RW SVT3 $k \neq 0$ Radiation

1 Background

1.1 Comoving $a(t)$

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) = -dt^2 + a(t)^2\tilde{g}_{ij}dx^i dx^j \tag{1.1}$$

$$G_{00} = -3ka^{-2} - 3\dot{a}^2 a^{-2}, \qquad G_{ij} = \tilde{g}_{ij}(k + \dot{a}^2 + 2a\ddot{a}) \tag{1.2}$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \qquad U_\mu = -\delta^0_\mu \tag{1.3}$$

$$T_{00} = \rho, \qquad T_{ij} = a^2(t)p\tilde{g}_{ij} \tag{1.4}$$

$$\Delta^{(0)}_{\mu\nu} = G_{\mu\nu} + T_{\mu\nu} = 0 \tag{1.5}$$

$$\Delta^{(0)}_{00} = \rho - 3ka^{-2} - 3\dot{a}^2 a^{-2}, \qquad \Delta^{(0)}_{ij} = \tilde{g}_{ij}(a^2 p + k + \dot{a}^2 + 2a\ddot{a}) \tag{1.6}$$

$$\rightarrow \quad \boxed{\rho = 3ka^{-2} + 3\dot{a}^2 a^{-2}} \qquad \boxed{p = -a^{-2}k - \dot{a}^2 a^{-2} - 2\ddot{a}a^{-1}} \tag{1.7}$$

1.2 Conformal $\Omega(\tau)$

$$ds^2 = \Omega^2(\tau)\tilde{g}_{\mu\nu}dx^\mu dx^\nu, \qquad \tilde{g}_{\mu\nu} = \mathrm{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) \tag{1.8}$$

$$G_{00} = -3k - 3\dot{\Omega}^2\Omega^{-2} \qquad G_{ij} = k\tilde{g}_{ij} - \dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 2\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij} \tag{1.9}$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\Omega^2\tilde{g}_{\mu\nu}, \qquad U_\mu = -\Omega\delta^0_\mu \qquad \text{[Evaluated in (1.8)]} \tag{1.10}$$

$$\Delta^{(0)}_{00} = -3k - 3\dot{\Omega}^2\Omega^{-2} + \Omega^2\rho \tag{1.11}$$

$$\rightarrow \quad \boxed{\rho = 3k\Omega^{-2} + 3\dot{\Omega}^2\Omega^{-4}} \tag{1.12}$$

$$\Delta^{(0)}_{ij} = k\tilde{g}_{ij} - \dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 2\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij} + \Omega^2 p\tilde{g}_{ij} \tag{1.13}$$

$$\rightarrow \quad \boxed{p = -k\Omega^{-2} + \dot{\Omega}^2\Omega^{-4} - 2\ddot{\Omega}\Omega^{-3}} \tag{1.14}$$

$$
\begin{aligned}
\nabla_\mu T^{\mu 0} &= \Omega^{-5}\left(\tilde{g}^{ab}T_{ab}\dot{\Omega} + T_{00}\dot{\Omega} + \dot{T}_{00}\Omega - \Omega\tilde{\nabla}_a T_0{}^a\right) \\
&= 3\dot{\Omega}\Omega^{-3}p + 3\dot{\Omega}\Omega^{-3}\rho + \Omega^{-2}\dot{\rho} \qquad (1.15)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu T^{\mu i} &= \Omega^{-5}\left(-2T_0{}^i\dot{\Omega} - \dot{T}_0{}^i\Omega + \Omega\tilde{\nabla}_a T^{ia}\right) \\
&= 0 \qquad (1.16)
\end{aligned}
$$

1.3 Radiation

Taking $\rho = 3p$ we find from (1.14) and (1.12),

$$\ddot{\Omega} = -k\Omega. \qquad (1.17)$$

Imposing initial condition $\Omega(0) = 0$ leads to solutions

$$\Omega = \begin{cases} A\tau & k = 0 \\ A\sin(\tau) & k = 1 \\ A\sinh(\tau) & k = -1. \end{cases} \qquad (1.18)$$

A perturbation of the energy momentum tensor

$$T_{\mu\nu} \;=\; p(4U_\mu U_\nu + p\Omega^2\tilde{g}_{\mu\nu}), \qquad (1.19)$$

is related to the perturbation of the generalized perfect fluid

$$T_{\mu\nu} \;=\; p(4U_\mu U_\nu + p\Omega^2\tilde{g}_{\mu\nu}), \qquad (1.20)$$

by the substitution

$$\delta\rho = 3\delta p. \qquad (1.21)$$

2 Field Equations (G.I. Form)

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = -\dot{\Omega}^{-1}\Omega\psi + B - \dot{E}, \qquad B_i - \dot{E}_i, \qquad E_{ij}, \qquad V_i \qquad (2.1)$$

$$V^{GI} = V - \Omega^2\dot{\Omega}^{-1}\psi \qquad (2.2)$$

$$\delta\rho^{GI} = \delta\rho - 12\dot{\Omega}^2\psi\Omega^{-4} + 6\ddot{\Omega}\psi\Omega^{-3} - 6k\psi\Omega^{-2} \qquad (2.3)$$

$$\delta p^{GI} = \delta p - 4\dot{\Omega}^2\psi\Omega^{-4} + 8\ddot{\Omega}\psi\Omega^{-3} + 2k\psi\Omega^{-2} - 2\dddot{\Omega}\dot{\Omega}^{-1}\psi\Omega^{-2} \qquad (2.4)$$

$$\Delta_{00} = \Omega^2\delta\rho^{GI} - 6\dot{\Omega}^2\Omega^{-2}\dot{\gamma} + 6\dot{\Omega}^2\Omega^{-2}\alpha + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \qquad (2.5)$$

