SVT3 RW Radiation, Conformal Flat $k = 0$

1 Four Velocity: Conformal Flat $\Omega(\tau)$

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \qquad \frac{dx^i}{dt} = 0 \tag{1.1}$$

$$d\tau^2 = -ds^2 = dt^2\left[1 - a^2(t)\left(\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right)\right] = dt^2 \tag{1.2}$$

$$U^\mu = \frac{dx^\mu}{d\tau} = \delta^\mu_0 \tag{1.3}$$

$$dp = \frac{dt}{a(t)}, \qquad p = \int_{t_0}^{t} \frac{dt}{a(t)} \tag{1.4}$$

$$ds^2 = a^2(p)\left(-dp^2 + dx^2 + dy^2 + dz^2\right), \qquad \frac{dx^i}{dp} = \frac{dx^i}{dt}\frac{dt}{dp} = 0 \tag{1.5}$$

$$d\tau^2 = a^2(p)dp^2\left[1 - \left(\frac{dx}{dp}\right)^2 - \left(\frac{dy}{dp}\right)^2 - \left(\frac{dz}{dp}\right)^2\right] = a^2(p)dp^2 \tag{1.6}$$

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dp}\frac{dp}{d\tau} = \frac{1}{a(p)}\delta^\mu_0 \tag{1.7}$$

$$U_\mu = -a(p)\delta^0_\mu \tag{1.8}$$

2 Background: Conformal Flat $\Omega(\tau)$

$$
\begin{aligned}
ds^2 &= \Omega^2(\tau)(-d\tau^2 + \tilde{g}_{ij}dx^i dx^j + f_{\mu\nu}dx^\mu dx^\nu) && (2.1)\\
\tilde{g}_{ij} &= \mathrm{diag}(1,1,1) \quad \text{or} \quad \mathrm{diag}(1, r^2, r^2\sin^2\theta) && (2.2)\\
G^{(0)}_{00} &= -3\frac{\dot{\Omega}^2}{\Omega^2}, \qquad G^{(0)}_{0i} = 0, \qquad G^{(0)}_{ij} = \tilde{g}_{ij}\left[2\frac{\ddot{\Omega}}{\Omega} - \frac{\dot{\Omega}^2}{\Omega^2}\right] && (2.3)\\
\kappa_4^2 T^{(0)}_{\mu\nu} &= p(4U_\mu U_\nu + \Omega^2\tilde{g}_{\mu\nu}) \qquad \text{[Evaluated in (2.1) coordinates, } U^\mu = \Omega^{-1}\delta^\mu_0\text{]} && (2.4)\\
\Delta^{(0)}_{\mu\nu} &\equiv G^{(0)}_{\mu\nu} + \kappa_4^2 T^{(0)}_{\mu\nu} = 0 && (2.5)\\
\Delta^{(0)}_{00} &= -3\frac{\dot{\Omega}^2}{\Omega^2} + 3\Omega^2 p \quad \Longrightarrow \quad \boxed{p = \frac{\dot{\Omega}^2}{\Omega^4}} && (2.6)\\
\Delta^{(0)}_{ij} &= 2\tilde{g}_{ij}\frac{\ddot{\Omega}}{\Omega} - \frac{\dot{\Omega}^2}{\Omega^2} + \Omega^2 p && (2.7)\\
\Delta^{(0)} &= \Omega^{-2}(-\Delta^{(0)}_{00} + \tilde{g}^{ab}\Delta^{(0)}_{ab}) = \Omega^{-2}(-G^{(0)}_{00} + \tilde{g}^{ab}G^{(0)}_{ab}) \\
&= 6\frac{\ddot{\Omega}}{\Omega^3}, \quad \Longrightarrow \quad \ddot{\Omega} = 0, \quad \Longrightarrow \quad \boxed{\Omega = A + B\tau} \quad \Longrightarrow \quad \boxed{p = \frac{B^2}{(A+B\tau)^4}} && (2.8)
\end{aligned}
$$

3 Fluctuations: Conformal Flat $\Omega(\tau) = \tau/2$

For $\Omega = \tau/2$, we have from (2.8) $A = 0$, $B = 1/2$ such that $p = 4\tau^{-4}$.

