Bach CF SVT4 v1

In a maximally symmetric background, the Bach tensor can be used to construct a fourth order covariant equation that relates $F_{\mu\nu}$ in terms of $h_{\mu\nu}$. Here we do the same, generalized now to an arbitrary conformal flat background geometry.

1 Background and Fluctuations

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{1.1}$$

$$R_{\lambda\mu\nu\kappa} = -\tfrac{1}{6}g_{\lambda\nu}g_{\mu\kappa}R + \tfrac{1}{6}g_{\lambda\kappa}g_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R_{\lambda\kappa} + \tfrac{1}{2}g_{\mu\kappa}R_{\lambda\nu} + \tfrac{1}{2}g_{\lambda\nu}R_{\mu\kappa} - \tfrac{1}{2}g_{\lambda\kappa}R_{\mu\nu} \tag{1.2}$$

$$W_{\mu\nu} = 0 \tag{1.3}$$

$$g^{\mu\nu}\delta W_{\mu\nu} = h_{\mu\nu}W^{\mu\nu} = 0 \tag{1.4}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{1.5}$$

$$\begin{aligned} K_{\mu\nu} &= h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}g^{\alpha\beta}h_{\alpha\beta} \\ &= -\tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha F + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \end{aligned} \tag{1.6}$$

$$g^{\mu\nu}F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad \nabla^\mu F_\mu = 0 \tag{1.7}$$

$$\begin{aligned} \delta W_{\mu\nu} = {} & \tfrac{1}{9}K_{\mu\nu}R^2 - \tfrac{1}{2}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} - \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + g_{\mu\nu}K^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha} \\ & + K_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta + \tfrac{1}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \tfrac{1}{3}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R + R_\nu{}^\alpha\nabla_\alpha\nabla_\beta K_\mu{}^\beta \\ & + R_\mu{}^\alpha\nabla_\alpha\nabla_\beta K_\nu{}^\beta - \tfrac{1}{2}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta - 2\nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} \\ & + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R \\ & - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + R_\nu{}^\alpha\nabla_\beta\nabla^\beta K_{\mu\alpha} + R_\mu{}^\alpha\nabla_\beta\nabla^\beta K_{\nu\alpha} + K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} + K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} \\ & + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha + \nabla_\beta R_{\nu\alpha}\nabla^\beta K_\mu{}^\alpha \\ & + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \nabla_\beta R_{\mu\alpha}\nabla^\beta K_\nu{}^\alpha - \tfrac{1}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\ & + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} \\ & + \tfrac{1}{2}\nabla_\alpha K^{\alpha\beta}\nabla_\mu R_{\nu\beta} - \tfrac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta + R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} \\ & - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha K_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} \\ & + \tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + \tfrac{1}{2}\nabla_\alpha K^{\alpha\beta}\nabla_\nu R_{\mu\beta} - \tfrac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta \\ & + R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha} - \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha K_\mu{}^\alpha - \tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} + \tfrac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\ & + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta} \end{aligned} \tag{1.8}$$

2 SVT4

$$\delta W_{\mu\nu} = X_{\mu\nu}(F) + Y_{\mu\nu}(F_\alpha) + Z_{\mu\nu}(F_{\alpha\beta}) \tag{2.1}$$

