QFT

Ch 3: The Dirac Field

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3.4 **Majorana fermions**

Recall from Eq. (3.40) that one can write a relativistic equation for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x)$, $a = 1, 2$.

(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0.$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordan equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

In the reduced Lorentz representation of two component left handed Weyl spinors, $SL(2,\mathbb{C})$, such a spin transforms as

$$\chi \to \Lambda_L\chi(\Lambda^{-1}x)$$
$$\chi^* \to \Lambda_L^*\chi^*(\Lambda^{-1}x)$$

with

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}},$$

where $S^{\mu\nu}$ is the antisymmetric tensor representing boosts and rotations

$$S^{0i} = -\frac{i}{2}\sigma^i; \qquad S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k.$$

The 4x4 matrix Λ is the real four vector Lorentz representation

$$\Lambda^\mu_\nu = e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}.$$

In order to determine how ψ_L transforms under a Lorentz boost, we will need an analogue to eq. 3.29. First note

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

In terms of the left and right spinor operators, we can write

$$\Lambda_{1/2} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}; \qquad \Lambda_{1/2}^{-1} = \begin{pmatrix} \Lambda_L^{-1} & 0 \\ 0 & \Lambda_R^{-1} \end{pmatrix}.$$

Now we make take eq. 3.29

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu_\nu\gamma^\nu$$

$$\Rightarrow \begin{pmatrix} \Lambda_L^{-1} & 0 \\ 0 & \Lambda_R^{-1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} = \Lambda^\mu_\nu\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}$$

Since Λ only acts on vector indices, we conclude that

$$\Lambda_L^{-1}\sigma^\mu\Lambda_R = \Lambda^\mu_\nu\sigma^\nu$$

$$\Lambda_R^{-1}\bar{\sigma}^\mu\Lambda_L = \Lambda^\mu_\nu\bar{\sigma}^\nu.$$

This is due to the block diagonal form of $S^{\mu\nu}$ in the Dirac representation. The last ingredient we will need is how the complex conjugate ψ_L^* transforms. Using the identity (three vectors here)

$$\sigma^2\sigma^* = -\sigma\sigma^2$$

we can show

$$\sigma^2\psi_L^* \to \sigma^2\Lambda_L^*\psi_L^* = \Lambda_R\sigma^2\psi_L^*.$$

Now compute the Majorana equation under a boost

$$\begin{aligned}
i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^* &\to i\bar{\sigma}^\mu\Lambda_L\left(\Lambda^{-1}\right)^\nu_\mu\partial_\nu\chi(\Lambda^{-1}x) - im\sigma^2\Lambda_L^*\chi^*(\Lambda^{-1}x)\\
&= i\Lambda_R\Lambda_R^{-1}\bar{\sigma}^\mu\Lambda_L\left(\Lambda^{-1}\right)^\nu_\mu\partial_\nu\chi(\Lambda^{-1}x) - im\Lambda_R\sigma^2\chi^*(\Lambda^{-1}x)\\
&= i\Lambda_R\Lambda^\mu_\sigma\bar{\sigma}^\sigma\left(\Lambda^{-1}\right)^\nu_\mu\partial_\nu\chi(\Lambda^{-1}x) - im\Lambda_R\sigma^2\chi^*(\Lambda^{-1}x)\\
&= \Lambda_R\left[i\bar{\sigma}^\nu\partial_\nu\chi(\Lambda^{-1}x) - im\sigma^2\chi^*(\Lambda^{-1}x)\right]\\
&= 0.
\end{aligned}$$

Before we show the Klein-Gordon field, note the extension of the identity

$$\sigma^2\sigma^{*\mu} = \bar{\sigma}^\mu\sigma^2$$

$$\sigma^2\sigma^\mu = \bar{\sigma}^{*\mu}\sigma^2.$$

Now take the field and its conjugate

$$\bar{\sigma}^\mu\partial_\mu\chi = m\sigma^2\chi^* \tag{1}$$

$$\bar{\sigma}^{*\nu}\partial_\nu\chi^* = -m\sigma^2\chi. \tag{2}$$

Solve for χ^* from (1)

$$\chi^* = \frac{\sigma^2}{m}\bar{\sigma}^\mu\partial_\mu\chi$$

insert into (2)

$$\bar{\sigma}^{*\nu}\sigma^2\bar{\sigma}^\mu\partial_\nu\partial_\mu\chi = -m^2\sigma^2\chi$$

use identity and eliminate σ^2

$$\sigma^\nu\bar{\sigma}^\mu\partial_\nu\partial_\mu\chi = -m^2\chi.$$

The matrices acting on derivatives sum to

$$\sigma^\nu\bar{\sigma}^\mu\partial_\nu\partial_\mu = g^{\nu\mu}\partial_\nu\partial_\mu,$$

this can be found by antisymmetry via the anticommutator $\{\sigma^i,\sigma^j\} = 2\delta_{ij}$. Thus we conclude

$$\sigma^\nu\bar{\sigma}^\mu\partial_\nu\partial_\mu = g^{\nu\mu}\partial_\nu\partial_\mu = \partial^\mu\partial_\mu$$

and we have

$$(\partial^\mu\partial_\mu + m^2)\chi = 0.$$

(b) Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma)^2_{ab}\chi^*_a\chi^*_b$; however, since σ^2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anticommuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anticommuting field, that is, as a field that takes as values *Grassmann numbers* which satisfy

$$\alpha\beta = -\beta\alpha \qquad \text{for any } \alpha, \beta.$$

Note that this relation implies that $\alpha = 0$ A Grassmann field $\zeta(x)$ can be expanded in a basis of functions as

$$\zeta(x) = \sum_n \alpha_n \phi_n(x),$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassman numbers to reverse the order:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*.$$

This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \, \left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right],$$

(where $\chi^\dagger = (\chi^*)^T$) is real ($S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

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