Cosmological Fluctuations in Standard and Conformal Gravity

Matthew Phelps

Doctoral Degree Final Examination

June 02, 2020

Overview

- Introduction and Formalism
  - Cosmological Geometries
  - Einstein Gravity
  - Perturbation Theory
  - Gauge Transformations
- Conformal Gravity

Comoving Robertson Walker geometry:

$$
\begin{aligned}
ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\
&= -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]
\end{aligned}
$$

3-Space Curvature Tensors,

$$
R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -3k\tilde{g}_{ij}, \qquad R = -6k
$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$
\tau = \int \frac{dt}{a(t)},
$$

$$
ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]
$$

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \qquad R_{ij} = -3k\tilde{g}_{ij}, \qquad R = -6k$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

set $k = 0$,

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]$$

Cosmological Geometries R.W. $k = \pm 1$

$k = 1$

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

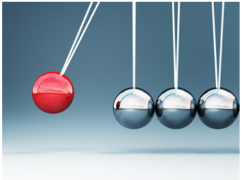
$k = -1$

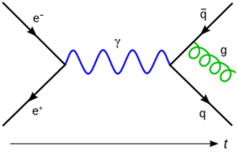
$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

Effective Physics Theories

- General Relativity

Length
and
Energy
Scale





Effective Physics Theories

- General Relativity
  - Geometric theory of gravitation

Length
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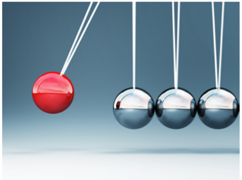
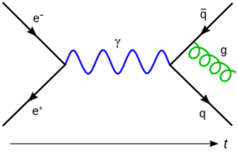
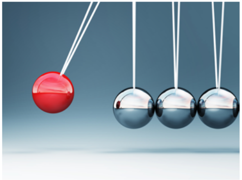


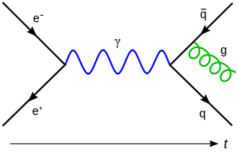


Effective Physics Theories

- General Relativity
  - Geometric theory of gravitation
  - Relativity of time

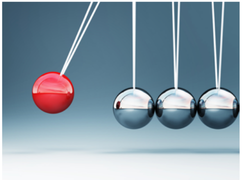
Length and Energy Scale

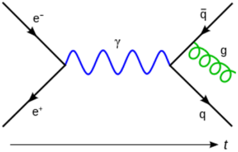




- General Relativity
  - Geometric theory of gravitation
  - Relativity of time
  - Black holes, cosmology, gps

Length and Energy Scale





Effective Physics Theories

- General Relativity
  - Geometric theory of gravitation
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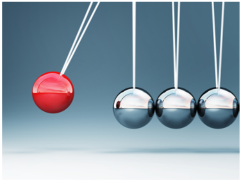
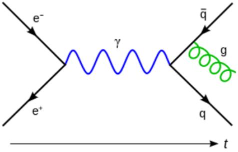
  - Black holes, cosmology, gps
- Newtonian Mechanics

Length and Energy Scale





Effective Physics Theories

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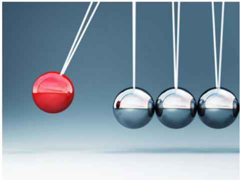
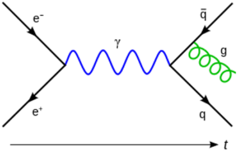
Length and Energy Scale





Effective Physics Theories

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  - Gravity as a force
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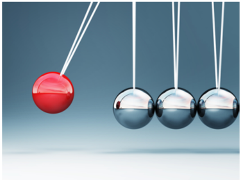
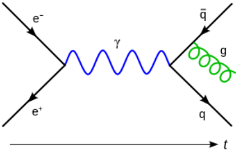
Length and Energy Scale





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  - Breaks down around $1\,\mu\mathrm{m}$

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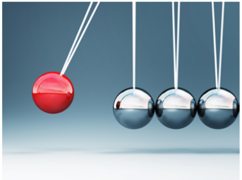

