

TABLE OF CONTENTS

1. Introduction	1
2. Formalism	2
2.1 Einstein Gravity	3
2.1.1 Fluctuations Around Flat in the Harmonic Gauge	7
2.2 Conformal Gravity	7
2.2.1 Conformal Invariance	11
2.2.2 Fluctuations Around Flat in the Transverse Gauge	16
2.2.3 On the Energy Momentum Tensor	18
2.3 Cosmological Geometries	24
3. Scalar, Vector, Tensor (SVT) Decomposition	27
3.1 SVT3	29
3.1.1 SVT3 in Terms of $h_{\mu\nu}$ in a Conformal Flat Background	30
3.1.2 SVT3 in Terms of the Traceless $k_{\mu\nu}$ in a Conformal Flat Background	32
3.1.3 Gauge Structure and Asymptotic Behavior	33
3.2 SVTD	42
3.2.1 Gauge Structure ($D = 4$)	47
3.3 Relating SVT3 to SVT4	49
3.4 Decomposition Theorem and Boundary Conditions	52

Chapter 1

Introduction

Chapter 2

Formalism

Before we can enter the discussion of the technical methods used to decompose and simplify the cosmological fluctuation equations, we must first introduce the necessary formalism describing the interaction of gravitation and matter. The general procedure, repeated for both standard and conformal gravity, consists of varying a classical gravitational action (a general coordinate scalar) with respect to the metric, with stationary solutions yielding the equations of motion. The metric is then decomposed into zeroth and first order contributions where we obtain the background and perturbed fluctuation equations, respectively. Serving as a prototypical example of what is to come, we illustrate the form of the fluctuation equations in their simplest configuration, namely within a source-less Minkowski background geometry. Following convention [1], we impose a standard gauge condition (e.g, the harmonic or transverse gauge), allowing us to solve the equations of motion exactly.

In the case of conformal gravity, there are particular properties not shared within Einstein gravity [2] that deserve special attention which are also explored

for appropriate values of ρ and p . Inspecting (2.58), we see that if $\omega^2 = 0$, $T_S^{\mu\nu} = 0$ and if $\omega^2 = \lambda^2 + k$, we can satisfy $T_S^{\mu\nu} = 0$ non-trivially if and only if k is negative. Thus, we proceed with k negative. In performing an incoherent averaging for T_S^{00} (recalling that we are taking $\omega = 0$ here), we obtain [11]

$$T_S^{00} = \frac{1}{6} \sum_{\ell, m} \left[\sum_{i=1}^3 \gamma^{ii} |\partial_i (g_{(-k)^{1/2}}^\ell Y_\ell^m(\theta, \phi))|^2 + k |g_{(-k)^{1/2}}^\ell Y_\ell^m(\theta, \phi)|^2 \right]. \quad (2.60)$$

It has been shown in [11] that the sum in (2.60) in fact vanishes identically. With scalar field modes providing a positive contribution to $T_S^{\mu\nu}$, the negative contributions of the gravitational field from its negative spatial curvature serve to cancel the scalar modes identically, resulting in a vanishing T_S^{00} . As regards the solutions to (2.57), with negative k these are determined to be associated Legendre functions. Despite $T_S^{\mu\nu}$ vanishing non-trivially, (2.57) still contains an infinite number of solutions, each labelled with a different ℓ and m . Hence, we shown that $T_S^{\mu\nu}$ admits of a non-trivial vacuum solution that can be obtained by taking an incoherent average over the spatial modes associated with the solutions of the scalar field.

While the choice of negative k may warrant concern in the standard treatment of gravitation and cosmology, where the universe geometry is phenomenologically taken as $k = 0$, in conformal gravity it poses no such restriction as evidenced in past work [3, 4, 12, 13, 14, 15]. In applications of conformal gravity to astrophysical and cosmological data it has been found that phenomenologically k should be negative. Specifically, in previous works within conformal cosmology

Chapter 3

Scalar, Vector, Tensor (SVT) Decomposition

In the field of perturbative cosmology, it is standard to first introduce a 3+1 decomposition of the metric perturbation followed by a decomposition into scalars, vectors, and tensors according to the underlying background 3-space. A la the SVT decomposition [18, 23, 24, 26], referred to as SVT3 with this work.

In this chapter, we first develop the SVT3 formalism by separately forming relations between SVT3 components and integrals over $h_{\mu\nu}$, as well as their inversions (i.e. higher derivative relations between $h_{\mu\nu}$ and SVT3 quantities.) In utilizing the SVT3 decomposition in conformal gravity, we also form the analogous relations between the traceless $K_{\mu\nu}$ and the requisite SVT3 components.

We then investigate the behavior of the SVT3 quantities under gauge transformations, and along with reference to the flat space gauge invariant Einstein tensor $\delta G_{\mu\nu}$, we construct the set of gauge invariant quantities. (In fact, such a gauge invariant construction lies behind the core utility of implementing the SVT3 decomposition in the first place). In forming the gauge invariants, we highlight their dependence upon underlying assumptions of being asymptotically

where $\tilde{\nabla}_i = \partial/\partial x^i$ and $\tilde{\nabla}^i = \delta^{ij}\tilde{\nabla}_j$ (with Latin indices) are defined with respect to the background three-space metric δ_{ij} . In addition, the SVT3 components within (3.2) are required to obey

$$\delta^{ij}\tilde{\nabla}_j B_i = 0, \quad \delta^{ij}\tilde{\nabla}_j E_i = 0, \quad E_{ij} = E_{ji}, \quad \delta^{jk}\tilde{\nabla}_k E_{ij} = 0, \quad \delta^{ij}E_{ij} = 0. \quad (3.3)$$

As written, (3.2) contains ten elements, whose transformations are defined with respect to the background spatial sector as four 3-dimensional scalars (ϕ, B, ψ, E), two transverse 3-dimensional vectors (B_i, E_i) each with two independent degrees of freedom, and one symmetric 3-dimensional transverse-traceless tensor (E_{ij}) with two degrees of freedom. A la, the scalar, vector, tensor (SVT) decomposition.

To implement the decomposition of $h_{\mu\nu}$ to the SVT3 form in (3.2), we utilize transverse and transverse-traceless projection operators as applied to tensor and vector components to yield a decomposition into scalars, vectors, and tensors. Both the 3+1 decomposition and projection operators have been derived in developed in detail within Appendix A.

3.1.1 SVT3 in Terms of $h_{\mu\nu}$ in a Conformal Flat Background

Following [19, 20] and making use of the projection operators in Appendix A, we express the ten degrees of freedom of the SVT3 components in a conformal to flat background in terms of the original fluctuations $h_{\mu\nu}$. First we introduce the 3-dimensional Green's function obeying

$$\delta^{ij}\tilde{\nabla}_i\tilde{\nabla}_j D^{(3)}(\mathbf{x} - \mathbf{y}) = \delta^3(\mathbf{x} - \mathbf{y}). \quad (3.4)$$

or $\delta(q)/q$. Thus, referring to (3.13) and (3.14), for spatially localized fluctuations comprising a single mode, the quantities ψ and E_{ij} given in (3.13) and (3.14) evaluate to

$$\begin{aligned}\psi &= e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \frac{[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^2}, \\ E_{ij} &= e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \left[\frac{[q^2 \epsilon_{ij}(q) - q_i q^k \epsilon_{kj}(q) - q_j q^k \epsilon_{ki}(q) + q_i q_j \delta^{k\ell} \epsilon_{k\ell}(q)]}{2q^2} \right. \\ &\quad \left. + \frac{(\delta_{ij} q^2 + q_i q_j)[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^4} \right].\end{aligned}\quad (3.17)$$

With application of $\tilde{\nabla}^j$, one may confirm the transverse relation $\tilde{\nabla}^j E_{ij} = 0$. To construct a wave packet, we sum over all modes viz. $h_{ij} = \sum_q a_q \epsilon_{ij}(q) e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$, to then obtain

$$\begin{aligned}\psi &= \sum_q a_q e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \frac{[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^2}, \\ E_{ij} &= \sum_q a_q e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t} \left[\frac{[q^2 \epsilon_{ij}(q) - q_i q^k \epsilon_{kj}(q) - q_j q^k \epsilon_{ki}(q) + q_i q_j \delta^{k\ell} \epsilon_{k\ell}(q)]}{2q^2} \right. \\ &\quad \left. + \frac{(\delta_{ij} q^2 + q_i q_j)[q^k q^\ell \epsilon_{k\ell}(q) - q^2 \delta^{k\ell} \epsilon_{k\ell}(q)]}{4q^4} \right],\end{aligned}\quad (3.18)$$

where again $\tilde{\nabla}^j E_{ij} = 0$. Since the set of all $e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$ plane waves is complete for fluctuations around flat, any mode can be expanded as a general sum $h_{ij} = \sum_q a_q \epsilon_{ij}(q) e^{i\mathbf{q}\cdot\mathbf{x}-i\omega(q)t}$, with it following that (3.18) then holds for the complete plane wave basis. Hence, by constructing the ψ and E_{ij} in a localized plane-wave basis, we confirm the transverse relation $\tilde{\nabla}^j E_{ij} = 0$ without encountering issues related to integration by parts.

While we have demonstrated the role asymptotic behavior plays within

Chapter 4

Construction and Solution of SVT Fluctuation Equations

4.1 SVT3

4.1.1 dS_4

In the SVT3 background de Sitter case one can write the background and fluctuation metric in the conformal to flat form

$$ds^2 = \frac{1}{H^2\tau^2} \left[(1 + 2\phi)d\tau^2 - 2(\tilde{\nabla}_i B + B_i)d\tau dx^i - [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right], \quad (4.1)$$

where the $\tilde{\nabla}_i$ denote derivatives with respect to the flat 3-space $\delta_{ij}dx^i dx^j$ metric.

In terms of the SVT3 form for the fluctuations the components of the perturbed $\delta G_{\mu\nu}$ are given by (see e.g. [19])

$$\begin{aligned} \delta G_{00} &= -\frac{6}{\tau}\dot{\psi} - \frac{2}{\tau}\tilde{\nabla}^2(\tau\psi + B - \dot{E}), \\ \delta G_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) + \frac{1}{\tau^2}\tilde{\nabla}_i(3B - 2\tau^2\dot{\psi} + 2\tau\phi) + \frac{3}{\tau^2}B_i, \\ \delta G_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[-2\tau^2\ddot{\psi} + 2\tau\dot{\phi} + 4\tau\dot{\psi} - 6\phi - 6\psi \right. \\ &\quad \left. + \tilde{\nabla}^2 \left(2\tau B - \tau^2\dot{B} + \tau^2\ddot{E} - 2\tau\dot{E} - \tau^2\phi + \tau^2\psi \right) \right] \end{aligned}$$

transverse-traceless wave packet

$$\begin{aligned}
E_{ij} &= \sum_{\mathbf{k}} \epsilon_{ij}(\mathbf{k}) \tau^2 [a_1(\mathbf{k}) j_1(k\tau) + b_1(\mathbf{k}) y_1(k\tau)] e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{k}} \epsilon_{ij}(\mathbf{k}) \left[a_1(\mathbf{k}) \left(\frac{\sin(k\tau)}{k^2} - \frac{\tau \cos(k\tau)}{k} \right) \right. \\
&\quad \left. + b_1(\mathbf{k}) \left(\frac{\cos(k\tau)}{k^2} + \frac{\tau \sin(k\tau)}{k} \right) \right], \tag{4.20}
\end{aligned}$$

and can choose the $a_1(\mathbf{k})$ and $b_1(\mathbf{k})$ coefficients to make the packet be as well-behaved at spatial infinity as desired. Finally, since according to (4.1) the full fluctuation is given not by E_{ij} but by $2E_{ij}/H^2\tau^2$, then with $\tau = e^{-Ht}/H$, through the $\cos(k\tau)/k^2$ term we find that at large comoving time E_{ij}/τ^2 behaves as e^{2Ht} , viz. the standard de Sitter fluctuation exponential growth.

