Astrophysics & Cosmology
HW 3

Matthew Phelps

6.1 Given that the sun has a composition of 70% hydrogen and a total mass of 2.0×10^{33} [gm], the number of hydrogen nuclei is

$$N = \frac{0.7(2.0 \times 10^{33})}{m_p} = 8.38 \times 10^{56}.$$

If all hydrogen is converted to helium, releasing an energy $E = 0.03m_pc^2$, the total energy supply of the sun is

$$E_{supply} = \frac{NE}{4} = \frac{0.7(2.0 \times 10^{33})(0.03c^2)}{4} = 9.45 \times 10^{51} \text{ [erg]}.$$

During the phase at which the sun undergoes hydrogen fusion in a shell, if this phase can use 13% of its total supply (the quantity given above) as it radiates at luminosity L_{sun}, then the total time this phase lasts is given by

$$t = \frac{E_{supply}}{L_{sun}} = \frac{9.45 \times 10^{51}}{3.9 \times 10^{33}} = 2.4 \times 10^{18} \text{ [sec]} = 7.7 \times 10^{10} \text{ [yr]}.$$

6.3 Referring to Dirac's treatment of quantum mechanics, start with

$$(m^2c^4 + p^2c^2)^{1/2} = \mathbf{a} \cdot \mathbf{p}c + bmc^2.$$

Square both sides,

$$m^2c^4 + p^2c^2 = (\mathbf{a} \cdot \mathbf{p}c)^2 + b^2m^2c^4 + 2bmc^2\mathbf{a} \cdot \mathbf{p}c. \tag{1}$$

Substitute

$$p^2 = p_x^2 + p_y^2 + p_z^2, \qquad \mathbf{a} \cdot \mathbf{p} = a_xp_x + a_yp_y + a_zp_z$$

so that (1) reads

$$\begin{aligned} m^2c^4 + c^2(p_x^2 + p_y^2 + p_z^2) = c^2[&a_x^2p_x^2 + a_y^2p_y^2 + a_z^2p_z^2 + (a_xa_y + a_ya_x)p_xp_y + (a_xa_z + a_za_x)p_xp_z + (a_ya_z + a_za_y)p_yp_z] \\ &+ b^2m^2c^4 + 2bmc^3(a_xp_x + a_yp_y + a_zp_z). \end{aligned}$$

Now we equate the coefficients of powers of p_i and mc^2 on each side. It is clear that

$$a_x^2 = a_y^2 = a_z^2 = 1$$

$$b^2 = 1$$

$$a_xb = a_yb = a_zb = 0$$

$$a_xa_y + a_ya_x = a_xa_z + a_za_x = a_ya_z + a_za_y = 0.$$

From the first two relations above, if a_i and b are ordinary numbers, then we conclude that

$$a_i = b = \pm 1.$$

This violates the next relation

$$a_x b = (\pm 1)(\pm 1) = \pm 1 \neq 0.$$

Hence the a_i and b cannot be scalars. However, the relations are satisfied by 4×4 matricies

$$a_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad a_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad a_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now verify the relations for a_i

$$a_x a_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_y a_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_z a_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_x a_y + a_y a_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 0$$

$$a_x a_z + a_z a_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 0$$

$$a_y a_z + a_z a_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = 0$$

6.7

$${}^3_2\mathrm{He} + {}^1_1\mathrm{H} \to {}^4_2\mathrm{He} + {}^0_1\bar{e} + {}^0_0\nu$$

6.8 Beta decay involves the conversion $n \to p + e^- + \bar{\nu}$ or $p \to n + \bar{e} + \nu$, which is a result of the weak interaction. Alpha decay involves the ejection of a ${}^4_2\mathrm{He}$ atom from the nucleus, governed by the strong and EM interaction. Gamma decay occurs when a system with positive binding energy undergoes fusion and excess energy is released in the form of a photon, or ${}^0_0\gamma$.