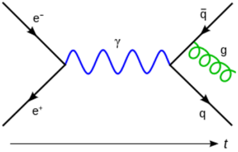

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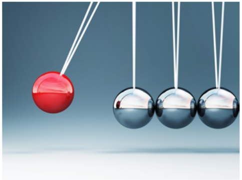
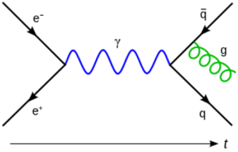
  - Breaks down around $1\,\mu\mathrm{m}$
- Quantum Theory

Length and Energy Scale





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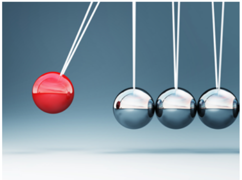
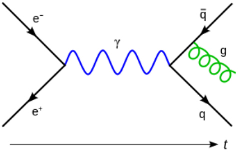
Length and Energy Scale





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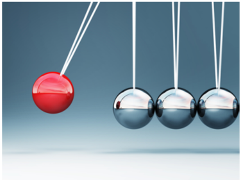
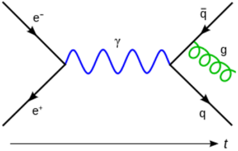
  - Quantum Mechanics
  - Quantum Field Theory

Length and Energy Scale





Effective Physics Theories

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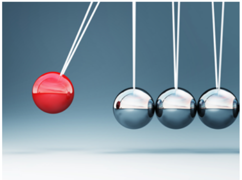
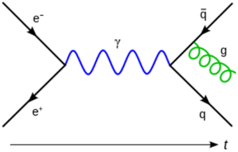
- Quantum Theory
  - Quantum Mechanics
  - Quantum Field Theory
  - Fundamental Particles

Length and Energy Scale





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  - Black holes, cosmology, gps
- Newtonian Mechanics
  - Gravity as a force
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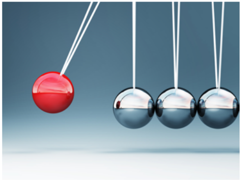
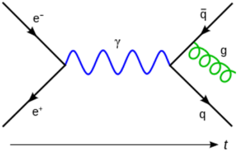
  - Breaks down around $1\ \mu\mathrm{m}$
- Quantum Theory
  - Quantum Mechanics
  - Quantum Field Theory
  - Fundamental Particles
  - Probabilistic flow of quantum states

Length and Energy Scale





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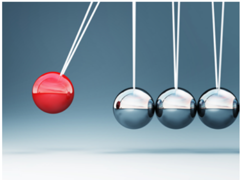
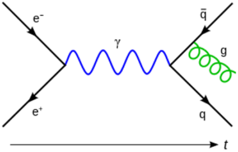
- Quantum Theory
  - Quantum Mechanics
  - Quantum Field Theory
  - Fundamental Particles
  - Probabilistic flow of quantum states
  - Discrete physics properties

Length and Energy Scale





- Independent of mass, force

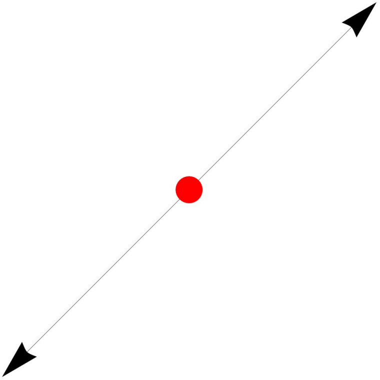
1D Kinematics

- Independent of mass, force
- Restrict to $D = 1$

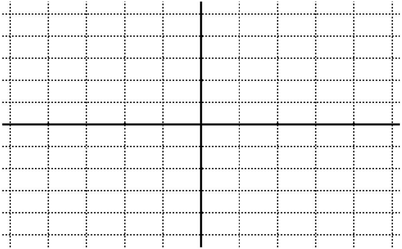
1D Kinematics

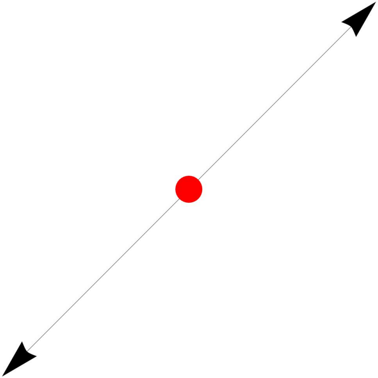
- Independent of mass, force
- Restrict to $D = 1$
- $\frac{d\mathbf{F}}{dt} = 0$