$$\begin{aligned}
X_{\mu\nu} = {} & \tfrac{5}{36} g_{\mu\nu} R^2 \nabla_\alpha \nabla^\alpha F - \tfrac{1}{6} R R_{\mu\nu} \nabla_\alpha \nabla^\alpha F + \tfrac{5}{36} g_{\mu\nu} R \nabla_\alpha R \nabla^\alpha F - \tfrac{1}{6} R_{\mu\nu} \nabla_\alpha R \nabla^\alpha F \\
& - \tfrac{5}{6} g_{\mu\nu} R^{\beta\gamma} \nabla_\alpha R_{\beta\gamma} \nabla^\alpha F + \tfrac{1}{2} R_\nu{}^\beta \nabla_\alpha R_{\mu\beta} \nabla^\alpha F + \tfrac{1}{3} R \nabla_\alpha R_{\mu\nu} \nabla^\alpha F + 2 R_\nu{}^\beta \nabla^\alpha F \nabla_\beta R_{\mu\alpha} \\
& + 2 R_\mu{}^\beta \nabla^\alpha F \nabla_\beta R_{\nu\alpha} - \tfrac{7}{18} g_{\mu\nu} R R^{\alpha\beta} \nabla_\beta \nabla_\alpha F - \nabla^\alpha \nabla_\nu F \nabla_\beta \nabla_\alpha R_\mu{}^\beta + \tfrac{1}{2} R_\mu{}^\alpha R_{\nu\alpha} \nabla_\beta \nabla^\beta F \\
& - \tfrac{1}{12} g_{\mu\nu} \nabla_\alpha \nabla^\alpha F \nabla_\beta \nabla^\beta R + \nabla^\alpha \nabla_\nu F \nabla_\beta \nabla^\beta R_{\mu\alpha} + \tfrac{1}{2} \nabla_\alpha \nabla^\alpha F \nabla_\beta \nabla^\beta R_{\mu\nu} + \nabla^\alpha \nabla_\mu F \nabla_\beta \nabla^\beta R_{\nu\alpha} \\
& + \tfrac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta \nabla^\beta \nabla_\alpha F - \nabla^\alpha R_{\mu\nu} \nabla_\beta \nabla^\beta \nabla_\alpha F - \tfrac{1}{12} g_{\mu\nu} \nabla^\alpha F \nabla_\beta \nabla^\beta \nabla_\alpha R \\
& + \tfrac{1}{2} \nabla^\alpha F \nabla_\beta \nabla^\beta \nabla_\alpha R_{\mu\nu} - \tfrac{5}{36} g_{\mu\nu} R_{\alpha\beta} \nabla^\alpha F \nabla^\beta R - \nabla_\beta \nabla_\alpha R_{\mu\nu} \nabla^\beta \nabla^\alpha F + \tfrac{1}{3} g_{\mu\nu} R^{\beta\gamma} \nabla^\alpha F \nabla_\gamma R_{\alpha\beta} \\
& + \tfrac{7}{6} g_{\mu\nu} R_\alpha{}^\gamma R^{\alpha\beta} \nabla_\gamma \nabla_\beta F + \tfrac{1}{3} g_{\mu\nu} \nabla^\beta \nabla^\alpha F \nabla_\gamma \nabla_\beta R_\alpha{}^\gamma - \tfrac{5}{12} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma F \\
& - \tfrac{1}{6} R_{\nu\alpha} \nabla^\alpha R \nabla_\mu F - \tfrac{1}{2} R^{\alpha\beta} \nabla_\beta R_{\nu\alpha} \nabla_\mu F - \tfrac{1}{3} R_{\nu\alpha} \nabla^\alpha F \nabla_\mu R - \tfrac{7}{6} R \nabla^\alpha F \nabla_\mu R_{\nu\alpha} \\
& + \nabla_\beta \nabla^\beta \nabla_\alpha F \nabla_\mu R_\nu{}^\alpha + 2 R_\alpha{}^\beta \nabla^\alpha F \nabla_\mu R_{\nu\beta} - \tfrac{5}{6} R R_\nu{}^\alpha \nabla_\mu \nabla_\alpha F + \tfrac{5}{2} R_\alpha{}^\beta R_\nu{}^\alpha \nabla_\mu \nabla_\beta F \\
& + \nabla^\beta \nabla^\alpha F \nabla_\mu \nabla_\beta R_{\nu\alpha} + \tfrac{1}{2} \nabla^\alpha F \nabla_\mu \nabla_\beta \nabla^\beta R_{\nu\alpha} - \tfrac{1}{2} R_{\mu\alpha} \nabla^\alpha R \nabla_\nu F + \tfrac{17}{36} R \nabla_\mu R \nabla_\nu F \\
