Electrodynamics II
HW 4

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1. A particle moving with velocity V dissociates "in flight" into two particles. Determine the relation between the angles of emergence of these particles and their energies.

   Define θ as the angle of emergence in the Lab Frame, and $\mathscr{E}_0$ energy in c.o.m. frame and $\mathscr{E}$ as lab frame energy. Then ($c = 1$ units)
$$\mathscr{E}_0 = \frac{\mathscr{E} - Vp\cos\theta}{\sqrt{1-V^2}}$$
$$\Rightarrow \cos\theta = \frac{\mathscr{E} - \mathscr{E}_0\sqrt{1-V^2}}{V\sqrt{\mathscr{E}^2 - m^2}} = \frac{\mathscr{E} - \mathscr{E}_0\sqrt{1-V^2}}{Vp}$$
   For $m_1 = m_2$ then $\theta_1 = \theta_2$ in the L.F. In c.o.m. frame, particles are separated by π (for $m_1 = m_2$).
   Taking our original equation
$$\mathscr{E}^2(1 - V^2\cos^2\theta) - 2\mathscr{E}\mathscr{E}_0\sqrt{1-v^2} + \mathscr{E}_0^2(1-V^2) + V^2m^2\cos^2\theta = 0$$
   solving for $\mathscr{E}$
$$\mathscr{E} = \frac{2\mathscr{E}_0\sqrt{1-V^2} \pm 2V\cos\theta\sqrt{\mathscr{E}_0^2(1-V^2) - m^2(1-V^2\cos^2\theta)}}{2(1-V^2\cos^2\theta)}.$$

2. For the collision of two particles of equal mass m, express $\mathscr{E}_1'$, $\mathscr{E}'$, χ in terms of the angle θ_1 of scattering in the L-system.
$$p_1p_1'\cos\theta_1 = \mathscr{E}_1'(\mathscr{E}_1 + m) - \mathscr{E}_1 m - m^2.$$
   Use $c = 1$ units. Now use the energy momentum relation
$$p_i^2 = \mathscr{E}_i^2 - m_i^2$$
$$\Rightarrow (\mathscr{E}_1^2 - m_1^2)(\mathscr{E}_1'^2 - m^2)\cos^2\theta_1 = (\mathscr{E}_1 + m)^2(\mathscr{E}_1' - m)^2$$
$$\Rightarrow \mathscr{E}_1'\cos^2\theta_1(\mathscr{E}_1 - m) + m_1\cos^2\theta_1(\mathscr{E}_1 - m) = \mathscr{E}_1'(\mathscr{E}_1 + m) - m(\mathscr{E}_1 + m)$$
$$\Rightarrow \mathscr{E}_1' = \frac{m_1\big((\mathscr{E}_1 + m) + (\mathscr{E}_1 - m)\cos^2\theta\big)}{(\mathscr{E}_1 + m) - (\mathscr{E}_1 - m)\cos^2\theta_1}.$$
   Using energy conservation
$$\mathscr{E}_2' = \mathscr{E}_1 + m - \mathscr{E}_1' = \mathscr{E}_1 + m - m\frac{\big((\mathscr{E}_1 + m_1) + (\mathscr{E}_1 - m)\cos^2\theta_1\big)}{(\mathscr{E}_1 + m) - (\mathscr{E}_1 - m)\cos^2\theta_1}$$

$$= m + \left[\frac{\mathscr{E}_1(2m) + \mathscr{E}_1(\mathscr{E}_1 - m)\sin^2\theta_1 - 2m\mathscr{E}_1 + m(\mathscr{E}_1 - m)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}\right]$$

$$\mathscr{E}_2 = m + \left[\frac{(\mathscr{E}_1^2 - m^2)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}\right]$$

Lastly, for χ

$$\mathscr{E}_1' = \mathscr{E}_1 - \frac{(\mathscr{E}_1 - m)}{2}(1 - \cos\chi)$$

$$\Rightarrow \mathscr{E}_1' - \mathscr{E}_1 - m = -m - \frac{(\mathscr{E}_1 - m)}{2}(1 - \cos\chi).$$