1D Motion: Analysis

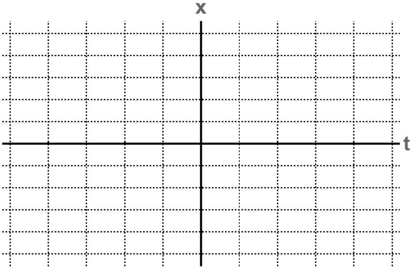


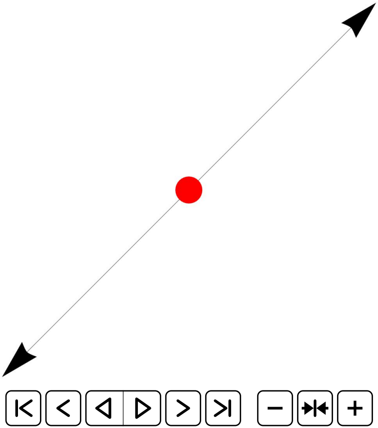
1D Motion: Analysis



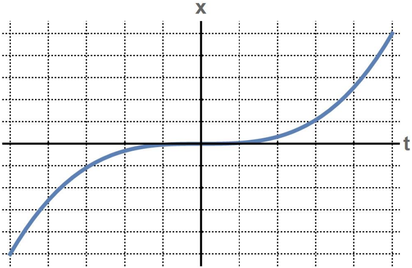


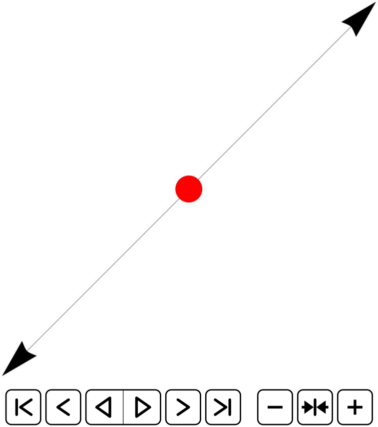
1D Motion: Analysis



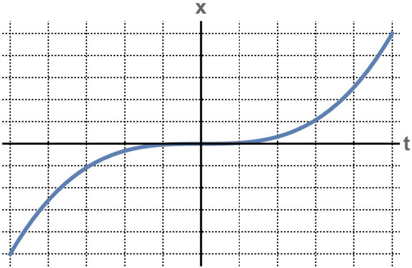


1D Motion: Analysis

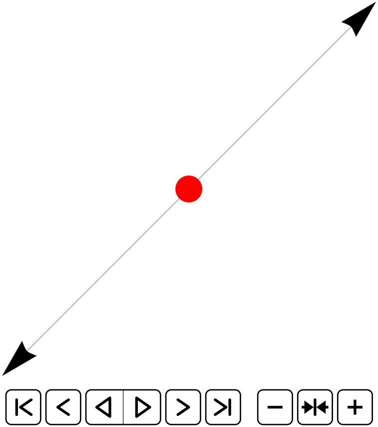




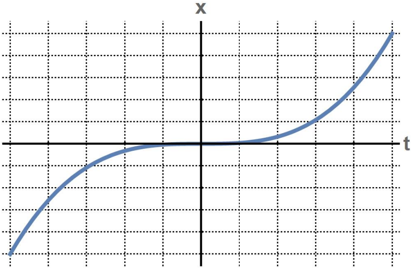
1D Motion: Analysis



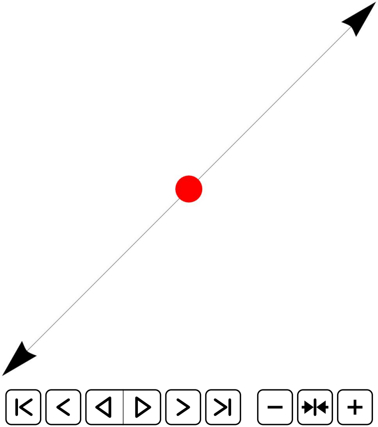
- Distance

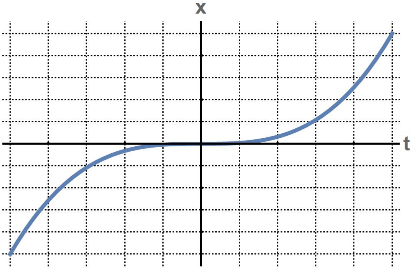


1D Motion: Analysis

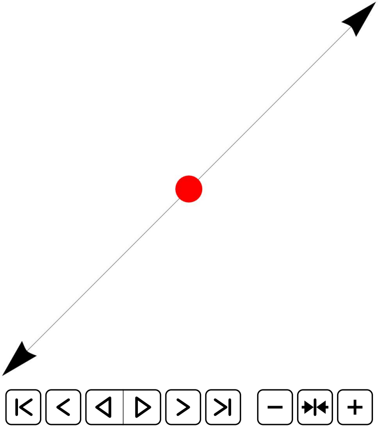


- Distance
- Position (Displacement)





- Distance
- Position (Displacement)