& - R^{\alpha\beta} \nabla_\mu R_{\alpha\beta} \nabla_\nu F - \tfrac{1}{12} \nabla_\mu \nabla_\alpha \nabla^\alpha R \nabla_\nu F + \tfrac{1}{12} R_{\mu\alpha} \nabla^\alpha F \nabla_\nu R + \tfrac{1}{36} R \nabla_\mu F \nabla_\nu R \\
& - \tfrac{1}{2} R_\mu{}^\beta \nabla^\alpha F \nabla_\nu R_{\alpha\beta} - \tfrac{1}{6} R \nabla^\alpha F \nabla_\nu R_{\mu\alpha} + \tfrac{1}{2} R_\alpha{}^\beta \nabla^\alpha F \nabla_\nu R_{\mu\beta} + \tfrac{1}{6} R R_\mu{}^\alpha \nabla_\nu \nabla_\alpha F \\
& - \tfrac{1}{3} \nabla^\alpha \nabla_\mu F \nabla_\nu \nabla_\alpha R - \tfrac{1}{6} \nabla_\mu R \nabla_\nu \nabla_\alpha \nabla^\alpha F - \tfrac{1}{2} R_\alpha{}^\beta R_\mu{}^\alpha \nabla_\nu \nabla_\beta F + \tfrac{1}{6} R^2 \nabla_\nu \nabla_\mu F \\
& - \tfrac{1}{2} R_{\alpha\beta} R^{\alpha\beta} \nabla_\nu \nabla_\mu F - \tfrac{1}{3} \nabla_\alpha \nabla^\alpha R \nabla_\nu \nabla_\mu F - \tfrac{1}{6} \nabla_\alpha \nabla^\alpha F \nabla_\nu \nabla_\mu R - \tfrac{1}{3} \nabla^\alpha F \nabla_\nu \nabla_\mu \nabla_\alpha R
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
Y_{\mu\nu} = {} & - \tfrac{1}{36} F^\alpha g_{\mu\nu} R \nabla_\alpha R - \tfrac{1}{6} F^\alpha R_{\mu\nu} \nabla_\alpha R - \tfrac{1}{4} F^\alpha R_\nu{}^\beta \nabla_\alpha R_{\mu\beta} + \tfrac{1}{2} F^\alpha R \nabla_\alpha R_{\mu\nu} - \tfrac{1}{4} F^\alpha R_\mu{}^\beta \nabla_\alpha R_{\nu\beta} \\
& + \tfrac{1}{4} R R_{\nu\alpha} \nabla^\alpha F_\mu - \tfrac{1}{4} R_{\alpha\beta} R_\nu{}^\beta \nabla^\alpha F_\mu + \tfrac{1}{4} R R_{\mu\alpha} \nabla^\alpha F_\nu - \tfrac{1}{4} R_{\alpha\beta} R_\mu{}^\beta \nabla^\alpha F_\nu - \tfrac{5}{24} F_\nu R_{\mu\alpha} \nabla^\alpha R \\
& - \tfrac{1}{24} F_\mu R_{\nu\alpha} \nabla^\alpha R - \tfrac{1}{4} F_\nu R^{\alpha\beta} \nabla_\beta R_{\mu\alpha} + \tfrac{3}{4} F^\alpha R_\nu{}^\beta \nabla_\beta R_{\mu\alpha} - \tfrac{1}{4} F_\mu R^{\alpha\beta} \nabla_\beta R_{\nu\alpha} + \tfrac{3}{4} F^\alpha R_\mu{}^\beta \nabla_\beta R_{\nu\alpha} \\
& - \tfrac{1}{2} \nabla^\alpha F_\nu \nabla_\beta \nabla_\alpha R_\mu{}^\beta - \tfrac{1}{2} \nabla^\alpha F_\mu \nabla_\beta \nabla_\alpha R_\nu{}^\beta + \tfrac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta \nabla^\beta F_\alpha - \nabla_\alpha R_{\mu\nu} \nabla_\beta \nabla^\beta F^\alpha \\
& - \tfrac{1}{12} F^\alpha g_{\mu\nu} \nabla_\beta \nabla^\beta \nabla_\alpha R + \tfrac{1}{2} F^\alpha \nabla_\beta \nabla^\beta \nabla_\alpha R_{\mu\nu} + \tfrac{1}{2} R_{\nu\alpha} \nabla_\beta \nabla^\beta \nabla^\alpha F_\mu + \tfrac{1}{2} R_{\mu\alpha} \nabla_\beta \nabla^\beta \nabla^\alpha F_\nu \\
