Astrophysics & Cosmology
HW 11

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15.1 The energy of a select galaxy of mass m lying at the edge of sphere of radius r of mass $M = 4/3\pi r^3\rho$ is

$$E = \frac{1}{2}m\dot{r}^2 - \frac{4\pi Gmr^2}{3}.$$

Setting $E = 0$ and using $\dot{r} = H_0 r$ we have

$$H_0^2 = \frac{8\pi G}{3}\rho.$$

or

$$\rho = \frac{3H_0^2}{8\pi G}.$$

Since this is the energy density such that $E = 0$, this is the critical energy density $\rho = \rho_c$. This density may be calculated at present time given

$$H_0 = 20 \text{ km/sec/million lyr}$$

which yields in CGS units

$$\rho_c \approx 8 \times 10^{-30} \text{ g cm}^{-3}.$$

To make an estimate of comparison of the energy density, we take the mass of the Local Group

$$M = 4 \times 10^{11} M_{sun}$$

spread over a volume of the M81 group

$$V = (15 \times 10^6)^3 \text{ lyr}^3.$$

This yields

$$\rho = \frac{M}{V} = 2.8 \times 10^{-31} \text{ g cm}^{-3}$$

which is very close but about a factor of 10 smaller than the critical density today.

15.2 Setting

$$r \approx 10^{10}/H_0 \approx 10^{10}/10^{-18} \text{ cm} \approx 10^{10} \text{ lyr}.$$

Referring to figure 14.23, distant ScI galaxies are at most 6×10^8 lyr. This is still an appreciable 100 times less than velocities required to begin approaching the speed of light. Thus it is likely that at these distances, H_0 still provides a good estimation for the distance as $v \ll c$.

15.3 For $E = 0$ we have

$$0 = \frac{1}{2}m\dot{r}^2 - \frac{GmM}{r}$$

in which m cancels. Then we rearrange

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r}}. \tag{1}$$

Since we measure red shifts and not blue shifts, the radius r is increasing over time and thus $\dot{r} > 0$. Hence the positive sign on the square root. Integrating (1) we have

$$\int_0^r dr\ \sqrt{r} = \int_0^t dt\ (2GM)^{1/2}$$
$$\frac{2}{3}r^{3/2} = (2GM)^{1/2}t$$

which implies

$$r = \frac{3}{2}(2GM)^{1/3}t^{2/3}.$$

It follow that at present time

$$H_0 r_0 = \dot{r}_0 = (2GM/r_0)^{1/2} = H_0\frac{3}{2}(2GM)^{1/3}t^{2/3}$$

or

$$H_0(3/2t_0)^{2/3}(2GM)^{1/3} = (2GM)^{1/3}(3/2t_0)^{1/3}$$
$$\frac{3}{2}H_0 t_0 = 1$$
$$t_0 = 2/3H_0^{-1}.$$

For $\dot{r} > \left(\frac{2GM}{r}\right)^{1/2}$ we have $E > 0$. Since $t \propto r^{3/2}$ and r increases at a faster rate than the marginally bound $E = 0$ we know that we must have

$$\frac{2}{3}H_0^{-1} < t_0 < t_{max} \qquad E > 0$$

where t_{max} is shown explicitly in Box 15.2 to take the value H_0^{-1}. For $E < 0$, $\dot{r} < \left(\frac{2GM}{r}\right)^{1/2}$ and the bound universe may only decrease to a lower bound of $t_0 = 0$. Thus

$$0 < t_0 < \frac{2}{3}H_0^{-1} \qquad E < 0.$$

15.4 Using $v = r_0 H_0$ and $M = 4/3\pi r_0^3\rho$ the energy equation may be written as

$$E = m(H_0^2/2 - 4\pi G\rho/3)r_0^2.$$

Setting this equal to 15.1 at arbitrary r (energy is conserverd)

$$m(H_0^2/2 - 4\pi G\rho/3)r_0^2 = \frac{1}{2}m\dot{r}^2 - \frac{GmM}{r}$$
$$(1-\Omega_0) = \left(\frac{\dot{r}}{H_0 r_0}\right)^2 - \frac{2GM}{H_0^2 r_0^2 r}$$
$$(1-\Omega_0) = \left(\frac{\dot{r}}{H_0 r_0}\right)^2 - \frac{\Omega_0 r_0}{r}$$