- Speed



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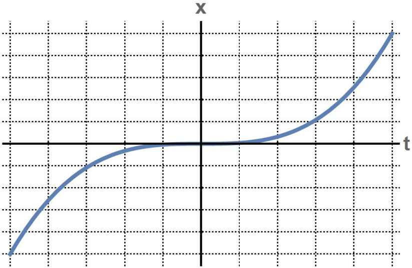
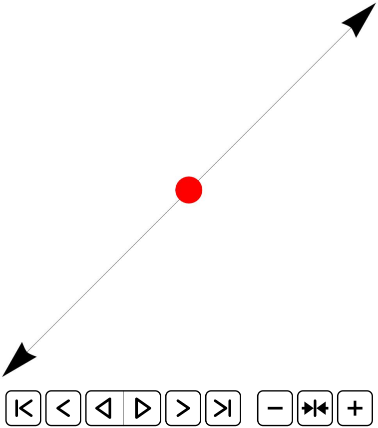
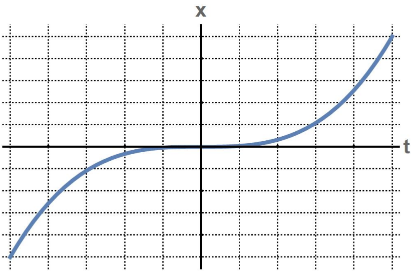


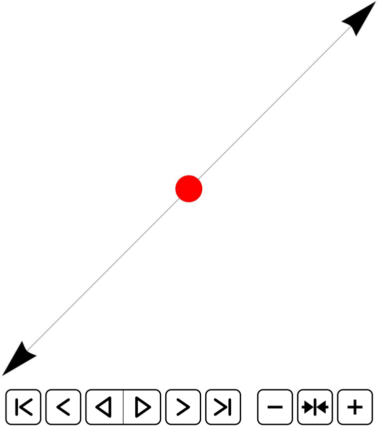
- Distance
- Position (Displacement)
- Speed
- Velocity



1D Motion: Average Velocity



$$v_{avg} = \frac{\Delta x}{\Delta t} \qquad |v| = \text{speed}$$



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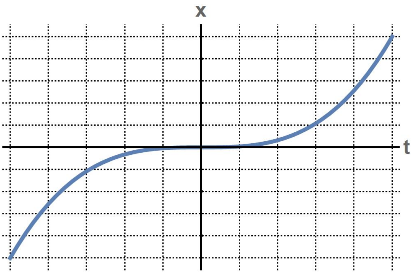
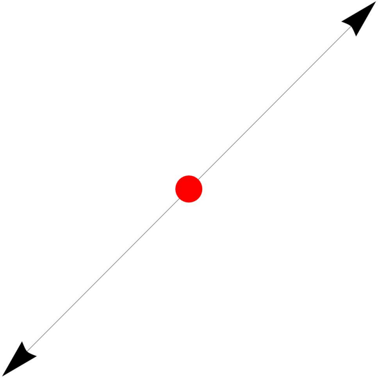


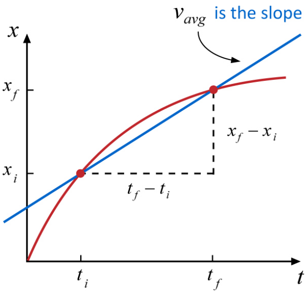


$$v_{avg} = \frac{\Delta x}{\Delta t} \qquad |v| = \text{speed}$$

What about $v(t)$?