Now use

$$\mathscr{E}_1' - \mathscr{E}_1 - m = -\mathscr{E}_0'$$

to arrive at

$$\mathscr{E}_2' = m + \frac{(\mathscr{E}_1 - m)}{2}(1 - \cos\chi).$$

$$\Rightarrow 2(\mathscr{E}_2' - m) = \frac{2(\mathscr{E}_1^2 - m^2)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1} = (\mathscr{E}_1 - m)(1 - \cos\chi)$$

Solve for χ

$$\cos\chi = 1 - \left[\frac{2(\mathscr{E}_1 + m)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}\right]$$

$$\Rightarrow \cos\chi = \frac{2m - (\mathscr{E}_1 + 3m)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}$$

3. Express the acceleration of a particle in terms of its velocity and the electric and magnetic field intensities.

Given the Lorentz force

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}(\mathbf{v} \times \mathbf{H})$$

substitute relation for $\mathbf{p}$

$$\mathbf{p} = \frac{\mathbf{v}\mathscr{E}_k}{c^2}, \qquad \mathscr{E}_k = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

then

$$\begin{aligned}\Rightarrow \frac{d\mathbf{p}}{dt} &= \dot{v}\frac{\mathscr{E}_k}{c^2} + \frac{v\dot{\mathscr{E}}_k}{c^2} \\ &= \dot{v}\left(\frac{\mathscr{E}_k}{c^2}\right) + \frac{v}{c^2}e\mathbf{E}\cdot\mathbf{v} \\ &= \dot{v}\left(\frac{\mathscr{E}_k}{c^2}\right) + \frac{e}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\end{aligned}$$

where $\dot{\mathscr{E}}_k = e\mathbf{E}\cdot\mathbf{v}$. Thus
we have

$$\dot{v} = \frac{e}{m}\sqrt{1 - \left(\frac{v}{c}\right)^2}\left(\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H}) - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right).$$

$$\begin{aligned}\dot{v}\left(\frac{\mathscr{E}_k}{c^2}\right) &= c\mathbf{E} + \frac{e}{c}(\mathbf{v} \times \mathbf{H}) - \frac{e}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}) \\ &= \frac{ec^2}{\mathscr{E}_k}\left(\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H}) - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right)\end{aligned}$$

Now using the energy relation

$$\mathscr{E}_k = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

we have

$$\dot{v} = \frac{e}{m}\sqrt{1-(v/c)^2}\left(\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{H}) - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right)$$

4. Determine the relativistic motion of a charge in electric and magnetic fields which are mutually perpendicular and equal in magnitude.

Using the lorentz force

$$\frac{dp}{dt} = e\mathbf{E} + \frac{e}{c}(\mathbf{v}\times\mathbf{H})$$

we have

$$\dot{p}_x = \frac{e}{c}Ev_y; \qquad \dot{p}_y = eE(1-\frac{v_x}{c}); \qquad \dot{p}_z = 0.$$

Now we use

$$\dot{\mathscr{E}}_k = e\mathbf{E}\cdot\mathbf{v}, \qquad \mathscr{E}_k = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

so

$$\dot{\mathscr{E}}_k = eEv_y, \qquad \dot{p}_x = \frac{\dot{\mathscr{E}}_k}{c} \Rightarrow \mathscr{E}_k - cp_x = const = \gamma$$

From this

$$\begin{aligned}\mathscr{E}_k^2 &= m^2c^4 + c^2(p_x^2+p_y^2+p_z^2)\\ &= c^2p_x^2 + c^2p_y^2 + \mathscr{E}^2\end{aligned}$$

where we have used

$$\mathscr{E}^2 = m^2c^4 + c^2p_z^2 = const.$$

We may rewrite this as

$$\begin{aligned}\mathscr{E}_k^2 - c^2p_x^2 &= c^2p_y^2 + \mathscr{E}^2\\ (\mathscr{E}_k - cp_x)(\mathscr{E}_k + cp_x) &= c^2p_y^2 + \mathscr{E}^2\\ \gamma(\mathscr{E}_k + cp_x) &= c^2p_y^2 + \mathscr{E}^2\end{aligned}$$

