x')]$$
$$+\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')[\partial_\tau\partial^\tau V^\sigma(x'') + \partial^\sigma\partial_\tau V^\tau(x'')]. \quad (C.38)$$

As constructed these functions then obey

$$\partial_\mu V^{T\,\mu\nu} = 0 \quad , \quad \partial_\mu V^{L\,\mu\nu} = \partial_\mu\partial^\mu V^\nu + \partial^\nu\partial_\mu V^\mu = \partial_\mu V^{\mu\nu} \quad , \qquad (C.39)$$

with $\partial_\mu V_\nu + \partial_\nu V_\mu$ thus being decomposed into two pieces, one of which is always transverse, and the other of which is always longitudinal no matter what the choice of V_μ.

Given these particular relations, for a gravitational fluctuation $h_{\mu\nu}$ we can replace $\partial_\mu V_\nu + \partial_\nu V_\mu$ by $h_{\mu\nu}$, and thus define longitudinal and transverse gravitational fluctuations according to

$$h^{L}_{\mu\nu} = \partial_\mu \int d^4x' D(x-x')\partial_\sigma h^\sigma{}_\nu(x') + \partial_\nu \int d^4x' D(x-x')\partial_\tau h^\tau{}_\mu(x')$$
$$-\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')\partial_\tau h^{\sigma\tau}(x'') \quad , \qquad (C.40)$$

$$h^{T}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \int d^4x' D(x-x')\partial_\sigma h^\sigma{}_\nu(x') - \partial_\nu \int d^4x' D(x-x')\partial_\tau h^\tau{}_\mu(x')$$
$$+\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')\partial_\tau h^{\sigma\tau}(x'') \quad . \qquad (C.41)$$

With these fluctuations being found to obey $\partial_\mu h^{L\,\mu\nu} = \partial_\mu h^{\mu\nu}$, $\partial_\nu h^{L\,\mu\nu} = \partial_\nu h^{\mu\nu}$ $\partial_\mu h^{T\,\mu\nu} = 0$, $\partial_\nu h^{T\,\mu\nu} = 0$, and with $\partial_\mu h^{L\,\mu\nu}$ never being able to vanish without $h^{L\,\mu\nu}$ itself vanishing, Eqs. (C.40) and (C.41) thus provide us with the decomposition of $h_{\mu\nu}$ into longitudinal and transverse parts which we desire.

Given this decomposition we can introduce longitudinal and transverse projectors $L_{\mu\nu\sigma\tau}$ and $T_{\mu\nu\sigma\tau}$ according to

$$L_{\mu\nu\sigma\tau} = \partial_\mu \int d^4x' D(x-x')\eta_{\nu\tau}\partial_\sigma + \partial_\nu \int d^4x' D(x-x')\eta_{\mu\sigma}\partial_\tau$$
$$-\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')\partial_\tau \quad , \qquad \text{(C.42)}$$

and

$$T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma}\eta_{\nu\tau} - \partial_\mu \int d^4x' D(x-x')\eta_{\nu\tau}\partial_\sigma - \partial_\nu \int d^4x' D(x-x')\eta_{\mu\sigma}\partial_\tau$$
$$+\partial_\mu\partial_\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')\partial_\tau \quad , \qquad \text{(C.43)}$$

as they effect

$$L_{\mu\nu\sigma\tau}h^{\sigma\tau} = h^{L}_{\mu\nu} \quad , \quad T_{\mu\nu\sigma\tau}h^{\sigma\tau} = h^{T}_{\mu\nu} \quad . \qquad \text{(C.44)}$$

As constructed, these projectors obey

$$T_{\mu\nu\sigma\tau}T^{\sigma\tau}{}_{\alpha\beta} = T_{\mu\nu\alpha\beta} \quad , \quad L_{\mu\nu\sigma\tau}L^{\sigma\tau}{}_{\alpha\beta} = L_{\mu\nu\alpha\beta} \quad ,$$
$$T_{\mu\nu\sigma\tau}L^{\sigma\tau}{}_{\alpha\beta} = 0 \quad , \quad L_{\mu\nu\sigma\tau}T^{\sigma\tau}{}_{\alpha\beta} = 0 \quad , \quad L_{\mu\nu\sigma\tau} + T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma}\eta_{\nu\tau} \qquad \text{(C.45)}$$

just as required for projectors.[6]

The key feature which makes the transverse fluctuations $h^{T}_{\mu\nu}$ of Eq. (C.41) so significant is that we can express the perturbative $\delta G_{\mu\nu}$ entirely in terms of them according to

$$\frac{1}{2}[\partial_\mu\partial_\nu h^{T} + \partial_\alpha\partial^\alpha h^{T}_{\mu\nu}] - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\partial^\sigma h^{T}$$
$$= \frac{1}{2}[\partial_\mu\partial_\nu h - \partial_\mu\partial_\lambda h^\lambda{}_\nu - \partial_\nu\partial_\lambda h^\lambda{}_\mu + \partial_\alpha\partial^\alpha h_{\mu\nu}] - \frac{1}{2}\eta_{\mu\nu}[\partial_\alpha\partial^\alpha h - \partial_\sigma\partial_\lambda h^{\sigma\lambda}]$$
$$= \delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R \quad , \qquad \text{(C.46)}$$

[6]The specific sequencing of derivatives indicated in Eqs. (C.42) and (C.43) is central to the establishment of these projector relations, with no need for any integration by parts being encountered. Additionally, implicit in our use of the above $L_{\mu\nu\sigma\tau}$ and $T_{\mu\nu\sigma\tau}$ is the requirement that they are to act only on objects which are symmetric in two contravariant indices (σ,τ), with an $h^{L}_{\mu\nu}$ or $h^{T}_{\mu\nu}$ constructed this way then automatically being symmetric in its (μ,ν) indices. Noting that as defined the $L_{\mu\nu\sigma\tau}$ projector, for instance, actually obeys the relations $L_{\mu\nu\sigma\tau}L^{\sigma\tau}{}_{\alpha\beta} = L_{\mu\nu\alpha\beta} + \partial_\nu \int D\partial_\mu\partial_\beta \int D\partial_\alpha - \partial_\nu \int D\partial_\mu\partial_\alpha \int D\partial_\beta + \partial_\mu\partial_\nu \int D\partial_\sigma \int D\partial^\sigma\partial_\alpha \int D\partial_\beta - \partial_\mu\partial_\nu \int D\partial_\sigma \int D\partial^\sigma\partial_\beta \int D\partial_\alpha$ and $L_{\mu\nu\sigma\tau}L^{\tau\sigma}{}_{\alpha\beta} = L_{\mu\nu\beta\alpha} + \partial_\mu \int D\partial_\nu\partial_\beta \int D\partial_\alpha - \partial_\mu \int D\partial_\nu\partial_\alpha \int D\partial_\beta$, a more general definition of the projector would be $\hat{L}_{\mu\nu\sigma\tau} = (1/2)[L_{\mu\nu\sigma\tau} + L_{\mu\nu\tau\sigma}]$, as it would obey $\hat{L}_{\mu\nu\sigma\tau}\hat{L}^{\sigma\tau}{}_{\alpha\beta} = \hat{L}_{\mu\nu\alpha\beta}$ regardless of the symmetry properties of the states on which it is to act.

where h^T is given by

$$h^T = \eta^{\alpha\beta} h^T_{\alpha\beta} = h - \partial_\nu \int d^4x' D(x-x')\partial_\sigma h^{\sigma\nu}(x') = \Pi_{\nu\sigma} h^{\nu\sigma} \quad . \tag{C.47}$$

Consequently, just like the analog Maxwell case, and as is of course to be anticipated since the energy-momentum tensor is transverse, we are able to write the perturbative Einstein equations $\delta G_{\mu\nu} = -\kappa_4^2 \delta T_{\mu\nu}$ entirely in terms of $h^T_{\mu\nu}$, to obtain the compact and convenient

$$\frac{1}{2}\partial_\alpha \partial^\alpha h^T_{\mu\nu} - \frac{1}{2}[\eta_{\mu\nu}\partial_\sigma \partial^\sigma - \partial_\mu \partial_\nu] h^T = -\kappa_4^2 \delta T_{\mu\nu} \quad . \tag{C.48}$$

As such, Eq. (C.48) is not only manifestly transverse, it is also of precisely the same generic form as that associated with the perturbative Einstein equations when written in the transverse gauge $\partial_\mu h^{\mu\nu} = 0$. Further, with Eqs. (C.47) and (C.48) entailing that

$$\partial_\sigma \partial^\sigma h^T = \partial_\sigma \partial^\sigma h - \partial_\sigma \partial_\tau h^{\sigma\tau} = \kappa_4^2 \delta T \quad , \tag{C.49}$$

on introducing[7]

$$\hat{\Pi}_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \partial_\nu \int d^4x' D(x-x') \quad , \tag{C.50}$$

and recalling Eq. (C.3), we can rewrite Eq. (C.48) as

$$\begin{aligned} &\frac{1}{2}\partial_\alpha \partial^\alpha h^T_{\mu\nu} + \frac{1}{2}\partial_\mu \partial_\nu \left[h^T - \int d^4x' D(x-x') h^T(x') \right] \\ &= \frac{1}{2}\partial_\alpha \partial^\alpha h^T_{\mu\nu} + \frac{1}{2}\partial_\mu \partial_\nu \int dS_\sigma [h^T(x')\partial^\sigma D(x-x') - D(x-x')\partial^\sigma h^T(x')] \\ &= -\kappa_4^2 \left[\delta T_{\mu\nu} - \frac{1}{2}\hat{\Pi}_{\mu\nu}\delta T \right] \quad . \end{aligned} \tag{C.51}$$

With $\partial_\alpha \partial^\alpha$ annihilating the $\int dS_\sigma$ surface term, in Eq. (C.51) transversality is still manifest.

In order to actually solve Eq. (C.51), great simplicity would be obtained if the $\int dS_\sigma$ surface term were to vanish, since Eq. (C.51) would then become diagonal in its (μ, ν) indices and thus readily solvable. Now at first glance this would appear unlikely since the relevant surface term is a 4-dimensional one which would initially appear to involve the behavior of h^T at large times as well as at large spatial distances, with the localized fluctuations of interest to gravitational perturbation theory being those which are only localized in space, and which might typically oscillate rather than be damped in time. However, the propagator which appears in Eq.

[7]The non-local operator $\hat{\Pi}_{\mu\nu}$ introduced here differs from the previously defined $\Pi_{\mu\nu}$ of Eq. (C.21) in the sequencing of derivatives, a sequencing which in general prevents $\hat{\Pi}_{\mu\nu}$ from obeying $\hat{\Pi}_{\mu\nu}\hat{\Pi}^\nu{}_\sigma = \hat{\Pi}_{\mu\sigma}$, except when it acts on functions for which one can integrate by parts without generating surface terms, cases for which the effects of $\hat{\Pi}_{\mu\nu}$ and $\Pi_{\mu\nu}$ would then be the same.

(C.51) is the retarded one of Eq. (16.75) which only involves the backward lightcone at time t, viz. only those times which obey $t' = t - |\bar{x} - \bar{x}'|$; with asymptotic $t' = -\infty$ then corresponding to asymptotic $|\bar{x}'|$ where a spatially localized fluctuation is in fact damped. To give a specific example of how the surface term might therefore be expected to behave, we note that for the particular choice $h^{T} = e^{-i\omega t}/r$, explicit evaluation reveals that the quantity $h^{T} - \int d^4x' D(x - x') h^{T}(x')$ actually vanishes identically. For spatial localized fluctuations then, the use of transverse projection gives us a readily tractable wave equation for the physical relevant transverse gravitational fluctuations $h^{T}_{\mu\nu}$.

With the $T_{\mu\nu\sigma\tau}$ and $L_{\mu\nu\sigma\tau}$ projectors decomposing both a general $h_{\mu\nu}$ and a general gauge function into separate transverse and longitudinal sectors, we note further that under a general gauge transformation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu \quad , \tag{C.52}$$

the transverse and longitudinal parts transform separately as

$$\bar{h}^{T}_{\mu\nu} = h^{T}_{\mu\nu} + V^{T}_{\mu\nu} \quad , \quad \bar{h}^{L}_{\mu\nu} = h^{L}_{\mu\nu} + V^{L}_{\mu\nu} \quad , \tag{C.53}$$

where $V^{T}_{\mu\nu}$ and $V^{L}_{\mu\nu}$ were introduced in Eqs. (C.38) and (C.37). Given these transformations we then find that

$$\begin{aligned}
\partial_\alpha \partial^\alpha \bar{h}^{T}_{\mu\nu} &= \partial_\alpha \partial^\alpha h^{T}_{\mu\nu} - 2\partial_\mu \partial_\nu \partial_\sigma V^\sigma \\
&\quad + \partial_\mu \partial_\nu \partial_\sigma \int d^4x' D(x - x')[\partial_\tau \partial^\tau V^\sigma(x') + \partial^\sigma \partial_\tau V^\tau(x')] \quad , \\
\partial_\mu \partial_\nu \bar{h}^{T} &= \partial_\mu \partial_\nu h^{T} + 2\partial_\mu \partial_\nu \partial_\sigma V^\sigma \\
&\quad - \partial_\mu \partial_\nu \partial_\sigma \int d^4x' D(x - x')[\partial_\tau \partial^\tau V^\sigma(x') + \partial^\sigma \partial_\tau V^\tau(x')] \quad , \\
\partial_\alpha \partial^\alpha \bar{h}^{T} &= \partial_\alpha \partial^\alpha h^{T} \quad ,
\end{aligned} \tag{C.54}$$

with the gauge invariance of Eq. (C.48) (and thus that of Eq. (C.51) which was derived from it purely by algebraic manipulation) thereby being confirmed. While Eq. (C.51) is gauge invariant, as we see from the transformations of Eq. (C.54), the individual terms on the left-hand side of Eq. (C.51) are not themselves separately gauge invariant, with gauge invariance requiring their interplay.

It is also important to note that our derivation of Eqs. (C.46) and (C.53) was a purely kinematic one which did not require the use of the equations of motion, with both of these relations holding even for non-stationary fields, and with $\delta G_{\mu\nu}$ being gauge invariant for any field configuration. Now as noted in Chapter 16, we can derive the source free region equation of motion $\delta G_{\mu\nu} = 0$ by variation of the action $S = \int d^4x h^{\mu\nu} \delta G_{\mu\nu}$ with respect to $h^{\mu\nu}$, with this action being gauge invariant under $\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ since $\delta G_{\mu\nu}$ is not only gauge invariant itself but because it also obeys the Bianchi identity $\partial_\mu \delta G^{\mu\nu} = 0$. However, since $\delta G_{\mu\nu}$ is transverse we can also write it as $\delta G_{\mu\nu} = T_{\mu\nu\sigma\tau} \delta G^{\sigma\tau}$, and because $T_{\mu\nu\sigma\tau}$

is a projector can thus write the action as $S = \int d^4x h^{T\,\mu\nu}\delta G_{\mu\nu}$. Then with $\delta G_{\mu\nu}$ being writable purely in terms of $h^{T}_{\mu\nu}$ in Eq. (C.46), we see that the action is not only writable purely in terms of $h^{T}_{\mu\nu}$, but because $V^{T}_{\mu\nu}$ is writable purely in the form of ∂_μ, ∂_ν type derivative terms in Eq. (C.38), it follows that the action is still gauge invariant. Consequently, writing the unconstrained action entirely in terms of $h^{T}_{\mu\nu}$ provides us with a gauge invariant starting point for quantizing gravity in which the longitudinal degrees of freedom have already been removed.[8]

C.3 Transverse-traceless projection for flat space tensor fields

Turning now to the construction of a projector $P_{\mu\nu\sigma\tau}$ which produces fluctuations which are both transverse and traceless, guided by the structure of the TT energy-momentum tensor which appears in Eq. (14.20), and noting the kinematic relation

$$\partial_\alpha\partial^\alpha\hat{\Pi}_{\mu\nu}h^{T} = \hat{\Pi}_{\mu\nu}\partial_\alpha\partial^\alpha h^{T} - \partial_\mu\partial_\nu\left[h^{T} - \int d^4x' D(x-x')\partial_\alpha\partial^\alpha h^{T}(x')\right] \quad , \tag{C.55}$$

use of the equations of motion of Eqs. (C.48) and (C.49) allows us to obtain

$$\begin{aligned}&\frac{1}{2}\partial_\alpha\partial^\alpha\left[h^{T}_{\mu\nu} - \frac{1}{3}\hat{\Pi}_{\mu\nu}h^{T}\right] + \frac{1}{3}\partial_\mu\partial_\nu\int dS_\sigma[h^{T}(x')\partial^\sigma D(x-x') - D(x-x')\partial^\sigma h^{T}(x')]\\ &= -\kappa_4^2\left[\delta T_{\mu\nu} - \frac{1}{3}\hat{\Pi}_{\mu\nu}\delta T\right] \quad ,\end{aligned} \tag{C.56}$$

an equation both sides of which are both transverse and traceless. Given Eq. (C.56), we can define projected transverse-traceless (PTT) and transverse non-traceless (TNT) components of $h^{T}_{\mu\nu}$ according to

$$\begin{aligned} h^{PTT}_{\mu\nu} &= h^{T}_{\mu\nu} - \frac{1}{3}\hat{\Pi}_{\mu\nu}h^{T} \quad ,\\ h^{TNT}_{\mu\nu} &= h^{T}_{\mu\nu} - h^{PTT}_{\mu\nu} = \frac{1}{3}\hat{\Pi}_{\mu\nu}h^{T} \quad ,\end{aligned} \tag{C.57}$$

and on introducing

$$P_{\mu\nu\sigma\tau} = T_{\mu\nu\sigma\tau} - \frac{1}{3}\hat{\Pi}_{\mu\nu}\Pi_{\sigma\tau} \quad , \quad Q_{\mu\nu\sigma\tau} = \frac{1}{3}\hat{\Pi}_{\mu\nu}\Pi_{\sigma\tau} \quad , \tag{C.58}$$

may rewrite Eq. (C.57) as

$$h^{PTT}_{\mu\nu} = P_{\mu\nu\sigma\tau}h^{\sigma\tau} \quad , \quad h^{TNT}_{\mu\nu} = Q_{\mu\nu\sigma\tau}h^{\sigma\tau} \quad , \tag{C.59}$$

[8]Moreover, within the surviving transverse sector, the term $\int d^4x h^{T}_{\mu\nu}h^{T\,\mu\nu}$ will serve as a gauge invariant counter term for the graviton self-energy, even though the general $\int d^4x h_{\mu\nu}h^{\mu\nu}$ itself would not be gauge invariant; with it being possible to augment the wave equation of Eq. (C.48) by terms of the form $\hat{\Pi}_{\mu\nu}\int D\partial_\alpha\partial^\alpha h^{T}$ without losing transversality or gauge invariance.

with $h^{PTT}_{\mu\nu}$ being traceless and with $\eta^{\mu\nu}h^{TNT}_{\mu\nu}$ being equal to h^{T}. As defined the PTT and TNT components obey the (coupled) wave equations

$$\begin{aligned}
&\frac{1}{2}\partial_\alpha\partial^\alpha h^{PTT}_{\mu\nu} + \frac{1}{3}\partial_\mu\partial_\nu \int dS_\sigma[h^{TNT}(x')\partial^\sigma D(x-x') - D(x-x')\partial^\sigma h^{TNT}(x')] \\
&= -\kappa_4^2\left[\delta\tau_{\mu\nu} - \frac{1}{3}\hat{\Pi}_{\mu\nu}\delta\tau\right] = -\kappa_4^2\delta\tau^{PTT}_{\mu\nu} \quad , \\
&\frac{1}{2}\partial_\alpha\partial^\alpha h^{TNT}_{\mu\nu} + \frac{1}{6}\partial_\mu\partial_\nu \int dS_\sigma[h^{TNT}(x')\partial^\sigma D(x-x') - D(x-x')\partial^\sigma h^{TNT}(x')] \\
&= \frac{\kappa_4^2}{6}\hat{\Pi}_{\mu\nu}\delta\tau = -\kappa_4^2\delta\tau^{TNT}_{\mu\nu} \quad ,
\end{aligned} \tag{C.60}$$

where in terms of the quantities $\delta\tau^{TT}_{\mu\nu}$ and $\delta\tau^{NT}_{\mu\nu}$ introduced in Eqs. (14.19) and (14.20), the PTT and TNT sources $\delta\tau^{PTT}_{\mu\nu}$ and $\delta\tau^{TNT}_{\mu\nu}$ are respectively given as $\delta\tau^{PTT}_{\mu\nu} = \delta\tau^{TT}_{\mu\nu}$ and $\delta\tau^{TNT}_{\mu\nu} = -(1/2)\delta\tau^{NT}_{\mu\nu}$.