4.1.2 Robertson Walker $k = 0$ Radiation Era

In comoving coordinates a spatially flat Robertson-Walker background metric takes the form $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$. In the radiation era where a perfect fluid pressure p and energy density ρ are related by $\rho = 3p$, the background energy-momentum tensor is given by the traceless

$$T_{\mu\nu} = p(4U_\mu U_\nu + g_{\mu\nu}), \tag{4.21}$$

where $g^{\mu\nu}U_\mu U_\nu = -1$, $U^0 = 1$, $U_0 = -1$, $U^i = 0$, $U_i = 0$. With this source the background Einstein equations $G_{\mu\nu} = -T_{\mu\nu}$ with $8\pi G = 1$ fix $a(t)$ to be $a(t) = t^{1/2}$. In conformal to flat coordinates we set $\tau = \int dt/t^{1/2} = 2t^{1/2}$, with the conformal factor being given by $\Omega(\tau) = \tau/2$. In conformal to flat coordinates

The τ dependence of $d_k(\tau)$ is thus given by $j_1(k\tau/\sqrt{3})/\tau$ and $y_1(k\tau/\sqrt{3})/\tau$, while the spatial dependence is given by plane waves. The general solution to (4.46) is thus given by

$$\begin{aligned} \alpha = & \frac{1}{\tau} \sum_{\mathbf{k}} [m_1(\mathbf{k})j_1(k\tau/\sqrt{3}) + n_1(\mathbf{k})y_1(k\tau/\sqrt{3})] e^{i\mathbf{k}\cdot\mathbf{x}} \\ & + \text{delta function terms,} \end{aligned} \quad (4.48)$$

where the delta function terms are solutions to $\tilde{\nabla}^4 \alpha = 0$. Finally, we recall that α and γ are related through $\tilde{\nabla}^4(\tau\alpha + 2\gamma) = 0$, with the coefficients thus obeying

$$m_1(\mathbf{k}) + 2a_1(\mathbf{k}) = 0, \quad n_1(\mathbf{k}) + 2b_1(\mathbf{k}) = 0. \quad (4.49)$$

Having determined α and γ , we can now determine $\delta p - 16\psi/\tau^4$ from the $\Delta_{00} = 0$ equation, and obtain

$$\begin{aligned} \delta p - \frac{16}{\tau^4} \psi &= -\frac{8}{\tau^4} (\alpha - \dot{\gamma}) - \frac{8}{3\tau^3} \tilde{\nabla}^2 \gamma \\ &= \sum_{\mathbf{k}} \left[\frac{8}{\tau^4} \left(\frac{2}{\tau} + \frac{\partial}{\partial \tau} \right) + \frac{8k^2}{3\tau^3} \right] [a_1(\mathbf{k})j_1(k\tau/\sqrt{3}) + b_1(\mathbf{k})y_1(k\tau/\sqrt{3})] e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\quad + \text{delta function terms} \\ &= \sum_{\mathbf{k}} a_1(\mathbf{k}) \left[\frac{8k}{\tau^4 \sqrt{3}} j_0(k\tau/\sqrt{3}) + \frac{8k^2}{3\tau^3} j_1(k\tau/\sqrt{3}) \right] \\ &\quad + \sum_{\mathbf{k}} b_1(\mathbf{k}) \left[\frac{8k}{\tau^4 \sqrt{3}} y_0(k\tau/\sqrt{3}) + \frac{8k^2}{3\tau^3} y_1(k\tau/\sqrt{3}) \right] \\ &\quad + \text{delta function terms.} \end{aligned} \quad (4.50)$$

To determine $B_i - \dot{E}_i$ we apply $\tilde{\nabla}^j$ to $\Delta_{ij} = 0$, to obtain

$$\tilde{\nabla}^2 \left[\frac{1}{\tau} (B_i - \dot{E}_i) + \frac{1}{2} (\dot{B}_i - \ddot{E}_i) \right] = \tilde{\nabla}_i \left[\frac{2}{\tau} (\dot{\alpha} - \ddot{\gamma}) + \frac{2}{3\tau} \tilde{\nabla}^2 \gamma \right], \quad (4.51)$$

$$f(\nu^2 = 0) = \chi, \chi^2. \quad (4.130)$$

For each $f(\chi)$ this would lead to solutions of the form

$$\begin{aligned} \hat{S}_0 &= \frac{1}{\sinh \chi} \frac{df}{d\chi}, \quad \hat{S}_1 = \frac{d\hat{S}_0}{d\chi}, \\ \hat{S}_2 &= \sinh \chi \frac{d}{d\chi} \left[\frac{\hat{S}_1}{\sinh \chi} \right], \quad \hat{S}_3 = \sinh^2 \chi \frac{d}{d\chi} \left[\frac{\hat{S}_2}{\sinh^2 \chi} \right], \dots \end{aligned} \quad (4.131)$$

However, on evaluating these expressions it can happen that some of these solutions vanish. Thus for $A_S = 0$ for instance where $f(\chi) = (\sinh \chi, \cosh \chi)$ the two solutions with $\ell = 0$ are $\cosh \chi / \sinh \chi$ and 1. However this would lead to the two solutions with $\ell = 1$ being $1/\sinh^2 \chi$ and 0. To address this point we note that suppose we have obtained some non-zero solution \hat{S}_ℓ . Then, a second solution of the form $\hat{f}_\ell(\chi)\hat{S}_\ell(\chi)$ may be found by inserting $\hat{f}_\ell(\chi)\hat{S}_\ell(\chi)$ into (4.126), to yield

$$\hat{S}_\ell \frac{d^2 \hat{f}_\ell}{d\chi^2} + 2\hat{S}_\ell \frac{\cosh \chi}{\sinh \chi} \frac{d\hat{f}_\ell}{d\chi} + 2 \frac{d\hat{S}_\ell}{d\chi} \frac{d\hat{f}_\ell}{d\chi} = 0, \quad (4.132)$$

which integrates to

$$\frac{d\hat{f}_\ell}{d\chi} = \frac{1}{\sinh^2 \chi \hat{S}_\ell^2}, \quad \hat{f}_\ell \hat{S}_\ell = \hat{S}_\ell \int \frac{d\chi}{\sinh^2 \chi \hat{S}_\ell^2}. \quad (4.133)$$

Thus for $\ell = 1$, from the non-trivial $A_S = 0$ solution $\hat{S}_1 = 1/\sinh^2 \chi$ we obtain a second solution of the form $\hat{f}_1 \hat{S}_1 = \cosh \chi / \sinh \chi - \chi / \sinh^2 \chi$. However, once we have this second solution we can then return to (4.131) and use it to obtain the subsequent solutions associated with higher ℓ values, since use of the chain in (4.131) only requires that at any point the elements in it are solutions regardless of how they may or may not have been found.

Having the form given in (4.133) is useful for another purpose, as it allows us to relate the behaviors of the solutions in the $\chi \rightarrow \infty$ and $\chi \rightarrow 0$ limits. Thus suppose that \hat{S}_ℓ behaves as $e^{\lambda\chi}$ and as χ^ℓ in these two limits. Then $\hat{f}_\ell \hat{S}_\ell$ must behave as $e^{-(\lambda+2)\chi}$ and $\chi^{-\ell-1}$ in the two limits. Alternatively, if \hat{S}_ℓ behaves as $e^{\lambda\chi}$ and as $\chi^{-\ell-1}$ in these two limits, then $\hat{f}_\ell \hat{S}_\ell$ must behave as $e^{-(\lambda+2)\chi}$ and χ^ℓ in the two limits. Comparing with (4.127), we note that if we set $\lambda = -1 \pm (1 - A_S)^{1/2}$ then consistently we find that $-(\lambda+2) = -1 \mp (1 - A_S)^{1/2}$. However, this analysis shows that we cannot directly identify which $\chi \rightarrow \infty$ behavior is associated with which $\chi \rightarrow 0$ behavior (the insertion of either χ^ℓ or $\chi^{-\ell-1}$ into (4.133) generates the other, with both behaviors thus being required in any $\hat{S}_\ell, \hat{f}_\ell \hat{S}_\ell$ pair), and to determine which is which we thus need to construct the asymptotic solutions directly.

For $A_S = 0$, $\nu = i$, the relevant $f(\nu^2)$ given in (4.130) are $\cosh \chi$ and $\sinh \chi$. Consequently, we find the first few $S_\ell^{(i)}$, $i = 1, 2$ solutions to $\tilde{\nabla}_a \tilde{\nabla}^a S = 0$ to be of the form

$$\begin{aligned}
\hat{S}_0^{(1)}(A_S = 0) &= \frac{\cosh \chi}{\sinh \chi}, & \hat{S}_0^{(2)}(A_S = 0) &= 1, \\
\hat{S}_1^{(1)}(A_S = 0) &= \frac{1}{\sinh^2 \chi}, & \hat{S}_1^{(2)}(A_S = 0) &= \frac{\cosh \chi}{\sinh \chi} - \frac{\chi}{\sinh^2 \chi}, \\
\hat{S}_2^{(1)}(A_S = 0) &= \frac{\cosh \chi}{\sinh^3 \chi}, & \hat{S}_2^{(2)}(A_S = 0) &= 1 + \frac{3}{\sinh^2 \chi} - \frac{3\chi \cosh \chi}{\sinh^3 \chi}, \\
\hat{S}_3^{(1)}(A_S = 0) &= \frac{4}{\sinh^2 \chi} + \frac{5}{\sinh^4 \chi}, \\
\hat{S}_3^{(2)}(A_S = 0) &= \frac{2 \cosh \chi}{\sinh \chi} + \frac{15 \cosh \chi}{\sinh^3 \chi} - \frac{12\chi}{\sinh^2 \chi} - \frac{15\chi}{\sinh^4 \chi}.
\end{aligned} \tag{4.134}$$

$$= 0. \quad (4.143)$$

On implementing this condition, the $(\chi, \theta, \phi) \equiv (1, 2, 3)$ components of $\tilde{\nabla}_a \tilde{\nabla}^a V^i$ take the form

$$\begin{aligned} \tilde{\nabla}_a \tilde{\nabla}^a V^1 &= V_1 \left(2 + \frac{2}{\sinh^2 \chi} \right) + \frac{4 \cosh \chi \partial_1 V_1}{\sinh \chi} + \partial_1 \partial_1 V_1 + \frac{\cos \theta \partial_2 V_1}{\sin \theta \sinh^2 \chi} + \frac{\partial_2 \partial_2 V_1}{\sinh^2 \chi} \\ &\quad + \frac{\partial_3 \partial_3 V_1}{\sin^2 \theta \sinh^2 \chi}, \\ \tilde{\nabla}_a \tilde{\nabla}^a V^2 &= V_2 \left(-\frac{2}{\sinh^4 \chi} + \frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{4 V_1 \cos \theta \cosh \chi}{\sin \theta \sinh^3 \chi} \\ &\quad + \frac{2 \cos \theta \partial_1 V_1}{\sin \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 V_2}{\sinh^2 \chi} + \frac{2 \cosh \chi \partial_2 V_1}{\sinh^3 \chi} + \frac{3 \cos \theta \partial_2 V_2}{\sin \theta \sinh^4 \chi} \\ &\quad + \frac{\partial_2 \partial_2 V_2}{\sinh^4 \chi} + \frac{\partial_3 \partial_3 V_2}{\sin^2 \theta \sinh^4 \chi}, \\ \tilde{\nabla}_a \tilde{\nabla}^a V^3 &= -\frac{2 V_3}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 V_3}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 V_3}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 V_3}{\sin^2 \theta \sinh^4 \chi} \\ &\quad + \frac{2 \cosh \chi \partial_3 V_1}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 V_2}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 V_3}{\sin^4 \theta \sinh^4 \chi}. \end{aligned} \quad (4.144)$$

To explore the structure of the $k = -1$ vector sector we seek solutions to

$$(\tilde{\nabla}_a \tilde{\nabla}^a + A_V) V_i = 0. \quad (4.145)$$

(Here V_i is to denote the full combinations of vector components that appear in (4.93) and (4.101).) In (4.145) we have introduced a generic vector sector constant A_V , whose values in (4.93) and (4.101) are $(2, -1, -2)$.

Conveniently, we find that the equation for V_1 involves no mixing with V_2 or V_3 , and can thus be solved directly. On setting $V_1(\chi, \theta, \phi) = g_{1,\ell}(\chi) Y_\ell^m(\theta, \phi)$, the equation for V_1 reduces to

$$\left[\frac{d^2}{d\chi^2} + 4 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + 2 + A_V + \frac{2}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} \right] g_{1,\ell} = 0. \quad (4.146)$$

$i = 1, 2$ solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)V_1 = 0$ of the form

$$\begin{aligned}
\hat{V}_0^{(1)}(A_V = 2) &= \frac{1}{\sinh^2 \chi}, & \hat{V}_0^{(2)}(A_V = 2) &= \frac{\chi}{\sinh^2 \chi}, \\
\hat{V}_1^{(1)}(A_V = 2) &= \frac{\cosh \chi}{\sinh^3 \chi}, & \hat{V}_1^{(2)}(A_V = 2) &= \frac{1}{\sinh^2 \chi} - \frac{\chi \cosh \chi}{\sinh^3 \chi}, \\
\hat{V}_2^{(1)}(A_V = 2) &= \frac{2}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi}, \\
\hat{V}_2^{(2)}(A_V = 2) &= \frac{3 \cosh \chi}{\sinh^3 \chi} - \frac{2\chi}{\sinh^2 \chi} - \frac{3\chi}{\sinh^4 \chi}, \\
\hat{V}_3^{(1)}(A_V = 2) &= \frac{2 \cosh \chi}{\sinh^3 \chi} + \frac{5 \cosh \chi}{\sinh^5 \chi}, \\
\hat{V}_3^{(2)}(A_V = 2) &= \frac{11}{\sinh^2 \chi} + \frac{15}{\sinh^4 \chi} - \frac{6\chi \cosh \chi}{\sinh^3 \chi} - \frac{15\chi \cosh \chi}{\sinh^5 \chi}. \quad (4.149)
\end{aligned}$$

The just as required by (4.147), the $\hat{V}_\ell^{(2)}(A_V = 2)$ solutions with $\ell \geq 1$ are bounded at $\chi = \infty$ and well-behaved at $\chi = 0$. Since they can thus not be excluded by boundary conditions at $\chi = \infty$ and $\chi = 0$ (though boundary conditions do exclude modes with $\ell = 0$), solutions to (4.93) and (4.101) do not become the vector sector solutions associated with (4.111). Thus if we implement (4.101) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)V_i = 0$, the decomposition theorem will fail in the vector sector for modes with $\ell \geq 1$. Thus an equation such as (4.101) will be solved by

$$(\tilde{\nabla}_a \tilde{\nabla}^a - 1)(\tilde{\nabla}_b \tilde{\nabla}^b - 2) \left[\frac{1}{2}(\dot{B}_i - \ddot{E}_i) + \dot{\Omega} \Omega^{-1}(B_i - \dot{E}_i) \right] = V_i, \quad (4.150)$$

and not by

$$\frac{1}{2}(\dot{B}_i - \ddot{E}_i) + \dot{\Omega} \Omega^{-1}(B_i - \dot{E}_i) = 0. \quad (4.151)$$

Thus (4.101) is solved by the χ dependence of $B_i - \dot{E}_i$ and not by its τ dependence, i.e., not by the $B_i - \dot{E}_i = 1/\Omega^2$ dependence on τ that one would have obtained

from the decomposition-theorem-required (4.151). This then raises the question of what does fix the τ dependence in the vector sector. We will address this issue below.

