Conformal to Flat Weyl in DeSitter Background

$$\Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$
$$\Omega^{-2} g^{\mu\nu} = \bar{g}^{\mu\nu}$$

Gauge condition:

$$\bar{\nabla}_N \bar{K}^{MN} = 4\Omega^{-1}\bar{K}^{MN}\partial_N\Omega.$$

Dropping the bars, this implies

$$\nabla_N K^N{}_M = 4\Omega^{-1}K^N{}_M\partial_N\Omega = 4\Omega^{-3}\eta^{NR}\partial_N\Omega K_{MR}$$

First order Weyl tensor:

$$\begin{aligned}\delta W_{MN} = &\frac{1}{2}[\nabla_A\nabla^A - 4H^2][\nabla_B\nabla^B - 2H^2]K_{MN} - \frac{1}{2}[\nabla_B\nabla^B - 4H^2][\nabla_M\nabla_L K^L{}_N + \nabla_N\nabla_L K^L{}_M]\\ &+ \frac{1}{6}[g_{MN}\nabla_A\nabla^A + 2\nabla_M\nabla_N - 6H^2 g_{MN}]\nabla_K\nabla_L K^{KL}\end{aligned}$$

Mannheim (56)

$$\begin{aligned}\nabla_B\nabla^B K_{MN} = &\Omega^{-1}\left(-2\partial_N\Omega\nabla_B K^B{}_M - 2\partial_M\Omega\nabla_B K^B{}_N\right)\\ &+ \Omega^{-2}\left(\eta^{BR}\partial_B\partial_R K_{MN}\right)\\ &+ \Omega^{-3}\Bigg(-2\eta^{BR}\partial_B\partial_R\Omega K_{MN} - 2\eta^{BR}\partial_R\Omega\partial_B K_{MN} + 2\eta^{BR}\partial_B\Omega\partial_M K_{NR} + 2\eta^{BR}\partial_B\Omega\partial_N K_{MR}\Bigg)\\ &+ \Omega^{-4}\Bigg(2\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR}\Bigg)\end{aligned}$$

Note the trace of an Ω^{-4} term has been dropped. We see that there are six symmetric terms in $\nabla_B\nabla^B K_{MN}$. Below is (56) with different indicies:

$$\begin{aligned}&\nabla_A\nabla^A K_{MN} =\\ &\qquad\Omega^{-1}\left[-2\partial_N\Omega\nabla_C K^C{}_M - 2\partial_M\Omega\nabla_C K^C{}_N\right]\\ &\qquad+ \Omega^{-2}\left[\eta^{CD}\partial_C\partial_D K_{MN}\right]\\ &\qquad+ \Omega^{-3}\Bigg[-2\eta^{CD}\partial_C\partial_D\Omega K_{MN} - 2\eta^{CD}\partial_D\Omega\partial_C K_{MN}\\ &\qquad\qquad+ 2\eta^{CD}\partial_C\Omega\partial_M K_{ND} + 2\eta^{CD}\partial_C\Omega\partial_N K_{MD}\Bigg]\\ &\qquad+ \Omega^{-4}\Bigg[2\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega K_{CD}\Bigg]\end{aligned}$$

Term 1:

$$K_{MN} \equiv -2(\Omega^{-1}\partial_N\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^B{}_N)$$

$$\begin{aligned}&\nabla_A\nabla^A[-2\Omega^{-1}(\partial_N\Omega\nabla_B K^B{}_M + \partial_M\Omega\nabla_B K^B{}_N)] =\\ &\quad\Omega^{-1}\left[4\partial_N\Omega\nabla_C(\Omega^{-1}\partial^C\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^{BC}) + 4\partial_M\Omega\nabla_C(\Omega^{-1}\partial_N\Omega\nabla_B K^{BC} + \Omega^{-1}\partial^C\Omega\nabla_B K^B{}_N)\right]\\ &\quad+ \Omega^{-2}\left[-2\eta^{CD}\partial_C\partial_D(\Omega^{-1}\partial_N\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^B{}_N)\right]\\ &\quad+ \Omega^{-3}\Bigg[4\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-1}\partial_N\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^B{}_N) + 4\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-1}\partial_N\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^B{}_N)\\ &\qquad- 4\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-1}\partial_N\Omega\nabla_B K^B{}_D + \Omega^{-1}\partial_D\Omega\nabla_B K^B{}_N) - 4\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-1}\partial_D\Omega\nabla_B K^B{}_M + \Omega^{-1}\partial_M\Omega\nabla_B K^B{}_D)\Bigg]\end{aligned}$$

$$+\,\Omega^{-4}\Big[-4\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-1}\partial_D\Omega\nabla_B K^B{}_C+\Omega^{-1}\partial_C\Omega\nabla_B K^B{}_D)\Big]$$

Term 2:

$$K_{MN}\equiv(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{MN})$$

$$\begin{aligned}
&\nabla_A\nabla^A(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{MN})=\\
&\qquad\Omega^{-1}\left[-2\partial_N\Omega\nabla_C(\Omega^{-2}\eta^{BR}\partial_B\partial_R K^C{}_M)-2\partial_M\Omega\nabla_C(\Omega^{-2}\eta^{BR}\partial_B\partial_R K^C{}_N)\right]\\
&\qquad+\Omega^{-2}\left[\eta^{CD}\partial_C\partial_D(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{MN})\right]\\
&\qquad+\Omega^{-3}\Big[-2\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{MN})-2\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{MN})\\
&\qquad\quad+2\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{DN})+2\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{DM})\Big]\\
&\qquad+\Omega^{-4}\Big[2\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-2}\eta^{BR}\partial_B\partial_R K_{CD})\Big]
\end{aligned}$$