With $\hat{\Pi}_{\mu\nu}$ and $\Pi_{\mu\nu}$ being found to obey the kinematic relations $T_{\mu\nu}{}^{\sigma\tau}\hat{\Pi}_{\sigma\tau} = \hat{\Pi}_{\mu\nu}$, $\Pi^{\mu\nu}T_{\mu\nu}{}^{\sigma\tau} = \Pi^{\sigma\tau}$ and $\Pi_{\mu\nu}\hat{\Pi}^{\mu\nu} = 3$, we immediately obtain first

$$T_{\mu\nu\sigma\tau}Q^{\sigma\tau}{}_{\alpha\beta} = Q_{\mu\nu\alpha\beta} \quad , \quad Q_{\mu\nu\sigma\tau}T^{\sigma\tau}{}_{\alpha\beta} = Q_{\mu\nu\alpha\beta} \quad , \quad Q_{\mu\nu\sigma\tau}Q^{\sigma\tau}{}_{\alpha\beta} = Q_{\mu\nu\alpha\beta} \quad , \tag{C.61}$$

and then

$$P_{\mu\nu\sigma\tau}Q^{\sigma\tau\alpha\beta} = 0 \quad , \quad Q_{\mu\nu\sigma\tau}P^{\sigma\tau\alpha\beta} = 0 \quad , \quad P_{\mu\nu\sigma\tau}P^{\sigma\tau}{}_{\alpha\beta} = P_{\mu\nu\alpha\beta} \quad , \tag{C.62}$$

with $P_{\mu\nu\sigma\tau}$ and $Q_{\mu\nu\sigma\tau}$ being orthogonal. The projectors $P_{\mu\nu\sigma\tau}$ and $Q_{\mu\nu\sigma\tau}$ thus break the $T_{\mu\nu\sigma\tau}$ projected sector into two independent subsectors, something which is to be anticipated since the six degrees of freedom in the $T_{\mu\nu\sigma\tau}$ sector can be decomposed into separate spin two (TT) and spin zero (TNT) components.[9]

As regards gauge transformations, if we conveniently write the transverse gauge function $V^{T}_{\mu\nu}$ in the form

$$V^{T}_{\mu\nu} = \partial_\mu\hat{V}_\nu + \partial_\nu\hat{V}_\mu \quad , \tag{C.63}$$

[9]Despite the fact that $\hat{\Pi}_{\mu\nu}$ is not itself a projector, in order to derive the relations obeyed by $P_{\mu\nu\sigma\tau}$ and $Q_{\mu\nu\sigma\tau}$ without needing to resort to integration by parts, it is crucial that the sequence of derivatives be precisely that associated with $\hat{\Pi}_{\mu\nu}\Pi_{\sigma\tau}$ (and not that of $\Pi_{\mu\nu}\Pi_{\sigma\tau}$, $\Pi_{\mu\nu}\hat{\Pi}_{\sigma\tau}$ or $\hat{\Pi}_{\mu\nu}\hat{\Pi}_{\sigma\tau}$), as applied in that particular order. Because of this ordering, the $P_{\mu\nu\sigma\tau}$ projector presented here differs from the $\hat{P}_{\mu\nu\sigma\tau} = (1/2)[\hat{\Pi}_{\mu\sigma}\hat{\Pi}_{\nu\tau} + \hat{\Pi}_{\mu\tau}\hat{\Pi}_{\nu\sigma}] - (1/3)\hat{\Pi}_{\mu\nu}\hat{\Pi}_{\sigma\tau}$ projector considered in [Anselmi (1998)] and [DeWolfe, Freedman, Gubser and Karch (2000)], as this latter one is only able to obey $\hat{P}_{\mu\nu\sigma\tau}\hat{P}^{\sigma\tau}{}_{\alpha\beta} = \hat{P}_{\mu\nu\alpha\beta}$ when it is restricted to act on functions for which no surface terms are generated in any integration by parts, a situation in which the effects of $P_{\mu\nu\sigma\tau}$ and $\hat{P}_{\mu\nu\sigma\tau}$ then would agree. Because the action of both of these projectors involves two non-local integrations, in passing we also note that for any $h_{\mu\nu}$, localized or not, the fourth derivative quantities $[\partial_\alpha\partial^\alpha]^2 P_{\mu\nu\sigma\tau}h^{\sigma\tau}$ and $[\partial_\alpha\partial^\alpha]^2\hat{P}_{\mu\nu\sigma\tau}h^{\sigma\tau}$ actually coincide.

where

$$\hat{V}_\mu = V_\mu - \int d^4x' D(x-x')[\partial_\sigma \partial^\sigma V_\mu(x') + \partial_\mu \partial_\sigma V^\sigma(x')] \quad ,$$
$$+\frac{1}{2}\partial_\mu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')[\partial_\tau \partial^\tau V^\sigma(x'') + \partial^\sigma \partial_\tau V^\tau(x'')] \ , \quad \text{(C.64)}$$

we can then define

$$V^{PTT}_{\mu\nu} = P_{\mu\nu\sigma\tau} V^{T\,\sigma\tau} = \partial_\mu \hat{V}_\nu + \partial_\nu \hat{V}_\mu - \frac{2}{3}\hat{\Pi}_{\mu\nu}\partial_\sigma \hat{V}^\sigma \quad , \qquad \text{(C.65)}$$

and

$$V^{TNT}_{\mu\nu} = Q_{\mu\nu\sigma\tau} V^{T\,\sigma\tau} = \frac{2}{3}\hat{\Pi}_{\mu\nu}\partial_\sigma \hat{V}^\sigma \quad . \qquad \text{(C.66)}$$

Then with the $P_{\mu\nu\sigma\tau}$ and $Q_{\mu\nu\sigma\tau}$ sectors being non-overlapping, $h^{PTT}_{\mu\nu}$ and $h^{TNT}_{\mu\nu}$ thus transform as

$$\bar{h}^{PTT}_{\mu\nu} = h^{PTT}_{\mu\nu} + V^{PTT}_{\mu\nu} \quad , \quad \bar{h}^{TNT}_{\mu\nu} = h^{TNT}_{\mu\nu} + V^{TNT}_{\mu\nu} \quad , \qquad \text{(C.67)}$$

under the general gauge transformation of Eq. (C.52). With $\hat{V}_\mu$ being found to obey the kinematic relation $\partial_\alpha \partial^\alpha \partial_\mu \hat{V}^\mu = 0$, the gauge invariance of the separate $h^{PTT}_{\mu\nu}$ and $h^{TNT}_{\mu\nu}$ wave equations of Eq. (C.60) is then readily established (with an interplay with the surface term again being needed). Finally, in cases where the fluctuations are localized enough for the surface term in Eq. (C.60) to actually vanish (a surface term which only depends on the TNT sector), the wave equations take the form

$$\frac{1}{2}\partial_\alpha \partial^\alpha h^{PTT}_{\mu\nu} = -\kappa_4^2 \delta\tau^{PTT}_{\mu\nu} \quad ,$$
$$\frac{1}{2}\partial_\alpha \partial^\alpha h^{TNT}_{\mu\nu} = -\kappa_4^2 \delta\tau^{TNT}_{\mu\nu} \quad , \qquad \text{(C.68)}$$

and are then readily soluble.[10]

[10] While the PTT wave equation has the same form as the TT wave equation given in Eq. (14.34), the TNT wave equation differs slightly in form from that of the NT wave equation given in Eq. (14.33) since this latter equation was derived in the harmonic rather than the transverse gauge. As regards the harmonic gauge we note in passing that if we define $H_{\mu\nu\sigma\tau} = (1/3)\eta_{\mu\nu}\int D[\eta_{\sigma\tau}\partial_\alpha\partial^\alpha - \partial_\sigma\partial_\tau] + (2/3)\partial_\mu\partial_\nu \int D \int D[\eta_{\sigma\tau}\partial_\alpha\partial^\alpha - \partial_\sigma\partial_\tau]$, we find that $H_{\mu\nu\sigma\tau}$ obeys the projector relation $H_{\mu\nu\sigma\tau}H^{\sigma\tau}{}_{\alpha\beta} = H_{\mu\nu\alpha\beta}$ while generating an $h^{H}_{\mu\nu} = H_{\mu\nu\sigma\tau}h^{\sigma\tau}$ which obeys $\partial_\sigma h^{H\,\sigma\nu} - (1/2)\partial^\nu h^{H} = 0$. (The function $h^{H}_{\mu\nu} = \hat{H}_{\mu\nu\sigma\tau}h^{\sigma\tau} = h_{\mu\nu} - \partial_\mu \int D[\partial_\sigma h^\sigma{}_\nu - (1/2)\partial_\nu h] - \partial_\nu \int D[\partial_\sigma h^\sigma{}_\mu - (1/2)\partial_\mu h]$ also obeys $\partial_\sigma h^{H\,\sigma\nu} - (1/2)\partial^\nu h^{H} = 0$, but the associated $\hat{H}_{\mu\nu\sigma\tau}$ (an $\hat{H}_{\mu\nu\sigma\tau}$ which possesses no double integral) is not found to be a projector.) Then with every solution $h_{\mu\nu}$ to the Einstein equations obeying the relation $\partial_\sigma\partial^\sigma h - \partial_\sigma\partial_\tau h^{\sigma\tau} = \kappa_4^2\delta\tau$ given in Eq. (C.49), we see that we may identify $h^{H}_{\mu\nu}$ with the quantity $h^{NT}_{\mu\nu}$ given in Eq. (14.32). With the transverse-traceless modes automatically obeying the harmonic gauge condition, both the gauge approach of Chapter 14 and the projection approach discussed here thus lead to Eq. (C.68) for localized transverse-traceless fluctuations.

As a technique transverse-traceless projection is not restricted to the Einstein equations. Rather, it can be applied in any covariant, pure metric theory of gravity, with a particularly interesting case being the fourth order conformal gravity theory since this particular theory is a conformal invariant one whose energy-momentum tensor is then automatically traceless.[11] Specifically, it is based on the Weyl action whose gravitational part is given by

$$S_W = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \quad , \tag{C.69}$$

where $C_{\lambda\mu\nu\kappa}$ is the Weyl tensor and α_g is the automatically dimensionless gravitational coupling constant of the theory; with the variation of S_W with respect to the metric yielding as its gravitational equation of motion

$$(-g)^{-1/2}\frac{\delta S_W}{\delta g_{\mu\nu}} = -2\alpha_g W^{\mu\nu} = -\frac{1}{2}T^{\mu\nu} \quad , \tag{C.70}$$

where $W^{\mu\nu}$ is given by

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} + R^{\mu\nu;\beta}{}_{;\beta} - R^{\mu\beta;\nu}{}_{;\beta} - R^{\nu\beta;\mu}{}_{;\beta} - 2R^{\mu\beta}R^{\nu}{}_{\beta} + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} \\ &\quad - \frac{2}{3}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} + \frac{2}{3}(R^{\alpha}{}_{\alpha})^{;\mu;\nu} + \frac{2}{3}R^{\alpha}{}_{\alpha}R^{\mu\nu} - \frac{1}{6}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^2 \quad . \end{aligned} \tag{C.71}$$

As constructed the tensor $W_{\mu\nu}$ kinematically obeys both the $W^{\mu\nu}{}_{;\mu} = 0$ covariant conservation condition and the $g^{\mu\nu}W_{\mu\nu} = 0$ tracelessness condition, which it would need to do since $T_{\mu\nu}$ is also covariantly conserved and traceless. On linearizing Eq. (C.70) around a flat spacetime background in the presence of a perturbation $\delta\tau^{TT}_{\mu\nu}$ which would necessarily have to be transverse and traceless also, we find that the fluctuations have to obey

$$\left[\frac{1}{2}(P_{\mu\sigma}P_{\nu\tau} + P_{\mu\tau}P_{\nu\sigma}) - \frac{1}{3}P_{\mu\nu}P_{\sigma\tau}\right]h^{\sigma\tau} = \frac{1}{2\alpha_g}\delta\tau^{TT}_{\mu\nu} \quad , \tag{C.72}$$

where $P_{\mu\nu} = \eta_{\mu\nu}\partial_\alpha\partial^\alpha - \partial_\mu\partial_\nu = \partial_\alpha\partial^\alpha\hat{\Pi}_{\mu\nu}$. Then, on recalling the definition of $h^{PTT}_{\mu\nu}$ given in Eq. (C.57), we find that Eq. (C.72) can be written very compactly as

$$[\partial_\alpha\partial^\alpha]^2 h^{PTT}_{\mu\nu} = \frac{1}{2\alpha_g}\delta\tau^{TT}_{\mu\nu} \quad . \tag{C.73}$$

With both sides of Eq. (C.73) being both transverse and traceless, we see that in the conformal theory, it is precisely the PTT piece of $h_{\mu\nu}$ which is relevant. While a general 10-component $h_{\mu\nu}$ would have four longitudinal and six transverse components, on noting that $[(P_{\mu\sigma}P_{\nu\tau} + P_{\mu\tau}P_{\nu\sigma})/2 - P_{\mu\nu}P_{\sigma\tau}/3][\eta^{\sigma\tau}h]$ just happens to vanish identically, we see that in Eq. (C.72) we can replace $h_{\mu\nu}$ by $K_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/4$ with $K_{\mu\nu}$ being traceless. In the conformal theory then only nine of

[11] This theory has been advocated as a candidate alternative to the standard theory, with it being able to naturally solve [Mannheim (1997); Mannheim (2001d)] the dark matter and cosmological constant problems which currently challenge the standard theory.

the ten components of an arbitrary $h_{\mu\nu}$ are relevant, and with the four longitudinal components being unphysical, only the five transverse-traceless components can ultimately play any physical role, and it is thus only they which appear in Eq. (C.73). Finally, recalling again that $\partial_\alpha\partial^\alpha\partial_\mu\hat{V}^\mu = 0$, from the transformations of Eq. (C.54) we see that the PTT wave equation of Eq. (C.73) is gauge invariant just as it should be, as would also be the action $S = \int d^4x h^{PTT\,\mu\nu}[\partial_\alpha\partial^\alpha]^2 h^{PTT}_{\mu\nu}$ from which it could be derived.

C.4 Transverse-traceless projection for curved space tensor fields

Turning now to the construction of the analogous tensor field projectors in a curved space background, this time we want to make the decomposition

$$h^{\mu\nu} = h^{T\,\mu\nu} + \nabla^\mu V^\nu + \nabla^\nu V^\mu \quad , \tag{C.74}$$

where $\nabla_\mu h^{T\,\mu\nu} = 0$, a condition which requires

$$\nabla_\mu h^{\mu\nu} = \nabla_\mu \nabla^\mu V^\nu + \nabla_\mu \nabla^\nu V^\mu \quad . \tag{C.75}$$

Thus, on defining a propagator which obeys

$$[g^\nu{}_\beta \nabla_\tau \nabla^\tau + \nabla_\beta \nabla^\nu] D^\beta{}_\sigma(x,x') = g^\nu{}_\sigma g^{-1/2}\delta^4(x-x') \quad . \tag{C.76}$$

we can solve for V^ν to obtain

$$V^\nu = \int d^4x' g^{1/2} D^\nu{}_\sigma(x,x') \nabla_\tau h^{\tau\sigma} \quad , \tag{C.77}$$

with its insertion into Eq. (C.74) allowing us to construct both the manifestly transverse state

$$h^{T\,\mu\nu} = h^{\mu\nu} - \nabla^\mu \int d^4x' g^{1/2} D^\nu{}_\sigma(x,x')\nabla_\tau h^{\tau\sigma} - \nabla^\nu \int d^4x' g^{1/2} D^\mu{}_\sigma(x,x')\nabla_\tau h^{\tau\sigma} \ , \tag{C.78}$$

and the strictly longitudinal state

$$h^{L\,\mu\nu} = \nabla^\mu \int d^4x' g^{1/2} D^\nu{}_\sigma(x,x')\nabla_\tau h^{\tau\sigma} + \nabla^\nu \int d^4x' g^{1/2} D^\mu{}_\sigma(x,x')\nabla_\tau h^{\tau\sigma} \tag{C.79}$$

which obeys $\nabla_\mu h^{L\,\mu\nu} = \nabla_\mu h^{\mu\nu}$, a state whose divergence can only vanish if $h^{L\,\mu\nu}$ itself vanishes. Equations (C.78) and (C.79) allow us to construct explicit projectors

$$\begin{aligned} T_{\mu\nu\sigma\tau} &= g_{\mu\sigma}g_{\nu\tau} - \nabla_\mu \int d^4x' g^{1/2} D_{\nu\sigma}(x,x')\nabla_\tau - \nabla_\nu \int d^4x' g^{1/2} D_{\mu\sigma}(x,x')\nabla_\tau \ , \\ L_{\mu\nu\sigma\tau} &= \nabla_\mu \int d^4x' g^{1/2} D_{\nu\sigma}(x,x')\nabla_\tau + \nabla_\nu \int d^4x' g^{1/2} D_{\mu\sigma}(x,x')\nabla_\tau \end{aligned} \tag{C.80}$$

which effect $T_{\mu\nu\sigma\tau}h^{\sigma\tau} = h^{T}_{\mu\nu}$ and $L_{\mu\nu\sigma\tau}h^{\sigma\tau} = h^{L}_{\mu\nu}$, with use of Eq. (C.76) allowing us to show that $T_{\mu\nu\sigma\tau}$ and $L_{\mu\nu\sigma\tau}$ do indeed obey the requisite projector relationships of Eq. (C.45).

While Eq. (C.78) does not immediately look like a generalization of the flat space $h^{T}_{\mu\nu}$ of Eq. (C.41) (and likewise for $h^{L}_{\mu\nu}$), we note that in the flat space case, taking the derivative of Eq. (C.75) yields

$$\partial_\mu\partial_\nu h^{\mu\nu} = 2\partial_\mu\partial^\mu\partial_\nu V^\nu \quad , \tag{C.81}$$

which admits of solution

$$\partial_\mu V^\mu = \frac{1}{2}\partial_\sigma \int d^4x' D(x-x')\partial_\tau h^{\sigma\tau} \quad , \tag{C.82}$$

where $D(x-x')$ is the flat space scalar propagator. Given Eq. (C.82) we can then obtain as solution to Eq. (C.75)

$$V^\nu = \int d^4x' D(x-x')\partial_\sigma h^{\sigma\nu} - \frac{1}{2}\partial^\nu \int d^4x' D(x-x')\partial_\sigma \int d^4x'' D(x'-x'')\partial_\tau h^{\sigma\tau} \ , \tag{C.83}$$

whose insertion into Eq. (C.74) then recovers Eq. (C.41). Finally, on comparing with Eq. (C.77) we find that we can write the flat space $D^{\beta}{}_{\sigma}(x,x')$ as

$$D^{\beta}{}_{\sigma}(x,x') = \eta^{\beta}{}_{\sigma}D(x-x') - \frac{1}{2}\frac{\partial}{\partial x_\beta}D(x-x')\frac{\partial}{\partial x'^\sigma}\int d^4x'' D(x'-x'') \quad , \tag{C.84}$$

with use of the distribution properties of delta functions allowing us to show that Eq. (C.84) is indeed a solution to Eq. (C.76) in the flat space case.[12]

As well as construct the general curved space transverse and longitudinal projectors, we note that because the longitudinal part of $h_{\mu\nu}$ has the same generic form as a general gauge transform, viz. $\bar{h}_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu$, it follows from the analysis of Appendix A that only $h^{T}_{\mu\nu}$ contributes to the general first order fluctuation equation $\Delta G_{\mu\nu} = \delta G_{\mu\nu} + \kappa_4^2\delta T_{\mu\nu} = -\kappa_4^2\delta\tau_{\mu\nu}$ around the $G_{\mu\nu} = -\kappa_4^2 T_{\mu\nu}$ background, with it thus taking the form

$$\begin{aligned}&\frac{1}{2}[\nabla_\mu\nabla_\nu h^{T} + R^{\sigma}{}_{\mu}h^{T}_{\sigma\nu} + R^{\sigma}{}_{\nu}h^{T}_{\sigma\mu} - 2R_{\mu\lambda\nu\sigma}h^{T\lambda\sigma} + \nabla_\alpha\nabla^\alpha h^{T}_{\mu\nu}] \\ &-\frac{1}{2}R^{\sigma}{}_{\sigma}h^{T}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}h^{T\alpha\beta} - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h^{T} + \kappa_4^2\delta T_{\mu\nu} = -\kappa_4^2\delta\tau_{\mu\nu} \quad ,\end{aligned} \tag{C.85}$$

an equation in which both sides are transverse with respect to the background.

Now while we do need the general curved space transverse projector to construct $h^{T}_{\mu\nu}$ via the two-index propagator $D^{\beta}{}_{\sigma}(x,x')$, in going from the 6-component

[12]Our ability to relate the flat space $D^{\beta}{}_{\sigma}(x,x')$ to the flat space $D(x-x')$ was contingent on the fact that for flat Cartesian coordinates both the scalar propagator and the covariant delta function $g^{-1/2}\delta^4(x-x')$ are only functions of the difference $(x-x')$, so that when any $\partial/\partial x$ type derivative acts on them we can replace it by $-\partial/\partial x'$. Equation (C.84) thus has no non-flat equivalent.

transverse fluctuation $h^T_{\mu\nu}$ to a 5-component transverse-traceless one we only need to remove the one h^T degree of freedom, and since h^T is a scalar, in some cases (at least) we will be able to effect this last step via a further projection which is based solely on a scalar propagator. And in fact, in our discussion of the dS_4 and AdS_4 branes, we found in Eq. (15.34) that for those particular branes the decomposition of the already transverse $S_{\mu\nu}$ into its TT and NT parts was precisely achieved via use of the scalar propagator given in Eq. (15.32). Consequently, we can anticipate that in cases where the background geometry has high enough symmetry, it might be possible to construct transverse-traceless states from transverse ones using scalar propagators alone. Thus modeled on Eq. (15.34) and using all the available background rank two tensors at our disposal, we try as a candidate

$$\begin{aligned} h^{PTT}_{\mu\nu} &= h^T_{\mu\nu} + [\lambda g_{\mu\nu} + \rho R g_{\mu\nu}]h^T \\ &+ [\alpha\nabla_\mu\nabla_\nu + \beta g_{\mu\nu} + \gamma R_{\mu\nu} + \kappa R g_{\mu\nu}]\int d^4x' g^{1/2}F(x,x')h^T(x') \quad , \end{aligned} \tag{C.86}$$

where $F(x,x')$ is a scalar propagator which obeys

$$\nabla_\nu\nabla^\nu F(x,x') - A(x)F(x,x') = g^{-1/2}\delta^4(x-x') \quad , \tag{C.87}$$

where the scalar $A(x)$ is to be determined, and α, β, γ, κ, λ and ρ are constants. Requiring that $h^{PTT}_{\mu\nu}$ be transverse entails that

$$\begin{aligned} &[\lambda\nabla_\mu + \rho R\nabla_\mu + \rho(\nabla_\mu R) + \alpha\nabla_\mu]h^T \\ &+ \left[\alpha[A\nabla_\mu + (\nabla_\mu A) - R^\nu{}_\mu\nabla_\nu] + \beta\nabla_\mu + \gamma R^\nu{}_\mu\nabla_\nu + \frac{1}{2}\gamma(\nabla_\mu R) + \kappa[R\nabla_\mu + (\nabla_\mu R]\right] \\ &\times \int d^4x' g^{1/2}F(x,x')h^T(x') = 0 \quad . \end{aligned} \tag{C.88}$$

Thus we must set

$$\lambda + \alpha = 0 \ , \ \rho = 0 \ , \ \alpha - \gamma = 0 \ , \ \alpha A + \beta + \kappa R = 0 \ , \ 2\alpha\nabla_\mu A + (\alpha + 2\kappa)\nabla_\mu R = 0 \ . \tag{C.89}$$

Finally, with the most general scalar A which can satisfy these equations being given by $A = p + qR$ where p and q are constants, we find that we additionally need to impose

$$\alpha p + \beta = 0 \ , \ (\alpha q + \kappa)R = 0 \ , \ \alpha\nabla_\mu R = 0 \quad , \tag{C.90}$$

unless R is equal to a constant R_0, in which case we would need to impose

$$\alpha p + \beta + (\alpha q + \kappa)R_0 = 0 \quad . \tag{C.91}$$

However, from Eq. (C.90) we see that since we cannot set $\alpha = 0$ (as that would lead to the vanishing of all the constants in Eq. (C.86)), we must instead set $\nabla_\mu R = 0$, to thus lead us back to the need to have R be a constant anyway. Similarly, the tracelessness of $h_{\mu\nu}^{PTT}$ requires

$$1+4\lambda+4\rho R+\alpha=0\ ,\ \alpha p+\alpha qR+4\beta+\gamma R+4\kappa R=0\ . \tag{C.92}$$

Consequently, when R is a constant, the most general allowed $h_{\mu\nu}^{PTT}=h_{\mu\nu}^{T}-h_{\mu\nu}^{TNT}$ takes the form

$$\begin{aligned} h_{\mu\nu}^{PTT} &= h_{\mu\nu}^{T}-\frac{1}{3}g_{\mu\nu}h^{T} \\ &+\frac{1}{3}\left[\nabla_\mu\nabla_\nu+R_{\mu\nu}-\frac{1}{3}Rg_{\mu\nu}\right]\int d^4x'g^{1/2}F(x,x')h^{T}(x')\ , \end{aligned} \tag{C.93}$$

where $F(x,x')$ obeys

$$\nabla_\mu\nabla^\nu F(x,x')-\frac{1}{3}RF(x,x')=g^{-1/2}\delta^4(x-x')\ . \tag{C.94}$$

Comparing with Eq. (15.34) we see that Eq. (C.94) nicely encompasses the previously discussed AdS_4/dS_4 cases where $\gamma=1/3$, $R_{\mu\nu}=-3kH^2g_{\mu\nu}$, $R=-12kH^2$, $F(x,x')=E(x,x')$, so that

$$h_{\mu\nu}^{PTT}=h_{\mu\nu}^{T}-\frac{1}{3}g_{\mu\nu}h^{T}+\frac{1}{3}[\nabla_\mu\nabla_\nu-g_{\mu\nu}H^2]\int d^4x'g^{1/2}E(x,x')h^{T}(x')\ . \tag{C.95}$$

However, Eq. (C.94) also encompasses any other geometry in which the Ricci scalar is a constant, some physically interesting examples of which are the standard Friedmann-Robertson-Walker cosmology in the high temperature radiation era phase where $R=0$, and the cosmology associated with conformal gravity [Mannheim (2001d)] where R is a constant at both high and low temperatures.

