

# RW SVT3 $k \neq 0$ v3

## 1 Background

### 1.1 Comoving $a(t)$

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) = -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \quad (1.1)$$

$$G_{00} = -3ka^{-2} - 3\dot{a}^2 a^{-2}, \quad G_{ij} = \tilde{g}_{ij}(k + \dot{a}^2 + 2a\ddot{a}) \quad (1.2)$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \quad U_\mu = -\delta_\mu^0 \quad (1.3)$$

$$T_{00} = \rho, \quad T_{ij} = a^2(t)p\tilde{g}_{ij} \quad (1.4)$$

$$\Delta_{\mu\nu}^{(0)} = G_{\mu\nu} + T_{\mu\nu} = 0 \quad (1.5)$$

$$\Delta_{00}^{(0)} = \rho - 3ka^{-2} - 3\dot{a}^2 a^{-2}, \quad \Delta_{ij}^{(0)} = \tilde{g}_{ij}(a^2 p + k + \dot{a}^2 + 2a\ddot{a}) \quad (1.6)$$

$$\rightarrow \boxed{\rho = 3ka^{-2} + 3\dot{a}^2 a^{-2}} \quad \boxed{p = -a^{-2}k - \dot{a}^2 a^{-2} - 2\ddot{a}a^{-1}} \quad (1.7)$$

### 1.2 Conformal $\Omega(\tau)$

$$ds^2 = \Omega^2(\tau) \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} = \text{diag} \left( -1, \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right) \quad (1.8)$$

$$G_{00} = -3k - 3\dot{\Omega}^2 \Omega^{-2} \quad G_{ij} = k\tilde{g}_{ij} - \dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} + 2\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij} \quad (1.9)$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\Omega^2 \tilde{g}_{\mu\nu}, \quad U_\mu = -\Omega \delta_\mu^0 \quad [\text{Evaluated in (1.8)}] \quad (1.10)$$

$$\Delta_{00}^{(0)} = -3k - 3\dot{\Omega}^2 \Omega^{-2} + \Omega^2 \rho \quad (1.11)$$

$$\rightarrow \boxed{\rho = 3k\Omega^{-2} + 3\dot{\Omega}^2 \Omega^{-4}} \quad (1.12)$$

$$\Delta_{ij}^{(0)} = k\tilde{g}_{ij} - \dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} + 2\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij} + \Omega^2 p \tilde{g}_{ij} \quad (1.13)$$

$$\rightarrow \boxed{p = -k\Omega^{-2} + \dot{\Omega}^2 \Omega^{-4} - 2\ddot{\Omega} \Omega^{-3}} \quad (1.14)$$

$$\begin{aligned}
\nabla_\mu T^{\mu 0} &= \Omega^{-5} \left( \tilde{g}^{ab} T_{ab} \dot{\Omega} + T_{00} \dot{\Omega} + \dot{T}_{00} \Omega - \Omega \tilde{\nabla}_a T_0^a \right) \\
&= 3\dot{\Omega} \Omega^{-3} p + 3\dot{\Omega} \Omega^{-3} \rho + \Omega^{-2} \dot{\rho}
\end{aligned} \tag{1.15}$$

$$\begin{aligned}
\nabla_\mu T^{\mu i} &= \Omega^{-5} \left( -2T_0^i \dot{\Omega} - \dot{T}_0^i \Omega + \Omega \tilde{\nabla}_a T^{ia} \right) \\
&= 0
\end{aligned} \tag{1.16}$$

## 2 Fluctuations

$$ds^2 = \Omega^2(\tau) [\tilde{g}_{\mu\nu} + f_{\mu\nu}] dx^\mu dx^\nu \tag{2.1}$$

$$\tilde{g}_{\mu\nu} = \text{diag} \left( -1, \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta \right) \tag{2.2}$$

$$f_{00} = -2\phi, \quad f_{0i} = \tilde{\nabla}_i B + B_i, \quad f_{ij} = -2\psi \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{2.3}$$

$$\delta T_{00} = \Omega^2 \delta \rho + (6k + 6\dot{\Omega}^2 \Omega^{-2}) \phi \tag{2.4}$$

