$\delta W_{\mu\nu}$ Residual Gauge v2

In the transverse gauge $\partial_\nu K^{\mu\nu} = 0$ in the Minkowski background the vacuum equation of motion for the traceless $K_{\mu\nu}$ is

$$\delta W_{\mu\nu} = \eta^{\alpha\beta}\eta^{\sigma\rho}\partial_\alpha\partial_\beta\partial_\sigma\partial_\rho K_{\mu\nu} = 0. \tag{1}$$

The momentum eigenstate solutions take the form

$$K_{\mu\nu} = A_{\mu\nu}e^{ikx} + n_\alpha x^\alpha B_{\mu\nu}e^{ikx} + \text{c.c.} \tag{2}$$

where $n_\alpha = (1, 0, 0, 0)$ and $k^\mu k_\mu = 0$. Following the transverse condition, the solution must obey

$$0 = (ik^\nu A_{\mu\nu} + n^\nu B_{\mu\nu})\, e^{ikx} + (ik^\nu B_{\mu\nu})\, n_\alpha x^\alpha e^{ikx} + \text{c.c.} \tag{3}$$

In addition to the tracelessness condition, to satisfy all x (noting that e^{ikx}, e^{-ikx}, te^{ikx} and te^{-ikx} are linearly independent), we set in (3) each coefficient preceding the space-time dependent function to zero, viz.

$$A^\mu{}_\mu = 0, \qquad B^\mu{}_\mu = 0, \qquad ik^\nu A_{\mu\nu} + n^\nu B_{\mu\nu} = 0, \qquad ik^\nu B_{\mu\nu} = 0. \tag{4}$$

We have a total of 10 conditions upon the 20 total components of $A_{\mu\nu}$ and $B_{\mu\nu}$. It is easy to check that these conditions (and also their implied conjugate expressions) satisfy our choice of transverse coordinate system and retain the tracelessness of $K_{\mu\nu}$. Under infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, $K_{\mu\nu}$ transforms as

$$K'_{\mu\nu} = K_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu + \frac{1}{2}g_{\mu\nu}\partial_\rho\epsilon^\rho. \tag{5}$$

We denote the change in $K_{\mu\nu}$ (Lie derivative) as the tensor

$$\Delta K_{\mu\nu} = -\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu + \frac{1}{2}g_{\mu\nu}\partial_\rho\epsilon^\rho. \tag{6}$$

Noting that $\Delta K_{\mu\nu}$ is manifestly traceless, in order to preserve the tranverse gauge condition $\partial_\mu K^{\mu\nu} = 0$, $\Delta K^{\mu\nu}$ must obey $\partial_\nu \Delta K^{\mu\nu} = 0$, viz.

$$0 = -\partial_\nu\partial^\nu\epsilon^\mu - \frac{1}{2}\partial^\mu\partial_\nu\epsilon^\nu. \tag{7}$$

We take the $\epsilon^\mu(x)$ to be of the plane wave form,

$$\epsilon^\mu(x) = iA^\mu e^{ikx} + iB^\mu n_\alpha x^\alpha e^{ikx} + \text{c.c.}, \tag{8}$$

which obeys the following relations:

$$\partial^\nu\epsilon^\mu = -k^\nu\left(A^\mu e^{ikx} + B^\mu n_\alpha x^\alpha e^{ikx}\right) + in^\nu\left(B^\mu e^{ikx}\right) + \text{c.c.} \tag{9}$$

$$\partial_\nu\partial^\nu\epsilon^\mu = -2k_\nu n^\nu\left(B^\mu e^{ikx}\right) + \text{c.c.}, \tag{10}$$

$$\partial_\mu\partial^\nu\epsilon^\mu = -ik_\mu k^\nu\left(A^\mu e^{ikx} + B^\mu n_\alpha x^\alpha e^{ikx}\right) - (k^\nu n_\mu + k_\mu n^\nu)\left[B^\mu e^{ikx}\right] + \text{c.c.}, \tag{11}$$

where for reference we also have the relation

$$\partial_\beta \partial^\beta (n_\alpha x^\alpha e^{ikx}) = 2i n_\alpha k^\alpha e^{ikx}. \tag{12}$$