& - \tfrac{7}{18} g_{\mu\nu} R R_{\alpha\beta} \nabla^\beta F^\alpha + \tfrac{7}{6} g_{\mu\nu} R_\alpha{}^\gamma R_{\beta\gamma} \nabla^\beta F^\alpha - R_{\alpha\beta} R_{\mu\nu} \nabla^\beta F^\alpha + \tfrac{1}{2} R_{\mu\beta} R_{\nu\alpha} \nabla^\beta F^\alpha \\
& + \tfrac{1}{2} R_{\mu\alpha} R_{\nu\beta} \nabla^\beta F^\alpha - \nabla_\beta \nabla_\alpha R_{\mu\nu} \nabla^\beta F^\alpha - \tfrac{1}{2} R_{\nu\beta} \nabla^\beta \nabla_\alpha \nabla^\alpha F_\mu - \tfrac{1}{2} R_{\mu\beta} \nabla^\beta \nabla_\alpha \nabla^\alpha F_\nu \\
& - \tfrac{1}{2} F^\alpha g_{\mu\nu} R^{\beta\gamma} \nabla_\gamma R_{\alpha\beta} + \tfrac{1}{3} g_{\mu\nu} \nabla^\beta F^\alpha \nabla_\gamma \nabla_\beta R_\alpha{}^\gamma - \tfrac{3}{4} R R_{\nu\alpha} \nabla_\mu F^\alpha + \tfrac{9}{4} R_{\alpha\beta} R_\nu{}^\beta \nabla_\mu F^\alpha \\
& + \nabla_\beta \nabla^\beta R_{\nu\alpha} \nabla_\mu F^\alpha + \tfrac{1}{12} R^2 \nabla_\mu F_\nu - \tfrac{1}{4} R_{\alpha\beta} R^{\alpha\beta} \nabla_\mu F_\nu - \tfrac{1}{6} \nabla_\alpha \nabla^\alpha R \nabla_\mu F_\nu + \tfrac{7}{24} F_\nu R \nabla_\mu R \\
& - \tfrac{1}{2} F^\alpha R_{\nu\alpha} \nabla_\mu R - \tfrac{1}{12} \nabla_\alpha \nabla^\alpha F_\nu \nabla_\mu R - \tfrac{1}{2} F_\nu R^{\alpha\beta} \nabla_\mu R_{\alpha\beta} + \tfrac{3}{2} F^\alpha R_\nu{}^\beta \nabla_\mu R_{\alpha\beta} - \tfrac{3}{4} F^\alpha R \nabla_\mu R_{\nu\alpha} \\
& + \tfrac{1}{2} \nabla_\beta \nabla^\beta F^\alpha \nabla_\mu R_{\nu\alpha} + \tfrac{5}{4} F^\alpha R_\alpha{}^\beta \nabla_\mu R_{\nu\beta} + \tfrac{1}{6} \nabla^\alpha F_\nu \nabla_\mu \nabla_\alpha R - \tfrac{1}{24} F_\nu \nabla_\mu \nabla_\alpha \nabla^\alpha R \\
& + \tfrac{1}{2} \nabla^\beta F^\alpha \nabla_\mu \nabla_\beta R_{\nu\alpha} + \tfrac{1}{4} F^\alpha \nabla_\mu \nabla_\beta \nabla^\beta R_{\nu\alpha} - \tfrac{3}{4} R R_{\mu\alpha} \nabla_\nu F^\alpha + \tfrac{9}{4} R_{\alpha\beta} R_\mu{}^\beta \nabla_\nu F^\alpha \\
& + \nabla_\beta \nabla^\beta R_{\mu\alpha} \nabla_\nu F^\alpha - \tfrac{1}{3} \nabla_\mu \nabla_\alpha R \nabla_\nu F^\alpha + \tfrac{1}{12} R^2 \nabla_\nu F_\mu - \tfrac{1}{4} R_{\alpha\beta} R^{\alpha\beta} \nabla_\nu F_\mu - \tfrac{1}{6} \nabla_\alpha \nabla^\alpha R \nabla_\nu F_\mu \\
& + \tfrac{17}{72} F_\mu R \nabla_\nu R - \tfrac{1}{3} F^\alpha R_{\mu\alpha} \nabla_\nu R - \tfrac{1}{12} \nabla_\alpha \nabla^\alpha F_\mu \nabla_\nu R - \tfrac{1}{2} F_\mu R^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + \tfrac{3}{2} F^\alpha R_\mu{}^\beta \nabla_\nu R_{\alpha\beta} \\
& - \tfrac{3}{4} F^\alpha R \nabla_\nu R_{\mu\alpha} + \tfrac{1}{2} \nabla_\beta \nabla^\beta F^\alpha \nabla_\nu R_{\mu\alpha} + \tfrac{5}{4} F^\alpha R_\alpha{}^\beta \nabla_\nu R_{\mu\beta} + \tfrac{1}{6} \nabla^\alpha F_\mu \nabla_\nu \nabla_\alpha R \\