$$
\begin{aligned}
\Delta_{0i} &= 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{\gamma} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\alpha + 2k\tilde{\nabla}_i\gamma + kQ_i + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a Q_i \\
&\quad + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V^{GI} \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{ij} &= 2\dot{\Omega}\Omega^{-1}\ddot{\gamma}\tilde{g}_{ij} + \Omega^2\delta p^{GI}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\alpha}\tilde{g}_{ij} + \dot{\gamma}(-2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij}) \\
&\quad + (2\dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} - 4\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij})\alpha - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\gamma + \tilde{\nabla}_j\tilde{\nabla}_i\alpha \\
&\quad + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\gamma + \tfrac{1}{2}\tilde{\nabla}_i\dot{Q}_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i Q_j + \tfrac{1}{2}\tilde{\nabla}_j\dot{Q}_i + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j Q_i - \ddot{E}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{E}_{ij}
\end{aligned}
$$

$$-2kE_{ij}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} \tag{2.7}$$

$$\begin{aligned} g^{\mu\nu}\Delta_{\mu\nu} &= 6\dot{\Omega}\Omega^{-3}\ddot{\gamma}+3\delta p^{GI}-\delta\rho^{GI}-6\dot{\Omega}\Omega^{-3}\dot{\alpha}+12\ddot{\Omega}\Omega^{-3}\dot{\gamma}-12\ddot{\Omega}\Omega^{-3}\alpha-2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha \\ &\quad -6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \end{aligned} \tag{2.8}$$

3 Field Equations $k=-1$ (G.I. Form)

$$\alpha = \phi+\psi+\dot{B}-\ddot{E}, \qquad \gamma=-\dot{\Omega}^{-1}\Omega\psi+B-\dot{E}, \qquad B_i-\dot{E}_i, \qquad E_{ij}, \qquad V_i \tag{3.1}$$

$$V^{GI} = V-A\cosh^{-1}\tau\sinh^2\tau\psi \tag{3.2}$$

$$\delta\rho^{GI} = \delta\rho-12A^{-2}\sinh^{-4}\tau\psi \qquad =3\delta p^{GI} \qquad \text{with} \qquad \delta\rho=3\delta p \tag{3.3}$$

$$\delta p^{GI} = \delta p-4A^{-2}\sinh^{-4}\tau\psi \tag{3.4}$$

$$\Delta_{00} = \dot{\gamma}\big(-6-6\sinh^{-2}\tau\big)+3A^2\sinh^2\tau\delta p^{GI}+\big(6+6\sinh^{-2}\tau\big)\alpha+2\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_a\tilde{\nabla}^a\gamma \tag{3.5}$$

$$\begin{aligned} \Delta_{0i} &= 2\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_i\dot{\gamma}-4A^{-1}\sinh^{-3}\tau\tilde{\nabla}_iV^{GI}-2\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_i\alpha \\ &\quad -2\tilde{\nabla}_i\gamma-4A^{-1}\sinh^{-3}\tau V_i-Q_i+\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^aQ_i \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Delta_{ij} &= \dot{\gamma}\tilde{g}_{ij}\big(2-2\sinh^{-2}\tau\big)+2\cosh\tau\ddot{\gamma}\tilde{g}_{ij}\sinh^{-1}\tau-2\cosh\tau\dot{\alpha}\tilde{g}_{ij}\sinh^{-1}\tau+A^2\tilde{g}_{ij}\sinh^2\tau\delta p^{GI} \\ &\quad +\tilde{g}_{ij}\big(-2+2\sinh^{-2}\tau\big)\alpha-\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\alpha-2\cosh\tau\tilde{g}_{ij}\sinh^{-1}\tau\tilde{\nabla}_a\tilde{\nabla}^a\gamma+\tilde{\nabla}_j\tilde{\nabla}_i\alpha \\ &\quad +2\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_j\tilde{\nabla}_i\gamma+\tfrac{1}{2}\tilde{\nabla}_i\dot{Q}_j+\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_iQ_j+\tfrac{1}{2}\tilde{\nabla}_j\dot{Q}_i \\ &\quad +\cosh\tau\sinh^{-1}\tau\tilde{\nabla}_jQ_i-\ddot{E}_{ij}-2\cosh\tau\sinh^{-1}\tau\dot{E}_{ij}+2E_{ij}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij} \end{aligned} \tag{3.7}$$

$$\begin{aligned} g^{\mu\nu}\Delta_{\mu\nu} &= 6A^{-2}\cosh\tau\ddot{\gamma}\sinh^{-3}\tau-6A^{-2}\cosh\tau\dot{\alpha}\sinh^{-3}\tau+12A^{-2}\dot{\gamma}\sinh^{-2}\tau-12A^{-2}\sinh^{-2}\tau\alpha \\ &\quad -2A^{-2}\sinh^{-2}\tau\tilde{\nabla}_a\tilde{\nabla}^a\alpha-6A^{-2}\cosh\tau\sinh^{-3}\tau\tilde{\nabla}_a\tilde{\nabla}^a\gamma \end{aligned} \tag{3.8}$$