For $A_V = -2$ we see that $\nu^2 = -4$ and that $f(\nu^2) = \cosh 2\chi, \sinh 2\chi$. However in the scalar case discussed above where $\nu^2 = A_S - 1$, ν^2 would also obey $\nu^2 = -4$ if $A_S = -3$. Thus for $A_V = -2$ we can obtain the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a - 2)V_1 = 0$ directly from (4.135), and after implementing $g_{1,\ell} = \alpha_\ell / \sinh \chi$ we obtain

$$\begin{aligned}
\hat{V}_0^{(1)}(A_V = -2) &= \frac{\cosh \chi}{\sinh \chi}, & \hat{V}_0^{(2)}(A_V = -2) &= 2 + \frac{1}{\sinh^2 \chi}, \\
\hat{V}_1^{(1)}(A_V = -2) &= 1, & \hat{V}_1^{(2)}(A_V = -2) &= 2 \frac{\cosh \chi}{\sinh \chi} - \frac{\cosh \chi}{\sinh^3 \chi}, \\
\hat{V}_2^{(1)}(A_V = -2) &= 2 \frac{\cosh \chi}{\sinh \chi} - \frac{3 \cosh \chi}{\sinh^3 \chi} + \frac{3\chi}{\sinh^4 \chi}, & \hat{V}_2^{(2)}(A_V = -2) &= \frac{1}{\sinh^4 \chi}, \\
\hat{V}_3^{(1)}(A_V = -2) &= 2 - \frac{5}{\sinh^2 \chi} - \frac{15}{\sinh^4 \chi} + \frac{15\chi \cosh \chi}{\sinh^5 \chi}, \\
\hat{V}_3^{(2)}(A_V = -2) &= \frac{\cosh \chi}{\sinh^5 \chi}.
\end{aligned} \tag{4.152}$$

As required by (4.147), the $\hat{V}_2^{(2)}(A_V = -2)$ and $\hat{V}_3^{(2)}(A_V = -2)$ solutions are bounded at $\chi = \infty$. However, they are not well-behaved at $\chi = 0$. Since they thus can be excluded by boundary conditions at $\chi = \infty$ and $\chi = 0$, if we implement (4.101) by $(\tilde{\nabla}_a \tilde{\nabla}^a - 2)V_i = 0$, the only allowed solution will be $V_i = 0$, and the decomposition theorem will then follow.

Finally, for $A_V = -1$, viz. $\nu = i\sqrt{3}$, $f(\nu^2) = e^{\chi\sqrt{3}}, e^{-\chi\sqrt{3}}$, the solutions to

$(\tilde{\nabla}_a \tilde{\nabla}^a - 1)V_1 = 0$ are of the form

$$\begin{aligned}
\hat{V}_0^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^2 \chi}, & \hat{V}_0^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^2 \chi}, \\
\hat{V}_1^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^3 \chi} [\sqrt{3} \sinh \chi - \cosh \chi], \\
\hat{V}_1^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^3 \chi} [-\sqrt{3} \sinh \chi - \cosh \chi], \\
\hat{V}_2^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^4 \chi} [3 - 3\sqrt{3} \cosh \chi \sinh \chi + 5 \sinh^2 \chi], \\
\hat{V}_2^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^4 \chi} [3 + 3\sqrt{3} \cosh \chi \sinh \chi + 5 \sinh^2 \chi], \\
\hat{V}_3^{(1)}(A_V = -1) &= \frac{e^{\chi\sqrt{3}}}{\sinh^5 \chi} \left[15\sqrt{3} \sinh \chi + 14\sqrt{3} \sinh^3 \chi - 15 \cosh \chi \right. \\
&\quad \left. - 24 \cosh \chi \sinh^2 \chi \right], \\
\hat{V}_3^{(2)}(A_V = -1) &= \frac{e^{-\chi\sqrt{3}}}{\sinh^5 \chi} \left[-15\sqrt{3} \sinh \chi - 14\sqrt{3} \sinh^3 \chi - 15 \cosh \chi \right. \\
&\quad \left. - 24 \cosh \chi \sinh^2 \chi \right].
\end{aligned} \tag{4.153}$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{V}_\ell^{(1)}(A_V = -1) - \hat{V}_\ell^{(2)}(A_V = -1)$ with $\ell \geq 1$ are well-behaved at $\chi = 0$. Thus if implement (4.101) by $(\tilde{\nabla}_a \tilde{\nabla}^a - 1)V_i = 0$, we are not forced to $V_i = 0$, with the decomposition theorem not then following in this sector.

The Tensor Sector

For $k = -1$ the transverse-traceless tensor sector modes need to satisfy

$$\begin{aligned}
\tilde{\gamma}^{ab} T_{ab} &= T_{11} + \frac{T_{22}}{\sinh^2 \chi} + \frac{T_{33}}{\sin^2 \theta \sinh^2 \chi} = 0, \\
\tilde{\nabla}_a T^{a1} &= -\frac{\cosh \chi T_{22}}{\sinh^3 \chi} - \frac{\cosh \chi T_{33}}{\sin^2 \theta \sinh^3 \chi} + \frac{\cos \theta T_{12}}{\sin \theta \sinh^2 \chi} + \frac{2 \cosh \chi T_{11}}{\sinh \chi} + \partial_1 T_{11}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial_2 T_{12}}{\sinh^2 \chi} + \frac{\partial_3 T_{13}}{\sin^2 \theta \sinh^2 \chi} = 0, \\
\tilde{\nabla}_a T^{a2} &= -\frac{\cos \theta T_{33}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta T_{22}}{\sin \theta \sinh^4 \chi} + \frac{2 \cosh \chi T_{12}}{\sinh^3 \chi} + \frac{\partial_1 T_{12}}{\sinh^2 \chi} + \frac{\partial_2 T_{22}}{\sinh^4 \chi} \\
& + \frac{\partial_3 T_{23}}{\sin^2 \theta \sinh^4 \chi} = 0, \\
\tilde{\nabla}_a T^{a3} &= \frac{\cos \theta T_{23}}{\sin^3 \theta \sinh^4 \chi} + \frac{2 \cosh \chi T_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 T_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_2 T_{23}}{\sin^2 \theta \sinh^4 \chi} \\
& + \frac{\partial_3 T_{33}}{\sin^4 \theta \sinh^4 \chi} = 0.
\end{aligned} \tag{4.154}$$

Under these conditions the components of $\tilde{\nabla}_a \tilde{\nabla}^a T^{ij}$ evaluate to

$$\begin{aligned}
\tilde{\nabla}_a \tilde{\nabla}^a T^{11} &= T_{11} \left(6 + \frac{6}{\sinh^2 \chi} \right) + \frac{6 \cosh \chi \partial_1 T_{11}}{\sinh \chi} + \partial_1 \partial_1 T_{11} + \frac{\cos \theta \partial_2 T_{11}}{\sin \theta \sinh^2 \chi} \\
& + \frac{\partial_2 \partial_2 T_{11}}{\sinh^2 \chi} + \frac{\partial_3 \partial_3 T_{11}}{\sin^2 \theta \sinh^2 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{22} &= \frac{4T_{22}}{\sinh^6 \chi} - \frac{4T_{22}}{\sin^2 \theta \sinh^6 \chi} + \frac{4T_{11}}{\sinh^4 \chi} - \frac{2T_{22}}{\sinh^4 \chi} - \frac{2T_{11}}{\sin^2 \theta \sinh^4 \chi} \\
& + \frac{2T_{11}}{\sinh^2 \chi} - \frac{2 \cosh \chi \partial_1 T_{22}}{\sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{22}}{\sinh^4 \chi} + \frac{4 \cosh \chi \partial_2 T_{12}}{\sinh^5 \chi} \\
& + \frac{\cos \theta \partial_2 T_{22}}{\sin \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 T_{22}}{\sinh^6 \chi} - \frac{4 \cos \theta \partial_3 T_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 T_{22}}{\sin^2 \theta \sinh^6 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{33} &= \frac{2T_{33}}{\sin^4 \theta \sinh^6 \chi} (1 - \sinh^2 \chi) + T_{11} \left(\frac{2}{\sin^4 \theta \sinh^4 \chi} + \frac{2}{\sin^2 \theta \sinh^2 \chi} \right) \\
& - \frac{4 \cos \theta \cosh \chi T_{12}}{\sin^3 \theta \sinh^5 \chi} - \frac{4 \cos \theta \partial_1 T_{12}}{\sin^3 \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_1 T_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{33}}{\sin^4 \theta \sinh^4 \chi} \\
& + \frac{4 \cos \theta \partial_2 T_{11}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta \partial_2 T_{33}}{\sin^5 \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 T_{33}}{\sin^4 \theta \sinh^6 \chi} + \frac{4 \cosh \chi \partial_3 T_{13}}{\sin^4 \theta \sinh^5 \chi} \\
& + \frac{\partial_3 \partial_3 T_{33}}{\sin^6 \theta \sinh^6 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{12} &= T_{12} \left(-\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{2 \cosh \chi \partial_1 T_{12}}{\sinh^3 \chi} + \frac{\partial_1 \partial_1 T_{12}}{\sinh^2 \chi} \\
& + \frac{2 \cosh \chi \partial_2 T_{11}}{\sinh^3 \chi} + \frac{\cos \theta \partial_2 T_{12}}{\sin \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 T_{12}}{\sinh^4 \chi} - \frac{2 \cos \theta \partial_3 T_{13}}{\sin^3 \theta \sinh^4 \chi} \\
& + \frac{\partial_3 \partial_3 T_{12}}{\sin^2 \theta \sinh^4 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{13} &= -\frac{2T_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{2 \cosh \chi \partial_1 T_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 \partial_1 T_{13}}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 T_{13}}{\sin^3 \theta \sinh^4 \chi}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial_2 \partial_2 T_{13}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_3 T_{11}}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 T_{12}}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 T_{13}}{\sin^4 \theta \sinh^4 \chi}, \\
\tilde{\nabla}_a \tilde{\nabla}^a T^{23} = & T_{23} \left(\frac{2(1 - \sinh^2 \chi)}{\sin^2 \theta \sinh^6 \chi} - \frac{1}{\sin^4 \theta \sinh^6 \chi} \right) + \frac{2 \cos \theta \partial_1 T_{13}}{\sin^3 \theta \sinh^4 \chi} \\
& - \frac{2 \cosh \chi \partial_1 T_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 T_{23}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_2 T_{13}}{\sin^2 \theta \sinh^5 \chi} + \frac{\cos \theta \partial_2 T_{23}}{\sin^3 \theta \sinh^6 \chi} \\
& + \frac{\partial_2 \partial_2 T_{23}}{\sin^2 \theta \sinh^6 \chi} + \frac{2 \cosh \chi \partial_3 T_{12}}{\sin^2 \theta \sinh^5 \chi} + \frac{2 \cos \theta \partial_3 T_{22}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 T_{23}}{\sin^4 \theta \sinh^6 \chi}.
\end{aligned} \tag{4.155}$$

Following our analysis of the vector sector, in the $k = -1$ tensor sector we seek solutions to

$$(\tilde{\nabla}_a \tilde{\nabla}^a + A_T) T_{ij} = 0. \tag{4.156}$$

(Here T_{ij} is to denote the full combination of tensor components that appears in (4.110).) In (4.156) we have introduced a generic tensor sector constant A_T , whose values in (4.110) are $(2, 3, 6)$. Conveniently, we find that the equation for T_{11} involves no mixing with any other components of T_{ij} , and can thus be solved directly. On setting $T_{11}(\chi, \theta, \phi) = h_{11,\ell}(\chi) Y_\ell^m(\theta, \phi)$, the equation for T_{11} reduces to

$$\left[\frac{d^2}{d\chi^2} + 6 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + 6 + \frac{6}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} + A_T \right] h_{11,\ell} = 0. \tag{4.157}$$

To determine the $\chi \rightarrow \infty$ and $\chi \rightarrow 0$ limits, we take the solutions to behave as $e^{\lambda\chi}$ (times an irrelevant polynomial in χ) and χ^n in these two limits. For (4.157) the limits give

$$\lambda^2 + 6\lambda + 6 + A_T, \quad \lambda = -3 \pm (3 - A_T)^{1/2},$$

$$\begin{aligned}
\hat{T}_2^{(2)}(A_T = 2) &= \frac{1}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi} - \frac{3\chi \cosh \chi}{\sinh^5 \chi}, \\
\hat{T}_3^{(1)}(A_T = 2) &= \frac{4}{\sinh^4 \chi} + \frac{5}{\sinh^6 \chi}, \\
\hat{T}_3^{(2)}(A_T = 2) &= \frac{2 \cosh \chi}{\sinh^3 \chi} + \frac{15 \cosh \chi}{\sinh^5 \chi} - \frac{12\chi}{\sinh^4 \chi} - \frac{15\chi}{\sinh^6 \chi}.
\end{aligned} \tag{4.160}$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{T}_\ell^{(2)}(A_T = 2)$ with $\ell \geq 2$ are well-behaved at $\chi = 0$. Thus if implement (4.110) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 2)T_{ij} = 0$, we are not forced to $T_{ij} = 0$, with the decomposition theorem not then following in the tensor sector.

