TT Projection Curved Space v4

1 Curved Space TT

1.1 SVT Decomposition

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \tag{1.1}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{1.2}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{1.3}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x') h \tag{1.4}$$

$$\begin{aligned} F_\mu &= W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \\ 2F_{\mu\nu} &= 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \end{aligned} \tag{1.5}$$

1.2 Conditions upon W_μ and Ψ

$$\Psi = \int g^{1/2} D(x,x') h \tag{1.6}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right] W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.7}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{1.8}$$

1.3 Isolating χ

According to (1.3),

$$\nabla^\sigma W_\sigma = D\left(\chi + \frac{1}{2(D-1)}h\right), \tag{1.9}$$

if we can find a derivative operator that acts upon $\nabla^\sigma W_\sigma$ to yield a relation proportional to (1.8), then we may be able to express derivatives onto χ as a function of $h_{\mu\nu}$. To fully invert χ, we also require any Ψ dependent term to be pre-fixed by a covariant box, $\nabla_\alpha\nabla^\alpha\Psi$. Inspection of (1.8) shows no foreseeable path to finding a relation meeting these requirements.

1.4 $h_{\mu\nu}(\chi, F, F_\mu)$

$$\nabla^\alpha\nabla^\beta h_{\alpha\beta} = -2\nabla_\alpha\nabla^\alpha\chi + 2\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F - \nabla_\alpha R\nabla^\alpha F - 2R^{\alpha\beta}\nabla_\beta\nabla_\alpha F - F^\alpha\nabla_\alpha R - 2R_{\alpha\beta}\nabla^\beta F^\alpha \quad (1.10)$$

$$\nabla_\alpha\nabla^\alpha h = -8\nabla_\alpha\nabla^\alpha\chi + 2\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F \quad (1.11)$$

$$\begin{aligned}\nabla_\sigma\nabla^\sigma\nabla^\alpha\nabla^\beta h_{\alpha\beta} = & -\tfrac{1}{12}F^\alpha R\nabla_\alpha R + \tfrac{3}{2}F^\alpha R^{\beta\gamma}\nabla_\alpha R_{\beta\gamma} - \nabla^\alpha R\nabla_\beta\nabla^\beta F_\alpha - \nabla^\alpha R\nabla_\beta\nabla^\beta\nabla_\alpha F - F^\alpha\nabla_\beta\nabla^\beta\nabla_\alpha R \\ & -\nabla^\alpha F\nabla_\beta\nabla^\beta\nabla_\alpha R - 2\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha\chi + \tfrac{4}{3}RR_{\alpha\beta}\nabla^\beta F^\alpha - 3R_\alpha{}^\gamma R_{\beta\gamma}\nabla^\beta F^\alpha - 2\nabla_\beta\nabla_\alpha R\nabla^\beta F^\alpha \\ & +\tfrac{5}{12}F^\alpha R_{\alpha\beta}\nabla^\beta R - 2\nabla_\beta\nabla_\alpha R\nabla^\beta\nabla^\alpha F - \tfrac{3}{2}F^\alpha R^{\beta\gamma}\nabla_\gamma R_{\alpha\beta} - 2\nabla^\beta F^\alpha\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\ & -2\nabla^\beta\nabla^\alpha F\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - 2R^{\alpha\beta}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha F + 2\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha F - 3R_{\alpha\beta}\nabla_\gamma\nabla^\gamma\nabla^\beta F^\alpha \\ & -4\nabla_\gamma\nabla_\beta\nabla_\alpha F\nabla^\gamma R^{\alpha\beta} + R_{\alpha\gamma}\nabla^\gamma\nabla_\beta\nabla^\beta F^\alpha + 2\nabla_\beta R_{\alpha\gamma}\nabla^\gamma\nabla^\beta F^\alpha - 6\nabla_\gamma R_{\alpha\beta}\nabla^\gamma\nabla^\beta F^\alpha \quad (1.12)\end{aligned}$$

