$g^{\mu\nu}\delta W^{(1)}_{\mu\nu}$ v2

1 RW $\Omega(\tau)$

$$
\begin{aligned}
ds^2 &= (g_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\tau)(\tilde{g}_{\mu\nu}+f_{\mu\nu})dx^\mu dx^\nu && (1.1)\\
\tilde{g}_{\mu\nu} &= \mathrm{diag}\left(-1,\frac{1}{1-kr^2},r^2,r^2\sin^2\theta\right) \qquad \tilde{\Gamma}^\lambda_{\alpha\beta}=\delta^\lambda_i\delta^j_\alpha\delta^k_\beta\tilde{\Gamma}^i_{jk} && (1.2)
\end{aligned}
$$

1.1 W_1

As evaluated in the background geometry of $\Omega^2(\tau)\tilde{g}_{\mu\nu}dx^\mu dx^\nu$.

$$
\begin{aligned}
W^{(1)}_{\mu\nu} &= \tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}^2\Omega^{-2}-2\tilde{R}\tilde{R}_{\mu\nu}\Omega^{-2}+2\tilde{g}_{\mu\nu}\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}-12\tilde{R}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-6\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}\\
&+6\tilde{g}_{\mu\nu}\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega-30\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega+96\tilde{g}_{\mu\nu}\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega\\
&-60\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega+12\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-60\tilde{g}_{\mu\nu}\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega\\
&+6\Omega^{-3}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{R}+6\Omega^{-3}\tilde{\nabla}_\mu\tilde{R}\tilde{\nabla}_\nu\Omega-12\tilde{R}\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega-168\Omega^{-5}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega\\
&+48\Omega^{-4}\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\Omega+48\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-2\Omega^{-2}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{R}+12\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\mu\Omega\\
&-12\Omega^{-3}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega+\tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}^2\Omega^{-2}-2\tilde{R}\tilde{R}_{\mu\nu}\Omega^{-2}+2\tilde{g}_{\mu\nu}\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}\\
&-12\tilde{R}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-6\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}+6\tilde{g}_{\mu\nu}\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\\
&-30\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega+96\tilde{g}_{\mu\nu}\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega-60\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega\\
&+12\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-60\tilde{g}_{\mu\nu}\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega+6\Omega^{-3}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{R}+6\Omega^{-3}\tilde{\nabla}_\mu\tilde{R}\tilde{\nabla}_\nu\Omega\\
&-12\tilde{R}\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega-168\Omega^{-5}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega+48\Omega^{-4}\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\Omega\\
&+48\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-2\Omega^{-2}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{R}+12\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\mu\Omega\\
&-12\Omega^{-3}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega+\tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}^2\Omega^{-2}-2\tilde{R}\tilde{R}_{\mu\nu}\Omega^{-2}+2\tilde{g}_{\mu\nu}\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}\\
&-12\tilde{R}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-6\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}+6\tilde{g}_{\mu\nu}\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\\
&-30\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega+96\tilde{g}_{\mu\nu}\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega-60\tilde{g}_{\mu\nu}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega\\
&+12\tilde{g}_{\mu\nu}\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-60\tilde{g}_{\mu\nu}\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega+6\Omega^{-3}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{R}+6\Omega^{-3}\tilde{\nabla}_\mu\tilde{R}\tilde{\nabla}_\nu\Omega\\
&-12\tilde{R}\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega-168\Omega^{-5}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega+48\Omega^{-4}\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\Omega\\
&+48\Omega^{-4}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-2\Omega^{-2}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{R}+12\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\nu\tilde{\nabla}_\mu\Omega-12\Omega^{-3}\tilde{\nabla}_\nu\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega \qquad (1.3)
\end{aligned}
$$

$$
\begin{aligned}
g^{\mu\nu}W^{(1)}_{\mu\nu} &= 6\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}-12\tilde{R}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-12\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}+12\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\\
&-108\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega+216\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega-144\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega\\
&+36\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-144\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega+6\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}-12\tilde{R}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\\
&-12\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}+12\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega-108\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega\\
&+216\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega-144\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega+36\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\\
&-144\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega+6\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\tilde{R}-12\tilde{R}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega-12\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\tilde{R}
\end{aligned}
$$

$$
\begin{aligned}
&+12\tilde{R}\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega - 108\Omega^{-4}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega + 216\Omega^{-5}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\Omega \\
&-144\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\Omega + 36\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega - 144\tilde{R}_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega \qquad (1.4)
\end{aligned}
$$

$$
= \quad 36\Omega^{-7}\Big[6\ddot{\Omega}\dot{\Omega}^2 - 3\ddot{\Omega}^2\Omega - 4\dddot{\Omega}\dot{\Omega}\Omega + 2k\dot{\Omega}^2\Omega + \ddddot{\Omega}\Omega^2 - 2k\ddot{\Omega}\Omega^2\Big] \qquad (1.5)
$$