For $A_T = 3$ we see that $\nu^2 = 0$. However in the vector case discussed above where $\nu^2 = A_V - 2$, ν^2 would also obey $\nu^2 = 0$ if $A_V = 2$. Thus for $A_T = 3$ we can obtain the solutions to $(\tilde{\nabla}_a \tilde{\nabla}^a + 3)T_{11} = 0$ directly from (4.149), and after implementing $h_\ell^{11} = \alpha_\ell / \sinh \chi$ we obtain

$$\begin{aligned}
\hat{T}_0^{(1)}(A_T = 3) &= \frac{1}{\sinh^3 \chi}, & \hat{T}_0^{(2)}(A_T = 3) &= \frac{\chi}{\sinh^3 \chi}, \\
\hat{T}_1^{(1)}(A_T = 3) &= \frac{\cosh \chi}{\sinh^4 \chi}, & \hat{T}_1^{(2)}(A_T = 3) &= \frac{1}{\sinh^3 \chi} - \frac{\chi \cosh \chi}{\sinh^4 \chi}, \\
\hat{T}_2^{(1)}(A_T = 3) &= \frac{2}{\sinh^3 \chi} + \frac{3}{\sinh^5 \chi}, \\
\hat{T}_2^{(2)}(A_T = 3) &= \frac{3 \cosh \chi}{\sinh^4 \chi} - \frac{2\chi}{\sinh^3 \chi} - \frac{3\chi}{\sinh^5 \chi}, \\
\hat{T}_3^{(1)}(A_T = 3) &= \frac{2 \cosh \chi}{\sinh^4 \chi} + \frac{5 \cosh \chi}{\sinh^6 \chi}, \\
\hat{T}_3^{(2)}(A_T = 3) &= \frac{11}{\sinh^3 \chi} + \frac{15}{\sinh^5 \chi} - \frac{6\chi \cosh \chi}{\sinh^4 \chi} - \frac{15\chi \cosh \chi}{\sinh^6 \chi}.
\end{aligned} \tag{4.161}$$

All of these solutions are bounded at $\chi = \infty$ and all $\hat{T}_\ell^{(2)}(A_T = 3)$ with $\ell \geq 2$ are well-behaved at $\chi = 0$. Thus if implement (4.110) by $(\tilde{\nabla}_a \tilde{\nabla}^a + 3)T_{ij} = 0$, we are

of vectors with tensors.

To see how to obtain such a needed common χ behavior we differentiate the scalar field (4.126) with respect to χ , to obtain

$$\begin{aligned} & \left[\frac{d^2}{d\chi^2} + 4 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + \frac{2}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} + 4 + A_S \right] \frac{dS_\ell}{d\chi} + 2A_S \frac{\cosh \chi}{\sinh \chi} S_\ell \\ & = 0. \end{aligned} \quad (4.162)$$

Comparing with the vector (4.146) we see that up to an overall normalization we can identify $dS_\ell/d\chi$ with the vector $g_{1,\ell}$ for modes that obey $A_S = 0$ and $A_V = 2$, so that these particular scalar and vector modes can interface. As a check, with the vector sector needing $\ell \geq 1$ we differentiate $\hat{S}_1^{(2)}(A_S = 0)$ to obtain

$$\begin{aligned} \frac{d}{d\chi} \hat{S}_1^{(2)}(A_S = 0) &= \frac{d}{d\chi} \left[\frac{\cosh \chi}{\sinh \chi} - \frac{\chi}{\sinh^2 \chi} \right] \\ &= -\frac{2}{\sinh^2 \chi} + \frac{2\chi \cosh \chi}{\sinh^3 \chi} = -2\hat{V}_1^{(2)}(A_V = 2). \end{aligned} \quad (4.163)$$

Similarly, if we differentiate the vector field (4.146) with respect to χ we obtain

$$\begin{aligned} & \left[\frac{d^2}{d\chi^2} + 6 \frac{\cosh \chi}{\sinh \chi} \frac{d}{d\chi} + 10 + A_V + \frac{6}{\sinh^2 \chi} - \frac{\ell(\ell+1)}{\sinh^2 \chi} \right] \frac{dg_{1,\ell}}{d\chi} \\ & + 2(2 + A_V) \frac{\cosh \chi}{\sinh \chi} g_{1,\ell} = 0. \end{aligned} \quad (4.164)$$

Comparing with the tensor (4.157) we see that up to an overall normalization we can identify $dg_{1,\ell}/d\chi$ with the tensor $h_{11,\ell}$ for modes that obey $A_V = -2$ and $A_T = 2$, so that these particular vector and tensor modes can interface. As a check, with the tensor sector needing $\ell \geq 2$ we differentiate $\hat{V}_2^{(1)}(A_V = -2)$ to

obtain

$$\begin{aligned}
\frac{d}{d\chi} \hat{V}_2^{(1)}(A_V = -2) &= \frac{d}{d\chi} \left[\frac{2 \cosh \chi}{\sinh \chi} - \frac{3 \cosh \chi}{\sinh^3 \chi} + \frac{3\chi}{\sinh^4 \chi} \right] \\
&= \frac{4}{\sinh^2 \chi} + \frac{12}{\sinh^4 \chi} - \frac{12\chi \cosh \chi}{\sinh^5 \chi} \\
&= 4\hat{T}_2^{(2)}(A_T = 2).
\end{aligned} \tag{4.165}$$

Thus while we can interface $A_S = 0$ and $A_V = 2$, we cannot interface $A_V = 2$ with any of the tensor modes. Rather, we must interface the $A_V = -2$ vector modes with the $A_T = 2$ tensor modes. With none of the scalar sector modes meeting the boundary conditions at both $\chi = \infty$ and $\chi = 0$ anyway, the scalar sector must satisfy $\Delta_{\mu\nu} = 0$ by itself, with the scalar term contribution to $\Delta_{\mu\nu} = 0$ then having to vanish, just as required of the decomposition theorem. However, in the vector and tensor sectors we can achieve a common χ behavior if we set $B_1 - \dot{E}_1 = p_1(\tau)\hat{V}_2^{(1)}(A_V = -2)$, $E_{11} = q_{11}(\tau)\hat{T}_2^{(2)}(A_T = 2)$, since then the $\Delta_{11} = 0$ equation reduces to

$$\begin{aligned}
\Delta_{11} &= \left[\frac{1}{\sinh^2 \chi} + \frac{3}{\sinh^4 \chi} - \frac{3\chi \cosh \chi}{\sinh^5 \chi} \right] \times \\
&\quad \left[8\dot{\Omega}\Omega^{-1}p_1(\tau) + 4\dot{p}_1(\tau) - \ddot{q}_{11}(\tau) + 2q_{11}(\tau) - 2\dot{\Omega}\Omega^{-1}\dot{q}_{11}(\tau) - 2q_{11}(\tau) \right] = 0.
\end{aligned} \tag{4.166}$$

This relation has a non-trivial solution of the form

$$4p_1(\tau) - \dot{q}_{11}(\tau) = \frac{1}{\Omega^2(\tau)}, \tag{4.167}$$

to thereby relate the τ dependencies of the vector and tensor sectors. With the other components of V_i and T_{ij} being constructed in a similar manner, as such

Fluctuations Around an Anti de Sitter Background

For an anti de Sitter background in four dimensions we have

$$\begin{aligned}
ds^2 &= \Omega^2(z) [dt^2 - dx^2 - dy^2 - dz^2] = -g_{\mu\nu} dx^\mu dx^\nu, \quad \Omega(z) = \frac{1}{Hz}, \\
R_{\lambda\mu\nu\kappa} &= -H^2(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}), \quad R_{\mu\kappa} = 3H^2 g_{\mu\kappa}, \quad R = 12H^2, \\
G_{\mu\nu} &= -3H^2 g_{\mu\nu}, \quad T_{\mu\nu} = 3H^2 g_{\mu\nu}.
\end{aligned} \tag{4.186}$$

We take the fluctuations to have the standard SVT3 form given in (??), viz.

$$\begin{aligned}
ds^2 &= -\Omega^2(z) (\eta_{\mu\nu} + f_{\mu\nu}) dx^\mu dx^\nu, \\
f_{00} &= -2\phi, \quad f_{0i} = \tilde{\nabla}_i B + B_i, \quad \tilde{\nabla}^i B_i = 0, \\
f_{ij} &= -2\psi\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \\
\tilde{\nabla}^i E_i &= 0, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0,
\end{aligned} \tag{4.187}$$

where $\tilde{\nabla}_i$ and $\tilde{\nabla}^i = \delta^{ij} \tilde{\nabla}_j$ are defined with respect to a flat three-dimensional background $\eta_{ij} dx^i dx^j = \delta_{ij} dx^i dx^j$.

On defining

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad \delta = \phi - \psi + \dot{B} - \ddot{E} + \frac{2}{z}(\tilde{\nabla}_3 E + E_3), \tag{4.188}$$

following some algebra we find that the components of

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} = \delta G_{\mu\nu} + 3\Omega^2 H^2 f_{\mu\nu} \tag{4.189}$$

are given by

$$g^{\mu\nu} \Delta_{\mu\nu} = -12H^2 \alpha - 3H^2 z^2 \ddot{\alpha} + 3H^2 z^2 \ddot{\delta} + 12H^2 \delta + H^2 z^2 \tilde{\nabla}^2 \alpha - 3H^2 z^2 \tilde{\nabla}^2 \delta$$

in a plane wave (4.256) reduces to

$$\ddot{F}_{00} - \frac{2}{\tau}\dot{F}_{00} + k^2 F_{00} = 0, \quad (4.257)$$

for the representative F_{00} component. The non-trivial solution to (4.257) is of the form

$$F_{00} = \sum_{\mathbf{k}} k^2 \tau^2 [c_{00}(\mathbf{k}) j_1(k\tau) + d_{00}(\mathbf{k}) y_1(k\tau)] e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (4.258)$$

The form for F_{00} given in (4.258) and its $F_{\mu\nu}$ analogs together with $\chi = 0$ thus constitute a non-trivial solution that corresponds to the decomposition theorem, so in this sense the decomposition theorem can be recovered, as it is a specific solution to the full evolution equations. However, there is no compelling reason to restrict the solutions to (4.245) to the trivial $A_{\mu\nu} = 0$, with it being (4.248), (4.252), (4.255) and (4.258) that provide the most general solution in the F_{00} sector and its analogs according to

$$\begin{aligned} F_{00} = & \sum_{\mathbf{k}} \frac{k^2}{2H^2} [-a_{00}(\mathbf{k}) \cos(k\tau) + b_{00}(\mathbf{k}) \sin(k\tau)] e^{i\mathbf{k}\cdot\mathbf{x}} \\ & + \sum_{\mathbf{k}} k^2 \tau^2 [c_{00}(\mathbf{k}) j_1(k\tau) + d_{00}(\mathbf{k}) y_1(k\tau)] e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \quad (4.259)$$

while at the same time (4.235) is the most general solution in the χ sector as constrained by (4.251). Moreover, in this solution we can choose the coefficients in (4.251) so that $F_{\mu\nu}$ and χ are localized in space. Thus no spatially asymptotic boundary coefficient could affect them. In fact suppose that we could have constrained the solutions by an asymptotic condition. We would need one that would

Inserting these solutions for α and F_{00} into $\Delta_{0i} = 0$ then yields

$$\ddot{F}_{0i} - \tilde{\nabla}^2 F_{0i} = - \sum_{\mathbf{k}} k k_i b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x} - ik\tau}, \quad (4.272)$$

with solution

$$F_{0i} = \sum_{\mathbf{k}} [a_{0i}(\mathbf{k}) + \tau b_{0i}(\mathbf{k})] e^{i\mathbf{k} \cdot \mathbf{x} - ik\tau}, \quad (4.273)$$

where

$$b_{0i}(\mathbf{k}) = -\frac{ik_i}{2} b_{\mathbf{k}}. \quad (4.274)$$

With F_{0i} obeying the transverse condition $\partial^i F_{0i} - \dot{F}_{00} = 0$, we obtain

$$ik^i a_{0i}(\mathbf{k}) = k^2 a_{\mathbf{k}} + \frac{3ik}{2} b_{\mathbf{k}}. \quad (4.275)$$

Finally, from $\Delta_{ij} = 0$ we obtain

$$\ddot{F}_{ij} - \frac{2}{\tau} \dot{F}_{ij} - \tilde{\nabla}^2 F_{ij} = \sum_{\mathbf{k}} [\delta_{ij} ik b_{\mathbf{k}} + 2k_i k_j a_{\mathbf{k}} - 2ik_i a_{0j} - 2ik_j a_{0i}] \frac{1}{\tau} e^{i\mathbf{k} \cdot \mathbf{x} - ik\tau}. \quad (4.276)$$

