3-Space Projectors v1

1 Flat δG_{ij}

$$
\begin{aligned}
\delta G_{00} &= \tfrac{1}{2}\nabla_a\nabla^a h_{00} + \tfrac{1}{2}\nabla_a\nabla^a h - \tfrac{1}{2}\nabla_b\nabla_a h^{ab} \\
&= -2\nabla_a\nabla^a\psi
\end{aligned}
$$

$$
\begin{aligned}
\delta G_{0i} &= -\tfrac{1}{2}\nabla_a\dot{h}_i{}^a + \tfrac{1}{2}\nabla_a\nabla^a h_{0i} + \tfrac{1}{2}\nabla_i\dot{h}_{00} + \tfrac{1}{2}\nabla_i\dot{h} - \tfrac{1}{2}\nabla_i\nabla_a h_0{}^a \\
&= -2\nabla_i\dot{\psi} + \tfrac{1}{2}\nabla_a\nabla^a B_i - \tfrac{1}{2}\nabla_a\nabla^a\dot{E}_i
\end{aligned}
$$

$$
\begin{aligned}
\delta G_{ij} &= -\tfrac{1}{2}\ddot{h}_{ij} + \tfrac{1}{2}\ddot{h}_{00}g_{ij} + \tfrac{1}{2}\ddot{h}g_{ij} - g_{ij}\nabla_a\dot{h}_0{}^a + \tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h + \tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} + \tfrac{1}{2}\nabla_i\dot{h}_{j0} \\
&\quad -\tfrac{1}{2}\nabla_i\nabla_a h_j{}^a + \tfrac{1}{2}\nabla_j\dot{h}_{i0} - \tfrac{1}{2}\nabla_j\nabla_a h_i{}^a + \tfrac{1}{2}\nabla_j\nabla_i h \\
&= -2\ddot{\psi}g_{ij} - g_{ij}\nabla_a\nabla^a\dot{B} + g_{ij}\nabla_a\nabla^a\ddot{E} - g_{ij}\nabla_a\nabla^a\phi + g_{ij}\nabla_a\nabla^a\psi + \nabla_j\nabla_i\dot{B} - \nabla_j\nabla_i\ddot{E} + \nabla_j\nabla_i\phi - \nabla_j\nabla_i\psi \\
&\quad +\tfrac{1}{2}\nabla_i\dot{B}_j - \tfrac{1}{2}\nabla_i\ddot{E}_j + \tfrac{1}{2}\nabla_j\dot{B}_i - \tfrac{1}{2}\nabla_j\ddot{E}_i - \ddot{E}_{ij} + \nabla_a\nabla^a E_{ij}
\end{aligned} \tag{1.1}
$$

2 3+1 Projectors

Recall the flat 3+1 projector

$$P_{\mu\nu} = \eta_{\mu\nu} + U_\mu U_\nu, \qquad U_\mu = -\delta^0_\mu, \qquad U^\mu = \delta^\mu_0. \tag{2.1}$$

In terms of the the flat space projectors, the splitting of the 3+1 components goes as

$$\rho = U^\sigma U^\tau T_{\sigma\tau} = T_{00}, \qquad q_i = -P_i{}^\sigma U^\tau T_{\sigma\tau} = -T_{0i}, \qquad \pi_{\mu\nu} = \left[\frac{1}{2}P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2}P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3}P_{\mu\nu}P^{\sigma\tau}\right]T_{\sigma\tau}, \tag{2.2}$$

in which it follows

$$\pi_{\mu\nu} = \pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}. \tag{2.3}$$

We recall the definition of Q_i as

$$Q_i = q_i - \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^i q_i. \tag{2.4}$$

This may be alternatively expressed as

$$Q_i = -T_{0i} + \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^j T_{0j} \tag{2.5}$$

Noting that π_{ij} is already traceless by construction, we may project out its transverse part and define $\pi^{T\theta}_{ij}$ as

$$
\begin{aligned}
\pi^{T\theta}_{ij} = \pi_{ij} &- \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^k\pi_{jk} - \tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}^k\pi_{ik} \\
&+ \tilde{\nabla}_i\tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}_k \int d^3z\ D(y-z)\tilde{\nabla}_l\pi^{kl}.
\end{aligned} \tag{2.6}
$$

Substituting in $\pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}$, we have

$$\begin{aligned}
\pi_{ij}^{T\theta} = & \left(T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}\right) - \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^k \left(T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}T_{mn}\right) \\
& - \tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}^k \left(T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}T_{mn}\right) \\
& + \tilde{\nabla}_i\tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}_k \ \int d^3z\ D(y-z)\tilde{\nabla}_l \left(T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}T_{mn}\right). \qquad (2.7)
\end{aligned}$$

In total, we may express relations (19) explicitly in terms of the components of the tensors as the following:

$$\bar{\rho} - \rho = \delta W_{00} - \delta T_{00} \qquad (2.8)$$

$$\bar{Q}_i - Q_i = -(\delta W_{0i} - \delta T_{0i}) + \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^j(\delta W_{0j} - \delta T_{0j}) \qquad (2.9)$$

$$\begin{aligned}
\bar{\pi}_{ij}^{T\theta} - \pi_{ij}^{T\theta} = & \left[\delta W_{ij} - \delta T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\left(\delta W_{kl} - \delta T_{kl}\right)\right] \\
& - \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^k \left[\delta W_{jk} - \delta T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right] \\
& - \tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}^k \left[\delta W_{ij} - \delta T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right] \\
& + \tilde{\nabla}_i\tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}_k \ \int d^3z\ D(y-z)\tilde{\nabla}_l \left[\delta W_{kl} - \delta T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right]. \qquad (2.10)
\end{aligned}$$

$$h_{\mu\nu}^{T} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu\nabla_\nu \int D\nabla^\sigma W_\sigma \qquad (2.11)$$

$$W_\mu = \int D\nabla^\sigma h_{\sigma\mu} \qquad (2.12)$$

$$h_{\mu\nu}^{T\theta} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu\nabla_\nu \int D(h + \nabla^\sigma W_\sigma) \qquad (2.13)$$

$$\begin{aligned}
h_{\mu\nu} = & \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu\nabla_\nu \int D(h + \nabla^\sigma W_\sigma)\right]}_{h_{\mu\nu}^{T\theta}} \\
& + \nabla_\mu W_\nu + \nabla_\nu W_\mu + \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) - \frac{1}{2}\nabla_\mu\nabla_\nu \int D(h + \nabla^\sigma W_\sigma) \qquad (2.14)
\end{aligned}$$

