Electrodynamics I
HW 6

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Due: April 30

1. A straight long wire and a ring of radius a lie in the same plane. The distance between the wire and the ring center is b. Find the mutual inductance L_{12} and interaction force $\mathbf{F}_{12}$, if the wire and ring currents are respectively I_1 and I_2.

   Ran out of time this week..

2. Determine the trajectory of an electron moving in the uniform electric $\mathbf{E} = E_y\hat{\mathbf{e}}_y + E_z\hat{\mathbf{e}}_z$ and magnetic $\mathbf{B} = B\hat{\mathbf{e}}_z$ fields. Initial conditions at $t = 0$: the electron radius vector $\mathbf{r} = 0$, and the electron velocity $\mathbf{v}(t = 0)$ is $\mathbf{v}_0 = v_{0x}\hat{\mathbf{e}}_x + v_{0z}\hat{\mathbf{e}}_z$.

   We can use the Lorentz force law to determine the trajectory of the electron. The law goes as

$$m\mathbf{a} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}).$$

   Taking the cross product

$$\mathbf{v} \times \mathbf{B} = v_y B\hat{\mathbf{x}} - v_x B\hat{\mathbf{y}}.$$

   Separating the components we have three differential equations

$$\ddot{x} = \frac{q}{m}\dot{y}B$$

$$\ddot{y} = \frac{q}{m}(E_y - \dot{x}B)$$

$$\ddot{z} = \frac{q}{m}E_z.$$

   The last equation is easily solved as

$$z(t) = \frac{q}{2m}E_z t^2 + v_z(0)t + z(0)$$

$$z(t) = \frac{E_z q}{2m}t^2 + v_{0z}t$$

   The other two equations are coupled together, but we can use a substitution to solve them. Lets set

$$u_1 = x + iy, \qquad u_2 = x - iy.$$

In terms of these new variables, our first differential equation for $\ddot{x}$ becomes

$$\ddot{u_1} + \ddot{u_2} = \frac{qB}{m} i(\dot{u_2} - \dot{u_1})$$

while the second differential equation for $\ddot{y}$ becomes

$$i(\ddot{u_2} - \ddot{u_1}) = \frac{2q}{m} E_y - \frac{qB}{m}(\dot{u_1} + \dot{u_2}).$$

Before we combine the equations, lets first multiply the $\ddot{x}$ equation by a factor of i. Our two equations are then

$$i(\ddot{u_1} + \ddot{u_2}) = \frac{qB}{m}(\dot{u_1} - \dot{u_2})$$

$$i(\ddot{u_2} - \ddot{u_1}) = \frac{2q}{m} E_y - \frac{qB}{m}(\dot{u_1} + \dot{u_2}).$$

Now we may add these equations together to get a differential equation of only one variable

$$2i\ddot{u_2} = \frac{2q}{m} E_y + \frac{2qB}{m}(-\dot{u_2}).$$

Simplifying

$$\ddot{u_2} = \frac{q}{m}(-iE_y + iB\dot{u_2}).$$

To make things easier, lets set $-\frac{q}{m}E_y \equiv a$ and $B \equiv b$ so that we have

$$\ddot{u_2} - ib\dot{u_2} = -ia.$$

As a inhomogeneous differential equation, we shall first find the homogeneous solution and then find the particular. For the equation

$$\ddot{u_2} - ib\dot{u_2} = 0$$

the solution is easily solved to be

$$u_2(t) = Ae^{ibt}.$$

Now in order to satisfy our inhomogeneous solution, we see that the differential equation is satisfied for the solution

$$u_2(t) = Ae^{ibt} - iat.$$

In terms of x and y we then have

$$u_2(t) = x(t) - iy(t) = \cos(bt) + i\sin(bt) - iat.$$

Equating real and imaginary parts,

$$x(t) = A\cos(bt)$$

$$y(t) = -A\sin(bt) - at$$

In terms of the original constants

$$x(t) = A\cos(Bt)$$

$$y(t) = -A\sin(Bt) + \frac{q}{m}E_y t$$

I think something went wrong here, because the initial conditions cannot be satisfied as it is written. It seems that it should be as $x \propto \sin(Bt)$ and $y \propto \cos(Bt)$ but I am not sure where the mistake lies