SVT Gauge Notes

Conformal to flat perturbed line element

$$ds^2 = \Omega^2 \left\{-(1+2\phi)d\tau^2 + (\partial_i B + B_i)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}]dx^i dx^j\right\}$$

where

$$\partial_i B^i = \partial_i E^i = 0, \qquad \partial_i E^{ij} = 0, \qquad \delta_{ij}E^{ij} = 0.$$

Minkowski Background

Under coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, the metric transforms as

$$g'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\rho}g^{\lambda\rho}(x)$$

which leads us to first order

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\nu.$$

Let us define

$$\epsilon_\mu = (\alpha, \partial_i\epsilon + \epsilon_i)$$

where the spatial vector has been decomposed into scalar ($\partial_i\epsilon$) and transverse ($\partial_i\epsilon^i = 0$) components. Now we form the transformations:

$$h'_{00} = h_{00} - 2\dot{\alpha}$$
$$\phi' = \phi + \dot{\alpha}$$

$$h'_{0i} = h_{0i} - \partial_i\dot{\epsilon} - \partial_i\alpha - \dot{\epsilon}_i$$
$$B' = B - \dot{\epsilon} - \alpha$$
$$B'_i = B_i - \dot{\epsilon}_i$$

$$h'_{ij} = h_{ij} - 2\partial_i\partial_j\epsilon - \partial_j\epsilon_i - \partial_i\epsilon_j$$
$$\psi' = \psi$$
$$E' = E - \epsilon$$
$$E'_i = E_i - \epsilon_i$$
$$E'_{ij} = E_{ij}.$$

Linear combinations of the perturbations can form gauge invariant quantities:

$$\phi_B = \phi + \dot{B} - \ddot{E}$$

$$\psi_B = \psi$$

$$(F_i)_B = \dot{E}_i - B_i$$
$$(E_{ij})_B = E_{ij}$$

K=0 RW background

$$\phi' = \phi - \dot{\alpha} - \left(\frac{\dot{a}}{a}\right)\alpha$$
$$B' = B + \alpha - \dot{\epsilon}$$
$$B_i' = B_i - \dot{\epsilon}_i$$
$$\psi' = \psi + \left(\frac{\dot{a}}{a}\right)\alpha$$
$$E' = E - \epsilon$$
$$E_i' = E_i - \epsilon_i$$
$$E_{ij}' = E_{ij}$$

Gauge Invariant Combinations are now:

$$\phi_B = \phi + \frac{\dot{a}}{a}\left(B - \dot{E}\right) + (\dot{B} - \ddot{E})$$
$$\psi_B = \psi - \frac{\dot{a}}{a}(B - \dot{E})$$
$$F_i = \dot{E}_i - B_i$$
$$E_{ij} = E_{ij}$$

Equation Decomposition

Bianchi Identities:
First lets see how they decouple using only Bianchi identities and the decomposition of $\delta G_{\mu\nu}$. The $\nu = 0$ component of the Bianchi identity $\eta^{\mu\alpha}\partial_\alpha \delta G_{\mu\nu}$ gives

$$-\partial_0 \delta G_{00} + \delta^{ij}\nabla_j \delta G_{0i} = 0.$$

Expressed in terms of the $\delta G_{\mu\nu}$ decomposition, this is

$$2\dot{\bar{\phi}} + \nabla^2 \bar{B} = 0.$$

Time Bianchi:

$$\boxed{\nabla^2 \bar{B} = -2\dot{\bar{\phi}}}. \tag{1}$$

Now set $\nu = i$, the Bianchi identity gives

$$-\partial_0 \delta G_{0i} + \delta^{jk}\nabla_j \delta G_{ik} = 0.$$

Space Bianchi:

$$\boxed{-(\nabla_i \dot{\bar{B}} + \dot{\bar{B}}_i) - 2\nabla_i \bar{\psi} + 2\nabla_i \nabla^2 \bar{E} + \nabla^2 \bar{E}_i = 0} \tag{2}$$

For later use we will also take the divergence of the above:

$$-\nabla^2(2\bar{\psi} + \dot{\bar{B}} - 2\nabla^2 \bar{E}) = 0.$$

Decompositions:
Now let us equate decompositions in $\delta G_{\mu\nu}$ to that of $h_{\mu\nu}$. Start with δG_{00}

$$\delta G_{00}: \quad \boxed{\bar{\phi} = \nabla^2 \psi} \quad \text{or} \quad \boxed{\nabla^2 \bar{B} = -2\nabla^2 \dot{\psi}} \tag{3}$$

Another scalar to solve for is the trace $\eta^{\mu\nu}\delta G_{\mu\nu}$

$$2\bar{\phi} - 6\bar{\psi} + 2\nabla^2 \bar{E} = 2\nabla^2 \psi + 2(\nabla^2 - 3\partial_0\partial_0)\psi - 2\nabla^2(\phi + \dot{B} - \ddot{E}).$$

Substituting eq. (3) we arrive at

$$\boxed{-6\bar{\psi} + 2\nabla^2 \bar{E} = -6\ddot{\psi} + 2\nabla^2 \psi - 2\nabla^2(\phi + \dot{B} - \ddot{E})} \tag{4}$$