if $h = 0$ then

$$\begin{aligned}
h_{\mu\nu} = & \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \frac{1}{2}g_{\mu\nu}\nabla^\sigma W_\sigma + \frac{1}{2}\nabla_\mu\nabla_\nu \int D\nabla^\sigma W_\sigma\right]}_{h_{\mu\nu}^{T\theta}} \\
& + \nabla_\mu \underbrace{\left(W_\nu - \nabla_\nu \int D\nabla^\sigma W_\sigma\right)}_{V_\nu^T} + \nabla_\nu \underbrace{\left(W_\mu - \nabla_\mu \int D\nabla^\sigma W_\sigma\right)}_{V_\mu^T} \\
& + \nabla_\mu\nabla_\nu \underbrace{\left(\frac{3}{2}\int D\nabla^\sigma W_\sigma\right)}_{2V} - g_{\mu\nu} \underbrace{\left(\frac{1}{2}\nabla^\sigma W_\sigma\right)}_{\frac{2}{3}\nabla^\alpha\nabla_\alpha V} \qquad (2.15)
\end{aligned}$$

$$h_{\mu\nu} = g_{\mu\nu}p + h^{T\theta}_{\mu\nu} + \nabla_\mu V^T_\nu + \nabla_\nu V^T_\mu + 2\nabla_\mu \nabla_\nu V - \frac{2}{3} g_{\mu\nu} \nabla_\alpha \nabla^\alpha V \tag{2.16}$$

$$V_\mu = W_\mu - \frac{1}{4}\nabla_\mu \int D\nabla^\sigma W_\sigma \tag{2.17}$$

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{3}\nabla^\sigma V_\sigma = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{1}{2} g_{\mu\nu}\nabla^\sigma W_\sigma - \frac{1}{2}\nabla_\mu \nabla_\nu \int D\nabla^\sigma W_\sigma \tag{2.18}$$

$$g_{\mu\nu} = P_{\mu\nu} - U_\mu U_\nu, \qquad P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \tag{2.19}$$

For a little more simplicity, we assume a $T_{\mu\nu}$ that is symmetric.

$$\begin{aligned}
T_{\mu\nu} &= g_\mu{}^\rho g_\nu{}^\sigma T_{\rho\sigma} \\
&= (P_\mu{}^\rho - U_\mu U^\rho)(P_\nu{}^\sigma - U_\nu U^\sigma) T_{\rho\sigma} \\
&= (P_\mu{}^\rho P_\nu{}^\sigma - P_\mu{}^\rho U_\nu U^\sigma - P_\nu{}^\sigma U_\mu U^\rho + U_\mu U_\nu U^\rho U^\sigma) T_{\rho\sigma} \\
&= U_\mu U_\nu \underbrace{U^\rho U^\sigma T_{\rho\sigma}}_{\rho} + U_\mu U_\nu \underbrace{\left(\tfrac{1}{3} P^{\rho\sigma} T_{\rho\sigma}\right)}_{p} + \underbrace{(P_{\mu\nu} - U_\mu U_\nu)}_{g_{\mu\nu}} \underbrace{\left(\tfrac{1}{3} P^{\rho\sigma} T_{\rho\sigma}\right)}_{p} \\
&\quad - U_\mu \underbrace{P_\nu{}^\sigma U^\rho T_{\rho\sigma}}_{-q_\nu} - U_\nu \underbrace{P_\mu{}^\rho U^\sigma T_{\rho\sigma}}_{-q_\mu} \\
&\quad \underbrace{\left(P_\mu{}^\rho P_\nu{}^\sigma - \tfrac{1}{3} P_{\mu\nu} P^{\rho\sigma}\right) T_{\rho\sigma}}_{\pi_{\mu\nu}}
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
U^\rho U^\sigma T_{\rho\sigma} &= T_{00} = \rho \\
\frac{1}{3} P^{\rho\sigma} T_{\rho\sigma} &= \frac{1}{3} g^{ij} T_{ij} = p \\
-U_\mu P_\nu{}^\sigma U^\rho T_{\rho\sigma} &= S_{\mu\nu} = T_{0i} = U_\mu q_\nu \\
-U_\nu P_\mu{}^\sigma U^\rho T_{\rho\sigma} &= S_{\mu\nu} = T_{i0} = U_\nu q_\mu \\
\left(P_\mu{}^\rho P_\nu{}^\sigma - \tfrac{1}{3} P_{\mu\nu} P^{\rho\sigma}\right) T_{\rho\sigma} &= T_{ij} - \frac{1}{3} g_{ij} g^{kl} T_{kl} = \pi_{ij}
\end{aligned} \tag{2.21}$$

Flat Einstein

$$\begin{aligned}
\delta G_{\mu\nu} &= \tfrac{1}{2}\nabla_\beta\nabla^\beta h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla^\beta h - \tfrac{1}{2}\nabla_\beta\nabla_\mu h_\nu{}^\beta - \tfrac{1}{2}\nabla_\beta\nabla_\nu h_\mu{}^\beta + \tfrac{1}{2}g_{\mu\nu}\nabla_\zeta\nabla_\beta h^{\beta\zeta} + \tfrac{1}{2}\nabla_\nu\nabla_\mu h \\
&= \Big[\tfrac{1}{2}g^\sigma{}_\mu g^\rho{}_\nu\nabla_\alpha\nabla^\alpha - \tfrac{1}{2}g_{\mu\nu}g^{\sigma\rho}\nabla_\alpha\nabla^\alpha - \tfrac{1}{2}g^\rho{}_\nu\nabla^\sigma\nabla_\mu - \tfrac{1}{2}g^\sigma{}_\mu\nabla^\rho\nabla_\nu \\
&\quad + \tfrac{1}{2}g_{\mu\nu}\nabla^\sigma\nabla^\rho + \tfrac{1}{2}g^{\sigma\rho}\nabla_\mu\nabla_\nu\Big]h_{\sigma\rho} \\
&= \hat{\mathcal{L}}^{\sigma\rho}{}_{\mu\nu}h_{\sigma\rho} \qquad (2.22)
\end{aligned}$$

$$\begin{aligned}
\delta G_{00} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{00}h_{\sigma\rho} \\
&= \tfrac{1}{2}\nabla_a\nabla^a(h_{00}+h) - \tfrac{1}{2}\nabla^a\nabla^b h_{ab} \\
&= \nabla^2 p - \tfrac{2}{3}\nabla^4 V
\end{aligned}$$

$$\begin{aligned}
\delta G_{0i} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{0i}h_{\sigma\rho} \\
&= -\tfrac{1}{2}\nabla^a\dot{h}_{ia} + \tfrac{1}{2}\nabla^2 h_{0i} + \tfrac{1}{2}\nabla_i\dot{h_{00}} + \tfrac{1}{2}\nabla_i\dot{h} - \tfrac{1}{2}\nabla_i\nabla^a h_{0a} \\
&= \nabla_i\left(\dot{p} - \tfrac{2}{3}\nabla^2\dot{V}\right) + \tfrac{1}{2}\nabla^2(Q_i - \dot{V}_i)
\end{aligned}$$

