

Weyl Tensor Simplifications

$\delta W_{\mu\nu}$ Trace Dependence (General)

In isolating the trace part of the substitution $h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}g_{\mu\nu}h$, the perturbed conformal tensor takes the form (after some simplification from Bianchi identity and a substitution like eq. 47 in Cosmology paper, also taking $g_{\mu\nu} \equiv g_{\mu\nu}^{(0)}$ hereonforth)

$$\begin{aligned} \delta W_{\mu\nu}(\frac{h}{4}g_{\mu\nu}) = & \frac{1}{24}g_{\mu\nu}R^2h - \frac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \frac{1}{6}RR_{\mu\nu}h + \frac{1}{2}R_{\alpha\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}h + \frac{1}{24}g_{\mu\nu}h\nabla_{\alpha}\nabla^{\alpha}R \\ & - \frac{1}{4}h\nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} - \frac{1}{4}\nabla_{\alpha}h\nabla^{\alpha}R_{\mu\nu} + \frac{1}{4}\nabla_{\alpha}h\nabla_{\beta}R_{\mu}{}^{\beta}{}_{\nu}{}^{\alpha} + \frac{1}{4}\nabla_{\alpha}h\nabla_{\nu}R_{\mu}{}^{\alpha} + \frac{1}{12}h\nabla_{\nu}\nabla_{\mu}R \end{aligned} \quad (1)$$

Using a once contracted bianchi identity,

$$\begin{aligned} \nabla^{\alpha}h\nabla_{\beta}R_{\mu}{}^{\beta}{}_{\nu}{}^{\alpha} &= -\nabla^{\alpha}h\nabla_{\beta}R^{\beta}{}_{\mu\nu\alpha} \\ &= \nabla^{\alpha}h\nabla_{\alpha}R_{\mu\nu} - \nabla^{\alpha}h\nabla_{\nu}R_{\mu\alpha}. \end{aligned} \quad (2)$$

(??) then becomes

$$\begin{aligned} \delta W_{\mu\nu} = & \frac{1}{24}g_{\mu\nu}R^2h - \frac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \frac{1}{6}RR_{\mu\nu}h + \frac{1}{2}R_{\alpha\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}h \\ & + \frac{1}{24}g_{\mu\nu}h\nabla_{\alpha}\nabla^{\alpha}R - \frac{1}{4}h\nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} + \frac{1}{12}h\nabla_{\nu}\nabla_{\mu}R \end{aligned} \quad (3)$$

Now note the form of $W_{\mu\nu}$

$$\begin{aligned} W_{\mu\nu} = & -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{6}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R \\ & + \nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} - \nabla_{\mu}\nabla^{\alpha}R_{\nu\alpha} - \nabla_{\nu}\nabla^{\alpha}R_{\mu\alpha} + \frac{2}{3}\nabla_{\nu}\nabla_{\mu}R \end{aligned} \quad (4)$$

Now use the Bianchi identity on (??) to bring it to

$$W_{\mu\nu} = -\frac{1}{6}g_{\mu\nu}R^2h + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h + \frac{2}{3}RR_{\mu\nu}h - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta}h - \frac{1}{6}g_{\mu\nu}h\nabla_{\alpha}\nabla^{\alpha}R + h\nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} - \frac{1}{3}h\nabla_{\nu}\nabla_{\mu}R \quad (5)$$

It becomes apparent that

$$\boxed{\delta W_{\mu\nu}(\frac{h}{4}g_{\mu\nu}) = -\frac{1}{4}hW_{\mu\nu}.} \quad (6)$$

This result matches eq. 1.13 in *Fluctuations_Summary_Matthew.pdf*, where it was proven in general using the conformal properties of $W_{\mu\nu}$.

$\delta W_{\mu\nu}$ (Flat)

If we perturb $W_{\mu\nu}$ in a flat background we arrive at the form of

$$\delta W_{\mu\nu} = 2\partial^{\lambda}\partial^{\kappa}\delta C_{\mu\lambda\nu\kappa} \quad (7)$$

Weyl quantities in a flat background $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\delta C^L{}_{MNK} = \delta R^L{}_{MNK} + \frac{1}{12}g^{AB}\delta R_{AB}(\delta^L{}_N g_{MK} - \delta^L{}_K g_{MN}) \quad (8)$$

$$-\frac{1}{3}(\delta^L_N \delta R_{MK} - \delta^L_K \delta R_{MN} - g_{MN} \delta R^L_K + g_{MK} \delta R^L_N)$$

$$\delta R^L_{MNK} = \delta \Gamma^L_{MN;K} - \delta \Gamma^L_{MK;N} \quad (9)$$

$$\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} (\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \quad (10)$$

Upon evaluating $\delta C_{\mu\lambda\nu\kappa}$ as defined above, after simplification the perturbed $W_{\mu\nu}$ reduces to

$$2\partial^\lambda \partial^\kappa \delta C_{\mu\lambda\nu\kappa} = \frac{1}{3} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\kappa \partial_\nu \partial_\mu K_{\alpha\lambda} + \frac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\alpha K_{\mu\nu} - \frac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\mu K_{\nu\alpha} \\ - \frac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\nu K_{\mu\alpha} + \frac{1}{6} \eta^{\alpha\kappa} \eta^{\gamma\eta} \eta^{\lambda\beta} \eta_{\mu\nu} \partial_\eta \partial_\gamma \partial_\beta \partial_\kappa K_{\alpha\lambda}. \quad (11)$$

This is equivalent to eq. (50) in Cosmology paper given as

$$\boxed{\delta W_{\mu\nu} = \frac{1}{2} \Pi^\rho_\mu \Pi^\sigma_\nu K_{\rho\sigma} - \frac{1}{6} \Pi_{\mu\nu} \Pi^{\rho\sigma} K_{\rho\sigma}} \quad (12)$$

where

$$\Pi_{\mu\nu} = \eta_{\mu\nu} \partial^\alpha \partial_\alpha - \partial_\mu \partial_\nu$$

and where one must keep in mind $\eta^{\alpha\beta} K_{\alpha\beta} = 0$.