With substitution $D = r/r_0$ and $\tau^* = H_0 t$ it follows that

$$\frac{dD}{d\tau^*} = \frac{1}{r_0 H_0}\dot{r}$$

and the energy equation yields

$$(1 - \Omega_0) = \left(\frac{dD}{d\tau^*}\right)^2 - \frac{\Omega_0}{D}. \tag{2}$$

We make another substitution

$$\xi = (|1 - \Omega_0|/\Omega_0)D$$

and

$$\tau = (|1 - \Omega_0|^{3/2}/\Omega_0)\tau^*.$$

We may rearrage (2) as

$$1 = \left(\frac{dD}{d\tau^*}\right)^2 \frac{1}{(1 - \Omega_0)} - \frac{\Omega_0}{(1 - \Omega_0)D}$$

Now with

$$\frac{d\xi}{d\tau} = \frac{1}{|1 - \Omega_0|^{1/2}} \frac{dD}{d\tau^*}$$

we can see that

$$\left(\frac{d\xi}{d\tau}\right)^2 - \frac{1}{\xi} = 1.$$

Here we have assumed that $0 < \Omega_0 < 1$. For $\Omega_0 > 1$, we have an overall change in sign, thus we can take absolute magnitudes and write the expression as

$$\left(\frac{d\xi}{d\tau}\right)^2 - \frac{1}{\xi} = \pm 1 \tag{3}$$

without any restriction on Ω_0. (3) yields the differential equation

$$d\tau = (\pm 1 + 1/\xi)^{-1/2}\, d\xi = \left(\frac{\xi}{1 \pm \xi}\right)^{1/2} d\xi$$

or

$$\tau = \int_0^\xi d\xi \, \left(\frac{\xi}{1 \pm \xi}\right)^{1/2}$$

for $\xi(0) = 0$. For $1 - \xi$ in the denominator $\xi \leq 1$ and we make substitution

$$\xi = \sin^2(\eta/2)$$

such that

$$d\xi = \sin(\eta/2)\cos(\eta/2)d\eta$$

and

$$\begin{aligned}
\tau &= \int d\eta \;\, \sin(\eta/2)\cos(\eta/2) \left(\frac{\sin^2(\eta/2)}{\cos^2(\eta/2)}\right)^{1/2} \\
&= \int d\eta \;\, \sin^2(\eta/2) \\
&= \int d\eta \; \frac{1}{2}(1 - \cos\eta) \\
\tau &= \frac{1}{2}(\eta - \sin\eta)
\end{aligned}$$

where we also note that $\xi(\eta = 0) = 0$.

For the positive denominator, $1 + \xi$ has $\xi > 0$ we have an unbound universe and we use the substitution

$$\xi = \sinh^2(\eta/2)$$

such that

$$d\xi = \sinh(\eta/2)\cosh(\eta/2)d\eta$$

and

$$\begin{aligned}
\tau &= \int d\eta \;\; \sinh(\eta/2)\cosh(\eta/2)\left(\frac{\sinh^2(\eta/2)}{\cosh^2(\eta/2)}\right)^{1/2} \\
&= \int d\eta \;\; \sinh^2(\eta/2) \\
&= \int d\eta \; \frac{1}{2}(\cosh\eta - 1) \\
\tau &= \frac{1}{2}(\sinh\eta - \eta)
\end{aligned}$$

where again $\xi(\eta = 0) = 0$.

For the bound universe $\xi = \frac{1}{2}(1 - \cos\eta)$ we see that at $\eta = \pi$, ξ is maximum, and assumes the value of one. Thus

$$\xi = 1 = \frac{|\Omega_0 - 1|}{\Omega_0}\frac{r_{max}}{r_0}$$

which leads to

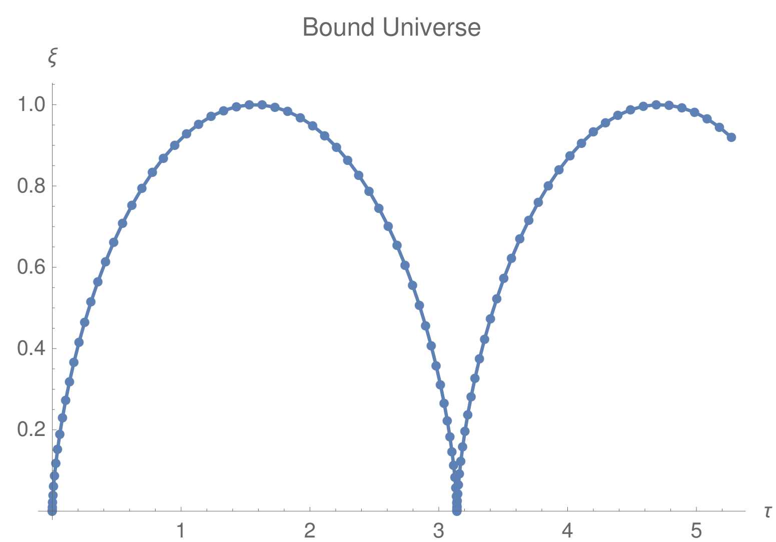
$$r_{max} = \frac{\Omega_0}{\Omega_0 - 1} r_0.$$