2 Max. Symmetric Space

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \quad (2.1)$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \quad (2.2)$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \quad (2.3)$$

$$F = \int g^{1/2}D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2}D(x,x')h \quad (2.4)$$

$$\begin{aligned}F_\mu &= W_\mu - \nabla_\mu\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma \\ 2F_{\mu\nu} &= 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \quad (2.5)\end{aligned}$$

In a space of maximal symmetry defined by

$$\begin{aligned}R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D), \quad (2.6)\end{aligned}$$

the conditions upon W_μ and Ψ reduce to

$$\Psi = \int g^{1/2}D(x,x')h \quad (2.7)$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)W_\nu + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{R}{D(D-1)}\nabla_\nu\Psi \quad (2.8)$$

$$\frac{2(D-1)}{D}\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\nabla^\sigma W_\sigma = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{R}{D(D-1)}\nabla_\alpha\nabla^\alpha\Psi \quad (2.9)$$

From (2.9), we may determine χ and F as

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\chi = \frac{1}{2(D-1)}\left[\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)h\right] \quad (2.10)$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\nabla_\beta\nabla^\beta F \quad = \quad \frac{D}{2(D-1)}\left(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{D}\nabla_\alpha\nabla^\alpha h\right). \tag{2.11}$$

To determine F_μ we apply $(\nabla_\alpha\nabla^\alpha + \frac{R}{D})$ to (2.8) to obtain the relation

$$\left(\nabla_\alpha\nabla^\alpha + \frac{R}{D}\right)\nabla^\sigma h_{\sigma\mu} - \frac{R}{D(D-1)}\nabla_\mu h \quad = \quad \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)\left(\nabla_\beta\nabla^\beta + \frac{R}{D}\right)W_\mu + \frac{D-2}{D}\nabla_\mu\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma. \tag{2.12}$$

As a result, we may obtain F_μ via

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)\left(\nabla_\beta\nabla^\beta + \frac{R}{D}\right)F_\mu \quad = \quad \left(\nabla_\alpha\nabla^\alpha + \frac{R}{D}\right)\nabla^\sigma h_{\sigma\mu} - \nabla_\mu\nabla^\alpha\nabla^\beta h_{\alpha\beta}. \tag{2.13}$$

With aide from the Bach tensor in $D = 4$, we may determine $F_{\mu\nu}$ in terms of $K_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}h$ as

$$\left(\nabla_\alpha\nabla^\alpha + \frac{R}{6}\right)\left(\nabla_\beta\nabla^\beta + \frac{R}{3}\right)F_{\mu\nu} \quad = \quad \delta W_{\mu\nu}(K_{\mu\nu}). \tag{2.14}$$

In $D = 3$ we have

$$\begin{aligned} 2\left(\nabla^2 + \frac{R}{3}\right)\left(\nabla^2 + \frac{R}{2}\right)F^{T\theta}_{ij} \quad = \quad & (\nabla^2 - 2k)(\nabla^2 - 3k)h_{ij} - \nabla^2\nabla_i\nabla^l h_{jl} - \nabla^2\nabla_j\nabla^l h_{il} + 3k\nabla_j\nabla^l h_{il} + 3k\nabla_i\nabla^l h_{jl} \\ & + \tfrac{1}{2}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl} + \tfrac{1}{2}g_{ij}\nabla^2\nabla^k\nabla^l h_{kl} - 2kg_{ij}\nabla^l\nabla^k h_{kl} + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2 + 4k)(g^{ab}h_{ab}) \\ & - \tfrac{1}{2}g_{ij}\nabla^2(\nabla^2 - 3k)(g^{ab}h_{ab}) - \tfrac{1}{2}g_{ij}k(\nabla^2 + 4k)(g^{ab}h_{ab}). \end{aligned} \tag{2.15}$$

2.1 Summary

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\chi \quad = \quad \frac{1}{2(D-1)}\left[\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)h\right] \tag{2.16}$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\nabla_\beta\nabla^\beta F \quad = \quad \frac{D}{2(D-1)}\left(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{D}\nabla_\alpha\nabla^\alpha h\right) \tag{2.17}$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)\left(\nabla_\beta\nabla^\beta + \frac{R}{D}\right)F_\mu \quad = \quad \left(\nabla_\alpha\nabla^\alpha + \frac{R}{D}\right)\nabla^\sigma h_{\sigma\mu} - \nabla_\mu\nabla^\alpha\nabla^\beta h_{\alpha\beta}, \tag{2.18}$$

with $F_{\mu\nu}$ given in terms of $h_{\mu\nu}$ in $D = 3$ and $D = 4$ according to (2.15) and (2.14) respectively.