Term 3:

$$K_{MN}\equiv-2(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{MN})$$

$$\begin{aligned}
&\nabla_A\nabla^A[-2(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{MN})]=\\
&\qquad\Omega^{-1}\left[4\partial_N\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K^C{}_M)+4\partial_M\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K^C{}_N)\right]\\
&\qquad+\Omega^{-2}\left[-2\eta^{CD}\partial_C\partial_D(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{MN})\right]\\
&\qquad+\Omega^{-3}\Big[4\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{MN})+4\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{MN})\\
&\qquad\quad-4\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{DN})-4\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{DM})\Big]\\
&\qquad+\Omega^{-4}\Big[-4\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-3}\eta^{BR}\partial_B\partial_R\Omega K_{CD})\Big]
\end{aligned}$$

Term 4:

$$K_{MN}\equiv-2(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{MN})$$

$$\begin{aligned}
&\nabla_A\nabla^A[-2(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{MN})]=\\
&\qquad\Omega^{-1}\left[4\partial_N\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K^C{}_M)+4\partial_M\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K^C{}_N)\right]\\
&\qquad+\Omega^{-2}\left[-2\eta^{CD}\partial_C\partial_D(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{MN})\right]\\
&\qquad+\Omega^{-3}\Big[4\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{MN})+4\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{MN})\\
&\qquad\quad-4\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{DN})-4\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{DM})\Big]\\
&\qquad+\Omega^{-4}\Big[-4\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-3}\eta^{BR}\partial_R\Omega\partial_B K_{CD})\Big]
\end{aligned}$$

Term 5:

$$K_{MN}\equiv2(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{MR})$$

$$
\begin{aligned}
&\nabla_A\nabla^A[2(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{MR})]=\\
&\qquad\Omega^{-1}\Big[-4\partial_N\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K^C{}_R+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial^C K_{MR})\\
&\qquad\quad-4\partial_M\Omega\nabla_C(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial^C K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K^C{}_R)\Big]\\
&\qquad+\Omega^{-2}\left[2\eta^{CD}\partial_C\partial_D(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{MR})\right]\\
&\qquad+\Omega^{-3}\Big[-4\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{MR})\\
&\qquad\quad-4\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{MR})\\
&\qquad\quad+4\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_D K_{NR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_N K_{DR})\\
&\qquad\quad+4\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_M K_{DR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_D K_{MR})\Big]\\
&\qquad+\Omega^{-4}\Big[4\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_C K_{DR}+\Omega^{-3}\eta^{BR}\partial_B\Omega\partial_D K_{CR})\Big]
\end{aligned}
$$

Term 6:

$$K_{MN}\equiv 2(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR})$$

$$
\begin{aligned}
&\nabla_A\nabla^A[2(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR})]=\\
&\qquad\Omega^{-1}\left[-4\partial_N\Omega\nabla_C(\Omega^{-4}\eta^{ER}\eta^{BT}\delta^C{}_M\partial_E\Omega\partial_T\Omega K_{BR})-4\partial_M\Omega\nabla_C(\Omega^{-4}\eta^{ER}\eta^{BT}\delta^C{}_N\partial_E\Omega\partial_T\Omega K_{BR})\right]\\
&\qquad+\Omega^{-2}\left[2\eta^{CD}\partial_C\partial_D(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR})\right]\\
&\qquad+\Omega^{-3}\Big[-4\eta^{CD}\partial_C\partial_D\Omega(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR})-4\eta^{CD}\partial_D\Omega\partial_C(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{MN}\partial_E\Omega\partial_T\Omega K_{BR})\\
&\qquad\quad+4\eta^{CD}\partial_C\Omega\partial_M(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{DN}\partial_E\Omega\partial_T\Omega K_{BR})+4\eta^{CD}\partial_C\Omega\partial_N(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{DM}\partial_E\Omega\partial_T\Omega K_{BR})\Big]\\
&\qquad+\Omega^{-4}\Big[4\eta^{DQ}\eta^{CX}\eta_{MN}\partial_Q\Omega\partial_X\Omega(\Omega^{-4}\eta^{ER}\eta^{BT}\eta_{CD}\partial_E\Omega\partial_T\Omega K_{BR})\Big]
\end{aligned}
$$

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}.$$

We will need an expression for the Christoffel symbol:

$$\Gamma^{\lambda}_{\mu\nu} = \Omega^{-1}\left(\delta^{\lambda}_{\nu}\partial_\mu\Omega + \delta^{\lambda}_{\mu}\partial_\nu\Omega - n_{\mu\nu}n^{\lambda\rho}\partial_\rho\Omega\right).$$