$\delta W_{\mu\nu}$ (General)

No Riemann (56 terms)

$$\delta W_{\mu\nu} = -\frac{1}{6} K_{\mu\nu} R^2 + \frac{1}{3} g_{\mu\nu} K^{\alpha\beta} R R_{\alpha\beta} + \frac{1}{2} K_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - g_{\mu\nu} K^{\alpha\beta} R_\alpha{}^\gamma R_{\beta\gamma} - \frac{2}{3} K^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + 2K^{\alpha\beta} R_{\mu\alpha} R_{\nu\beta} \\ + \frac{1}{3} R \nabla_\alpha \nabla^\alpha K_{\mu\nu} - \frac{1}{6} K_{\mu\nu} \nabla_\alpha \nabla^\alpha R - \frac{1}{3} R \nabla_\alpha \nabla_\mu K_\nu{}^\alpha - \frac{1}{2} \nabla_\alpha \nabla_\mu \nabla_\beta \nabla^\beta K_\nu{}^\alpha - \frac{1}{3} R \nabla_\alpha \nabla_\nu K_\mu{}^\alpha - \frac{1}{2} \nabla_\alpha \nabla_\nu \nabla_\beta \nabla^\beta K_\mu{}^\alpha \\ - \frac{1}{6} \nabla_\alpha K_{\mu\nu} \nabla^\alpha R + \frac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta} \nabla_\beta R_{\mu\nu} + \frac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \frac{2}{3} R_{\mu\nu} \nabla_\beta \nabla_\alpha K^{\alpha\beta} + R_\nu{}^\alpha \nabla_\beta \nabla_\alpha K_\mu{}^\beta \\ - R^{\alpha\beta} \nabla_\beta \nabla_\alpha K_{\mu\nu} + R_\mu{}^\alpha \nabla_\beta \nabla_\alpha K_\nu{}^\beta + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\beta \nabla_\alpha R - K^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} - R_\nu{}^\alpha \nabla_\beta \nabla^\beta K_{\mu\alpha} - R_\mu{}^\alpha \nabla_\beta \nabla^\beta K_{\nu\alpha} \\ + \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha K_{\mu\nu} - \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\mu K_\nu{}^\alpha - \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\nu K_\mu{}^\alpha + R_\nu{}^\alpha \nabla_\beta \nabla_\mu K_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\mu K_{\nu\alpha} + K^{\alpha\beta} \nabla_\beta \nabla_\mu R_{\nu\alpha} \\ + \frac{1}{2} \nabla_\beta \nabla_\mu \nabla_\alpha \nabla^\beta K_\nu{}^\alpha + \frac{1}{2} \nabla_\beta \nabla_\mu \nabla_\alpha \nabla_\nu K^{\alpha\beta} + R_\mu{}^\alpha \nabla_\beta \nabla_\nu K_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\nu K_{\mu\alpha} + K^{\alpha\beta} \nabla_\beta \nabla_\nu R_{\mu\alpha} + \frac{1}{2} \nabla_\beta \nabla_\nu \nabla_\alpha \nabla^\beta K_\mu{}^\alpha \\ + \frac{1}{2} \nabla_\beta \nabla_\nu \nabla_\alpha \nabla_\mu K^{\alpha\beta} + \nabla_\alpha R_{\nu\beta} \nabla^\beta K_\mu{}^\alpha - \nabla_\beta R_{\nu\alpha} \nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta} \nabla^\beta K_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha} \nabla^\beta K_\nu{}^\alpha - g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla_\beta K_\alpha{}^\gamma \\ + \frac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma K_{\alpha\beta} + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} + \frac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha K^{\alpha\beta} + \frac{1}{3} g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma K^{\alpha\beta} + \frac{1}{6} \nabla^\alpha R \nabla_\mu K_{\nu\alpha} \\ + \nabla_\alpha K^{\alpha\beta} \nabla_\mu R_{\nu\beta} - \frac{1}{6} \nabla_\mu R_{\alpha\beta} \nabla_\nu K^{\alpha\beta} + \frac{1}{6} \nabla^\alpha R \nabla_\nu K_{\mu\alpha} - \frac{1}{6} \nabla_\mu K^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + \nabla_\alpha K^{\alpha\beta} \nabla_\nu R_{\mu\beta} - \frac{2}{3} R^{\alpha\beta} \nabla_\nu \nabla_\mu K_{\alpha\beta} \\ - \frac{2}{3} K^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} - \frac{2}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \frac{1}{4} h W_{\mu\nu} (g_{\mu\nu}^{(0)})$$

With Riemann (57 terms)