From the figures below, we will find that the behavior is the same towards the origin because a series expansion around $\eta = 0$ up to leading order in η gives the same for the bounded and unbounded universe, that is

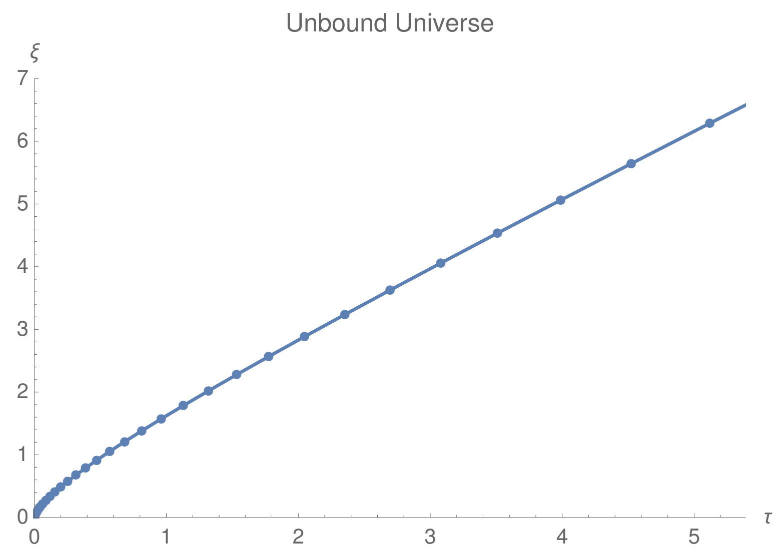
$$\xi \approx \eta^2$$

$$\tau \approx \frac{x^3}{3}.$$

Below we have included sample tables, though the graphs generated used many more points.



τ	ξ	η
0.	0.	0.
0.0792645	0.229849	1.
0.545351	0.708073	2.
1.42944	0.994996	3.
2.3784	0.826822	4.
2.97946	0.358169	5.
3.13971	0.0199149	6.
3.17151	0.123049	7.
3.50532	0.57275	8.
4.29394	0.955565	9.
5.27201	0.919536	10.



τ	ξ	η
0.	0.	0.
0.0876006	0.27154	1.
0.81343	1.3811	2.
3.50894	4.53383	3.
11.645	13.1541	4.
34.6016	36.605	5.
97.8566	100.358	6.
270.658	273.659	7.
741.239	744.74	8.
2021.27	2025.27	9.
5501.62	5506.12	10.

15.8 Starting with the Freidmann equation

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} = -\frac{c^2}{R^2}$$

we substitute $\rho = \frac{M}{2\pi^2 R^3}$ and multiply through by $\frac{1}{2}R^2$ to arrive at

$$\frac{1}{2}\dot{R}^2 - \frac{2G}{3\pi}\left(2\pi^2 \rho R^2\right) = -\frac{1}{2}c^2$$

or

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{3\pi R} = -\frac{1}{2}c^2. \tag{4}$$

Compared to eq 15.1

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}$$

both involve time derivatives of a radius R governed by a gravtiational interaction, however, the Freidmann equation makes no reference to the intrinsic mass of an object and differs in coeffiecents of kinetic and potential - like terms. Also, the energy constant is given by c^2 in the Freidmann equation. We now introduce auxially variable

$$\xi = \frac{3\pi c^2}{4GM}R$$

$$\tau = \frac{3\pi c^3}{4GM}t$$

such that

$$\frac{d\xi}{d\tau} = \frac{\dot{R}}{c}.$$

Multiplying (4) through by $2/c^2$, we now see that (4) may be written in terms of ξ and τ as