$$\begin{aligned}
\delta G_{ij} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{ij}h_{\sigma\rho} \\
&= \tfrac{1}{2}(\nabla^2 - \partial_t^2)h_{ij} + \tfrac{1}{2}g_{ij}(\ddot{h}_{00} + \ddot{h}) + \tfrac{1}{2}g_{ij}\nabla^2 h + \tfrac{1}{2}\nabla_i\nabla_j h - g_{ij}\nabla^a\dot{h}_{0a} \\
&\quad + \tfrac{1}{2}g_{ij}\nabla^a\nabla^b h_{ab} + \tfrac{1}{2}(\nabla_i\dot{h}_{j0} + \nabla_j\dot{h}_{i0}) - \tfrac{1}{2}(\nabla_i\nabla^a h_{ja} + \nabla_j\nabla^a h_{ia}) \\
&= g_{ij}(\ddot{p} - \tfrac{2}{3}\nabla^2\ddot{V}) - \tfrac{1}{2}g_{ij}\nabla^2(p - \tfrac{2}{3}\nabla^2 V) + \tfrac{1}{2}g_{ij}\nabla^2(\rho - 2\dot{Q} + 2\ddot{V}) \\
&\quad + \tfrac{1}{2}\nabla_i\nabla_j(p - \tfrac{2}{3}\nabla^2 V) - \tfrac{1}{2}\nabla_i\nabla_j(\rho - 2\dot{Q} + 2\ddot{V}) \\
&\quad + \tfrac{1}{2}\nabla_i(\dot{Q}_j - \ddot{V}_j) + \tfrac{1}{2}\nabla_j(\dot{Q}_i - \ddot{V}_i) - \ddot{V}_{ij} + \nabla^2 V_{ij} \qquad (2.23)
\end{aligned}$$

$$\ddot{p}g_{ij} + \tfrac{1}{3}g_{ij}\nabla_a\nabla^a\ddot{V} - g_{ij}\nabla_a\nabla^a\dot{Q} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a p + \tfrac{1}{2}g_{ij}\nabla_a\nabla^a\rho \qquad (2.24)$$

$$+\tfrac{1}{3}g_{ij}\nabla_b\nabla^b\nabla_a\nabla^a V - \nabla_j\nabla_i\ddot{V} + \nabla_j\nabla_i\dot{Q} + \tfrac{1}{2}\nabla_j\nabla_i p - \tfrac{1}{2}\nabla_j\nabla_i\rho - \tfrac{1}{3}\nabla_j\nabla_i\nabla_a\nabla^a V \qquad (2.25)$$

Conformal Part

$$\begin{aligned}
\delta G^{\Omega(x)}_{\mu\nu} &= -2h_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega - g_{\mu\nu}\Omega^{-1}\nabla_\alpha\Omega\nabla^\alpha h + \Omega^{-1}\nabla_\alpha h_{\mu\nu}\nabla^\alpha\Omega + h_{\mu\nu}\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega + 2g_{\mu\nu}\Omega^{-1}\nabla^\alpha\Omega\nabla_\beta h_\alpha{}^\beta \\
&\quad - g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha\Omega\nabla^\beta\Omega + 2g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta\nabla^\alpha\Omega - \Omega^{-1}\nabla^\alpha\Omega\nabla_\mu h_{\nu\alpha} - \Omega^{-1}\nabla^\alpha\Omega\nabla_\nu h_{\mu\alpha} \\
&= \Omega^{-1}\Big[-2g^\sigma{}_\mu g^\rho{}_\nu\nabla_\alpha\nabla^\alpha\Omega - g_{\mu\nu}g^{\sigma\rho}\nabla_\alpha\Omega\nabla^\alpha + g^\sigma{}_\mu g^\rho{}_\nu\nabla_\alpha\Omega\nabla^\alpha + 2g_{\mu\nu}\nabla^\sigma\Omega\nabla^\rho + 2g_{\mu\nu}\nabla^\sigma\nabla^\rho \\
&\quad - g^\rho{}_\nu\nabla^\sigma\Omega\nabla_\mu - g^\sigma{}_\mu\nabla^\rho\Omega\nabla_\nu\Big] + \Omega^{-2}\Big[g^\sigma{}_\mu g^\rho{}_\nu\nabla_\alpha\Omega\nabla^\alpha\Omega - g_{\mu\nu}\nabla^\rho\Omega\nabla^\sigma\Omega\Big]h_{\sigma\rho} \\
&= \hat{\mathcal{J}}^{\sigma\rho}{}_{\mu\nu}h_{\sigma\rho} \qquad (2.26)
\end{aligned}$$

3 $\delta G_{\mu\nu}$

Bianchi Identity $\nabla^\mu \delta G_{\mu\nu} = 0$

$$\begin{aligned} \dot\rho &= \nabla^2 Q \\ \dot Q_i + \nabla_i \dot Q &= \nabla_i p + \tfrac{4}{3}\nabla^2\nabla_i V + \nabla^2 V_i \end{aligned} \tag{3.1}$$

$$\begin{aligned} \delta G_{00} &= \nabla^2(p - \tfrac{2}{3}\nabla^2 V) \\ \delta G_{0i} &= \nabla_i(\dot p - \tfrac{2}{3}\nabla^2 \dot V) + \tfrac{1}{2}\nabla^2(Q_i - \dot V_i) \\ \delta G_{ij} &= g_{ij}(\ddot p - \tfrac{2}{3}\nabla^2 \ddot V) - \tfrac{1}{2}g_{ij}\nabla^2(p - \tfrac{2}{3}\nabla^2 V) + \tfrac{1}{2}g_{ij}\nabla^2(\rho - 2\dot Q + 2\ddot V) \\ &\quad + \tfrac{1}{2}\nabla_i\nabla_j(p - \tfrac{2}{3}\nabla^2 V) - \tfrac{1}{2}\nabla_i\nabla_j(\rho - 2\dot Q + 2\ddot V) \\ &\quad + \tfrac{1}{2}\nabla_i(\dot Q_j - \ddot V_j) + \tfrac{1}{2}\nabla_j(\dot Q_i - \ddot V_i) - \ddot V_{ij} + \nabla^2 V_{ij} \\ g^{ij}\delta G_{ij} &= 3(\ddot p - \tfrac{2}{3}\nabla^2 \ddot V) - \nabla^2(p - \tfrac{2}{3}\nabla^2 V) + \nabla^2(\rho - 2\dot Q + 2\ddot V) \end{aligned} \tag{3.2}$$