$$\begin{aligned}
\delta T_{0i} &= (-k + \dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_i B \\
&\quad + (-4\dot{\Omega}^2 \Omega^{-3} + 2\ddot{\Omega} \Omega^{-2} - 2k \Omega^{-1}) \tilde{\nabla}_i V + (-k + \dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega} \Omega^{-1}) B_i \\
&\quad + (-4\dot{\Omega}^2 \Omega^{-3} + 2\ddot{\Omega} \Omega^{-2} - 2k \Omega^{-1}) V_i
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\delta T_{ij} &= \Omega^2 \delta p \tilde{g}_{ij} + (2k \tilde{g}_{ij} - 2\dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} + 4\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij}) \psi \\
&\quad + (-2k + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_i \tilde{\nabla}_j E + (-k + \dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_i E_j \\
&\quad + (-k + \dot{\Omega}^2 \Omega^{-2} - 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_j E_i + (-2k + 2\dot{\Omega}^2 \Omega^{-2} - 4\ddot{\Omega} \Omega^{-1}) E_{ij}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
g^{\mu\nu} \delta T_{\mu\nu} &= 3\delta p - \delta \rho + (-6\dot{\Omega}^2 \Omega^{-4} - 6k \Omega^{-2}) \phi + (-6\dot{\Omega}^2 \Omega^{-4} + 12\ddot{\Omega} \Omega^{-3} + 6k \Omega^{-2}) \psi \\
&\quad + (2\dot{\Omega}^2 \Omega^{-4} - 4\ddot{\Omega} \Omega^{-3} - 2k \Omega^{-2}) \tilde{\nabla}_a \tilde{\nabla}^a E
\end{aligned} \tag{2.7}$$

$$\delta G_{00} = 6\dot{\Omega} \Omega^{-1} \dot{\psi} - 6k\phi - 6k\psi + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a B - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} - 2\tilde{\nabla}_a \tilde{\nabla}^a \psi \tag{2.8}$$

$$\begin{aligned}
\delta G_{0i} &= (3k - \dot{\Omega}^2 \Omega^{-2} + 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_i B - 2k \tilde{\nabla}_i \dot{E} - 2\tilde{\nabla}_i \dot{\psi} - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_i \phi + (2k - \dot{\Omega}^2 \Omega^{-2} + 2\ddot{\Omega} \Omega^{-1}) B_i \\
&\quad - k \dot{E}_i + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^a B_i - \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E}_i
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\delta G_{ij} &= -2\dot{\psi} \tilde{g}_{ij} - 2\dot{\Omega} \Omega^{-1} \dot{\phi} \tilde{g}_{ij} - 4\dot{\Omega} \Omega^{-1} \dot{\psi} \tilde{g}_{ij} + (2\dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} - 4\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij}) \phi \\
&\quad + (2\dot{\Omega}^2 \Omega^{-2} \tilde{g}_{ij} - 4\ddot{\Omega} \Omega^{-1} \tilde{g}_{ij}) \psi - 2\dot{\Omega} \Omega^{-1} \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a B - \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \dot{B} + \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \ddot{E} \\
&\quad + 2\dot{\Omega} \Omega^{-1} \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} - \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \phi + \tilde{g}_{ij} \tilde{\nabla}_a \tilde{\nabla}^a \psi + 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i B + \tilde{\nabla}_j \tilde{\nabla}_i \dot{B} - \tilde{\nabla}_j \tilde{\nabla}_i \ddot{E} \\
&\quad - 2\dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \tilde{\nabla}_i \dot{E} + (2k - 2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_j \tilde{\nabla}_i E + \tilde{\nabla}_j \tilde{\nabla}_i \phi - \tilde{\nabla}_j \tilde{\nabla}_i \psi + \dot{\Omega} \Omega^{-1} \tilde{\nabla}_i B_j \\
&\quad + \frac{1}{2} \tilde{\nabla}_i \dot{B}_j - \frac{1}{2} \tilde{\nabla}_i \ddot{E}_j - \dot{\Omega} \Omega^{-1} \tilde{\nabla}_i \dot{E}_j + (k - \dot{\Omega}^2 \Omega^{-2} + 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_i E_j + \dot{\Omega} \Omega^{-1} \tilde{\nabla}_j B_i + \frac{1}{2} \tilde{\nabla}_j \dot{B}_i \\
&\quad - \frac{1}{2} \tilde{\nabla}_j \ddot{E}_i - \dot{\Omega} \Omega^{-1} \tilde{\nabla}_j \dot{E}_i + (k - \dot{\Omega}^2 \Omega^{-2} + 2\ddot{\Omega} \Omega^{-1}) \tilde{\nabla}_j E_i - \ddot{E}_{ij} - 2\dot{\Omega} \Omega^{-1} \dot{E}_{ij} \\
&\quad + (-2\dot{\Omega}^2 \Omega^{-2} + 4\ddot{\Omega} \Omega^{-1}) E_{ij} + \tilde{\nabla}_a \tilde{\nabla}^a E_{ij}
\end{aligned} \tag{2.10}$$

$$g^{\mu\nu} \delta G_{\mu\nu} = -6\Omega^{-2} \dot{\psi} - 6\dot{\Omega} \Omega^{-3} \dot{\phi} - 18\dot{\Omega} \Omega^{-3} \dot{\psi} + (6\dot{\Omega}^2 \Omega^{-4} - 12\ddot{\Omega} \Omega^{-3} + 6k \Omega^{-2}) \phi$$

$$\begin{aligned}
& + (6\dot{\Omega}^2\Omega^{-4} - 12\ddot{\Omega}\Omega^{-3} + 6k\Omega^{-2})\psi - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} \\
& + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} + (-2\dot{\Omega}^2\Omega^{-4} + 4\ddot{\Omega}\Omega^{-3} + 2k\Omega^{-2})\tilde{\nabla}_a\tilde{\nabla}^a E - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi \quad (2.11)
\end{aligned}$$

