Astrophysics & Cosmology
HW 4

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7.1 Starting with the result

$$P = nv_x p_x$$

with

$$p_x \approx hn^{1/3}$$

and

$$v_x = p_x/m_e$$

we have

$$P = np_x^2/m_e = \frac{h^2 n_e^{5/3}}{m_e}.$$

For a density of ions of atomic number Z and weight A, from neutrality we have

$$Zn_+ = n_e$$

and the density is then

$$\rho = Am_p n_+.$$

Substituting the above condition into the neutrality equation,

$$n_e = Zn_+ = Z\frac{\rho}{Am_p}.$$

The precise formula for the electron degeneracy pressure is

$$P_e = 0.0485\frac{h^2 n_e^{5/3}}{m_e}.$$

Substituting the relation for n_e into this we have

$$P_e = 0.0485\frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\rho}{m_p}\right)^{5/3}.$$

For typical white dwarf conditions, we have

$$\rho = 10^6 \text{ gm cm}^{-3}, \qquad T = 10^7 \text{ K}, \qquad \frac{Z}{A} = 0.5.$$

Under these conditions, the thermal pressure is

$$P_T = n_e kT = \frac{1}{2}\frac{\rho}{m_p}kT = 4.13 \times 10^{20} \text{ Ba}$$

Compare this to the degeneracy pressure

$$P_e = 0.0485\frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\rho}{m_p}\right)^{5/3} = 3.14 \times 10^{22} \text{ Ba}.$$

We see that the electron degeneracy pressure is greater by about a factor of 100.

For a gas of fermionic ions, the degeneracy pressure is given the same, except for n_+ number density of ions wit now a total atomic weight Am_p. Thus

$$P_+ = 0.0485 h^2 \frac{n_+^{5/3}}{Am_p}$$

This pressure is reduced compared to the electron degeneracy pressure because the number density is reduced $n_+ = n_e/Z$ and because the ions are more massive $m_+ = Am_p$. Similarly, the ion thermal pressure n_+kT will be less than the electron thermal pressure.

7.2 From the last problem we showed that the electron degeneracy pressure is given as

$$P_e = 0.0485 \frac{h^2}{m_e} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{\rho}{m_p}\right)^{5/3}.$$

In order to calculate the degeneracy pressure at the center of the star, we must evaluate the number density n_e at the center. This is given as

$$(n_e)_c = \frac{Z}{A} \frac{\rho_c}{m_p}$$

(the above is also in 7.1, coming from neutrality condition). Substituting this in, the central degeneracy pressure is then

$$(P_e)_c = 0.0485 \frac{h^2}{m_e} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{\rho_c}{m_p}\right)^{5/3}.$$

Equating this pressure equal to the gravitational pressure

$$P_c = 0.770 \frac{GM^2}{R^4}$$

we have

$$0.0485 \frac{h^2}{m_e} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{\rho_c}{m_p}\right)^{5/3} = 0.770 \frac{GM^2}{R^4}.$$

The central density can be expressed as

$$\rho_c = 1.43 \frac{M}{R^3}$$

so that we have

$$0.0485 \frac{h^2}{m_e} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{1.43M}{m_p}\right)^{5/3} \frac{1}{R^5} = 0.770 \frac{GM^2}{R^4}.$$

Solving for R we have

$$R = 0.114 \frac{h^2}{G m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} M^{-1/3}.$$

Taking $\frac{Z}{A} = 0.5$, we compute this R for two masses

$$R(0.5M_{sun}) = 1.1 \times 10^9 \text{ cm}$$

$$R(M_{sun}) = 8.8 \times 10^8 \text{ cm}.$$

We see that the larger mass has a smaller radius. In fact, the radius of the earth and the radius of a star with electron degeneracy pressure equal to the gravitational pressure is of the same order of magnitude of the radius of the earth $R = 7 \times 10^8$ cm.

7.3 Relativistically, the electron degeneracy pressure is given by the max value at $v = c$

$$(P_e)_{rel} = 0.123hcn_e^{4/3}.$$

In order for the relativistic pressure to equal the non-relativistic degen. pressure we used prior, we have

$$0.123hcn_e^{4/3} = 0.0485h^2\frac{n_e^{5/3}}{m_e}.$$

This gives a n_e of

$$n_e = \left(2.54\frac{m_e c}{h}\right)^3 = 1.15 \times 10^{30} \text{ cm}^{-3}.$$

At this value of n_e the typical velocity is then

$$v_x = p_x/m_e \approx \frac{hn_e^{1/3}}{m_e} = 7.6 \times 10^{10} \text{ cm sec}^{-1} = 2.54c.$$

From 7.2, the central density is given as

$$\rho_c = 1.43\frac{M}{R^3}.$$

Using

$$M = M_{sun}, \qquad R(M_{sun}) = 8.8 \times 10^8$$

this gives

$$\rho_c = 4.2 \times 10^6$$

and so

$$(n_e)_c = \frac{Z}{A}\frac{\rho_c}{m_p} = 1.25 \times 10^{30} \text{ cm}^{-3}.$$

The density at the center of the white dwarf of one solar mass is nearly equal in magnitude to the relativistic electron density.