1.2 $g^{\mu\nu}\delta W^{(1)}_{\mu\nu}$

$$
\begin{aligned}
g^{\mu\nu}\delta W^{(1)}_{\mu\nu} \;=\; & 12\Omega^{-4}\Big[-3\ddddot{\psi} + \psi(3k^2 - 48\ddot{\Omega}\dot{\Omega}^2\Omega^{-3} + 15\ddot{\Omega}^2\Omega^{-2} + 30\dddot{\Omega}\dot{\Omega}\Omega^{-2} - 6k\dot{\Omega}^2\Omega^{-2} - 6\ddddot{\Omega}\Omega^{-1}) \\
&+\ddot{\phi}(12\dot{\Omega}^2\Omega^{-2} - 12\ddot{\Omega}\Omega^{-1}) + \ddot{\psi}(6k + 30\dot{\Omega}^2\Omega^{-2} - 12\ddot{\Omega}\Omega^{-1}) \\
&+\phi(3k^2 - 84\ddot{\Omega}\dot{\Omega}^2\Omega^{-3} + 33\ddot{\Omega}^2\Omega^{-2} + 54\dddot{\Omega}\dot{\Omega}\Omega^{-2} - 18k\dot{\Omega}^2\Omega^{-2} - 12\ddddot{\Omega}\Omega^{-1} + 12k\ddot{\Omega}\Omega^{-1}) \\
&+\dot{\phi}(-18\dot{\Omega}^3\Omega^{-3} + 54\ddot{\Omega}\dot{\Omega}\Omega^{-2} - 18\dddot{\Omega}\Omega^{-1} + 6k\dot{\Omega}\Omega^{-1}) \\
&+\dot{\psi}(-54\dot{\Omega}^3\Omega^{-3} + 90\ddot{\Omega}\dot{\Omega}\Omega^{-2} - 18\dddot{\Omega}\Omega^{-1} + 6k\dot{\Omega}\Omega^{-1}) - 3\dddot{\phi}\dot{\Omega}\Omega^{-1} - 3\dddot{\psi}\dot{\Omega}\Omega^{-1} \\
&+(-18\dot{\Omega}^3\Omega^{-3} + 30\ddot{\Omega}\dot{\Omega}\Omega^{-2} - 6\dddot{\Omega}\Omega^{-1} + 6k\dot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b B + (10\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b\dot{B} \\
&-\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{B} - \tilde{\nabla}_b\tilde{\nabla}^b\dddot{B} + \tilde{\nabla}_b\tilde{\nabla}^b\dddot{E} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{E} + (-10\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b\dot{E} \\
&-\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\phi} + 5\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\psi} + (18\dot{\Omega}^3\Omega^{-3} - 30\ddot{\Omega}\dot{\Omega}\Omega^{-2} + 6\dddot{\Omega}\Omega^{-1} - 6k\dot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b\dot{E} \\
&+5\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\dot{\phi} + 5\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\dot{\psi} \\
&+(-k^2 + 16\ddot{\Omega}\dot{\Omega}^2\Omega^{-3} - 5\ddot{\Omega}^2\Omega^{-2} - 10\dddot{\Omega}\dot{\Omega}\Omega^{-2} + 6k\dot{\Omega}^2\Omega^{-2} + 2\ddddot{\Omega}\Omega^{-1} - 4k\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b E \\
&+(-2\dot{\Omega}^2\Omega^{-2} + 8\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b\phi + (-6k + 4\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_b\tilde{\nabla}^b\psi + 3\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a B \\
&+\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} - \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} - 3\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} + \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\phi - 2\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\psi\Big] \qquad (1.6)
\end{aligned}
$$

1.2.1 $k = 0$

A solution to (1.5) $g^{\mu\nu}W^{(1)}_{\mu\nu} = 0$ for $k = 0$ is $\Omega(\tau) = \tau$. The gauge invariants are then

$$
\alpha \quad = \quad \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \psi - \tau^{-1}(B - \dot{E}). \qquad (1.7)
$$

Evaluating (1.6), we find

$$
\begin{aligned}
g^{\mu\nu}\delta W^{(1)}_{\mu\nu} \;=\; & -216\dot{\alpha}\tau^{-7} - 432\dot{\gamma}\tau^{-7} + 144\ddot{\alpha}\tau^{-6} + 216\ddot{\gamma}\tau^{-6} - 36\dddot{\alpha}\tau^{-5} - 36\dddot{\gamma}\tau^{-4} - 12\tau^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\alpha} \\
&+72\tau^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\gamma} + 60\tau^{-5}\tilde{\nabla}_b\tilde{\nabla}^b\dot{\alpha} - 24\tau^{-6}\tilde{\nabla}_b\tilde{\nabla}^b\alpha + 72\tau^{-6}\tilde{\nabla}_b\tilde{\nabla}^b\gamma + 12\tau^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha \\
&-36\tau^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\gamma. \qquad (1.8)
\end{aligned}
$$