4 Gauge Invariance

Under $x^\mu \to x^\mu - \epsilon^\mu(x)$

$$\Delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \tag{4.1}$$

where

$$\epsilon_0 = -T, \qquad \epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j \epsilon_j}_{L_i} + \nabla_i \underbrace{\int D\nabla^j \epsilon_j}_{L} \tag{4.2}$$

$$\begin{aligned} \Delta_\epsilon g_{00} &= -2\dot T \\ \Delta_\epsilon g_{0i} &= -\nabla_i T + \dot L_i + \nabla_i \dot L \\ \Delta_\epsilon g_{ij} &= 2\nabla_i\nabla_j L + \nabla_i L_j + \nabla_j L_i \end{aligned} \tag{4.3}$$

$$\begin{aligned} \bar\rho &= \rho - 2\dot T \\ \bar Q_i + \nabla_i \bar Q &= Q_i + \nabla_i Q + \nabla_i(\dot L - T) + \dot L_i \\ g_{ij}\bar p + 2\nabla_i\nabla_j \bar V - \tfrac{2}{3}g_{ij}\nabla^2 \bar V + \nabla_i \bar V_j + \nabla_j \bar V_i + \bar V_{ij} &= g_{ij} p + 2\nabla_i\nabla_j V - \tfrac{2}{3}g_{ij}\nabla^2 V + \nabla_i V_j + \nabla_j V_i + V_{ij} \\ &\quad + 2\nabla_i\nabla_j L + \nabla_i L_j + \nabla_j L_i \end{aligned} \tag{4.4}$$

$$\begin{aligned} \bar\rho &= \rho - 2\dot T \\ \bar Q &= Q + \dot L - T \\ \bar Q_i &= Q_i + \dot L_i \\ \bar V_i &= V_i + L_i \\ \bar p &= p + \frac{2}{3}\nabla^2 L \\ \bar V &= V + L \\ \bar V_{ij} &= V_{ij} \end{aligned} \tag{4.5}$$

$$\rho - 2\dot{Q} + 2\ddot{V}, \qquad p - \tfrac{2}{3}\nabla^2 V, \qquad Q_i - \dot{V}_i, \qquad V_{ij} \tag{4.6}$$

$$\begin{aligned}
\underbrace{\bar{h}_{00}}_{\bar{\rho}} &= \underbrace{h_{00}}_{\rho} - 2\dot{T} \\
\underbrace{\int D\nabla^j \bar{h}_{0j}}_{\bar{Q}} &= \underbrace{\int D\nabla^j h_{0j}}_{Q} + \underbrace{\int D\nabla^2(\dot{L}-T)}_{(\dot{L}-T)^L} \\
\underbrace{\bar{h}_{0i} - \nabla_i \int D\nabla^j \bar{h}_{0j}}_{\bar{Q}_i} &= \underbrace{h_{0i} - \nabla_i \int D\nabla^j h_{0j}}_{Q_i} + \dot{L}_i + \underbrace{\nabla_i(\dot{L}-T) - \nabla_i \int D\nabla^2(\dot{L}-T)}_{\nabla_i(\dot{L}-T)^T} \\
\underbrace{g^{ij}\bar{h}_{ij}}_{3\bar{p}} &= \underbrace{g^{ij}h_{ij}}_{3p} + 2\nabla^2 L \\
\underbrace{\frac{3}{4}\int D\nabla^k \bar{W}_k}_{\bar{V}} &= \underbrace{\frac{3}{4}\int D\nabla^k W_k}_{V} + \frac{3}{4}\int D\nabla^k \int D\nabla^2(\tfrac{4}{3}\nabla_k L + L_k) \\
\underbrace{\bar{W}_i - \nabla_i \int D\nabla^j \bar{W}_j}_{\bar{V}_i} &= \underbrace{W_i - \nabla_i \int D\nabla^j W_j}_{V_i} + \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2(\tfrac{4}{3}\nabla_j L + L_j)
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\Delta_\epsilon k_{ij} &= 2\nabla_i\nabla_j L - \tfrac{2}{3} g_{ij}\nabla^2 L + \nabla_i L_j + \nabla_j L_i \\
\Delta_\epsilon \nabla^j k_{ij} &= \nabla^2(\tfrac{4}{3}\nabla_i L + L_i) \\
\Delta_\epsilon W_i &= \int D\nabla^j k_{ij} = \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i)
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\bar{V}_{ij} - V_{ij} &= \Delta_\epsilon \left[k_{ij} - \nabla_i W_j - \nabla_j W_i + \tfrac{1}{2} g_{ij}\nabla^k W_k + \tfrac{1}{2}\nabla_i\nabla_j \int D\nabla^k W_k \right] \\
&= 2\nabla_i\nabla_j L - \tfrac{2}{3} g_{ij}\nabla^2 L + \nabla_i L_j + \nabla_j L_i - \nabla_i \int D\nabla^2(\tfrac{4}{3}\nabla_j L + L_j) - \nabla_j \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i) \\
&\quad + \tfrac{1}{2} g_{ij}\nabla^k \int D\nabla^2(\tfrac{4}{3}\nabla_k L + L_k) + \tfrac{1}{2}\nabla_i\nabla_j \int D\nabla^k \int D\nabla^2(\tfrac{4}{3}\nabla_k L + L_k)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\bar{\rho} &= \rho - 2\dot{T} \\
\bar{p} &= p + \frac{2}{3}\nabla^2 L \\
\bar{Q} &= Q + \int D\nabla^2(\dot{L}-T) \\
\bar{Q}_i &= Q_i + \dot{L}_i + \nabla_i(\dot{L}-T) - \nabla_i \int D\nabla^2(\dot{L}-T) \\
\bar{V} &= V + \frac{3}{4}\int D\nabla^k \int D\nabla^2(\tfrac{4}{3}\nabla_k L + L_k) \\
\bar{V}_i &= V_i + \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2(\tfrac{4}{3}\nabla_j L + L_j)
\end{aligned} \tag{4.10}$$

Appendix A Bach Tensor

$$\begin{aligned}
W_{\mu\nu} &= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R_{\mu}{}^{\alpha}R_{\nu\alpha} \\
&\quad -\tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \nabla_\alpha\nabla^\alpha R_{\mu\nu} + 2\nabla_\beta\nabla_\alpha R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} + \tfrac{2}{3}\nabla_\nu\nabla_\mu R \\
&= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \tfrac{1}{3}\nabla_\nu\nabla_\mu R \\
&= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R_{\mu}{}^{\alpha}R_{\nu\alpha} \\
&\quad -\tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\alpha\nabla_\mu R_{\nu}{}^{\alpha} - \nabla_\alpha\nabla_\nu R_{\mu}{}^{\alpha} + \tfrac{2}{3}\nabla_\nu\nabla_\mu R \\
\nabla_\alpha\nabla^\alpha G^{T\theta}_{\mu\nu} &= \nabla_\alpha\nabla^\alpha G_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}\nabla_\rho\nabla^\rho g^{\alpha\beta}G_{\alpha\beta} + \tfrac{1}{3}\nabla_\mu\nabla_\nu g^{\alpha\beta}G_{\alpha\beta} \\
&= \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \tfrac{1}{3}\nabla_\mu\nabla_\nu R \\
\text{remaining} &= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\
&= -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R_{\mu}{}^{\alpha}R_{\nu\alpha} \\
&\quad -\nabla_\alpha\nabla_\mu R_{\nu}{}^{\alpha} - \nabla_\alpha\nabla_\nu R_{\mu}{}^{\alpha} + \nabla_\mu\nabla_\nu R
\end{aligned} \tag{A.1}$$