### 3 Field Equations

$$\Delta_{\mu\nu} \equiv \delta G_{\mu\nu} + \delta T_{\mu\nu} \quad (3.1)$$

$$\Delta_{00} = \Omega^2\delta\rho + 6\dot{\Omega}\Omega^{-1}\dot{\psi} + 6\dot{\Omega}^2\Omega^{-2}\phi - 6k\psi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi \quad (3.2)$$

$$\begin{aligned}
\Delta_{0i} = & 2k\tilde{\nabla}_i B - 2k\tilde{\nabla}_i\dot{E} - 2\tilde{\nabla}_i\dot{\psi} + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi + k B_i - k\dot{E}_i \\
& + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \quad (3.3)
\end{aligned}$$

$$\begin{aligned}
\Delta_{ij} = & -2\ddot{\psi}\tilde{g}_{ij} + \Omega^2\delta p\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\phi}\tilde{g}_{ij} - 4\dot{\Omega}\Omega^{-1}\dot{\psi}\tilde{g}_{ij} + (2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} - 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij})\phi + 2k\tilde{g}_{ij}\psi \\
& - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi \\
& + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\tilde{\nabla}_j E + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} \\
& - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j \\
& + \frac{1}{2}\tilde{\nabla}_i\dot{B}_j - \frac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \frac{1}{2}\tilde{\nabla}_j\dot{B}_i - \frac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i - \ddot{E}_{ij} \\
& - 2\dot{\Omega}\Omega^{-1}\dot{E}_{ij} - 2kE_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} \quad (3.4)
\end{aligned}$$

$$\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} = & -6\Omega^{-2}\ddot{\psi} + 3\delta p - \delta\rho - 6\dot{\Omega}\Omega^{-3}\dot{\phi} - 18\dot{\Omega}\Omega^{-3}\dot{\psi} - 12\ddot{\Omega}\Omega^{-3}\phi + 12k\Omega^{-2}\psi - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B \\
& - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi \quad (3.5)
\end{aligned}$$

### 4 Field Equations (G.I. Form)

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad \gamma = -\dot{\Omega}^{-1}\Omega\psi + B - \dot{E}, \quad B_i - \dot{E}_i, \quad E_{ij}, \quad V_i \quad (4.1)$$

$$V^{GI} = V - \Omega^2\dot{\Omega}^{-1}\psi \quad (4.2)$$

$$\delta\rho^{GI} = \delta\rho - 12\dot{\Omega}^2\psi\Omega^{-4} + 6\ddot{\Omega}\psi\Omega^{-3} - 6k\psi\Omega^{-2} \quad (4.3)$$

$$\delta p^{GI} = \delta p - 4\dot{\Omega}^2\psi\Omega^{-4} + 8\ddot{\Omega}\psi\Omega^{-3} + 2k\psi\Omega^{-2} - 2\ddot{\Omega}\dot{\Omega}^{-1}\psi\Omega^{-2} \quad (4.4)$$

$$\Delta_{00} = \Omega^2\delta\rho^{GI} - 6\dot{\Omega}^2\Omega^{-2}\dot{\gamma} + 6\dot{\Omega}^2\Omega^{-2}\alpha + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \quad (4.5)$$

$$\begin{aligned}
\Delta_{0i} = & 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{\gamma} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\alpha + 2k\tilde{\nabla}_i\gamma + kQ_i + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a Q_i \\
& + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V^{GI} \quad (4.6)
\end{aligned}$$

$$\begin{aligned}
\Delta_{ij} = & 2\dot{\Omega}\Omega^{-1}\dot{\gamma}\tilde{g}_{ij} + \Omega^2\delta p^{GI}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\alpha}\tilde{g}_{ij} + \dot{\gamma}(-2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij}) \\
& + (2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} - 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij})\alpha - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\gamma + \tilde{\nabla}_j\tilde{\nabla}_i\alpha \\
& + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\gamma + \frac{1}{2}\tilde{\nabla}_i\dot{Q}_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i Q_j + \frac{1}{2}\tilde{\nabla}_j\dot{Q}_i + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j Q_i - \ddot{E}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{E}_{ij} \\
& - 2kE_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} \quad (4.7)
\end{aligned}$$

$$\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} &= 6\dot{\Omega}\Omega^{-3}\dot{\gamma} + 3\delta p^{GI} - \delta\rho^{GI} - 6\dot{\Omega}\Omega^{-3}\dot{\alpha} + 12\ddot{\Omega}\Omega^{-3}\dot{\gamma} - 12\ddot{\Omega}\Omega^{-3}\alpha - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha \\
&\quad - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\gamma
\end{aligned} \tag{4.8}$$