1D Motion: Instantaneous Velocity

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x(t_f) - x(t_i)}{t_f - t_i}$$

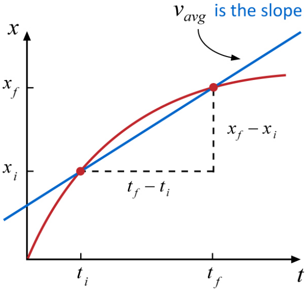


1D Motion: Instantaneous Velocity

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x(t_f) - x(t_i)}{t_f - t_i}$$

$$t_f = t_i + \Delta t$$

$$v_{avg} = \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t}$$



1D Motion: Instantaneous Velocity

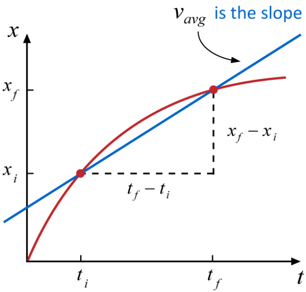
$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x(t_f) - x(t_i)}{t_f - t_i}$$

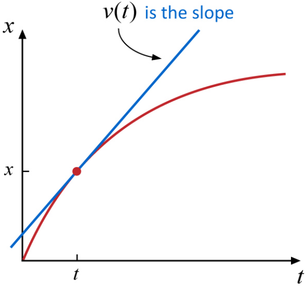
$$t_f = t_i + \Delta t$$

$$v_{avg} = \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t}$$

$$v(t_i) = \lim_{\Delta t \to 0} \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} = \left.\frac{dx}{dt}\right|_{t=t_i}$$

$$v = \frac{dx}{dt}$$





1D Motion: Average Acceleration

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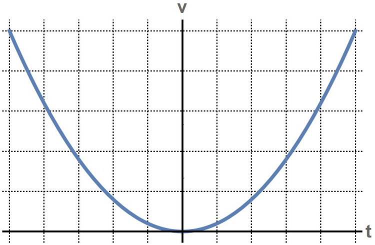
What about $a(t)$?



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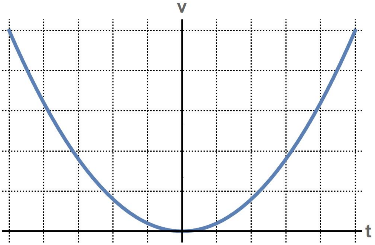
What about $a(t)$?

$$t_f = t_i + \Delta t$$

$$a_{avg} = \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t}$$



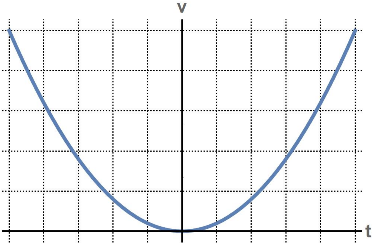
1D Motion: Average Acceleration

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v(t_f) - v(t_i)}{t_f - t_i}$$

What about $a(t)$?

$$t_f = t_i + \Delta t$$

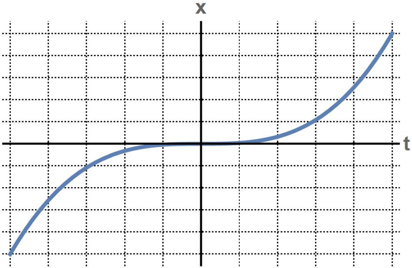
$$a_{avg} = \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t}$$



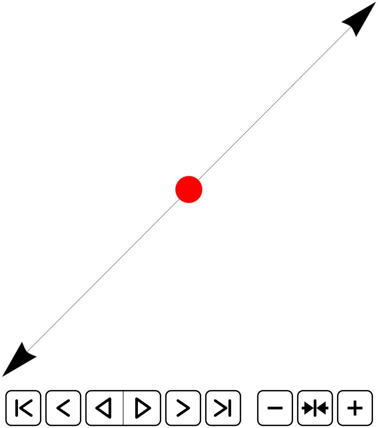
$$a(t_i) = \lim_{\Delta t \to 0} \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t} = \left.\frac{dv}{dt}\right|_{t=t_i}$$

$$a = \frac{dv}{dt}$$

1D Motion: Vocabulary



- Distance
- Displacement
- Speed
- Average Velocity
- Instantaneous Velocity
- Average Acceleration
- Instantaneous Acceleration



A particle's trajectory is given by the function

$$x(t) = (21 + 22t - 6t^2),$$

with t, x in SI units. What is the average velocity from $t = 1$ to $t = 3$?