4 Field Equations $k=1$ (G.I. Form)

$$\alpha = \phi+\psi+\dot{B}-\ddot{E}, \qquad \gamma=-\dot{\Omega}^{-1}\Omega\psi+B-\dot{E}, \qquad B_i-\dot{E}_i, \qquad E_{ij}, \qquad V_i \tag{4.1}$$

$$V^{GI} = V-A\cos^{-1}\tau\sin^2\tau\psi \tag{4.2}$$

$$\delta\rho^{GI} = \delta\rho-12A^{-2}\sin^{-4}\tau\psi \qquad =3\delta p^{GI} \qquad \text{with} \qquad \delta\rho=3\delta p \tag{4.3}$$

$$\delta p^{GI} = \delta p-4A^{-2}\sin^{-4}\tau\psi \tag{4.4}$$

$$\Delta_{00} = \dot{\gamma}\big(6-6\sin^{-2}\tau\big)+3A^2\sin^2\tau\delta p^{GI}+\big(-6+6\sin^{-2}\tau\big)\alpha+2\cos\tau\sin^{-1}\tau\tilde{\nabla}_a\tilde{\nabla}^a\gamma \tag{4.5}$$

$$\begin{aligned} \Delta_{0i} &= 2\cos\tau\sin^{-1}\tau\tilde{\nabla}_i\dot{\gamma}-4A^{-1}\sin^{-3}\tau\tilde{\nabla}_iV^{GI}-2\cos\tau\sin^{-1}\tau\tilde{\nabla}_i\alpha+2\tilde{\nabla}_i\gamma-4A^{-1}\sin^{-3}\tau V_i+Q_i \\ &\quad +\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^aQ_i \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
\Delta_{ij} &= \dot{\gamma}\tilde{g}_{ij}\big(-2-2\sin^{-2}\tau\big)+2\cos\tau\ddot{\gamma}\tilde{g}_{ij}\sin^{-1}\tau-2\cos\tau\dot{\alpha}\tilde{g}_{ij}\sin^{-1}\tau+A^{2}\tilde{g}_{ij}\sin^{2}\tau\delta p^{GI}\\
&+\tilde{g}_{ij}\big(2+2\sin^{-2}\tau\big)\alpha-\tilde{g}_{ij}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\alpha-2\cos\tau\tilde{g}_{ij}\sin^{-1}\tau\tilde{\nabla}_{a}\tilde{\nabla}^{a}\gamma+\tilde{\nabla}_{j}\tilde{\nabla}_{i}\alpha\\
&+2\cos\tau\sin^{-1}\tau\tilde{\nabla}_{j}\tilde{\nabla}_{i}\gamma+\tfrac{1}{2}\tilde{\nabla}_{i}\dot{Q}_{j}+\cos\tau\sin^{-1}\tau\tilde{\nabla}_{i}Q_{j}+\tfrac{1}{2}\tilde{\nabla}_{j}\dot{Q}_{i}+\cos\tau\sin^{-1}\tau\tilde{\nabla}_{j}Q_{i}-\ddot{E}_{ij}\\
&-2\cos\tau\sin^{-1}\tau\dot{E}_{ij}-2E_{ij}+\tilde{\nabla}_{a}\tilde{\nabla}^{a}E_{ij} \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} &= 6A^{-2}\cos\tau\ddot{\gamma}\sin^{-3}\tau-6A^{-2}\cos\tau\dot{\alpha}\sin^{-3}\tau-12A^{-2}\dot{\gamma}\sin^{-2}\tau+12A^{-2}\sin^{-2}\tau\alpha\\
&-2A^{-2}\sin^{-2}\tau\tilde{\nabla}_{a}\tilde{\nabla}^{a}\alpha-6A^{-2}\cos\tau\sin^{-3}\tau\tilde{\nabla}_{a}\tilde{\nabla}^{a}\gamma \qquad (4.8)
\end{aligned}
$$

5 Conservation

Variations are with respect to background (1.8).

$$
\begin{aligned}
\delta(\nabla_{\mu}A^{\mu\nu}) &= \Omega^{-4}\tilde{\nabla}_{\alpha}\delta A^{\nu\alpha}+\tfrac{1}{2}A^{(0)\nu}{}_{\alpha}\Omega^{-4}\tilde{\nabla}^{\alpha}f+2\delta A^{\nu}{}_{\alpha}\Omega^{-5}\tilde{\nabla}^{\alpha}\Omega-2A^{(0)\nu\beta}f_{\alpha\beta}\Omega^{-5}\tilde{\nabla}^{\alpha}\Omega\\
&+A^{(0)\beta}{}_{\beta}f^{\nu}{}_{\alpha}\Omega^{-5}\tilde{\nabla}^{\alpha}\Omega-2A^{(0)}{}_{\alpha}{}^{\beta}f^{\nu}{}_{\beta}\Omega^{-5}\tilde{\nabla}^{\alpha}\Omega-f^{\nu\alpha}\Omega^{-4}\tilde{\nabla}_{\beta}A^{(0)}{}_{\alpha}{}^{\beta}-f^{\alpha\beta}\Omega^{-4}\tilde{\nabla}_{\beta}A^{(0)\nu}{}_{\alpha}\\
&-A^{(0)\nu\alpha}\Omega^{-4}\tilde{\nabla}_{\beta}f_{\alpha}{}^{\beta}-\tfrac{1}{2}A^{(0)\alpha\beta}\Omega^{-4}\tilde{\nabla}^{\nu}f_{\alpha\beta}-\delta A^{\alpha}{}_{\alpha}\Omega^{-5}\tilde{\nabla}^{\nu}\Omega+A^{(0)\alpha\beta}f_{\alpha\beta}\Omega^{-5}\tilde{\nabla}^{\nu}\Omega \qquad (5.1)
\end{aligned}
$$