## 5 Conservation

Variations are with respect to background (1.8).

$$\begin{aligned}
\delta(\nabla_\mu A^{\mu\nu}) &= \Omega^{-4}\tilde{\nabla}_\alpha\delta A^{\nu\alpha} + \frac{1}{2}A^{(0)\nu}{}_\alpha\Omega^{-4}\tilde{\nabla}^\alpha f + 2\delta A^\nu{}_\alpha\Omega^{-5}\tilde{\nabla}^\alpha\Omega - 2A^{(0)\nu\beta}f_{\alpha\beta}\Omega^{-5}\tilde{\nabla}^\alpha\Omega \\
&\quad + A^{(0)\beta}{}_\beta f^\nu{}_\alpha\Omega^{-5}\tilde{\nabla}^\alpha\Omega - 2A^{(0)}{}_\alpha{}^\beta f^\nu{}_\beta\Omega^{-5}\tilde{\nabla}^\alpha\Omega - f^{\nu\alpha}\Omega^{-4}\tilde{\nabla}_\beta A^{(0)}{}_\alpha{}^\beta - f^{\alpha\beta}\Omega^{-4}\tilde{\nabla}_\beta A^{(0)\nu}{}_\alpha \\
&\quad - A^{(0)\nu\alpha}\Omega^{-4}\tilde{\nabla}_\beta f_\alpha{}^\beta - \frac{1}{2}A^{(0)\alpha\beta}\Omega^{-4}\tilde{\nabla}^\nu f_{\alpha\beta} - \delta A^\alpha{}_\alpha\Omega^{-5}\tilde{\nabla}^\nu\Omega + A^{(0)\alpha\beta}f_{\alpha\beta}\Omega^{-5}\tilde{\nabla}^\nu\Omega
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
\delta(\nabla_\mu T^{\mu 0}) &= \delta T^a{}_a\dot{\Omega}\Omega^{-5} + \delta T_{00}\dot{\Omega}\Omega^{-5} - T^{ab}\dot{\Omega}f_{ab}\Omega^{-5} + T^a{}_a\dot{\Omega}f_{00}\Omega^{-5} + 2T_{00}\dot{\Omega}f_{00}\Omega^{-5} - 2T_0{}^a\dot{\Omega}f_{0a}\Omega^{-5} \\
&\quad + \delta\dot{T}_{00}\Omega^{-4} + \frac{1}{2}T^{ab}\dot{f}_{ab}\Omega^{-4} + \frac{3}{2}T_{00}\dot{f}_{00}\Omega^{-4} - 2T_0{}^a\dot{f}_{0a}\Omega^{-4} + \frac{1}{2}T_{00}\dot{f}\Omega^{-4} + 2\dot{T}_{00}f_{00}\Omega^{-4} \\
&\quad - 2\dot{T}_0{}^af_{0a}\Omega^{-4} - \Omega^{-4}\tilde{\nabla}_a\delta T_0{}^a - f_0{}^a\Omega^{-4}\tilde{\nabla}_aT_{00} - f_{00}\Omega^{-4}\tilde{\nabla}_aT_0{}^a - T_{00}\Omega^{-4}\tilde{\nabla}_af_0{}^a \\
&\quad - \frac{1}{2}T_0{}^a\Omega^{-4}\tilde{\nabla}_af + f_0{}^a\Omega^{-4}\tilde{\nabla}_bT_a{}^b + T_0{}^a\Omega^{-4}\tilde{\nabla}_bf_a{}^b + f_{ab}\Omega^{-4}\tilde{\nabla}^bT_0{}^a
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&= \Omega^{-2}\dot{\delta\rho} + 3\dot{\Omega}\Omega^{-3}\delta p + 3\dot{\Omega}\Omega^{-3}\delta\rho + (-12\dot{\Omega}^2\Omega^{-6} + 6\ddot{\Omega}\Omega^{-5} - 6k\Omega^{-4})\dot{\psi} \\
&\quad + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^aB + (4\dot{\Omega}^2\Omega^{-6} - 2\ddot{\Omega}\Omega^{-5} + 2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\
&\quad + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a\tilde{\nabla}^aV
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&= \Omega^{-2}\dot{\delta\rho}^{GI} + 3\dot{\Omega}\Omega^{-3}\delta p^{GI} + 3\dot{\Omega}\Omega^{-3}\delta\rho^{GI} \\
&\quad + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a\gamma + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}_a\tilde{\nabla}^aV^{GI}
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