Appendix B Einstein Related to Weyl

$$\begin{aligned}
I_G &= \int d^4x\sqrt{g}\left(G_{\mu\nu}G^{\mu\nu} - \tfrac{1}{3}(g^{\alpha\beta}G_{\alpha\beta})^2\right) \\
&= I_{G_2} - \frac{1}{3}I_{G_1} = \int d^4x\sqrt{g}G_{\mu\nu}G^{\mu\nu} - \frac{1}{3}\int d^4x\sqrt{g}(g^{\alpha\beta}G_{\alpha\beta})^2
\end{aligned} \tag{B.1}$$

Recalling

$$\delta(\sqrt{g}) = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \tag{B.2}$$

we have

$$\begin{aligned}
W^{\mu\nu}_{(2)} &= \frac{1}{\sqrt{g}}\frac{\delta I_{G_2}}{\delta g_{\mu\nu}} = \frac{1}{2}g^{\mu\nu}G_{\alpha\beta}G^{\alpha\beta} + \frac{\delta}{\delta g_{\mu\nu}}\left(G_{\alpha\beta}G^{\alpha\beta}\right) \\
\frac{\delta}{\delta g_{\mu\nu}}\left(G_{\alpha\beta}G^{\alpha\beta}\right) &= -2\delta g_{\mu\nu}G^{\mu}{}_{\alpha}G^{\nu\alpha} + 2\delta G_{\mu\nu}G^{\mu\nu}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
2G^{\mu\nu}\delta G_{\mu\nu} &= 2G^{\mu\nu}(\delta R_{\mu\nu} - \tfrac{1}{2}\delta g_{\mu\nu}R + \tfrac{1}{2}g_{\mu\nu}\delta g_{\alpha\beta}R^{\alpha\beta} - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\delta R_{\alpha\beta}) \\
&= 2\underbrace{\left(G^{\mu\nu} - \tfrac{1}{2}g_{\alpha\beta}G^{\alpha\beta}g^{\mu\nu}\right)}_{\bar{G}^{\mu\nu}}\delta R_{\mu\nu} + (g_{\alpha\beta}G^{\alpha\beta}R^{\mu\nu} - G^{\mu\nu}R)\delta g_{\mu\nu}
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\delta R_{\mu\nu} &= \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda} - \nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} \\
\delta\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2}g^{\lambda\rho}[\nabla_\nu\delta g_{\mu\rho} + \nabla_\mu\delta g_{\nu\rho} - \nabla_\rho\delta g_{\mu\nu}]
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
2\bar{G}^{\mu\nu}\delta R_{\mu\nu} &= 2\left[\nabla_\nu(\bar{G}^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) - \nabla_\nu\bar{G}^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} - \nabla_\lambda(\bar{G}^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) + \nabla_\lambda\bar{G}^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right] \\
&= \nabla_\alpha\nabla^\mu\bar{G}^{\alpha\nu} - \nabla_\alpha\nabla^\alpha\bar{G}^{\mu\nu} - \nabla^\nu\nabla_\alpha\bar{G}^{\mu\alpha} + \nabla^\mu\nabla_\alpha\bar{G}^{\nu\alpha} \\
&= \left(g^{\mu\nu}\nabla_\alpha\nabla_\beta\bar{G}^{\alpha\beta} - \nabla_\alpha\nabla^\nu\bar{G}^{\mu\alpha} - \nabla_\alpha\nabla^\mu\bar{G}^{\alpha\nu} + \nabla_\alpha\nabla^\alpha\bar{G}^{\mu\nu}\right)\delta g_{\mu\nu}
\end{aligned} \tag{B.6}$$

Hence we have for $W^{\mu\nu}_{(2)}$

$$\begin{aligned}
W^{\mu\nu}_{(2)} &= \frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{g}G^{\mu\nu}G_{\mu\nu} \\
&= \int d^4x\sqrt{g}\Big(\frac{1}{2}g^{\mu\nu}G_{\alpha\beta}G^{\alpha\beta} - 2G^{\mu}{}_{\alpha}G^{\nu\alpha} + g_{\alpha\beta}G^{\alpha\beta}R^{\mu\nu} - G^{\mu\nu}R \\
&\quad + g^{\mu\nu}\nabla_\alpha\nabla^\beta\bar{G}^{\alpha\beta} - \nabla_\alpha\nabla^\nu\bar{G}^{\mu\alpha} - \nabla_\alpha\nabla^\mu\bar{G}^{\alpha\nu} + \nabla_\alpha\nabla^\alpha\bar{G}^{\mu\nu}\Big)\delta g_{\mu\nu} \\
&= \frac{1}{2}g^{\mu\nu}G_{\alpha\beta}G^{\alpha\beta} - 2G^{\mu}{}_{\alpha}G^{\nu\alpha} + g_{\alpha\beta}G^{\alpha\beta}R^{\mu\nu} - G^{\mu\nu}R - g^{\mu\nu}\nabla_\alpha\nabla^\alpha G + \nabla^\mu\nabla^\nu G \\
&\quad + \nabla_\alpha\nabla^\alpha G^{\mu\nu} - \nabla_\alpha\nabla^\mu G^{\alpha\nu} - \nabla_\alpha\nabla^\nu G^{\mu\alpha}
\end{aligned} \tag{B.7}$$

$$\delta(g^{\alpha\beta}G_{\alpha\beta})^2 = -2g^{\alpha\beta}G_{\alpha\beta}G^{\mu\nu}\delta g_{\mu\nu} + 2g^{\alpha\beta}G_{\alpha\beta}g^{\mu\nu}\delta G_{\mu\nu} \tag{B.8}$$

$$2Gg^{\mu\nu}\delta G_{\mu\nu} = -2Gg^{\mu\nu}\delta R_{\mu\nu} + (4GR^{\mu\nu} - Gg^{\mu\nu}R)\delta g_{\mu\nu} \tag{B.9}$$

$$-2Gg^{\mu\nu}\delta R_{\mu\nu} = (-2\nabla_\alpha\nabla^\alpha G + 2\nabla^\mu\nabla^\nu G)\delta g_{\mu\nu} \tag{B.10}$$

