Quantum Mechanics III
HW 1

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1.1 A collection of vectors $v_1, ..., v_N$ is *linearly independent* if the only way to make the zero vector as a linear combination of these vectors is to have all zero coefficients: $\lambda_1 v_1 + ... + \lambda_N v_N = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = ... = \lambda_N = 0$. Otherwise the set of vectors is linearly dependent. Show that

(a) A set of vectors is linearly dependent precisely when (at least) one of the vectors may be written as a linear combination of the others.

(b) If the set of vectors $v_1, ..., v_N$ is orthonormal, it is linearly independent.

(a) Take the set of vectors $\{v_n\}$ such that one vector v_i is linearly dependent

$$v_i = \sum_{j \neq i} c_j v_j, \;\; c_j \in \mathbb{C}$$

Now form the zero vector as a linear combination of the vectors in the set (with $\lambda_i \in \mathbb{C}$)

$$\lambda_1 v_1 + ... + \lambda_i v_i + ... + \lambda_N v_N = \sum_i \lambda_j v_j = 0$$

which upon substitution of v_i becomes

$$\sum_{j \neq i} (c_j + \lambda_j) v_j = 0.$$

For $v_j \neq 0$, we must then have $\lambda_j = -c_j$, hence

$$\forall \; \lambda_j \; \exists \; \{\lambda_k\} \subset \{\lambda_j\} : \sum_j \lambda_j v_j = 0, \{\lambda_k\} \neq 0$$

(I may be abusing notation here; I wish to say that for all λ_j there exists a subset $\{\lambda_k\}$ such that $\sum_j \lambda_j = 0$ and $\{\lambda_k\} \neq 0$.)
Since the coefficients are not uniquely zero in forming the zero vector, the set must be linearly dependent. The same argument may be repeated if more than one vector may be written as a linear combination of others.

(b) The decomposition of a vector v lying within the space spanned by $\{v_N\}$ is given as

$$v = \sum_i \lambda_i v_i$$

where

$$\lambda_i = (v_i, v).$$

Given the orthonormal set $\{v_N\}$ we may attempt to decompose any given vector v_j within this set as

$$v_j = \sum_i (v_i, v_j)v_i = \sum_i \delta_{ij} v_i = v_j$$

Hence no vector within the set $\{v_N\}$ can be decomposed as linear combination of other vectors. When we form the zero vector as a linear combination of vectors within the set for arbitrary $\lambda_i \in \mathbb{C}$

$$\sum_i \lambda_i v_i = 0$$

we see that the only unique solution for $v_i \neq 0$ is $\forall i, \lambda_i = 0$. Hence the orthonormal set is linearly independent.

1.3 Show that the set of vectors $\mathscr{P}_a = \{u \in \mathscr{H} : Au = au\}$, which contains all eigenvectors of A corresponding to the eigenvalue a and the zero vector, is a subspace of the quantum mechanical Hilbert space $\mathscr{H}$.

$\forall c \in \mathbb{C}$

$$A(cu) = c(Au) = a(cu)$$

and $\forall u_1, u_2 \in \mathscr{P}_a$

$$A(u_1 + u_2) = Au_1 + Au_2 = a(2u).$$

Hence the space $\mathscr{P}_a$ is closed under addition and multiplication by a scalar. If we include the zero vector, then $\mathscr{P}_a$ is a linear vector space and since its elements are necessarily vectors of the Hilbert spa ce, it is thus a subspace of $\mathscr{H}$.

1.4 Suppose that the operator A is invertible.

(a) Show that the inverse is unique.

(b) Show that if either $AX = 1$ or $XA = 1$, then $X = A^{-1}$.

(a) Assume two inverse operators $\bar{A}^{-1}$ and A^{-1} such that

$$A\bar{A}^{-1} = \bar{A}^{-1}A = 1; \qquad AA^{-1} = A^{-1}A = 1$$

then

$$\bar{A}^{-1}A = A^{-1}A \Rightarrow (\bar{A}^{-1} - A^{-1})A = 0 \Rightarrow \bar{A}^{-1} = A^{-1}.$$

Hence the inverse operator in unique.

(b) For $AX = 1$ we have, given the invertibility of A,

$$AX = 1 = AA^{-1} \Rightarrow AX = AA^{-1} \Rightarrow A(X - A^{-1}) = 0 \Rightarrow X = A^{-1}.$$

Similarly for $XA = 1$

$$XA = 1 = A^{-1}A \Rightarrow (X - A^{-1})A = 0 \Rightarrow X = A^{-1}.$$

1.5 a

(a) Show by induction that if $[A, B] = 0$, then $[A^n, B] = 0$, $n = 0, 1, 2, ...$

(b) Define a function f via its power series, $f(z) = \sum_k b_k z^k$, and the corresponding function of the operator A (at least formally) using the same power series, $f(A) = \sum_k b_k A^k$. Show that if $[A, B] = 0$, then also $[f(A), B] = 0$ and $[f(A), g(B)] = 0$ for any functions f and g. Thus, arbitrary functions of two commuting operators also commute.

(a) Basis:

$$[A^0, B] = [\mathbb{1}, B] = 0.$$

Induction:

Assume $[A^n, B] = 0$, then

$$[A^{n+1}, B] = [A, B]A^n + A[A^n, B] = 0.$$

(b)

$$[f(A), B] = [\sum b_k A^k, B] = \sum b_k [A^k, B] = 0$$

where the proof from (a) was used.

Define $g(B) = \sum_l c_l B^l$, then

$$\begin{aligned} [f(A), g(B)] &= \sum_l c_l [f(A), B^l] \\ &= \sum_l c_l \left([f(A), B]B^{l-1} + B[f(A), B]B^{l-2} + B^2[f(A), B]B^{l-3} + ... + B^{l-1}[f(A), B]\right) \\ &= 0 \end{aligned}$$

due to $[f(A), B] = 0$. Thus

$$[f(A), g(B)] = 0.$$

1.8 Show that the operators xp and px are not hermitian, but the "symmetrized" product $\frac{1}{2}(xp + px)$ is.

$$xp - (xp)^\dagger = xp - p^\dagger x^\dagger = xp - px = [x, p] = i\hbar$$

$$px - (px)^\dagger = px - xp = -i\hbar$$

$$\frac{1}{2}(xp + px) - \frac{1}{2}(xp + px)^\dagger = \frac{1}{2}(xp + px) - \frac{1}{2}(px + xp) = 0$$