\delta(\nabla_\mu T^{\mu i}) &= -2\delta T_0{}^i\dot{\Omega}\Omega^{-5} + 2T_0{}^a\dot{\Omega}f_a{}^i\Omega^{-5} - 2T_0{}^i\dot{\Omega}f_{00}\Omega^{-5} + 2T^i{}_a\dot{\Omega}f_0{}^a\Omega^{-5} - T^a{}_a\dot{\Omega}f_0{}^i\Omega^{-5} \\
&\quad - T_{00}\dot{\Omega}f_0{}^i\Omega^{-5} - \delta\dot{T}_0{}^i\Omega^{-4} - T_0{}^if_{00}\Omega^{-4} + T^i{}_af_0{}^a\Omega^{-4} - \frac{1}{2}T_0{}^if\Omega^{-4} + \dot{T}_0{}^af_a{}^i\Omega^{-4} \\
&\quad - \dot{T}_0{}^if_{00}\Omega^{-4} + \dot{T}_0{}^af_0{}^a\Omega^{-4} - \dot{T}_{00}f_0{}^i\Omega^{-4} + \Omega^{-4}\tilde{\nabla}_a\delta T^{ia} + f_0{}^i\Omega^{-4}\tilde{\nabla}_aT_0{}^a + f_0{}^a\Omega^{-4}\tilde{\nabla}_aT_0{}^i \\
&\quad + T_0{}^i\Omega^{-4}\tilde{\nabla}_af_0{}^a + \frac{1}{2}T^i{}_a\Omega^{-4}\tilde{\nabla}^af - f^{ia}\Omega^{-4}\tilde{\nabla}_bT_a{}^b - f^{ab}\Omega^{-4}\tilde{\nabla}_bT^i{}_a - T^{ia}\Omega^{-4}\tilde{\nabla}_bf_a{}^b \\
&\quad - \frac{1}{2}T^{ab}\Omega^{-4}\tilde{\nabla}^if_{ab} - \frac{1}{2}T_{00}\Omega^{-4}\tilde{\nabla}^if_{00} + T_0{}^a\Omega^{-4}\tilde{\nabla}^if_{0a}
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&= \Omega^{-2}\tilde{\nabla}^i\delta p + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}^i\dot{V} \\
&\quad + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^iV \\
&\quad + (4\dot{\Omega}^2\Omega^{-6} - 2\ddot{\Omega}\Omega^{-5} + 2k\Omega^{-4})\tilde{\nabla}^i\phi + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\dot{V}^i \\
&\quad + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})V^i
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&= \Omega^{-2}\tilde{\nabla}^i\delta p^{GI} + (-4\dot{\Omega}^2\Omega^{-6} + 2\ddot{\Omega}\Omega^{-5} - 2k\Omega^{-4})\tilde{\nabla}^i\dot{\gamma} \\
&\quad + (4\dot{\Omega}^2\Omega^{-6} - 2\ddot{\Omega}\Omega^{-5} + 2k\Omega^{-4})\tilde{\nabla}^i\alpha + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\dot{V}^i \\
&\quad + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})V^i + (4\dot{\Omega}^2\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2k\Omega^{-5})\tilde{\nabla}^i\dot{V}^{GI} \\
&\quad + (-4\dot{\Omega}^3\Omega^{-8} + 8\ddot{\Omega}\dot{\Omega}\Omega^{-7} - 2\ddot{\Omega}\Omega^{-6} + 2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^iV^{GI}
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\delta(\nabla_\mu G^{\mu 0}) &= \delta G^a{}_a\dot{\Omega}\Omega^{-5} + \delta G_{00}\dot{\Omega}\Omega^{-5} - 2B^aG_{0a}\dot{\Omega}\Omega^{-5} - 2G^{ab}\dot{\Omega}E_{ab}\Omega^{-5} - 2G^a{}_a\dot{\Omega}\phi\Omega^{-5} - 4G_{00}\dot{\Omega}\phi\Omega^{-5} \\
&\quad + 2G^a{}_a\dot{\Omega}\psi\Omega^{-5} + \delta G_{00}\dot{\Omega}\Omega^{-4} - 2B^a\dot{G}_{0a}\Omega^{-4} - 2\dot{B}^aG_{0a}\Omega^{-4} + G^{ab}\dot{E}_{ab}\Omega^{-4} - 2G_{00}\dot{\phi}\Omega^{-4}
\end{aligned}$$