In the particular case of AdS_4 Eq. (C.85) takes the form

$$\frac{1}{2}[\nabla_\alpha\nabla^\alpha+2H^2]h_{\mu\nu}^{T}+\frac{1}{2}[\nabla_\mu\nabla_\nu+H^2g_{\mu\nu}-g_{\mu\nu}\nabla_\alpha\nabla^\alpha]h^{T}=-\kappa_4^2\delta\tau_{\mu\nu}\ , \tag{C.96}$$

from which it follows that

$$[\nabla_\alpha\nabla^\alpha-3H^2]h^{T}=\kappa_4^2\delta\tau\ , \tag{C.97}$$

Use of Eq. (C.97) allows us to manipulate $\delta\tau_{\mu\nu}^{PTT}$ into the form

$$\begin{aligned} \delta\tau_{\mu\nu}^{PTT} &= \delta\tau_{\mu\nu}-\frac{1}{3}g_{\mu\nu}\delta\tau+\frac{1}{3}[\nabla_\mu\nabla_\nu-H^2g_{\mu\nu}]\int d^4x'g^{1/2}E\delta\tau \\ &= \delta\tau_{\mu\nu}-\frac{1}{3}g_{\mu\nu}\delta\tau+\frac{H^2}{3\kappa_4^2}[\nabla_\mu\nabla_\nu-H^2g_{\mu\nu}]\int d^4x'g^{1/2}Eh^{T} \\ &+\frac{1}{3\kappa_4^2}[\nabla_\mu\nabla_\nu-H^2g_{\mu\nu}]\int d^4x'g^{1/2}E[\nabla_\alpha\nabla^\alpha-4H^2]h^{T}\ , \end{aligned} \tag{C.98}$$

from which, on setting $h^{T} = h^{TNT}$, it follows that the fluctuations $h_{\mu\nu}^{PTT}$ and $h_{\mu\nu}^{TNT}$ obey

$$\begin{aligned}
&\frac{1}{2}[\nabla_\alpha\nabla^\alpha + 2H^2]h_{\mu\nu}^{PTT} + \kappa_4^2\delta\tau_{\mu\nu}^{PTT} \\
&= -\frac{1}{3}[\nabla_\mu\nabla_\nu - H^2 g_{\mu\nu}]\left[h^{TNT} - \int d^4x' g^{1/2} E[\nabla_\alpha\nabla^\alpha - 4H^2]h^{TNT}\right] \\
&= -\frac{1}{3}[\nabla_\mu\nabla_\nu - H^2 g_{\mu\nu}]\left[\int dS_\alpha g^{1/2}[h^{TNT}\nabla^\alpha E - E\nabla^\alpha h^{TNT}\right] \quad , \qquad \text{(C.99)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}[\nabla_\alpha\nabla^\alpha + 2H^2]h_{\mu\nu}^{TNT} + \kappa_4^2\delta\tau_{\mu\nu}^{TNT} + \frac{1}{2}[\nabla_\mu\nabla_\nu + H^2 g_{\mu\nu} - g_{\mu\nu}\nabla_\alpha\nabla^\alpha]h^{TNT} \\
&= \frac{1}{3}[\nabla_\mu\nabla_\nu - H^2 g_{\mu\nu}]\left[\int dS_\alpha g^{1/2}[h^{TNT}\nabla^\alpha E - E\nabla^\alpha h^{TNT}\right] \quad . \qquad \text{(C.100)}
\end{aligned}$$

For fluctuations for which the surface term in Eq. (C.99) vanishes, we thus recover the TT mode equation given in Eq. (15.46).

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Appendix D

Fluctuation Energy-Momentum Tensor

D.1 General second order Einstein-Hilbert action

In order to construct an appropriate brane-world fluctuation mode energy-momentum tensor, it is useful to recall [Bak, Cangemi and Jackiw (1994)] that via an integration by parts the Einstein-Hilbert action can be brought to the form

$$
\begin{aligned}
I_{EH} &= -\frac{1}{2\kappa_4^2}\int g^{1/2}R \\
&= -\frac{1}{2\kappa_4^2}\int g^{1/2}g^{\mu\nu}\left(\partial_\nu\Gamma^\beta_{\mu\beta} - \partial_\beta\Gamma^\beta_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}\right) \\
&= \frac{1}{2\kappa_4^2}\int g^{1/2}\Gamma^\beta_{\mu\beta}\left(\Gamma^\alpha_{\nu\alpha}g^{\mu\nu} - \Gamma^\mu_{\nu\alpha}g^{\alpha\nu} - \Gamma^\nu_{\nu\alpha}g^{\mu\alpha}\right) \\
&\quad - \frac{1}{2\kappa_4^2}\int g^{1/2}\Gamma^\beta_{\mu\nu}\left(\Gamma^\alpha_{\beta\alpha}g^{\mu\nu} - \Gamma^\mu_{\beta\alpha}g^{\alpha\nu} - \Gamma^\nu_{\beta\alpha}g^{\mu\alpha}\right) \\
&\quad - \frac{1}{2\kappa_4^2}\int g^{1/2}g^{\mu\nu}\left(\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}\right) + \text{surface terms} \\
&= \frac{1}{2\kappa_4^2}\int g^{1/2}g^{\mu\nu}\left(\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}\right) + \text{surface terms}\ . \qquad \text{(D.1)}
\end{aligned}
$$

The utility of Eq. (D.1) is that in a determination of the equations of motion via a functional variation of the metric in which its variation is held fixed on the surfaces, these particular surface terms play no role. Consequently, for variational purposes the surface terms can be discarded, with variation of the extremely convenient to work with but seemingly non-covariant action $(1/2\kappa_4^2)\int g^{1/2}g^{\mu\nu}(\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta})$ then actually yielding precisely the same covariant equations of motion as are obtained via variation of the original $-(1/2\kappa_4^2)\int g^{1/2}R$. For $g_{\mu\nu} + h_{\mu\nu}$ fluctuations around a general non-flat background with metric $g_{\mu\nu}$ and Christoffel symbol $\Gamma^\alpha_{\mu\nu}$, the term in the action which is second order in a fluctuation $h_{\mu\nu}$ can readily be obtained from Eq. (D.1) by using the perturbed covariant metric $g_{\mu\nu} + h_{\mu\nu}$, the associated perturbed contravariant metric $g^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\sigma h^{\sigma\nu}$ and the associated perturbed measure $g^{1/2}(1+h/2+h^2/8-h_{\mu\nu}h^{\mu\nu}/4)$ through second order,[1] to yield

$$
\begin{aligned}
I^{(2)}_{EH} &= \frac{1}{2\kappa_4^2}\int g^{1/2}g^{\mu\nu}\left(\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\nu\alpha}-\Gamma^{(1)\alpha}_{\mu\nu}\Gamma^{(1)\beta}_{\alpha\beta}\right)\\
&+\frac{1}{2\kappa_4^2}\int g^{1/2}g^{\mu\nu}\left(\Gamma^{(2)\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{(2)\beta}_{\nu\alpha}-\Gamma^{(2)\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{(2)\beta}_{\alpha\beta}\right)\\
&+\frac{1}{16\kappa_4^2}\int g^{1/2}\left[8h^{\mu}{}_{\sigma}h^{\sigma\nu}-4hh^{\mu\nu}+(h^2-2h_{\sigma\tau}h^{\sigma\tau})g^{\mu\nu}\right]\left(\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right)\\
&+\frac{1}{4\kappa_4^2}\int g^{1/2}\left(g^{\mu\nu}h-2h^{\mu\nu}\right)\left(\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\nu\alpha}-\Gamma^{(1)\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{(1)\beta}_{\alpha\beta}\right), \quad (D.2)
\end{aligned}
$$

where

$$\Gamma^{(1)\alpha}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(\nabla_\nu h_{\beta\mu}+\nabla_\mu h_{\beta\nu}-\nabla_\beta h_{\mu\nu}\right) \quad . \qquad (D.3)$$

Comparing Eq. (D.2) with the direct second order expansion of $I^{(2)}_{EH}$ around a background geometry with metric $g_{\mu\nu}$ and Ricci tensor $R_{\mu\nu}$, viz.

$$
\begin{aligned}
I^{(2)}_{EH} &= -\frac{1}{2\kappa_4^2}\int d^4x g^{1/2}[g^{\mu\nu}R^{(2)}_{\mu\nu}-h^{\mu\nu}R^{(1)}_{\mu\nu}+h^{\mu}{}_{\sigma}h^{\sigma\nu}R_{\mu\nu}]\\
&-\frac{1}{4\kappa_4^2}\int d^4x g^{1/2}h[g^{\mu\nu}R^{(1)}_{\mu\nu}-h^{\mu\nu}R_{\mu\nu}]\\
&-\frac{1}{16\kappa_4^2}\int d^4x g^{1/2}(h^2-2h_{\alpha\beta}h^{\alpha\beta})g^{\mu\nu}R_{\mu\nu} \quad , \qquad (D.4)
\end{aligned}
$$

we see that when calculated in a background which is momentarily taken to be flat, unlike Eq. (D.4) which would still involve a second order term (viz. $R^{(2)}_{\mu\nu}$), Eq. (D.2) would only involve first order terms (viz. the $\Gamma^{(1)}\Gamma^{(1)}$ product terms which appear in the first of its integrals); with the flat background $I^{(2)}_{EH}$ then readily integrating by parts to

$$
\begin{aligned}
I^{(2)}_{EH} &= \frac{1}{8\kappa_4^2}\int d^4x h^{\mu\nu}(\partial_\mu\partial_\nu h-\partial_\mu\partial_\alpha h^{\alpha}{}_{\nu}-\partial_\nu\partial_\alpha h^{\alpha}{}_{\mu}+\partial_\alpha\partial^\alpha h_{\mu\nu}\\
&\qquad -\eta_{\mu\nu}\partial_\alpha\partial^\alpha h+\eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}) \quad . \qquad (D.5)
\end{aligned}
$$

We recognize Eq. (D.5) to be of the form

$$I^{(2)}_{EH}=\frac{1}{4\kappa_4^2}\int d^4x h^{\mu\nu}\delta G_{\mu\nu} \quad , \qquad (D.6)$$

where $\delta G_{\mu\nu}$ is the first order change in the Einstein tensor in a flat background.

As such, Eq. (D.6) is noteworthy for several reasons. First, its stationary variation with respect to $h^{\mu\nu}$ leads to $\delta G_{\mu\nu}=0$, the standard source-free region flat background fluctuation equation. Second, with $\delta G_{\mu\nu}$ kinematically obeying

[1] For a metric $g_{\mu\nu}+h_{\mu\nu}$, its determinant is given through second order in $h_{\mu\nu}$ by $g+c_1h+c_2h^2+c_3h_{\mu\nu}h^{\mu\nu}$ where the numerical coefficients c_i can be determined by working in a basis in which $g_{\mu\nu}+h_{\mu\nu}$ is diagonal, to yield $c_1=1$, $c_2=1/2$, $c_3=-1/2$.

$\partial_\mu \delta G^{\mu\nu} = 0$ and with it remaining unchanged when $h_{\mu\nu}$ is replaced by $\bar{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$, as written the action of Eq. (D.6) is gauge invariant.[2] And finally, Eq. (D.6) admits of the immediate curved space generalization

$$I_{EH}^{(2)} = \frac{1}{4\kappa_4^2} \int d^4x g^{1/2} h^{\mu\nu} \Delta G_{\mu\nu} \ , \tag{D.7}$$

where $\Delta G_{\mu\nu}$ is given in Eq. (13.7), with the variation of Eq. (D.7) with respect to $h^{\mu\nu}$ then leading to $\Delta G_{\mu\nu} = 0$. With the action of Eq. (D.7) being gauge invariant under the general $\bar{h}_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$, its variation with respect to the background $g_{\mu\nu}$ would then allow us to construct a fluctuation energy-momentum tensor which would automatically be covariantly conserved.

D.2 Second order fluctuation energy-momentum tensor in AdS_5

Rather than vary the analog AdS_5 second order Einstein-Hilbert action with respect to the background g_{MN}, precisely because $I_{EH}^{(2)}$ is gauge invariant, we can simplify the problem by first putting $I_{EH}^{(2)}$ into the RS gauge introduced in Chapter 14, to thereby yield the 5-dimensional TT sector mode effective action[3]

$$\begin{aligned} I_{EH}^{(2)} &= \frac{1}{4\kappa_5^2} \int d^5x g^{1/2} h^{TT\, MN} \Delta G^{TT}_{MN} \\ &= \frac{1}{8\kappa_5^2} \int d^5x g^{1/2} g^{MA} g^{NB} h^{TT}_{AB} \left(g^{CD} \nabla_C \nabla_D h^{TT}_{MN} + 2b^2 h^{TT}_{MN} \right) \end{aligned} \tag{D.8}$$

given in Eq. (16.63). To now determine the variation of this action under a change δg_{MN} in the background metric in which h^{TT}_{MN} is kept fixed, we note that with $\nabla_C \nabla_D h^{TT}_{MN}$ being given by

$$\begin{aligned} \nabla_C \nabla_D h^{TT}_{MN} &= \partial_C (\nabla_D h^{TT}_{MN}) - \Gamma^E_{CD} \nabla_E h^{TT}_{MN} - \Gamma^E_{CM} \nabla_D h^{TT}_{EN} - \Gamma^E_{CN} \nabla_D h^{TT}_{ME} \\ &= \partial_C (\partial_D h^{TT}_{MN} - \Gamma^F_{DM} h^{TT}_{FN} - \Gamma^F_{DN} h^{TT}_{MF}) \\ &\quad - \Gamma^E_{CD} (\partial_E h^{TT}_{MN} - \Gamma^F_{EM} h^{TT}_{FN} - \Gamma^F_{EN} h^{TT}_{MF}) \\ &\quad - \Gamma^E_{CM} (\partial_D h^{TT}_{EN} - \Gamma^F_{DE} h^{TT}_{FN} - \Gamma^F_{DN} h^{TT}_{EF}) \\ &\quad - \Gamma^E_{CN} (\partial_D h^{TT}_{ME} - \Gamma^F_{DM} h^{TT}_{FE} - \Gamma^F_{DE} h^{TT}_{MF}) \ , \end{aligned} \tag{D.9}$$

[2]While asymptotically badly behaved gauge functions would lead to the generation of surface terms in an integration by parts in $\int d^4x [\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu] \delta G_{\mu\nu}$, such surface terms would not contribute in a functional variation of the action in which end points are kept fixed.

[3]As noted in Chapter 16, AdS_5 modes which are 5-dimensional axial gauge and 4-dimensional TT (viz. RS gauge modes) are also 5-dimensional TT; with use of Eq. (13.10) showing that in all six of the maximally 4-symmetric brane worlds of interest to us, the source-free variant of the axial gauge TT mode equation of motion given in Eq. (16.1) is writable as $\nabla_C \nabla^C h^{TT}_{MN} + 2b^2 h^{TT}_{MN} = 0$.

the change in $\nabla_C\nabla_D h^{TT}_{MN}$ is then given by

$$\delta(\nabla_C\nabla_D h^{TT}_{MN}) = -\partial_C[\delta(\Gamma^F_{DM})h^{TT}_{FN} + \delta(\Gamma^F_{DN})h^{TT}_{MF}]$$
$$- \delta(\Gamma^E_{CD})(\partial_E h^{TT}_{MN} - \Gamma^F_{EM}h^{TT}_{FN} - \Gamma^F_{EN}h^{TT}_{MF}) + \Gamma^E_{CD}[\delta(\Gamma^F_{EM})h^{TT}_{FN} + \delta(\Gamma^F_{EN})h^{TT}_{MF}]$$
$$- \delta(\Gamma^E_{CM})(\partial_D h^{TT}_{EN} - \Gamma^F_{DE}h^{TT}_{FN} - \Gamma^F_{DN}h^{TT}_{EF}) + \Gamma^E_{CM}[\delta(\Gamma^F_{DE})h^{TT}_{FN} + \delta(\Gamma^F_{DN})h^{TT}_{EF}]$$
$$- \delta(\Gamma^E_{CN})(\partial_D h^{TT}_{ME} - \Gamma^F_{DM}h^{TT}_{FE} - \Gamma^F_{DE}h^{TT}_{MF})$$
$$+ \Gamma^E_{CN}[\delta(\Gamma^F_{DM})h^{TT}_{FE} + \delta(\Gamma^F_{DE})h^{TT}_{MF}] \ , \tag{D.10}$$

where

$$\delta\Gamma^A_{MN} = \frac{1}{2}g^{AB}(\nabla_M\delta g_{BN} + \nabla_N\delta g_{BM} - \nabla_B\delta g_{MN}) \ . \tag{D.11}$$

With the first order $\delta\Gamma^A_{MN}$ transforming as a true tensor with respect to the background, Eq. (D.10) can be written entirely in terms of covariant derivatives, with it being found to take the compact form

$$\begin{aligned}\delta(\nabla_C\nabla_D h^{TT}_{MN}) = & -\nabla_C(\delta\Gamma^F_{DM}h_{FN}) - \nabla_C(\delta\Gamma^F_{DN}h_{FM}) \\ & -\delta\Gamma^E_{CD}\nabla_E h_{MN} - \delta\Gamma^E_{CM}\nabla_D h_{EN} - \delta\Gamma^E_{CN}\nabla_D h_{EM} \ ,\end{aligned} \tag{D.12}$$

a relation which greatly facilitates the analysis below. With the variations of $g^{1/2}$ and g^{MN} induced by δg_{PQ} being respectively given by $(1/2)g^{1/2}g^{PQ}\delta g_{PQ}$ and $-(1/2)[g^{MP}g^{NQ} + g^{MQ}g^{NP}]\delta g_{PQ}$, functional variation of Eq. (D.8) with respect to the background metric then yields

$$\begin{aligned}T^{PQ} &= \frac{2}{g^{1/2}}\frac{\delta I^{(2)}_{EH}}{\delta g_{PQ}} = \frac{1}{8\kappa_5^2}g^{PQ}h^{TT\,MN}[\nabla_C\nabla^C h^{TT}_{MN} + 2b^2 h^{TT}_{MN}] \\ &- \frac{1}{8\kappa_5^2}[h^{TT\,QA}\nabla_C\nabla^C h^{TT\,P}{}_A + h^{TT\,PA}\nabla_C\nabla^C h^{TT\,Q}{}_A + 4b^2 h^{TT\,QA}h^{TT\,P}{}_A \\ &+ h^{TT\,MN}\nabla^P\nabla^Q h^{TT}_{MN} + h^{TT\,MN}\nabla^Q\nabla^P h^{TT}_{MN}] + X^{PQ} \ ,\end{aligned} \tag{D.13}$$

where X^{PQ} denotes the contribution due to the $\delta(\nabla_C\nabla_D h^{TT}_{MN})$ variation. Before evaluating this X^{PQ} however, we note that for modes which obey the equation of motion $\nabla_C\nabla^C h^{TT}_{MN} + 2b^2 h^{TT}_{MN} = 0$, Eq. (D.13) simplifies to

$$T^{PQ} = -\frac{1}{8\kappa_5^2}[h^{TT\,MN}\nabla^P\nabla^Q h^{TT}_{MN} + h^{TT\,MN}\nabla^Q\nabla^P h^{TT}_{MN}] + X^{PQ} \ . \tag{D.14}$$

With use of Eq. (D.12) the X^{PQ} term can be evaluated. To do this while being as general as possible, we initially introduce a general four index function J^{MCND} which is for the moment only required to possess $J^{MCND} = J^{NCND}$ symmetry, in the presence of which $J^{MCND}\delta(\nabla_C\nabla_D h^{TT}_{MN})$ then evaluates to

$$\begin{aligned}&J^{MCND}\delta(\nabla_C\nabla_D h^{TT}_{MN}) = \\ &J^{MCND}[-2\nabla_C(\delta\Gamma^F_{DM}h_{FN}) - 2\delta\Gamma^E_{CM}\nabla_D h_{EN} - \delta\Gamma^E_{CD}\nabla_E h_{MN}] \ .\end{aligned} \tag{D.15}$$

Given Eqs. (D.11) and (D.15), we find, following some algebra, that the contribution Y^{PQ} of the variation $\delta(\nabla_C\nabla_D h^{TT}_{MN})$ to the general quantity $\delta\int d^5x g^{1/2}J^{MCND}\nabla_C\nabla_D h^{TT}_{MN}/g^{1/2}\delta g_{PQ}$ is given by[4]

$$\begin{aligned}
Y^{PQ} = & -\frac{1}{2}\nabla_M[h^{TT\,P}{}_N\nabla_C J^{MCNQ}] - \frac{1}{2}\nabla_M[h^{TT\,Q}{}_N\nabla_C J^{MCNP}] \\
& -\frac{1}{2}\nabla_M[h^{TT\,P}{}_N\nabla_C J^{QCNM}] - \frac{1}{2}\nabla_M[h^{TT\,Q}{}_N\nabla_C J^{PCNM}] \\
& +\frac{1}{2}\nabla_M[h^{TT\,M}{}_N\nabla_C J^{PCNQ}] + \frac{1}{2}\nabla_M[h^{TT\,M}{}_N\nabla_C J^{QCNP}] \\
& +\frac{1}{2}\nabla_M[J^{MQNC}\nabla_C h^{TT\,P}{}_N] + \frac{1}{2}\nabla_M[J^{MPNC}\nabla_C h^{TT\,Q}{}_N] \\
& +\frac{1}{2}\nabla_M[J^{QMNC}\nabla_C h^{TT\,P}{}_N] + \frac{1}{2}\nabla_M[J^{PMNC}\nabla_C h^{TT\,Q}{}_N] \\
& -\frac{1}{2}\nabla_M[J^{PQNC}\nabla_C h^{TT\,M}{}_N] - \frac{1}{2}\nabla_M[J^{QPNC}\nabla_C h^{TT\,M}{}_N] \\
& +\frac{1}{4}\nabla_M[J^{CQNM}\nabla^P h^{TT}_{CN}] + \frac{1}{4}\nabla_M[J^{CPNM}\nabla^Q h^{TT}_{CN}] \\
& +\frac{1}{4}\nabla_M[J^{CMNQ}\nabla^P h^{TT}_{CN}] + \frac{1}{4}\nabla_M[J^{CMNP}\nabla^Q h^{TT}_{CN}] \\
& -\frac{1}{4}\nabla_M[J^{CPNQ}\nabla^M h^{TT}_{CN}] - \frac{1}{4}\nabla_M[J^{CQNP}\nabla^M h^{TT}_{CN}] \ .
\end{aligned} \qquad \text{(D.16)}$$

For the particular case now in which J^{MCND} is given by $h^{TT\,MN}g^{CD}$, Y^{PQ} then takes the form

$$\begin{aligned}
Y^{PQ} &= -\nabla_M[h^{TT\,P}{}_N\nabla^Q h^{TT\,MN}] - \nabla_M[h^{TT\,Q}{}_N\nabla^P h^{TT\,MN}] \\
&+ \nabla_M[h^{TT\,M}{}_N\nabla^Q h^{TT\,PN}] + \nabla_M[h^{TT\,M}{}_N\nabla^P h^{TT\,QN}] + \frac{1}{2}\nabla^Q[h^{TT\,CN}\nabla^P h^{TT}_{CN}] \\
&+ \frac{1}{2}\nabla^P[h^{TT\,CN}\nabla^Q h^{TT}_{CN}] - \frac{1}{2}g^{PQ}\nabla_M[h^{TT\,CN}\nabla^M h^{TT}_{CN}] \ .
\end{aligned} \qquad \text{(D.17)}$$

Some simplification of Eq. (D.17) can be achieved by using the $\nabla_M h^{TT\,MN}=0$, $g_{MN}h^{TT\,MN}=0$, $\nabla_C\nabla^C h^{TT}_{MN}+2b^2h^{TT}_{MN}=0$ properties that $h^{TT\,MN}$ obeys, and by noting that since interchange of orders of covariant derivatives entails that

$$\nabla_K\nabla_N h^{TT}_{LM} - \nabla_N\nabla_K h^{TT}_{LM} = h^{TT\,S}{}_M R_{LSNK} + h^{TT\,S}{}_L R_{MSNK} \qquad \text{(D.18)}$$

---

[4]We are being a little more general here than we need to be since this same analysis would apply to a variation which would involve any J^{MCND} which obeys $J^{MCND}=J^{NCMD}$. Such a family would include $J^{MCND}=(1/2)[g^{MN}h^{CD}+g^{CD}h^{MN}+(1/2)g^{MC}g^{ND}h+(1/2)g^{MD}g^{NC}h-g^{MC}h^{ND}-g^{NC}h^{MD}-g^{MN}g^{CD}h]$, a function which is found to obey $\nabla_D\nabla_C J^{MCND}=\delta R^{MN}-(1/2)g^{MN}g^{AB}\delta R_{AB}$. With the $\delta R^{MN}-(1/2)g^{MN}g^{AB}\delta R_{AB}$ combination of terms being precisely that combination which appears in ΔG_{MN} of Eq. (13.7), and with $\delta\int d^5x g^{1/2}J^{MCND}\nabla_C\nabla_D h^{TT}_{MN}/\delta g_{PQ}$ being equal to $\delta\int d^5x g^{1/2}h^{TT}_{MN}\nabla_D\nabla_C J^{MCND}/\delta g_{PQ}$, use of Eq. (D.15) would allow us to vary the general $I^{(2)}_{EH}$ without needing to make any gauge choice or restrict to any specific set of modes such as the TT ones at all.