$${}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} \to {}^{13}_7\mathrm{N} + {}^0_0\gamma \quad \text{EM}$$
$${}^{13}_7\mathrm{N} \to {}^{13}_6\mathrm{C} + {}^0_1\bar{e} + {}^0_0\nu \quad \text{Weak}$$
$${}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} \to {}^{14}_7\mathrm{N} + {}^0_0\gamma \quad \text{EM}$$

$$
\begin{aligned}
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + {}^{0}_{0}\nu \quad \text{EM} \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + {}^{0}_{1}\bar{e} + {}^{0}_{0}\nu \quad \text{Weak} \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} \quad \text{Strong}
\end{aligned}
$$

6.9 The total energy on the one-body problem mass m is the sum of kinetic and potential. The only appreciable potential forces at this scale are those due to Coulomb repulsion. Thus, infinitely far away, all energy is kinetic

$$E(r=\infty) = \frac{1}{2}mv^2.$$

According to classical mechanics, a particle will come in from infinity, and reflect off the Coulomb barrier at some potential $V(r)$. At the barrier, all energy is converted into potential and the particle has zero translational energy. Penetration of the barrier would violate energy conservation. The point at which the potential is equal to the total energy is

$$\frac{q_1 q_2}{r} = \frac{1}{2}mv^2.$$

For a given velocity, the particle hits the barrier at a radius

$$r = \frac{2q_1 q_2}{mv^2}.$$

However, according to quantum mechanics, there exists a finite probability of penetrating the barrier due to the uncertainty principle. The penetration probability is proportional to

$$P_1 \propto \exp\left(-2\pi^2 r/\lambda\right) = \exp\left(-4\pi^2 q_1 q_2/hv\right).$$

Higher velocities yield higher probabilities for penetration. From statistical mechanics, the velocity of particles in thermodynamic equilibrium is given by the probability distribution

$$P_2 \propto \exp\left(-mv^2/2kT\right).$$

Thus the probabilty for a particle is maximized by finding the maximum of $P_1 P_2$. This occurs at velocity

$$v = \left(\frac{4\pi^2 q_1 q_2 kT}{hm}\right)^{1/3}.$$

For a proton proton reaction at $T = 1.5 \times 10^7$ K, the radius at which penetration is most likely occurs at

$$r = \frac{2e^2}{mv^2} = \frac{4e^2}{m_p}\left(\frac{8\pi^2 e^2 kT}{hm_p}\right)^{-2/3} = 2.4 \times 10^{-11} \text{ cm}.$$

This is about 10^2 larger than the approximate "size" of the neutron/proton.

The maximum probability occurs by substituting the v given above into $P_1 P_2$:

$$
\begin{aligned}
P_1 P_2(v_{max}) &= \exp(-4\pi q_1 q_2/hv)\exp(-mv^2/2kT) = \exp\left(\frac{-8\pi^2 q_1 q_2 kT - mhv^3}{2kThv}\right) \\
&= \exp\left(\frac{-8\pi^2 q_1 q_2 kT - 4\pi^2 q_1 q_2 kT}{2kT(4\pi^2 q_1 q_2 kT)^{1/3}(hm)^{-1/3}}\right) \\
&= \exp\left[-\frac{3}{2}(4\pi^2 q_1 q_2/h)^{2/3}(m/kT)^{1/3}\right]
\end{aligned}
$$