$$\begin{aligned}
& -G^a{}_a \dot{\psi} \Omega^{-4} - 3G_{00} \dot{\psi} \Omega^{-4} + 2kG_0^a E_a \Omega^{-4} - 4\dot{G}_{00} \phi \Omega^{-4} - 2G_0^a \dot{\Omega} \Omega^{-5} \tilde{\nabla}_a B - 2\dot{G}_0^a \Omega^{-4} \tilde{\nabla}_a B \\
& - 2G_0^a \Omega^{-4} \tilde{\nabla}_a \dot{B} - \Omega^{-4} \tilde{\nabla}_a \delta G_0^a - B^a \Omega^{-4} \tilde{\nabla}_a G_{00} + 2\phi \Omega^{-4} \tilde{\nabla}_a G_0^a - 2\psi \Omega^{-4} \tilde{\nabla}_a G_0^a \\
& + 2kG_0^a \Omega^{-4} \tilde{\nabla}_a E - G_0^a \Omega^{-4} \tilde{\nabla}_a \phi + G_0^a \Omega^{-4} \tilde{\nabla}_a \psi - G_{00} \Omega^{-4} \tilde{\nabla}_a \tilde{\nabla}^a B + G_{00} \Omega^{-4} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} \\
& - \Omega^{-4} \tilde{\nabla}_a G_{00} \tilde{\nabla}^a B + B^a \Omega^{-4} \tilde{\nabla}_b G_a{}^b + \Omega^{-4} \tilde{\nabla}^a B \tilde{\nabla}_b G_a{}^b + G_0^a \Omega^{-4} \tilde{\nabla}_b \tilde{\nabla}^b E_a \\
& + G_0^a \Omega^{-4} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a E + 2E_{ab} \Omega^{-4} \tilde{\nabla}^b G_0^a + \Omega^{-4} \tilde{\nabla}_a E_b \tilde{\nabla}^b G_0^a + \Omega^{-4} \tilde{\nabla}_b E_a \tilde{\nabla}^b G_0^a \\
& + 2\Omega^{-4} \tilde{\nabla}_b \tilde{\nabla}_a E \tilde{\nabla}^b G_0^a + G_{ab} \Omega^{-4} \tilde{\nabla}^b \dot{E}^a - 2G_{ab} \dot{\Omega} \Omega^{-5} \tilde{\nabla}^b E^a + G_{ab} \Omega^{-4} \tilde{\nabla}^b \tilde{\nabla}^a \dot{E} \\
& - 2G_{ab} \dot{\Omega} \Omega^{-5} \tilde{\nabla}^b \tilde{\nabla}^a E
\end{aligned} \tag{5.8}$$