(R_{LSNK} being given by Eq. (13.15) in the AdS_5 brane-world cases of interest), the quantity $\nabla_M[h^{TT\,P}{}_N\nabla^Q h^{TT\,MN}]$ can be re-expressed as

$$\begin{aligned}\nabla_M[h^{TT\,P}{}_N\nabla^Q h^{TT\,MN}] &= \nabla_M h^{TT\,P}{}_N\nabla^Q h^{TT\,MN} \\ &\quad - h^{TT\,P}{}_N h^{TT\,SN}R_S{}^Q + h^{TT\,P}{}_N h^{TT\,SM}R^N{}_S{}^Q{}_M \ ,\end{aligned} \tag{D.19}$$

so that Y^{PQ} can be simplified to

$$\begin{aligned}Y^{PQ} &= -\nabla_M h^{TT\,P}{}_N\nabla^Q h^{TT\,MN} - \nabla_M h^{TT\,Q}{}_N\nabla^P h^{TT\,MN} \\ &+ h^{TT\,P}{}_N h^{TT\,SN}R_S{}^Q - h^{TT\,P}{}_N h^{TT\,SM}R^N{}_S{}^Q{}_M \\ &+ h^{TT\,Q}{}_N h^{TT\,SN}R_S{}^P - h^{TT\,Q}{}_N h^{TT\,SM}R^N{}_S{}^P{}_M \\ &+ h^{TT\,M}{}_N\nabla_M\nabla^Q h^{TT\,PN} + h^{TT\,M}{}_N\nabla_M\nabla^P h^{TT\,QN} + \frac{1}{2}\nabla^Q[h^{TT\,CN}\nabla^P h^{TT}_{CN}] \\ &+ \frac{1}{2}\nabla^P[h^{TT\,CN}\nabla^Q h^{TT}_{CN}] - \frac{1}{2}g^{PQ}\nabla_M h^{TT\,CN}\nabla^M h^{TT}_{CN} + b^2 g^{PQ}h^{TT\,CN}h^{TT}_{CN} \ .\end{aligned} \tag{D.20}$$

Finally, with X^{PQ} being given by $X^{PQ} = 2(1/8\kappa_5^2)Y^{PQ} = Y^{PQ}/4\kappa_5^2$ when $J^{MCND} = h^{TT\,MN}g^{CD}$, we find that the T^{PQ} which ultimately results then takes the form given by the AdS_5 TT fluctuation mode energy-momentum tensor introduced in Eq. (16.64), viz.

$$\begin{aligned}T^{PQ} = \frac{1}{8\kappa_5^2}[&2\nabla^P h^{TT\,AB}\nabla^Q h^{TT}_{AB} + 2b^2 g^{PQ}h^{TT\,AB}h^{TT}_{AB} - g^{PQ}\nabla^S h^{TT\,AB}\nabla_S h^{TT}_{AB} \\ &-2\nabla^P h^{TT\,AB}\nabla_B h^{TT\,Q}{}_A - 2\nabla^Q h^{TT\,AB}\nabla_B h^{TT\,P}{}_A \\ &+2h^{TT\,P}{}_N h^{TT\,SN}R_S{}^Q - 2h^{TT\,P}{}_N h^{TT\,SM}R^N{}_S{}^Q{}_M \\ &+2h^{TT\,Q}{}_N h^{TT\,SN}R_S{}^P - 2h^{TT\,Q}{}_N h^{TT\,SM}R^N{}_S{}^P{}_M \\ &+2h^{TT\,B}{}_A\nabla_B\nabla^Q h^{TT\,PA} + 2h^{TT\,B}{}_A\nabla_B\nabla^P h^{TT\,QA}] \ .\end{aligned} \tag{D.21}$$

Equation (D.21) is our main result.

D.3 Conservation of the second order energy-momentum tensor

While T^{PQ} has to obey $\nabla_P T^{PQ} = 0$ since it was constructed via variation with respect to the background metric of an action which was a general coordinate scalar with respect to the same background, it is nonetheless still instructive to show the covariant conservation directly. Thus as an example for demonstration purposes we shall consider the particularly simple case in which the background is taken to be just pure AdS_5 itself, viz. a case in which we momentarily ignore the Z_2 symmetry. In the pure AdS_5 case the Riemann tensor of Eq. (13.15) simplifies to

$R_{MLNK} = -b^2(g_{MK}g_{LN} - g_{MN}g_{LK})$, Eq. (D.21) takes the form

$$\begin{aligned} T^{PQ} = & \frac{1}{8\kappa_5^2}[2\nabla^P h^{TT\,AB}\nabla^Q h^{TT}_{AB} + 2b^2 g^{PQ} h^{TT\,AB} h^{TT}_{AB} - g^{PQ}\nabla^S h^{TT\,AB}\nabla_S h^{TT}_{AB} \\ & -2\nabla^P h^{TT\,AB}\nabla_B h^{TT\,Q}{}_A - 2\nabla^Q h^{TT\,AB}\nabla_B h^{TT\,P}{}_A + 20b^2 h^{TT\,PA}\nabla_P h^{TT\,Q}{}_A \\ & +2h^{TT\,B}{}_A\nabla_B\nabla^Q h^{TT\,PA} + 2h^{TT\,B}{}_A\nabla_B\nabla^P h^{TT\,QA}] \ , \end{aligned} \tag{D.22}$$

and Eq. (D.18) reduces to

$$\begin{aligned} & \nabla_K\nabla_N h^{TT}_{LM} - \nabla_N\nabla_K h^{TT}_{LM} \\ & = -b^2 g_{LK} h^{TT}_{NM} + b^2 g_{LN} h^{TT}_{KM} - b^2 g_{MK} h^{TT}_{NL} + b^2 g_{MN} h^{TT}_{KL} \ . \end{aligned} \tag{D.23}$$

On differentiating Eq. (D.22) we obtain

$$\begin{aligned} & 8\kappa_5^2\nabla_P T^{PQ} = 2\nabla^P h^{TT\,AB}[\nabla_P\nabla^Q h^{TT}_{AB} - \nabla^Q\nabla_P h^{TT}_{AB}] \\ & + 4b^2 h^{TT\,AB}\nabla_B h^{TT\,Q}{}_A - 2\nabla_P h^{TT\,AB}[\nabla_P\nabla_B h^{TT\,Q}{}_A - \nabla_B\nabla_P h^{TT\,Q}{}_A] \\ & - 2\nabla^Q h^{TT\,AB}\nabla_P\nabla_B h^{TT\,P}{}_A + 20b^2 h^{TT\,PA}\nabla_P h^{TT\,Q}{}_A \\ & + 2h^{TT\,AB}\nabla_P\nabla_B\nabla^Q h^{TT\,P}{}_A + 2h^{TT\,AB}\nabla_P\nabla_B\nabla^P h^{TT\,Q}{}_A \ , \end{aligned} \tag{D.24}$$

with use of Eq. (D.23) allowing us to then obtain

$$\begin{aligned} 8\kappa_5^2\nabla_P T^{PQ} = & \ 26b^2 h^{TT}_{PA}\nabla^P h^{TT\,QA} + 12b^2 h^{TT}_{PA}\nabla^Q h^{TT\,PA} \\ & +2h^{TT\,AB}\nabla_P\nabla_B\nabla^Q h^{TT\,P}{}_A + 2h^{TT\,AB}\nabla_P\nabla_B\nabla^P h^{TT\,Q}{}_A \ . \end{aligned} \tag{D.25}$$

On noting that

$$\begin{aligned} & \nabla_P\nabla_K\nabla_N h^{TT}_{LM} - \nabla_K\nabla_P\nabla_N h^{TT}_{LM} \\ & = \nabla_N h^{TT\,S}{}_M R_{LSKP} + \nabla_N h^{TT\,S}{}_L R_{MSKP} + \nabla^S h^{TT}_{LM} R_{NSKP} \\ & = -b^2 g_{LP}\nabla_N h^{TT}_{KM} + b^2 g_{LK}\nabla_N h^{TT}_{PM} - b^2 g_{MP}\nabla_N h^{TT}_{KL} \\ & \quad + b^2 g_{MK}\nabla_N h^{TT}_{PL} - b^2 g_{NP}\nabla_K h^{TT}_{LM} + b^2 g_{NK}\nabla_P h^{TT}_{LM} \end{aligned} \tag{D.26}$$

in the pure AdS_5 case, we find that the quantity $2h^{TT\,AB}\nabla_P\nabla_B\nabla^Q h^{TT\,P}{}_A + 2h^{TT\,AB}\nabla_P\nabla_B\nabla^P h^{TT\,Q}{}_A$ evaluates to

$$\begin{aligned} & 2h^{TT\,AB}\nabla_P\nabla_B\nabla^Q h^{TT\,P}{}_A + 2h^{TT\,AB}\nabla_P\nabla_B\nabla^P h^{TT\,Q}{}_A \\ & = 2h^{TT\,AB}\nabla_B\nabla_P\nabla^Q h^{TT\,P}{}_A + 2h^{TT\,AB}\nabla_B\nabla_P\nabla^P h^{TT\,Q}{}_A \\ & \quad - 12b^2 h^{TT}_{PA}\nabla^P h^{TT\,QA} - 12b^2 h^{TT}_{PA}\nabla^Q h^{TT\,PA} \\ & = -26b^2 h^{TT}_{PA}\nabla^P h^{TT\,QA} - 12b^2 h^{TT}_{PA}\nabla^Q h^{TT\,PA} \ . \end{aligned} \tag{D.27}$$

We thus confirm that $\nabla_P T^{PQ}$ is indeed zero, so that the fluctuation bilinear T^{PQ} can indeed be thought of as being the energy-momentum tensor associated with a gravitational wave propagating in AdS_5.

D.4 Brane-world second order energy-momentum tensor

To extend the above analysis to the brane world, we introduce the Z_2 symmetry and return to the general metric of the form $ds^2 = dw^2 + e^{2A(|w|)}q_{\mu\nu}(x^\lambda)dx^\mu dx^\nu$ given in Eq. (13.4). For such a metric the evaluation of the TT T^{PQ} of Eq. (D.21) in the 5-dimensional axial gauge is straightforward though lengthy, and through use of Eq. (13.26) is found to yield

$$\begin{aligned} T_{\mu\nu} = \frac{1}{8\kappa_5^2}\Big[& 2g^{\alpha\delta}g^{\beta\gamma}\tilde{h}^{TT}_{\delta\gamma;\mu}\tilde{h}^{TT}_{\alpha\beta;\nu} - g_{\mu\nu}g^{\alpha\delta}g^{\beta\gamma}g^{\sigma\tau}\tilde{h}^{TT}_{\delta\gamma;\sigma}\tilde{h}^{TT}_{\alpha\beta;\tau} \\ & -g_{\mu\nu}g^{\alpha\delta}g^{\beta\gamma}(\partial_w - 2A')h^{TT}_{\delta\gamma}(\partial_w - 2A')h^{TT}_{\alpha\beta} + 2(b^2 - A'^2)g_{\mu\nu}h^{TT\,\alpha\beta}h^{TT}_{\alpha\beta} \\ & +4(A'' + 4A'^2 - 4b^2)h^{TT\,\alpha}{}_{\mu}h^{TT}_{\alpha\nu} + 4A'g^{\alpha\beta}\partial_w\left(h^{TT}_{\alpha\mu}h^{TT}_{\beta\nu}\right) \\ & -2g^{\alpha\delta}g^{\beta\gamma}(\tilde{h}^{TT}_{\delta\gamma;\mu}\tilde{h}^{TT}_{\alpha\nu;\beta} + \tilde{h}^{TT}_{\delta\gamma;\nu}\tilde{h}^{TT}_{\alpha\mu;\beta}) + 2h^{TT\,\alpha\beta}(\tilde{h}^{TT}_{\mu\alpha;\nu;\beta} + \tilde{h}^{TT}_{\nu\alpha;\mu;\beta})\Big] \quad , \end{aligned} \tag{D.28}$$

$$\begin{aligned} T_{5\mu} = \frac{1}{4\kappa_5^2}\Big[& g^{\alpha\delta}g^{\beta\gamma}\tilde{h}^{TT}_{\delta\gamma;\mu}\partial_w h^{TT}_{\alpha\beta} - g^{\alpha\delta}g^{\beta\gamma}\tilde{h}^{TT}_{\mu\alpha;\beta}\partial_w h^{TT}_{\delta\gamma} \\ & +h^{TT\,\alpha\beta}\partial_w\tilde{h}^{TT}_{\mu\alpha;\beta} - 2A'h^{TT\,\alpha\beta}\tilde{h}^{TT}_{\mu\alpha;\beta} - 2A'h^{TT\,\alpha\beta}\tilde{h}^{TT}_{\alpha\beta;\mu}\Big] \quad , \end{aligned} \tag{D.29}$$

where the tildes denote evaluation in a geometry with metric $q_{\mu\nu}$ and the primes denote derivatives with respect to w. Use of the source-free variant of Eq. (16.1) and Eq. (B.13) then enable us to rewrite $T_{\mu\nu}$ as

$$\begin{aligned} T_{\mu\nu} = \frac{1}{8\kappa_5^2}\Big[& 2g^{\alpha\delta}g^{\beta\gamma}\tilde{h}^{TT}_{\delta\gamma;\mu}\tilde{h}^{TT}_{\alpha\beta;\nu} - g_{\mu\nu}g^{\alpha\delta}g^{\beta\gamma}g^{\sigma\tau}\tilde{h}^{TT}_{\delta\gamma;\sigma}\tilde{h}^{TT}_{\alpha\beta;\tau} \\ & -g_{\mu\nu}g^{\alpha\delta}g^{\beta\gamma}\partial_w\left(h^{TT}_{\delta\gamma}\left(\partial_w - 2A'\right)h^{TT}_{\alpha\beta}\right) - g_{\mu\nu}e^{-2A}h^{TT\,\alpha\beta}\tilde{\nabla}_\sigma\tilde{\nabla}^\sigma h^{TT}_{\alpha\beta} \\ & +16\left(A'^2 - b^2\right)h^{TT\,\alpha}{}_{\mu}h^{TT}_{\alpha\nu} + 4g^{\alpha\beta}\partial_w\left(A'h^{TT}_{\alpha\mu}h^{TT}_{\beta\nu}\right) \\ & -2g^{\alpha\delta}g^{\beta\gamma}(\tilde{h}^{TT}_{\delta\gamma;\mu}\tilde{h}^{TT}_{\alpha\nu;\beta} + \tilde{h}^{TT}_{\delta\gamma;\nu}\tilde{h}^{TT}_{\alpha\mu;\beta}) + 2h^{TT\,\alpha\beta}(\tilde{h}^{TT}_{\mu\alpha;\nu;\beta} + \tilde{h}^{TT}_{\nu\alpha;\mu;\beta})\Big] \quad , \end{aligned} \tag{D.30}$$

a form which is particularly convenient for the fluctuation mode normalization analysis given in Chapter 16. Its use, for instance, in the $M_4^{\pm}$ brane-world cases where $A = \mp b|w|$, $dA/dw = \mp b\epsilon(w)$, yields a $T_{\mu\nu}$ of the form

$$\begin{aligned} T_{\mu\nu} = \frac{1}{8\kappa_5^2}\Big[& 2e^{-4A}\eta^{\alpha\delta}\eta^{\beta\gamma}\partial_\mu h^{TT}_{\delta\gamma}\partial_\nu h^{TT}_{\alpha\beta} - e^{-4A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\eta^{\sigma\tau}\partial_\sigma h^{TT}_{\delta\gamma}\partial_\tau h^{TT}_{\alpha\beta} \\ & -e^{-2A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\partial_w\left(h^{TT}_{\delta\gamma}\left(\partial_w - 2A'\right)h^{TT}_{\alpha\beta}\right) \\ & -e^{-4A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\eta^{\sigma\tau}h^{TT}_{\delta\gamma}\partial_\sigma\partial_\tau h^{TT}_{\alpha\beta} + 4e^{-2A}\eta^{\alpha\beta}\partial_w\left(A'h^{TT}_{\alpha\mu}h^{TT}_{\beta\nu}\right) \\ & -2e^{-4A}\eta^{\alpha\delta}\eta^{\beta\gamma}(\partial_\mu h^{TT}_{\delta\gamma}\partial_\beta h^{TT}_{\alpha\nu} + \partial_\nu h^{TT}_{\delta\gamma}\partial_\beta h^{TT}_{\alpha\mu}) \\ & +2e^{-4A}\eta^{\alpha\delta}\eta^{\beta\gamma}h^{TT}_{\delta\gamma}\left(\partial_\beta\partial_\nu h^{TT}_{\mu\alpha} + \partial_\beta\partial_\mu h^{TT}_{\nu\alpha}\right)\Big] \quad , \end{aligned} \tag{D.31}$$

an expression which can itself immediately be reduced to

$$
\begin{aligned}
T_{\mu\nu} = \frac{1}{8\kappa_5^2}\Big[& 2e^{-4A}\eta^{\alpha\delta}\eta^{\beta\gamma}\partial_\mu \overset{TT}{h}_{\delta\gamma}\,\partial_\nu \overset{TT}{h}_{\alpha\beta} - e^{-4A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\eta^{\sigma\tau}\partial_\sigma \overset{TT}{h}_{\delta\gamma}\,\partial_\tau \overset{TT}{h}_{\alpha\beta} \\
& - e^{-2A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\partial_w\left(\overset{TT}{h}_{\delta\gamma}\left(\partial_w - 2A'\right)\overset{TT}{h}_{\alpha\beta}\right) \\
& - e^{-4A}\eta_{\mu\nu}\eta^{\alpha\delta}\eta^{\beta\gamma}\eta^{\sigma\tau}\overset{TT}{h}_{\delta\gamma}\,\partial_\sigma\partial_\tau \overset{TT}{h}_{\alpha\beta} + 4e^{-2A}\eta^{\alpha\beta}\partial_w\left(A'\overset{TT}{h}_{\alpha\mu}\overset{TT}{h}_{\beta\nu}\right)\Big] \qquad \text{(D.32)}
\end{aligned}
$$

in any TT mode solution of the generic form $\overset{TT}{h}_{\mu\nu} = f_m(|w|)e_{\mu\nu}(p^\lambda, m)e^{ip\cdot x}$ where $(p^0)^2 - \bar{p}^2 = m^2$, $p_\alpha e^{\alpha\beta} = 0$, $e^\alpha{}_\alpha = 0$, since for such TT modes relations such as $\overset{TT}{h}{}^{\alpha\beta}\partial_\beta \overset{TT}{h}_{\mu\alpha} = \overset{TT}{h}{}^{\alpha\beta} ip_\beta \overset{TT}{h}_{\mu\alpha} = 0$ hold identically. Similarly, when evaluated in such modes, the $M_4^\pm$ brane-world $T_{5\mu}$ reduces to

$$
T_{5\mu} = \frac{e^{-4A}}{4\kappa_5^2}\eta^{\alpha\delta}\eta^{\beta\gamma}\partial_\mu \overset{TT}{h}_{\delta\gamma}\left(\partial_w - 2A'\right)\overset{TT}{h}_{\alpha\beta} \ . \qquad \text{(D.33)}
$$

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Appendix E

Retarded Propagators for Pure AdS and Pure dS Spacetimes

E.1 Retarded propagator for pure AdS spacetimes

For an n-dimensional pure AdS_n space (viz. one with no Z_2 symmetry and no brane) with metric

$$ds^2 = dx^2 + e^{2Hx}[-dt^2 + dx_2^2 + ... + dx_{n-1}^2] \quad , \tag{E.1}$$

the wave equation for a massive scalar field $\phi(x)$ is given by

$$\begin{aligned}\left[\Box_S - m^2\right]\phi(x) &= \left[\partial_x^2 + H(n-1)\partial_x \right. \\ &\quad \left. + e^{-2Hx}(-\partial_t^2 + \partial_2^2 + ... + \partial_{n-1}^2) - m^2\right]\phi(x) = 0 \quad ,\end{aligned} \tag{E.2}$$

where $\Box_S = g^{-1/2}\partial_\mu(g^{1/2}\partial^\mu)$ and m is real. The transformation $y = e^{-Hx}/H$, and the substitution $\phi(x) = y^{(n-1)/2}f(y)$ brings Eq. (E.2) to the form

$$\begin{aligned}\left[\Box_S - m^2\right]\phi(x) &= H^2 y^{(n+3)/2}\left[\partial_y^2 + \frac{1}{y}\partial_y - \frac{(n-1)^2}{4y^2}\right. \\ &\quad \left. -\partial_t^2 + \partial_2^2 + ... + \partial_{n-1}^2 - \frac{m^2}{H^2y^2}\right] f(y) = 0 \quad .\end{aligned} \tag{E.3}$$

With Eq. (E.3) being in the form of a Bessel equation, $\phi(x)$ is thus given by

$$\phi(x) = y^{(n-1)/2}e^{-ip^0t + ip^2x_2 + ... + ip^{n-1}x_{n-1}}J_\sigma(p^1y) \quad , \tag{E.4}$$

where

$$\sigma = \left(\frac{(n-1)^2}{4} + \frac{m^2}{H^2}\right)^{1/2} \quad , \quad (p^1)^2 = (p^0)^2 - (p^2)^2 - .... - (p^{n-1})^2 \quad ; \tag{E.5}$$

and with the range of x being the full $-\infty \leq x \leq \infty$, we have restricted to the $J_\sigma(p^1e^{-Hx}/H)$ Bessel function since it is the only one of the Bessel functions which is regular over the entire range. Given these basis modes and the standard $y > 0$, $y' > 0$, $Re[\sigma] > 0$ Bessel function orthonormality relation

$$\int_0^\infty d\hat{k}\hat{k}J_\sigma(\hat{k}y)J_\sigma(\hat{k}y') = \frac{1}{y}\delta(y - y') \quad , \tag{E.6}$$

the scalar propagator

$$D_S(x,x',m) = \frac{(e^{-Hx}e^{-Hx'})^{(n-1)/2}}{(2\pi)^{n-1}H}\int_{-\infty}^{\infty} dp^0 dp^2 ... dp^{n-1}\int_0^{\infty} d\hat{k}\hat{k}J_\sigma(\hat{k}e^{-Hx}/H)$$
$$\times J_\sigma(\hat{k}e^{-Hx'}/H)\frac{e^{-ip^0(t-t')+ip^2(x_2-x_2')+....+ip^{n-1}(x_{n-1}-x'_{n-1})}}{[(p^0)^2-(p^2)^2-....-(p^{n-1})^2-\hat{k}^2+i\epsilon\epsilon(p^0)]} \tag{E.7}$$

is then readily checked to obey the scalar Green's function equation

$$[\Box_S - m^2]D_S(x,x',m) = e^{-(n-1)Hx}\delta^n(x-x') \tag{E.8}$$

appropriate to the AdS_4 background.