And for $W^{\mu\nu}_{(1)}$

$$\begin{aligned}
W^{\mu\nu}_{(1)} &= \frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{g}(g^{\alpha\beta}G_{\alpha\beta})^2 \\
&= \int d^4x\sqrt{g}\left(\tfrac{1}{2}g^{\mu\nu}G^2 - 2GG^{\mu\nu} + 4GR^{\mu\nu} - g^{\mu\nu}GR - 2\nabla_\alpha\nabla^\alpha G + 2\nabla^\mu\nabla^\nu G\right)\delta g_{\mu\nu}
\end{aligned} \tag{B.11}$$

$$W^{\mu\nu}_{(1)} = \tfrac{1}{2}g^{\mu\nu}G^2 - 2GG^{\mu\nu} + 4GR^{\mu\nu} - g^{\mu\nu}GR - 2\nabla_\alpha\nabla^\alpha G + 2\nabla^\mu\nabla^\nu G \tag{B.12}$$

$$\begin{aligned}
W^{\mu\nu}_{(2)} &= \frac{1}{2}g^{\mu\nu}G_{\alpha\beta}G^{\alpha\beta} - 2G^{\mu}{}_{\alpha}G^{\nu\alpha} + g_{\alpha\beta}G^{\alpha\beta}R^{\mu\nu} - G^{\mu\nu}R - g^{\mu\nu}\nabla_\alpha\nabla^\alpha G + \nabla^\mu\nabla^\nu G \\
&\quad + \nabla_\alpha\nabla^\alpha G^{\mu\nu} - \nabla_\alpha\nabla^\mu G^{\alpha\nu} - \nabla_\alpha\nabla^\nu G^{\mu\alpha}
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
W^{(2)}_{\mu\nu} - \tfrac{1}{3}W^{(1)}_{\mu\nu} &= \tfrac{1}{2}g_{\mu\nu}G^{\alpha\beta}G_{\alpha\beta} - 2G_{\mu\alpha}G_{\nu}{}^{\alpha} - \tfrac{1}{3}GR_{\mu\nu} - RG_{\mu\nu} + \tfrac{2}{3}GG_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}G^2 \\
&\quad - \tfrac{1}{3}g_{\mu\nu}\nabla_\alpha\nabla^\alpha G + \tfrac{1}{3}\nabla_\mu\nabla_\nu G + \nabla_\alpha\nabla^\alpha G_{\mu\nu} - \nabla^\alpha\nabla_\mu G_{\alpha\nu} - \nabla^\alpha\nabla_\nu G_{\mu\alpha}
\end{aligned} \tag{B.14}$$

$$h^{T\theta}_{\mu\nu} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\left[\nabla_\mu\nabla_\nu - \frac{1}{6}Rg_{\mu\nu}\right]\int D(h + \nabla^\sigma W_\sigma) \tag{B.15}$$

$$\nabla^2 G^{T\theta}_{\mu\nu} = \nabla^2 G_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\nabla^2 G + \frac{1}{3}\nabla^2\left[\nabla_\mu\nabla_\nu - \frac{1}{12}Rg_{\mu\nu}\right]\int DG \tag{B.16}$$

$$\begin{aligned}
\nabla^\mu W_{\mu\nu} &= G^{\alpha\beta}\nabla_\nu G_{\alpha\beta} - 2G_{\mu\alpha}\nabla^\mu G_\nu{}^\alpha - \tfrac{1}{3}\nabla^\mu GR_{\mu\nu} - \tfrac{1}{6}G\nabla_\nu R - \nabla^\mu RG_{\mu\nu} + \tfrac{2}{3}\nabla^\mu GG_{\mu\nu} \\
&\quad - \tfrac{1}{3}G\nabla_\nu G - \tfrac{1}{3}\nabla^\mu\nabla^2 G + \tfrac{1}{3}\nabla^2\nabla_\nu G + \nabla^\mu\nabla^2 G_{\mu\nu} - \nabla^\mu\nabla^\alpha\nabla_\mu G_{\alpha\nu} - \nabla^\mu\nabla^\alpha\nabla_\nu G_{\mu\alpha}
\end{aligned} \tag{B.17}$$

If $R = const$

$$\nabla^\mu W_{\mu\nu} = G^{\alpha\beta}\nabla_\nu G_{\alpha\beta} - 2G_{\mu\alpha}\nabla^\mu G_\nu{}^\alpha + \nabla^\mu\nabla^2 G_{\mu\nu} - \nabla^\mu\nabla^\alpha\nabla_\mu G_{\alpha\nu} - \nabla^\mu\nabla^\alpha\nabla_\nu G_{\mu\alpha} \tag{B.18}$$

B.1 New Result

$$\Delta_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{6} g_{\mu\nu} R \tag{B.19}$$

$$g^{\alpha\beta}\Delta_{\alpha\beta} = \tfrac{1}{3} R \tag{B.20}$$

$$W_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu}\Delta_{\alpha\beta}\Delta^{\alpha\beta} + \Delta_{\mu\nu}\Delta - 2\Delta^{\alpha\beta}R_{\mu\alpha\nu\beta} + \nabla_\alpha\nabla^\alpha\Delta_{\mu\nu} - \nabla_\nu\nabla_\mu\Delta \tag{B.21}$$

$$= \tfrac{1}{2} g_{\mu\nu}\Delta_{\alpha\beta}\Delta^{\alpha\beta} - 2\Delta_\mu{}^\alpha\Delta_{\nu\alpha} + \nabla_\alpha\nabla^\alpha\Delta_{\mu\nu} - \nabla_\alpha\nabla_\mu\Delta_\nu{}^\alpha - \nabla_\alpha\nabla_\nu\Delta_\mu{}^\alpha + \nabla_\nu\nabla_\mu\Delta \tag{B.22}$$

Working in a conformal to flat geometry, we also have

$$\begin{aligned} W_{\mu\nu} = {} & \tfrac{1}{2} g_{\mu\nu}\Delta_{\alpha\beta}\Delta^{\alpha\beta} + \Delta_{\mu\nu}\Delta - \tfrac{1}{3}\Delta_{\mu\nu}R + \tfrac{1}{3} g_{\mu\nu}\Delta R - g_{\mu\nu}\Delta^{\alpha\beta}R_{\alpha\beta} + \Delta_\nu{}^\alpha R_{\mu\alpha} - \Delta R_{\mu\nu} + \Delta_\mu{}^\alpha R_{\nu\alpha} \\ & + \nabla_\alpha\nabla^\alpha\Delta_{\mu\nu} - \nabla_\nu\nabla_\mu\Delta \end{aligned} \tag{B.23}$$

Appendix C SVT Traditional Form

Applied to 3 dimensions.

$$\begin{aligned} h_{\mu\nu} = {} & \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2} g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu\nabla_\nu \int D(h + \nabla^\sigma W_\sigma)\right]}_{2E^{T\theta}_{\mu\nu}} \\ & + \nabla_\mu \underbrace{\left(W_\nu - \nabla_\nu \int D\nabla^\sigma W_\sigma\right)}_{E_\nu} + \nabla_\nu \underbrace{\left(W_\mu - \nabla_\mu \int D\nabla^\sigma W_\sigma\right)}_{E_\mu} \\ & - 2 g_{\mu\nu} \underbrace{\left(\tfrac{1}{4}\nabla^\sigma W_\sigma - \tfrac{1}{4}h\right)}_{\psi} + 2\nabla_\mu\nabla_\nu \underbrace{\int D\left(\tfrac{3}{4}\nabla^\sigma W_\sigma - \tfrac{1}{4}h\right)}_{E} \end{aligned} \tag{C.1}$$