& - \tfrac{1}{3} \nabla_\mu F^\alpha \nabla_\nu \nabla_\alpha R - \tfrac{1}{24} F_\mu \nabla_\nu \nabla_\alpha \nabla^\alpha R + \tfrac{1}{2} \nabla^\beta F^\alpha \nabla_\nu \nabla_\beta R_{\mu\alpha} + \tfrac{1}{4} F^\alpha \nabla_\nu \nabla_\beta \nabla^\beta R_{\mu\alpha} \\
& - \tfrac{1}{3} F^\alpha \nabla_\nu \nabla_\mu \nabla_\alpha R
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
Z_{\mu\nu} = {} & \tfrac{2}{9} F_{\mu\nu} R^2 - F^{\alpha\beta} g_{\mu\nu} R R_{\alpha\beta} - F_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 2 F^{\alpha\beta} g_{\mu\nu} R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3} F_\nu{}^\alpha R R_{\mu\alpha} + 2 F_\nu{}^\alpha R_{\alpha\beta} R_\mu{}^\beta \\
& + \tfrac{2}{3} F^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + 2 F^{\alpha\beta} R_{\mu\alpha} R_{\nu\beta} - \tfrac{2}{3} F_{\mu\nu} \nabla_\alpha \nabla^\alpha R - \nabla_\alpha F_{\mu\nu} \nabla^\alpha R - 2 R^{\alpha\beta} \nabla_\beta \nabla_\alpha F_{\mu\nu} \\
& + \tfrac{1}{3} F^{\alpha\beta} g_{\mu\nu} \nabla_\beta \nabla_\alpha R - 2 F^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} + 2 R_\nu{}^\alpha \nabla_\beta \nabla^\beta F_{\mu\alpha} + 2 R_\mu{}^\alpha \nabla_\beta \nabla^\beta F_{\nu\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+2F_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha}+2F_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha}+\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F_{\mu\nu}+2\nabla_\alpha R_{\nu\beta}\nabla^\beta F_\mu{}^\alpha\\
&+2\nabla_\beta R_{\nu\alpha}\nabla^\beta F_\mu{}^\alpha+2\nabla_\alpha R_{\mu\beta}\nabla^\beta F_\nu{}^\alpha+2\nabla_\beta R_{\mu\alpha}\nabla^\beta F_\nu{}^\alpha-\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma F_{\alpha\beta}\\
&-\tfrac{2}{3}F^{\alpha\beta}g_{\mu\nu}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}-\tfrac{4}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma F^{\alpha\beta}-2\nabla_\beta R_{\nu\alpha}\nabla_\mu F^{\alpha\beta}+\tfrac{1}{3}\nabla^\alpha R\nabla_\mu F_{\nu\alpha}\\
&+2R^{\alpha\beta}\nabla_\mu\nabla_\beta F_{\nu\alpha}-2\nabla_\beta R_{\mu\alpha}\nabla_\nu F^{\alpha\beta}+\tfrac{2}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu F^{\alpha\beta}+\tfrac{1}{3}\nabla^\alpha R\nabla_\nu F_{\mu\alpha}+\tfrac{2}{3}\nabla_\mu F^{\alpha\beta}\nabla_\nu R_{\alpha\beta}\\
&+2R^{\alpha\beta}\nabla_\nu\nabla_\beta F_{\mu\alpha}-\tfrac{4}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu F_{\alpha\beta}+\tfrac{2}{3}F^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta}
\end{aligned}
\tag{2.4}
$$