$$= 0 \tag{5.9}$$

$$\begin{aligned}
\delta(\nabla_\mu G^{\mu i}) = & -2\delta G_0^i \dot{\Omega} \Omega^{-5} - B^i G_a{}^a \dot{\Omega} \Omega^{-5} + 2B^a G_a{}^i \dot{\Omega} \Omega^{-5} - B^i G_{00} \dot{\Omega} \Omega^{-5} + 4G_0^a \dot{\Omega} E_a{}^i \Omega^{-5} \\
& + 4G_0^i \dot{\Omega} \phi \Omega^{-5} - 4G_0^i \dot{\Omega} \psi \Omega^{-5} - \delta \dot{G}_0^i \Omega^{-4} + B^a \dot{G}_a{}^i \Omega^{-4} - B^i \dot{G}_{00} \Omega^{-4} + \dot{B}^a G_a{}^i \Omega^{-4} \\
& + G_0^i \dot{\phi} \Omega^{-4} + 3G_0^i \dot{\psi} \Omega^{-4} + 2\dot{G}_0^a E_a{}^i \Omega^{-4} - 2kG_a{}^i E^a \Omega^{-4} + 2\dot{G}_0^i \phi \Omega^{-4} - 2\dot{G}_0^i \psi \Omega^{-4} \\
& + \Omega^{-4} \tilde{\nabla}_a \delta G^{ia} + 4\psi \Omega^{-4} \tilde{\nabla}_a G^{ia} + B^i \Omega^{-4} \tilde{\nabla}_a G_0^a + B^a \Omega^{-4} \tilde{\nabla}_a G_0^i + 2G_0^a \dot{\Omega} \Omega^{-5} \tilde{\nabla}_a E^i \\
& + \dot{G}_0^a \Omega^{-4} \tilde{\nabla}_a E^i + G_0^i \Omega^{-4} \tilde{\nabla}_a \tilde{\nabla}^a B - G_0^i \Omega^{-4} \tilde{\nabla}_a \tilde{\nabla}^a \dot{E} + 2G_a{}^i \dot{\Omega} \Omega^{-5} \tilde{\nabla}^a B + \dot{G}_a{}^i \Omega^{-4} \tilde{\nabla}^a B \\
& + \Omega^{-4} \tilde{\nabla}_a G_0^i \tilde{\nabla}^a B + G_a{}^i \Omega^{-4} \tilde{\nabla}^a \dot{B} - 2kG_a{}^i \Omega^{-4} \tilde{\nabla}^a E + G_a{}^i \Omega^{-4} \tilde{\nabla}^a \phi - G_a{}^i \Omega^{-4} \tilde{\nabla}^a \psi \\
& - 2E^{ia} \Omega^{-4} \tilde{\nabla}_b G_a{}^b - \Omega^{-4} \tilde{\nabla}^a E^i \tilde{\nabla}_b G_a{}^b - 2E^{ab} \Omega^{-4} \tilde{\nabla}_b G_a{}^i - G_a{}^i \Omega^{-4} \tilde{\nabla}_b \tilde{\nabla}^b E^a \\
& - G_a{}^i \Omega^{-4} \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}^a E - \Omega^{-4} \tilde{\nabla}_a G_b{}^i \tilde{\nabla}^b E^a - \Omega^{-4} \tilde{\nabla}_b G_a{}^i \tilde{\nabla}^b E^a - 2\Omega^{-4} \tilde{\nabla}_b G_a{}^i \tilde{\nabla}^b \tilde{\nabla}^a E \\
& - G_a{}^i \dot{\Omega} \Omega^{-5} \tilde{\nabla}^i B - G_{00} \dot{\Omega} \Omega^{-5} \tilde{\nabla}^i B - \dot{G}_{00} \Omega^{-4} \tilde{\nabla}^i B + \Omega^{-4} \tilde{\nabla}_a G_0^a \tilde{\nabla}^i B + G_0^a \Omega^{-4} \tilde{\nabla}^i B_a \\
& - G^{ab} \Omega^{-4} \tilde{\nabla}^i E_{ab} + 2G_0^a \dot{\Omega} \Omega^{-5} \tilde{\nabla}^i E_a + \dot{G}_0^a \Omega^{-4} \tilde{\nabla}^i E_a - \Omega^{-4} \tilde{\nabla}_b G_a{}^b \tilde{\nabla}^i E^a + G_{00} \Omega^{-4} \tilde{\nabla}^i \phi \\
& + G_a{}^a \Omega^{-4} \tilde{\nabla}^i \psi + G_0^a \Omega^{-4} \tilde{\nabla}^i \tilde{\nabla}_a B + 4G_0^a \dot{\Omega} \Omega^{-5} \tilde{\nabla}^i \tilde{\nabla}_a E + 2\dot{G}_0^a \Omega^{-4} \tilde{\nabla}^i \tilde{\nabla}_a E \\
& - 2\Omega^{-4} \tilde{\nabla}_b G_a{}^b \tilde{\nabla}^i \tilde{\nabla}^a E - G_{ab} \Omega^{-4} \tilde{\nabla}^i \tilde{\nabla}^b E^a - G_{ab} \Omega^{-4} \tilde{\nabla}^i \tilde{\nabla}^b \tilde{\nabla}^a E
\end{aligned} \tag{5.10}$$

