

4D SVT dS₄ Einstein v3

1 $h_{\mu\nu}$ Decomposition

Curvature Tensors:

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D) \end{aligned} \tag{1.1}$$

Covariant Commutation:

$$\begin{aligned} [\nabla^\sigma \nabla_\nu] W_\sigma &= -R_\nu{}^\sigma W_\sigma = -\frac{R}{D} W_\nu \\ [\nabla^\mu \nabla_\mu, \nabla_\nu] V &= -R_\nu{}^\mu \nabla_\mu V = -\frac{R}{D} \nabla_\nu V \\ [\nabla^2, \nabla_\mu \nabla_\nu] V &= \frac{2g_{\mu\nu}R}{D(D-1)} \nabla^2 V - \frac{2R}{D-1} \nabla_\mu \nabla_\nu V \end{aligned} \tag{1.2}$$

Decomposition:

$$\begin{aligned} h_{\mu\nu} &= h_{\mu\nu}^{T\theta} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1} (\nabla^\sigma W_\sigma - h) \\ &\quad + \frac{2-D}{D-1} \left(\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)} \right) \int D(x, x') \nabla^\sigma W_\sigma - \frac{1}{D-1} \left(\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)} \right) \int D(x, x') h \end{aligned} \tag{1.3}$$

$$\begin{aligned} \left(\nabla_\alpha \nabla^\alpha - \frac{R}{D-1} \right) D(x, x') &= g^{-1/2} \delta^4(x - x') \\ \nabla^\mu h_{\mu\nu} &= \left(\nabla_\alpha \nabla^\alpha - \frac{R}{D} \right) W_\nu \end{aligned} \tag{1.4}$$

With the box-like operator mixing indices of W_ν , the particular integral solution for W_ν involves a bi-tensor Green's function $F_{\sigma\rho'}$ which obeys

$$\left(\nabla^\alpha \nabla_\alpha - \frac{R}{D} \right) F_{\sigma\rho'}(x, x') = g_{\sigma\rho'} g^{-1/2} \delta^4(x - x'). \tag{1.5}$$

Here $g_{\sigma\rho'}$ represents a parallel propagator, defined in terms of Vierbeins e_μ^a :

$$g^{\alpha'}{}_\beta(x, x') = e_a^{\alpha'}(x') e_\beta^a(x), \quad g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b. \tag{1.6}$$

In terms of (1.5), W_ν has particular solution

$$W_\nu = \int F_{\nu}{}^{\rho'}(x, x') \nabla^{\sigma'} h_{\rho'\sigma'}. \tag{1.7}$$

To construct a transverse vector E_μ , split W_μ

$$\begin{aligned}
W_\mu &= \underbrace{W_\mu - \nabla_\mu \int A(x, x') \nabla^\sigma W_\sigma}_{E_\mu} + \nabla_\mu \int A(x, x') \nabla^\sigma W_\sigma \\
\nabla_\alpha \nabla^\alpha A(x, x') &= g^{-1/2} \delta^4(x - x')
\end{aligned} \tag{1.8}$$

With $h_{\mu\nu}^{T\theta} = 2E_{\mu\nu}$, (1.3) may be expressed as

$$\begin{aligned}
h_{\mu\nu} &= 2E_{\mu\nu}^{T\theta} + \nabla_\mu E_\nu + \nabla_\nu E_\mu - \frac{g_{\mu\nu}}{D-1} (\nabla^\sigma W_\sigma - h) + 2\nabla_\mu \nabla_\nu \int A(x, x') \nabla^\sigma W_\sigma \\
&\quad + \frac{1}{D-1} \left(\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu} R}{D(D-1)} \right) \int D(x, x') [(2-D) \nabla^\sigma W_\sigma - h].
\end{aligned} \tag{1.9}$$

We may simplify this to

$$\begin{aligned}
h_{\mu\nu} &= 2E_{\mu\nu}^{T\theta} + \nabla_\mu E_\nu + \nabla_\nu E_\mu \\
&\quad + 2\nabla_\mu \nabla_\nu \left(\int A(x, x') \nabla^\sigma W_\sigma + \frac{1}{2(D-1)} \int D(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \right) \\
&\quad - \frac{2g_{\mu\nu}}{2(D-1)} \left(\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)} \int D(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \right).
\end{aligned} \tag{1.10}$$

SVT Definitions:

$$\begin{aligned}
2E_{\mu\nu}^{T\theta} &= h_{\mu\nu}^{T\theta} \\
E_\mu &= W_\mu - \nabla_\mu \int A(x, x') \nabla^\sigma W_\sigma \\
E &= \int A(x, x') \nabla^\sigma W_\sigma + \frac{1}{2(D-1)} \int D(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \\
\psi &= \frac{1}{2(D-1)} \left(\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)} \int D(x, x') [(2-D) \nabla^\sigma W_\sigma - h] \right)
\end{aligned} \tag{1.11}$$

In the flat space limit, $A(x, x') = D(x, x')$ and we have

$$\begin{aligned}
2E_{\mu\nu}^{T\theta} &= h_{\mu\nu}^{T\theta} \\
E_\mu &= W_\mu - \nabla_\mu \int D(x, x') \nabla^\sigma W_\sigma \\
E &= \frac{1}{2(D-1)} \int D(x, x') [D \nabla^\sigma W_\sigma - h] \\
\psi &= \frac{1}{2(D-1)} (\nabla^\sigma W_\sigma - h),
\end{aligned} \tag{1.12}$$

a form that coincides with *Localization_Condition_Matthew* (2.1).