Now form the Riemann tensor

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= g_{\lambda\rho}(\partial_\kappa\Gamma^{\rho}_{\mu\nu} - \partial_\nu\Gamma^{\rho}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\rho}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\rho}_{\nu\eta}) \\ &= \Omega\left(\eta_{\lambda\nu}\partial_\mu\partial_\kappa\Omega + \eta_{\kappa\mu}\partial_\nu\partial_\lambda\Omega - \eta_{\mu\nu}\partial_\lambda\partial_\kappa\Omega - \eta_{\kappa\lambda}\partial_\mu\partial_\nu\Omega\right) + \eta_{\mu\kappa}\eta_{\lambda\nu}\partial_\alpha\Omega\partial^\alpha\Omega - \eta_{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial^\alpha\Omega \\ &\quad + 2\eta_{\mu\nu}\partial_\kappa\Omega\partial_\lambda\Omega - 2\eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega - 2\eta_{\kappa\mu}\partial_\lambda\Omega\partial_\nu\Omega + 2\eta_{\kappa\lambda}\partial_\mu\Omega\partial_\nu\Omega \end{aligned}$$

Conformal transformation:

$$\Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$
$$\Omega^{-2} g^{\mu\nu} = \bar{g}^{\mu\nu}$$
$$h^{\mu}{}_{\nu} = g^{(0)}_{\rho\nu}h^{\mu\rho} = (\Omega^{-2}\bar{g}^{(0)}_{\rho\nu})(\Omega^2\bar{h}^{\mu\rho}) = \bar{h}^{\mu}{}_{\nu}$$

The following will be useful within our gauge transformation:

$$\Gamma^{\lambda}_{\mu\nu} = \bar{\Gamma}^{\lambda}_{\mu\nu} - \Omega^{-1}\left(\delta^{\lambda}_{\nu}\partial_\mu\Omega + \delta^{\lambda}_{\mu}\partial_\nu\Omega - n_{\mu\nu}n^{\lambda\rho}\partial_\rho\Omega\right)$$

$$\begin{aligned} \nabla_\mu h^{\mu}{}_{\nu} - \frac{1}{2}\nabla_\nu h^{\mu}{}_{\mu} &= \partial_\mu h^{\mu}{}_{\nu} + \Gamma^{\mu}_{\mu\rho}h^{\rho}{}_{\nu} - \Gamma^{\rho}_{\mu\nu}h^{\mu}{}_{\rho} - \frac{1}{2}\partial_\nu h^{\mu}{}_{\mu} \\ &= \partial_\mu\bar{h}^{\mu}{}_{\nu} + \bar{\Gamma}^{\mu}_{\mu\rho}\bar{h}^{\rho}{}_{\nu} - \bar{\Gamma}^{\rho}_{\mu\nu}h^{\mu}{}_{\rho} - \frac{1}{2}\partial_\nu\bar{h}^{\mu}{}_{\mu} - 4\Omega^{-1}\bar{h}^{\rho}{}_{\nu}\partial_\rho\Omega + \Omega^{-1}\bar{h}^{\mu}{}_{\rho}\left(\delta^{\rho}_{\nu}\partial_\mu\Omega + \delta^{\rho}_{\mu}\partial_\nu\Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha\Omega\right) \\ &= \bar{\nabla}_\mu\bar{h}^{\mu}{}_{\nu} - \frac{1}{2}\bar{\nabla}_\nu\bar{h}^{\mu}{}_{\mu} - 4\Omega^{-1}\bar{h}^{\rho}{}_{\nu}\partial_\rho\Omega + \Omega^{-1}\bar{h}^{\mu}{}_{\rho}\left(\delta^{\rho}_{\nu}\partial_\mu\Omega + \delta^{\rho}_{\mu}\partial_\nu\Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha\Omega\right) \\ &= \bar{\nabla}_\mu\bar{h}^{\mu}{}_{\nu} - \frac{1}{2}\bar{\nabla}_\nu\bar{h}^{\mu}{}_{\mu} - 4\Omega^{-1}\bar{h}^{\rho}{}_{\nu}\partial_\rho\Omega + \Omega^{-1}\bar{h}^{\mu}{}_{\mu}\partial_\nu\Omega \end{aligned}$$

In a conformal to flat space, we need to work in the gauge

$$\bar{\nabla}_\mu\bar{h}^{\mu}{}_{\nu} - \frac{1}{2}\bar{\nabla}_\nu\bar{h}^{\mu}{}_{\mu} = 4\Omega^{-1}\bar{h}^{\rho}{}_{\nu}\partial_\rho\Omega - \Omega^{-1}h^{\mu}{}_{\mu}\partial_\nu\Omega$$

Perturbation of Ricci Tensor:

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda} \equiv S_{\mu\nu}$$
$$\delta R_{\mu\nu} = \delta S_{\mu\nu}$$

Weinberg (10.9.3)

$$\begin{aligned} \delta R_{\mu\nu} &= (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} \\ &= \frac{1}{2}g^{\lambda\rho}\left[(h_{\lambda\rho})_{;\mu;\nu} - (h_{\rho\mu})_{;\nu;\lambda} - (h_{\rho\nu})_{;\mu;\lambda} + (h_{\mu\nu})_{;\rho;\lambda}\right] \\ &= \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\lambda\nabla_\nu h^{\lambda}{}_{\mu} - \nabla_\lambda\nabla_\mu h^{\lambda}{}_{\nu} + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) \end{aligned}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\lambda\nabla_\nu h^{\lambda}{}_{\mu} - \nabla_\lambda\nabla_\mu h^{\lambda}{}_{\nu} + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right)$$
$$\bar{\nabla}_\mu\bar{h}^{\mu}{}_{\nu} - \frac{1}{2}\bar{\nabla}_\nu\bar{h}^{\mu}{}_{\mu} = 4\Omega^{-1}\bar{h}^{\rho}{}_{\nu}\partial_\rho\Omega - \Omega^{-1}h^{\mu}{}_{\mu}\partial_\nu\Omega$$