$$\delta W_{\mu\nu} = -\frac{1}{6} K_{\mu\nu} R^2 + \frac{1}{3} g_{\mu\nu} K^{\alpha\beta} R R_{\alpha\beta} + \frac{1}{2} K_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} K_\nu{}^\alpha R R_{\mu\alpha} - \frac{2}{3} K^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + \frac{1}{3} K_\mu{}^\alpha R R_{\nu\alpha} - g_{\mu\nu} K^{\alpha\beta} R^{\gamma\eta} R_{\alpha\gamma\beta\eta} \\ - \frac{2}{3} K^{\alpha\beta} R R_{\mu\alpha\nu\beta} - K_\nu{}^\alpha R^{\beta\gamma} R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta} R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta} R_{\alpha\gamma\beta\eta} R_\mu{}^\gamma{}_\nu{}^\eta - K_\mu{}^\alpha R^{\beta\gamma} R_{\nu\beta\alpha\gamma} + \frac{1}{3} R \nabla_\alpha \nabla^\alpha K_{\mu\nu} \\ - \frac{1}{6} K_{\mu\nu} \nabla_\alpha \nabla^\alpha R - \frac{1}{6} \nabla_\alpha K_{\mu\nu} \nabla^\alpha R + \frac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta} \nabla_\beta R_{\mu\nu} + \frac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \frac{2}{3} R_{\mu\nu} \nabla_\beta \nabla_\alpha K^{\alpha\beta} \\ + \frac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\alpha K_\mu{}^\beta - R^{\alpha\beta} \nabla_\beta \nabla_\alpha K_{\mu\nu} + \frac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\alpha K_\nu{}^\beta + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\beta \nabla_\alpha R - K^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} + \frac{1}{2} K_\nu{}^\alpha \nabla_\beta \nabla^\beta R_{\mu\alpha} \\ + \frac{1}{2} K_\mu{}^\alpha \nabla_\beta \nabla^\beta R_{\nu\alpha} + \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha K_{\mu\nu} - \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\mu \nabla_\alpha K_\nu{}^\alpha - \frac{1}{2} \nabla_\beta \nabla^\beta \nabla_\nu \nabla_\alpha K_\mu{}^\alpha - g_{\mu\nu} R^{\alpha\beta} \nabla_\beta \nabla_\gamma K_\alpha{}^\gamma \\ - \frac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\mu K_\alpha{}^\beta - \frac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\nu K_\alpha{}^\beta + \nabla_\alpha R_{\nu\beta} \nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta} \nabla^\beta K_\nu{}^\alpha + \frac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma K_{\alpha\beta} - 2R_{\mu\alpha\nu\beta} \nabla_\gamma \nabla^\gamma K^{\alpha\beta} \\ + \frac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} - K^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\mu\alpha\nu\beta} + \frac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha K^{\alpha\beta} + \frac{1}{3} g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma K^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta} \nabla^\gamma K^{\alpha\beta} \\ + R_{\mu\beta\nu\gamma} \nabla^\gamma \nabla_\alpha K^{\alpha\beta} + R_{\mu\gamma\nu\beta} \nabla^\gamma \nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha} \nabla_\mu K^{\alpha\beta} + \frac{1}{6} \nabla^\alpha R \nabla_\mu K_{\nu\alpha} - \frac{1}{3} R \nabla_\mu \nabla_\alpha K_\nu{}^\alpha + R^{\alpha\beta} \nabla_\mu \nabla_\beta K_{\nu\alpha} \\ - \nabla_\beta R_{\mu\alpha} \nabla_\nu K^{\alpha\beta} + \frac{1}{3} \nabla_\mu R_{\alpha\beta} \nabla_\nu K^{\alpha\beta} + \frac{1}{6} \nabla^\alpha R \nabla_\nu K_{\mu\alpha} + \frac{1}{3} \nabla_\mu K^{\alpha\beta} \nabla_\nu R_{\alpha\beta} - \frac{1}{3} R \nabla_\nu \nabla_\alpha K_\mu{}^\alpha + R^{\alpha\beta} \nabla_\nu \nabla_\beta K_{\mu\alpha} \\ - \frac{2}{3} R^{\alpha\beta} \nabla_\nu \nabla_\mu K_{\alpha\beta} + \frac{1}{3} K^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} + \frac{1}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \frac{1}{4} h W_{\mu\nu} (g_{\mu\nu}^{(0)})$$

$\delta C_{\lambda\mu\nu\kappa}$ and Trace Dependence (General)

$$\begin{aligned}\delta C_{\lambda\mu\nu\kappa} = & -\frac{1}{6}g_{\mu\nu}K_{\kappa\lambda}R + \frac{1}{6}g_{\lambda\nu}K_{\kappa\mu}R + \frac{1}{6}g_{\kappa\mu}K_{\lambda\nu}R - \frac{1}{6}g_{\kappa\lambda}K_{\mu\nu}R - \frac{1}{6}g_{\kappa\mu}g_{\lambda\nu}K^{\alpha\beta}R_{\alpha\beta} + \frac{1}{6}g_{\kappa\lambda}g_{\mu\nu}K^{\alpha\beta}R_{\alpha\beta} + \frac{1}{2}K_{\mu\nu}R_{\kappa\lambda} \\ & - \frac{1}{2}K_{\lambda\nu}R_{\kappa\mu} - \frac{1}{2}K_{\kappa\mu}R_{\lambda\nu} + \frac{1}{2}K_{\kappa\lambda}R_{\mu\nu} + K_{\lambda}^{\alpha}R_{\kappa\nu\mu\alpha} + \frac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}K_{\kappa\lambda} - \frac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla^{\alpha}K_{\kappa\mu} - \frac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla^{\alpha}K_{\lambda\nu} \\ & + \frac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla_{\kappa}K_{\lambda}^{\alpha} + \frac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla_{\kappa}K_{\mu}^{\alpha} - \frac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla_{\lambda}K_{\kappa}^{\alpha} + \frac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla_{\lambda}K_{\nu}^{\alpha} + \frac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla_{\mu}K_{\kappa}^{\alpha} \\ & - \frac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla_{\mu}K_{\nu}^{\alpha} + \frac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla_{\nu}K_{\lambda}^{\alpha} - \frac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla_{\nu}K_{\mu}^{\alpha} - \frac{1}{6}g_{\kappa\mu}g_{\lambda\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \frac{1}{6}g_{\kappa\lambda}g_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} \\ & - \frac{1}{2}\nabla_{\kappa}\nabla_{\lambda}K_{\mu\nu} + \frac{1}{2}\nabla_{\kappa}\nabla_{\mu}K_{\lambda\nu} + \frac{1}{2}\nabla_{\kappa}\nabla_{\nu}K_{\lambda\mu} - \frac{1}{2}\nabla_{\nu}\nabla_{\kappa}K_{\lambda\mu} + \frac{1}{2}\nabla_{\nu}\nabla_{\lambda}K_{\kappa\mu} - \frac{1}{2}\nabla_{\nu}\nabla_{\mu}K_{\kappa\lambda} + \frac{1}{4}hC_{\lambda\mu\nu\kappa}\end{aligned}$$

where the trace dependent terms are

$$\begin{aligned}\delta C_{\lambda\mu\nu\kappa}(\frac{h}{4}g_{\mu\nu}^{(0)}) &= \frac{1}{24}g_{\kappa\mu}g_{\lambda\nu}Rh - \frac{1}{24}g_{\kappa\lambda}g_{\mu\nu}Rh + \frac{1}{8}g_{\mu\nu}R_{\kappa\lambda}h - \frac{1}{8}g_{\lambda\nu}R_{\kappa\mu}h - \frac{1}{8}g_{\kappa\mu}R_{\lambda\nu}h + \frac{1}{8}g_{\kappa\lambda}R_{\mu\nu}h - \frac{1}{4}R_{\kappa\nu\lambda\mu}h \\ &= \frac{1}{4}hC_{\lambda\mu\nu\kappa}.\end{aligned}\quad (13)$$