C.1 Gauge Invariance

Under $x^\mu \to x^\mu - \epsilon^\mu(x)$

$$\Delta_\epsilon h_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu \tag{C.2}$$

where

$$\epsilon_0 = -T, \qquad \epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j\epsilon_j}_{L_i} + \nabla_i \underbrace{\int D\nabla^j\epsilon_j}_{L} \tag{C.3}$$

$$\begin{aligned} \Delta_\epsilon h_{00} &= -2\dot{T} \\ \Delta_\epsilon h_{0i} &= -\nabla_i T + \dot{L}_i + \nabla_i\dot{L} \\ \Delta_\epsilon h_{ij} &= 2\nabla_i\nabla_j L + \nabla_i L_j + \nabla_j L_i \\ \Delta_\epsilon(\nabla^j h_{ij}) &= 2\nabla^2\nabla_i L + \nabla^2 L_i \\ \Delta_\epsilon W_i &= \int D\nabla^2(2\nabla_i L + L_i) \\ \Delta_\epsilon(g^{ij}h_{ij}) &= 2\nabla^2 L \end{aligned} \tag{C.4}$$

$$\Delta_\epsilon \tag{C.5}$$

$$
\begin{aligned}
\underbrace{\bar{h}_{00}}_{-2\bar{\phi}} &= \underbrace{h_{00}}_{-2\phi} -2\dot{T} \\
\underbrace{\int D\nabla^j \bar{h}_{0j}}_{\bar{B}} &= \underbrace{\int D\nabla^j h_{0j}}_{B} + \underbrace{\int D\nabla^2(\dot{L}-T)}_{(\dot{L}-T)^L} \\
\underbrace{\bar{h}_{0i} - \nabla_i \int D\nabla^j \bar{h}_{0j}}_{\bar{B}_i} &= \underbrace{h_{0i} - \nabla_i \int D\nabla^j h_{0j}}_{B_i} + \dot{L}_i + \underbrace{\nabla_i(\dot{L}-T) - \nabla_i \int D\nabla^2(\dot{L}-T)}_{\nabla_i(\dot{L}-T)^T} \\
\underbrace{\tfrac{1}{4}\nabla^i \bar{W}_i - \tfrac{1}{4}g^{ij}\bar{h}_{ij}}_{\bar{\psi}} &= \underbrace{\tfrac{1}{4}\nabla^i W_i - \tfrac{1}{4}g^{ij}h_{ij}}_{\psi} - \tfrac{1}{2}\nabla^2 L + \tfrac{1}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) \\
\underbrace{\int D(\tfrac{3}{4}\nabla^i \bar{W}_i - \tfrac{1}{4}g^{ij}\bar{h}_{ij})}_{\bar{E}} &= \underbrace{\int D(\tfrac{3}{4}\nabla^i W_i - \tfrac{1}{4}g^{ij}h_{ij})}_{E} + \int D\left(\tfrac{3}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) - \tfrac{1}{2}\nabla^2 L\right) \\
\underbrace{\bar{W}_i - \nabla_i \int D\nabla^j \bar{W}_j}_{\bar{E}_i} &= \underbrace{W_i - \nabla_i \int D\nabla^j W_j}_{E_i} + \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2(\tfrac{4}{3}\nabla_j L + L_j) \\
2\bar{E}_{ij} - 2E_{ij} &= 2\nabla_i\nabla_j L + \nabla_i L_j + \nabla_j L_i - \nabla_i \int D\nabla^2(2\nabla_j L + L_j) - \nabla_j \int D\nabla^2(2\nabla_i L + L_i) \\
&\quad -\tfrac{1}{2}g_{ij}\left(2\nabla^2 L - \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right) \\
&\quad +\tfrac{1}{2}\nabla_i\nabla_j \int D\left(2\nabla^2 L + \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right)
\end{aligned}
\tag{C.6}
$$

We may also include the trace condition

$$
-6\bar{\psi} + 2\nabla^2 \bar{E} = -6\psi + 2\nabla^2 E + 2\nabla^2 L \tag{C.7}
$$

From integrating the identity

$$
\nabla^2 D\phi = D\nabla^2\phi + \nabla_i\left(\nabla^i\phi D - \nabla^i D\phi\right), \tag{C.8}
$$

we may decompose a general scalar ϕ into its harmonic (T) and non-harmonic (L) pieces viz

$$
\phi = \underbrace{\int_V D\nabla^2\phi}_{\phi^L} + \underbrace{\oint_{\partial V} dS_i\left(D\nabla^i\phi - \nabla^i D\phi\right)}_{\phi^T}. \tag{C.9}
$$

The harmonic ϕ^T is defined only upon the boundary surface with $\nabla^2\phi^T$ vanishing identically for any ϕ and with $\nabla^2\phi^L = 0$ only vanishing for $\phi^L = 0$. From (C.9) we see that if we require

1. $\phi(x) = 0 \quad \text{for} \quad x \in \partial V$
2. $\nabla_i D(x,y) = 0 \quad \text{for} \quad x \in \partial V$

then we may always use $\phi = \int D\nabla^2\phi$. By definition of the Green's function equation

$$
\nabla^2 D(x,y) = \delta(x-y) \tag{C.10}
$$

we may add to $D(x,y)$ a two-point function $F(x,y)$ that satisfies $\nabla^2 F(x,y) = 0$ (i.e. a harmonic F). Such an F must also be entirely defined on the boundary and thus we may use this freedom to construct a $D(x,y)$ such that $\nabla_i D(x,y) = 0$ for $x \in \partial V$.

The above conditions correspond to Dirichlet boundary conditions, however we may instead impose Neumann boundary conditions and use F to construct a Green's function that vanishes on the boundary itself. As expected from a PDE, the solution of the general $\nabla^2\phi = \rho$ has to include boundary conditions.

Rexpressing (C.6)

$$
\begin{aligned}
\bar{\phi} &= \phi + \dot{T} \\
\bar{B} &= B + \int D\nabla^2(\dot{L} - T) \\
\bar{B}_i &= B_i + \dot{L}_i + \nabla_i(\dot{L} - T) - \nabla_i \int D\nabla^2(\dot{L} - T) \\
\bar{\psi} &= \psi - \tfrac{1}{2}\nabla^2 L + \tfrac{1}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) \\
\bar{E} &= E + \int D\left(\tfrac{3}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) - \tfrac{1}{2}\nabla^2 L\right) \\
\bar{E}_i &= E_i + \int D\nabla^2(\tfrac{4}{3}\nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2(\tfrac{4}{3}\nabla_j L + L_j) \\
\bar{E}_{ij} &= E_{ij} + \nabla_i\nabla_j L + \tfrac{1}{2}\nabla_i L_j + \tfrac{1}{2}\nabla_j L_i - \tfrac{1}{2}\nabla_i \int D\nabla^2(2\nabla_j L + L_j) - \tfrac{1}{2}\nabla_j \int D\nabla^2(2\nabla_i L + L_i) \\
&\quad - \tfrac{1}{4}g_{ij}\left(2\nabla^2 L - \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right) \\
&\quad + \tfrac{1}{4}\nabla_i\nabla_j \int D\left(2\nabla^2 L + \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right) \qquad \text{(C.11)}
\end{aligned}
$$