$$\left(\frac{d\xi}{d\tau}\right)^2 - \frac{1}{\xi} = -1.$$

This is the same differential equation solved in 15.4, with solutions

$$\xi = \frac{1}{2}(1 - \cos\eta), \qquad \tau = \frac{1}{2}(\eta - \sin\eta).$$

We see that ξ takes its maximum value at $\xi(\pi) = 1$. It then follows that the maximum radius is

$$R_{max} = \frac{4GM}{3\pi c^2}.$$

The time elapsed from big bang to big squeeze is $\tau(\xi = 0)$. Since ξ reaches its second zero at $\eta = 2\pi$, this yields

$$\tau(2\pi) = \pi = \frac{3\pi c^3}{4GM}t$$

or

$$t = \frac{4GM}{3c^3}.$$

For $M = 10^{24}M_{sun}$ we have

$$R_{max} \approx 6 \times 10^{10} \text{ lyr}$$

$$t_{squeeze} = 6.5 \times 10^{18} \text{ sec} = 2 \times 10^{11} \text{ yr}.$$

We may use the hubble relation

$$H_0 = \frac{\dot{R}_0}{R_0}$$

to express (4) as

$$H_0^2 - \frac{4GM}{3\pi}\frac{1}{R_0^3} = -\frac{c^2}{R_0^2}.$$

Solving this cubic equation in terms of H_0 and other constants given, we have

$$R_0 = 2.04 \times 10^{28} \text{ cm} \approx 2 \times 10^{10} \text{ lyr}.$$

This yields a ξ_0 of

$$\xi_0 = \frac{3\pi c^2}{4GM} R_0 = 0.326.$$

With $\xi_0 = \frac{1}{2}(1 - \cos\eta_0)$ this gives

$$\eta_0 = 1.21536$$

and thus

$$\tau_0 = \frac{1}{2}(\eta_0 - \sin\eta_0) = 0.1389$$

and finally

$$t_0 = \frac{4GM}{3\pi c^3}\tau_0 = 2.9 \times 10^{17} \text{ sec} = 9.2 \times 10^9 \text{ yr}.$$

Compare this to

$$t_0 = \frac{2}{3} H_0^{-1} = 10^{10} \text{ yr}$$

and we see that the estimate above is very close the numerical calculation given by GR.

15.9 The invariant volume element (for spatial metrics) is

$$dV = \sqrt{g} d^n x$$

where n denotes the dimension and g is the determinant from the matrix g_{ij} in the generalized line element

$$dl^2 = g_{ij} dx^i dx^j.$$

Using this form, we may find the "volume" of a closed surface in one, two, or three dimensions.
For $n = 1$

$$g_{ij} = R^2 \qquad g = R^2$$

$$\int dx^1 \sqrt{g} = \int_0^{2\pi} d\phi R = 2\pi R.$$

For $n = 2$

$$dl^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$g = R^2 \sin^2\theta$$

$$\int dx^1 dx^2 \sqrt{g} = \int_0^{\pi} d\theta \ R \int_0^{2\pi} d\phi \ R\sin\theta = 4\pi R^2$$

For $n = 3$

$$dl^2 = R^2[d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\phi^2]$$

$$g = R^2(sin^2\psi \sin^2\psi \sin^2\theta)$$

$$\int dx^1 dx^2 dx^3 \sqrt{g} = \int_0^{\pi} d\psi \ R \int_0^{\pi} d\phi \ R\sin\theta \int_0^{2\pi} d\psi \ R\sin\psi\sin\theta$$
$$= (R\pi)(2\pi R^2) = 2\pi^2 R^3.$$

We note that in the last problem, we used

$$\rho = M/V = M/(2\pi^2 R^3)$$

which is the proper volume to use in a space with positive curvature.

To put our line element in a more suitiable form, we make the coordinate transformation

$$r = R \sin \psi$$

such that

$$dr = R \cos \psi d\psi.$$

Now

$$(1 - r^2/R^2)^{-1} dr^2 = (1 - \cos^2 \psi)^{-1} R^2 \cos^2 \psi d\psi^2 = R^2 d\psi^2.$$

Thus we replace $R^2 d\psi^2$ in the line element

$$dl^2 = R^2[d\psi^2 + \sin^2 \psi d\theta^2 + \sin^2 \psi \sin^2 \theta d\phi^2]$$

to yield

$$dl^2 = \frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

This is the line element of 3-space with uniform positive curvature.