In the non-zero mass case the propagator of Eq. (E.7) is somewhat unusual in that even though it genuinely is a bona fide solution to the massive scalar field Green's function equation, its denominator is not of the usual $(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2-m^2+i\epsilon\epsilon(p^0)$ form familiar from flat spacetime massive propagators, with the only dependence on the mass parameter being in the index σ of the Bessel functions (a phenomenon which we had encountered even earlier with the basis mode solutions given in Eq. (E.4)). Now while it is in fact possible to rewrite the propagator in a form in which the standard massive denominator does appear, it can only be done at a price. Specifically, if we define $\hat{k}$ as $\hat{k}=((p^1)^2+m^2)^{1/2}$ where p^1 is now some general parameter which does not obey Eq. (E.5), we can rewrite Eq. (E.7) as

$$D_S(x,x',m) =$$
$$\frac{(e^{-Hx}e^{-Hx'})^{(n-1)/2}}{(2\pi)^{n-1}H}\int_{-\infty}^{\infty} dp^0 dp^2 ... dp^{n-1}\left[\int_0^{\infty}+\int_{im}^{0}\right] dp^1 p^1 J_\sigma(\hat{k}e^{-Hx}/H)$$
$$\times J_\sigma(\hat{k}e^{-Hx'}/H)\frac{e^{-ip^0(t-t')+ip^2(x_2-x_2')+....+ip^{n-1}(x_{n-1}-x'_{n-1})}}{[(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2-m^2+i\epsilon\epsilon(p^0)]}, \tag{E.9}$$

but see that in so doing, we generate an unconventional range for the p^1 integration which includes a region in which p^1 is pure imaginary. While this region still yields a contribution to $D_S(x,x',m)$ which is real (the integrand and integration measure are quadratic in p^1) and, for $m \neq 0$, non-zero, we need to show that such a region will not contribute when we let the AdS_4 curvature parameter H go to zero, since we must then recover the standard flat spacetime massive scalar propagator. Since we also of course want to be assured that when we let H go to zero, we recover the standard flat spacetime massless propagator as well, we need to discuss the limit for both the cases. Since the analysis is different in the two cases, we analyze them separately.

Thus, for the massive case first, we note that in the $H \to 0$ limit the index of each Bessel function becomes infinite. In such a limit the behavior of each Bessel function will depend on the magnitude of its argument, with the relevant limits being

$$J_\sigma\left(\frac{\sigma}{\cos\beta}\right) \to \left(\frac{1}{2\pi\sigma\tan\beta}\right)^{1/2}\left(e^{i(\sigma\tan\beta-\sigma\beta-\pi/4)}+e^{-i(\sigma\tan\beta-\sigma\beta-\pi/4)}\right) \ ,$$
$$J_\sigma\left(\frac{\sigma}{\cosh\alpha}\right) \to \frac{e^{\sigma\tanh\alpha-\sigma\alpha}}{(2\pi\sigma\tanh\alpha)^{1/2}} \ , \tag{E.10}$$

dependent on whether the argument divided by σ is smaller or greater than one. With the arguments being of the form $\hat{k}e^{-Hx}/H$ and $\hat{k}e^{-Hx'}/H$, and with σ being given by Eq. (E.5), to leading order in small H we can write the arguments as $\sigma\hat{k}/m$. Thus, when $\hat{k}$ is greater than m we set $\cos\beta = m/\hat{k}$ and use the first expansion in Eq. (E.10), and when $\hat{k}$ is less than m we set $\cosh\alpha = m/\hat{k}$ and use the second expansion in Eq. (E.10). With the division of the $\hat{k}$ integration range into regions with $\hat{k} > m$ and $\hat{k} < m$ also being a division of the $p^1 = (\hat{k}^2 - m^2)^{1/2}$ integration range into regions where p^1 is purely real or purely imaginary, we see that this division precisely parallels that given in Eq. (E.10). With it being the second of the expansions in Eq. (E.10) which corresponds to the pure imaginary range for p^1, and with α (a quantity which is defined as being positive in Eq. (E.10)) always being greater than $\tanh\alpha$, we see that in this region the $e^{-\sigma\alpha}$ factor then causes the large σ limit of $J_\sigma(\sigma\hat{k}/m)$ to vanish, with the pure imaginary p^1 region contribution to Eq. (E.9) then being completely suppressed when $H \to 0$. For the $\hat{k} > m$ region, the small H limit of $J_\sigma(\hat{k}e^{-Hx}/H)$ is given by $(H/2\pi p^1)^{1/2}(e^{i\psi(x)} + e^{-i\psi(x)})$ where $\psi = p^1/H - (m/H)\arctan(p^1/m) - p^1x - \pi/4$; so that after discarding all terms which oscillate away, the propagator of Eq. (E.9) is then found to limit to

$$\begin{aligned}
&D_S(x,x',m) = \\
&\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} dp^0dp^2...dp^{n-1}\int_0^\infty dp^1\left(e^{-ip^1(x-x')}+e^{ip^1(x-x')}\right) \\
&\times \frac{e^{-ip^0(t-t')+ip^2(x_2-x_2')+....+ip^{n-1}(x_{n-1}-x_{n-1}')}}{[(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2-m^2+i\epsilon\epsilon(p^0)]} \\
&= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} dp^n \frac{e^{-ip^0(t-t')+ip^1(x-x')+....+ip^{n-1}(x_{n-1}-x_{n-1}')}}{[(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2-m^2+i\epsilon\epsilon(p^0)]} \ .
\end{aligned} \tag{E.11}$$

We recognize Eq. (E.11) as being none other than the standard massive flat spacetime scalar propagator just as required.

As regards the $H \to 0$ limit of the massless propagator, we note that for this case the Bessel function index is given by $\sigma = (n-1)/2$, a quantity which, unlike its massive counterpart, does not get large when $H \to 0$. Rather, in this case the relevant $H \to 0$ limit of the integrand of Eq. (E.7) is given by the large argument behavior of Bessel functions with $\sigma = (n-1)/2$ and $\hat{k} = p^1$, to yield a behavior of the form

$$\frac{1}{H}J_{(n-1)/2}\left(\frac{p^1e^{-Hx}}{H}\right)J_{(n-1)/2}\left(\frac{p^1e^{-Hx'}}{H}\right)$$
$$\to \frac{1}{2\pi p^1}\left(e^{i\psi(x)}+e^{-i\psi(x)}\right)\left(e^{i\psi(x')}+e^{-i\psi(x')}\right) \ , \qquad \text{(E.12)}$$

where $\psi(x) = p^1/H - p^1x - \pi$. With only the $(e^{-ip^1(x-x')} + e^{-ip^1(x'-x)})/2\pi p^1$ piece of the integrand then not oscillating away, the massless AdS_4 propagator does indeed become the flat spacetime scalar propagator in the limit in which $H \to 0$.

As well as the massless case, a second situation in which the σ index does not become large in the small H limit is one in which the masses themselves are of order H, a situation that we actually met in the discrete $P^2_\nu[\tanh(b|w|-\sigma)]$ sector of the AdS_4^+ brane world discussed in Chapter 19. For this case then the limit follows that of the massless propagator. In fact even in the non-small H case, for fields with masses which are small enough (i.e. which are small regardless of whether H itself is large or small) to cause the imaginary p^1 region contribution in Eq. (E.9) to be negligible, Eq. (E.9) can to a good approximation then be replaced by

$$D_S(x,x',m) =$$
$$\frac{(e^{-Hx}e^{-Hx'})^{(n-1)/2}}{(2\pi)^{n-1}H}\int_{-\infty}^{\infty}dp^0dp^2...dp^{n-1}\int_0^\infty dp^1p^1J_\sigma(\hat{k}e^{-Hx}/H)$$
$$\times J_\sigma(\hat{k}e^{-Hx'}/H)\frac{e^{-ip^0(t-t')+ip^2(x_2-x_2')+....+ip^{n-1}(x_{n-1}-x'_{n-1})}}{[(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2-m^2+i\epsilon\epsilon(p^0)]} \ , \qquad \text{(E.13)}$$

with the massless scalar field AdS_4 propagator itself simplifying to

$$D_S(x,x',m=0) =$$
$$\frac{(e^{-Hx}e^{-Hx'})^{(n-1)/2}}{(2\pi)^{n-1}H}\int_{-\infty}^{\infty}dp^0dp^2...dp^{n-1}\int_0^\infty dp^1p^1J_{(n-1)/2}(\hat{k}e^{-Hx}/H)$$
$$\times J_{(n-1)/2}(\hat{k}e^{-Hx'}/H)\frac{e^{-ip^0(t-t')+ip^2(x_2-x_2')+....+ip^{n-1}(x_{n-1}-x'_{n-1})}}{[(p^0)^2-(p^1)^2-(p^2)^2-....-(p^{n-1})^2+i\epsilon\epsilon(p^0)]} \qquad \text{(E.14)}$$

no matter how large or small H might be.

Before proceeding to an actual evaluation of the exact pure AdS_4 scalar propagator of Eq. (E.7), we need to first show that when evaluated over a retarded contour the propagator is actually causal. To this end we follow the analog discussion given in Chapter 17, and since we only need to explore the behavior in the 2-dimensional (t,x) space in order to check for causality, we introduce a source $J(x,t) = \delta(x)\delta(t)$ which only depends on these particular two coordinates. In the presence of such a source the solution to $[\Box_S - m^2]\phi(x) = J(x,t)$ is readily given as

$$\phi(x,t) = \int d^n x' e^{(n-1)Hx'} D_S(x,x',m)\delta(x')\delta(t')$$
$$= \frac{e^{-(n-1)Hx/2}}{2\pi H}\int_{-\infty}^{\infty} dp^0 \int_0^{\infty} d\hat{k}\hat{k}\frac{J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}/H)e^{-ip^0 t}}{[(p^0)^2 - \hat{k}^2]} \quad , \tag{E.15}$$

and we need to show that it only takes support in $Ht \geq 1 - e^{-Hx}$ when $x > 0$ (or in $Ht \geq e^{-Hx} - 1$ when $x < 0$). On closing the contour in the upper half p^0 plane where the retarded contour possesses no singularities, Eq. (E.15) is given as the negative of the same integral as evaluated counter-clockwise along a half circle with large radius P. Thus, on setting $p^0 = Pe^{i\theta}$ where $0 < \theta < \pi$, we obtain

$$\phi(x,t) = -\frac{e^{-(n-1)Hx/2}iP}{2\pi H}\int_0^{\pi} d\theta \int_0^{\infty} d\hat{k}\frac{e^{i\theta}\hat{k}J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}/H)e^{-iPe^{i\theta}t}}{[P^2e^{2i\theta} - \hat{k}^2]} \quad . \tag{E.16}$$

Recalling the known integral

$$\int_0^{\infty} d\hat{k}\frac{\hat{k}J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}/H)}{[\hat{k}^2 + \mu^2]}$$
$$= \theta(x)I_\sigma(\mu e^{-Hx}/H)K_\sigma(\mu/H) + \theta(-x)I_\sigma(\mu/H)K_\sigma(\mu e^{-Hx}/H) \tag{E.17}$$

which holds in $\mathrm{Re}[\mu] > 0$, $\mathrm{Re}[\sigma] > -1$, on setting $\mu = -iPe^{i\theta}$ (so that $\mathrm{Re}[\mu] > 0$ along the circle in the upper half p^0 plane), we obtain for $x > 0$

$$\phi(x > 0, t) = \frac{e^{-(n-1)Hx/2}iP}{2\pi H}$$
$$\times \int_0^{\pi} d\theta e^{i\theta} I_\sigma(-iPe^{i\theta}e^{-Hx}/H)K_\sigma(-iPe^{i\theta}/H)e^{-iPe^{i\theta}t} \quad . \tag{E.18}$$

Then, with the large $|z|$ asymptotic behavior

$$I_\sigma(z) \to \left(\frac{1}{2\pi z}\right)^{1/2}(e^z + e^{-z+i\pi(\sigma+1/2)})\left[1 + O\left(\frac{1}{z}\right)\right] \quad ,$$
$$K_\sigma(z) \to \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right] \tag{E.19}$$

(the large $|z|$ limit for $I_\sigma(z)$ is valid for $-\pi/2 < \arg z < 3\pi/2$), in the $0 < \theta < \pi$ region we obtain the leading term

$$\phi(x > 0, t) = -\frac{e^{-(n-2)Hx/2}}{4\pi}$$
$$\times \int_0^{\pi} d\theta [e^{-iPe^{i\theta}e^{-Hx}/H} + e^{i\pi(\sigma+1/2)}e^{iPe^{i\theta}e^{-Hx}/H}]e^{+iPe^{i\theta}/H}e^{-iPe^{i\theta}t} \quad . \tag{E.20}$$

To evaluate Eq. (E.20) and its non-leading terms we need to evaluate integrals of the form

$$f(\alpha) = \int_0^{\pi} d\theta \exp(-iP\alpha e^{i\theta}) \quad , \quad g(\alpha) = \frac{i}{P}\int_0^{\pi} d\theta e^{-i\theta}\exp(-iP\alpha e^{i\theta}) \quad ,$$
$$h(\alpha) = -\frac{1}{P^2}\int_0^{\pi} d\theta e^{-2i\theta}\exp(-iP\alpha e^{i\theta}) \quad . \tag{E.21}$$

To evaluate $f(\alpha)$ we note that its derivative readily evaluates to

$$f'(\alpha) = \frac{2\sin(P\alpha)}{\alpha} \quad . \tag{E.22}$$

With the $P \to \infty$ limit of $\sin(P\alpha)/\alpha$ being given by

$$\frac{\sin(P\alpha)}{\alpha} \to \pi\delta(\alpha) \quad , \tag{E.23}$$

and with $f(0) = \pi$, $f(\alpha)$ is thus given by

$$f(\alpha) = \pi + 2\pi\int_0^{\alpha} d\alpha\delta(\alpha) = \pi + 2\pi[\theta(\alpha) - \theta(0)] \quad , \tag{E.24}$$

which, with $\theta(0) = 1/2$, yields

$$f(\alpha) = 2\pi\theta(\alpha) \quad . \tag{E.25}$$

As regards $g(\alpha)$, we note that its derivative with respect to α is given by

$$g'(\alpha) = \int_0^{\pi} d\theta \exp(-iP\alpha e^{i\theta}) = 2\pi\theta(\alpha) \quad , \tag{E.26}$$

and with $g(0)$ being given by $g(0) = 2/P \to 0$, $g(\alpha)$ itself is thus given by

$$g(\alpha) = 2\pi\alpha\theta(\alpha) \quad . \tag{E.27}$$

The same generic structure continues to hold for the subsequent terms in the expansion, with the next one behaving as $h(\alpha) = \pi\alpha^2\theta(\alpha)$, and so on.

Given these integrals, Eq. (E.20) now readily evaluates to

$$\phi(x > 0, t) = -\frac{e^{-(n-2)Hx/2}}{4\pi}$$
$$\times \left[\theta(t - 1/H + e^{-Hx}/H) + e^{i\pi(\sigma+1/2)}\theta(t - 1/H - e^{-Hx}/H)\right] \quad , \tag{E.28}$$

together with non-leading terms that behave as $(t - 1/H + e^{-Hx}/H)\theta(t - 1/H + e^{-Hx}/H)$ and so on. With both step functions in Eq. (E.28) only taking support on or inside the $x > 0$ AdS_n lightcone, and with such support regions requiring t to be positive when x is positive, the above propagator is thus seen to be both causal and retarded, just as desired.

To evaluate the contribution along the half circle in the lower half p^0 plane we set $\mu = iPe^{i\theta}$ (so that $\mathrm{Re}[\mu] > 0$ along the circle in the lower half p^0 plane), with the analog of Eq. (E.18) then taking the form

$$\begin{aligned}\phi(x>0,t) &= \frac{e^{-(n-1)Hx/2}iP}{2\pi H} \\ &\times \int_{2\pi}^{\pi} d\theta e^{i\theta} I_\sigma(iPe^{i\theta}e^{-Hx}/H)K_\sigma(iPe^{i\theta}/H)e^{-iPe^{i\theta}t} \qquad \text{(E.29)}\end{aligned}$$

along the lower half circle. The transformation $\theta \to \theta - \pi$ brings Eq. (E.29) to the form

$$\begin{aligned}\phi(x>0,t) &= \frac{e^{-(n-1)Hx/2}iP}{2\pi H} \\ &\times \int_{0}^{\pi} d\theta e^{i\theta} I_\sigma(-iPe^{i\theta}e^{-Hx}/H)K_\sigma(-iPe^{i\theta}/H)e^{iPe^{i\theta}t} \quad , \text{(E.30)}\end{aligned}$$

an expression which immediately evaluates to

$$\begin{aligned}\phi(x>0,t) &= \frac{e^{-(n-2)Hx/2}}{4\pi} \\ &\times \left[\theta(-t-1/H+e^{-Hx}/H)+e^{i\pi(\sigma+1/2)}\theta(-t-1/H-e^{-Hx}/H)\right] \quad . \text{(E.31)}\end{aligned}$$

With neither of the two step functions in Eq. (E.31) having any support at all in the $t>0$, $x>0$ region, the AdS_n causal propagator can thus be determined entirely from the contribution of its lower half p^0 plane singularities.

To explicitly evaluate this singular contribution (something we can actually carry through analytically for any source), we note first that with the n-dimensional metric of Eq. (E.1) having an induced metric along any fixed x slice which is an $(n-1)$-dimensional Minkowski metric, we can evaluate the propagator by setting all of the induced metric coordinates $x_i - x_i'$ equal to zero except for $t-t'$, and then use the $(n-1)$-dimensional covariance to restore the dependence on these coordinates by replacing $(t-t')$ by $[(t-t')^2-(x_2-x_2')^2-...-(x_{n-1}-x_{n-1}')^2]^{1/2}$ in the expression which ultimately results. With this simplification, and with the integrand of Eq. (E.7) only possessing poles in the lower half p^0 plane, the singular contribution is given by

$$\begin{aligned}D_S(x,x',m) &= -\frac{(e^{-Hx}e^{-Hx'})^{(n-1)/2}}{(2\pi)^{n-2}H}\int_{-\infty}^{\infty}dp^2...dp^{n-1}\int_0^{\infty}d\hat{k}\hat{k}\frac{\sin E_p(t-t')}{E_p} \\ &\times J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}e^{-Hx'}/H) \quad , \qquad \text{(E.32)}\end{aligned}$$

where $E_p = [\hat{k}^2+(p^2)^2+..+(p^{n-1})^2]^{1/2}$. With Eq. (E.32) holding for general n, for the massless $n=5$ case σ then takes the value $\sigma=2$, in complete generic accord with structure of the singular part of the M_4^+ brane-world propagator given in Eq. (17.25), with comparison of Eqs. (17.25) and (E.32) revealing the difference between pure AdS_5 and brane-world AdS_5.

The general $dp^2...dp^{n-1}$ integration in Eq. (E.32) is readily performed and, on setting $(p^2)^2 + (p^3)^2 = s^2$, yields in the illustrative $n = 4$ case

$$
\begin{aligned}
&D_S(x, x', m) \\
&= -\frac{(e^{-Hx}e^{-Hx'})^{3/2}}{2H\pi}\int_0^\infty d\hat{k}\hat{k}\int_{\hat{k}^2}^\infty ds \sin s(t-t')J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}e^{-Hx'}/H) \\
&= -\frac{(e^{-Hx}e^{-Hx'})^{3/2}}{2H\pi(t-t')}\int_0^\infty d\hat{k}\hat{k}\cos\hat{k}(t-t')J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}e^{-Hx'}/H) \\
&= -\frac{(e^{-Hx}e^{-Hx'})^{3/2}}{2H\pi(t-t')}\frac{d}{dt}\int_0^\infty d\hat{k}\sin\hat{k}(t-t')J_\sigma(\hat{k}e^{-Hx}/H)J_\sigma(\hat{k}e^{-Hx'}/H) \\
&\equiv -\frac{(e^{-Hx}e^{-Hx'})^{3/2}}{2H\pi(t-t')}\frac{d}{dt}I(t-t', x, x') \ .
\end{aligned}
\tag{E.33}
$$

Now the $I(t - t', x, x')$ integral in Eq. (E.33) is known in completely closed form, being given (for $e^{-Hx'} > e^{-Hx}$ for definitiveness) by

$$
I(t-t', x, x') = 0 \ ; \ \ e^{-Hx'} - e^{-Hx} > H(t-t') > 0 \ , \tag{E.34}
$$

$$
\begin{aligned}
I(t-t', x, x') &= \frac{H}{2e^{-Hx/2}e^{-Hx'/2}}P_{\sigma-1/2}(z) \ ; \\
&\quad e^{-Hx'} + e^{-Hx} > H(t-t') > e^{-Hx'} - e^{-Hx} \ ,
\end{aligned}
\tag{E.35}
$$

$$
\begin{aligned}
I(t-t', x, x') &= -\frac{H\cos\sigma\pi}{\pi e^{-Hx/2}e^{-Hx'/2}}Q_{\sigma-1/2}(-z) \ ; \\
&\quad H(t-t') > e^{-Hx'} + e^{-Hx} \ ,
\end{aligned}
\tag{E.36}
$$

where

$$
z = \frac{[e^{-2Hx} + e^{-2Hx'} - H^2(t-t')^2]}{2e^{-Hx}e^{-Hx'}} \ , \tag{E.37}
$$

with the switch between the $P_{\sigma-1/2}(z)$ and $Q_{\sigma-1/2}(z)$ associated Legendre functions occurring at the branch point they possess at argument $z = -1$. As we see from Eq. (E.34), $I(t - t', x, x')$ vanishes outside the AdS_4 lightcone where $e^{-Hx'} - e^{-Hx} > H(t-t')$, and with it thus behaving as a theta function of $H(t-t') - (e^{-Hx'} - e^{-Hx})$, the time derivative needed for Eq. (E.33) thus behaves as a theta function plus a delta function of the same argument, with the propagator thereby not taking support outside the AdS_4 lightcone. As constructed then, the retarded propagator of Eq. (E.33) is again confirmed to be completely causal, and when used in a pure AdS_4 spacetime would lead to propagation of an initial disturbance within its Cauchy development alone. The construction of the causal propagator given here thus serves to complement the treatment given in [Avis, Isham and Storey (1978)] of propagation in a globally non-hyperbolic AdS_4 spacetime.