Referring to Mannheim (35), we may use the covariant interchange identity to express the Ricci variation as

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\nu\nabla_\lambda h^{\lambda}{}_{\mu} - \nabla_\mu\nabla_\lambda h^{\lambda}{}_{\nu} + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^{\sigma}{}_{\rho}R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho}R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu}R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu}R_{\rho\sigma\mu\lambda}\right).$$

Substituting our gauge choice for the middle two covariant derivative terms

$$\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \nabla_\nu\nabla_\lambda h^\lambda{}_\mu - \nabla_\mu\nabla_\lambda h^\lambda{}_\nu + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&= \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \frac{1}{2}\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \frac{1}{2}\nabla_\mu\nabla_\nu h^\lambda{}_\lambda + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&\quad - \nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)\\
&= \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&\quad - \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)
\end{aligned}$$

From here we would like to evaluate the Riemann tensor for a conformal to flat metric. From Weinberg (6.1.5) we have

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\partial_\kappa\partial_\mu g_{\lambda\nu} - \partial_\kappa\partial_\lambda g_{\mu\nu} - \partial_\mu\partial_\nu g_{\lambda\kappa} + \partial_\nu\partial_\lambda g_{\mu\kappa}\right) + g_{\rho\sigma}\left(\Gamma^\rho_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right).$$

$$\begin{aligned}
\Omega^2 g_{\mu\nu} &= \bar{g}_{\mu\nu}\\
\Omega^{-2} g^{\mu\nu} &= \bar{g}^{\mu\nu}
\end{aligned}$$

We will need an expression for the Christoffel symbol:

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \frac{g^{\lambda\rho}}{2}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\\
&= n^{\lambda\rho}\Omega^{-1}\left(\eta_{\rho\nu}\partial_\mu\Omega + \eta_{\rho\mu}\partial_\nu\Omega - \eta_{\mu\nu}\partial_\rho\Omega\right)\\
&= \Omega^{-1}\left(\delta^\lambda_\nu\partial_\mu\Omega + \delta^\lambda_\mu\partial_\nu\Omega - n_{\mu\nu}n^{\lambda\rho}\partial_\rho\Omega\right)
\end{aligned}$$

$$\begin{aligned}
\Gamma^\rho_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\mu\nu} &= \Omega^{-2}\left(\delta^\rho_\nu\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\nu\Omega - n_{\nu\lambda}n^{\rho\sigma}\partial_\sigma\Omega\right)\left(\delta^\sigma_\mu\partial_\kappa\Omega + \delta^\sigma_\kappa\partial_\mu\Omega - n_{\mu\kappa}n^{\sigma\alpha}\partial_\alpha\Omega\right)\\
&\quad - \Omega^{-2}\left(\delta^\rho_\kappa\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\kappa\Omega - n_{\kappa\lambda}n^{\rho\alpha}\partial_\alpha\Omega\right)\left(\delta^\sigma_\mu\partial_\nu\Omega + \delta^\sigma_\nu\partial_\mu\Omega - \eta_{\mu\nu}\eta^{\sigma\beta}\partial_\beta\Omega\right)
\end{aligned}$$

$$\begin{aligned}
\Omega^2\Gamma^\rho_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} &= \left(\delta^\rho_\nu\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\nu\Omega - n_{\nu\lambda}n^{\rho\sigma}\partial_\sigma\Omega\right)\left(\delta^\sigma_\mu\partial_\kappa\Omega + \delta^\sigma_\kappa\partial_\mu\Omega - n_{\mu\kappa}n^{\sigma\alpha}\partial_\alpha\Omega\right)\\
&= \left(\delta^\rho_\nu\delta^\sigma_\mu\partial_\lambda\Omega\partial_\kappa\Omega + \delta^\rho_\nu\delta^\sigma_\kappa\partial_\lambda\Omega\partial_\mu\Omega - \delta^\rho_\nu\eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\lambda\Omega\partial_\alpha\Omega\right)\\
&\quad + \left(\delta^\rho_\lambda\delta^\sigma_\mu\partial_\nu\Omega\partial_\kappa\Omega + \delta^\rho_\lambda\delta^\sigma_\kappa\partial_\nu\Omega\partial_\mu\Omega - \delta^\rho_\lambda\eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\nu\Omega\partial_\alpha\Omega\right)\\
&\quad - \eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\alpha\Omega\left(\delta^\rho_\nu\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\nu\Omega - n_{\nu\lambda}n^{\rho\sigma}\partial_\sigma\Omega\right)
\end{aligned}$$

