Special Gauge v3 Matthew

The perturbed Einstein tensor $\delta G_{\mu\nu}(h_{\mu\nu})$ evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{1}$$

is calculated as

$$\begin{aligned}\delta G_{\mu\nu} = {} & \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{2}$$

When calculated explicitly in the Cartesian coordinate system, we see that each tensor component is far away from being diagonal in the perturbation components $h_{\mu\nu}$. In order to solve these equations, we seek to find a gauge that allows the equations to become diagonalized. To this end, we may impose the most general gauge as

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega \tag{3}$$

where J, P, and R are constant coefficients that we vary. Upon taking $J=-2$, $P=\frac{1}{2}$, and $R=1$, the fluctuation equations take a form diagonal in $h_{\mu\nu}$ up to its trace and $\delta T^\lambda{}_\lambda$ ($J=-2$ specifically, otherwise terms like h_{0i} will appear in δG_{ii}). With this choice of coefficients, the gauge is expressed as

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + \tfrac{1}{2}\partial_\nu h + \Omega^{-1}h\partial_\nu\Omega. \tag{4}$$

Within this gauge, the trace of the Einstein tensor evaluates to

$$g^{\mu\nu}\delta G_{\mu\nu} = (-10\Omega^{-4}\dot\Omega^2 + 6\Omega^{-3}\ddot\Omega)h_{00} + (-4\Omega^{-4}\dot\Omega^2 + \Omega^{-3}\ddot\Omega - \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. \tag{5}$$

the perturbed Einstein tensor has been calculated as :

$$\begin{aligned}\delta G_{00} = {} & (3\Omega^{-2}\dot\Omega^2 - \Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot\Omega\partial_0)h_{00} + (\tfrac{3}{2}\Omega^{-2}\dot\Omega^2 - \tfrac{1}{2}\Omega^{-1}\ddot\Omega + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ & - \Omega^{-1}\dot\Omega\partial_0)h.\end{aligned} \tag{6}$$

$$\begin{aligned}\delta G_{11} = {} & (-2\Omega^{-2}\dot\Omega^2 + \Omega^{-1}\ddot\Omega)h_{00} + (-\Omega^{-2}\dot\Omega^2 + 2\Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot\Omega\partial_0)h_{11} \\ & + (-\tfrac{1}{2}\Omega^{-2}\dot\Omega^2 - \tfrac{1}{2}\Omega^{-1}\ddot\Omega - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h.\end{aligned} \tag{7}$$

$$\begin{aligned}\delta G_{22} = {} & (-2\Omega^{-2}\dot\Omega^2 + \Omega^{-1}\ddot\Omega)h_{00} + (-\Omega^{-2}\dot\Omega^2 + 2\Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot\Omega\partial_0)h_{22} \\ & + (-\tfrac{1}{2}\Omega^{-2}\dot\Omega^2 - \tfrac{1}{2}\Omega^{-1}\ddot\Omega - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h.\end{aligned} \tag{8}$$

$$\begin{aligned}\delta G_{33} = {} & (-2\Omega^{-2}\dot\Omega^2 + \Omega^{-1}\ddot\Omega)h_{00} + (-\Omega^{-2}\dot\Omega^2 + 2\Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot\Omega\partial_0)h_{33} \\ & + (-\tfrac{1}{2}\Omega^{-2}\dot\Omega^2 - \tfrac{1}{2}\Omega^{-1}\ddot\Omega - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h.\end{aligned} \tag{9}$$

$$\delta G_{01} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{01} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_1 h. \tag{10}$$

$$\delta G_{02} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{02} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_2 h. \tag{11}$$

$$\delta G_{03} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{03} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_3 h. \tag{12}$$

$$\delta G_{12} = (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{12}. \tag{13}$$

$$\delta G_{13} = (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{13}. \tag{14}$$

$$\delta G_{23} = (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{23}. \tag{15}$$

We can compactify the notation:

$$\begin{aligned}\delta G_{00} &= (3\Omega^{-2}\dot{\Omega}^2 - \Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{00} + (\tfrac{3}{2}\Omega^{-2}\dot{\Omega}^2 - \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ &\quad - \Omega^{-1}\dot{\Omega}\partial_0)h. &(16)\\ \delta G_{0i} &= (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{0i} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_i h. &(17)\\ \delta G_{ij} &= \eta_{ij}(-2\Omega^{-2}\dot{\Omega}^2 + \Omega^{-1}\ddot{\Omega})h_{00} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)h_{ij} \\ &\quad + \eta_{ij}(-\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 - \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. &(18)\end{aligned}$$

Now we take the deSitter background $\Omega(\tau) = \frac{1}{H\tau}$ and evaluate it in the same gauge. The trace of the Einstein tensor evaluates to

$$g^{\mu\nu}\delta G_{\mu\nu} = 2H^2 h_{00} + (-2H^2 - \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu)h. \tag{19}$$

The tensor perturbations in this geometry simplify further, as the extraneous h_{00} terms cancel according to $(-2\Omega^{-2}\dot{\Omega}^2 + \Omega^{-1}\ddot{\Omega}) = 0$ in dS_4.

$$\delta G_{00} = (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{00} + (\tfrac{1}{2}\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h. \tag{20}$$

$$\delta G_{11} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{11} + (-\tfrac{3}{2}\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. \tag{21}$$

$$\delta G_{22} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{22} + (-\tfrac{3}{2}\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. \tag{22}$$

$$\delta G_{33} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{33} + (-\tfrac{3}{2}\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. \tag{23}$$

$$\delta G_{01} = (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{01} + \tfrac{1}{2}\tau^{-1}\partial_1 h. \tag{24}$$

$$\delta G_{02} = (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{02} + \tfrac{1}{2}\tau^{-1}\partial_2 h. \tag{25}$$

$$\delta G_{03} = (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{03} + \tfrac{1}{2}\tau^{-1}\partial_3 h. \tag{26}$$

$$\delta G_{12} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{12}. \tag{27}$$

$$\delta G_{13} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{13}. \tag{28}$$

$$\delta G_{23} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{23}. \tag{29}$$

Again, we compactify the notation:

$$\begin{aligned}\delta G_{00} &= (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{00} + (\tfrac{1}{2}\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h. &(30)\\ \delta G_{0i} &= (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{0i} + \tfrac{1}{2}\tau^{-1}\partial_i h. &(31)\\ \delta G_{ij} &= (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)h_{ij} + \eta_{ij}(-\tfrac{3}{2}\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu)h. &(32)\end{aligned}$$