E.2 Retarded propagator for pure dS spacetimes

An analysis similar to the above allows us to construct the pure dS_4 retarded propagator as well. Here the metric is given by

$$ds^2 = -dt^2 + e^{2Ht}[dx^2 + dy^2 + dz^2] \quad , \tag{E.38}$$

with the wave equation for a massive scalar field $\phi(x)$ being given by

$$\begin{aligned}\left[\Box_S - m^2\right]\phi(x) &= [-\partial_t^2 - 3H\partial_t \\ &+ e^{-2Ht}(\partial_x^2 + \partial_y^2 + \partial_z^2) - m^2]\phi(x) = 0 \quad .\end{aligned} \tag{E.39}$$

The transformation $y = e^{-Ht}/H$, and the substitution $\phi(x) = y^{3/2}f(y)$ brings Eq. (E.39) to the form

$$\begin{aligned}\left[\Box_S - m^2\right]\phi(x) &= H^2 y^{7/2}\left[-\partial_y^2 - \frac{1}{y}\partial_y + \frac{9}{4y^2}\right. \\ &\left. +\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{m^2}{H^2y^2}\right] f(y) = 0 \quad .\end{aligned} \tag{E.40}$$

With Eq. (E.40) being in the form of a Bessel equation, $\phi(x)$ is thus given by

$$\phi(x) = y^{3/2}e^{ip^1x + ip^2y + ip^3z}Z_\sigma(p^0y) \quad , \tag{E.41}$$

where

$$\sigma = \left(\frac{9}{4} - \frac{m^2}{H^2}\right)^{1/2} \quad , \quad (p^0)^2 = (p^1)^2 + (p^2)^2 + (p^3)^2 \quad , \tag{E.42}$$

and where $Z_\sigma(p^0e^{-Ht}/H)$ is an appropriate Bessel function which will be chosen below. (Since t will correspond to the observation time of the propagator, Bessel functions which only diverge at the later time $t \to \infty$ will not be forbidden from appearing in it.)

Noting that the identity

$$\begin{aligned}&(k^2 - q^2)zZ_\sigma(kz)\hat{Z}_\sigma(qz) \\ &= \frac{d}{dz}\left(zZ_\sigma(kz)\frac{d\hat{Z}_\sigma(qz)}{dz} - z\frac{dZ_\sigma(kz)}{dz}\hat{Z}_\sigma(qz)\right)\end{aligned} \tag{E.43}$$

holds for any pair of Bessel functions, for a particular combination of Hankel functions (viz. precisely the combination which will turn out below to be the one needed for causality in the dS_4 case) we would immediately be able to set

$$\begin{aligned}&\frac{1}{4}\int_{-\infty}^{\infty} d\hat{k}\hat{k}H_\sigma^{(1)}(\hat{k}y)H_\sigma^{(2)}(\hat{k}y') \\ &= \mathrm{Lim}_{\Lambda\to\infty}\left(\frac{1}{\pi(yy')^{1/2}}\frac{\sin\Lambda(y - y')}{(y - y')}\right) = \frac{1}{y}\delta(y - y')\end{aligned} \tag{E.44}$$

if there were no singularities within the integration region. However, unlike the analogous $J_\sigma(\hat{k}y)$ functions which were used in the AdS_4 case, the dS_4 Hankel functions diverge at zero argument. To address this concern we consider the special case of the massless mode, viz. one for which $\sigma = 3/2$. In this particular case the Hankel functions are known in closed form, and yield

$$\begin{aligned}
&\frac{1}{4}\int_{-\infty}^{\infty} d\hat{k}\hat{k}H^{(1)}_{3/2}(\hat{k}y)H^{(2)}_{3/2}(\hat{k}y') \\
&= \frac{1}{2\pi(yy')^{3/2}}\int_{-\infty}^{\infty} d\hat{k}\left[yy' + i\frac{(y'-y)}{\hat{k}} + \frac{1}{\hat{k}^2}\right] e^{-i(y'-y)\hat{k}} \ . \qquad \text{(E.45)}
\end{aligned}$$

On giving a meaning to the singularity at $\hat{k} = 0$ by treating the integral as a contour integral in the complex $\hat{k}$ plane (which it would anyway have to be once we set $\hat{k} = ((p^0)^2 - m^2)^{1/2}$ below), we obtain

$$\begin{aligned}
&\frac{1}{4}\int_{-\infty}^{\infty} d\hat{k}\hat{k}H^{(1)}_{3/2}(\hat{k}y)H^{(2)}_{3/2}(\hat{k}y') \\
&= \frac{1}{(yy')^{3/2}}\left[yy'\delta(y'-y) + (y'-y)\theta(y'-y) - (y'-y)\theta(y'-y)\right] \\
&= \frac{1}{y}\delta(y-y') \qquad \text{(E.46)}
\end{aligned}$$

just as needed. With a similar pattern being found for the other half-integral σ combinations [for $\sigma = 1/2$ for instance we have $(1/4)\int_{-\infty}^{\infty} d\hat{k}\hat{k}H^{(1)}_{1/2}(\hat{k}y)H^{(2)}_{1/2}(\hat{k}y') = (1/2\pi)\int_{-\infty}^{\infty} d\hat{k}e^{i\hat{k}(y-y')}/(yy')^{1/2} = \delta(y-y')/y$], and with the $H^{(1)}_\sigma(\hat{k}y)H^{(2)}_\sigma(\hat{k}y')$ integrand not actually diverging at $\hat{k} = 0$ at all in the pure imaginary σ region associated with masses which obey $m^2 > 9H^2/4$ (cases where the Hankel functions then behave as $z^{\pm i(m^2/H^2-9/4)^{1/2}}$ at small z), when understood as being defined as a contour integral, we can thus establish the validity of the orthonormality relations given in Eq. (E.44).

Given these relations, the scalar propagator

$$\begin{aligned}
D_S(x,x',m) &= \frac{(e^{-Ht}e^{-Ht'})^{3/2}}{4(2\pi)^3H}\int_{-\infty}^{\infty} d^3p\int_{-\infty}^{\infty} d\hat{k}\hat{k}H^{(1)}_\sigma\left(\frac{\hat{k}e^{-Ht}}{H}\right)H^{(2)}_\sigma\left(\frac{\hat{k}e^{-Ht'}}{H}\right) \\
&\times \frac{e^{ip^1(x-x')+ip^2(y-y')+ip^3(z-z')}}{[\hat{k}^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 + i\epsilon\epsilon(p^0)]} \qquad \text{(E.47)}
\end{aligned}$$

(as defined for a general integration variable $\hat{k} = ((p^0)^2 - m^2)^{1/2}$ where p^0 does not obey Eq. (E.42)) is then readily checked to obey

$$\left[\Box_S - m^2\right] D_S(x,x',m) = e^{-3Ht}\delta^4(x-x') \qquad \text{(E.48)}$$

exactly for arbitrary m and H, large or small, with the dependence of the propagator on the mass parameter again being only in the index of the Bessel functions.

With the Hankel functions behaving asymptotically as

$$H^{(1)}_{\sigma}(z) \to \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\pi\sigma/2-\pi/4)}\left[1-\frac{(4\sigma^2-1)}{8iz}+O\left(\frac{1}{z^2}\right)\right] \ ,$$
$$H^{(2)}_{\sigma}(z') \to \left(\frac{2}{\pi z'}\right)^{1/2} e^{-i(z'-\pi\sigma/2-\pi/4)}\left[1+\frac{(4\sigma^2-1)}{8iz'}+O\left(\frac{1}{z'^2}\right)\right] \quad \text{(E.49)}$$

as $|z|\to\infty$, $|z'|\to\infty$, the Hankel function combination we have specifically chosen has leading behavior

$$H^{(1)}_{\sigma}\left(\frac{\hat{k}e^{-Ht}}{H}\right)H^{(2)}_{\sigma}\left(\frac{\hat{k}e^{-Ht'}}{H}\right) \to \frac{2He^{Ht/2}e^{Ht'/2}}{\pi\hat{k}}$$
$$\times \exp\left(\frac{i\hat{k}(e^{-Ht}-e^{-Ht'})}{H}\right)\left[1-\frac{(4\sigma^2-1)H}{8i\hat{k}}(e^{Ht}-e^{Ht'})\right] \quad \text{(E.50)}$$

as $\hat{k}_I \to \pm\infty$. Consequently, this particular combination is precisely the one which for $t > t'$ yields an exponentially falling time dependence along a large half circle in the lower half $\hat{k}$ plane, viz. just the behavior one would want for a retarded propagator.

While the propagator of Eq. (E.47) is thus a good potential candidate for the pure dS_4 retarded propagator we seek, to test for this explicitly by evaluating it as a contour integral is not totally straightforward since as well as contain poles in the lower half $\hat{k}$ plane, there are also branch cuts at $\hat{k} = 0$ due to the multiple-valuedness of the Hankel functions. To bypass this difficulty, we instead close the contour in the upper half $\hat{k}$ plane where the integrand possesses no singularities; with the Eq. (E.47) integral then being given by the negative of the same integral as evaluated counter-clockwise around a large half circle in the upper half plane. Since the integrand of Eq. (E.47) has no poles in the upper half $\hat{k}$ plane, the d^3p integration can be performed directly. The relevant part of the integration over the direction of $\bar{p}$ yields

$$\int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{[\hat{k}^2-\bar{p}^2]} = -\frac{4\pi}{r}\int_0^{\infty} dp \frac{p\sin(pr)}{[\bar{p}^2-\hat{k}^2]} \quad \text{(E.51)}$$

where $r = |\bar{x}|$; and on defining a quantity $\mu = -i\hat{k} = -i\hat{k}_R + \hat{k}_I$ whose real part is positive in the upper half $\hat{k}$ plane, the integration over the magnitude $p = |\bar{p}|$ is then found to yield the familiar converging Yukawa potential form

$$\int d^3p \frac{e^{i\bar{p}\cdot\bar{x}}}{[\hat{k}^2-\bar{p}^2]} = -\frac{4\pi}{r}\int_0^{\infty} dp \frac{p\sin(pr)}{[\bar{p}^2+\mu^2]} = -\frac{2\pi^2}{r}e^{-\mu r} \quad \text{(E.52)}$$

associated with a particle whose mass obeys $\mathrm{Re}[\mu] > 0$. Along a large circle of radius P in the upper half $\hat{k}$ plane we can set $\hat{k} = Pe^{i\theta}$ where $0 \le \theta \le \pi$, with the

dS_4 scalar propagator thus being given by

$$D_S(x,x',m)=\frac{(e^{-Ht}e^{-Ht'})^{3/2}P^2}{16\pi Hr}$$
$$\times\int_0^{\pi}d\theta ie^{2i\theta}H_\sigma^{(1)}\left(\frac{Pe^{i\theta}e^{-Ht}}{H}\right)H_\sigma^{(2)}\left(\frac{Pe^{i\theta}e^{-Ht'}}{H}\right)\exp\left(iPe^{i\theta}|\bar{x}-\bar{x}'|\right) \quad (E.53)$$

as evaluated at $P=\infty$. From Eq. (E.50) the leading contribution to $D_S(x,x',m)$ is thus given by

$$\begin{aligned}&D_S(x,x',m)\\&=\frac{e^{-Ht}e^{-Ht'}P}{8\pi^2 r}\int_0^{\pi}d\theta ie^{i\theta}\exp\left(-iP\alpha e^{i\theta}\right)\left(1-\frac{(2H^2-m^2)}{2iPe^{i\theta}H}(e^{Ht}-e^{Ht'})\right)\\&=-\frac{e^{-Ht}e^{-Ht'}}{4\pi^2 r}\frac{\sin(P\alpha)}{\alpha}+\frac{(m^2-2H^2)}{16\pi^2 rH}\left(e^{-Ht'}-e^{-Ht}\right)f(\alpha)\ ,\end{aligned} \quad (E.54)$$

where

$$\alpha=\frac{e^{-Ht'}}{H}-\frac{e^{-Ht}}{H}-|\bar{x}-\bar{x}'|\ , \quad (E.55)$$

and where $f(\alpha)$ is given in Eq. (E.21). Thus given Eq. (E.25), the leading contribution to the dS_4 scalar propagator immediately evaluates to

$$D_S(x,x',m)=-\frac{e^{-Ht}e^{-Ht'}}{4\pi r}\delta(\alpha)+\frac{(m^2-2H^2)}{8\pi rH}\left(e^{-Ht'}-e^{-Ht}\right)\theta(\alpha)\ . \quad (E.56)$$

Similarly, the next leading terms in Eq. (E.56) as obtained from the expansion of the Hankel functions given in Eq. (E.49) will be proportional to $g(\alpha)=2\pi\alpha\theta(\alpha)$ and $h(\alpha)=\pi\alpha^2\theta(\alpha)$, and so on. With the dS_4 radial timelike and null geodesics which obey $dt^2\geq e^{2Ht}dr^2$ being given by $\alpha=e^{-Ht'}/H-e^{-Ht}/H-|\bar{x}-\bar{x}'|\geq 0$, we thus see that the dS_4 scalar propagator of Eq. (E.56) is indeed retarded,[1] and that it does not take support outside the dS_4 lightcone. It is thus the retarded, causal dS_4 propagator we seek.[2]

[1]The positivity of $|\bar{x}-\bar{x}'|$ and the constraints on α imposed by the $\delta(\alpha)$ and $\theta(\alpha)$ terms entail that t cannot be earlier than t'.

[2]As a check on our calculation we note that when we set both m and H to zero in Eq. (E.56), we immediately recover the M_4 free massless retarded scalar propagator given in Eq. (17.36) as $D_{\rm RET}(x,m=0)=-\delta(t-|\bar{x}|)/4\pi|\bar{x}|=-\theta(t)\delta(t-|\bar{x}|)/4\pi|\bar{x}|$, with the upper half $\hat{k}$ plane calculation given here thus paralleling the upper half p^0 plane one given for $D_{\rm RET}(x,m=0)$ in Chapter 17.

E.3 Retarded propagator for pure M_4 spacetime

A straightforward way to evaluate the retarded propagator of pure M_4, viz.

$$D(x,m;\mathrm{RET}) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4p \frac{e^{ip\cdot x}}{[(p^0)^2 - p^2 - m^2 + i\epsilon\epsilon(p^0)]} \ , \tag{E.57}$$

where $p^2 = (p^1)^2 + (p^2)^2 + (p^3)^2$, is to first evaluate the singular part associated with closing the p^0 contour below the real p^0 axis. Poles appear at $p^0 = \pm(p^2 + m^2)^{1/2}$, and their residues generate the singular contribution

$$\begin{aligned} D(x,m;\mathrm{SING}) &= -\frac{1}{2\pi^2 r}\int_0^\infty \frac{pdp}{(p^2+m^2)^{1/2}}\sin(pr)\sin\left(t(p^2+m^2)^{1/2}\right) \\ &= \frac{1}{2\pi^2 r}\frac{\partial}{\partial r}\int_0^\infty \frac{dp}{(p^2+m^2)^{1/2}}\cos(pr)\sin\left(t(p^2+m^2)^{1/2}\right) \ . \end{aligned} \tag{E.58}$$

With the needed integral being given in the $t > 0$ and $t < 0$ cases by

$$\begin{aligned} &\int_0^\infty \frac{dp}{(p^2+m^2)^{1/2}}\cos(pr)\sin\left(t(p^2+m^2)^{1/2}\right) = \frac{\pi}{2}\theta(t-r)J_0\left(m(t^2-r^2)^{1/2}\right) \ , \\ &\int_0^\infty \frac{dp}{(p^2+m^2)^{1/2}}\cos(pr)\sin\left(t(p^2+m^2)^{1/2}\right) = -\frac{\pi}{2}\theta(-t-r)J_0\left(m(t^2-r^2)^{1/2}\right) \ , \end{aligned} \tag{E.59}$$

on setting $\delta(-t-r) = \delta(t+r)$, Eq. (E.58) immediately evaluates to

$$\begin{aligned} D(x,m;\mathrm{SING}) &= -\frac{1}{4\pi r}\delta(t-r) + \frac{m}{4\pi(t^2-r^2)^{1/2}}\theta(t-r)J_1\left(m(t^2-r^2)^{1/2}\right) \\ &+ \frac{1}{4\pi r}\delta(t+r) - \frac{m}{4\pi(t^2-r^2)^{1/2}}\theta(-t-r)J_1\left(m(t^2-r^2)^{1/2}\right) \ . \end{aligned} \tag{E.60}$$

With $D(x,m;\mathrm{RET})$ being given in terms of lower half p^0 plane singular and circle at infinity contributions as (in the notation of Chapter 17) $D(x,m;\mathrm{SING}) = D(x,m;\mathrm{RET}) + D(x,m;\mathrm{LHP}) = -D(x,m;\mathrm{UHP}) + D(x,m;\mathrm{LHP})$, with (for completeness) the advanced M_4 scalar propagator $D(x,m;\mathrm{ADV})$ being given by the lower half p^0 plane circle at infinity contribution alone, and with both propagators having to limit to their $m = 0$ counterparts given in Chapter 17, we can thus identify $D(x,m;\mathrm{RET})$ and $D(x,m;\mathrm{ADV})$ as

$$\begin{aligned} D(x,m;\mathrm{RET}) &= -D(x,m;\mathrm{UHP}) \\ &= -\frac{1}{4\pi r}\delta(t-r) + \frac{m}{4\pi(t^2-r^2)^{1/2}}\theta(t-r)J_1\left(m(t^2-r^2)^{1/2}\right) \ , \\ D(x,m;\mathrm{ADV}) &= -D(x,m;\mathrm{LHP}) \\ &= -\frac{1}{4\pi r}\delta(t+r) + \frac{m}{4\pi(t^2-r^2)^{1/2}}\theta(-t-r)J_1\left(m(t^2-r^2)^{1/2}\right) \ . \end{aligned} \tag{E.61}$$

Appendix F

Embedding $M(2,1)$ Branes in AdS_4

F.1 M_3^+ and M_3^- brane-world background geometries

In order to provide additional insight into the causal structure of the M_4^+ and M_4^- brane worlds described in Chapters 17 and 18, it is very instructive to consider brane worlds analogous to them with one lower spatial dimension, since for such cases it turns out that the mode wave functions can then be expressed entirely in terms of elementary functions. We thus consider the embedding of $M(2,1)$ branes of positive or negative tension in AdS_4, viz. the M_3^+ and M_3^- brane worlds described by the generic metric

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= dw^2 + e^{2A(w)}\eta_{ij}dx^i dx^j \\ &= dw^2 + e^{2A(w)}(dx^2 + dy^2 - dt^2) \quad , \end{aligned} \tag{F.1}$$

where we use (μ, ν) to denote the four-dimensional coordinates (w, x, y, t), and (i, j) to denote the three-dimensional $M(2,1)$ coordinates (x, y, t). For this set-up the Einstein equations which determine $A(w)$ are given by

$$G_{\mu\nu} = -\kappa_4^2 T_{\mu\nu} \tag{F.2}$$

with energy-momentum tensor

$$T_{\mu\nu} = -\Lambda_4 g_{\mu\nu} - \lambda\eta_{ij}\delta(w)\delta^i_\mu\delta^j_\nu \quad . \tag{F.3}$$

For the metric of Eq. (F.1) the non-vanishing components of the Riemann and Ricci tensors are readily found to evaluate to (primes denote derivatives with respect to w)

$$\begin{aligned} &R^w{}_{iwj} = [A'' + A'^2]g_{ij} \quad , \quad R^\ell{}_{ijk} = A'^2[\delta^\ell_j g_{ik} - \delta^\ell_k g_{ij}] \quad , \\ &R_{ij} = [A'' + 3A'^2]g_{ij} \quad , \quad R_{ww} = 3[A'' + A'^2] \quad . \end{aligned} \tag{F.4}$$

With the background Einstein equations taking the form

$$\begin{aligned} G_{ij} &= -[2A'' + 3A'^2]g_{ij} = \kappa_4^2\Lambda_4 g_{ij} + \kappa_4^2\lambda\delta(w)\eta_{ij} \quad , \\ G_{ww} &= -3A'^2 = \kappa_4^2\Lambda_4 \quad , \end{aligned} \tag{F.5}$$

the metric coefficient $A(w)$ is thus given by

$$A(w) = -\frac{\kappa_4^2\lambda|w|}{4} \quad . \tag{F.6}$$

Thus, on defining

$$H = \frac{\kappa_4^2|\lambda|}{4} \quad , \tag{F.7}$$

we obtain

$$A = -\epsilon(\lambda)H|w| \quad , \quad \kappa_4^2\Lambda_4 = -3H^2 \quad , \tag{F.8}$$

with the various cosmological constants thus being related by

$$16\Lambda_4 + 3\kappa_4^2\lambda^2 = 0 \quad , \tag{F.9}$$

a relation which is in complete analog to the M_4 brane-world fine-tuning relation given in Eq. (2.15). The above equations thus describe the $M_3^{\pm}$ brane-world backgrounds.