(A) $2\,\mathrm{m/s}$
(B) $-4\,\mathrm{m/s}$
(C) $-2\,\mathrm{m/s}$
(D) $-8\,\mathrm{m/s}$
(E) $8\,\mathrm{m/s}$

iClicker 1

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(C) $\boxed{-2\,\mathrm{m/s}}$

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(E) $8\,\mathrm{m/s}$

$$\begin{aligned} v_{avg} &= \frac{x(3) - x(1)}{3 - 1} \\ &= \frac{33 - 37}{2} \\ &= -2\,\mathrm{m/s} \end{aligned}$$

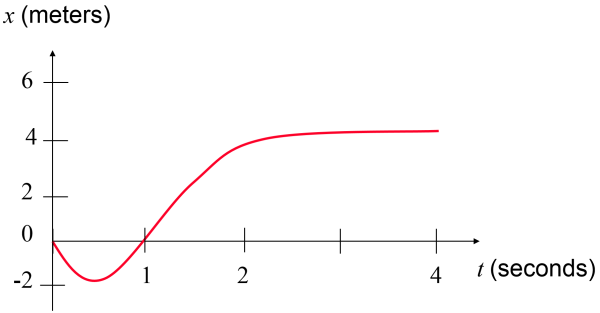
What is the velocity (instantaneous) at $t = 4$?

(A) $4\,\mathrm{m/s}$

(B) $0\,\mathrm{m/s}$

(C) $1\,\mathrm{m/s}$

(D) Not enough information



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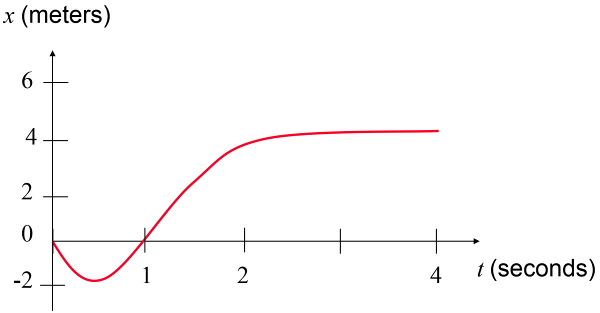


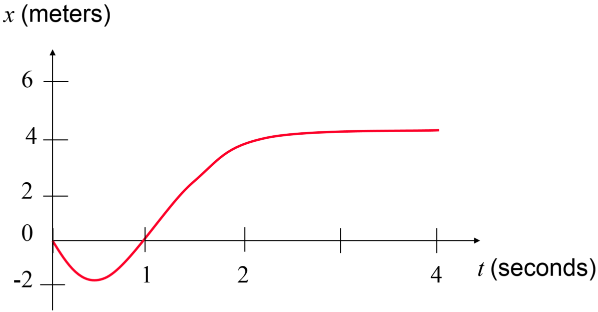
What is the average velocity over the first 4 seconds?

(A) $-2\,\mathrm{m/s}$

(B) $4\,\mathrm{m/s}$

(C) $1\,\mathrm{m/s}$

(D) Not enough information



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What is the average velocity over the first 4 seconds?

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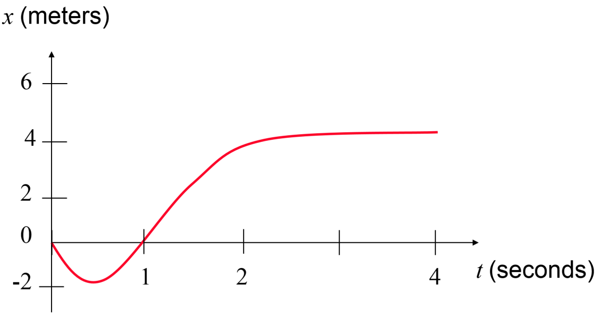