2 $\delta T_{\mu\nu}$ Decomposition

For a conserved $\delta T_{\mu\nu}$ we take $W_\mu = 0$.

$$\begin{aligned}
\delta T_{\mu\nu} &= \delta T_{\mu\nu}^{T\theta} + \frac{g_{\mu\nu}}{D-1} \delta T - \frac{1}{D-1} \left(\nabla_\mu \nabla_\nu - \frac{g_{\mu\nu} R}{D(D-1)} \right) \int D(x, x') \delta T \\
6\bar{\chi} &= \int D(x, x') \delta T \\
2\bar{E}_{\mu\nu} &= \delta T_{\mu\nu}^{T\theta} \\
6(\nabla_\alpha \nabla^\alpha + 4k)\bar{\chi} &= \delta T \\
\delta T_{\mu\nu} &= 2(\nabla_\alpha \nabla^\alpha g_{\mu\nu} + 3k g_{\mu\nu} - \nabla_\mu \nabla_\nu) \bar{\chi} + 2\bar{E}_{\mu\nu}
\end{aligned} \tag{2.1}$$

3 dS₄ Background and Fluctuations

$$\begin{aligned}
G_{\mu\nu}^{(0)} &= 3kg_{\mu\nu} \\
R_{\lambda\mu\nu\kappa}^{(0)} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\
R_{\mu\kappa}^{(0)} &= -3kg_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\
R^{(0)} &= -12k \\
\\
ds^2 &= (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
\\
\delta G_{\mu\nu} &= 2kh_{\mu\nu} - \frac{1}{2}kg_{\mu\nu}h + \frac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \frac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha \\
&\quad - \frac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \frac{1}{2}\nabla_\nu\nabla_\mu h \\
\\
\delta G &= \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h \\
\\
h_{\mu\nu} &= -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu E_\nu + \nabla_\nu E_\mu + 2E_{\mu\nu} \\
\\
\delta G_{\mu\nu} &= 4kE_{\mu\nu} + \nabla_\alpha\nabla^\alpha E_{\mu\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + 3k\nabla_\mu E_\nu + 3k\nabla_\nu E_\mu + 6k\nabla_\nu\nabla_\mu F - 2\nabla_\nu\nabla_\mu\chi \\
\\
\nabla^\mu\delta G_{\mu\nu} &= 3k\nabla^\mu h_{\mu\nu} \\
&= -6k\nabla_\nu\chi + 18k^2\nabla_\nu F + 6k\nabla_\nu\nabla_\alpha\nabla^\alpha F + 9k^2E_\nu + 3k\nabla_\alpha\nabla^\alpha E_\nu \\
\\
-\kappa_4^2 T_{\mu\nu}^{(0)} &= 3kg_{\mu\nu} \\
\\
-\kappa_4^2 \delta T_{\mu\nu}^{(b)} &= 3kh_{\mu\nu} \\
&= -6kg_{\mu\nu}\chi + 6k\nabla_\mu\nabla_\nu F + 3k\nabla_\mu E_\nu + 3k\nabla_\nu E_\mu + 6kE_{\mu\nu} \\
\\
\nabla^\mu(\delta G_{\mu\nu} + \kappa_4^2 \delta T_{\mu\nu}^{(b)}) &= 0 \\
\\
\delta T_{\mu\nu} &= \delta T_{\mu\nu}^{(b)} + \delta T_{\mu\nu}^{(s)} \\
\\
-\kappa_4^2 \delta T_{\mu\nu}^{(s)} &= 2(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu)\bar{\chi} + 2\bar{E}_{\mu\nu}, \quad -\kappa_4^2\nabla^\mu\delta T_{\mu\nu}^{(s)} = 0
\end{aligned} \tag{3.1}$$

4 SVT Separation

$$\begin{aligned}
\delta G_{\mu\nu} + \kappa_4^2 \delta T_{\mu\nu}^{(b)} &= -\kappa_4^2 \delta T_{\mu\nu}^{(s)} \\
2(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu)\chi + (\nabla_\alpha\nabla^\alpha - 2k)E_{\mu\nu} &= 2(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu)\bar{\chi} + 2\bar{E}_{\mu\nu}
\end{aligned} \tag{4.1}$$

Trace (4.1):

$$6(\nabla_\alpha\nabla^\alpha + 4k)\chi = 6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} \tag{4.2}$$