Note that this is opposite in sign compared to the trace dependence of $\delta W_{\mu\nu}$. To see this, under conformal transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$$

we know the Weyl tensor is invariant

$$C^{\lambda}_{\mu\nu\kappa} \rightarrow C^{\lambda}_{\mu\nu\kappa}.$$

This is equivalent to

$$g^{\lambda\alpha}C_{\alpha\mu\nu\kappa} \rightarrow \Omega^{-2}(x)g^{\lambda\alpha}\tilde{C}_{\alpha\mu\nu\kappa},$$

and thus for the quantity to remain invariant, the covariant Weyl tensor must transform as

$$C_{\lambda\mu\nu\kappa} \rightarrow \Omega^2(x)C_{\lambda\mu\nu\kappa}.$$

The conformal symmetry applies to the trace as the following:

$$\begin{aligned}C_{\lambda\mu\nu\kappa}\left(1 + \frac{h}{4}\right)g_{\mu\nu}^{(0)} &= \left(1 + \frac{h}{4}\right)C_{\lambda\mu\nu\kappa}(g_{\mu\nu}^{(0)}) \\ C_{\lambda\mu\nu\kappa}(g_{\mu\nu}^{(0)}) + \delta C_{\lambda\mu\nu\kappa}(\frac{h}{4}g_{\mu\nu}^{(0)}) &= C_{\lambda\mu\nu\kappa}(g_{\mu\nu}^{(0)}) + \frac{h}{4}C_{\lambda\mu\nu\kappa}(g_{\mu\nu}^{(0)})\end{aligned}$$

and therefore

$$\boxed{\delta C_{\lambda\mu\nu\kappa}\left(\frac{h}{4}g_{\mu\nu}^{(0)}\right) = \frac{h}{4}C_{\lambda\mu\nu\kappa}(g_{\mu\nu}^{(0)})}. \quad (14)$$

Latest Format

$$W_{\mu\nu}^{(1)} = \frac{1}{2}g_{\mu\nu}R^2 - 2RR_{\mu\nu} + 2g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R - 2\nabla_{\nu}\nabla_{\mu}R. \quad (15)$$

$$W_{\mu\nu}^{(2)} = \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + \nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} - \nabla_{\mu}\nabla^{\alpha}R_{\nu\alpha} - \nabla_{\nu}\nabla^{\alpha}R_{\mu\alpha}. \quad (16)$$

$$\begin{aligned}W_{\mu\nu} = & -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{6}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + \nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} \\ & - \nabla_{\mu}\nabla^{\alpha}R_{\nu\alpha} - \nabla_{\nu}\nabla^{\alpha}R_{\mu\alpha} + \frac{2}{3}\nabla_{\nu}\nabla_{\mu}R.\end{aligned}\quad (17)$$