We can thus set

$$F_{ij} = \sum_{\mathbf{k}} [a_{ij}(\mathbf{k}) + \tau b_{ij}(\mathbf{k})] e^{i\mathbf{k} \cdot \mathbf{x} - ik\tau}, \quad (4.277)$$

where

$$2ika_{ij}(\mathbf{k}) - 2b_{ij}(\mathbf{k}) = \delta_{ij} ik b_{\mathbf{k}} + 2k_i k_j a_{\mathbf{k}} - 2ik_i a_{0j} - 2ik_j a_{0i}. \quad (4.278)$$

With F_{ij} obeying the transverse and traceless conditions $\partial^j F_{ij} = \dot{F}_{0i}$, $\delta^{ij} F_{ij} - F_{00} = 0$, we obtain

$$ik^j a_{ij}(\mathbf{k}) = -ika_{0i}(\mathbf{k}) - \frac{ik_i}{2} b_{\mathbf{k}}, \quad ik^j b_{ij}(\mathbf{k}) = -\frac{kk_i}{2} b_{\mathbf{k}},$$

Chapter 5

Constructing and Imposing Gauge Conditions

In the context of conformal gravity, we continue work done in obtaining solutions to the cosmological fluctuation equations [2] by constructing a gauge condition that is conformally invariant. Referred to as the conformal gauge, we find that in backgrounds that are conformal to flat, the conformal gravity cosmological fluctuation equations can be brought to an exceedingly simple form, comprised only of a single term. This calculation requires a series of many steps which are given in detail within the chapter, consisting of first motivating and constructing the conformal gauge, composing the fluctuation equations and using curvature identities and covariant derivatives to reduce its form, and finally imposing the conformal gauge itself, with expansion of covariant derivatives into flat Minkowski partial derivatives to yield (5.32), one of the seminal results of this thesis. Using the conformal properties detailed in Sect. 2.2.1, we find the extra degree of symmetry afforded by conformal invariance provides significant simplifications regarding the trace and moreover allows a very straightforward treatment of the entire cosmological fluctuations.

5.1.2 $\delta W_{\mu\nu}$ in an Arbitrary Background

Composing the Fluctuation Equations

With the conformal gauge in hand, we proceed in a sequence of steps in order to implement it with the fluctuation equations. Prior to perturbing $W_{\mu\nu}$ we present a useful identity

$$\nabla_\beta \nabla_\nu T_{\lambda\mu} = \nabla_\nu \nabla_\beta T_{\lambda\mu} + R_{\lambda\sigma\nu\beta} T_{\mu}^{\sigma} - R_{\sigma\mu\nu\beta} T_{\lambda}^{\sigma}, \quad (5.11)$$

which holds for any rank two tensor. We then express $W^{\mu\nu}$ as

$$\begin{aligned} W_{\mu\nu} = & -\frac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta R_{\alpha}^{\alpha} + \nabla_\beta\nabla^\beta R_{\mu\nu} - \frac{1}{3}\nabla_\mu\nabla_\nu R_{\alpha}^{\alpha} - R^{\beta\sigma}R_{\sigma\mu\beta\nu} \\ & - R^{\beta\sigma}R_{\sigma\nu\beta\mu} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}R_{\alpha}^{\alpha}R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}(R_{\alpha}^{\alpha})^2. \end{aligned} \quad (5.12)$$

We set the metric as the most general $g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ denotes an arbitrary background metric and $\delta g_{\mu\nu} = h_{\mu\nu}$ an arbitrary and general fluctuation. Upon perturbing $W^{\mu\nu}$ we then obtain

$$\begin{aligned} \delta W_{\mu\nu}(h_{\mu\nu}) = & \frac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}h^{\alpha\beta}R_{\alpha}^{\gamma}R_{\beta\gamma} - \frac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \frac{1}{3}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta}R \\ & - \frac{1}{6}h_{\mu\nu}R^2 + h^{\alpha\beta}R_{\alpha}^{\gamma}R_{\mu\beta\nu\gamma} + h^{\alpha\beta}R_{\alpha}^{\gamma}R_{\mu\gamma\nu\beta} - \frac{1}{6}h_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R - h^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} \\ & + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R + \frac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} + \frac{1}{3}h^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R_{\alpha\beta} + \frac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}h_{\mu\nu} \\ & + R_{\mu\beta\nu\gamma}\nabla_{\alpha}\nabla^{\gamma}h^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla_{\alpha}\nabla^{\gamma}h^{\alpha\beta} - \frac{1}{3}R\nabla_{\alpha}\nabla_{\mu}h_{\nu}^{\alpha} - \frac{1}{3}R\nabla_{\alpha}\nabla_{\nu}h_{\mu}^{\alpha} \\ & - \frac{1}{6}\nabla_{\alpha}h_{\mu\nu}\nabla^{\alpha}R + \frac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}h_{\alpha}^{\beta} - \nabla_{\alpha}h^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} - \frac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} \\ & + \frac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} + \frac{1}{2}R_{\nu}^{\alpha}\nabla_{\beta}\nabla_{\alpha}h_{\mu}^{\beta} - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}h_{\mu\nu} + \frac{1}{2}R_{\mu}^{\alpha}\nabla_{\beta}\nabla_{\alpha}h_{\nu}^{\beta} \end{aligned}$$

5.1.3 $\delta W_{\mu\nu}$ in a Conformal to Flat Minkowski Background

Implementing the Conformal Gauge Condition

With (5.24) now being ready for insertion of the conformal gauge, we evaluate (5.24) in the conformal to flat background given in (2.23) recalling that we take $\Omega(x)$ to be a completely general and arbitrary spacetime function. In the $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ background the gauge condition $\nabla_\nu K^{\mu\nu} = \frac{1}{2}K^{\mu\nu}g^{\alpha\beta}\partial_\nu g_{\alpha\beta}$ takes the form

$$\begin{aligned}
 \nabla_\nu K^{\mu\nu} - \frac{1}{2}K^{\mu\nu}\Omega^{-2}\eta^{\alpha\beta}\eta_{\alpha\beta}\partial_\nu\Omega^2 &= \nabla_\nu K^{\mu\nu} - 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \\
 &= \partial_\nu K^{\mu\nu} + 6\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega - 4\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega \\
 &= \partial_\nu K^{\mu\nu} + 2\Omega^{-1}K^{\mu\nu}\partial_\nu\Omega = \Omega^{-2}\partial_\nu(\Omega^2 K^{\mu\nu}) \\
 &= 0.
 \end{aligned} \tag{5.26}$$

We can factor out a contribution of $\Omega^2(x)$ from the fluctuation by setting $K^{\mu\nu} = \Omega^{-2}(x)k^{\mu\nu}$ and $K_{\mu\nu} = \Omega^2(x)k_{\mu\nu}$. For clarification, here indices on $k^{\mu\nu}$ and $k_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}k^{\alpha\beta}$ are raised and lowered with $\eta_{\mu\nu}$. As a result, (5.26) can thus be expressed as the familiar transverse form $\partial_\nu k^{\mu\nu} = 0$, noting that the conformal gauge condition is such that there is no longer a dependence upon the conformal factor.

¹ Before we also evaluate the (5.24) within a conformal Minkowski background in

¹ Within (5.26) we note that we have taken $\Omega(x)$ to be a general function of the coordinates so that we additionally encompass the special case of Robertson-Walker geometries with arbitrary spatial curvature k . As given in Appendix B, $\Omega(x)$ will only depend on the time coordinate t for $k = 0$, while for $k \neq 0$, $\Omega(x)$ will depend on both t and the radial coordinate r .

(B.4) will thus grow as t^4 itself as modified by the prefactor in (5.41), and thus as t^4 for the lightlike case (the solutions to $\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta[\Omega^{-2}(x)K_{\mu\nu}] = 0$ as given in (5.37) are lightlike), and as t^2 for the timelike case. With the $ds^2 = 0$ light cone being both general coordinate invariant and conformal invariant, lightlike modes associated with the (B.15) metric will transform into lightlike modes associated with the metric (B.4). A t^4 growth for lightlike modes is a quite substantial growth rate, a growth rate that is not achievable in standard Einstein gravity if one uses the same radiation matter source.

Since in transforming from one coordinate system to another the transformation is effected by

$$K_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} K'_{\alpha\beta}, \quad (5.42)$$

the transformations between the (p', r') , (p, χ) and (t, r) coordinate systems are effected by

$$\begin{aligned} \frac{\partial p'}{\partial p} &= \frac{\partial r'}{\partial \chi} = \frac{1 + \cosh p \cosh \chi}{[\cosh p + \cosh \chi]^2}, & \frac{\partial p'}{\partial \chi} &= \frac{\partial r'}{\partial p} = -\frac{\sinh p \sinh \chi}{[\cosh p + \cosh \chi]^2}, \\ \frac{\partial p}{\partial t} &= \frac{1}{La(t)}, & \frac{\partial \chi}{\partial r} &= \frac{1}{L \cosh \chi}. \end{aligned} \quad (5.43)$$

Discussion of (5.43) depends on whether $p = \chi$ or $p \gg \chi$. Since according to (5.34) $t = d \sinh p$ when $La(t) = (d^2 + t^2)^{1/2}$, at large t we see that when $p = \chi$ (i.e. both p and χ then being large) we have

$$\frac{\partial p'}{\partial p} = \frac{\partial r'}{\partial \chi} = 1, \quad \frac{\partial p'}{\partial \chi} = \frac{\partial r'}{\partial p} = 1, \quad \frac{\partial p}{\partial t} = \frac{1}{t}, \quad \frac{\partial \chi}{\partial r} = \frac{d}{Lt}. \quad (5.44)$$

$$\times \left[\tanh[(p+\chi)/2] - \tanh[(p-\chi)/2] \right] \frac{\exp[-iqp']}{\sin^2 \theta} q r' j_0(qr'),$$

With r' being given by $r' = (\tanh[(p+\chi)/2] - \tanh[(p-\chi)/2])/2 = \sinh \chi / (\cosh p + \cosh \chi)$, we can thus set:

$$K_{\theta\theta}(t, r, \theta, \phi) = L^2 a^2(t) \left[\alpha_{\theta\theta}^{(-)} + \beta_{\theta\theta}^{(-)} \frac{\sinh p}{\cosh p + \cosh \chi} \right] \sinh^2 \chi \frac{\exp[-iqp']}{\sin^2 \theta} q j_0(qr'). \quad (5.48)$$

Thus, for the lightlike $p = \chi$ case (where $r' \rightarrow 1/2$ at large p) $K_{\theta\theta}(t, r, \theta, \phi)$ grows like t^4 , while for the timelike $p \gg \chi$ case (where $r' \rightarrow 0$ at large p) $K_{\theta\theta}(t, r, \theta, \phi)$ grows like t^2 , just as found above. (The timelike t^2 growth had been incorrectly given in [2].) Interestingly, in [2] we had only considered a particular class of solutions, namely those with $K_{tt} = 0$, $K_{tr} = 0$, $K_{t\theta} = 0$ and $K_{t\phi} = 0$. We now see that for both $p = \chi$ and $p \gg \chi$, all of the modes that obey the synchronous condition $K_{0\mu} = 0$ are non-leading, with the angular modes respectively being leading or coleading in the two cases.

5.3 Imposing Specific Gauges within $\delta G_{\mu\nu}$

With background plus fluctuation metric of the form $ds^2 = \Omega^2(x)[dt^2 - \delta_{ij}dx^i dx^j] - \Omega^2(x)f_{\mu\nu}dx^\mu dx^\nu$, the fluctuation in the Einstein tensor is given by

$$\begin{aligned} \delta G_{\mu\nu} = & -\frac{1}{2}\tilde{\nabla}_\alpha \tilde{\nabla}_\mu f_\nu^\alpha - \frac{1}{2}\tilde{\nabla}_\alpha \tilde{\nabla}_\nu f_\mu^\alpha - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\alpha f \tilde{\nabla}_\beta \Omega + \eta^{\alpha\beta}\Omega^{-1}\tilde{\nabla}_\alpha f_{\mu\nu} \tilde{\nabla}_\beta \Omega \\ & + \eta^{\beta\alpha}f_{\mu\nu}\Omega^{-2}\tilde{\nabla}_\alpha \Omega \tilde{\nabla}_\beta \Omega - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\tilde{\nabla}_\beta \tilde{\nabla}_\alpha f + \frac{1}{2}\eta_{\mu\nu}\tilde{\nabla}_\beta \tilde{\nabla}_\alpha f^{\alpha\beta} + \frac{1}{2}\eta^{\alpha\beta}\tilde{\nabla}_\beta \tilde{\nabla}_\alpha f_{\mu\nu} \\ & + 2\eta_{\mu\nu}f^{\alpha\beta}\Omega^{-1}\tilde{\nabla}_\beta \tilde{\nabla}_\alpha \Omega - 2\eta^{\alpha\beta}f_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\beta \tilde{\nabla}_\alpha \Omega + 2\eta^{\alpha\gamma}\eta_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\beta f_\alpha^\beta \tilde{\nabla}_\gamma \Omega \end{aligned}$$