7.4 Substituing

$$n_e = \frac{Z}{A}\frac{\rho}{m_p}$$

into

$$(P_e)_{rel} = 0.123hcn_e^{4/3}.$$

we have

$$(P_e)_{rel} = 0.123hc\left(\frac{Z}{A}\right)^{4/3}\left(\frac{\rho}{m_p}\right)^{4/3}.$$

For the self-gravitating sphere at hydrostatic equilibrium we have

$$P_c = 11\frac{GM^2}{R^4}, \qquad \rho_c = 54.2\frac{3M}{4\pi R^3}.$$

Equating the central gravitational pressure to the electron degeneracy pressure (relativistic)

$$11\frac{GM^2}{R^4} = 0.123hc\left(\frac{Z}{Am_p}\right)^{4/3}\left(54.2\frac{3M}{4\pi R^3}\right)^{4/3}.$$

We see that we have the same power of R. We solve for M

$$M = 0.198\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Z}{Am_p}\right)^2.$$

For $Z/A = 1/2$, we may compute the Chandrasekhar mass as

$$M = 2.9 \times 10^{33} \text{ gm} = 1.45 M_{sun}$$

7.5 The critical temperature at which the thermal energy per ion is equal to its electrostatic energy is given by setting the two equal

$$Z^2 e^2 n_+^{1/3} = kT.$$

For a mean density of helium $A = 4$ and

$$\rho = A m_p n_+ = 4 m_p n_+ = 10^6 \text{ gm cm}^{-3}.$$

Thus

$$n_+ = 10^6 / 4m_p = 1.5 \times 10^{29} \text{ cm}^{-3}.$$

The critical temperature is then for $Z = 2$

$$T = \frac{Z^2 e^2 n_+^{1/3}}{k} = 3.5 \times 10^7 \text{ K}.$$

This critical temperature corresponds to the temperature at which the star is stable and does not undergo any thermonuclear reactions. Any increase in temperature will allow nuclear fusion reactions - at $T = 10^8$ K, we will have high enough temperature to undergo the triple alpha processes converting helium into carbon.

7.6 In a neutron star, protons are combined with electrons to produce neutrons. Thus the number of neutrons will be equal to the number of protons, and we will assume $m_p \approx m_n$. As fermions, they will have a degeneracy pressure equal to

$$P_p = 0.0485 \frac{h^2 n_p^{5/3}}{m_p}.$$

Now with

$$n_p = \rho / m_p$$

we have

$$P_p = 0.0485 \frac{h^2}{m_p} \left(\frac{\rho}{m_p} \right)^{5/3}$$

With $\rho_c = 1.43 \frac{M}{R^3}$, we set the central degeneracy pressure equal to the gravitational pressure

$$0.770 \frac{GM^2}{R^4} = 0.0485 \frac{h^2}{m_p^{8/3}} \left(\frac{M}{R^3} \right)^{5/3}$$

and solve for R

$$R = 0.114 \frac{h^2}{G m_p^{8/3}} M^{-1/3}.$$

7.12 Here we are deriving heuristically the relation of the temperature of black holes

$$kT = \frac{hc^3}{16\pi^2 GM}.$$

From the uncertaintiy principle we have particle-antiparticle pair production from vacuum fluctuations

$$\Delta E \Delta t \approx \frac{\hbar}{2}.$$

If these pairs travel a separation distance approximately equal to half the circumferance of the black hole, then

$$c\Delta t/2 \approx 2\pi GM/c^2$$

$$\Delta t = \frac{4\pi GM}{c^3}.$$

Then the thermal energy can be written as

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{hc^3}{16\pi^2 GM}$$

or

$$kT = \frac{hc^3}{16\pi^2 GM}.$$

For a black hole with mass $M = M_{sun}$

$$T(M_{sun}) = 6.2 \times 10^{-8} \text{ K}$$

and for $M = 10^{15}$ gm

$$T(M = 10^{15}) = 1.2 \times 10^{11} \text{ K}.$$

Mini black holes have extremely high temperatures. The Schwarzchild radius of the miniblack hole is

$$R_{sch} = \frac{2GM}{c^2} = 1.5 \times 10^{-13} \text{ cm}.$$

This is about the same as the classical electron radius.