General, no Bianchi, no explicit covariant derivative commutation

80 Terms

$$\begin{aligned}
\delta W_{\mu\nu}(h_{\mu\nu}) = & -\frac{1}{6}h_{\mu\nu}R^2 + \frac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}h^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\beta\gamma} - \frac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\
& + 2h^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\mu\gamma\nu\beta} + \frac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}h_{\mu\nu} - \frac{1}{6}h_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + R_{\mu\beta\nu\gamma}\nabla_{\alpha}\nabla^{\gamma}h^{\alpha\beta} \\
& + R_{\mu\gamma\nu\beta}\nabla_{\alpha}\nabla^{\gamma}h^{\alpha\beta} - \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\alpha}\nabla_{\mu}h^{\beta}{}_{\beta} - \frac{1}{3}R\nabla_{\alpha}\nabla_{\mu}h_{\nu}{}^{\alpha} - \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\alpha}\nabla_{\nu}h^{\beta}{}_{\beta} \\
& - \frac{1}{3}R\nabla_{\alpha}\nabla_{\nu}h_{\mu}{}^{\alpha} - \frac{1}{12}g_{\mu\nu}\nabla_{\alpha}h^{\beta}{}_{\beta}\nabla^{\alpha}R + \frac{1}{3}\nabla_{\alpha}h_{\mu\nu}\nabla^{\alpha}R + \frac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}h_{\alpha}{}^{\beta} \\
& - \nabla^{\alpha}h_{\mu\nu}\nabla_{\beta}R_{\alpha}{}^{\beta} - \nabla_{\alpha}h^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} - \frac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} \\
& + \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}h^{\gamma}{}_{\gamma} + \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}h_{\mu}{}^{\beta} - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}h_{\mu\nu} + \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}h_{\nu}{}^{\beta} \\
& + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R - h^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla^{\beta}h^{\alpha}{}_{\alpha} + \frac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla^{\beta}h^{\alpha}{}_{\alpha} \\
& - \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}h_{\mu\alpha} - \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}h_{\nu\alpha} + \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla^{\alpha}h_{\mu\nu} - \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\mu}h_{\nu}{}^{\alpha} \\
& - \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\nu}h_{\mu}{}^{\alpha} + \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\nu}\nabla_{\mu}h^{\alpha}{}_{\alpha} - \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\mu}h_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\mu}h_{\nu\alpha} \\
& - \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\nu}h_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\nu}h_{\mu\alpha} + \frac{1}{2}\nabla_{\beta}R_{\mu\nu}\nabla^{\beta}h^{\alpha}{}_{\alpha} + \nabla_{\alpha}R_{\nu\beta}\nabla^{\beta}h_{\mu}{}^{\alpha} \\
& - \nabla_{\beta}R_{\nu\alpha}\nabla^{\beta}h_{\mu}{}^{\alpha} + \nabla_{\alpha}R_{\mu\beta}\nabla^{\beta}h_{\nu}{}^{\alpha} - \nabla_{\beta}R_{\mu\alpha}\nabla^{\beta}h_{\nu}{}^{\alpha} - g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla_{\beta}h_{\alpha}{}^{\gamma} \\
& + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}h_{\alpha\beta} - R_{\mu\alpha\nu\beta}\nabla_{\gamma}\nabla^{\gamma}h^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} \\
& + \frac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla^{\beta}h^{\alpha}{}_{\alpha} + \frac{1}{3}g_{\mu\nu}\nabla_{\gamma}R_{\alpha\beta}\nabla^{\gamma}h^{\alpha\beta} \\
& - R_{\mu\beta\nu\gamma}\nabla^{\gamma}\nabla^{\beta}h^{\alpha}{}_{\alpha} - \frac{1}{3}\nabla^{\alpha}R\nabla_{\mu}h_{\nu\alpha} + \nabla_{\beta}R_{\alpha}{}^{\beta}\nabla_{\mu}h_{\nu}{}^{\alpha} + \nabla_{\alpha}h^{\alpha\beta}\nabla_{\mu}R_{\nu\beta} \\
& - \frac{1}{2}\nabla^{\beta}h^{\alpha}{}_{\alpha}\nabla_{\mu}R_{\nu\beta} - \frac{1}{2}\nabla_{\mu}\nabla_{\alpha}\nabla_{\beta}\nabla^{\beta}h_{\nu}{}^{\alpha} + R_{\nu}{}^{\alpha}\nabla_{\mu}\nabla_{\beta}h_{\alpha}{}^{\beta} + h^{\alpha\beta}\nabla_{\mu}\nabla_{\beta}R_{\nu\alpha} \\
& + \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}\nabla^{\beta}h_{\nu}{}^{\alpha} + \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}\nabla_{\nu}h^{\alpha\beta} - \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\nabla^{\beta}\nabla_{\nu}h^{\alpha}{}_{\alpha} + \frac{1}{2}R^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}h_{\alpha\beta} \\
& - \frac{1}{6}\nabla_{\mu}R_{\alpha\beta}\nabla_{\nu}h^{\alpha\beta} - \frac{1}{3}\nabla^{\alpha}R\nabla_{\nu}h_{\mu\alpha} + \nabla_{\beta}R_{\alpha}{}^{\beta}\nabla_{\nu}h_{\mu}{}^{\alpha} - \frac{1}{6}\nabla_{\mu}h^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta} \\
& + \nabla_{\alpha}h^{\alpha\beta}\nabla_{\nu}R_{\mu\beta} - \frac{1}{2}\nabla^{\beta}h^{\alpha}{}_{\alpha}\nabla_{\nu}R_{\mu\beta} - \frac{1}{2}\nabla_{\nu}\nabla_{\alpha}\nabla_{\beta}\nabla^{\beta}h_{\mu}{}^{\alpha} + R_{\mu}{}^{\alpha}\nabla_{\nu}\nabla_{\beta}h_{\alpha}{}^{\beta} \\
& + h^{\alpha\beta}\nabla_{\nu}\nabla_{\beta}R_{\mu\alpha} + \frac{1}{2}\nabla_{\nu}\nabla_{\beta}\nabla_{\alpha}\nabla^{\beta}h_{\mu}{}^{\alpha} + \frac{1}{2}\nabla_{\nu}\nabla_{\beta}\nabla_{\alpha}\nabla_{\mu}h^{\alpha\beta} - \frac{1}{2}\nabla_{\nu}\nabla_{\beta}\nabla^{\beta}\nabla_{\mu}h^{\alpha}{}_{\alpha} \\
& - \frac{7}{6}R^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}h_{\alpha\beta} + \frac{1}{3}R\nabla_{\nu}\nabla_{\mu}h^{\alpha}{}_{\alpha} - \frac{2}{3}h^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}R_{\alpha\beta} - \frac{2}{3}\nabla_{\nu}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} \\
& + \frac{2}{3}\nabla_{\nu}\nabla_{\mu}\nabla_{\beta}\nabla^{\beta}h^{\alpha}{}_{\alpha}.
\end{aligned} \tag{18}$$

