Coordinate Transformation RW SVT3

1 Background

$$
\begin{aligned}
ds^2 &= (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\tau)(\tilde{g}_{\mu\nu} + f_{\mu\nu})dx^\mu dx^\nu && (1.1)\\
\tilde{g}_{\mu\nu} &= \mathrm{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) && (1.2)
\end{aligned}
$$

$$
x^\mu(\tau, r, \theta, \phi) \rightarrow x'^\mu(T, R, \theta, \phi) \qquad (1.3)
$$

$$
ds^2 = (g'_{\mu\nu} + h'_{\mu\nu})dx'^\mu dx'^\nu = \Omega'^2(T,R)(\tilde{g}'_{\mu\nu} + f'_{\mu\nu})dx'^\mu dx'^\nu \qquad (1.4)
$$

$$
\tilde{g}'_{\mu\nu} = \mathrm{diag}\left(-1, 1, R^2, R^2\sin^2\theta\right) \qquad (1.5)
$$

$$
h_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}h'_{\alpha\beta} \qquad (1.6)
$$

$$
\begin{aligned}
h_{00} &= \left(\frac{\partial T}{\partial \tau}\right)^2 h'_{00} + 2\left(\frac{\partial T}{\partial \tau}\right)\left(\frac{\partial R}{\partial \tau}\right)h'_{0r} + \left(\frac{\partial R}{\partial \tau}\right)^2 h'_{rr}\\
h_{0r} &= \left(\frac{\partial T}{\partial \tau}\right)\left(\frac{\partial T}{\partial r}\right)h'_{00} + \left[\left(\frac{\partial T}{\partial \tau}\right)\left(\frac{\partial T}{\partial r}\right) + \left(\frac{\partial T}{\partial \tau}\right)\left(\frac{\partial R}{\partial r}\right)\right]h'_{0r}\\
h_{0\theta} &= \left(\frac{\partial T}{\partial \tau}\right)h'_{0\theta} + \left(\frac{\partial R}{\partial \tau}\right)h'_{r\theta}\\
h_{0\phi} &= \left(\frac{\partial T}{\partial \tau}\right)h'_{0\phi} + \left(\frac{\partial R}{\partial \tau}\right)h'_{r\phi}\\
h_{rr} &= \left(\frac{\partial T}{\partial r}\right)^2 h'_{00} + 2\left(\frac{\partial T}{\partial r}\right)\left(\frac{\partial R}{\partial r}\right)h'_{0r} + \left(\frac{\partial R}{\partial r}\right)^2 h'_{rr}\\
h_{r\theta} &= \left(\frac{\partial T}{\partial r}\right)h'_{00} + \left(\frac{\partial R}{\partial r}\right)h'_{r\theta}\\
h_{r\phi} &= \left(\frac{\partial T}{\partial r}\right)h'_{00} + \left(\frac{\partial R}{\partial r}\right)h'_{r\phi}\\
h_{\theta\phi} &= h'_{\theta\phi}\\
h_{\theta\theta} &= h'_{\theta\theta}\\
h_{\phi\phi} &= h'_{\phi\phi} && (1.7)
\end{aligned}
$$

1.1 Identities

A, B, ρ, p are functions only of coordinate x^0.

$$
U^\alpha U^\beta \nabla_\alpha F \nabla_\beta A = -\nabla^\alpha F \nabla_\alpha A \qquad (1.8)
$$

$$
\begin{aligned}
F^\alpha U_\alpha U^\beta \nabla_\beta A &= -F^\alpha \nabla_\alpha A && (1.9) \\
U^\alpha \nabla_\alpha U^\mu &= 0 && (1.10) \\
\nabla_\mu U_\nu &= \dot{\Omega}\Omega^{-2}(g_{\mu\nu} + U_\mu U_\nu) && (1.11)
\end{aligned}
$$

