Lecture 8

02/17/2016

recap

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_\omega(\vec{r}\,')\, e^{-i\vec{k}\vec{r}\,'} d^3r'$$

$$e^{-i\vec{k}\vec{r}\,'} = 1 - i\vec{k}\vec{r}\,' + \frac{i^2(\vec{k}\vec{r}\,')^2}{2!} + \dots$$

$\boxed{r'/\lambda \ll 1}$

electric dipole:

$$\boxed{\vec{A}^{(\vec{p})}_\omega = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_\omega(\vec{r}\,')\, d^3r'} \Leftarrow \text{leading term}$$

magnetic dipole and electric quadrupole:

$$\boxed{\vec{A}^{(2)}_\omega = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}_\omega(\vec{r}\,')(\vec{e}_k\vec{r}\,')\, d^3r'}$$

We considered the leading term of expansion $\vec{A}^{(p)}_\omega$ and has divided a second term $\vec{A}^{(2)}_\omega$ in two parts:

$\vec{A}^{(2)}_\omega = \vec{A}^{(Q)}_\omega + \vec{A}^{(m)}_\omega$, where we introduced:

(a) the electric quadrupole field

$$\vec{A}^{(Q)}_\omega = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{1}{2}\int\{(\vec{e}_k\vec{r}\,')\vec{J}_\omega + (\vec{e}_k\vec{J}_\omega)\vec{r}\,'\}\, d^3r'$$

and

(b) the magnetic dipole field

$$\vec{A}^{(m)}_\omega = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\frac{(\vec{r}\,'\times\vec{J}_\omega)\times\vec{e}_k}{2}\, d^3r' =$$

$$= \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}(\vec{e}_k\times\vec{m}_\omega)$$

where

$$\boxed{\vec{m} = \frac{1}{2}\int \vec{r}\,'\times\vec{j}_\omega(\vec{r}\,')\, d^3r'}$$

is a magnetic moment

This is the component of magnetic moment, which is perpendicular to the k-vector

$$-\vec{e}_k(\vec{e}_k\vec{m}_\omega) + \vec{m}_\omega\underbrace{(\vec{e}_k\cdot\vec{e}_k)}_{=1}$$

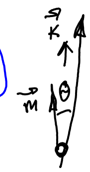
(The electric dipole field was
$$\vec{A}^{(\vec{p})}_\omega = -\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{p}_\omega\ )$$

Magnetic fied: $\vec{B}_\omega = \vec{\nabla}\times\vec{A}_\omega = i\vec{k}\times\vec{A}_\omega = \frac{i^2k^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{e}_k\times(\vec{e}_k\times\vec{m}_\omega) = \frac{\mu_0k^2}{4\pi}(\vec{e}_k\times\vec{m}_\omega)\times\vec{e}_k\frac{e^{ikr}}{r}$

Electric field: $\vec{E}_\omega = c(\vec{B}_\omega\times\vec{e}_k)$

(from the Maxwell eq.)

$\boxed{\vec{B}_\omega \perp \vec{k}}$

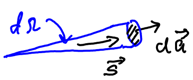


Poyting vector: $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \vec{e}_k\, c\,\frac{\vec{B}_\omega^2}{\mu_0}\frac{\cos^2(kr-\omega t)}{r^2} = \vec{e}_k\, c\,\frac{\mu_0k^4}{16\pi^2}m_\omega^2\sin^2\theta\cdot\frac{\cos^2(kr-\omega t)}{r^2}$

Intensity of the magnetic moment emisson:

$$I(\theta) = \frac{\vec{S}\,d\vec{a}}{d\Omega} = \frac{S\,r^2\cancel{d\Omega}}{\cancel{d\Omega}} = \frac{\mu_0ck^4}{16\pi^2}m_\omega^2\sin^2\theta\cos^2\phi = \frac{\mu_0}{4\pi}|\ddot{\vec{m}}|^2\cdot\sin^2\theta\cdot\cos^2\phi$$

$(\phi = kr-\omega t)$,



Quadrupole term:

$$\vec{A}^{(Q)}_{\omega} = -\frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\underbrace{\frac{1}{2}\int\left\{(\vec{e}_k\vec{r}\,')\vec{J}_\omega + (\vec{e}_k\vec{J}_\omega)\vec{r}\,'\right\}d^3r'}_{-\frac{i\omega}{2}\int \vec{r}\,'(\vec{e}_k\vec{r}\,')\rho_\omega(\vec{r}\,')\,d^3r'}$$

$\rho_\omega(\vec{r})$ - the density of electric charge

$-i\omega\rho_\omega(\vec{r}) + \vec{\nabla}\vec{J} = 0$

$$\boxed{\vec{A}^{(Q)}_{\omega} = -\frac{ck^2\mu_0}{8\pi}\frac{e^{ikr}}{r}\int \vec{r}\,'(\vec{e}_k\vec{r}\,')\rho_\omega(\vec{r}\,')\,d^3r'}$$

$$\vec{B}_\omega = \vec{\nabla}\times\vec{A}_\omega = -\frac{ick^3}{8\pi}\mu_0\frac{e^{ikr}}{r}\underbrace{\int(\vec{e}_k\times\vec{r}\,')(\vec{e}_k\cdot\vec{r}\,')\rho_\omega(\vec{r}\,')\,d^3r'}$$

This integral: $\vec{e}_k\times\int\vec{r}\,'(\vec{e}_k\vec{r}\,')\rho(\vec{r}\,')\,d^3r' = \frac{1}{3}\,\vec{e}_k\times\vec{Q}(\vec{e}_k)$

$$Q_\alpha(\vec{e}_k) = \sum_\beta Q_{\alpha\beta}\cdot e_{k,\beta}$$

vector $\vec{Q} = (Q_x, Q_y, Q_z)$

The "quadrupole vector" $\vec{Q}(\hat{e}_k)$ is constructed using components of the quadrupole momentum $Q_{\alpha\beta}$

$$\boxed{Q_{\alpha\beta} = \int(3x_\alpha x_\beta - \delta_{\alpha\beta}r^2)\rho(\vec{x})\,d^3x}$$

Components of vector $\vec{Q}_k$

$\vec{Q}_k = (Q_x, Q_y, Q_z)$

$$\left(Q_\alpha = \sum_\beta Q_{\alpha\beta}e_{k,\beta}\right)$$

$$\begin{cases}\vec{B}_\omega = \vec{\nabla}\times\vec{A}_\omega = -\frac{i\mu_0 ck^3}{24\pi}\frac{e^{ikr}}{r}(\vec{e}_k\times\vec{Q}_k)\\ \vec{E}_\omega = c(\vec{B}_\omega\times\vec{e}_k)\end{cases}$$

$$\boxed{\langle I^{(Q)}(\theta)\rangle_t = \frac{dP}{d\Omega} = \frac{\mu_0\,\omega^6}{1152\pi^2c^3}\left|[\vec{e}_k\times\vec{Q}_k]\times\vec{e}_k\right|^2}$$

Time-averaged angular intensity

Simplest example:

$Q_{zz} = Q_0$

$Q_{xx} = Q_{yy} = -\frac{Q_0}{2}$

$$\boxed{\langle I^{(Q)}(\theta)\rangle_t = \frac{dP}{d\Omega} = \frac{\mu_0\omega^6Q_0^2}{512\pi^2}\sin^2\theta\cdot\cos^2\theta}$$

$(Q_{xy} = Q_{zx} = Q_{zy} = 0)$