64 Terms

$$\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) = & -\frac{1}{6}K_{\mu\nu}R^2 + \frac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\beta\gamma} - \frac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\
& + 2K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\mu\gamma\nu\beta} + \frac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} - \frac{1}{6}K_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + R_{\mu\beta\nu\gamma}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} \\
& + R_{\mu\gamma\nu\beta}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} - \frac{1}{3}R\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} - \frac{1}{3}R\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} + \frac{1}{3}\nabla_{\alpha}K_{\mu\nu}\nabla^{\alpha}R \\
& + \frac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}K_{\alpha}{}^{\beta} - \nabla^{\alpha}K_{\mu\nu}\nabla_{\beta}R_{\alpha}{}^{\beta} - \nabla_{\alpha}K^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} \\
& - \frac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\mu}{}^{\beta} - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}K_{\mu\nu} + \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\nu}{}^{\beta} \\
& + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R - K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} - \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\mu\alpha} - \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\nu\alpha} \\
& + \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} - \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} - \frac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \frac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\mu}K_{\alpha}{}^{\beta} \\
& + R^{\alpha\beta}\nabla_{\beta}\nabla_{\mu}K_{\nu\alpha} - \frac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\nu}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\nu}K_{\mu\alpha} + \nabla_{\alpha}R_{\nu\beta}\nabla^{\beta}K_{\mu}{}^{\alpha} \\
& - \nabla_{\beta}R_{\nu\alpha}\nabla^{\beta}K_{\mu}{}^{\alpha} + \nabla_{\alpha}R_{\mu\beta}\nabla^{\beta}K_{\nu}{}^{\alpha} - \nabla_{\beta}R_{\mu\alpha}\nabla^{\beta}K_{\nu}{}^{\alpha} - g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla_{\beta}K_{\alpha}{}^{\gamma} \\
& + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}K_{\alpha\beta} - R_{\mu\alpha\nu\beta}\nabla_{\gamma}\nabla^{\gamma}K^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} \\
& + \frac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \frac{1}{3}g_{\mu\nu}\nabla_{\gamma}R_{\alpha\beta}\nabla^{\gamma}K^{\alpha\beta} - \frac{1}{3}\nabla^{\alpha}R\nabla_{\mu}K_{\nu\alpha} + \nabla_{\beta}R_{\alpha}{}^{\beta}\nabla_{\mu}K_{\nu}{}^{\alpha} \\
& + \nabla_{\alpha}K^{\alpha\beta}\nabla_{\mu}R_{\nu\beta} - \frac{1}{2}\nabla_{\mu}\nabla_{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\nu}{}^{\alpha} + R_{\nu}{}^{\alpha}\nabla_{\mu}\nabla_{\beta}K_{\alpha}{}^{\beta} + K^{\alpha\beta}\nabla_{\mu}\nabla_{\beta}R_{\nu\alpha} \\
& + \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}\nabla^{\beta}K_{\nu}{}^{\alpha} + \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}\nabla_{\nu}K^{\alpha\beta} + \frac{1}{2}R^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}K_{\alpha\beta} - \frac{1}{6}\nabla_{\mu}R_{\alpha\beta}\nabla_{\nu}K^{\alpha\beta} \\
& - \frac{1}{3}\nabla^{\alpha}R\nabla_{\nu}K_{\mu\alpha} + \nabla_{\beta}R_{\alpha}{}^{\beta}\nabla_{\nu}K_{\mu}{}^{\alpha} - \frac{1}{6}\nabla_{\mu}K^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta} + \nabla_{\alpha}K^{\alpha\beta}\nabla_{\nu}R_{\mu\beta} \\
& - \frac{1}{2}\nabla_{\nu}\nabla_{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\mu}{}^{\alpha} + R_{\mu}{}^{\alpha}\nabla_{\nu}\nabla_{\beta}K_{\alpha}{}^{\beta} + K^{\alpha\beta}\nabla_{\nu}\nabla_{\beta}R_{\mu\alpha} + \frac{1}{2}\nabla_{\nu}\nabla_{\beta}\nabla_{\alpha}\nabla^{\beta}K_{\mu}{}^{\alpha}
\end{aligned}$$

$$+ \frac{1}{2} \nabla_\nu \nabla_\beta \nabla_\alpha \nabla_\mu K^{\alpha\beta} - \frac{7}{6} R^{\alpha\beta} \nabla_\nu \nabla_\mu K_{\alpha\beta} - \frac{2}{3} K^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} - \frac{2}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha K^{\alpha\beta}. \quad (19)$$

31 Terms

$$\begin{aligned} \delta W_{\mu\nu}(\frac{h}{4}g_{\mu\nu}^{(0)}) = & -\frac{1}{24}g_{\mu\nu}R^2h + \frac{1}{12}g_{\mu\nu}RR^\alpha_\alpha h - \frac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \frac{1}{6}R^\alpha_\alpha R_{\mu\nu}h + \frac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h \\ & - \frac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\nabla_\alpha\nabla^\alpha h + \frac{1}{4}R_{\mu\nu}\nabla_\alpha\nabla^\alpha h - \frac{1}{4}R_{\mu}{}^\beta{}_\nu{}^\beta\nabla_\alpha\nabla^\alpha h \\ & + \frac{1}{4}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h + \frac{1}{24}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R + \frac{1}{12}g_{\mu\nu}\nabla_\alpha R^\beta_\beta\nabla^\alpha h - \frac{1}{4}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h \\ & - \frac{1}{4}g_{\mu\nu}\nabla^\alpha h\nabla_\beta R_{\alpha}{}^\beta + \frac{1}{24}g_{\mu\nu}h\nabla_\beta\nabla^\beta R^\alpha_\alpha + \frac{1}{6}g_{\mu\nu}R^\alpha_\alpha\nabla_\beta\nabla^\beta h - \frac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h \\ & + \frac{1}{4}\nabla_\alpha R_{\nu}{}^\alpha\nabla_\mu h + \frac{1}{4}h\nabla_\mu\nabla_\alpha R_{\nu}{}^\alpha - \frac{1}{4}\nabla_\mu\nabla_\alpha\nabla^\alpha\nabla_\nu h - \frac{1}{12}\nabla_\mu h\nabla_\nu R - \frac{1}{24}\nabla_\mu h\nabla_\nu R^\alpha_\alpha \\ & + \frac{1}{4}\nabla_\alpha R_{\mu}{}^\alpha\nabla_\nu h - \frac{1}{12}\nabla_\mu R\nabla_\nu h - \frac{1}{24}\nabla_\mu R^\alpha_\alpha\nabla_\nu h + \frac{1}{4}h\nabla_\nu\nabla_\alpha R_{\mu}{}^\alpha - \frac{1}{4}\nabla_\nu\nabla_\alpha\nabla^\alpha\nabla_\mu h \\ & - \frac{1}{6}h\nabla_\nu\nabla_\mu R^\alpha_\alpha + \frac{1}{6}R\nabla_\nu\nabla_\mu h - \frac{1}{6}R^\alpha_\alpha\nabla_\nu\nabla_\mu h + \frac{1}{4}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \end{aligned} \quad (20)$$