F.2 Fluctuations in the M_3^+ and M_3^- brane worlds

For the M_3^+ and M_3^- brane-world fluctuations, we note that in the axial gauge where $h_{wi} = 0$, the analog of Eq. (13.27) is given as

$$\begin{aligned} \delta R_{ww} &= \frac{1}{2}h'' + A'h' \quad , \quad \delta R_{wi} = \frac{1}{2}\partial_w[h_{;i} - h^j{}_{i;j}] \quad , \\ \delta R_{ij} &= \frac{1}{2}[h''_{ij} - A'h'_{ij} + 4A'^2h_{ij} + A'g_{ij}h'] + e^{-2A}\widetilde{\delta R}_{ij} \quad , \end{aligned} \tag{F.10}$$

where $\widetilde{\delta R}_{ij}$ is given by

$$\widetilde{\delta R}_{ij} = \frac{1}{2}\left[\left(\partial_x^2 + \partial_y^2 - \partial_t^2\right)h_{ij} + \eta^{k\ell}\partial_i\partial_j h_{k\ell} - \eta^{k\ell}\partial_i\partial_k h_{\ell j} - \eta^{k\ell}\partial_j\partial_k h_{\ell i}\right] \tag{F.11}$$

in this particular case. From Eq. (13.7) it thus follows that when a small perturbation $\delta\tau_{ij} = S_{ij}\delta(w)$ is introduced on the brane, the axial gauge TT fluctuations which obey $h^{\mu\nu}{}_{;\nu} = 0$, $g_{\mu\nu}h^{\mu\nu} = 0$ (viz. $\partial_j h^{ij} = 0$, $\eta_{ij}h^{ij} = 0$ in this particular case) thus obey the wave equation

$$\begin{aligned} \Delta G_{ij}^{TT} = &\frac{1}{2}\left[\frac{\partial^2}{\partial w^2} + \epsilon(\lambda)H\epsilon(w)\frac{\partial}{\partial w} + 4\epsilon(\lambda)H\delta(w) - 2H^2\right. \\ &\left. + e^{-2A}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}\right)\right]h_{ij}^{TT} = -\kappa_4^2 S_{ij}^{TT}\delta(w) \quad . \end{aligned} \tag{F.12}$$

Hence, for modes which depend on $|w|$, the wave equation takes the form

$$\frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2}+\epsilon(\lambda)H\frac{\partial}{\partial|w|}-2H^2+e^{-2A}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial t^2}\right)\right]h_{ij}^{TT}=0 \ , \quad \text{(F.13)}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|}+2\epsilon(\lambda)H\right]h_{ij}^{TT}=-\kappa_4^2 S_{ij}^{TT}\delta(w) \ , \quad \text{(F.14)}$$

to thus define the model.

F.3 M_3^+ brane-world mode basis

Basis mode solutions to Eq. (F.13) are readily found. For M_3^+ first, we find, in terms of the mass parameter

$$m^2=(p^0)^2-(p^1)^2-(p^2)^2 \ , \quad \text{(F.15)}$$

that the $m^2\neq 0$ solutions take the form

$$h_{ij}^{TT}=e_{ij}^{TT}e^{-ip^0t+ip^1x+ip^2y}e^{-H|w|/2}\left[\alpha_m J_{3/2}(me^{H|w|}/H)+\beta_m Y_{3/2}(me^{H|w|}/H)\right] \ , \quad \text{(F.16)}$$

where e_{ij}^{TT} is a TT polarization tensor. With the Bessel functions obeying

$$\begin{aligned}\left(\frac{d}{d|w|}+2H\right)&\left[e^{-H|w|/2}Z_{3/2}(me^{H|w|}/H)\right]\\ &=e^{-H|w|/2}\left(\frac{d}{d|w|}+\frac{3H}{2}\right)Z_{3/2}(me^{H|w|}/H)\\ &=me^{H|w|/2}Z_{1/2}(me^{H|w|}/H) \ ,\end{aligned} \quad \text{(F.17)}$$

$m^2\neq 0$ solutions to Eq. (F.13) will satisfy the source-free M_3^+ junction condition

$$\delta(w)\left[\frac{\partial}{\partial|w|}+2H\right]h_{ij}^{TT}=0 \quad \text{(F.18)}$$

provided the α_m and β_m coefficients obey

$$\alpha_m J_{1/2}(m/H)+\beta_m Y_{1/2}(m/H)=0 \ , \quad \text{(F.19)}$$

i.e. provided they obey

$$\alpha_m\sin(m/H)-\beta_m\cos(m/H)=0 \ . \quad \text{(F.20)}$$

For the $M(2,1)$ graviton with $m^2=0$, the solution which obeys both Eqs. (F.13) and (F.18) is given as the warp factor wave function

$$h_{ij}^{TT}=e_{ij}^{TT}e^{-ip^0t+ip^1x+ip^2y}\beta_0 e^{-2H|w|} \ . \quad \text{(F.21)}$$

On defining

$$g_m(|w|) = \alpha_m J_{3/2}(me^{H|w|}/H) + \beta_m Y_{3/2}(me^{H|w|}/H) \tag{F.22}$$

and using the wave equation, we find that any two solutions to the wave equation identically obey the relation

$$(m_1^2 - m_2^2)e^{-2A} g_{m_1} g_{m_2} = \frac{d}{d|w|}\left[g_{m_1}\left(\frac{d}{d|w|} + \frac{3H}{2}\right)g_{m_2} - g_{m_2}\left(\frac{d}{d|w|} + \frac{3H}{2}\right)g_{m_1}\right] . \tag{F.23}$$

With the junction condition of Eq. (F.18) being rewritable as

$$\delta(w)\left[\frac{d}{d|w|} + \frac{3H}{2}\right] g_m(|w|) = 0 \ , \tag{F.24}$$

we see that pairs of modes which obey

$$g_{m_1}\left(\frac{d}{d|w|} + \frac{3H}{2}\right)g_{m_2} - g_{m_2}\left(\frac{d}{d|w|} + \frac{3H}{2}\right)g_{m_1} \to 0 \quad \text{as} \quad |w| \to \infty \tag{F.25}$$

(by damping of the modes or by rapid oscillation) will then be orthogonal with respect to the e^{-2A} measure. Consequently, we can take as normalization condition for the modes

$$2\int_0^\infty d|w| e^{2H|w|} g_m(|w|) g_{m'}(|w|) = \delta_{m,m'} \ . \tag{F.26}$$

With the relevant Bessel functions being known in completely closed form as

$$J_{3/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left(\frac{\sin z}{z} - \cos z\right) \ , \quad Y_{3/2}(z) = -\left(\frac{2}{\pi z}\right)^{1/2}\left(\frac{\cos z}{z} + \sin z\right) \ ,$$

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z \ , \quad Y_{1/2}(z) = -\left(\frac{2}{\pi z}\right)^{1/2} \cos z \ , \tag{F.27}$$

the normalized spectrum of M_3^+ thus consists of a bound state normalized massless graviton and a continuum normalized KK continuum of modes with masses which begin at $m^2 = 0^+$ and wave functions which obey $\beta_m/\alpha_m = \tan(m/H)$.

F.4 Causality in the M_3^+ brane world

In analog to Eq. (17.30), solutions to M_3^+ brane-world fluctuation equation of Eq. (F.12) can be written in the form

$$\begin{aligned} h_{ij}^{TT} &= -\frac{\kappa_4^2}{(2\pi)^3}\int d^3x' d^3p e^{ip\cdot(x-x')} e^{-H|w|/2} \\ &\times \frac{[\alpha_q J_{3/2}(qe^{H|w|}/H) + \beta_q Y_{3/2}(qe^{H|w|}/H)]}{q[\alpha_q J_{1/2}(q/H) + \beta_q Y_{1/2}(q/H)]} S_{ij}^{TT}(x') \ , \end{aligned} \tag{F.28}$$

where $q^2 = (p^0)^2 - (p^1)^2 - (p^2)^2$. For the typical source $S^{TT}_{ij}(x') = A_{ij}\delta(t')$ with constant A_{ij}, Eq. (F.28) reduces to

$$h^{TT}_{ij} = -\frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{-H|w|/2} \frac{[\alpha_q J_{3/2}(p^0 e^{H|w|}/H) + \beta_q Y_{3/2}(p^0 e^{H|w|}/H)]}{p^0[\alpha_q J_{1/2}(p^0/H) + \beta_q Y_{1/2}(p^0/H)]}, \tag{F.29}$$

where now $q = p^0$. As in Chapter 17, we locate all the singularities of the integrand in the lower half p^0 plane (the retarded contour prescription), and then evaluate Eq. (F.29) as an integral along a half circle in the upper half p^0 plane with large radius P. For the outgoing wave $J_{3/2} + iY_{3/2}$ based combination this yields, in the notation of Eq. (17.48),

$$\begin{aligned}
&h^{TT}_{ij}(\mathrm{RET}; J_{3/2} + iY_{3/2}) \\
&= -\frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{-H|w|} \frac{\left(He^{-H|w|} - ip^0\right)}{(p^0)^2} \exp[ip^0(e^{H|w|} - 1)/H] \\
&= \frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_0^{\pi} d\theta \left(1 + i\frac{He^{-H|w|}}{Pe^{i\theta}}\right) e^{-iPe^{i\theta}\alpha} \\
&= \kappa_4^2 A_{ij} e^{-H|w|}\theta(\alpha)\left(1 + He^{-H|w|}\alpha\right) \\
&= \kappa_4^2 A_{ij} e^{-2H|w|}\theta(\alpha)\,(Ht + 1) \quad ,
\end{aligned} \tag{F.30}$$

where

$$\alpha = \frac{1}{H}(Ht - e^{H|w|} + 1) \quad , \tag{F.31}$$

and where use has been made of Eqs. (E.25) and (E.27). With the M_3^+ brane-world lightcone and its interior being given by $\alpha \geq 0$, as we thus see, the $J_{3/2} + iY_{3/2}$ based propagator is fully retarded and fully causal, just as desired.

This same propagator can also be evaluated using a lower half plane contour, to yield, again in the notation of Eq. (17.48),

$$\begin{aligned}
&h^{TT}_{ij}(\mathrm{RET}; J_{3/2} + iY_{3/2}) - h^{TT}_{ij}(\mathrm{SING}; J_{3/2} + iY_{3/2}) \\
&= \frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_{2\pi}^{\pi} d\theta \left(1 + i\frac{He^{-H|w|}}{Pe^{i\theta}}\right) e^{-iPe^{i\theta}\alpha} \\
&= -\frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_0^{\pi} d\theta \left(1 - i\frac{He^{-H|w|}}{Pe^{i\theta}}\right) e^{iPe^{i\theta}\alpha} \\
&= -\kappa_4^2 A_{ij} e^{-H|w|}\theta(-\alpha)\left(1 + He^{-H|w|}\alpha\right) \\
&= -\kappa_4^2 A_{ij} e^{-2H|w|}\theta(-\alpha)\,(Ht + 1) \quad .
\end{aligned} \tag{F.32}$$

Then, on combining Eqs. (F.30) and (F.32) we obtain

$$\begin{aligned} h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) &= \kappa_4^2 A_{ij} e^{-2H|w|}(Ht+1) \quad , \\ h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) &= \theta(\alpha) h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) \quad , \end{aligned} \tag{F.33}$$

in close analogy to Eqs. (17.49) and (17.50), with $h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2})$ not taking support outside the lightcone even while $h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2})$ (a contour integral over all the normalizable graviton and KK modes) does.[1]

The treatment of the ingoing wave $J_{3/2}-iY_{3/2}$ based combination is analogous, and yields

$$\begin{aligned} & h_{ij}^{TT}(\mathrm{RET}; J_{3/2}-iY_{3/2}) \\ &= -\frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{-H|w|}\frac{(He^{-H|w|}+ip^0)}{(p^0)^2}\exp[-ip^0(e^{H|w|}-1)/H] \\ &= \frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_0^{\pi} d\theta\left(-1+i\frac{He^{-H|w|}}{Pe^{i\theta}}\right)e^{-iPe^{i\theta}\beta} \\ &= \kappa_4^2 A_{ij} e^{-H|w|}\theta(\beta)\left(-1+He^{-H|w|}\beta\right) \\ &= \kappa_4^2 A_{ij} e^{-2H|w|}\theta(\beta)(Ht-1) \quad , \end{aligned} \tag{F.34}$$

where

$$\beta = \frac{1}{H}(Ht + e^{H|w|}-1) \quad . \tag{F.35}$$

As we see, and in close analogy to the $J_2 - iY_2$ based propagator of M_4^-, the $J_{3/2} - iY_{3/2}$ based M_3^+ propagator is neither retarded nor causal, leaving the $J_{3/2}+iY_{3/2}$ based one as the relevant one for M_3^+.

Additionally, in the lower half plane we obtain

$$\begin{aligned} & h_{ij}^{TT}(\mathrm{RET}; J_{3/2}-iY_{3/2}) - h_{ij}^{TT}(\mathrm{SING}; J_{3/2}-iY_{3/2}) \\ &= \frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_{2\pi}^{\pi} d\theta\left(-1+i\frac{He^{-H|w|}}{Pe^{i\theta}}\right)e^{-iPe^{i\theta}\beta} \\ &= -\frac{\kappa_4^2 A_{ij} e^{-H|w|}}{2\pi}\int_0^{\pi} d\theta\left(-1-i\frac{He^{-H|w|}}{Pe^{i\theta}}\right)e^{iPe^{i\theta}\beta} \\ &= -\kappa_4^2 A_{ij} e^{-H|w|}\theta(-\beta)\left(-1+He^{-H|w|}\beta\right) \\ &= -\kappa_4^2 A_{ij} e^{-2H|w|}\theta(-\beta)(Ht-1) \quad . \end{aligned} \tag{F.36}$$

[1] On defining $\hat{D} = (1/2)\left[\partial_w^2 + H\epsilon(w)\partial_w + 4H\delta(w) - 2H^2 + e^{2H|w|}(\partial_x^2+\partial_y^2-\partial_t^2)\right]$, it can explicitly be checked that $h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) = \kappa_4^2 A_{ij} e^{-2H|w|}\theta(\alpha)(Ht+1)$ of Eq. (F.30) does indeed obey the inhomogeneous $\hat{D}h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) = -\kappa_4^2 S_{ij}^{TT}\delta(w)$ just as it should. However, at the same time it can also explicitly be checked that the quantity $h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) = \kappa_4^2 A_{ij} e^{-2H|w|}(Ht+1)$ obeys the homogeneous $\hat{D}h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) = 0$, the result announced in Chapter 17.

Then, on combining Eqs. (F.34) and (F.36) we obtain

$$\begin{aligned} h_{ij}^{TT}(\text{SING}; J_{3/2} - iY_{3/2}) &= \kappa_4^2 A_{ij} e^{-2H|w|}(Ht-1) \ , \\ h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2}) &= \theta(\beta) h_{ij}^{TT}(\text{SING}; J_{3/2} - iY_{3/2}) \ , \end{aligned} \tag{F.37}$$

to again confirm the lack of causality of $h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2})$ in the M_3^+ brane world.

For our $S_{ij}^{TT}(x') = A_{ij}\delta(t')$ source, upper half p^0 plane evaluation of the pure $J_{3/2}$ and $Y_{3/2}$ based propagators of M_3^+ brane world yields

$$\begin{aligned} &h_{ij}^{TT}(\text{RET}; J_{3/2}) \\ &= -\frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{-2H|w|} \frac{\left[H\sin(p^0 e^{H|w|}/H) - p^0 e^{H|w|}\cos(p^0 e^{H|w|}/H)\right]}{(p^0)^2 \sin(p^0/H)} \\ &= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi}\int_0^{\pi} d\theta e^{-iPe^{i\theta}t} \\ &\times \frac{\left[H\sin(Pe^{i\theta}e^{H|w|}/H) - Pe^{i\theta}e^{H|w|}\cos(Pe^{i\theta}e^{H|w|}/H)\right]}{Pe^{i\theta}\sin(Pe^{i\theta}/H)} \ , \end{aligned} \tag{F.38}$$

$$\begin{aligned} &h_{ij}^{TT}(\text{RET}; Y_{3/2}) \\ &= -\frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{-2H|w|} \frac{\left[H\cos(p^0 e^{H|w|}/H) + p^0 e^{H|w|}\sin(p^0 e^{H|w|}/H)\right]}{(p^0)^2 \cos(p^0/H)} \\ &= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi}\int_0^{\pi} d\theta e^{-iPe^{i\theta}t} \\ &\times \frac{\left[H\cos(Pe^{i\theta}e^{H|w|}/H) + Pe^{i\theta}e^{H|w|}\sin(Pe^{i\theta}e^{H|w|}/H)\right]}{Pe^{i\theta}\cos(Pe^{i\theta}/H)} \ . \end{aligned} \tag{F.39}$$

To evaluate these propagators we note that when $0 < \theta < \pi$ the quantity $\exp(iPe^{i\theta}) = \exp(iP\cos\theta - P\sin\theta)$ is very small in the large P limit. Consequently, we can power series expand the denominators in Eqs. (F.38) and (F.39) by setting $\sin(Pe^{i\theta}/H) = (i/2)\exp(-iPe^{i\theta}/H)\left[1 - \exp(2iPe^{i\theta}/H)\right]$ and $\cos(Pe^{i\theta}/H) = (1/2)\exp(-iPe^{i\theta}/H)\left[1 + \exp(2iPe^{i\theta}/H)\right]$, to obtain, following some algebra and the use of Eqs. (E.25) and (E.27), the power series expressions

$$\begin{aligned} h_{ij}^{TT}(\text{RET}; J_{3/2}) = \ &\kappa_4^2 A_{ij} e^{-2H|w|}[(Ht-1)[\theta(\beta) - \theta(\hat{\alpha})] \\ &+(Ht-3)[\theta(\beta - 2/H) - \theta(\hat{\alpha} - 2/H)] \\ &+(Ht-5)[\theta(\beta - 4/H) - \theta(\hat{\alpha} - 4/H)] + ...] \ , \end{aligned} \tag{F.40}$$

$$\begin{aligned} h_{ij}^{TT}(\text{RET}; Y_{3/2}) = \ &\kappa_4^2 A_{ij} e^{-2H|w|}[(Ht-1)[\theta(\beta) + \theta(\hat{\alpha})] \\ &-(Ht-3)[\theta(\beta - 2/H) + \theta(\hat{\alpha} - 2/H)] \\ &+(Ht-5)[\theta(\beta - 4/H) + \theta(\hat{\alpha} - 4/H)] - ....] \ , \end{aligned} \tag{F.41}$$

where β is given in Eq. (F.35) and $\hat{\alpha}$ is given by

$$\hat{\alpha} = \frac{1}{H}(Ht - e^{H|w|} - 1) \ . \tag{F.42}$$

Then, with the M_3^+ lightcone and its interior being given by $Ht \geq e^{H|w|} - 1$, both of the above expressions are found to take support outside the lightcone, to once again confirm that the $J_{3/2} + iY_{3/2}$ based propagator is the unique causal propagator for M_3^+ brane world.

The evaluation of $h_{ij}^{TT}(\text{RET}; J_{3/2})$ and $h_{ij}^{TT}(\text{RET}; Y_{3/2})$ via lower half p^0 plane contour integration is analogous, and is found to only require the replacement of t by $-t$ in the above, viz.

$$\begin{aligned}
&h_{ij}^{TT}(\text{RET}; J_{3/2}) - h_{ij}^{TT}(\text{SING}; J_{3/2}) \\
&= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi} \int_{2\pi}^{\pi} d\theta e^{-iPe^{i\theta}t} \\
&\quad \times \frac{[H\sin(Pe^{i\theta}e^{H|w|}/H) - Pe^{i\theta}e^{H|w|}\cos(Pe^{i\theta}e^{H|w|}/H)]}{Pe^{i\theta}\sin(Pe^{i\theta}/H)} \\
&= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi} \int_{0}^{\pi} d\theta e^{iPe^{i\theta}t} \\
&\quad \times \frac{[H\sin(Pe^{i\theta}e^{H|w|}/H) - Pe^{i\theta}e^{H|w|}\cos(Pe^{i\theta}e^{H|w|}/H)]}{Pe^{i\theta}\sin(Pe^{i\theta}/H)} \\
&= -\kappa_4^2 A_{ij} e^{-2H|w|}[(Ht+1)[\theta(-\alpha) - \theta(-\hat{\beta})] \\
&\quad + (Ht+3)[\theta(-\alpha - 2/H) - \theta(-\hat{\beta} - 2/H)] \\
&\quad + (Ht+5)[\theta(-\alpha - 4/H) - \theta(-\hat{\beta} - 4/H)] + ...] \ ,
\end{aligned} \tag{F.43}$$

$$\begin{aligned}
&h_{ij}^{TT}(\text{RET}; Y_{3/2}) - h_{ij}^{TT}(\text{SING}; Y_{3/2}) \\
&= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi} \int_{2\pi}^{\pi} d\theta e^{-iPe^{i\theta}t} \\
&\quad \times \frac{[H\cos(Pe^{i\theta}e^{H|w|}/H) + Pe^{i\theta}e^{H|w|}\sin(Pe^{i\theta}e^{H|w|}/H)]}{Pe^{i\theta}\cos(Pe^{i\theta}/H)} \\
&= \frac{i\kappa_4^2 A_{ij} e^{-2H|w|}}{2\pi} \int_{0}^{\pi} d\theta e^{iPe^{i\theta}t} \\
&\quad \times \frac{[H\cos(Pe^{i\theta}e^{H|w|}/H) + Pe^{i\theta}e^{H|w|}\sin(Pe^{i\theta}e^{H|w|}/H)]}{Pe^{i\theta}\cos(Pe^{i\theta}/H)} \\
&= -\kappa_4^2 A_{ij} e^{-2H|w|}[(Ht+1)[\theta(-\alpha) + \theta(-\hat{\beta})] \\
&\quad - (Ht+3)[\theta(-\alpha - 2/H) + \theta(-\hat{\beta} - 2/H)] \\
&\quad + (Ht+5)[\theta(-\alpha - 4/H) + \theta(-\hat{\beta} - 4/H)] - ....] \ ,
\end{aligned} \tag{F.44}$$

where α is given in Eq. (F.31) and $\hat{\beta}$ is given by

$$\hat{\beta} = \frac{1}{H}(Ht + e^{H|w|} + 1) \quad . \tag{F.45}$$

As well as $h_{ij}^{TT}(\mathrm{RET}; J_{3/2})$ and $h_{ij}^{TT}(\mathrm{RET}; Y_{3/2})$ not being causal, from Eqs. (F.43) and (F.44) we find that $h_{ij}^{TT}(\mathrm{SING}; J_{3/2})$ and $h_{ij}^{TT}(\mathrm{SING}; Y_{3/2})$ are non-causal also, with there being no relations in these two sectors which are analog to Eqs. (F.33) and (F.37).