Chapter 6

Conclusions

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Appendix A

SVT by Projection

A.1 The 3 + 1 Decomposition

The standard covariant 3+1 decomposition of a symmetric rank two tensor $T_{\mu\nu}$ in a 4-dimensional geometry with metric $g_{\mu\nu}$ is constructed by introducing a 4-vector U^μ that obeys $g_{\mu\nu}U^\mu U^\nu = -1$ and a projector

$$P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \quad (\text{A.1})$$

that obeys

$$U_\mu P^{\mu\nu} = 0, \quad P_{\mu\nu} P^{\mu\nu} = g_{\mu\nu} P^{\mu\nu} = 3, \quad P_{\mu\sigma} P^\sigma{}_\nu = P_{\mu\nu}. \quad (\text{A.2})$$

In terms of the projector we can write

$$\begin{aligned} T_{\mu\nu} &= g_\mu{}^\sigma g_\nu{}^\tau T_{\sigma\tau} = P_\mu{}^\sigma P_\nu{}^\tau T_{\sigma\tau} - U_\mu U^\sigma P_\nu{}^\tau T_{\sigma\tau} \\ &\quad - P_\mu{}^\sigma U_\nu U^\tau T_{\sigma\tau} + U_\mu U_\nu U^\sigma U^\tau T_{\sigma\tau}. \end{aligned} \quad (\text{A.3})$$

On introducing

$$\begin{aligned} \rho &= U^\sigma U^\tau T_{\sigma\tau}, \quad p = \frac{1}{3} P^{\sigma\tau} T_{\sigma\tau}, \quad q_\mu = -P_\mu{}^\sigma U^\tau T_{\sigma\tau}, \\ \pi_{\mu\nu} &= \left[\frac{1}{2} P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2} P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3} P_{\mu\nu} P^{\sigma\tau} \right] T_{\sigma\tau}, \end{aligned} \quad (\text{A.4})$$

which obey

$$U^\mu q_\mu = 0, \quad U^\nu \pi_{\mu\nu} = 0, \quad \pi_{\mu\nu} = \pi_{\nu\mu}, \quad g^{\mu\nu} \pi_{\mu\nu} = P^{\mu\nu} \pi_{\mu\nu} = 0, \quad (\text{A.5})$$

we can rewrite $T_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} + U_\mu q_\nu + U_\nu q_\mu + \pi_{\mu\nu}, \quad (\text{A.6})$$

a familiar form that may for instance be found in [26]. As constructed, the ten-component $T_{\mu\nu}$ has been covariantly decomposed into two one-component 4-scalars, one three-component 4-vector that is orthogonal to U_μ and one five-component traceless, rank two tensor that is also orthogonal to U_μ .

A.2 Vector Fields

One of the key virtues of the SVT3 and SVT4 formalisms is that both of them enable us to construct fluctuation equations that are gauge invariant. Thus while a general $h_{\mu\nu}$ has ten components, because of the freedom to impose four coordinate transformations only six of the components are physical. Both the SVT3 and SVT4 formalisms then automatically provide fluctuation equations that only depend on six combinations of the terms in the SVT3 or SVT4 expansions of $h_{\mu\nu}$. Since the SVT3 and SVT4 components are related to the components of $h_{\mu\nu}$ via integral relations such as that given in (3.16), viz.

$$B = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}, \quad B_i = h_{0i} - \tilde{\nabla}_i \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^i h_{0i}, \quad (\text{A.7})$$

the SVT3 and SVT4 components are intrinsically non-local, with their very existence requiring that the associated integrals exist. Thus asymptotic boundary conditions are built into their very existence. Interestingly, as discussed in [21] and [19], there is another way to implement gauge invariance using non-local operators, namely to use a projection operator approach. We now show that this approach is equivalent to the SVT approach.

To discuss the application of the projection operator approach to rank two tensors such as $h_{\mu\nu}$ we first apply it to a four-dimensional gauge field A_μ . Thus in analog to (A.7) we set

$$\begin{aligned} A_\mu &= A_\mu^T + \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha = A_\mu^T + A_\mu^L, \\ A_\mu^T &= A_\mu - \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha, \quad A_\mu^L = \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha, \end{aligned} \quad (\text{A.8})$$

where $\partial_\mu \partial^\mu D(x - x') = \delta^4(x - x')$, where A_μ^T obeys the transverse condition $\partial^\mu A_\mu^T = 0$, and where A_μ^L is longitudinal. The utility of this expansion is that under $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ the transverse A_μ^T transforms as

$$\begin{aligned} A_\mu^T &\rightarrow A_\mu + \partial_\mu \chi - \partial_\mu \int d^4x' D(x - x') \partial^\alpha A_\alpha - \partial_\mu \int d^4x' D(x - x') \partial^\alpha \partial_\alpha \chi \\ &= A_\mu^T, \end{aligned} \quad (\text{A.9})$$

assuming integration by parts. Thus with integration by parts the transverse A_μ^T is automatically gauge invariant. In addition we note that A_μ^T obeys

$$\partial_\nu \partial^\nu A_\mu^T = \partial_\nu \partial^\nu A_\mu - \partial_\mu \partial^\nu A_\nu = \partial^\nu F_{\nu\mu}. \quad (\text{A.10})$$

Thus just as with the use of the non-local SVT formalism for gravity, the use of the non-local A_μ^T enables us to write the Maxwell equations entirely in terms of

gauge-invariant quantities. With A_μ^L being the derivative of a scalar function it is pure gauge, and thus cannot appear in the gauge-invariant Maxwell equations. Moreover, while there may be an integration by parts issue for A_μ^T , there is none for $\partial_\nu \partial^\nu A_\mu^T$ as it is equal to the gauge-invariant quantity $\partial_\nu \partial^\nu A_\mu - \partial_\mu \partial^\alpha A_\alpha$, just as it must be since the Maxwell equations are gauge invariant. In the SVT language, with (A.7) and (A.9) only involving scalars and vectors, we can think of (A.7) as an SV3 decomposition of the 3-component h_{0i} , and (A.9) as an SV4 decomposition of the 4-component A_μ .

An alternate way of understanding these results is to introduce a projection operator

$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial}{\partial x^\mu} \int d^4x' D(x-x') \frac{\partial}{\partial x'^\nu}, \quad (\text{A.11})$$

as we can then rewrite A_μ^T as

$$A_\mu^T = \Pi_{\mu\nu} A^\nu. \quad (\text{A.12})$$

As introduced, $\Pi_{\mu\nu}$ obeys the projector algebra relations

$$\begin{aligned} \Pi_{\mu\nu} \Pi^\nu{}_\sigma &= \Pi_{\mu\sigma}, \\ \Pi_{\mu\nu} A^{T\nu} &= A_\mu^T - \partial_\mu \int d^4x' D(x-x') \partial_\nu A^{T\nu}(x') = A_\mu^T, \\ \Pi_{\mu\nu} A^{L\nu} &= \partial_\mu \int d^4x' D(x-x') \partial_\nu A^\nu(x') - \partial_\mu \int d^4x' D(x-x') \times \\ &\quad \partial_\nu \partial^\nu \int d^4x'' D(x'-x'') \partial_\sigma A^\sigma(x'') = 0. \end{aligned} \quad (\text{A.13})$$

In the SVT4 language we set $A_\mu = A_\mu^T + \partial_\mu A$, and can thus identify

$$A_\mu^T = \Pi_{\mu\nu} A^\nu, \quad A_\mu^L = \partial_\mu A = (\eta_{\mu\nu} - \Pi_{\mu\nu}) A^\nu. \quad (\text{A.14})$$

For vector fields the SVT formalism is thus equivalent to the projector formalism. Having now established this equivalence for vector fields, we turn now to tensor fields.

A.3 Transverse and Longitudinal Projection Operators for Flat Spacetime Tensor Fields

For tensor fields we introduce 4-dimensional flat spacetime transverse and longitudinal projection operators [19, 21]:

$$T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma} \eta_{\nu\tau} - \partial_\mu \int d^4x' D(x-x') \eta_{\nu\tau} \partial_\sigma - \partial_\nu \int d^4x' D(x-x') \eta_{\mu\sigma} \partial_\tau$$

$$\begin{aligned}
& + \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\tau, \\
L_{\mu\nu\sigma\tau} & = \partial_\mu \int d^4 x' D(x-x') \eta_{\nu\tau} \partial_\sigma + \partial_\nu \int d^4 x' D(x-x') \eta_{\mu\sigma} \partial_\tau \\
& - \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\tau.
\end{aligned} \tag{A.15}$$

As constructed, these projectors obey a standard projector algebra

$$\begin{aligned}
T_{\mu\nu\sigma\tau} T_{\alpha\beta}^{\sigma\tau} & = T_{\mu\nu\alpha\beta}, \quad L_{\mu\nu\sigma\tau} L_{\alpha\beta}^{\sigma\tau} = L_{\mu\nu\alpha\beta}, \\
T_{\mu\nu\sigma\tau} L_{\alpha\beta}^{\sigma\tau} & = 0, \quad L_{\mu\nu\sigma\tau} T_{\alpha\beta}^{\sigma\tau} = 0, \quad L_{\mu\nu\sigma\tau} + T_{\mu\nu\sigma\tau} = \eta_{\mu\sigma} \eta_{\nu\tau}.
\end{aligned} \tag{A.16}$$

In terms of these projectors we define transverse and longitudinal components $h_{\mu\nu}^T$ and $h_{\mu\nu}^L$ of $h_{\mu\nu}$ according to

$$\begin{aligned}
T_{\mu\nu\sigma\tau} h^{\sigma\tau} & = h_{\mu\nu}^T = h_{\mu\nu} - \partial_\mu \int d^4 x' D(x-x') \partial_\sigma h^\sigma{}_\nu(x') \\
& \quad - \partial_\nu \int d^4 x' D(x-x') \partial_\kappa h^\kappa{}_\mu(x') \\
& \quad + \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\kappa h^{\sigma\kappa}(x''), \\
L_{\mu\nu\sigma\tau} h^{\sigma\tau} & = h_{\mu\nu}^L = \partial_\mu \int d^4 x' D(x-x') \partial_\sigma h^\sigma{}_\nu(x') + \partial_\nu \int d^4 x' D(x-x') \partial_\kappa h^\kappa{}_\mu(x') \\
& \quad - \partial_\mu \partial_\nu \int d^4 x' D(x-x') \partial_\sigma \int d^4 x'' D(x'-x'') \partial_\kappa h^{\sigma\kappa}(x'').
\end{aligned} \tag{A.17}$$

Assuming integration by parts these components obey

$$\partial_\nu h^{T\mu\nu} = 0, \quad \partial_\nu h^{L\mu\nu} = \partial_\nu h^{\mu\nu}. \tag{A.18}$$

With $h_{\mu\nu}^T$ transforming as $h_{\mu\nu}^T \rightarrow h_{\mu\nu}^T$ under $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ as long as we can integrate by parts, we see that, as introduced, $h_{\mu\nu}^T$ is both transverse and gauge invariant.

On evaluation we obtain

$$\begin{aligned}
\frac{1}{2} [\partial_\mu \partial_\nu h^T + \partial_\alpha \partial^\alpha h_{\mu\nu}^T] - \frac{1}{2} \eta_{\mu\nu} \partial_\sigma \partial^\sigma h^T & = \frac{1}{2} [\partial_\mu \partial_\nu h - \partial_\mu \partial_\lambda h^\lambda{}_\nu - \partial_\nu \partial_\lambda h^\lambda{}_\mu \\
& \quad + \partial_\alpha \partial^\alpha h_{\mu\nu}] - \frac{1}{2} \eta_{\mu\nu} [\partial_\alpha \partial^\alpha h - \partial_\sigma \partial_\lambda h^{\sigma\lambda}],
\end{aligned} \tag{A.19}$$

where h^T is given by

$$h^T = \eta^{\alpha\beta} h_{\alpha\beta}^T = h - \partial_\nu \int d^4 x' D(x-x') \partial_\sigma h^{\sigma\nu}(x'), \tag{A.20}$$

with $h = \eta^{\alpha\beta} h_{\alpha\beta}$. On recognizing the right-hand side of (A.19) as $\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R = \delta G_{\mu\nu}$, we obtain

$$\delta G_{\mu\nu} = \frac{1}{2}[\partial_\mu\partial_\nu h^T + \partial_\alpha\partial^\alpha h_{\mu\nu}^T] - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\partial^\sigma h^T. \quad (\text{A.21})$$

We thus write the perturbed Einstein tensor entirely in terms of the non-local, gauge invariant, six degree of freedom $h_{\mu\nu}^T$.

To make contact with the SVT4 expansion we insert

$$h_{\mu\nu} = -2\eta_{\mu\nu}\chi + 2\partial_\mu\partial_\nu F + \partial_\mu F_\nu + \partial_\nu F_\mu + 2F_{\mu\nu} \quad (\text{A.22})$$

into $h_{\mu\nu}^T$, to obtain

$$h_{\mu\nu}^T = -2\eta_{\mu\nu}\chi + 2F_{\mu\nu} + 2\partial_\mu\partial_\nu \int d^4D(x-x')\chi(x'), \quad h^T = -6\chi. \quad (\text{A.23})$$

With $\delta G_{\mu\nu}$ being written in terms of the projected $h_{\mu\nu}^T$, we see that it is written in terms of the SVT4 $F_{\mu\nu}$ and χ . However as written, $h_{\mu\nu}^T$ contains an integral term in (A.23). To eliminate it we extend transverse projection to transverse-traceless projection.