To relate vectors to vectors, we look at δG_{0i} and its Laplacian,

$$\delta G_{0i}: \quad \nabla_i \bar{B} + \bar{B}_i = -2\nabla_i \dot{\psi} + \frac{1}{2}\nabla^2(B_i - \dot{E}_i)$$

$$\nabla_i \nabla^2 \bar{B} + \nabla^2 \bar{B}_i = -2\nabla_i \nabla^2 \dot{\psi} + \frac{1}{2}\nabla^4(B_i - \dot{E}_i).$$

Now substitute eq. (3) into the above, giving

$$\boxed{\nabla^2 \bar{B}_i = \frac{1}{2}\nabla^4(B_i - \dot{E}_i)} \tag{5}$$

Now for the spatial components δG_{ij} we have

$$\begin{aligned}
&-2\bar{\psi}\delta_{ij} + 2\nabla_i\nabla_j \bar{E} + \nabla_i \bar{E}_j + \nabla_j \bar{E}_i + 2\bar{E}_{ij} = \\
&\quad -2\ddot{\psi}\delta_{ij} - (\nabla_i\nabla_j - \delta_{ij}\nabla^2)\psi + (\nabla_i\nabla_j - \nabla^2\delta_{ij})(\phi + \dot{B} - \ddot{E}) + \frac{1}{2}\nabla_i(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\nabla_j(\dot{B}_i - \ddot{E}_i) + \Box E_{ij}.
\end{aligned}$$

We may form a scalar by taking $\delta^{ij}\delta G_{ij}$ or $\nabla^i\nabla^j\delta G_{ij}$. The former is the trace equation, while the later is

$$\boxed{-2\nabla^2\bar{\psi} + 2\nabla^4 \bar{E} = -2\nabla^2\ddot{\psi}}. \tag{6}$$

This is equivalent to the equation we found for the divergence of the Bianchi space identity. We can try to form a vector by taking the divergence $\nabla^j\delta G_{ij}$

$$-2\nabla_i\bar{\psi} + 2\nabla_i\nabla^2\bar{E} + \nabla^2\bar{E}_i = -2\ddot{\nabla}_i\psi + \frac{1}{2}\nabla^2(\dot{B}_i - \ddot{E}_i)$$

If we further take the Laplacian we can subtitute eq. (6) in to give

$$\boxed{\nabla^4 \bar{E}_i = \frac{1}{2}\nabla^4(\dot{B}_i - \ddot{E}_i)} \tag{7}$$

In order to equate tensor components, scalar and vector parts must cancel. For the vector portions to cancel according to eq. (7), we must take $\nabla^4\delta G_{ij}$. This gives

$$\begin{aligned}
&-2\nabla^4\bar{\psi}\delta_{ij} + 2\nabla_i\nabla_j\nabla^4\bar{E} + \nabla_i\nabla^4\bar{E}_j + \nabla_j\nabla^4\bar{E}_i + 2\nabla^4\bar{E}_{ij} = \\
&\quad -2\nabla^4\ddot{\psi}\delta_{ij} - (\nabla_i\nabla_j - \delta_{ij}\nabla^2)\nabla^4\psi + (\nabla_i\nabla_j - \nabla^2\delta_{ij})\nabla^4(\phi + \dot{B} - \ddot{E}) + \frac{1}{2}\nabla_i\nabla^4(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\nabla_j\nabla^4(\dot{B}_i - \ddot{E}_i) + \nabla^4\Box E_{ij}
\end{aligned}$$

in which the vector portion of the equation is

$$\nabla_i\nabla^4\bar{E}_j + \nabla_j\nabla^4\bar{E}_i = \frac{1}{2}\nabla_i\nabla^4(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\nabla_j\nabla^4(\dot{B}_i - \ddot{E}_i).$$

This is satisfied by eq. (7) and hence the vectors will drop out. However, the scalar portions

$$-2\nabla^4\bar{\psi}\delta_{ij} + 2\nabla_i\nabla_j\nabla^4\bar{E} = -2\nabla^4\ddot{\psi}\delta_{ij} - (\nabla_i\nabla_j - \delta_{ij}\nabla^2)\nabla^4\psi + (\nabla_i\nabla_j - \nabla^2\delta_{ij})\nabla^4(\phi + \dot{B} - \ddot{E})$$

will not cancel using substitutions eq. (4) and (6). Thus I am not sure how to relate tensor components to each other.

Pure Gauge/ Gauge Definition

After decomposing the metric into SVT as

$$ds^2 = -(1+2\phi)d\tau^2 + (\partial_i B + B_i)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}]dx^i dx^j$$

there exist transformations that leave the metric invariant. These are

$$B \to B + p$$

$$B_i \to B_i - \nabla_i p$$

$$E \to E + q$$

$$E_i \to E_i - \nabla_i q + 2r_i$$

$$E_{ij} \to E_{ij} - \nabla_i r_j - \nabla_j r_i.$$

In order to preserve transverness in B_i, E_i and E_{ij} we require

$$\nabla^2 p = 0$$

$$\nabla^2 q = 0$$

$$\nabla^2 r_i = 0$$

$$\nabla^i r_i = 0.$$

The above total to four indepedent components, as expected from the four gauge depedent coordinate transformations. In forming the decomposition of the metric, we choose scalars, vectors, and tensors such that q, p, and r_i are zero, or in other words $B' = B + p$ etc.