thus

$$\mathscr{E}_k + cp_x = \frac{1}{\gamma}(c^2p_y^2 + \mathscr{E}^2)$$

Forging onward, we use the relations

$$\mathscr{E}_k = \frac{\gamma}{2} + \frac{c^2p_y^2 + \mathscr{E}^2}{2\gamma}$$

$$p_x = -\frac{\gamma}{2c} + \frac{c^2p_y^2 + \mathscr{E}^2}{2\alpha c}$$

$$\dot{p}_y = eE(1 - v_x/c) \Rightarrow eE(\mathscr{E}_k - cp_x) = eE\gamma$$

and we integrate over time

$$eE\gamma\int dt = \int dp_y\ \left(\frac{\gamma}{2} + \frac{\mathscr{E}^2}{2\gamma} + \frac{c^2p_y^2}{2\gamma}\right)$$

$$\Rightarrow eE\gamma t = p_y\left(\frac{\gamma}{2} + \frac{\mathscr{E}^2}{2\gamma}\right) + \frac{c^2 p_y^3}{6\gamma}$$

$$\Rightarrow 2eE\gamma t = p_y\left(1 + \frac{\mathscr{E}^2}{\gamma^2}\right) + \frac{c^2 p_y^3}{3\alpha^2}.$$

Transforming our variables

$$\frac{dx}{dt} = \frac{c^2 p_x}{\mathscr{E}_k} \Rightarrow dt = \frac{\mathscr{E}_k \; dp_y}{eE\gamma}$$

$$\begin{aligned}\Rightarrow dx &= \frac{c^2 p_x}{\mathscr{E}_k} dt \\ &= \frac{c^2}{eE\gamma}\left(-\frac{\gamma}{2c} + \frac{\mathscr{E}^2}{2\gamma c} + \frac{c^2 p_y^2}{2\gamma c}\right) dp_y\end{aligned}$$

Now integrate

$$x = \frac{c^2}{eEx}\left(\frac{E^2}{2\gamma c} - \frac{\gamma}{2c}\right) p_y + \frac{c^2 p_y^3 c^2}{6\gamma c E e\gamma}$$

$$x = \frac{c}{2eE}\left(\frac{\mathscr{E}^2}{\gamma^2} - 1\right) p_y + \frac{c^3 p_y^3}{6\alpha^2 eE}.$$

Now we repeat the same procedure to find both y and z. Starting with

$$\frac{dy}{dt} = \frac{c^2 p_y}{\mathscr{E}_k} \Rightarrow dy = \frac{c^2}{\mathscr{E}_k} p_y \frac{\mathscr{E}_k}{eE\gamma} dp_y \Rightarrow dt = \frac{\mathscr{E}_k \; dp_y}{eEx}$$

this leads to

$$y = \frac{c^2 p_y^2}{2\gamma eE}.$$

As for the z component

$$\frac{dz}{dt} = \frac{c^2 p_z}{\mathscr{E}_k} \Rightarrow dz = \frac{p_z c^2}{\mathscr{E}_k} \frac{\mathscr{E}_k dp_y}{eE\gamma}$$

which leads to

$$z = \frac{p_z c^2 p_y}{eE\gamma}.$$

5. Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2 / c)}$$

This is an alternative way to derive the parallel-velocity addition law.

First boost to S'

$$x_0' = \gamma_1(x_0 - \beta_1 x_1), \qquad x_1' = \gamma_1(x_1 - \beta_1 x_0)$$

where we have

$$\gamma_1 = \frac{1}{\sqrt{1 - (v_1/c)^2}}, \quad \beta_1 = v_1/c$$

and likewise for a boost to S"

$$x_0'' = \gamma_2(x_0' - \beta_2 x_1'), \qquad x_1'' = \gamma_2(x_1' - \beta_2 x_0')$$

where again

$$\gamma_2 = \frac{1}{\sqrt{1-(v_2/c)^2}}, \quad \beta_2 = v_2/c.$$

Substitute these into each other

$$x_0'' = \gamma_2\gamma_1\left((1+\beta_2\beta_1)x_0 - (\beta_1+\beta_2)x_1\right)$$

$$x_1'' = \gamma_2\gamma_1\left((1+\beta_2\beta_1)x_1 - (\beta_1+\beta_2)x_0\right).$$