When evaluated in an arbitrary conformal to flat geometry, we find $X_{\mu\nu}(F)$ and $Y_{\mu\nu}(F_\alpha)$ vanish (as to be expected by D.O.F. counting). Thus we are left with $\delta W_{\mu\nu}=Z_{\mu\nu}(F_{\alpha\beta})$:

$$
\begin{aligned}
\delta W_{\mu\nu} \;=\;& \tfrac{2}{9}F_{\mu\nu}R^2-F^{\alpha\beta}g_{\mu\nu}RR_{\alpha\beta}-F_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}+2F^{\alpha\beta}g_{\mu\nu}R_\alpha{}^\gamma R_{\beta\gamma}-\tfrac{2}{3}F_\nu{}^\alpha RR_{\mu\alpha}+2F_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta\\
&+\tfrac{2}{3}F^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}+2F^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta}-\tfrac{2}{3}F_{\mu\nu}\nabla_\alpha\nabla^\alpha R-\nabla_\alpha F_{\mu\nu}\nabla^\alpha R-2R^{\alpha\beta}\nabla_\beta\nabla_\alpha F_{\mu\nu}\\
&+\tfrac{1}{3}F^{\alpha\beta}g_{\mu\nu}\nabla_\beta\nabla_\alpha R-2F^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu}+2R_\nu{}^\alpha\nabla_\beta\nabla^\beta F_{\mu\alpha}+2R_\mu{}^\alpha\nabla_\beta\nabla^\beta F_{\nu\alpha}\\
&+2F_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha}+2F_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha}+\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F_{\mu\nu}+2\nabla_\alpha R_{\nu\beta}\nabla^\beta F_\mu{}^\alpha\\
&+2\nabla_\beta R_{\nu\alpha}\nabla^\beta F_\mu{}^\alpha+2\nabla_\alpha R_{\mu\beta}\nabla^\beta F_\nu{}^\alpha+2\nabla_\beta R_{\mu\alpha}\nabla^\beta F_\nu{}^\alpha-\tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma F_{\alpha\beta}\\
&-\tfrac{2}{3}F^{\alpha\beta}g_{\mu\nu}\nabla_\gamma\nabla^\gamma R_{\alpha\beta}-\tfrac{4}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma F^{\alpha\beta}-2\nabla_\beta R_{\nu\alpha}\nabla_\mu F^{\alpha\beta}+\tfrac{1}{3}\nabla^\alpha R\nabla_\mu F_{\nu\alpha}\\
&+2R^{\alpha\beta}\nabla_\mu\nabla_\beta F_{\nu\alpha}-2\nabla_\beta R_{\mu\alpha}\nabla_\nu F^{\alpha\beta}+\tfrac{2}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu F^{\alpha\beta}+\tfrac{1}{3}\nabla^\alpha R\nabla_\nu F_{\mu\alpha}+\tfrac{2}{3}\nabla_\mu F^{\alpha\beta}\nabla_\nu R_{\alpha\beta}\\
&+2R^{\alpha\beta}\nabla_\nu\nabla_\beta F_{\mu\alpha}-\tfrac{4}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu F_{\alpha\beta}+\tfrac{2}{3}F^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta}.
\end{aligned}
\tag{2.5}
$$

Consequently, we may express $F_{\mu\nu}$ as a function of $h_{\mu\nu}$ by equating (2.5) to (1.8).

If the background is maximally symmetric then (2.5) becomes

$$
\delta W_{\mu\nu} = \tfrac{1}{18}F_{\mu\nu}R^2+\tfrac{1}{2}R\nabla_\alpha\nabla^\alpha F_{\mu\nu}+\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F_{\mu\nu} \tag{2.6}
$$

$$
= \left(\nabla_\alpha\nabla^\alpha+\frac{R}{6}\right)\left(\nabla_\beta\nabla^\beta+\frac{R}{3}\right)F_{\mu\nu}. \tag{2.7}
$$

3 Comments

Equation (2.5) is the lowest derivative order relation between $F_{\mu\nu}$ and $h_{\mu\nu}$ (without integrals) in a conformal flat background. Equation (2.5) is also gauge invariant, with the Bach tensor vanishing in a conformal flat background.

In the limit of maximal symmetry, we recover the most reduced relation between $F_{\mu\nu}$ and $h_{\mu\nu}$, one that we had previously determined by applying appropriate derivatives to the $h^{T\theta}_{\mu\nu}$ integral decomposition.