Lecture 2

01/23/2012

Wave equations for Potentials (Chapter J6)

$$\vec\nabla \vec E = \rho/\varepsilon_0 \qquad \vec E = -\frac{\partial \vec A}{\partial t} - \vec\nabla\varphi$$

$$\vec B = \vec\nabla \times \vec A$$

$$-\vec\nabla \frac{\partial \vec A}{\partial t} - \vec\nabla(\vec\nabla\varphi) = \rho/\varepsilon_0$$

$$\vec\nabla \times \vec B = \vec\nabla \times (\vec\nabla \times \vec A) = \mu_0 j + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial A}{\partial t} - \vec\nabla\varphi\right)$$

$$\vec\nabla(\vec\nabla \vec A) - \vec\nabla^2 \vec A = \mu_0 j - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\varphi$$

$$\vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec j + \vec\nabla\left(\underbrace{\vec\nabla A + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}}_{\text{Lorenz gauge}}\right)$$

Static solutions

$$\vec A = \frac{\mu_0}{4\pi}\int \frac{j(r')\,d^3r'}{|r-r'|}$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d^3r'}{|r-\bar r'|}$$

Wave equation for the Lorentz Gauge

$$\boxed{\begin{aligned}\vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} &= -\mu_0 \vec j \\ \vec\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} &= -\rho/\varepsilon_0\end{aligned}}$$

$$\boxed{\vec\nabla \vec A + \frac{1}{c^2}\frac{\partial \vec A}{\partial t} = 0}$$

because $\vec\nabla \vec A + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$

$$\vec\nabla \frac{\partial \vec A}{\partial t} = \frac{\partial}{\partial t}\vec\nabla\vec A = -\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$$

For Coulomb gauge: $\vec\nabla \cdot \vec A = 0$!!

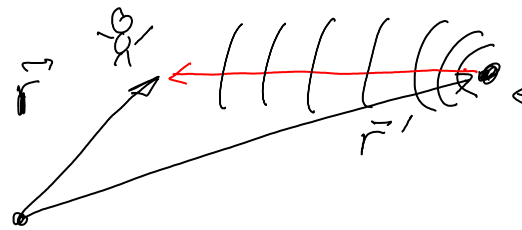
$$\vec\nabla^2\phi = -\rho/\varepsilon_0 \Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec r')\,d^3r'}{|\bar r - \bar r'|}$$

$$\vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec j + \frac{1}{c^2}\vec\nabla\frac{\partial\phi}{\partial t} \qquad \rho = \rho(\vec r, t)$$

$$\vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec j - \frac{\varepsilon_0\mu_0}{4\pi\varepsilon_0}\int\frac{\dot\rho(\vec r', t)(\bar r - \bar r')}{|\bar r - \bar r'|^3}d^3r'$$

$$\boxed{\vec\nabla^2 \vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec j - \frac{\mu_0}{4\pi}\int\frac{\dot\rho(\vec r', t)(\vec r - \vec r')}{|\bar r - \bar r'|^3}d^3r'}$$

Solution of the inhomogeneous wave equation: Qualitative Approach.



$\vec{j}(\vec{r}\,',t)$

$\delta(\vec{r}-\vec{r}\,')\cdot\delta(t-t')$

"point source"

wave equation:

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}(\vec{r},t)$$

non-homogeneous differential equation

$$\vec{A}(\vec{r},t)=\vec{A}_{in}(\vec{r},t)+\vec{A}_{non}(\vec{r},t)$$

$$\boxed{\vec{\nabla}^2\vec{A}_{in}-\frac{1}{c^2}\frac{\partial^2\vec{A}_{in}}{\partial t}=0}$$

homogeneous equation with the solution $\vec{A}_{in}$

$$\boxed{\vec{\nabla}^2\vec{A}_{non}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}(\vec{r},t)}$$

solution of this non-homogeneous equation is $\vec{A}_{non}$

$\mu_0\vec{j}(\vec{r},t)\leftarrow$ source of $\vec{A}_{non}$

Physics of $\vec{A}_{non}$:

$$\vec{A}_{non}(\vec{r},t)=\mu_0\int\frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\,|\vec{r}-\vec{r}\,'|}\vec{j}\,'(\vec{r}\,',t')\,dt'\,d^3r'$$

$$\Rightarrow\quad\boxed{\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\int\frac{\vec{j}\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$