$$
\begin{aligned}
\delta(\nabla_{\mu}T^{\mu 0}) &= \delta T^{a}{}_{a}\dot{\Omega}\Omega^{-5}+\delta T_{00}\dot{\Omega}\Omega^{-5}-T^{ab}\dot{\Omega}f_{ab}\Omega^{-5}+T^{a}{}_{a}\dot{\Omega}f_{00}\Omega^{-5}+2T_{00}\dot{\Omega}f_{00}\Omega^{-5}-2T_{0}{}^{a}\dot{\Omega}f_{0a}\Omega^{-5}\\
&+\delta\dot{T}_{00}\Omega^{-4}+\tfrac{1}{2}T^{ab}\dot{f}_{ab}\Omega^{-4}+\tfrac{3}{2}T_{00}\dot{f}_{00}\Omega^{-4}-2T_{0}{}^{a}\dot{f}_{0a}\Omega^{-4}+\tfrac{1}{2}T_{00}\dot{f}\Omega^{-4}+2\dot{T}_{00}f_{00}\Omega^{-4}\\
&-2\dot{T}_{0}{}^{a}f_{0a}\Omega^{-4}-\Omega^{-4}\tilde{\nabla}_{a}\delta T_{0}{}^{a}-f_{0}{}^{a}\Omega^{-4}\tilde{\nabla}_{a}T_{00}-f_{00}\Omega^{-4}\tilde{\nabla}_{a}T_{0}{}^{a}-T_{00}\Omega^{-4}\tilde{\nabla}_{a}f_{0}{}^{a}\\
&-\tfrac{1}{2}T_{0}{}^{a}\Omega^{-4}\tilde{\nabla}_{a}f+f_{0}{}^{a}\Omega^{-4}\tilde{\nabla}_{b}T_{a}{}^{b}+T_{0}{}^{a}\Omega^{-4}\tilde{\nabla}_{b}f_{a}{}^{b}+f_{ab}\Omega^{-4}\tilde{\nabla}^{b}T_{0}{}^{a} \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega^{-2}\dot{\delta\rho}+3\dot{\Omega}\Omega^{-3}\delta p+3\dot{\Omega}\Omega^{-3}\delta\rho+(-12\dot{\Omega}^{2}\Omega^{-6}+6\ddot{\Omega}\Omega^{-5}-6k\Omega^{-4})\dot{\psi}\\
&+(-4\dot{\Omega}^{2}\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}_{a}\tilde{\nabla}^{a}B+(4\dot{\Omega}^{2}\Omega^{-6}-2\ddot{\Omega}\Omega^{-5}+2k\Omega^{-4})\tilde{\nabla}_{a}\tilde{\nabla}^{a}\dot{E}\\
&+(4\dot{\Omega}^{2}\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}_{a}\tilde{\nabla}^{a}V \qquad (5.3)
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega^{-2}\dot{\delta\rho}^{GI}+3\dot{\Omega}\Omega^{-3}\delta p^{GI}+3\dot{\Omega}\Omega^{-3}\delta\rho^{GI}\\
&+(-4\dot{\Omega}^{2}\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}_{a}\tilde{\nabla}^{a}\gamma+(4\dot{\Omega}^{2}\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}_{a}\tilde{\nabla}^{a}V^{GI} \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
\delta(\nabla_{\mu}T^{\mu i}) &= -2\delta T_{0}{}^{i}\dot{\Omega}\Omega^{-5}+2T_{0}{}^{a}\dot{\Omega}f^{i}{}_{a}\Omega^{-5}-2T_{0}{}^{i}\dot{\Omega}f_{00}\Omega^{-5}+2T^{i}{}_{a}\dot{\Omega}f_{0}{}^{a}\Omega^{-5}-T^{a}{}_{a}\dot{\Omega}f_{0}{}^{i}\Omega^{-5}\\
&-T_{00}\dot{\Omega}f_{0}{}^{i}\Omega^{-5}-\delta\dot{T}_{0}{}^{i}\Omega^{-4}-T_{0}{}^{i}\dot{f}_{00}\Omega^{-4}+T^{i}{}_{a}\dot{f}_{0}{}^{a}\Omega^{-4}-\tfrac{1}{2}T_{0}{}^{i}\dot{f}\Omega^{-4}+\dot{T}_{0}{}^{a}f^{i}{}_{a}\Omega^{-4}\\
&-\dot{T}_{0}{}^{i}f_{00}\Omega^{-4}+\dot{T}^{i}{}_{a}f_{0}{}^{a}\Omega^{-4}-\dot{T}_{00}f_{0}{}^{i}\Omega^{-4}+\Omega^{-4}\tilde{\nabla}_{a}\delta T^{ia}+f_{0}{}^{i}\Omega^{-4}\tilde{\nabla}_{a}T_{0}{}^{a}+f_{0}{}^{a}\Omega^{-4}\tilde{\nabla}_{a}T_{0}{}^{i}\\
&+T_{0}{}^{i}\Omega^{-4}\tilde{\nabla}_{a}f_{0}{}^{a}+\tfrac{1}{2}T^{i}{}_{a}\Omega^{-4}\tilde{\nabla}^{a}f-f^{ia}\Omega^{-4}\tilde{\nabla}_{b}T_{a}{}^{b}-f^{ab}\Omega^{-4}\tilde{\nabla}_{b}T^{i}{}_{a}-T^{ia}\Omega^{-4}\tilde{\nabla}_{b}f_{a}{}^{b}\\
&-\tfrac{1}{2}T^{ab}\Omega^{-4}\tilde{\nabla}^{i}f_{ab}-\tfrac{1}{2}T_{00}\Omega^{-4}\tilde{\nabla}^{i}f_{00}+T_{0}{}^{a}\Omega^{-4}\tilde{\nabla}^{i}f_{0a} \qquad (5.5)
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega^{-2}\tilde{\nabla}^{i}\delta p+(4\dot{\Omega}^{2}\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}^{i}\dot{V}\\
&+(-4\dot{\Omega}^{3}\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^{i}V\\
&+(4\dot{\Omega}^{2}\Omega^{-6}-2\ddot{\Omega}\Omega^{-5}+2k\Omega^{-4})\tilde{\nabla}^{i}\phi+(4\dot{\Omega}^{2}\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\dot{V}^{i}\\
&+(-4\dot{\Omega}^{3}\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})V^{i} \qquad (5.6)
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega^{-2}\tilde{\nabla}^{i}\delta p^{GI}+(-4\dot{\Omega}^{2}\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}^{i}\dot{\gamma}\\
&+(4\dot{\Omega}^{2}\Omega^{-6}-2\ddot{\Omega}\Omega^{-5}+2k\Omega^{-4})\tilde{\nabla}^{i}\alpha+(4\dot{\Omega}^{2}\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\dot{V}^{i}
\end{aligned}
$$