2 Fluctuations

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{2.1}$$

$$g^{\mu\nu}F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad \nabla^\mu F_\mu = 0 \tag{2.2}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{2.3}$$

$$g^{\alpha\beta}h_{\alpha\beta} = -8\chi + 2\nabla_\alpha\nabla^\alpha F \tag{2.4}$$

$$\begin{aligned}\nabla^\mu h_{\mu\nu} &= 2U^\alpha U_\nu(p+\rho)\nabla_\alpha F + (-p+\rho)\nabla_\nu F - 2\nabla_\nu\chi + 2\nabla_\nu\nabla_\alpha\nabla^\alpha F + \tfrac{1}{2}F_\nu(-p+\rho) \\ &\quad + F^\alpha U_\alpha U_\nu(p+\rho) + \nabla_\alpha\nabla^\alpha F_\nu\end{aligned} \tag{2.5}$$

$$\begin{aligned}\nabla^\mu\nabla^\nu h_{\mu\nu} &= (-p+\rho)\nabla_\alpha\nabla^\alpha F - 2\nabla_\alpha\nabla^\alpha\chi + (-3\nabla_\alpha p - \nabla_\alpha\rho)\nabla^\alpha F + 2U^\alpha(p+\rho)\nabla_\alpha F\nabla_\beta U^\beta \\ &\quad + 2U^\alpha U^\beta(p+\rho)\nabla_\beta\nabla_\alpha F + 2\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F + 2U^\alpha U^\beta(p+\rho)\nabla_\beta F_\alpha \\ &\quad + F^\alpha\big(-3\nabla_\alpha p - \nabla_\alpha\rho + 2U_\alpha(p+\rho)\nabla_\beta U^\beta\big)\end{aligned} \tag{2.6}$$

$$U^\mu U^\nu h_{\mu\nu} = 2\chi + 2U^\alpha U^\beta\nabla_\beta\nabla_\alpha F + 2U^\alpha U^\beta\nabla_\beta F_\alpha + 2F_{\alpha\beta}U^\alpha U^\beta \tag{2.7}$$

$$(U^\mu U^\nu + g^{\mu\nu})h_{\mu\nu} = -6\chi + 2\nabla_\alpha\nabla^\alpha F + 2U^\alpha U^\beta\nabla_\beta\nabla_\alpha F + 2U^\alpha U^\beta\nabla_\beta F_\alpha + 2F_{\alpha\beta}U^\alpha U^\beta \tag{2.8}$$

$$(U^\mu\nabla^\nu + U^\nu\nabla^\mu)h_{\mu\nu} = -4U^\alpha\nabla_\alpha\chi + 4U^\alpha\nabla_\beta\nabla^\beta\nabla_\alpha F - F^\alpha U_\alpha(3p+\rho) + 2U^\alpha\nabla_\beta\nabla^\beta F_\alpha \tag{2.9}$$

$$\begin{aligned}\Delta'_{\mu\nu} &= 2(p+\rho)U_\mu U_\nu U^\alpha\nabla_\alpha V^{GI} + (\tfrac{2}{3}g_{\mu\nu} + \tfrac{2}{3}U_\mu U_\nu)\nabla_\alpha\nabla^\alpha\chi + (\tfrac{2}{3}g_{\mu\nu}U^\alpha U^\beta + \tfrac{8}{3}U_\mu U_\nu U^\alpha U^\beta)\nabla_\beta\nabla_\alpha\chi \\ &\quad + (p+\rho)U_\nu\nabla_\mu V^{GI} + (p+\rho)U_\mu\nabla_\nu V^{GI} - 2\nabla_\nu\nabla_\mu\chi + 2(p+\rho)U_\mu U_\nu U^\alpha V_\alpha + (p+\rho)U_\nu V_\mu \\ &\quad + (p+\rho)U_\mu V_\nu + \big(\tfrac{2}{9}\rho g_{\mu\nu}U^\alpha U^\beta - \tfrac{4}{9}(9p+7\rho)U_\mu U_\nu U^\alpha U^\beta\big)F_{\alpha\beta} - 2(p+\rho)U_\nu U^\alpha F_{\mu\alpha} - \tfrac{2}{3}\rho F_{\mu\nu} \\ &\quad - 2(p+\rho)U_\mu U^\alpha F_{\nu\alpha} + \nabla_\alpha\nabla^\alpha F_{\mu\nu} + (-\tfrac{1}{3}g_{\mu\nu}U^\alpha U^\beta - \tfrac{4}{3}U_\mu U_\nu U^\alpha U^\beta)\nabla_\gamma\nabla^\gamma F_{\alpha\beta}\end{aligned} \tag{2.10}$$