$$ds^2 = \Omega^2(\tau)(-d\tau^2 + \tilde{g}_{ij}dx^i dx^j + f_{\mu\nu}dx^\mu dx^\nu) \tag{3.1}$$

$$\tilde{g}_{ij} = \mathrm{diag}(1,1,1) \quad \text{or} \quad \mathrm{diag}(1, r^2, r^2\sin^2\theta) \tag{3.2}$$

$$f_{00} = -2\phi, \qquad f_{0i} = \tilde{\nabla}_i B + B_i, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{3.3}$$

$$\delta U_i = V_i + \tilde{\nabla}_i V, \qquad \tilde{g}^{ij}\tilde{\nabla}_i V_j = 0 \tag{3.4}$$

$$\kappa_4^2 \delta T_{\mu\nu} = \delta p(4U_\mu U_\nu + \Omega^2\tilde{g}_{\mu\nu}) + 4\tau^{-4}\left(4\delta U_\mu U_\nu + 4U_\mu\delta U_\nu + \Omega^2 f_{\mu\nu}\right), \qquad U^\mu = \Omega^{-1}\delta_0^\mu \tag{3.5}$$

$$\kappa_4^2\delta T_{00} = -16\delta U_0\tau^{-3} + \tfrac{3}{4}\delta p\tau^2 - 2\tau^{-2}\phi \tag{3.6}$$

$$\kappa_4^2\delta T_{0i} = \tau^{-2}\tilde{\nabla}_i B - 8\tau^{-3}\tilde{\nabla}_i V - 8V_i\tau^{-3} + B_i\tau^{-2} \tag{3.7}$$

$$\kappa_4^2\delta T_{ij} = \tfrac{1}{4}\tilde{g}_{ij}\delta p\tau^2 - 2\tilde{g}_{ij}\tau^{-2}\psi + 2\tau^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + \tau^{-2}\tilde{\nabla}_i E_j + \tau^{-2}\tilde{\nabla}_j E_i + 2E_{ij}\tau^{-2} \tag{3.8}$$

$$\Omega^{-2}(\tilde{g}^{ab}\kappa_4^2\delta T_{ab}) = 3\delta p - 24\tau^{-4}\psi + 8\tau^{-4}\tilde{\nabla}_a\tilde{\nabla}^a E \tag{3.9}$$

$$\kappa_4^2 g^{\mu\nu}\delta T_{\mu\nu} = 64\delta U_0\tau^{-5} + 8\tau^{-4}\phi - 24\tau^{-4}\psi + 8\tau^{-4}\tilde{\nabla}_s\tilde{\nabla}^s E \tag{3.10}$$

$$\delta G_{00} = 6\dot{\psi}\tau^{-1} + 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi \tag{3.11}$$

$$\delta G_{0i} = -\tau^{-2}\tilde{\nabla}_i B - 2\tilde{\nabla}_i\dot{\psi} - 2\tau^{-1}\tilde{\nabla}_i\phi - B_i\tau^{-2} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \tag{3.12}$$

$$\begin{aligned}
\delta G_{ij} = {} & -2\ddot{\psi}\tilde{g}_{ij} - 2\dot{\phi}\tilde{g}_{ij}\tau^{-1} - 4\dot{\psi}\tilde{g}_{ij}\tau^{-1} + 2\tilde{g}_{ij}\tau^{-2}\phi + 2\tilde{g}_{ij}\tau^{-2}\psi - 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B \\
& -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi \\
& +2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} - 2\tau^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
& +\tau^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \tau^{-1}\tilde{\nabla}_i\dot{E}_j - \tau^{-2}\tilde{\nabla}_i E_j + \tau^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i \\
& -\tau^{-1}\tilde{\nabla}_j\dot{E}_i - \tau^{-2}\tilde{\nabla}_j E_i - \ddot{E}_{ij} - 2E_{ij}\tau^{-2} - 2\dot{E}_{ij}\tau^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\Omega^{-2}\tilde{g}^{ab}\delta G_{ab} = {} & -24\dot{\phi}\tau^{-3} - 48\dot{\psi}\tau^{-3} - 24\ddot{\psi}\tau^{-2} + 24\tau^{-4}\phi + 24\tau^{-4}\psi - 16\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} \\
& +8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 16\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 8\tau^{-4}\tilde{\nabla}_a\tilde{\nabla}^a E - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
g^{\mu\nu}\delta G_{\mu\nu} = {} & \Omega^{-2}(-\delta G_{00} + \tilde{g}^{ab}\delta G_{ab}) \\
= {} & -24\dot{\phi}\tau^{-3} - 72\dot{\psi}\tau^{-3} - 24\ddot{\psi}\tau^{-2} + 24\tau^{-4}\phi + 24\tau^{-4}\psi - 24\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} \\
& +8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 24\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 8\tau^{-4}\tilde{\nabla}_a\tilde{\nabla}^a E - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 16\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi
\end{aligned} \tag{3.15}$$