$$+(-4\dot{\Omega}^3\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})V^i+(4\dot{\Omega}^2\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}^i\dot{V}^{GI}$$
$$+(-4\dot{\Omega}^3\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^iV^{GI} \tag{5.7}$$

$$\begin{aligned}
\delta(\nabla_\mu G^{\mu 0}) \;=\;& \delta G^a{}_a\dot{\Omega}\Omega^{-5}+\delta G_{00}\dot{\Omega}\Omega^{-5}-2B^aG_{0a}\dot{\Omega}\Omega^{-5}-2G^{ab}\dot{\Omega}E_{ab}\Omega^{-5}-2G^a{}_a\dot{\Omega}\phi\Omega^{-5}-4G_{00}\dot{\Omega}\phi\Omega^{-5}\\
&+2G^a{}_a\dot{\Omega}\psi\Omega^{-5}+\delta\dot{G}_{00}\Omega^{-4}-2B^a\dot{G}_{0a}\Omega^{-4}-2\dot{B}^aG_{0a}\Omega^{-4}+G^{ab}\dot{E}_{ab}\Omega^{-4}-2G_{00}\dot{\phi}\Omega^{-4}\\
&-G^a{}_a\dot{\psi}\Omega^{-4}-3G_{00}\dot{\psi}\Omega^{-4}+2kG_0{}^aE_a\Omega^{-4}-4\dot{G}_{00}\phi\Omega^{-4}-2G_0{}^a\dot{\Omega}\Omega^{-5}\tilde{\nabla}_aB-2\dot{G}_0{}^a\Omega^{-4}\tilde{\nabla}_aB\\
&-2G_0{}^a\Omega^{-4}\tilde{\nabla}_a\dot{B}-\Omega^{-4}\tilde{\nabla}_a\delta G_0{}^a-B^a\Omega^{-4}\tilde{\nabla}_aG_{00}+2\phi\Omega^{-4}\tilde{\nabla}_aG_0{}^a-2\psi\Omega^{-4}\tilde{\nabla}_aG_0{}^a\\
&+2kG_0{}^a\Omega^{-4}\tilde{\nabla}_aE-G_0{}^a\Omega^{-4}\tilde{\nabla}_a\phi+G_0{}^a\Omega^{-4}\tilde{\nabla}_a\psi-G_{00}\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^aB+G_{00}\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}\\
&-\Omega^{-4}\tilde{\nabla}_aG_{00}\tilde{\nabla}^aB+B^a\Omega^{-4}\tilde{\nabla}_bG_a{}^b+\Omega^{-4}\tilde{\nabla}^aB\tilde{\nabla}_bG_a{}^b+G_0{}^a\Omega^{-4}\tilde{\nabla}_b\tilde{\nabla}^bE_a\\
&+G_0{}^a\Omega^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_aE+2E_{ab}\Omega^{-4}\tilde{\nabla}^bG_0{}^a+\Omega^{-4}\tilde{\nabla}_aE_b\tilde{\nabla}^bG_0{}^a+\Omega^{-4}\tilde{\nabla}_bE_a\tilde{\nabla}^bG_0{}^a\\
&+2\Omega^{-4}\tilde{\nabla}_b\tilde{\nabla}_aE\tilde{\nabla}^bG_0{}^a+G_{ab}\Omega^{-4}\tilde{\nabla}^b\dot{E}^a-2G_{ab}\dot{\Omega}\Omega^{-5}\tilde{\nabla}^bE^a+G_{ab}\Omega^{-4}\tilde{\nabla}^b\tilde{\nabla}^a\dot{E}\\
&-2G_{ab}\dot{\Omega}\Omega^{-5}\tilde{\nabla}^b\tilde{\nabla}^aE
\end{aligned} \tag{5.8}$$