For $\chi = \bar{\chi}$, (4.1) becomes

$$(\nabla_\alpha\nabla^\alpha - 2k)E_{\mu\nu} = 2\bar{E}_{\mu\nu} \tag{4.3}$$

5 SVT4 Covariant vs. Conformal

Perturbing on the background source, we have in SV4 covariant (barred quantities):

$$\Delta = 6(\nabla_\alpha \nabla^\alpha + 4k)\bar{\chi} \quad (5.1)$$

$$\Delta_{\mu\nu} = 2(\nabla_\alpha \nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu \nabla_\nu)\bar{\chi} + (\nabla_\alpha \nabla^\alpha - 2k)\bar{E}_{\mu\nu} \quad (5.2)$$

This may be compared to SVT4 conformal (unbarred quantities):

$$\begin{aligned} \Delta_{\mu\nu} = & \tau^{-2}(6\ddot{F}\tilde{g}_{\mu\nu} + 2\dot{\chi}\tilde{g}_{\mu\nu}\tau + 6\tilde{g}_{\mu\nu}\chi + 2\tilde{g}_{\mu\nu}\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 2\tilde{g}_{\mu\nu}\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi + 2\tau U_\nu\tilde{\nabla}_\mu\chi + 2\tau U_\mu\tilde{\nabla}_\nu\chi \\ & - 2\tau\tilde{\nabla}_\nu\tilde{\nabla}_\mu\dot{F} - 2\tau^2\tilde{\nabla}_\nu\tilde{\nabla}_\mu\chi + 6\dot{E}^\alpha\tilde{g}_{\mu\nu}U_\alpha + 2\tilde{g}_{\mu\nu}\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha - 2\tau U^\alpha\tilde{\nabla}_\nu\tilde{\nabla}_\mu E_\alpha \\ & + 2\dot{E}_{\mu\nu}\tau + 6E_{\alpha\beta}\tilde{g}_{\mu\nu}U^\alpha U^\beta + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha E_{\mu\nu} - 2\tau U^\alpha\tilde{\nabla}_\mu E_{\nu\alpha} - 2\tau U^\alpha\tilde{\nabla}_\nu E_{\mu\alpha}) \end{aligned} \quad (5.3)$$

$$\Delta = k(24\ddot{F} + 12\dot{\chi}\tau + 24\chi + 6\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi + 24\dot{E}^\alpha U_\alpha + 6\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha + 24E_{\alpha\beta}U^\alpha U^\beta) \quad (5.4)$$

Evaluating (5.4) in conformal flat coordinates

$$6(\nabla_\alpha \nabla^\alpha + 4k)\bar{\chi} = 6k(2\dot{\chi}\tau + 4\bar{\chi} + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi}) \quad (5.5)$$

Equating (5.1) to (5.4)

$$\begin{aligned} 12\dot{\chi}\tau + 24\bar{\chi} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi} = & 24\ddot{F} + 12\dot{\chi}\tau + 24\chi + 6\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi \\ & + 24\dot{E}^\alpha U_\alpha + 6\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha + 24E_{\alpha\beta}U^\alpha U^\beta \end{aligned} \quad (5.6)$$

$$6(\nabla_\alpha \nabla^\alpha + 4k)\bar{\chi} = \nabla^\alpha \nabla^\beta h_{\alpha\beta} - (\nabla_\alpha \nabla^\alpha + 3k)h \quad (5.7)$$

Various Relations:

$$\bar{\chi} = \frac{1}{6} \int g^{1/2} D(x, x') [\nabla^\alpha \nabla^\beta h_{\alpha\beta} - (\nabla_\alpha \nabla^\alpha + 3k)h] \quad (5.8)$$

$$\left(\nabla_\alpha \nabla^\alpha - \frac{R}{D-1} \right) D(x, x') = g^{-1/2} \delta^4(x - x') \quad (5.9)$$

$$\Omega^4 k(2\tau\partial_0 + 4 + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha)D(x, x') = \delta^4(x - x') = \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha F(x, x') \quad (5.10)$$

$$\nabla_\alpha \nabla^\alpha \bar{\chi} = k(2\dot{\chi}\tau + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi}) \quad (5.11)$$

$$\begin{aligned} \nabla^\alpha \nabla^\beta h_{\alpha\beta} - (\nabla_\alpha \nabla^\alpha + 3k)h = & -\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f - f\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega - 3\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha f + 2f\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega \\ & + 6\Omega^{-3}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta f_\alpha{}^\beta + \Omega^{-2}\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f^{\alpha\beta} + 4f^{\alpha\beta}\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\Omega + 4f_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega \\ = & -3k\dot{f}\tau + 12kf_{\alpha\beta}U^\alpha U^\beta - k\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f \\ & + 6k\tau U^\alpha\tilde{\nabla}_\beta f_\alpha{}^\beta + k\tau^2\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f^{\alpha\beta} \end{aligned} \quad (5.12)$$