$$= \exp\left[-\left(\frac{T_0}{T}\right)^{1/3}\right]$$

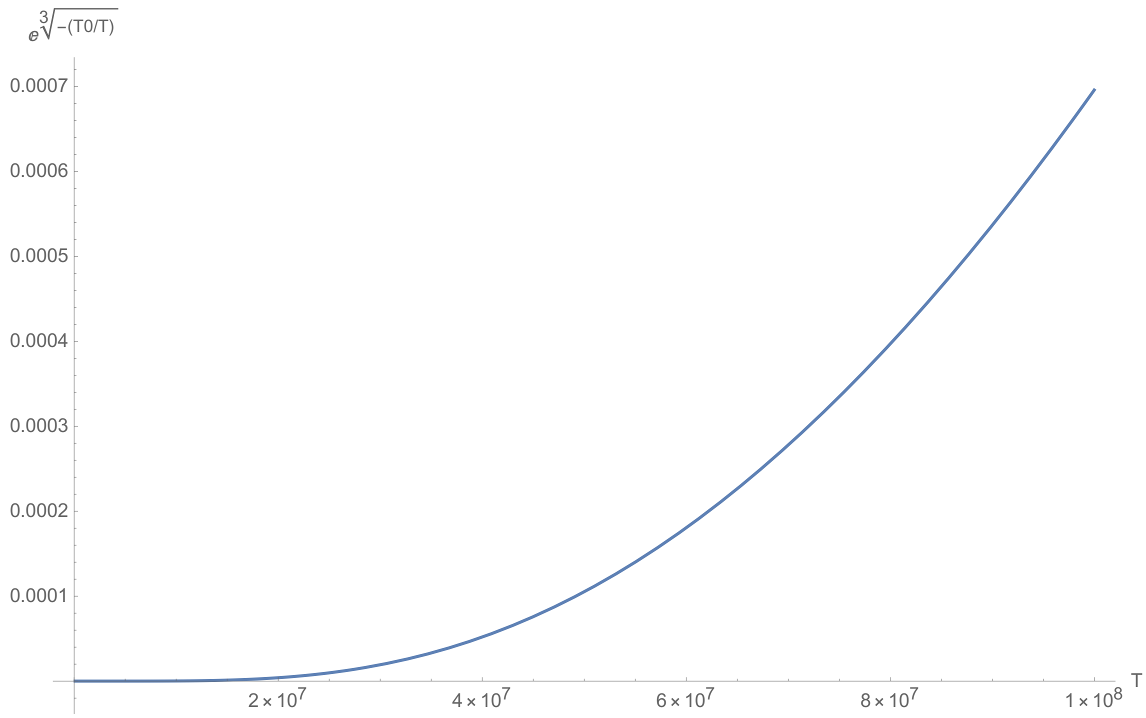
where

$$T_0 = (3/2)^3(4\pi^2 q_1 q_2/h)^2(m/k).$$

For hydrogen reactions T_0 is given by

$$T_0 = (3/2)^3(4\pi^2 e^2/h)^2(m_p/2k) = 3.84 \times 10^{10} \text{ K}.$$

Plotting the resulting probability, we have



It looks the the probability for fusion is negligible until we reach temperatures greater than $T = 2 \times 10^7$ K. At we increase up to $T = 10^8$ K, we approach a probability of about 1/1000. The increase as a function of T is exponential.

6.13 Dimensional analysis of the following quantities:

$$G = \text{gm}^{-1}\ \text{cm}^3\ \text{s}^{-2}$$

$$h = \text{gm cm}^2\ \text{s}^{-1}$$

$$c = \text{cm s}^{-1}$$

To get the plank mass we can form the combination

$$m_{plank} = \left(\frac{hc}{G}\right)^{1/2} = 5.4 \times 10^{-5} \text{ gm}.$$

This is 10^{19} times more massive than the proton!

6.14 The kinetic energy of car of mass $m = 2 \times 10^6$ gm moving at speed $v = 1.2 \times 10^3$ cm sec^{-1} is

$$E = \frac{1}{2}mv^2 = 1.44 \times 10^{12} \text{ erg.}$$

Meanwhile the rest mass energy of a proton is

$$m_pc^2 = 1.5 \times 10^{-3} \text{ erg.}$$

In order for a car to have the same energy as $10^19m_pc^2$ (energy per particle for "Supergravity", or Plank mass) it must travel at a velocity

$$v = 1.2 \times 10^5 \text{ cm sec}^{-1} \approx 3000 \text{ mph.}$$