1.2.2 $k = 1$

A solution to (1.5) $g^{\mu\nu}W^{(1)}_{\mu\nu} = 0$ for $k = 1$ is $\Omega(\tau) = \cos(\tau)$. The gauge invariants are then

$$
\alpha \quad = \quad \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \psi + \frac{\sin\tau}{\cos\tau}(B - \dot{E}). \qquad (1.9)
$$

Evaluating (1.6), we find

$$
\begin{aligned}
g^{\mu\nu}\delta W^{(1)}_{\mu\nu} \;=\; & -36(\cos\tau)^{-4}\dddot{\gamma} + 36(\cos\tau)^{-5}\ddot{\alpha}\sin\tau + \ddot{\alpha}\big(144(\cos\tau)^{-4} + 144(\cos\tau)^{-6}\sin^2\tau\big) \\
&+\ddot{\gamma}\big(72(\cos\tau)^{-4} + 216(\cos\tau)^{-6}\sin^2\tau\big) + \dot{\alpha}\big(360(\cos\tau)^{-5}\sin\tau + 216(\cos\tau)^{-7}\sin^3\tau\big) \\
&+\dot{\gamma}\big(432(\cos\tau)^{-5}\sin\tau + 432(\cos\tau)^{-7}\sin^3\tau\big) + \big(144(\cos\tau)^{-4} + 144(\cos\tau)^{-6}\sin^2\tau\big)\alpha
\end{aligned}
$$

$$
\begin{aligned}
&-12(\cos\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\alpha}+72(\cos\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\gamma}-60(\cos\tau)^{-5}\sin\tau\tilde{\nabla}_b\tilde{\nabla}^b\dot{\alpha}\\
&+\big(-96(\cos\tau)^{-4}-24(\cos\tau)^{-6}\sin^2\tau\big)\tilde{\nabla}_b\tilde{\nabla}^b\alpha+\big(72(\cos\tau)^{-4}+72(\cos\tau)^{-6}\sin^2\tau\big)\tilde{\nabla}_b\tilde{\nabla}^b\gamma\\
&+12(\cos\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha-36(\cos\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\gamma
\end{aligned}
\tag{1.10}
$$

1.2.3 $k=-1$

A solution to (1.5) $g^{\mu\nu}W^{(1)}_{\mu\nu}=0$ for $k=-1$ is $\Omega(\tau)=\cosh(\tau)$. The gauge invariants are then

$$
\alpha \;=\; \phi+\psi+\dot{B}-\ddot{E},\qquad \gamma=\psi-\frac{\sinh\tau}{\cosh\tau}(B-\dot{E}).
\tag{1.11}
$$

Evaluating (1.6), we find

$$
\begin{aligned}
g^{\mu\nu}\delta W^{(1)}_{\mu\nu} \;=\;& -36(\cosh\tau)^{-4}\dddot{\gamma}-36(\cosh\tau)^{-5}\ddot{\alpha}\sinh\tau+\ddot{\alpha}\big(-144(\cosh\tau)^{-4}+144(\cosh\tau)^{-6}\sinh^2\tau\big)\\
&+\ddot{\gamma}\big(-72(\cosh\tau)^{-4}+216(\cosh\tau)^{-6}\sinh^2\tau\big)\\
&+\dot{\gamma}\big(432(\cosh\tau)^{-5}\sinh\tau-432(\cosh\tau)^{-7}\sinh^3\tau\big)\\
&+\dot{\alpha}\big(360(\cosh\tau)^{-5}\sinh\tau-216(\cosh\tau)^{-7}\sinh^3\tau\big)\\
&+\big(144(\cosh\tau)^{-4}-144(\cosh\tau)^{-6}\sinh^2\tau\big)\alpha-12(\cosh\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\alpha}+72(\cosh\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\ddot{\gamma}\\
&+60(\cosh\tau)^{-5}\sinh\tau\tilde{\nabla}_b\tilde{\nabla}^b\dot{\alpha}+\big(96(\cosh\tau)^{-4}-24(\cosh\tau)^{-6}\sinh^2\tau\big)\tilde{\nabla}_b\tilde{\nabla}^b\alpha\\
&+\big(-72(\cosh\tau)^{-4}+72(\cosh\tau)^{-6}\sinh^2\tau\big)\tilde{\nabla}_b\tilde{\nabla}^b\gamma+12(\cosh\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\alpha\\
&-36(\cosh\tau)^{-4}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\gamma
\end{aligned}
\tag{1.12}
$$