4 Field Equations: Conformal Flat $\Omega(\tau) = \tau/2$

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} + \kappa_4^2 \delta T_{\mu\nu} \tag{4.1}$$

$$\Delta_{00} = -16\delta U_0 \tau^{-3} + 6\dot{\psi}\tau^{-1} + \tfrac{3}{4}\delta p \tau^2 - 2\tau^{-2}\phi + 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a \dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi \tag{4.2}$$

$$\Delta_{0i} = -2\tilde{\nabla}_i\dot{\psi} - 8\tau^{-3}\tilde{\nabla}_i V - 2\tau^{-1}\tilde{\nabla}_i\phi - 8V_i\tau^{-3} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a \dot{E}_i \tag{4.3}$$

$$\begin{aligned}\Delta_{ij} = {} & -2\ddot{\psi}\tilde{g}_{ij} - 2\dot{\phi}\tilde{g}_{ij}\tau^{-1} - 4\dot{\psi}\tilde{g}_{ij}\tau^{-1} + \tfrac{1}{4}\tilde{g}_{ij}\delta p\tau^2 + 2\tilde{g}_{ij}\tau^{-2}\phi - 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B \\ & -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi \\ & +2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\ & +\tau^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \tau^{-1}\tilde{\nabla}_i\dot{E}_j + \tau^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \tau^{-1}\tilde{\nabla}_j\dot{E}_i \\ & -\ddot{E}_{ij} - 2\dot{E}_{ij}\tau^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}\end{aligned} \tag{4.4}$$

$$\begin{aligned}\Omega^{-2}\tilde{g}^{ab}\Delta_{ab} = {} & 3\delta p - 24\dot{\phi}\tau^{-3} - 48\dot{\psi}\tau^{-3} - 24\ddot{\psi}\tau^{-2} + 24\tau^{-4}\phi - 16\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} \\ & +8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 16\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi\end{aligned} \tag{4.5}$$

$$\begin{aligned}g^{\mu\nu}\Delta_{\mu\nu} = {} & 64\delta U_0\tau^{-5} - 24\dot{\phi}\tau^{-3} - 72\dot{\psi}\tau^{-3} - 24\ddot{\psi}\tau^{-2} + 32\tau^{-4}\phi - 24\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} \\ & +8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 24\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 16\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi\end{aligned} \tag{4.6}$$

5 Field Equations (Simplified): Conformal Flat $\Omega(\tau) = \tau/2$

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = -\left(\frac{\Omega}{\dot{\Omega}}\right)\psi + B - \dot{E} \tag{5.1}$$

$$\Delta_{00} = -16\delta U_0\tau^{-3} - 6\dot{\gamma}\tau^{-2} + 6\alpha\tau^{-2} + \tfrac{3}{4}\delta p\tau^2 - 8\tau^{-2}\phi - 12\tau^{-2}\psi + 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \tag{5.2}$$

$$\Delta_{0i} = 2\tau^{-1}\tilde{\nabla}_i\dot{\gamma} - 2\tau^{-1}\tilde{\nabla}_i\alpha - 8\tau^{-3}\tilde{\nabla}_i V + 4\tau^{-1}\tilde{\nabla}_i\psi - 8V_i\tau^{-3} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \tag{5.3}$$

$$\begin{aligned}\Delta_{ij} = {} & -2\dot{\gamma}\tilde{g}_{ij}\tau^{-2} + 2\tilde{g}_{ij}\alpha\tau^{-2} + 2\ddot{\gamma}\tilde{g}_{ij}\tau^{-1} - 2\dot{\alpha}\tilde{g}_{ij}\tau^{-1} + \tfrac{1}{4}\tilde{g}_{ij}\delta p\tau^2 - 4\tilde{g}_{ij}\tau^{-2}\psi - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\alpha \\ & -2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\gamma + \tilde{\nabla}_j\tilde{\nabla}_i\alpha + 2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\gamma \\ & +\tau^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j - \tau^{-1}\tilde{\nabla}_i\dot{E}_j + \tau^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \tau^{-1}\tilde{\nabla}_j\dot{E}_i \\ & -\ddot{E}_{ij} - 2\dot{E}_{ij}\tau^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}\end{aligned} \tag{5.4}$$

$$\Omega^{-2}\tilde{g}^{ab}\Delta_{ab} = 3\delta p - 24\dot{\gamma}\tau^{-4} + 24\alpha\tau^{-4} + 24\ddot{\gamma}\tau^{-3} - 24\dot{\alpha}\tau^{-3} - 48\tau^{-4}\psi - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 16\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \tag{5.5}$$

$$g^{\mu\nu}\Delta_{\mu\nu} = 64\delta U_0\tau^{-5} + 24\ddot{\gamma}\tau^{-3} - 24\dot{\alpha}\tau^{-3} + 32\tau^{-4}\phi - 8\tau^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 24\tau^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\gamma \tag{5.6}$$