$$= 0 \tag{5.11}$$

$$\begin{aligned}
\delta(\nabla_\mu \Delta^{\mu 0}) = & \Omega^{-2} \delta \rho^{GI} + 3\dot{\Omega} \Omega^{-3} \delta p^{GI} + 3\dot{\Omega} \Omega^{-3} \delta \rho^{GI} \\
& + (-4\dot{\Omega}^2 \Omega^{-6} + 2\ddot{\Omega} \Omega^{-5} - 2k\Omega^{-4}) \tilde{\nabla}_a \tilde{\nabla}^a \gamma + (4\dot{\Omega}^2 \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2k\Omega^{-5}) \tilde{\nabla}_a \tilde{\nabla}^a V^{GI}
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\delta(\nabla_\mu \Delta^{\mu i}) = & \Omega^{-2} \tilde{\nabla}^i \delta p^{GI} + (-4\dot{\Omega}^2 \Omega^{-6} + 2\ddot{\Omega} \Omega^{-5} - 2k\Omega^{-4}) \tilde{\nabla}^i \dot{\gamma} \\
& + (4\dot{\Omega}^2 \Omega^{-6} - 2\ddot{\Omega} \Omega^{-5} + 2k\Omega^{-4}) \tilde{\nabla}^i \alpha + (4\dot{\Omega}^2 \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2k\Omega^{-5}) \dot{V}^i \\
& + (-4\dot{\Omega}^3 \Omega^{-8} + 8\ddot{\Omega} \dot{\Omega} \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2\dot{\Omega} k \Omega^{-6}) V^i + (4\dot{\Omega}^2 \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2k\Omega^{-5}) \tilde{\nabla}^i \dot{V}^{GI} \\
& + (-4\dot{\Omega}^3 \Omega^{-8} + 8\ddot{\Omega} \dot{\Omega} \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2\dot{\Omega} k \Omega^{-6}) \tilde{\nabla}^i V^{GI}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\nabla_i \delta(\nabla_\mu \Delta^{\mu i}) = & \Omega^{-2} \tilde{\nabla}_a \tilde{\nabla}^a \delta p^{GI} + (-4\dot{\Omega}^2 \Omega^{-6} + 2\ddot{\Omega} \Omega^{-5} - 2k\Omega^{-4}) \tilde{\nabla}_a \tilde{\nabla}^a \dot{\gamma} \\
& + (4\dot{\Omega}^2 \Omega^{-6} - 2\ddot{\Omega} \Omega^{-5} + 2k\Omega^{-4}) \tilde{\nabla}_a \tilde{\nabla}^a \alpha + (4\dot{\Omega}^2 \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2k\Omega^{-5}) \tilde{\nabla}_a \tilde{\nabla}^a \dot{V}^{GI} \\
& + (-4\dot{\Omega}^3 \Omega^{-8} + 8\ddot{\Omega} \dot{\Omega} \Omega^{-7} - 2\ddot{\Omega} \Omega^{-6} + 2\dot{\Omega} k \Omega^{-6}) \tilde{\nabla}_a \tilde{\nabla}^a V^{GI}
\end{aligned} \tag{5.14}$$

Computationally, we find that  $\delta(\nabla_\mu G^{\mu\nu})$  evaluates to zero as expected from the Bianchi identity and that  $\delta(\nabla_\mu \Delta^{\mu\nu}) = \delta(\nabla_\mu T^{\mu\nu})$ . This is the perturbed covariant conservation condition for a RW perfect fluid in analogy to (1.15).

## Appendix A Possibly Useful Relations

$$\tilde{\nabla}^i \Delta_{0i} = 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\alpha + 2k\tilde{\nabla}_a\tilde{\nabla}^a\gamma + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_a\tilde{\nabla}^a V^G \quad (\text{A.1})$$

$$\begin{aligned} \tilde{g}^{ij}\Delta_{ij} &= 6\dot{\Omega}\Omega^{-1}\dot{\gamma} + 3\Omega^2\delta p^{GI} - 6\dot{\Omega}\Omega^{-1}\dot{\alpha} + (-6\dot{\Omega}^2\Omega^{-2} + 12\ddot{\Omega}\Omega^{-1})\dot{\gamma} + (6\dot{\Omega}^2\Omega^{-2} - 12\ddot{\Omega}\Omega^{-1})\alpha \\ &\quad - 2\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 4\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \tilde{\nabla}^i\Delta_{ij} &= 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{\gamma} + \Omega^2\tilde{\nabla}_j\delta p^{GI} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{\alpha} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\dot{\gamma} \\ &\quad + (2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\alpha + 4\dot{\Omega}k\Omega^{-1}\tilde{\nabla}_j\gamma + k\dot{Q}_j + 2\dot{\Omega}k\Omega^{-1}Q_j + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{Q}_j \\ &\quad + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^aQ_j \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \tilde{\nabla}^i\tilde{\nabla}^j\Delta_{ij} &= 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + \Omega^2\tilde{\nabla}_a\tilde{\nabla}^a\delta p^{GI} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{\alpha} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} \\ &\quad + (2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_a\tilde{\nabla}^a\alpha + 4\dot{\Omega}k\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \tilde{\nabla}_a\tilde{\nabla}^a\Delta_{ij} &= 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + \Omega^2\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\delta p^{GI} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{\alpha} \\ &\quad + (-2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij})\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + (2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} - 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij})\tilde{\nabla}_a\tilde{\nabla}^a\alpha + \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i\alpha \\ &\quad + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i\gamma - \tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\gamma + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i\dot{Q}_j \\ &\quad + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_iQ_j + \frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\dot{Q}_i + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_jQ_i - \tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} \\ &\quad - 2k\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} + \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b\Delta_{ij} &= 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + \Omega^2\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\delta p^{GI} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{\alpha} \\ &\quad - 2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma} + 2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha \\ &\quad - 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha + \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i\alpha + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\tilde{\nabla}_i\gamma \\ &\quad - \tilde{g}_{ij}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\gamma + \frac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i\dot{Q}_j \\ &\quad + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_iQ_j + \frac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\dot{Q}_i + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_jQ_i - \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} \\ &\quad - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} - 2k\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} + \tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} \end{aligned} \quad (\text{A.6})$$