$$\begin{aligned}
\Omega^2\Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\mu\nu} &= \left(\delta^\rho_\kappa\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\kappa\Omega - n_{\kappa\lambda}n^{\rho\alpha}\partial_\alpha\Omega\right)\left(\delta^\sigma_\mu\partial_\nu\Omega + \delta^\sigma_\nu\partial_\mu\Omega - \eta_{\mu\nu}\eta^{\sigma\beta}\partial_\beta\Omega\right)\\
&= \left(\delta^\rho_\kappa\delta^\sigma_\mu\partial_\lambda\Omega\partial_\nu\Omega + \delta^\rho_\kappa\delta^\sigma_\nu\partial_\lambda\Omega\partial_\mu\Omega - \delta^\rho_\kappa\eta_{\mu\nu}\eta^{\sigma\beta}\partial_\lambda\Omega\partial_\beta\Omega\right)\\
&\quad + \left(\delta^\rho_\lambda\delta^\sigma_\mu\partial_\kappa\Omega\partial_\nu\Omega + \delta^\rho_\lambda\delta^\sigma_\nu\partial_\kappa\Omega\partial_\mu\Omega - \delta^\rho_\kappa\eta_{\mu\nu}\eta^{\sigma\beta}\partial_\lambda\Omega\partial_\beta\Omega\right)\\
&\quad - \eta_{\mu\nu}\eta^{\sigma\beta}\partial_\beta\Omega\left(\delta^\rho_\kappa\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\kappa\Omega - n_{\kappa\lambda}n^{\rho\alpha}\partial_\alpha\Omega\right)
\end{aligned}$$

$$\begin{aligned}
g_{\rho\sigma}\Gamma^\rho_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} &= g_{\rho\sigma}[\left(\delta^\rho_\nu\delta^\sigma_\mu\partial_\lambda\Omega\partial_\kappa\Omega + \delta^\rho_\nu\delta^\sigma_\kappa\partial_\lambda\Omega\partial_\mu\Omega - \delta^\rho_\nu\eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\lambda\Omega\partial_\alpha\Omega\right)\\
&\quad + \left(\delta^\rho_\lambda\delta^\sigma_\mu\partial_\nu\Omega\partial_\kappa\Omega + \delta^\rho_\lambda\delta^\sigma_\kappa\partial_\nu\Omega\partial_\mu\Omega - \delta^\rho_\lambda\eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\nu\Omega\partial_\alpha\Omega\right)\\
&\quad - \eta_{\mu\kappa}\eta^{\sigma\alpha}\partial_\alpha\Omega\left(\delta^\rho_\nu\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\nu\Omega - n_{\nu\lambda}n^{\rho\sigma}\partial_\sigma\Omega\right)]\\
&= \eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega + \eta_{\nu\kappa}\partial_\lambda\Omega\partial_\mu\Omega - \eta_{\mu\kappa}\partial_\lambda\Omega\partial_\nu\Omega\\
&\quad + \eta_{\lambda\mu}\partial_\nu\Omega\partial_\kappa\Omega + \eta_{\lambda\kappa}\partial_\nu\Omega\partial_\mu\Omega - \eta_{\mu\kappa}\partial_\nu\Omega\partial_\lambda\Omega\\
&\quad - 2\eta_{\mu\kappa}\partial_\nu\Omega\partial_\lambda\Omega + 4\eta_{\mu\kappa}\eta_{\nu\lambda}\partial_\sigma\Omega\partial^\sigma\Omega
\end{aligned}$$

$$\begin{aligned}
&= \eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega + \eta_{\nu\kappa}\partial_\lambda\Omega\partial_\mu\Omega + \eta_{\lambda\mu}\partial_\nu\Omega\partial_\kappa\Omega + \eta_{\lambda\kappa}\partial_\nu\Omega\partial_\mu\Omega \\
&\quad - 4\eta_{\mu\kappa}\partial_\nu\Omega\partial_\lambda\Omega + 4\eta_{\mu\kappa}\eta_{\nu\lambda}\partial_\sigma\Omega\partial^\sigma\Omega
\end{aligned}$$

$$\begin{aligned}
g_{\rho\sigma}\Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\mu\nu} &= g_{\rho\sigma}\big[\left(\delta^\rho_\kappa\delta^\sigma_\mu\partial_\lambda\Omega\partial_\nu\Omega + \delta^\rho_\kappa\delta^\sigma_\nu\partial_\lambda\Omega\partial_\mu\Omega - \delta^\rho_\kappa\eta_{\mu\nu}\eta^{\sigma\beta}\partial_\lambda\Omega\partial_\beta\Omega\right) \\
&\quad + \left(\delta^\rho_\lambda\delta^\sigma_\mu\partial_\kappa\Omega\partial_\nu\Omega + \delta^\rho_\lambda\delta^\sigma_\nu\partial_\kappa\Omega\partial_\mu\Omega - \delta^\rho_\kappa\eta_{\mu\nu}\eta^{\sigma\beta}\partial_\lambda\Omega\partial_\beta\Omega\right) \\
&\quad - \eta_{\mu\nu}\eta^{\sigma\beta}\partial_\beta\Omega\left(\delta^\rho_\kappa\partial_\lambda\Omega + \delta^\rho_\lambda\partial_\kappa\Omega - n_{\kappa\lambda}n^{\rho\alpha}\partial_\alpha\Omega\right)\big] \\
&= \eta_{\mu\kappa}\partial_\lambda\Omega\partial_\nu\Omega + \eta_{\kappa\nu}\partial_\lambda\Omega\partial_\mu\Omega - \eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega \\
&\quad + \eta_{\lambda\mu}\partial_\kappa\Omega\partial_\nu\Omega + \eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega - \eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega \\
&\quad - 2\eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega + 4\eta_{\mu\nu}\eta_{\lambda\kappa}\partial_\sigma\Omega\partial^\sigma\Omega \\
&= \eta_{\mu\kappa}\partial_\lambda\Omega\partial_\nu\Omega + \eta_{\kappa\nu}\partial_\lambda\Omega\partial_\mu\Omega + \eta_{\lambda\mu}\partial_\kappa\Omega\partial_\nu\Omega + \eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega \\
&\quad - 4\eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega + 4\eta_{\mu\nu}\eta_{\lambda\kappa}\partial_\sigma\Omega\partial^\sigma\Omega
\end{aligned}$$