The transformation to original frame S goes as

$$x_0'' = \gamma(x_0 - \beta x_1)$$

$$x_1'' = \gamma(x_1 - \beta x_0)$$

where

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}, \quad \beta = v/c.$$

Matching the coefficients from the equations above

$$\gamma_2\gamma_1(1+\beta_2\beta_1) = \gamma$$

$$\Rightarrow \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(v_2/c)^2}}\frac{1}{\sqrt{1-(v_1/c)^2}}\left(1+\frac{v_2v_1}{c^2}\right).$$

Now solve for v

$$v = \sqrt{c^2 - \frac{(1-(v_2/c)^2(1-(v_1/c)^2)}{(1+v_2v_1/c^2)^2}}$$

$$\Rightarrow v = \frac{v_1+v_2}{1+(v_1v_2/c^2)}$$

6. A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\mathbf{a}_{||} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{||}$$

$$\mathbf{a}_\perp = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2}\times(\mathbf{a}'\times\mathbf{u}')\right)$$

Let us perform a boost in the x direction

$$x^0 = \gamma(x'^0 + \beta x'), \qquad x = \gamma(x' + \beta x'^0).$$

Then for the K frame

$$\mathbf{u} = c\frac{\partial \mathbf{x}}{\partial x^0}, \qquad \mathbf{a} = c\frac{\partial \mathbf{u}}{\partial x^0}$$

and for the K' frame

$$\mathbf{u}' = c\frac{\partial \mathbf{x}'}{\partial x'^0}, \qquad \mathbf{a}' = c\frac{\partial \mathbf{u}'}{\partial x'^0}.$$

It follows that

$$\frac{dx^0}{dx'^0} = \gamma(1 + \beta u'_x/c)$$

and

$$\frac{dx'^0}{dx} = \frac{1}{\gamma(1 + \beta u'_x/c)}$$

and so

$$\begin{aligned} u_x &= c\frac{\partial x}{\partial x^0} \\ &= c\frac{dx'^0}{dx^0}\frac{dx}{dx'^0} \\ &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx'^0}\gamma(x' + \beta x'^0) \\ &= \frac{u'_x + c\beta}{1 + \beta u'_x/c}. \end{aligned}$$

As for the y component

$$\begin{aligned} u_y &= c\frac{dy}{dx^0} \\ &= c\frac{dx'^0}{dx^0}\frac{dy}{dx'^0} \\ &= \frac{c}{\gamma(1+\beta u'_x/c}\left(\frac{u'_y}{c}\right) \\ &= \frac{u'_y}{\gamma(1+\beta u'_x/c)}. \end{aligned}$$

Now from using

$$\beta u'_x = \beta \cdot \mathbf{u}'$$

and that $\mathbf{x}$ and $\mathbf{v}$ are parallel (and perpendicular to $\mathbf{y}$) we have

$$\mathbf{u}_{||} = \frac{\mathbf{u}'_{||} + c\beta}{1 + \beta \cdot \mathbf{u}'/c}$$

$$\mathbf{u}_{\perp} = \frac{\mathbf{u}'_{\perp}}{\gamma(1 + \beta \cdot \mathbf{u}'/c)}.$$

To find the accelerations

$$\begin{aligned} a_x &= c\frac{du_x}{dx^0} \\ &= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx'^0}\frac{u'_x + c\beta}{1+\beta u'_x/c} \\ &= \frac{(1-\beta^2)a'_x}{\gamma(1+\beta u'_x/c)^3} \\ &= \frac{a'_x}{\gamma^3(1+\beta u'_x/c)^3} \end{aligned}$$

and similarly for a_y we have

$$a_y = \frac{a'_y + \beta(u'_x a'_y - u'_y a'_x)/c}{\gamma^2(1+\beta u'_x/c)^3}$$