$$= 0 \tag{5.9}$$

$$\begin{aligned}
\delta(\nabla_\mu G^{\mu i}) \;=\;& -2\delta G_0{}^i\dot{\Omega}\Omega^{-5}-B^iG^a{}_a\dot{\Omega}\Omega^{-5}+2B^aG^i{}_a\dot{\Omega}\Omega^{-5}-B^iG_{00}\dot{\Omega}\Omega^{-5}+4G_0{}^a\dot{\Omega}E^i{}_a\Omega^{-5}\\
&+4G_0{}^i\dot{\Omega}\phi\Omega^{-5}-4G_0{}^i\dot{\Omega}\psi\Omega^{-5}-\delta\dot{G}_0{}^i\Omega^{-4}+B^a\dot{G}^i{}_a\Omega^{-4}-B^i\dot{G}_{00}\Omega^{-4}+\dot{B}^aG^i{}_a\Omega^{-4}\\
&+G_0{}^i\dot{\phi}\Omega^{-4}+3G_0{}^i\dot{\psi}\Omega^{-4}+2\dot{G}_0{}^aE^i{}_a\Omega^{-4}-2kG^i{}_aE^a\Omega^{-4}+2\dot{G}_0{}^i\phi\Omega^{-4}-2\dot{G}_0{}^i\psi\Omega^{-4}\\
&+\Omega^{-4}\tilde{\nabla}_a\delta G^{ia}+4\psi\Omega^{-4}\tilde{\nabla}_aG^{ia}+B^i\Omega^{-4}\tilde{\nabla}_aG_0{}^a+B^a\Omega^{-4}\tilde{\nabla}_aG_0{}^i+2G_0{}^a\dot{\Omega}\Omega^{-5}\tilde{\nabla}_aE^i\\
&+\dot{G}_0{}^a\Omega^{-4}\tilde{\nabla}_aE^i+G_0{}^i\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^aB-G_0{}^i\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}+2G^i{}_a\dot{\Omega}\Omega^{-5}\tilde{\nabla}^aB+\dot{G}^i{}_a\Omega^{-4}\tilde{\nabla}^aB\\
&+\Omega^{-4}\tilde{\nabla}_aG_0{}^i\tilde{\nabla}^aB+G^i{}_a\Omega^{-4}\tilde{\nabla}^a\dot{B}-2kG^i{}_a\Omega^{-4}\tilde{\nabla}^aE+G^i{}_a\Omega^{-4}\tilde{\nabla}^a\phi-G^i{}_a\Omega^{-4}\tilde{\nabla}^a\psi\\
&-2E^{ia}\Omega^{-4}\tilde{\nabla}_bG_a{}^b-\Omega^{-4}\tilde{\nabla}^aE^i\tilde{\nabla}_bG_a{}^b-2E^{ab}\Omega^{-4}\tilde{\nabla}_bG^i{}_a-G^i{}_a\Omega^{-4}\tilde{\nabla}_b\tilde{\nabla}^bE^a\\
&-G^i{}_a\Omega^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}^aE-\Omega^{-4}\tilde{\nabla}_aG^i{}_b\tilde{\nabla}^bE^a-\Omega^{-4}\tilde{\nabla}_bG^i{}_a\tilde{\nabla}^bE^a-2\Omega^{-4}\tilde{\nabla}_bG^i{}_a\tilde{\nabla}^b\tilde{\nabla}^aE\\
&-G^a{}_a\dot{\Omega}\Omega^{-5}\tilde{\nabla}^iB-G_{00}\dot{\Omega}\Omega^{-5}\tilde{\nabla}^iB-\dot{G}_{00}\Omega^{-4}\tilde{\nabla}^iB+\Omega^{-4}\tilde{\nabla}_aG_0{}^a\tilde{\nabla}^iB+G_0{}^a\Omega^{-4}\tilde{\nabla}^iB_a\\
&-G^{ab}\Omega^{-4}\tilde{\nabla}^iE_{ab}+2G_0{}^a\dot{\Omega}\Omega^{-5}\tilde{\nabla}^iE_a+\dot{G}_0{}^a\Omega^{-4}\tilde{\nabla}^iE_a-\Omega^{-4}\tilde{\nabla}_bG_a{}^b\tilde{\nabla}^iE^a+G_{00}\Omega^{-4}\tilde{\nabla}^i\phi\\
&+G^a{}_a\Omega^{-4}\tilde{\nabla}^i\psi+G_0{}^a\Omega^{-4}\tilde{\nabla}^i\tilde{\nabla}_aB+4G_0{}^a\dot{\Omega}\Omega^{-5}\tilde{\nabla}^i\tilde{\nabla}_aE+2\dot{G}_0{}^a\Omega^{-4}\tilde{\nabla}^i\tilde{\nabla}_aE\\
&-2\Omega^{-4}\tilde{\nabla}_bG_a{}^b\tilde{\nabla}^i\tilde{\nabla}^aE-G_{ab}\Omega^{-4}\tilde{\nabla}^i\tilde{\nabla}^bE^a-G_{ab}\Omega^{-4}\tilde{\nabla}^i\tilde{\nabla}^b\tilde{\nabla}^aE
\end{aligned} \tag{5.10}$$

$$= 0 \tag{5.11}$$

$$\begin{aligned}
\delta(\nabla_\mu\Delta^{\mu 0}) \;=\;& \Omega^{-2}\delta\dot{\rho}^{GI}+3\dot{\Omega}\Omega^{-3}\delta p^{GI}+3\dot{\Omega}\Omega^{-3}\delta\rho^{GI}\\
&+(-4\dot{\Omega}^2\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a\gamma+(4\dot{\Omega}^2\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}_a\tilde{\nabla}^aV^{GI}
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\delta(\nabla_\mu\Delta^{\mu i}) \;=\;& \Omega^{-2}\tilde{\nabla}^i\delta p^{GI}+(-4\dot{\Omega}^2\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}^i\dot{\gamma}\\
&+(4\dot{\Omega}^2\Omega^{-6}-2\ddot{\Omega}\Omega^{-5}+2k\Omega^{-4})\tilde{\nabla}^i\alpha+(4\dot{\Omega}^2\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\dot{V}^i\\
&+(-4\dot{\Omega}^3\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})V^i+(4\dot{\Omega}^2\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}^i\dot{V}^{GI}\\
&+(-4\dot{\Omega}^3\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}^iV^{GI}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\nabla_i\delta(\nabla_\mu\Delta^{\mu i}) \;=\;& \Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\delta p^{GI}+(-4\dot{\Omega}^2\Omega^{-6}+2\ddot{\Omega}\Omega^{-5}-2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a\dot{\gamma}\\
&+(4\dot{\Omega}^2\Omega^{-6}-2\ddot{\Omega}\Omega^{-5}+2k\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a\alpha+(4\dot{\Omega}^2\Omega^{-7}-2\ddot{\Omega}\Omega^{-6}+2k\Omega^{-5})\tilde{\nabla}_a\tilde{\nabla}^a\dot{V}^{GI}\\
&+(-4\dot{\Omega}^3\Omega^{-8}+8\ddot{\Omega}\dot{\Omega}\Omega^{-7}-2\dddot{\Omega}\Omega^{-6}+2\dot{\Omega}k\Omega^{-6})\tilde{\nabla}_a\tilde{\nabla}^aV^{GI}
\end{aligned} \tag{5.14}$$

Computationally, we find that $\delta(\nabla_\mu G^{\mu\nu})$ evaluates to zero as expected from the Bianchi identity and that $\delta(\nabla_\mu \Delta^{\mu\nu}) = \delta(\nabla_\mu T^{\mu\nu})$. This is the perturbed covariant conservation condition for a RW perfect fluid in analogy to (1.15).