Calculus: Derivatives

Given $f(x)$,

$$\frac{df}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

$$f(x) = x^n \qquad \frac{df}{dx} = nx^{n-1}$$

$$f(x) = \sin(x) \qquad \frac{df}{dx} = \cos x$$

$$f(x) = \cos(x) \qquad \frac{df}{dx} = -\sin x$$

Calculus: Integrals

Definite:

$$\int_a^b f(x)\ dx$$

$$\begin{aligned}\int_a^b x^n\ dx &= \frac{x^{n+1}}{n+1}\bigg|_b^a \\ &= \frac{a^{n+1}}{(n+1)} - \frac{b^{n+1}}{(n+1)}\end{aligned}$$

Indefinite:

$$\int f(x)\ dx$$

$$\int x^n\ dx = \frac{x^{n+1}}{(n+1)} + C$$

1D Kinematic Equations: Derivation

- Fundamental Theorem of Calculus
- $\mathbf{a} =$ constant
- Boundaries

1D Kinematic Equations

Arbitrary Time Interval:

$$v(t_f) - v(t_i) = a(t_f - t_i)$$

$$x(t_f) - x(t_i) = v(t_i) + \frac{1}{2}a(t_f - t_i)^2$$

1D Kinematic Equations

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Initial Time Interval:

$$t_i = 0, \quad t_f = t$$
$$v(0) = v_0, \quad x(0) = x_0$$

$$\boxed{v(t) = v_0 + at}$$

$$\boxed{x(t) = x_0 + v_0 t + \frac{1}{2}at^2}$$

1D Kinematic Equations

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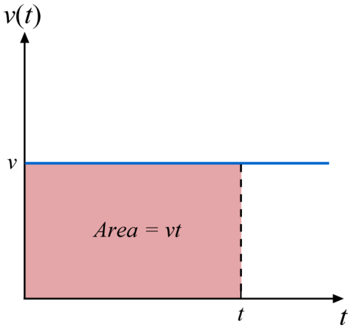
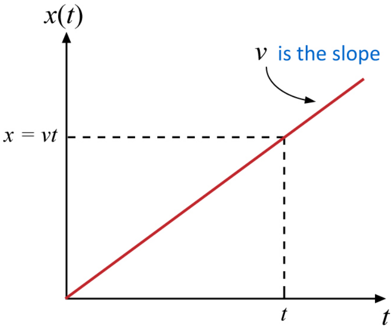
$$\boxed{x(t) = x_0 + v_0 t + \frac{1}{2}at^2}$$

Solve for common t:

$$\boxed{2a(x - x_0) = v^2 - v_0^2}$$

Example: Velocity Integration





$$\int_a^b v(t)\ dt = \int_a^b \left(\frac{dx}{dt}\right)\ dt$$
$$= x(b) - x(a)$$

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The position of a particle is given by

$$x(t) = 6t - 3t^2.$$

Determine the x position at which the particle's velocity is zero.

(A) $3\,\mathrm{m}$

(B) $2\,\mathrm{m}$

(C) $-1\,\mathrm{m}$

(D) $-3\,\mathrm{m}$

iClicker 4

The position of a particle is given by

$$x(t) = 6t - 3t^2.$$

Determine the x position at which the particle's velocity is zero.

$$v = \frac{dx}{dt} = 6 - 6t$$

- (A) $\boxed{3\,\mathrm{m}}$
- (B) $2\,\mathrm{m}$
- (C) $-1\,\mathrm{m}$
- (D) $-3\,\mathrm{m}$

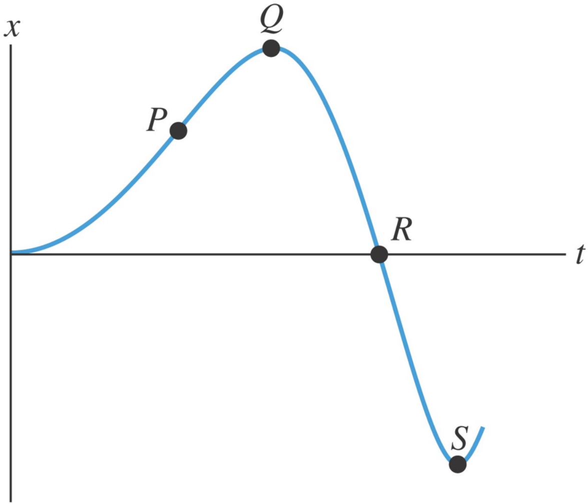
$$6 - 6t = 0$$

$$t = 1$$

$$x(1) = 3\,\mathrm{m}$$

At which point is the speed the greatest?