General, with Bianchi, no explicit covariant derivative commutation

52 Terms

$$\begin{aligned} \delta W_{\mu\nu}(K_{\mu\nu}) = & -\frac{1}{6}K_{\mu\nu}R^2 + \frac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_{\alpha}{}^\gamma R_{\beta\gamma} - \frac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\ & + 2K^{\alpha\beta}R_{\alpha}{}^\gamma R_{\mu\gamma\nu\beta} + \frac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \frac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R + R_{\mu\beta\nu\gamma}\nabla_\alpha\nabla^\gamma K^{\alpha\beta} \\ & + R_{\mu\gamma\nu\beta}\nabla_\alpha\nabla^\gamma K^{\alpha\beta} - \frac{1}{3}R\nabla_\alpha\nabla_\mu K_{\nu}{}^\alpha - \frac{1}{3}R\nabla_\alpha\nabla_\nu K_{\mu}{}^\alpha - \frac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R \\ & + \frac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_{\alpha}{}^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \frac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\ & + \frac{1}{2}R_{\nu}{}^\alpha\nabla_\beta\nabla_\alpha K_{\mu}{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \frac{1}{2}R_{\mu}{}^\alpha\nabla_\beta\nabla_\alpha K_{\nu}{}^\beta + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R \\ & - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} - \frac{1}{2}R_{\nu}{}^\alpha\nabla_\beta\nabla^\beta K_{\mu\alpha} - \frac{1}{2}R_{\mu}{}^\alpha\nabla_\beta\nabla^\beta K_{\nu\alpha} + \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} \\ & - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu K_{\nu}{}^\alpha - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu K_{\mu}{}^\alpha - \frac{1}{2}R_{\nu}{}^\alpha\nabla_\beta\nabla_\mu K_{\alpha}{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} \\ & - \frac{1}{2}R_{\mu}{}^\alpha\nabla_\beta\nabla_\nu K_{\alpha}{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_{\mu}{}^\alpha - \nabla_\beta R_{\nu\alpha}\nabla^\beta K_{\mu}{}^\alpha \\ & + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_{\nu}{}^\alpha - \nabla_\beta R_{\mu\alpha}\nabla^\beta K_{\nu}{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta K_{\alpha}{}^\gamma + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} \\ & - R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\ & + \frac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \frac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} + \frac{1}{2}R^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta} \\ & + \frac{1}{2}K^{\alpha\beta}\nabla_\mu\nabla_\nu R_{\alpha\beta} - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \frac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \frac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} \\ & + \frac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \frac{7}{6}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} - \frac{1}{6}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta}. \end{aligned} \quad (21)$$

15 Terms

$$\begin{aligned} \delta W_{\mu\nu}(\frac{h}{4}g_{\mu\nu}^{(0)}) = & \frac{1}{24}g_{\mu\nu}R^2h - \frac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \frac{1}{6}RR_{\mu\nu}h + \frac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h + \frac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R \\ & - \frac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} + \frac{1}{4}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h - \frac{1}{4}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h - \frac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h + \frac{1}{4}\nabla^\alpha h\nabla_\mu R_{\nu\alpha} \\ & + \frac{1}{4}R_{\nu}{}^\alpha\nabla_\mu\nabla_\alpha h + \frac{1}{4}\nabla^\alpha h\nabla_\nu R_{\mu\alpha} + \frac{1}{4}R_{\mu}{}^\alpha\nabla_\nu\nabla_\alpha h + \frac{1}{12}h\nabla_\nu\nabla_\mu R - \frac{1}{4}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h. \end{aligned} \quad (22)$$

With Covariant Derivative Commutation and Bianchi

71 Terms

$$\begin{aligned} \delta W_{\mu\nu}(h_{\mu\nu}) = & -\frac{1}{6}h_{\mu\nu}R^2 + \frac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}h_{\nu}{}^\alpha RR_{\mu\alpha} - \frac{1}{2}h_{\nu}{}^\alpha R_{\alpha\beta}R_{\mu}{}^\beta \\ & - \frac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \frac{1}{3}h_{\mu}{}^\alpha RR_{\nu\alpha} + h^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \frac{1}{2}h_{\mu}{}^\alpha R_{\alpha\beta}R_{\nu}{}^\beta - g_{\mu\nu}h^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\ & - \frac{2}{3}h^{\alpha\beta}RR_{\mu\alpha\nu\beta} - h_{\nu}{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2h^{\alpha\beta}R_{\alpha}{}^\gamma R_{\mu\gamma\nu\beta} + 2h^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_{\mu}{}^\gamma{}_\nu{}^\eta \\ & - h_{\mu}{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \frac{1}{3}R\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{6}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \frac{1}{2}R_{\nu}{}^\alpha\nabla_\alpha\nabla_\beta h_{\mu}{}^\beta \\ & + \frac{1}{2}R_{\mu}{}^\alpha\nabla_\alpha\nabla_\beta h_{\nu}{}^\beta - \frac{1}{2}R_{\nu}{}^\alpha\nabla_\alpha\nabla_\mu h_{\nu}{}^\beta - \frac{1}{2}R_{\mu}{}^\alpha\nabla_\alpha\nabla_\nu h_{\nu}{}^\beta - \frac{1}{12}g_{\mu\nu}\nabla_\alpha h_{\nu}{}^\beta\nabla^\alpha R \\ & - \frac{1}{6}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R + \frac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_{\alpha}{}^\beta - \nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta} \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} + \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h^\gamma_\gamma - R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu} + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R \\
& - h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla^\beta h^\alpha_\alpha + \frac{2}{3}R_{\mu\nu}\nabla_\beta\nabla^\beta h^\alpha_\alpha + \frac{1}{2}h_\nu^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} \\
& + \frac{1}{2}h_\mu^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha h_\nu^\alpha - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha h_\mu^\alpha \\
& - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma h_\alpha^\gamma + \frac{1}{2}\nabla_\beta R_{\mu\nu}\nabla^\beta h^\alpha_\alpha + \nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu^\alpha \\
& + \frac{1}{2}\nabla^\beta h^\alpha_\alpha\nabla_\gamma R_\mu^\gamma{}_{\nu\beta} + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
& - h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta h^\alpha_\alpha \\
& + \frac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma h^{\alpha\beta} + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha h^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha h^{\alpha\beta} \\
& - \nabla_\beta R_{\nu\alpha}\nabla_\mu h^{\alpha\beta} + \frac{1}{6}\nabla^\alpha R\nabla_\mu h_{\nu\alpha} - \frac{1}{2}\nabla^\beta h^\alpha_\alpha\nabla_\mu R_{\nu\beta} - \frac{1}{3}R\nabla_\mu\nabla_\alpha h_\nu^\alpha \\
& - \frac{1}{2}R_\nu^\alpha\nabla_\mu\nabla_\beta h_\alpha^\beta + R^{\alpha\beta}\nabla_\mu\nabla_\beta h_{\nu\alpha} - \nabla_\beta R_{\mu\alpha}\nabla_\nu h^{\alpha\beta} + \frac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} \\
& + \frac{1}{6}\nabla^\alpha R\nabla_\nu h_{\mu\alpha} + \frac{1}{3}\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \frac{1}{3}R\nabla_\nu\nabla_\alpha h_\mu^\alpha - \frac{1}{2}R_\mu^\alpha\nabla_\nu\nabla_\beta h_\alpha^\beta \\
& + R^{\alpha\beta}\nabla_\nu\nabla_\beta h_{\mu\alpha} - \frac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} + \frac{1}{3}R\nabla_\nu\nabla_\mu h^\alpha_\alpha + \frac{1}{3}h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\
& + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} + \frac{1}{6}\nabla_\nu\nabla_\mu\nabla_\beta\nabla^\beta h^\alpha_\alpha.
\end{aligned} \tag{23}$$