$$g_{\rho\sigma}(\Gamma^\rho_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}) = 5\eta_{\mu\nu}\partial_\lambda\Omega\partial_\kappa\Omega - 5\eta_{\mu\kappa}\partial_\nu\Omega\partial_\lambda\Omega + \eta_{\lambda\kappa}\partial_\nu\Omega\partial_\mu\Omega + \eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega + 4\eta_{\mu\kappa}\eta_{\nu\lambda}\partial_\sigma\Omega\partial^\sigma\Omega - 4\eta_{\mu\nu}\eta_{\lambda\kappa}\partial_\sigma\Omega\partial^\sigma\Omega$$

$$\Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$

$$\Omega^{-2} g^{\mu\nu} = \bar{g}^{\mu\nu}$$

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\kappa\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}.$$

We will need an expression for the Christoffel symbol:

$$\Gamma^\lambda_{\mu\nu} = \Omega^{-1}\left(\delta^\lambda_\nu\partial_\mu\Omega + \delta^\lambda_\mu\partial_\nu\Omega - n_{\mu\nu}n^{\lambda\rho}\partial_\rho\Omega\right).$$

Now form the Riemann tensor

$$\begin{aligned}
R_{\lambda\mu\nu\kappa} &= g_{\lambda\rho}(\partial_\kappa\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\rho_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\rho_{\nu\eta}) \\
&= \Omega\left(\eta_{\lambda\nu}\partial_\mu\partial_\kappa\Omega + \eta_{\kappa\mu}\partial_\nu\partial_\lambda\Omega - \eta_{\mu\nu}\partial_\lambda\partial_\kappa\Omega - \eta_{\kappa\lambda}\partial_\mu\partial_\nu\Omega\right) + \eta_{\mu\kappa}\eta_{\lambda\nu}\partial_\alpha\Omega\partial^\alpha\Omega - \eta_{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial^\alpha\Omega \\
&\quad + 2\eta_{\mu\nu}\partial_\kappa\Omega\partial_\lambda\Omega - 2\eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega - 2\eta_{\kappa\mu}\partial_\lambda\Omega\partial_\nu\Omega + 2\eta_{\kappa\lambda}\partial_\mu\Omega\partial_\nu\Omega
\end{aligned}$$

$$\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\
&\quad - \nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)
\end{aligned}$$

$$\begin{aligned}
\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) &= 4\Omega^{-1}\left(\nabla_\nu\bar{h}^\rho{}_\mu\partial_\rho\Omega + \bar{h}^\rho{}_\mu\nabla_\nu\nabla_\rho\Omega - \Omega^{-1}\bar{h}^\rho{}_\mu\partial_\nu\Omega\partial_\rho\Omega\right) \\
&\quad - \Omega^{-1}\left(\partial_\nu\bar{h}^\lambda{}_\lambda\partial_\mu\Omega + \bar{h}^\lambda{}_\lambda\nabla_\nu\nabla_\mu\Omega - \Omega^{-1}\bar{h}^\lambda{}_\lambda\partial_\nu\Omega\partial_\mu\Omega\right) \\
&= 4\Omega^{-3}\eta^{\rho\kappa}\left(\nabla_\nu\bar{h}_{\kappa\mu}\partial_\rho\Omega + \bar{h}_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega - \Omega^{-1}\bar{h}_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega\right) \\
&\quad - \Omega^{-3}\eta^{\lambda\kappa}\left(\partial_\nu\bar{h}_{\kappa\lambda}\partial_\mu\Omega + \bar{h}_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega - \Omega^{-1}\bar{h}_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right) \\
&= \Omega^{-3}\left(4\eta^{\rho\kappa}\nabla_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega\right) \\
&\quad \Omega^{-4}\left(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right)
\end{aligned}$$

$$\nabla_\nu h_{\kappa\mu} = \partial_\nu h_{\kappa\mu} + \Omega^{-1}\left(\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega + \eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega - h_{\mu\nu}\partial_\kappa\Omega - h_{\kappa\nu}\partial_\mu\Omega - 2h_{\kappa\mu}\partial_\nu\Omega\right)$$

$$\nabla_\nu \nabla_\rho \Omega = \partial_\rho \partial_\nu \Omega + \Omega^{-1}(\eta^{\alpha\beta}\eta_{\nu\rho}\partial_\alpha\Omega\partial_\beta\Omega - 2\partial_\nu\Omega\partial_\rho\Omega)$$

$$\begin{aligned}
\nabla_\nu \left(4\Omega^{-1}\bar{h}^\rho{}_\mu \partial_\rho \Omega - \Omega^{-1}\bar{h}^\lambda{}_\lambda \partial_\mu \Omega\right) = & \\
& \Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right) \\
& + \Omega^{-4}(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega \\
& \quad + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\mu\nu}\partial_\kappa\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\
& \quad - 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\nu\rho}h_{\kappa\mu}\partial_\alpha\Omega\partial_\beta\Omega - 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega \\
& \quad - \eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega + 2\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega) \\
& \\
= & \quad \Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right) \\
& + \Omega^{-4}(3\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\
& \quad - 16\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega)
\end{aligned}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda \nabla^\lambda h_{\mu\nu} + \frac{1}{2} g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)$$
$$- \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu \partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda \partial_\mu \Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda \partial_\nu \Omega\right)$$