The transverse condition per (7) then takes the form

$$0 = 2k_\nu n^\nu \left(B^\mu e^{ikx}\right) + \frac{1}{2} i k_\nu k^\mu \left(A^\nu e^{ikx} + B^\nu n_\alpha x^\alpha e^{ikx}\right) + \frac{1}{2}(k^\mu n_\nu + k_\nu n^\mu)\left[B^\nu e^{ikx}\right] + \text{c.c. .} \tag{13}$$

To hold for arbitrary x, we have the two separate conditions,

$$2k_\nu n^\nu B^\mu + \frac{1}{2} i k_\nu k^\mu A^\nu + \frac{1}{2}(k^\mu n_\nu + k_\nu n^\mu) B^\nu = 0, \qquad \frac{1}{2} i k_\nu k^\mu B^\nu = 0. \tag{14}$$

For arbitrary k^μ, the second condition in 14 implies $k_\nu B^\nu = 0$. As such, the remaining condition is

$$2k_\nu n^\nu B^\mu + \frac{1}{2} k^\mu n_\nu B^\nu + \frac{1}{2} i k_\nu k^\mu A^\nu = 0. \tag{15}$$

Let us now take a wave propagating in the z direction, with wavevector

$$k^\mu = (k, 0, 0, k), \qquad k_\mu = (-k, 0, 0, k). \tag{16}$$

The transverse condition $\partial^\mu \Delta K_{\mu\nu}$ then entails

$$B_0 = -B_3, \qquad B_0 = \frac{i}{5} k(A_0 + A_3), \qquad B_1 = B_2 = 0. \tag{17}$$

We see that the specific form of $\epsilon^\mu(x)$ comprises of four independent components, here chosen as B_0, A_0, A_1, and A_2. The dependencies are:

$$B_1 = B_2 = 0, \qquad B_3 = -B_0, \qquad A_3 = -A_0 - \frac{5i}{k} B_0. \tag{18}$$

For the tensor polarizations $A_{\mu\nu}$ and $B_{\mu\nu}$ the transverse relations take the form

$$B^\mu{}_\mu = A^\mu{}_\mu = 0, \qquad B_{0\mu} = -B_{3\mu}, \qquad ik(A_{\mu 0} + A_{\mu 3}) = B_{0\mu}. \tag{19}$$

Although this would appear to be 10 total constraints, the condition $B_{00} = -B_{30}$ reduces the equation

$$ik(A_{\mu 0} + A_{\mu 3}) = B_{0\mu}, \tag{20}$$

from 4 to 3 conditions, namely

$$ik(A_{10} + A_{13}) = B_{01}, \qquad ik(A_{20} + A_{23}) = B_{02}, \qquad A_{00} + 2A_{03} + A_{33} = 0. \tag{21}$$

We will take 11 the independent components as

$$B_{00}, B_{01}, B_{02}, B_{11}, B_{12}, A_{00}, A_{01}, A_{02}, A_{11}, A_{22}, A_{12}. \tag{22}$$

In order to arrive at the following choice of independent components for $B_{\mu\nu}$, we utliize the gauge conditions which lead us to following dependencies:

$$B_{33} = -B_{03} = B_{00}, \qquad B_{23} = -B_{02}, \qquad B_{13} = -B_{01}, \qquad B_{22} = -B_{11}. \tag{23}$$

As for $A_{\mu\nu}$, the dependencies are:

$$A_{13} = -\frac{i}{k} B_{01} - A_{01}, \quad A_{23} = -\frac{i}{k} B_{02} - A_{02}, \quad A_{33} = A_{00} - A_{11} - A_{22}, \quad A_{03} = -A_{00} + \frac{1}{2}(A_{11} + A_{22}). \tag{24}$$