Appendix A Possibly Useful Relations

$$\tilde\nabla^i\Delta_{0i} = 2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\dot\gamma - 2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\alpha + 2k\tilde\nabla_a\tilde\nabla^a\gamma + (-4\dot\Omega^2\Omega^{-3} + 2\ddot\Omega\Omega^{-2} - 2k\Omega^{-1})\tilde\nabla_a\tilde\nabla^a V^{GI} \quad (A.1)$$

$$\begin{aligned}\tilde g^{ij}\Delta_{ij} &= 6\dot\Omega\Omega^{-1}\ddot\gamma + 3\Omega^2\delta p^{GI} - 6\dot\Omega\Omega^{-1}\dot\alpha + (-6\dot\Omega^2\Omega^{-2} + 12\ddot\Omega\Omega^{-1})\dot\gamma + (6\dot\Omega^2\Omega^{-2} - 12\ddot\Omega\Omega^{-1})\alpha \\ &\quad -2\tilde\nabla_a\tilde\nabla^a\alpha - 4\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\gamma \end{aligned}\quad (A.2)$$

$$\begin{aligned}\tilde\nabla^i\Delta_{ij} &= 2\dot\Omega\Omega^{-1}\tilde\nabla_j\ddot\gamma + \Omega^2\tilde\nabla_j\delta p^{GI} - 2\dot\Omega\Omega^{-1}\tilde\nabla_j\dot\alpha + (-2\dot\Omega^2\Omega^{-2} + 4\ddot\Omega\Omega^{-1})\tilde\nabla_j\dot\gamma \\ &\quad +(2k + 2\dot\Omega^2\Omega^{-2} - 4\ddot\Omega\Omega^{-1})\tilde\nabla_j\alpha + 4\dot\Omega k\Omega^{-1}\tilde\nabla_j\gamma + k\dot Q_j + 2\dot\Omega k\Omega^{-1}Q_j + \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a\dot Q_j \\ &\quad +\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a Q_j \end{aligned}\quad (A.3)$$

$$\begin{aligned}\tilde\nabla^i\tilde\nabla^j\Delta_{ij} &= 2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\ddot\gamma + \Omega^2\tilde\nabla_a\tilde\nabla^a\delta p^{GI} - 2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\dot\alpha + (-2\dot\Omega^2\Omega^{-2} + 4\ddot\Omega\Omega^{-1})\tilde\nabla_a\tilde\nabla^a\dot\gamma \\ &\quad +(2k + 2\dot\Omega^2\Omega^{-2} - 4\ddot\Omega\Omega^{-1})\tilde\nabla_a\tilde\nabla^a\alpha + 4\dot\Omega k\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\gamma \end{aligned}\quad (A.4)$$

$$\begin{aligned}\tilde\nabla_a\tilde\nabla^a\Delta_{ij} &= 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_a\tilde\nabla^a\ddot\gamma + \Omega^2\tilde g_{ij}\tilde\nabla_a\tilde\nabla^a\delta p^{GI} - 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_a\tilde\nabla^a\dot\alpha \\ &\quad +(-2\dot\Omega^2\Omega^{-2}\tilde g_{ij} + 4\ddot\Omega\Omega^{-1}\tilde g_{ij})\tilde\nabla_a\tilde\nabla^a\dot\gamma + (2\dot\Omega^2\Omega^{-2}\tilde g_{ij} - 4\ddot\Omega\Omega^{-1}\tilde g_{ij})\tilde\nabla_a\tilde\nabla^a\alpha + \tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\tilde\nabla_i\alpha \\ &\quad +2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\tilde\nabla_i\gamma - \tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\alpha - 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\gamma + \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_i\dot Q_j \\ &\quad +\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_i Q_j + \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\dot Q_i + \dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j Q_i - \tilde\nabla_a\tilde\nabla^a\ddot E_{ij} - 2\dot\Omega\Omega^{-1}\tilde\nabla_a\tilde\nabla^a\dot E_{ij} \\ &\quad -2k\tilde\nabla_a\tilde\nabla^a E_{ij} + \tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a E_{ij} \end{aligned}\quad (A.5)$$

$$\begin{aligned}\tilde\nabla_a\tilde\nabla^a\tilde\nabla_b\tilde\nabla^b\Delta_{ij} &= 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\ddot\gamma + \Omega^2\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\delta p^{GI} - 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\dot\alpha \\ &\quad -2\dot\Omega^2\Omega^{-2}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\dot\gamma + 4\ddot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\dot\gamma + 2\dot\Omega^2\Omega^{-2}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\alpha \\ &\quad -4\ddot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\alpha + \tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\tilde\nabla_i\alpha + 2\dot\Omega\Omega^{-1}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\tilde\nabla_i\gamma \\ &\quad -\tilde g_{ij}\tilde\nabla_c\tilde\nabla^c\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\alpha - 2\dot\Omega\Omega^{-1}\tilde g_{ij}\tilde\nabla_c\tilde\nabla^c\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\gamma + \tfrac{1}{2}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_i\dot Q_j \\ &\quad +\dot\Omega\Omega^{-1}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_i Q_j + \tfrac{1}{2}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j\dot Q_i + \dot\Omega\Omega^{-1}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\tilde\nabla_j Q_i - \tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\ddot E_{ij} \\ &\quad -2\dot\Omega\Omega^{-1}\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a\dot E_{ij} - 2k\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a E_{ij} + \tilde\nabla_c\tilde\nabla^c\tilde\nabla_b\tilde\nabla^b\tilde\nabla_a\tilde\nabla^a E_{ij} \end{aligned}\quad (A.6)$$