59 Terms

$$\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) = & -\frac{1}{6}K_{\mu\nu}R^2 + \frac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}K_\nu^\alpha RR_{\mu\alpha} - \frac{1}{2}K_\nu^\alpha R_{\alpha\beta}R_\mu^\beta \\
& - \frac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \frac{1}{3}K_\mu^\alpha RR_{\nu\alpha} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \frac{1}{2}K_\mu^\alpha R_{\alpha\beta}R_\nu^\beta - g_{\mu\nu}K^{\alpha\beta}R^\gamma{}_\gamma R_{\alpha\gamma\beta\eta} \\
& - \frac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} - K_\nu^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_\alpha^\gamma R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu^\gamma{}_\nu{}^\eta \\
& - K_\mu^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \frac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \frac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \frac{1}{2}R_\nu^\alpha\nabla_\alpha\nabla_\beta K_\mu^\beta \\
& + \frac{1}{2}R_\mu^\alpha\nabla_\alpha\nabla_\beta K_\nu^\beta - \frac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \frac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} \\
& + \frac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \frac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R \\
& - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \frac{1}{2}K_\nu^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} + \frac{1}{2}K_\mu^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} \\
& - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha K_\nu^\alpha - \frac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha K_\mu^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha^\gamma + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu^\alpha \\
& + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu^\alpha + \frac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} + \frac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
& - K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} + \frac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \frac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} \\
& - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta} + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} \\
& + \frac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} - \frac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu^\alpha - \frac{1}{2}R_\nu^\alpha\nabla_\mu\nabla_\beta K_\alpha^\beta + R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} \\
& - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \frac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \frac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \frac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} \\
& - \frac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu^\alpha - \frac{1}{2}R_\mu^\alpha\nabla_\nu\nabla_\beta K_\alpha^\beta + R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha} - \frac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} \\
& + \frac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + \frac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta}.
\end{aligned} \tag{24}$$

$$\delta W_{\mu\nu}(\frac{h}{4}g_{\mu\nu}^{(0)}) = -\frac{1}{4}hW_{\mu\nu}(g_{\mu\nu}^{(0)}) \tag{25}$$

Weyl Tensor Flat

$$\begin{aligned}
\delta C_{\lambda\mu\nu\kappa} = & \frac{1}{4}\eta_{\mu\nu}\partial_\alpha\partial^\alpha K_{\kappa\lambda} - \frac{1}{4}\eta_{\lambda\nu}\partial_\alpha\partial^\alpha K_{\kappa\mu} - \frac{1}{4}\eta_{\kappa\mu}\partial_\alpha\partial^\alpha K_{\lambda\nu} + \frac{1}{4}\eta_{\kappa\lambda}\partial_\alpha\partial^\alpha K_{\mu\nu} \\
& - \frac{1}{6}\eta_{\kappa\mu}\eta_{\lambda\nu}\partial_\beta\partial_\alpha K^{\alpha\beta} + \frac{1}{6}\eta_{\kappa\lambda}\eta_{\mu\nu}\partial_\beta\partial_\alpha K^{\alpha\beta} - \frac{1}{4}\eta_{\mu\nu}\partial_\kappa\partial_\alpha K_\lambda^\alpha + \frac{1}{4}\eta_{\lambda\nu}\partial_\kappa\partial_\alpha K_\mu^\alpha \\
& + \frac{1}{2}\partial_\kappa\partial_\mu K_{\lambda\nu} - \frac{1}{4}\eta_{\mu\nu}\partial_\lambda\partial_\alpha K_\kappa^\alpha + \frac{1}{4}\eta_{\kappa\mu}\partial_\lambda\partial_\alpha K_\nu^\alpha - \frac{1}{2}\partial_\lambda\partial_\kappa K_{\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\nu K_{\kappa\mu} \\
& + \frac{1}{4}\eta_{\lambda\nu}\partial_\mu\partial_\alpha K_\kappa^\alpha - \frac{1}{4}\eta_{\kappa\lambda}\partial_\mu\partial_\alpha K_\nu^\alpha - \frac{1}{2}\partial_\mu\partial_\nu K_{\kappa\lambda} + \frac{1}{4}\eta_{\kappa\mu}\partial_\nu\partial_\alpha K_\lambda^\alpha - \frac{1}{4}\eta_{\kappa\lambda}\partial_\nu\partial_\alpha K_\mu^\alpha.
\end{aligned} \tag{26}$$