A.4 Transverse-Traceless Projection Operators for Flat Spacetime Tensor Fields

In [21] and [19] two further projectors were introduced

$$\begin{aligned} Q_{\mu\nu\sigma\tau} &= \frac{1}{3} \left[\eta_{\mu\nu} - \partial_\mu\partial_\nu \int d^4x' D(x-x') \right] \left[\eta_{\sigma\tau} - \partial'_\sigma \int d^4x'' D(x'-x'') \partial''_\tau \right], \\ P_{\mu\nu\sigma\tau} &= T_{\mu\nu\sigma\tau} - Q_{\mu\nu\sigma\tau}. \end{aligned} \quad (\text{A.24})$$

They obey the projector algebra

$$\begin{aligned} T_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} &= Q_{\mu\nu\alpha\beta}, \quad Q_{\mu\nu\sigma\tau} T^{\sigma\tau}_{\alpha\beta} = Q_{\mu\nu\alpha\beta}, \quad Q_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} = Q_{\mu\nu\alpha\beta}, \\ P_{\mu\nu\sigma\tau} Q^{\sigma\tau}_{\alpha\beta} &= 0, \quad Q_{\mu\nu\sigma\tau} P^{\sigma\tau}_{\alpha\beta} = 0, \quad P_{\mu\nu\sigma\tau} P^{\sigma\tau}_{\alpha\beta} = P_{\mu\nu\alpha\beta}. \end{aligned} \quad (\text{A.25})$$

The projector $P_{\mu\nu\sigma\tau}$ projects out the traceless piece of $h_{\mu\nu}^T$, while $Q_{\mu\nu\sigma\tau}$ projects out its complement, and they implement

$$P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^T = h_{\mu\nu}^{T\theta}, \quad Q_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^T = h_{\mu\nu}^T - h_{\mu\nu}^{T\theta}, \quad (\text{A.26})$$

with $h_{\mu\nu}^{T\theta}$ being both traceless and transverse. With $Q_{\mu\nu}{}^{\sigma\tau}$ implementing $Q_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^L = 0$, $P_{\mu\nu}{}^{\sigma\tau}$ implements $P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau}^L = 0$ as well, to thus implement

$$P_{\mu\nu}{}^{\sigma\tau} h_{\sigma\tau} = h_{\mu\nu}^{T\theta}. \quad (\text{A.27})$$

$P_{\mu\nu\sigma\tau}$ is thus a traceless projector not just for the transverse $h_{\mu\nu}^T$ but for the full $h_{\mu\nu}$ as well. We can thus introduce its complementary projection operator $U_{\mu\nu\sigma\tau} = \eta_{\mu\sigma}\eta_{\nu\tau} - P_{\mu\nu\sigma\tau}$, as it obeys

$$\begin{aligned} P_{\mu\nu\sigma\tau} U^{\sigma\tau\alpha\beta} &= 0, \quad U_{\mu\nu\sigma\tau} P^{\sigma\tau\alpha\beta} = 0, \quad U_{\mu\nu\sigma\tau} U^{\sigma\tau}_{\alpha\beta} = U_{\mu\nu\alpha\beta}, \\ U_{\mu\nu}^{\sigma\tau} h_{\sigma\tau} &= h_{\mu\nu} - h_{\mu\nu}^{T\theta} = h_{\mu\nu}^{L\theta} + \frac{1}{3} \eta_{\mu\nu} \eta^{\sigma\tau} h_{\sigma\tau} \\ &\quad - \frac{1}{3} \partial_\mu \partial_\nu \int d^4 y D(x-y) \eta^{\sigma\tau} h_{\sigma\tau}, \end{aligned} \quad (\text{A.28})$$

Given (A.26) and (A.24) we obtain

$$h_{\mu\nu}^{T\theta} = h_{\mu\nu}^T - \frac{1}{3} \eta_{\mu\nu} \eta^{\sigma\kappa} h_{\sigma\kappa}^T + \frac{1}{3} \partial_\mu \partial_\nu \int d^4 y D(x-y) \eta^{\sigma\kappa} h_{\sigma\kappa}^T, \quad (\text{A.29})$$

Inserting (A.23) into (A.29) yields

$$h_{\mu\nu}^{T\theta} = 2F_{\mu\nu}, \quad (\text{A.30})$$

with χ dropping out. Finally, in terms of $h_{\mu\nu}^{T\theta}$ we can rewrite (A.21) as

$$\delta G_{\mu\nu} = \frac{1}{2} \partial_\alpha \partial^\alpha h_{\mu\nu}^{T\theta} - \frac{1}{3} \eta_{\mu\nu} \partial_\sigma \partial^\sigma h^T + \frac{1}{3} \partial_\mu \partial_\nu h^T. \quad (\text{A.31})$$

Then with

$$F_{\mu\nu} = \frac{1}{2} h_{\mu\nu}^{T\theta}, \quad \chi = -\frac{1}{6} h^T, \quad (\text{A.32})$$

we can rewrite (A.31) as

$$\delta G_{\mu\nu} = \partial_\alpha \partial^\alpha F_{\mu\nu} + 2\eta_{\mu\nu} \partial_\alpha \partial^\alpha \chi - 2\partial_\mu \partial_\nu \chi. \quad (\text{A.33})$$

We recognize (A.33) as the expression for $\delta G_{\mu\nu}$ as given in (3.40) when $D = 4$, and with $h_{\mu\nu}^T$ and thus $h_{\mu\nu}^{T\theta}$ and h^T being gauge invariant, we confirm that given integration by parts $F_{\mu\nu}$ and χ are gauge invariant, just as noted in Sec. ???. Thus with (A.32) we establish the equivalence of the SVT4 decomposition and the projection operator technique.

As a further example of this equivalence we note that for conformal gravity fluctuations around a flat spacetime background (4.297) takes the form

$$\begin{aligned} \delta W_{\mu\nu} &= \frac{1}{2} \left(\partial_\sigma \partial^\sigma \partial_\tau \partial^\tau K_{\mu\nu} - \partial_\sigma \partial^\sigma \partial_\mu \partial^\alpha K_{\alpha\nu} - \partial_\sigma \partial^\sigma \partial_\nu \partial^\alpha K_{\alpha\mu} \right. \\ &\quad \left. + \frac{2}{3} \partial_\mu \partial_\nu \partial^\alpha \partial^\beta K_{\alpha\beta} + \frac{1}{3} \eta_{\mu\nu} \partial_\sigma \partial^\sigma \partial^\alpha \partial^\beta K_{\alpha\beta} \right), \end{aligned} \quad (\text{A.34})$$

where all derivatives are four-dimensional derivatives with respect to a flat Minkowski metric, and where $K_{\mu\nu}$ is given by $K_{\mu\nu} = h_{\mu\nu} - (1/4)\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}$. Inserting (A.17) and (A.29) into (A.34) yields

$$\delta W_{\mu\nu} = \frac{1}{2}\partial_\sigma\partial^\sigma\partial_\tau\partial^\tau h_{\mu\nu}^{T\theta}. \quad (\text{A.35})$$

With the insertion of (A.22) into (A.34) yielding

$$\delta W_{\mu\nu} = \partial_\sigma\partial^\sigma\partial_\tau\partial^\tau F_{\mu\nu}, \quad (\text{A.36})$$

(cf. (4.299) with $\Omega = 1$), we recover (A.30), and again confirm the equivalence of the SVT4 decomposition and the projection operator technique.

A.5 Transverse and Longitudinal Projection Operators for Curved Spacetime Tensor Fields

For curved spacetime with background metric $g_{\mu\nu}$ it is convenient to define a 2-index propagator

$$[g^\nu_\beta\nabla_\tau\nabla^\tau + \nabla_\beta\nabla^\nu]D^\beta_\sigma(x, x') = g^\nu_\sigma(-g)^{-1/2}\delta^4(x - x'). \quad (\text{A.37})$$

In terms of it we introduce [21]

$$\begin{aligned} T_{\mu\nu\sigma\tau} &= g_{\mu\sigma}g_{\nu\tau} - \nabla_\mu \int d^4x' (-g)^{1/2} D_{\nu\sigma}(x, x') \nabla_\tau \\ &\quad - \nabla_\nu \int d^4x' (-g)^{1/2} D_{\mu\sigma}(x, x') \nabla_\tau, \\ L_{\mu\nu\sigma\tau} &= \nabla_\mu \int d^4x' (-g)^{1/2} D_{\nu\sigma}(x, x') \nabla_\tau + \nabla_\nu \int d^4x' (-g)^{1/2} D_{\mu\sigma}(x, x') \nabla_\tau. \end{aligned} \quad (\text{A.38})$$

These projection operators close on the projector algebra given in (A.16). As such, they effect $T_{\mu\nu\sigma\tau}h^{\sigma\tau} = h_{\mu\nu}^T$, and $L_{\mu\nu\sigma\tau}h^{\sigma\tau} = h_{\mu\nu}^L$, where

$$h_{\mu\nu}^T = h_{\mu\nu} - \nabla_\mu \int d^4x' (-g)^{1/2} D^\nu_\sigma(x, x') \nabla_\tau h^{\sigma\tau} - \nabla_\nu \int d^4x' (-g)^{1/2} D^\mu_\sigma(x, x') \nabla_\tau h^{\sigma\tau}, \quad (\text{A.39})$$

$$h_{\mu\nu}^L = \nabla_\mu \int d^4x' (-g)^{1/2} D^\nu_\sigma(x, x') \nabla_\tau h^{\sigma\tau} + \nabla_\nu \int d^4x' (-g)^{1/2} D^\mu_\sigma(x, x') \nabla_\tau h^{\sigma\tau}. \quad (\text{A.40})$$

The utility of constructing these projected states is that under a gauge transformation $h_{\mu\nu}$ transforms into $h_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu$. However, we see that this is precisely the structure of $h_{\mu\nu}^L$. The longitudinal component of $h_{\mu\nu}$ can thus be removed by a gauge transformation, and the fluctuation Einstein equations can only depend on the 6-component $h_{\mu\nu}^T$. However, unlike the flat background case where one can write $\delta G_{\mu\nu}$ itself entirely in terms of $h_{\mu\nu}^T$, in the curved background case there must be a background $T_{\mu\nu}$, and thus it is only in the full $\delta G_{\mu\nu} + 8\pi G \delta T_{\mu\nu}$ that the metric fluctuations can be described entirely by $h_{\mu\nu}^T$. If we introduce a quantity $\delta T_{\mu\nu}^T$ in which the dependence on ϵ_μ has been excluded (i.e. under a gauge transformation $\delta T_{\mu\nu} \rightarrow \delta T_{\mu\nu}^T$ plus a function of ϵ_μ , and this function of ϵ_μ cancels against an identical function of ϵ_μ in $\delta G_{\mu\nu}$), then following the commuting of some derivatives, the fluctuation equations take the form [21]

$$\begin{aligned} \delta G_{\mu\nu} + 8\pi G \delta T_{\mu\nu} &= \frac{1}{2} [\nabla_\mu \nabla_\nu h^T + R^\sigma{}_\mu h^T_{\sigma\nu} + R^\sigma{}_\nu h^T_{\sigma\mu} - 2R_{\mu\lambda\nu\sigma} h^{T\lambda\sigma} + \nabla_\alpha \nabla^\alpha h^T_{\mu\nu}] \\ &- \frac{1}{2} R^\sigma{}_\sigma h^T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} h^{T\alpha\beta} - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla^\alpha h^T + 8\pi G \delta T_{\mu\nu}^T = 0. \end{aligned} \quad (\text{A.41})$$

The SVT4 fluctuations around a de Sitter background as given in (4.225) to (4.228) and around a general Robertson-Walker background as given in (4.295) are special cases of (A.41), with the only metric fluctuations that appear in (4.228) and (4.295) being $F_{\mu\nu}$ and χ , viz. just the six degrees of freedom associated with $h_{\mu\nu}^T$.