A.1 Commutations

$$[\nabla^2\nabla_i, \nabla_i\nabla^2]A_j = 2k(\nabla_i A_j + \nabla_j A_i - g_{ij}\nabla^k A_k) \quad (A.7)$$

$$[\nabla^2\nabla_i, \nabla^i\nabla^2]A_i = -2k\nabla^k A_k \quad (A.8)$$

$$[\nabla^2\nabla_i\nabla_j, \nabla_i\nabla_j\nabla^2]S = 2k(3\nabla_i\nabla_j S - g_{ij}\nabla^2 S) \quad (A.9)$$

$$(\nabla^2 - 3k)(\nabla_i\nabla_j + kg_{ij}) = (\nabla_i\nabla_j - kg_{ij})(\nabla^2 + 3k) \quad (A.10)$$

$$\begin{aligned}(\nabla^2 - 2k)(\nabla^2 - 3k)(\nabla_i W_j + \nabla_j W_i) &= -4kg_{ij}(2\nabla^2 + k)\nabla^k W_k + \nabla_j\nabla^2(\nabla^2 + 2k)W_i + k\nabla_j(\nabla^2 + 2k)W_i \\ &\quad +\nabla_i\nabla^2(\nabla^2 + 2k)W_j + k\nabla_i(\nabla^2 + 2k)W_j \end{aligned}\quad (A.11)$$

$$\begin{aligned}\nabla_i\nabla^2(\nabla^2 + 2k)W_j &= \nabla^2\nabla_i(\nabla^2 + 2k)W_j - 2k\nabla_j(\nabla^2 + 2k)W_i - 2k\nabla_i(\nabla^2 + 2k)W_j \\ &\quad +2kg_{ij}(\nabla^2 + 4k)\nabla^k W_k \end{aligned}\quad (A.12)$$

$$
\begin{aligned}
(\nabla^2-2k)(\nabla^2-3k)(\nabla_i W_j+\nabla_j W_i) &= \nabla^2\nabla_i(\nabla^2+2k)W_j+\nabla^2\nabla_j(\nabla^2+2k)W_i-3k\nabla_j(\nabla^2+2k)W_i-3k\nabla_i(\nabla^2+2k)W_j \\
&\quad -4kg_{ij}\nabla^2\nabla^k W_k+12k^2 g_{ij}\nabla^k W_k \qquad \text{(A.13)}
\end{aligned}
$$

$$
(\nabla^2+4k)\nabla^k W_k = \nabla^k\nabla^l h_{kl} \qquad \text{(A.14)}
$$

$$
h_{ij}^{T\theta} = h_{ij}-\nabla_i W_j-\nabla_j W_i+\frac{g_{ij}}{2}(\nabla^k W_k-h)+\frac{1}{2}(\nabla_i\nabla_j+kg_{ij})\int D(\nabla^k W_k+h) \qquad \text{(A.15)}
$$

$$
\begin{aligned}
&(\nabla^2-2k)(\nabla^2-3k)\left[\frac{g_{ij}}{2}(\nabla^k W_k-h)+\frac{1}{2}(\nabla_i\nabla_j+kg_{ij})\int D(\nabla^k W_k+h)\right] \\
= \quad & \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)\nabla^k W_k+\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2+4k)\nabla^k W_k \\
& -6kg_{ij}\nabla^2\nabla^k W_k+4k^2 g_{ij}\nabla^k W_k+\tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h-\tfrac{1}{2}g_{ij}\nabla^4 h+kg_{ij}\nabla^2 h-2k^2 g_{ij}h \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
& (\nabla^2-2k)(\nabla^2-3k)h_{ij}^{T\theta} \\
= \quad & (\nabla^2-2k)(\nabla^2-3k)h_{ij}-\nabla^2\nabla_i(\nabla^2+2k)W_j-\nabla^2\nabla_j(\nabla^2+2k)W_i+3k\nabla_j(\nabla^2+2k)W_i+3k\nabla_i(\nabla^2+2k)W_j \\
& +\tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)\nabla^k W_k+\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2+4k)\nabla^k W_k-2kg_{ij}(\nabla^2+4k)\nabla^k W_k \\
& +\tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h-\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)h-\tfrac{1}{2}k(\nabla^2+4k)h \qquad \text{(A.17)} \\
= \quad & (\nabla^2-2k)(\nabla^2-3k)h_{ij}-\nabla^2\nabla_i\nabla^l h_{jl}-\nabla^2\nabla_j\nabla^l h_{il}+3k\nabla_j\nabla^l h_{il}+3k\nabla_i\nabla^l h_{jl}+\tfrac{1}{2}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl}+\tfrac{1}{2}g_{ij}\nabla^2\nabla^k\nabla^l h_{kl} \\
& -2kg_{ij}\nabla^l\nabla^k h_{kl}+\tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h-\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)h-\tfrac{1}{2}g_{ij}k(\nabla^2+4k)h \qquad \text{(A.18)}
\end{aligned}
$$