$$\begin{aligned}\frac{1}{2} g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \\ \Omega^{-3}\Big(\eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\partial_\mu\Omega + 2\eta^{\alpha\beta}h_{\mu\alpha}\partial_\beta\partial_\nu\Omega \\ - \eta^{\alpha\beta}\eta^{\eta\gamma}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\eta\partial_\beta\Omega - \eta^{\alpha\beta}h_{\alpha\beta}\partial_\nu\partial_\mu\Omega\Big) \\ + \Omega^{-4}\Big(- \eta^{\alpha\eta}\eta^{\gamma\beta}\eta_{\mu\nu}h_{\beta\gamma}\partial_\alpha\Omega\partial_\eta\Omega + 2\eta^{\alpha\kappa}h_{\mu\nu}\partial_\alpha\Omega\partial_\kappa\Omega \\ - 4\eta^{\alpha\rho}h_{\nu\alpha}\partial_\rho\Omega\partial_\mu\Omega - 4\eta^{\alpha\eta}h_{\mu\alpha}\partial_\eta\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}h_{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega \\ + 2\eta^{\alpha\lambda}\eta^{\beta\rho}\eta_{\mu\nu}h_{\alpha\beta}\partial_\lambda\Omega\partial_\rho\Omega\Big)\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu \partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda \partial_\mu \Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda \partial_\nu \Omega\right) = \\ \Omega^{-3}\left(2\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 2\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \frac{1}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right) \\ + \Omega^{-4}\Big(\frac{3}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 2\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 2\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\ - 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\Big) \\ + (\mu \leftrightarrow \nu)\end{aligned}$$

$$\frac{1}{2}\Box\left(\Omega^{-2}\bar{h}_{\mu\nu}\right) = \frac{1}{2}\Omega^{-2}\Box\bar{h}_{\mu\nu} - \Omega^{-3}\bar{h}_{\mu\nu}\Box\Omega$$

$$
\begin{aligned}
\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} &+ \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\
&- \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\{& \\
&[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} + [\Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\nu} - \partial_\lambda\Gamma^\kappa_{\rho\nu} - \Gamma^\kappa_{\rho\nu}\partial_\lambda - \Gamma^\kappa_{\lambda\nu}\partial_\rho]h_{\kappa\mu} \\
&+ [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\mu} - \partial_\lambda\Gamma^\kappa_{\rho\mu} - \Gamma^\kappa_{\rho\mu}\partial_\lambda - \Gamma^\kappa_{\lambda\mu}\partial_\rho]h_{\kappa\nu}\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda}) &= \frac{1}{2}g^{\lambda\rho}(h_{\sigma\rho}R^\sigma{}_{\nu\mu\lambda}) \\
&= \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\rho}
\end{aligned}
$$

$$
\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda}) = \frac{1}{2}g^{\lambda\rho}[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}
$$

$$
\begin{aligned}
\frac{1}{2}g^{\lambda\rho}(-h^\sigma{}_\mu R_{\rho\sigma\nu\lambda}) = \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\mu R_{\sigma\rho\nu\lambda}) &= \frac{1}{2}g^{\lambda\rho}(h_{\sigma\mu}R^\sigma{}_{\rho\nu\lambda}) \\
&= \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu}
\end{aligned}
$$

$$
\frac{1}{2}g^{\lambda\rho}(-h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}
$$

$$
\begin{aligned}
\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}\big\{& \\
&[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho} \\
&+ [\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu} \\
&+ [\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}\big\}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right) =& \\
\Omega^{-3}\Bigg(2\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 2\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \frac{1}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\Bigg)& \\
+ \Omega^{-4}\Bigg(\frac{3}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 2\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 2\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega& \\
- 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\Bigg)& \\
+ (\mu \leftrightarrow \nu)&
\end{aligned}
$$

$$
\frac{1}{2}\Box\left(\Omega^{-2}\bar{h}_{\mu\nu}\right) = \frac{1}{2}\Omega^{-2}\Box\bar{h}_{\mu\nu} - \Omega^{-3}\bar{h}_{\mu\nu}\Box\Omega
$$

Gauge:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu}\bar{h}^\mu{}_\rho$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\ &+ \frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \frac{1}{2}\nabla_\nu(\Gamma^\sigma_{\rho\mu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\mu)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\{ \\ &[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} + [\Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\nu} - \partial_\lambda\Gamma^\kappa_{\rho\nu} - \Gamma^\kappa_{\rho\nu}\partial_\lambda - \Gamma^\kappa_{\lambda\nu}\partial_\rho]h_{\kappa\mu} \\ &+ [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\mu} - \partial_\lambda\Gamma^\kappa_{\rho\mu} - \Gamma^\kappa_{\rho\mu}\partial_\lambda - \Gamma^\kappa_{\lambda\mu}\partial_\rho]h_{\kappa\nu}\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}\big\{ \\ &[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho} \\ &+ [\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu} \\ &+ [\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}\big\}\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) \\ &\quad = \frac{1}{2}g^{\lambda\rho}\big\{[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} \\ &\qquad + [2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho} \\ &\qquad + [-\partial_\nu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\nu}\partial_\lambda - \Gamma^\sigma_{\lambda\nu}\partial_\rho]h_{\sigma\mu} \\ &\qquad + [-\partial_\mu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\mu}\partial_\lambda - \Gamma^\sigma_{\lambda\mu}\partial_\rho]h_{\sigma\nu}\big\}\end{aligned}$$

