$g^{\mu\nu}\delta G_{\mu\nu}$ Radiation v2

1   Conformal Flat $\Omega(x)$

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(x)(\tilde{g}_{\mu\nu} + f_{\mu\nu})dx^\mu dx^\nu \tag{1.1}$$

$$\tilde{g}_{\mu\nu} = \mathrm{diag}\left(-1, 1, r^2, r^2\sin^2\theta\right) \qquad \tilde{\Gamma}^\lambda_{\alpha\beta} = \delta^\lambda_i \delta^j_\alpha \delta^k_\beta \tilde{\Gamma}^i_{jk} \tag{1.2}$$

All subsequent equations hold for any flat $\tilde{g}_{\mu\nu}$, i.e. any $\tilde{g}_{\mu\nu}$ such that the corresponding curvature tensors vanish.

1.1 $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \tag{1.3}$$

$$\begin{aligned}
g^{\mu\nu}G_{\mu\nu} &= -R \\
&= -\tilde{R}\Omega^{-2} - 6\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega \\
&= 6\ddot{\Omega}\Omega^{-3} - 6\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega
\end{aligned} \tag{1.4}$$

1.2 $\delta(g^{\mu\nu}G_{\mu\nu})$

We calculate $\delta(g^{\mu\nu}G_{\mu\nu}) = -h^{\mu\nu}G^{(0)}_{\mu\nu} + g^{\mu\nu}\delta G_{\mu\nu}$ as this is the perturbed equation that follows directly from (1.4). Additional remarks on the trace are in Trace Gauge Invariance.

$$\begin{aligned}
\delta(g^{\mu\nu}G_{\mu\nu}) = {}& -6\dot{\phi}\dot{\Omega}\Omega^{-3} - 18\dot{\psi}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\phi\Omega^{-3} - 6\ddot{\psi}\Omega^{-2} - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} \\
& + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi - 12\psi\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega \\
& - 12\Omega^{-3}\tilde{\nabla}_a\dot{\Omega}\tilde{\nabla}^a B - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\dot{B} - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\phi + 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\psi \\
& + 6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E + 12\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E - 12B^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{B}^a\Omega^{-3}\tilde{\nabla}_a\Omega \\
& + 6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a + 12\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega
\end{aligned} \tag{1.5}$$

We substitute the gauge invariants and null trace condition ($g^{\mu\nu}G_{\mu\nu} = 0 \implies \tilde{\nabla}_a\tilde{\nabla}^a\Omega = \ddot{\Omega}$),

$$\begin{aligned}
\alpha &= \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \psi - \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega], \\
Q_i &= B_i - \dot{E}_i, \qquad E_{ij}.
\end{aligned} \tag{1.6}$$

The perturbed trace $\delta(g^{\mu\nu}G_{\mu\nu})$ is then expressed entirely in terms of the gauge invariants as

$$\begin{aligned}
\delta(g^{\mu\nu}G_{\mu\nu}) = {}& -6\dot{\alpha}\dot{\Omega}\Omega^{-3} - 12\dot{\gamma}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\alpha\Omega^{-3} - 6\ddot{\gamma}\Omega^{-2} - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \\
& - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\alpha + 12\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\gamma - 12Q^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{Q}^a\Omega^{-3}\tilde{\nabla}_a\Omega + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega.
\end{aligned} \tag{1.7}$$

Defining the gauge invariants instead as

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \phi - \psi + \dot{B} - \ddot{E} + 2\Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega],$$

$$Q_i \;=\; B_i - \dot{E}_i, \qquad E_{ij}. \tag{1.8}$$

then (1.5) becomes

$$
\begin{aligned}
\delta(g^{\mu\nu}G_{\mu\nu}) \;=\;& -12\dot{\alpha}\dot{\Omega}\Omega^{-3} + 6\dot{\gamma}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\alpha\Omega^{-3} - 3\ddot{\alpha}\Omega^{-2} + 3\ddot{\gamma}\Omega^{-2} + \Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 3\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \\
& -6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\gamma - 12Q^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} - 6\dot{Q}^a\Omega^{-3}\tilde{\nabla}_a\Omega + 12E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega.
\end{aligned} \tag{1.9}
$$

1.3   Trace Gauge Invariance

With the transformation of the first order $\delta G_{\mu\nu}$ behaving as

$$\Delta_\epsilon[\delta G_{\mu\nu}] \;=\; G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + G^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda G_{\mu\nu}\epsilon^\lambda, \tag{1.10}$$

upon taking the trace, we have

$$g^{\mu\nu}\Delta_\epsilon[\delta G_{\mu\nu}] \;=\; 2G^{\lambda\mu}\nabla_\mu\epsilon_\lambda + \nabla_\lambda G^\mu{}_\mu \epsilon^\lambda. \tag{1.11}$$

The above indicates that a vanishing $G^\mu{}_\mu$ alone does not ensure $g^{\mu\nu}\delta G_{\mu\nu}$ is gauge invariant. However, we may subtract from (1.10) the contribution $h^{\mu\nu}G_{\mu\nu}$, which transforms as

$$G_{\mu\nu}\Delta_\epsilon[h^{\mu\nu}] \;=\; 2G^{\lambda\mu}\nabla_\mu\epsilon_\lambda. \tag{1.12}$$

Thus the quantity that is invariant under gauge transformation (assuming only that $G^\lambda{}_\lambda = 0$) is in fact

$$\Delta_\epsilon[\delta(g^{\mu\nu}G_{\mu\nu})] = \Delta_\epsilon[-h^{\mu\nu}G_{\mu\nu} + g^{\mu\nu}\delta G_{\mu\nu}] = 0. \tag{1.13}$$

If the background trace vanishes, it must be the perturbation of the full trace (and not just $g^{\mu\nu}\delta G_{\mu\nu}$) that is then gauge invariant.