Finally

$$a_{||} = a_x$$

so

$$a_{||} = \frac{\mathbf{a}'_{||}}{\gamma^3(1+\beta\cdot\mathbf{u}'/c)^3}.$$

For the perpendicular component

$$a_\perp = \frac{a'_\perp + \mathbf{a}'(\beta\cdot\mathbf{u}') - u'(\beta\cdot\mathbf{a}')/c}{\gamma^2(1+\beta\cdot\mathbf{u}'/c)^3}$$

and so

$$a_\perp = \frac{a'_\perp + \beta\times(\mathbf{a}'\times\mathbf{u}')/c}{\gamma^2(1+\beta\cdot\mathbf{u}'/c)^3}$$

7. A particle of mass M and 4-momentum P decays into two particles of masses m_1 and m_2.

   (a) Use the conservation of energy and momentum in the form, $p_2 = P - p_1$, and the invariance of scalar products of 4-vectors to show that the total energy of the first particle in the rest frame of the decaying particle is

   $$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

   and that E_2 is obtained by interchanging m_1 and m_2.

   (b) Show that the *kinetic energy* T_i of the ith particle in the same frame is

   $$T_i = \Delta M\left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)$$

   where $\Delta M = M - m_1 - m_2$ is the mass excess or Q value of the process.

   (c) The charged pi-meson (M = 139.6MeV) decays into a mu-meson (m_1 = 105.7 MeV) and a neutrino (m_2 = 0). Calculate the kinetic energies of the mu-meson and the neutrino in the pi-meson's rest frame. The unique kinetic energy of the muon is the signature of a two-body decay. It entered importantly in the discovery of the pi-meson in photographic emulsions by Powell and his coworkers in 1947.

   (a) In the center of mass frame

   $$E = E_1 + E_2 = Mc^2$$

   and

   $$\mathbf{p} - \mathbf{p}_1 = \mathbf{p}_2 \Rightarrow \mathbf{p}_2 = -\mathbf{p}_1.$$

   Thus

   $$(E_1, \mathbf{p}_1) + (E_2, \mathbf{p}_2) = (Mc^2 - E_1, -\mathbf{p}_1)$$

   and using the energy momentum relation

   $$E_1^2 - E_2^2 = c^4(m_1^2 - m_2^2)$$
   $$(E_1 + E_2)(E_1 - E_2) = c^4(m_1^2 - m_2^2)$$

   $$\Rightarrow E_1 - E_2 = c^4\frac{(m_1^2 - m_2^2)}{Mc^2}$$
   $$\Rightarrow E_1 + E_2 = Mc^2$$

   Add these last two equations together

   $$E_1 = \frac{c^2(m_1^2 - m_2^2 + M^2)}{2M}$$

or subtract the two

$$E_2 = \frac{c^2(M^2 + m_2^2 - m_1^2)}{2M}.$$

(b) Employing the $c = 1$ convenient units

$$\begin{aligned}
T_1 &= E_1 - m_1 \\
&= \frac{m_1^2 - m_2^2 - M^2 - 2m_1 M}{2M} \\
&= \frac{M^2 - m_1^2 - m_2^2 - 2m_1 M}{2m} \\
&= \frac{\Delta M}{2m}(M - m_1 - m_2) \\
&= \frac{\Delta M}{2M}\left(M - m_1 + m_2 - \Delta M + M - m_1 - m_2\right) \\
&= \frac{\Delta M}{2M}(2M - 2m_1 - \Delta M) \\
&= \Delta M \left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right)
\end{aligned}$$

thus

$$T_i = \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right).$$

(c) From part (a)

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} = 109.8\ (MeV)$$

and so for the mu-meson

$$T_1 = E_1 - m_1 = 4.1\ (MeV).$$

As for the pi-meson

$$E_1 = 29.8\ (MeV)$$

and so again

$$T_2 = E_2 - m_2 = 29.8\ (MeV).$$