$$\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) = \frac{1}{2}g^{\lambda\rho}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma} - \Gamma^\rho_{\rho\sigma}h_{\lambda\nu})$$

$$\frac{1}{2}g^{\lambda\rho}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma}) = \frac{1}{2}g^{\lambda\rho}(h_{\lambda\sigma}\nabla_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\nabla_\mu h_{\lambda\sigma})$$

$$\begin{aligned}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) &\equiv \nabla_\mu T_\nu = \partial_\mu T_\nu - \Gamma^\lambda_{\mu\nu}T_\lambda \\ &= \partial_\mu\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\rho\lambda}h^\rho{}_\sigma\end{aligned}$$

Gauge:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu}\bar{h}^\mu{}_\rho$$

We can also write this as:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + (\bar{\nabla}_\mu - \partial_\mu)\bar{h}^\mu{}_\nu$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\ &+ \frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \frac{1}{2}\nabla_\nu(\Gamma^\sigma_{\rho\mu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\mu)\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) \\ &\quad = \frac{1}{2}g^{\lambda\rho}\big\{[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} \\ &\qquad + [2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho} \\ &\qquad + [-\partial_\nu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\nu}\partial_\lambda - \Gamma^\sigma_{\lambda\nu}\partial_\rho]h_{\sigma\mu} \\ &\qquad + [-\partial_\mu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\mu}\partial_\lambda - \Gamma^\sigma_{\lambda\mu}\partial_\rho]h_{\sigma\nu}\big\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) = &\frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\lambda\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\lambda\kappa} + \Gamma^\sigma_{\mu\nu}\Gamma^\kappa_{\kappa\lambda})h_{\sigma\rho} - (\partial_\mu\Gamma^\sigma_{\sigma\lambda} + \Gamma^\sigma_{\sigma\lambda}\partial_\mu)h_{\rho\nu}\right] \\ &+ \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\lambda\nu}h_{\rho\sigma} - \Gamma^\sigma_{\sigma\lambda}h_{\rho\nu}\right)\end{aligned}$$

$$\begin{aligned}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) \equiv \nabla_\mu T_\nu &= \partial_\mu T_\nu - \Gamma^\lambda_{\mu\nu}T_\lambda \\ &= \partial_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\rho\lambda}h^\rho{}_\sigma\end{aligned}$$

$$\frac{1}{2}\nabla_\mu(\nabla_\rho - \partial_\rho)h^\rho{}_\nu = \frac{1}{2}g^{\lambda\rho}[\nabla_\mu\nabla_\rho h_{\lambda\nu} - \nabla_\mu(\partial_\rho g^{\lambda\rho}h_{\lambda\nu})]$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\ &- \frac{1}{2}\nabla_\mu(\nabla_\lambda - \partial_\lambda)h^\lambda{}_\nu - \frac{1}{2}\nabla_\nu(\nabla_\lambda - \partial_\lambda)h^\lambda{}_\mu\end{aligned}$$

$$\delta R_{\mu\nu} = \frac{1}{2}(\nabla_\lambda\nabla^\lambda h_{\mu\nu} - \nabla_\lambda\nabla_\mu h^\lambda{}_\nu - \nabla_\lambda\nabla_\nu h^\lambda{}_\mu) - \nabla_\mu\partial_\lambda h^\lambda{}_\nu - \nabla_\nu\partial_\lambda h^\lambda{}_\mu$$

$$\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma = \Omega^{-1}h^\sigma{}_\sigma\nabla_\nu\Omega$$

$$\Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu = 4\Omega^{-1}h^\rho{}_\nu\nabla_\rho\Omega$$

$$\begin{aligned}\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) &= \frac{1}{2}\left[\partial_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa}h^\rho{}_\sigma - \partial_\mu(\Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h^\rho{}_\kappa\right] \\ &= \frac{1}{2}\left[(\partial_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa})h^\rho{}_\sigma - (\partial_\mu\Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\sigma\rho}\partial_\mu)h^\rho{}_\nu + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h^\rho{}_\kappa\right] \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa})h_{\lambda\sigma} - (\partial_\mu\Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\sigma\rho}\partial_\mu)h_{\lambda\nu} + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h_{\lambda\kappa}\right] \\ &\quad + \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma} - \Gamma^\sigma_{\sigma\rho}h_{\lambda\nu}\right) \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\lambda\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\lambda\kappa} + \Gamma^\sigma_{\mu\nu}\Gamma^\kappa_{\kappa\lambda})h_{\sigma\rho} - (\partial_\mu\Gamma^\sigma_{\sigma\lambda} + \Gamma^\sigma_{\sigma\lambda}\partial_\mu)h_{\rho\nu}\right] \\ &\quad + \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\lambda\nu}h_{\rho\sigma} - \Gamma^\sigma_{\sigma\lambda}h_{\rho\nu}\right)\end{aligned}$$