The form for the transformation (Lie derivative) onto $K_{\mu\nu}$ is

$$\begin{aligned}\Delta K_{\mu\nu} = &\left[k_\nu A_\mu + k_\mu A_\nu - i\left(n_\nu B_\mu + n_\mu B_\nu\right) - \frac{1}{2} g_{\mu\nu} A^\alpha k_\alpha + \frac{i}{2} g_{\mu\nu} n_\alpha B^\alpha\right] e^{ikx} \\ &+ \left[k_\nu B_\mu + k_\mu B_\nu\right] n_\alpha x^\alpha e^{ikx}.\end{aligned} \tag{25}$$

It will be useful to evaluate this for different components:

$$\begin{aligned}
&\Delta K_{00} = \left[-2kA_0 + \frac{1}{2}k(A_0 + A_3) - \frac{3i}{2}B_0\right] e^{ikx} - \left[2kB_0\right] n_\alpha x^\alpha e^{ikx} \\
&\Delta K_{01} = -kA_1 e^{ikx}, \qquad \Delta K_{02} = -kA_2 e^{ikx} \\
&\Delta K_{03} = \left[-kA_3 + kA_0 - iB_3\right] e^{ikx} - \left[2kB_3\right] n_\alpha x^\alpha e^{ikx} \\
&\Delta K_{11} = \Delta K_{22} = \left[-\frac{1}{2}k(A_0 + A_3) - \frac{i}{2}B_0\right] e^{ikx}, \qquad \Delta K_{12} = 0 \\
&\Delta K_{13} = [kA_1] e^{ikx}, \qquad \Delta K_{23} = [kA_2] e^{ikx} \\
&\Delta K_{33} = \left[2kA_3 - \frac{1}{2}k(A_0 + A_3) - \frac{i}{2}B_0\right] e^{ikx} + \left[2kB_3\right] n_\alpha x^\alpha e^{ikx}.
\end{aligned} \tag{26}$$

We express the gauge variables (A_μ and B_μ) in terms of the 4 independent components, and compute the total transformation on each polarization tensor. For $A_{\mu\nu} \to A'_{\mu\nu}$ and $B_{\mu\nu} \to B'_{\mu\nu}$, we have

$$\begin{aligned}
A'_{00} &= A_{00} - 2kA_0 - 4iB_0 & B'_{00} &= B_{00} - 2kB_0 \\
A'_{01} &= A_{01} - kA_1 & B'_{01} &= B_{01} \\
A'_{02} &= A_{02} - kA_2 & B'_{02} &= B_{02} \\
A'_{03} &= A_{03} + 2kA_0 + 6iB_0 & B'_{03} &= B_{03} + 2kB_0 \\
A'_{11} &= A_{11} + 2iB_0 & B'_{11} &= B_{11} \\
A'_{22} &= A_{22} + 2iB_0 & B'_{22} &= B_{22} \\
A'_{33} &= A_{33} - 2kA_0 - 8iB_0 & B'_{33} &= B_{33} - 2kB_0 \\
A'_{12} &= A_{12} & B'_{12} &= B_{12} \\
A'_{13} &= A_{13} + kA_1 & B'_{13} &= B_{13} \\
A'_{23} &= A_{23} + kA_2 & B'_{23} &= B_{23}.
\end{aligned} \tag{27}$$

If we filter the above such that we are only looking at the independent components, this becomes

$$\begin{aligned}
A'_{00} &= A_{00} - 2kA_0 - 4iB_0 & B'_{00} &= B_{00} - 2kB_0 \\
A'_{01} &= A_{01} - kA_1 & B'_{01} &= B_{01} \\
A'_{02} &= A_{02} - kA_2 & B'_{02} &= B_{02} \\
A'_{11} &= A_{11} + 2iB_0 & B'_{11} &= B_{11} \\
A'_{22} &= A_{22} + 2iB_0 & & \\
A'_{12} &= A_{12} & B'_{12} &= B_{12}.
\end{aligned} \tag{28}$$

We see that of the 11 independent components, A_{12}, B_{01}, B_{02}, and B_{12} are gauge invariant. For the $B_{\mu\nu}$ modes, this corresponds to a helicity $+2$ tensor mode and helicity $+1$ vector mode. However, it appears there is only one gauge invariant quantity, A_{12}, which grows as e^{ikx}.