SVT4 δU_μ Covariant Comment

From (2.5) and (2.6) in RW_k_Ein_SVT4_v4_Matthew we have the perturbed EM tensor:

$$\begin{aligned}\delta T_{\mu\nu} &= \delta p\tilde{g}_{\mu\nu}\Omega^2 + \delta p U_\mu U_\nu \Omega^2 + \delta\rho U_\mu U_\nu \Omega^2 - 2\tilde{g}_{\mu\nu} p\chi\Omega^2 + 2p\Omega^2\tilde{\nabla}_\mu\tilde{\nabla}_\nu F + \delta U_\nu p U_\mu \Omega^2 + \delta U_\mu p U_\nu \Omega^2 \\ &\quad + \delta U_\nu U_\mu \rho\Omega^2 + \delta U_\mu U_\nu \rho\Omega^2 + p\Omega^2\tilde{\nabla}_\mu F_\nu + p\Omega^2\tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu} p\Omega^2 \end{aligned} \tag{0.1}$$

$$\begin{aligned}g^{\mu\nu}\delta T_{\mu\nu} &= 3\delta p - \delta\rho - 6p\chi + 2\rho\chi + 2p\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha F + 2pU^\alpha U^\beta\tilde{\nabla}_\beta\tilde{\nabla}_\alpha F + 2U^\alpha U^\beta\rho\tilde{\nabla}_\beta\tilde{\nabla}_\alpha F + 2pU^\alpha U^\beta\tilde{\nabla}_\beta F_\alpha \\ &\quad + 2U^\alpha U^\beta\rho\tilde{\nabla}_\beta F_\alpha + 2F_{\alpha\beta} p U^\alpha U^\beta + 2F_{\alpha\beta}U^\alpha U^\beta\rho. \end{aligned} \tag{0.2}$$

The perturbed four-velocity obeys the kinematic relation,

$$U^\mu \delta U_\mu = \tfrac{1}{2} U^\mu U^\nu f_{\mu\nu} \tag{0.3}$$

resulting from $\delta(\tilde{g}^{\mu\nu}U_\mu U_\nu) = 0$.

When forming the trace $g^{\mu\nu}\delta T_{\mu\nu}$ we get terms that go as $\delta U_\alpha U^\alpha$ - a form that is readily available to implement(0.3). As a result, we were able to bring $g^{\mu\nu}\Delta_{\mu\nu}$ into an entirely covariant gauge invariant form.

However, within $\delta T_{\mu\nu}$ itself, we have terms of the form

$$\Omega^2(\rho + p)(\delta U_\mu U_\nu + U_\mu \delta U_\nu). \tag{0.4}$$

For $\mu\nu = (0,0)$ this becomes $-2\Omega^2(\rho+p)\delta U_0$, for $\mu\nu = (0,i)$ this becomes $-\Omega^2(\rho+p)\delta U_i$, while for $\mu\nu = (i,j)$ it vanishes. Hence, for an explicit choice of components, we need to substitute a kinematic relation for δU_0, while for others we seek to express δU_i as $V_i + \tilde{\nabla}_i V$.

Is it possible to incorporate the kinematic identity while retaining full covariance?