A.6 D-dimensional SVTD Transverse-Traceless Projection Operators for Curved Spacetime Tensor Fields

Rather than generalize the general curved spacetime transverse and longitudinal projection technique to the general transverse-traceless case, we have instead found it more convenient to generalize the SVTD discussion given in Secs. ?? and ?? to general curved spacetime background fluctuations. To this end we take $h_{\mu\nu}$ to be of the form:

$$h_{\mu\nu} = 2F_{\mu\nu} + W_{\mu\nu} + S_{\mu\nu}, \quad (\text{A.42})$$

where

$$\begin{aligned} W_{\mu\nu} &= \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu} \nabla^\alpha W_\alpha, \\ S_{\mu\nu} &= \frac{1}{D-1} (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.43})$$

with $D(x, x')$ obeying

$$\nabla_\alpha \nabla^\alpha D^{(D)}(x, x') = [-g(x)]^{-1/2} \delta^{(D)}(x - x'). \quad (\text{A.44})$$

From (A.43) we obtain

$$g^{\mu\nu} W_{\mu\nu} = 0, \quad g^{\mu\nu} S_{\mu\nu} = h, \quad (\text{A.45})$$

$$\nabla^\nu h_{\mu\nu} = \nabla^\nu W_{\mu\nu} + \nabla^\nu S_{\mu\nu} \quad (\text{A.46})$$

as the conditions that $F_{\mu\nu}$ be transverse and traceless. From (A.46) we obtain

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \nabla_\alpha \nabla_\nu - \frac{2}{D} \nabla_\nu \nabla_\alpha \right] W^\alpha &= \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1} (\nabla_\nu \nabla_\alpha \nabla^\alpha \\ &\quad - \nabla_\alpha \nabla^\alpha \nabla_\nu) \times \\ &\quad \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.47})$$

and by commuting derivatives can rewrite (A.47) as

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \left(\frac{D-2}{D} \right) \nabla_\nu \nabla_\alpha - R_{\nu\alpha} \right] W^\alpha &= \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1} R_{\nu\alpha} \nabla^\alpha \times \\ &\quad \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'). \end{aligned} \quad (\text{A.48})$$

To solve for W_μ it is convenient to use the bitensor formalism in which we define $G_\alpha^{(D)\beta}(x, x') = e_\alpha^a(x) e_a^\beta(x')$ where the D-dimensional $e_\alpha^a(x)$ vierbeins obey $g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x)$, with a and b referring to a fixed D-dimensional basis. With this bitensor definition $e_\alpha^a(x)$ and $e_a^\beta(x')$ are acting in separate spaces, but at $x = x'$ we obtain $G_\alpha^{(D)\beta}(x, x) = g_\alpha^\beta(x)$. On the introducing the propagator that satisfies

$$\begin{aligned} \left[g_{\nu\alpha} \nabla_\beta \nabla^\beta + \left(\frac{D-2}{D} \right) \nabla_\nu \nabla_\alpha - R_{\nu\alpha} \right] D_{(D)}^{\alpha\gamma}(x, x') &= G_\nu^{(D)\gamma}(x, x') [-g(x')]^{-1/2} \times \\ &\quad \delta^{(D)}(x - x'), \end{aligned} \quad (\text{A.49})$$

we can solve for W_μ as

$$W_\mu(x) = \int d^D x' [-g(x')]^{1/2} D_\mu^{(D)\sigma}(x, x') \left[\nabla_{x'}^\rho h_{\sigma\rho}(x') - \frac{1}{D-1} R_{\sigma\rho}(x') \nabla_{x'}^\rho \times \right.$$

$$\int d^D x'' [-g(x'')]^{1/2} D^{(D)}(x', x'') h(x'') \Big]. \quad (\text{A.50})$$

Next we decompose W_μ into transverse and longitudinal components viz.

$$\begin{aligned} W_\mu &= W_\mu^T + W_\mu^L = F_\mu + \nabla_\mu H, \quad \nabla^\mu F_\mu = 0, \\ H &= \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \nabla^\sigma W_\sigma(x'), \end{aligned} \quad (\text{A.51})$$

with $h_{\mu\nu}$ then taking the form

$$\begin{aligned} h_{\mu\nu} &= 2F_{\mu\nu} + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2\nabla_\mu \nabla_\nu H - \frac{2}{D} g_{\mu\nu} \nabla_\alpha \nabla^\alpha H \\ &+ \frac{1}{D-1} (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'). \end{aligned} \quad (\text{A.52})$$

Upon further defining

$$\begin{aligned} F &= H - \frac{1}{2(D-1)} \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \\ \chi &= \frac{1}{D} \nabla_\alpha \nabla^\alpha H - \frac{1}{2(D-1)} \nabla_\alpha \nabla^\alpha \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') h(x'), \end{aligned} \quad (\text{A.53})$$

we may express $h_{\mu\nu}$ in the SVTD form:

$$h_{\mu\nu} = -2g_{\mu\nu} \chi + 2\nabla_\mu \nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}, \quad (\text{A.54})$$

where

$$\begin{aligned} \chi &= \frac{1}{D} \nabla^\sigma W_\sigma - \frac{1}{2(D-1)} h, \\ F_\mu &= W_\mu^T = W_\mu - \nabla_\mu \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \nabla^\sigma W_\sigma(x'), \\ F &= \int d^D x' [-g(x')]^{1/2} D^{(D)}(x, x') \left(\nabla^\sigma W_\sigma(x') - \frac{1}{2(D-1)} h(x') \right), \\ 2F_{\mu\nu} &= h_{\mu\nu} + 2g_{\mu\nu} \chi - 2\nabla_\mu \nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu. \end{aligned} \quad (\text{A.55})$$

We thus generalize the SVTD approach to the arbitrary D-dimensional curved spacetime background.

Appendix B

Conformal to Flat Cosmological Geometries

B.1 Robertson-Walker $k = 0$

In order to apply (5.32) to cosmology we need to write the Robertson-Walker and de Sitter background geometries in a conformal to flat Minkowski form. For a $k = 0$ Robertson-Walker background the comoving coordinate system metric takes the form

$$ds^2(\text{comoving}) = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \quad (\text{B.1})$$

The straightforward introduction of the conformal time

$$d\tau = \int \frac{dt}{a(t)} \quad (\text{B.2})$$

then allows us to write the conformal time metric as

$$ds^2(\text{conformal time}) = a^2(\tau)[d\tau^2 - dx^2 - dy^2 - dz^2]. \quad (\text{B.3})$$

B.2 Robertson-Walker $k > 0$

For a $k > 0$ or a $k < 0$ Robertson-Walker background the comoving and conformal time coordinate system metrics take the form

$$\begin{aligned} ds^2(\text{comoving}) &= dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \\ ds^2(\text{conformal time}) &= a^2(\tau) \left[d\tau^2 - \frac{dr^2}{1 - kr^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \right]. \end{aligned} \quad (\text{B.4})$$

To bring the RW geometries with non-zero k to a conformal to flat form requires coordinate transformations that involve both τ and r . For the $k > 0$ case first, it is convenient to set $k = 1/L^2$, and introduce $\sin \chi = r/L$, with the conformal time metric given in (B.4) then taking the form

$$ds^2 = L^2 a^2(p) [dp^2 - d\chi^2 - \sin^2 \chi d\theta^2 - \sin^2 \chi \sin^2 \theta d\phi^2], \quad (\text{B.5})$$

where $p = \tau/L$. Following e.g. [2] we introduce

$$\begin{aligned} p' + r' &= \tan[(p + \chi)/2], & p' - r' &= \tan[(p - \chi)/2], \\ p' &= \frac{\sin p}{\cos p + \cos \chi}, & r' &= \frac{\sin \chi}{\cos p + \cos \chi}, \end{aligned} \quad (\text{B.6})$$

so that

$$dp'^2 - dr'^2 = \frac{1}{4}[dp^2 - d\chi^2] \sec^2[(p + \chi)/2] \sec^2[(p - \chi)/2], \quad (\text{B.7})$$

$$\begin{aligned} \frac{1}{4}(\cos p + \cos \chi)^2 &= \cos^2[(p + \chi)/2] \cos^2[(p - \chi)/2] \\ &= \frac{1}{[1 + (p' + r')^2][1 + (p' - r')^2]}. \end{aligned} \quad (\text{B.8})$$

With these transformations the $k > 0$ line element then takes the conformal to flat form

$$ds^2 = \frac{4L^2 a^2(p)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [dp'^2 - dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2 \theta d\phi^2]. \quad (\text{B.9})$$

To bring the spatial sector of (B.9) to Cartesian coordinates we set $x' = r' \sin \theta \cos \phi$, $y' = r' \sin \theta \sin \phi$, $z' = r' \cos \theta$ and thus bring the line element to the form

$$ds^2 = L^2 a^2(p) (\cos p + \cos \chi)^2 [dp'^2 - dx'^2 - dy'^2 - dz'^2], \quad (\text{B.10})$$

where now $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. With these transformations (B.10) is now in the form given in (2.23).

B.3 Robertson-Walker $k < 0$

For the $k < 0$ case, it is convenient to set $k = -1/L^2$, and introduce $\sinh \chi = r/L$, with the conformal time metric given in (B.4) then taking the form

$$ds^2 = L^2 a^2(p) [dp^2 - d\chi^2 - \sinh^2 \chi d\theta^2 - \sinh^2 \chi \sin^2 \theta d\phi^2], \quad (\text{B.11})$$

where $p = \tau/L$. Next we introduce

$$\begin{aligned} p' + r' &= \tanh[(p + \chi)/2], & p' - r' &= \tanh[(p - \chi)/2], \\ p' &= \frac{\sinh p}{\cosh p + \cosh \chi}, & r' &= \frac{\sinh \chi}{\cosh p + \cosh \chi}, \end{aligned} \quad (\text{B.12})$$

so that

$$dp'^2 - dr'^2 = \frac{1}{4}[dp^2 - d\chi^2] \text{sech}^2[(p + \chi)/2] \text{sech}^2[(p - \chi)/2],$$

$$\begin{aligned}
\frac{1}{4}(\cosh p + \cosh \chi)^2 &= \cosh^2[(p + \chi)/2] \cosh^2[(p - \chi)/2] \\
&= \frac{1}{[1 - (p' + r')^2][1 - (p' - r')^2]}.
\end{aligned} \tag{B.13}$$

With these transformations the line element takes the conformal to flat form

$$ds^2 = \frac{4L^2 a^2(p)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [dp'^2 - dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2 \theta d\phi^2]. \tag{B.14}$$

The spatial sector can then be written in Cartesian form

$$ds^2 = L^2 a^2(p) (\cosh p + \cosh \chi)^2 [dp'^2 - dx'^2 - dy'^2 - dz'^2], \tag{B.15}$$

where again $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. We note that in transforming from (B.4) to (B.10) or to (B.15) we have only made coordinate transformations and not made any conformal transformation.

B.4 dS_4 and AdS_4 Background Solutions

While the conformal to flat Minkowski structures given in (B.3), (B.10) and (B.15) are purely kinematical, the explicit form of $a(t)$ can be determined once a dynamics has been specified. Thus in regard to a de Sitter or anti-de Sitter cosmology, a de Sitter or an anti-de Sitter geometry is just a particular case of a Robertson-Walker geometry in which $a(t)$ has a specific assigned value for each possible choice of spatial 3-curvature k . On writing the maximally 4-symmetric geometry condition $R_{\mu\nu} = -3\alpha g_{\mu\nu}$ in Robertson-Walker form one obtains

$$\dot{a}^2(t) + k = \alpha a^2(t). \tag{B.16}$$

(In terms of the scalar field model described in (2.48) – (2.52) we have $K = \alpha = -2\lambda_S S_0^2$.) Here α is positive for de Sitter and negative for anti-de Sitter. Allowable solutions to (B.16) depend on the values of α and k , and are of the form (see e.g. [?])

$$\begin{aligned}
a(t, \alpha > 0, k < 0) &= \left(-\frac{k}{\alpha}\right)^{1/2} \sinh(\alpha^{1/2}t), \\
a(t, \alpha > 0, k = 0) &= a(t = 0) \exp(\alpha^{1/2}t), \\
a(t, \alpha > 0, k > 0) &= \left(\frac{k}{\alpha}\right)^{1/2} \cosh(\alpha^{1/2}t), \\
a(t, \alpha = 0, k < 0) &= (-k)^{1/2}t, \\
a(t, \alpha < 0, k < 0) &= \left(\frac{k}{\alpha}\right)^{1/2} \sin((-\alpha)^{1/2}t).
\end{aligned} \tag{B.17}$$

In these solutions (B.3), (B.10), and (B.15) all apply to a de Sitter or an anti-de Sitter cosmology.

B.5 dS_4 and AdS_4 Background Solutions - Radiation Era

For Robertson-Walker cosmologies we note that with slight modification we can extend the scalar field model given above to include a perfect fluid, with the energy-momentum tensor then being given by [8]

$$T_S^{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} - \frac{1}{6}S_0^2 \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha \right) - g^{\mu\nu}\lambda_S S_0^4, \quad (\text{B.18})$$

with the background conformal cosmology still obeying $T_S^{\mu\nu} = 0$ since the background Robertson-Walker geometry continues to obey $W_{\mu\nu} = 0$. On taking the perfect fluid energy-momentum tensor to be traceless radiation (viz. $\rho = 3p$, $\rho = A/a^4(t)$, $A > 0$) as needed in the early universe, and with $\alpha = -2\lambda_S S_0^2$ as before, the evolution equation takes the form

$$\dot{a}^2 + k = \alpha a^2 - \frac{2A}{S_0^2 a^2}, \quad (\text{B.19})$$

with allowed solutions to the cosmology being given by [?]

$$\begin{aligned} a(t, \alpha > 0, k < 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} - \frac{k\beta}{\alpha} \sinh^2(\alpha^{1/2}t) \right)^{1/2}, \\ a(t, \alpha > 0, k = 0, A > 0) &= \left(-\frac{A}{\lambda_S S_0^4} \right)^{1/4} \cosh^{1/2}(2\alpha^{1/2}t), \\ a(t, \alpha > 0, k > 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} + \frac{k\beta}{\alpha} \cosh^2(\alpha^{1/2}t) \right)^{1/2}, \\ a(t, \alpha = 0, k < 0, A > 0) &= \left(-\frac{2A}{kS_0^2} - kt^2 \right)^{1/2}, \\ a(t, \alpha < 0, k < 0, A > 0) &= \left(-\frac{k(\beta - 1)}{2\alpha} + \frac{k\beta}{\alpha} \sin^2((-\alpha)^{1/2}t) \right)^{1/2}, \end{aligned} \quad (\text{B.20})$$

where $\beta = (1 + 8A\alpha/k^2 S_0^2)^{1/2}$.