(A) P
(B) Q
(C) R
(D) S



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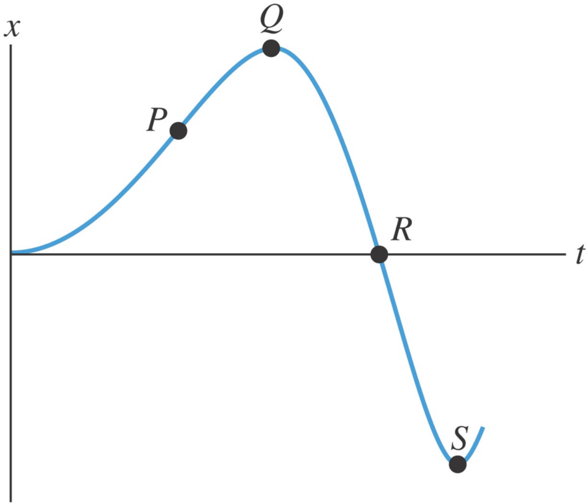
At which point is the speed the greatest?

(A) P

(B) Q

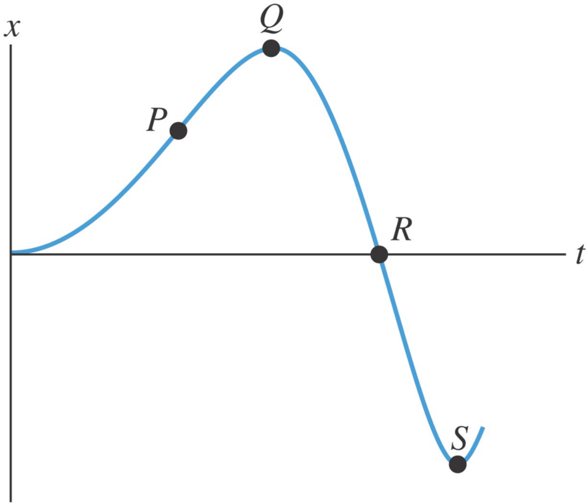
(C) $\boxed{\text{R}}$

(D) S



At which point does the object have maximum positive velocity?

(A) P
(B) Q
(C) R
(D) S



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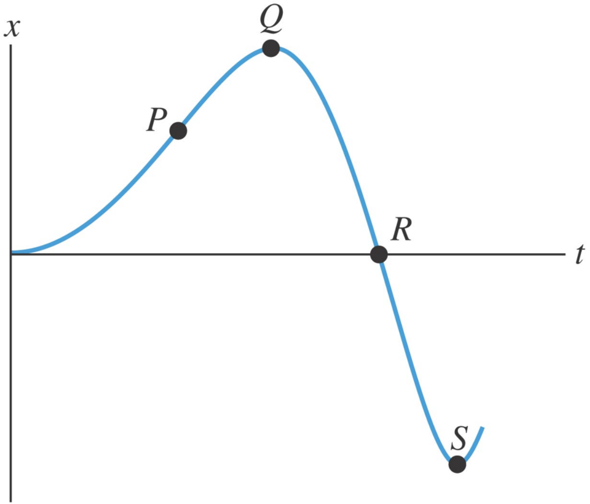

Example: Vertical Ball Throw

A ball is thrown upward vertically from an initial height of $5\,\mathrm{m}$ and with an initial velocity of $10\,\mathrm{m/s}$. Determine $x(t)$, $v(t)$, and the maximum height the ball reaches.

What time does it hit the ground?

Reminders

- 1D Kinematics Homework due Tuesday Sep. 4 (11:59PM)
- Vectors and 2D Kinematics Prelecture and Checkpoint due Wednesday Sep. 5 (11:25 AM)
- Lab begins week of Sep. 10
  - Lab Notebook
  - Safety Glasses
  - Ruler/Calculator