F.5 Causality in the M_3^- brane world

For the divergent warp factor M_3^- brane world where the modes obey

$$\frac{1}{2}\left[\frac{\partial^2}{\partial|w|^2} - H\frac{\partial}{\partial|w|} - 2H^2 + e^{-2H|w|}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}\right)\right] h_{ij}^{TT} = 0 \quad , \tag{F.46}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2H\right] h_{ij}^{TT} = 0 \quad , \tag{F.47}$$

the solutions are of the form

$$h_{ij}^{TT} = e_{ij}^{TT}\, e^{-ip^0t+ip^1x+ip^2y} e^{H|w|/2}\left[\alpha_m J_{3/2}(me^{-H|w|}/H) + \beta_m Y_{3/2}(me^{-H|w|}/H)\right] \quad , \tag{F.48}$$

with α_m and β_m coefficients which obey

$$\alpha_m J_{1/2}(m/H) + \beta_m Y_{1/2}(m/H) = 0 \quad , \tag{F.49}$$

viz.

$$\alpha_m \sin(m/H) - \beta_m \cos(m/H) = 0 \quad , \tag{F.50}$$

together with a massless graviton with wave function $\beta_0 e^{2H|w|}$. With normalization condition

$$2\int_0^\infty d|w| e^{-2H|w|} g_m(|w|) g_{m'}(|w|) = \delta_{m,m'} \quad , \tag{F.51}$$

this time we find that none of the $\beta_m e^{H|w|/2} Y_{3/2}(me^{-H|w|}/H)$ type modes or the massless graviton are normalizable, but that the $\alpha_m e^{H|w|/2} J_{3/2}(me^{-H|w|}/H)$ type modes are. Hence in the normalizable sector these modes must satisfy the junction condition on their own, with the allowed normalizable modes having to obey $\sin(m/H) = 0$, to yield the discrete mass spectrum[2]

$$m = n\pi H \quad , \quad n = \pm 1, \pm 2, \pm 3, ... \quad , \tag{F.52}$$

[2]Since $J_{3/2}(me^{-H|w|}/H)$ vanishes identically when $m = 0$, there is no solution with $n = 0$.

with the modes with $g_m(|w|) = \alpha_m J_{3/2}(me^{-H|w|}/H)$ being normalized to one if we set

$$
\begin{aligned}
\frac{1}{\alpha_m^2} &= 2\int_0^{\infty} d|w|e^{-2H|w|}J_{3/2}^2(me^{-H|w|}/H) \\
&= \frac{e^{-2H|w|}}{H}\left[J_{1/2}(me^{-H|w|}/H)J_{5/2}(me^{-H|w|}/H) - J_{3/2}^2(me^{-H|w|}/H)\right]\Big|_0^{\infty} \\
&= \frac{1}{H}J_{3/2}^2(m/H) = \frac{2}{\pi m} \ .
\end{aligned} \tag{F.53}
$$

With these modes being orthonormal, they satisfy the completeness relation

$$
\sum_m g_m(|w|)g_m(|w'|) = e^{2H|w|}\delta(w-w') \tag{F.54}
$$

as well. Consequently, in terms of these modes and the $M(2,1)$ retarded scalar propagator

$$
D(x,m) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3p \frac{e^{ip\cdot x}}{[(p^0)^2 - (p^1)^2 - (p^2)^2 - m^2 + i\epsilon\epsilon(p^0)]} \ , \tag{F.55}
$$

one can construct a propagator

$$
G^{TT}(x,x',w,w') = \sum_m e^{H|w|/2}g_m(|w|)e^{H|w'|/2}g_m(|w'|)D(x-x',m) \ , \tag{F.56}
$$

with the completeness relation of Eq. (F.54) ensuring that

$$
h_{ij}^{TT}(x,w) = -2\kappa_4^2\int d^3x' G^{TT}(x,x',w,0)S_{ij}^{TT}(x') \tag{F.57}
$$

is then an exact solution to Eq. (F.12).

While the restriction to normalizable modes does lead to an exact solution to the M_3^- brane-world wave equation without approximation, to test for causality and retardation[3] we must return to the analysis of Hankel function based propagators, with the M_3^- analogs of Eqs. (F.28) and (F.29) being given as

$$
\begin{aligned}
h_{ij}^{TT} &= \frac{\kappa_4^2}{(2\pi)^3}\int d^3x' d^3p e^{ip\cdot(x-x')}e^{H|w|/2} \\
&\quad\times\frac{[\alpha_q J_{3/2}(qe^{-H|w|}/H) + \beta_q Y_{3/2}(qe^{-H|w|}/H)]}{q[\alpha_q J_{1/2}(q/H) + \beta_q Y_{1/2}(q/H)]}S_{ij}^{TT}(x') \ ,
\end{aligned} \tag{F.58}
$$

$$
h_{ij}^{TT} = \frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t}e^{H|w|/2}\frac{[\alpha_q J_{3/2}(p^0e^{-H|w|}/H) + \beta_q Y_{3/2}(p^0e^{-H|w|}/H)]}{p^0[\alpha_q J_{1/2}(p^0/H) + \beta_q Y_{1/2}(p^0/H)]} \ , \tag{F.59}
$$

[3] Even though the $D(x,m)$ propagator is causal and retarded, it is only causal and retarded on $M(2,1)$ sectionings of M_3^- with fixed $|w|$.

with the change in overall sign compared to Eqs. (F.28) and (F.29) being due to the change in sign of the brane tension. The evaluation of the $J_{3/2}+iY_{3/2}$ and $J_{3/2}-iY_{3/2}$ based propagators via integration along a half circle in the upper half p^0 plane proceeds just as in the M_3^+ case, to yield for M_3^-

$$\begin{aligned}
&h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) \\
&= \frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{H|w|}\frac{\left(He^{H|w|}-ip^0\right)}{(p^0)^2}\exp[ip^0(e^{-H|w|}-1)/H] \\
&= -\frac{\kappa_4^2 A_{ij} e^{H|w|}}{2\pi}\int_0^{\pi} d\theta\left(1+i\frac{He^{H|w|}}{Pe^{i\theta}}\right)e^{-iPe^{i\theta}\alpha} \\
&= -\kappa_4^2 A_{ij} e^{H|w|}\theta(\alpha)\left(1+He^{H|w|}\alpha\right) \\
&= -\kappa_4^2 A_{ij} e^{2H|w|}\theta(\alpha)\left(Ht+1\right) \quad ,
\end{aligned} \tag{F.60}$$

where

$$\alpha = \frac{1}{H}(Ht - e^{-H|w|}+1) \quad , \tag{F.61}$$

and

$$\begin{aligned}
&h_{ij}^{TT}(\mathrm{RET}; J_{3/2}-iY_{3/2}) \\
&= \frac{\kappa_4^2 A_{ij}}{2\pi}\int_{-\infty}^{+\infty} dp^0 e^{-ip^0 t} e^{H|w|}\frac{\left(He^{H|w|}+ip^0\right)}{(p^0)^2}\exp[-ip^0(e^{-H|w|}-1)/H] \\
&= -\frac{\kappa_4^2 A_{ij} e^{H|w|}}{2\pi}\int_0^{\pi} d\theta\left(-1+i\frac{He^{H|w|}}{Pe^{i\theta}}\right)e^{-iPe^{i\theta}\beta} \\
&= -\kappa_4^2 A_{ij} e^{H|w|}\theta(\beta)\left(-1+He^{H|w|}\beta\right) \\
&= -\kappa_4^2 A_{ij} e^{2H|w|}\theta(\beta)\left(Ht-1\right) \quad ,
\end{aligned} \tag{F.62}$$

where

$$\beta = \frac{1}{H}(Ht + e^{-H|w|}-1) \quad . \tag{F.63}$$

So now, with the M_3^- brane-world lightcone and its interior being given by $\beta \geq 0$, we see that this time it is the $J_{3/2}-iY_{3/2}$ based propagator which is causal and retarded, rather than the $J_{3/2}+iY_{3/2}$ based one.

Lower half p^0 plane evaluation of these propagators is also as before, to yield

$$\begin{aligned}
&h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) - h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) \\
&= \kappa_4^2 A_{ij} e^{2H|w|}\theta(-\alpha)\left(Ht+1\right) \quad ,
\end{aligned} \tag{F.64}$$

and

$$\begin{aligned}
h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) &= -\kappa_4^2 A_{ij} e^{2H|w|}\left(Ht+1\right) \quad , \\
h_{ij}^{TT}(\mathrm{RET}; J_{3/2}+iY_{3/2}) &= \theta(\alpha) h_{ij}^{TT}(\mathrm{SING}; J_{3/2}+iY_{3/2}) \quad ,
\end{aligned} \tag{F.65}$$

together with

$$\begin{aligned} &h_{ij}^{TT}(\mathrm{RET}; J_{3/2} - iY_{3/2}) - h_{ij}^{TT}(\mathrm{SING}; J_{3/2} - iY_{3/2}) \\ &\quad = \kappa_4^2 A_{ij} e^{2H|w|}\theta(-\beta)\,(Ht-1) \quad , \end{aligned} \tag{F.66}$$

and

$$\begin{aligned} h_{ij}^{TT}(\mathrm{SING}; J_{3/2} - iY_{3/2}) &= -\kappa_4^2 A_{ij} e^{2H|w|}\,(Ht-1) \quad , \\ h_{ij}^{TT}(\mathrm{RET}; J_{3/2} - iY_{3/2}) &= \theta(\beta) h_{ij}^{TT}(\mathrm{SING}; J_{3/2} - iY_{3/2}) \quad . \end{aligned} \tag{F.67}$$

Again, we confirm the causality of $h_{ij}^{TT}(\mathrm{RET}; J_{3/2} - iY_{3/2})$ in the M_3^- brane world, while also noting that despite its lack of normalizability, and in complete analog to the M_4^- massless graviton, the M_3^- massless graviton contributes as a pole in $h_{ij}^{TT}(\mathrm{SING}; J_{3/2} - iY_{3/2})$ which has a well-behaved, finite residue.[4]

While the treatment of the $J_{3/2}$ and $Y_{3/2}$ based M_3^- propagators proceeds completely analogously to the M_3^+ case, in this M_3^- sector a surprise awaits us. Specifically, these particular propagators are found to evaluate to

$$\begin{aligned} h_{ij}^{TT}(\mathrm{RET}; J_{3/2}) = \; & -\kappa_4^2 A_{ij} e^{2H|w|}[(Ht-1)[\theta(\beta) - \theta(\alpha - 2/H)] \\ & +(Ht-3)[\theta(\beta - 2/H) - \theta(\alpha - 4/H)] \\ & +(Ht-5)[\theta(\beta - 4/H) - \theta(\alpha - 6/H)] + ...] \quad , \end{aligned} \tag{F.68}$$

and

$$\begin{aligned} h_{ij}^{TT}(\mathrm{RET}; J_{3/2}) \; & - h_{ij}^{TT}(\mathrm{SING}; J_{3/2}) \\ & = \kappa_4^2 A_{ij} e^{2H|w|}[(Ht+1)[\theta(-\alpha) - \theta(-\beta - 2/H)] \\ & \quad + (Ht+3)[\theta(-\alpha - 2/H) - \theta(-\beta - 4/H)] \\ & \quad + (Ht+5)[\theta(-\alpha - 4/H) - \theta(-\beta - 6/H)] - ....] \quad , \end{aligned} \tag{F.69}$$

together with

$$\begin{aligned} h_{ij}^{TT}(\mathrm{RET}; Y_{3/2}) = \; & -\kappa_4^2 A_{ij} e^{2H|w|}[(Ht-1)[\theta(\beta) + \theta(\alpha - 2/H)] \\ & -(Ht-3)[\theta(\beta - 2/H) + \theta(\alpha - 4/H)] \\ & +(Ht-5)[\theta(\beta - 4/H) + \theta(\alpha - 6/H)] + ...] \quad , \end{aligned} \tag{F.70}$$

and

$$\begin{aligned} h_{ij}^{TT}(\mathrm{RET}; Y_{3/2}) \; & - h_{ij}^{TT}(\mathrm{SING}; Y_{3/2}) \\ & = \kappa_4^2 A_{ij} e^{2H|w|}[(Ht+1)[\theta(-\alpha) + \theta(-\beta - 2/H)] \\ & \quad - (Ht+3)[\theta(-\alpha - 2/H) + \theta(-\beta - 4/H)] \\ & \quad + (Ht+5)[\theta(-\alpha - 4/H) + \theta(-\beta - 6/H)] - ....] \quad . \end{aligned} \tag{F.71}$$

[4]The $q \to 0$ limit of the $e^{H|w|/2}[\alpha_q J_{3/2}(qe^{-H|w|}/H) + \beta_q Y_{3/2}(qe^{-H|w|}/H)]/q[\alpha_q J_{1/2}(q/H) + \beta_q Y_{1/2}(q/H)]$ term in the integrand in Eq. (F.58) is given as $He^{2H|w|}/q^2$ as long as $\beta_q \neq 0$.

Inspection of the various step functions which appear in these relations reveals that $h_{ij}^{TT}(\text{RET}; J_{3/2})$ and $h_{ij}^{TT}(\text{RET}; Y_{3/2})$ both vanish identically when t is negative, with both of the $J_{3/2}$ and $Y_{3/2}$ based propagators thus being fully retarded. Additionally, the right-hand sides of Eqs. (F.69) and (F.71) both vanish identically when t is positive. And thus, since $h_{ij}^{TT}(\text{RET}; J_{3/2})$ and $h_{ij}^{TT}(\text{RET}; Y_{3/2})$ are retarded, we obtain as identities

$$\begin{aligned} h_{ij}^{TT}(\text{RET}; J_{3/2}) &= \theta(t) h_{ij}^{TT}(\text{SING}; J_{3/2}) \ , \\ h_{ij}^{TT}(\text{RET}; Y_{3/2}) &= \theta(t) h_{ij}^{TT}(\text{SING}; Y_{3/2}) \ , \end{aligned} \tag{F.72}$$

relations which while similar in format to Eqs. (F.65) and (F.67), nonetheless differ from them quite substantially in their step function domains. As regards the causal properties of the propagators, we recall that in the M_3^- brane world the null geodesic through the point $|w| = 0$, $t = 0$ is given by $H|t| = 1 - e^{-H|w|}$. With the parameters α and β being related by $\alpha - 2/H = \beta - 2e^{-H|w|}/H$, and with $e^{-H|w|}$ always being less than or equal to one, the region exterior to the forward (viz. $t > 0$) M_3^- lightcone is thus given by $\beta < 0$, $\alpha - 2/H < 0$. Then, with the right-hand sides of Eqs. (F.68) and (F.70) both vanishing under such conditions, we see that $h_{ij}^{TT}(\text{RET}; J_{3/2})$ and $h_{ij}^{TT}(\text{RET}; Y_{3/2})$ (quantities which are only non-zero when $t \geq 0$) are both causal. Moreover, because of Eq. (F.72), $h_{ij}^{TT}(\text{SING}; J_{3/2})$ and $h_{ij}^{TT}(\text{SING}; Y_{3/2})$ both then vanish outside the forward M_3^- lightcone too. With the exterior to the backward ($t < 0$) M_3^- lightcone being given by $-t + e^{-H|w|}/H - 1/H < 0$, we see this region is given by $-\alpha < 0$, $-\beta - 2/H = -t + e^{-H|w|} - 1/H - 2e^{-H|w|} < 0$, with it then following from Eqs. (F.69) and (F.71) that $h_{ij}^{TT}(\text{SING}; J_{3/2})$ and $h_{ij}^{TT}(\text{SING}; Y_{3/2})$ both vanish outside the backward M_3^- lightcone too. Thus all four quantities $h_{ij}^{TT}(\text{RET}; J_{3/2})$, $h_{ij}^{TT}(\text{RET}; Y_{3/2})$, $h_{ij}^{TT}(\text{SING}; J_{3/2})$ and $h_{ij}^{TT}(\text{SING}; Y_{3/2})$ never take support anywhere outside the M_3^- lightcone at all, and apart from taking support within it, they actually take support right on it as well.[5] Comparing these results with the $J_{3/2} - iY_{3/2}$ sector, we see that while $h_{ij}^{TT}(\text{SING}; J_{3/2} - iY_{3/2})$ is not itself causal (in this sector it is only $h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2})$ which is causal and not its singular part), in the $J_{3/2}$ and $Y_{3/2}$ sectors both $h_{ij}^{TT}(\text{SING}; J_{3/2})$ and $h_{ij}^{TT}(\text{SING}; Y_{3/2})$ actually are causal. Comparing also with the M_3^+ sector, we recall that the M_3^+ and M_3^- brane worlds each possess one orthonormal mode basis (the massless graviton plus KK continuum mode basis in the M_3^+ case, and the discrete $J_{3/2}$ basis in the M_3^- case), with each such basis possessing a conventional completeness relation of the generic form given in Eq. (16.73). Nonetheless, despite this similarity, the M_3^+ $h_{ij}^{TT}(\text{SING}; J_{3/2} + iY_{3/2})$ is not causal, while the M_3^- $h_{ij}^{TT}(\text{SING}; J_{3/2})$ is.

[5]Despite the absence of any massless pole in $h_{ij}^{TT}(\text{SING}; J_{3/2})$, the $h_{ij}^{TT}(\text{RET}; J_{3/2})$ propagator nonetheless still takes support on the M_3^- lightcone. The reason for this is that the entire tower of massive $J_{3/2}(n\pi e^{-H|w|})$ modes [viz. massive from the perspective of the 3-dimensional $M(2,1)$] *collectively* form a *single* solution to Eq. (F.12), a wave equation which describes the propagation of one 4-dimensional massless AdS_4 graviton. The absence of massless poles in $h_{ij}^{TT}(\text{SING}; J_{3/2})$ does not inhibit propagation of information at the speed of light.

Now we found earlier that in the M_3^- brane world the $h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2})$ propagator was also causal, and so we seem to have no less than three candidate causal M_3^- propagators (the $J_{3/2} - iY_{3/2}$, $J_{3/2}$ and $Y_{3/2}$ based ones). Since there cannot possibly be more than one causal propagator (the solution to the Cauchy initial value problem is unique), we need to clarify the situation. To this end we note that a null signal which leaves $|w| = 0$ at $t = 0$ first reaches $|w| = \infty$ at $Ht = 1$, and since $0 \leq e^{-H|w|} \leq 1$, the time interval $0 < Ht < 1$ thus corresponds to

$$
\begin{aligned}
0 < (Ht + e^{-H|w|}) < 2 \quad &, \quad -1 < (Ht - e^{-H|w|}) < 1 \quad , \\
0 < \alpha < 2/H \quad &, \quad -1/H < \beta < 1/H \quad .
\end{aligned}
\tag{F.73}
$$

Within this time interval then we therefore obtain

$$
h_{ij}^{TT}(\text{RET}; J_{3/2}) = h_{ij}^{TT}(\text{RET}; Y_{3/2}) = -\kappa_4^2 A_{ij} e^{2H|w|}(Ht - 1)\theta(\beta) \quad , \tag{F.74}
$$

to discover that in this time interval not only do these two propagators both take the same value, they both take exactly the same value as that found for $h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2})$ in Eq. (F.62). However, in addition we note that within the subsequent time interval $1 < Ht < 3$ during which the null signal returns to the brane and then goes back again to the boundary, i.e. within

$$
\begin{aligned}
1 < (Ht + e^{-H|w|}) < 4 \quad &, \quad 0 < (Ht - e^{-H|w|}) < 3 \quad , \\
1 < \alpha < 4/H \quad &, \quad 0 < \beta < 3/H \quad ,
\end{aligned}
\tag{F.75}
$$

we obtain

$$
\begin{aligned}
h_{ij}^{TT}(\text{RET}; J_{3/2}) = & -\kappa_4^2 A_{ij} e^{2H|w|}[(Ht - 1)[\theta(\beta) - \theta(\alpha - 2/H)] \\
& + (Ht - 3)\theta(\beta - 2/H)] \quad , \\
h_{ij}^{TT}(\text{RET}; Y_{3/2}) = & -\kappa_4^2 A_{ij} e^{2H|w|}[(Ht - 1)[\theta(\beta) + \theta(\alpha - 2/H)] \\
& - (Ht - 3)\theta(\beta - 2/H)] \quad .
\end{aligned}
\tag{F.76}
$$

Thus, in this particular time interval not only do $h_{ij}^{TT}(\text{RET}; J_{3/2})$ and $h_{ij}^{TT}(\text{RET}; Y_{3/2})$ differ from each other, both of them differ from $h_{ij}^{TT}(\text{RET}; J_{3/2} - iY_{3/2})$ as it continues to be given by Eq. (F.62).

To resolve this seeming conflict we apply the wave operator

$$
D = \frac{1}{2}\left[\frac{\partial^2}{\partial w^2} - H\epsilon(w)\frac{\partial}{\partial w} - 4H\delta(w) - 2H^2 - e^{-2H|w|}\frac{\partial^2}{\partial t^2}\right] \tag{F.77}
$$

of Eq. (F.12) to each of the individual terms in Eq. (F.76), to obtain

$$
\begin{aligned}
D\left[e^{2H|w|}(Ht - 1)\theta(\beta)\right] &= \delta(w)\delta(t) \quad , \\
D\left[e^{2H|w|}(Ht - 1)\theta(\alpha - 2/H)\right] &= \delta(w)\delta(t - 2/H) \quad , \\
D\left[e^{2H|w|}(Ht - 3)\theta(\beta - 2/H)\right] &= \delta(w)\delta(t - 2/H) \quad .
\end{aligned}
\tag{F.78}
$$

As we see, while the action of D on the common first term of $h_{ij}^{TT}(\mathrm{RET}; J_{3/2})$ and $h_{ij}^{TT}(\mathrm{RET}; Y_{3/2})$ gives back the wave equation in the form

$$D\left[-\kappa_4^2 A_{ij} e^{2H|w|}(Ht-1)\theta(\beta)\right] = -\kappa_4^2 A_{ij}\delta(w)\delta(t) = -\kappa_4^2 S_{ij}^{TT}\delta(w) \ , \qquad \text{(F.79)}$$

its action on their next two terms gives zero, viz.

$$D\left[e^{2H|w|}(Ht-1)\theta(\alpha-2/H) - e^{2H|w|}(Ht-3)\theta(\beta-2/H)\right] = 0 \ . \qquad \text{(F.80)}$$

With Eq. (F.79) also applying to the action of D on $h_{ij}^{TT}(\mathrm{RET}; J_{3/2} - iY_{3/2})$ as given in Eq. (F.62), we see that it is only the $e^{2H|w|}(Ht-1)\theta(\beta)$ term which is associated with the physical $\delta(t)$ source, with the differences between the three propagators residing only in combinations of terms which satisfy the homogeneous, source-free wave equation, combinations which nonetheless are fully retarded and fully causal. With none of these specific combinations thus being of any physical consequence, the response of the system to an $S_{ij}^{TT} = A_{ij}\delta(t)$ source on the brane is thus unambiguously given without approximation as

$$h_{ij}^{TT} = -\kappa_4^2 A_{ij} e^{2H|w|}(Ht-1)\theta(\beta) \qquad \text{(F.81)}$$

at all times.

As well as provide a completely soluble model for studying causality issues in the brane world, our analysis of M_3^- shows that normalizability of the modes is not a particularly relevant criterion for globally non-hyperbolic brane worlds. Firstly, even with a restriction to the $J_{3/2}$ sector, the presence of the $e^{2H|w|}$ factor in the $h_{ij}^{TT}(\mathrm{RET}; J_{3/2})$ propagator shows that localization of gravity to the brane is not achieved anyway. In fact, the presence of such a diverging factor entails the lack of stability of M_3^- to small perturbations, even normalizable ones. Moreover, once the $h_{ij}^{TT}(\mathrm{RET}; J_{3/2} - iY_{3/2})$ propagator is shown to be causal, the behavior of the $h_{ij}^{TT}(\mathrm{RET}; J_{3/2})$ propagator is then highly constrained. Specifically, with the $J_{3/2} - iY_{3/2}$ propagator being based on non-normalizable modes, it should be expected to produce fluctuations which diverge at large $|w|$, and it indeed does. Then, if any other propagator is to be causal too, then it must produce the same response to a source as the $J_{3/2} - iY_{3/2}$ based propagator. Thus once the $J_{3/2} - iY_{3/2}$ propagator is causal, the only way that the $J_{3/2}$ propagator could avoid generating divergent fluctuations would be by not being causal at all. Finally, with the $J_{3/2} - iY_{3/2}$ based propagator describing the full content of the response of theory to a local source of the type exhibited in Eq. (F.81), its pole and cut structure describes the physically observable states of the theory, with it not being permissible to leave the non-normalizable massless graviton out.

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