If we now restrict to gauge transformations that vanish asymptotically, we may then utilize $\phi = \int D\nabla^2\phi$ and the gauge structure becomes the familiar

$$
\begin{aligned}
\bar{\phi} &= \phi + \dot{T} \\
\bar{B} &= B + \dot{L} - T \\
\bar{\psi} &= \psi \\
\bar{E} &= E + L \\
\bar{B}_i &= B_i + \dot{L}_i \\
\bar{E}_i &= E_i + \dot{L}_i \\
\bar{E}_{ij} &= E_{ij} \qquad \text{(C.12)}
\end{aligned}
$$

with gauge invariant quantities

$$
\bar{\psi} = \psi, \qquad \bar{\phi} + \dot{\bar{B}} - \ddot{\bar{E}} = \phi + \dot{B} - \ddot{E}, \qquad \bar{B}_i - \dot{\bar{E}}_i = B_i - \dot{E}_i, \qquad \bar{E}_{ij} = E_{ij}. \qquad \text{(C.13)}
$$

$$
\begin{aligned}
\delta G_{00} &= -2\nabla^2\psi \\
\delta G_{0i} &= -2\nabla_i\dot{\psi} + \tfrac{1}{2}\nabla^2(B_i - \dot{E}_i) \\
\delta G_{ij} &= -2g_{ij}\ddot{\psi} + g_{ij}\nabla^2\psi - \nabla_i\nabla_j\psi - g_{ij}\nabla^2(\phi + \dot{B} - \ddot{E}) + \nabla_i\nabla_j(\phi + \dot{B} - \ddot{E}) \\
&\quad + \tfrac{1}{2}\nabla_i(\dot{B}_j - \ddot{E}_j) + \tfrac{1}{2}\nabla_j(\dot{B}_i - \ddot{E}_i) + \nabla^2 E_{ij} - \ddot{E}_{ij} \\
g^{ij}\delta G_{ij} &= -6\ddot{\psi} + 2\nabla^2\psi - 2\nabla^2(\phi + \dot{B} - \ddot{E}) \qquad \text{(C.14)}
\end{aligned}
$$

For the Weyl case, in taking the transverse vector component of δW_{0i}, we will incur terms that depend on the boundary as

$$\nabla^2\Psi - \int D\nabla^4\Psi = \oint dS^i \left[\nabla_i D\nabla^2\Psi - D\nabla_i\nabla^2\Psi\right] \tag{C.15}$$

whereby we see that is must be $\nabla^2\Psi$ that vanishes on the boundary rather than Ψ itself.

5 Curved Space Transverse Projector

Recall the flat space transverse projector

$$h^T_{\mu\nu} \quad = \quad h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu\nabla_\nu \int D\nabla_\sigma W^\sigma \tag{5.1}$$

where $W_\mu = \int D\nabla^\sigma h_{\mu\sigma}$ and we define D as the Green's function that obeys

$$\nabla_\alpha\nabla^\alpha D(x,x') \quad = \quad \sqrt{g}\delta^3(x-x') \tag{5.2}$$

To generalize this to curved space, we must introduce a two index Green's function $D_{\mu\alpha'}(x,x')$ (i.e. a bi-tensor). In this way, W_μ will be defined as

$$W_\mu = \int D_\mu{}^{\sigma'}(x,x')\nabla^{\rho'}h_{\sigma'\rho'}. \tag{5.3}$$

For a manifold with non-vanishing Riemann tensor, Vierbiens are position dependent.

$$h^T_{\mu\nu} \quad = \quad h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu\nabla_\nu \int D_{\sigma'}{}^{\rho''}\nabla_{\rho''}W_{\sigma''} \tag{5.4}$$

$$\begin{aligned}\nabla^\sigma\nabla_\mu W_\sigma &= \nabla_\mu\nabla^\sigma W_\sigma - R_\mu{}^\sigma W_\sigma \\ \nabla^\sigma\nabla_\mu\nabla_\sigma A &= \nabla_\mu\nabla^\sigma\nabla_\sigma A - R_\mu{}^\sigma\nabla_\sigma A\end{aligned} \tag{5.5}$$

$$h^T_{\mu\nu} \quad = \quad h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu\nabla_\nu \int D\nabla^{\sigma'}W_{\sigma'} \tag{5.6}$$

$$(\nabla_\alpha\nabla^\alpha - \frac{R}{D})D_\sigma{}^{\rho'}(x,x') \quad = \quad g_\sigma{}^{\rho'}\sqrt{g}\delta^3(x,x') \tag{5.7}$$

Now using

$$\begin{aligned}g^\kappa{}_{\rho'}g^{\rho'}{}_\sigma &= \delta^\kappa{}_\sigma \\ g^\sigma{}_{\rho'}g^{\rho'}{}_\sigma &= D\end{aligned} \tag{5.8}$$

we may relate the two index $D_\sigma{}^{\rho'}$ to a scalar $D(x,x')$ by

$$g^\sigma{}_{\rho'}D_\sigma{}^{\rho'} \quad \equiv \quad DD(x,x') \tag{5.9}$$

such that $D(x,x')$ obeys

$$(\nabla_\alpha\nabla^\alpha - \frac{R}{D}) \quad = \quad \sqrt{g}\delta^3(x,x') \tag{5.10}$$

$$W_\mu = \int D_\mu{}^{\rho'}(x,x')\nabla^{\sigma'}h_{\rho'\sigma'} \tag{5.11}$$

$$h^T_{\mu\nu} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu \nabla_\nu \int D \nabla^{\sigma'} W_{\sigma'} \tag{5.12}$$

$$\begin{aligned} h^T &= g^{\rho\sigma} h^T_{\rho\sigma} \\ &= h - 2\nabla^\sigma W_\sigma + \nabla_\alpha \nabla^\alpha \int D \nabla^{\sigma'} W_{\sigma'} \end{aligned} \tag{5.13}$$

$$h^{T\theta}_{\mu\nu} = h^T_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} h^T + \frac{1}{d-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{d(d-1)} \right] \int F h^T \tag{5.14}$$

For the special case that $h_{\mu\nu}$ is apriori transverse, then it follows that $W_\mu = 0$ and $h^T = h$ and thus the transverse traceless component takes the simple form

$$h^{T\theta}_{\mu\nu} = h_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} h + \frac{1}{d-1} \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \frac{R}{d(d-1)} \right] \int F h^T \tag{5.15}$$

$$h^{T\theta}_{\mu\nu} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu \nabla_\nu \int D \nabla^{\sigma'} W_{\sigma'} - \frac{1}{d-1} g_{\mu\nu} \tag{5.16}$$

$$h_{\mu\nu} = \underbrace{h^T_{\mu\nu}}_{h^{TT}_{\mu\nu} + h^{TNT}_{\mu\nu}} + \underbrace{h^L_{\mu\nu}}_{h^{LT}_{\mu\nu} + h^{LNT}_{\mu\nu}} \tag{5.17}$$