Appendix A Curved Space TT Decomposition

Assume $h_{\mu\nu}$ to be of the form:

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \underbrace{\left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\alpha W_\alpha\right)}_{W_{\mu\nu}} + \underbrace{\frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi}_{S_{\mu\nu}} \tag{A.1}$$

Taking the trace of (A.1), we find the vector sector $W_{\mu\nu}$ is decoupled from the trace and Ψ can easily be inverted,

$$g^{\mu\nu}W_{\mu\nu} = 0 \tag{A.2}$$

$$g^{\mu\nu}S_{\mu\nu} = \nabla_\alpha\nabla^\alpha\Psi = h \qquad \to \Psi = \int g^{1/2} D(x,x')h \tag{A.3}$$

Taking the divergence of (A.1), we have

$$\nabla^\mu h_{\mu\nu} = \nabla^\mu W_{\mu\nu} + \nabla^\mu S_{\mu\nu}(h) \tag{A.4}$$

By substituting (A.3), the above serves to define an equation for W_μ in terms of h and $h_{\mu\nu}$, namely

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h \tag{A.5}$$

Commuting derivatives, (A.5) can be expressed in the equivalent forms,

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \nabla_\alpha\nabla_\nu - \frac{2}{D}\nabla_\nu\nabla_\alpha\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h, \tag{A.6}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}R_{\nu\alpha}\nabla^\alpha\int g^{1/2}D(x,x')h. \tag{A.7}$$

Similar to (**??**), the requisite Green's function that solves W_α is a bi-tensor defined as

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]D^{\alpha\gamma'} = g^{\alpha\gamma'}g^{-1/2}\delta^{(D)}(x,x'). \tag{A.8}$$

Hence, W_μ takes the form

$$W_\mu = \int g^{1/2}D_\mu{}^{\sigma'}\left[\nabla^{\rho'}h_{\sigma'\rho'} - \frac{1}{D-1}R_{\sigma'\rho'}\nabla^{\rho'}\int g^{1/2}D(x',x'')h\right]. \tag{A.9}$$

Appendix B SVTD Decomposition

Starting with

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi, \tag{B.1}$$

we decompose W_μ into transverse and longitudinal components viz.

$$W_\mu = \underbrace{W_\mu - \nabla_\mu\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{F_\mu} + \nabla_\mu\underbrace{\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{H}. \tag{B.2}$$

Setting $h^{T\theta}_{\mu\nu} = 2F_{\mu\nu}$, (B.1) becomes

$$h_{\mu\nu} = 2F_{\mu\nu} + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2\nabla_\mu\nabla_\nu H - \frac{2}{D}g_{\mu\nu}\nabla_\alpha\nabla^\alpha H + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi. \tag{B.3}$$

Upon further defining

$$F = H - \frac{1}{2(D-1)}\Psi \tag{B.4}$$

$$\chi = \frac{1}{D}\nabla_\alpha\nabla^\alpha H - \frac{1}{2(D-1)}\nabla_\alpha\nabla^\alpha\Psi, \tag{B.5}$$

we may express (B.1) as the desired SVTD form:

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{B.6}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{B.7}$$

$$F = \int g^{1/2}D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2}D(x,x')h \tag{B.8}$$

$$F_\mu = W_\mu - \nabla_\mu \int g^{1/2}D(x,x')\nabla^\sigma W_\sigma \tag{B.9}$$

$$2F_{\mu\nu} = 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \tag{B.10}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{B.11}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{B.12}$$