$\delta W_{\mu\nu}$ 3+1 in RW

Within the metric

$$-ds^2 = -d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

any Christoffel symbol with a time index will vanish. Therefore, in looking at the Riemann tensor

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} \tag{2}$$

we note that any time component of the Riemann tensor will vanish, i.e. only spatial indices are nonzero. If we construct the four velocity projector

$$U^{\mu\nu} = u^\mu u^\nu, \qquad \text{where} \quad u^\mu = \frac{dx^\mu}{d\tau} = (1,0,0,0), \tag{3}$$

and the 3-space projector

$$P_{\mu\nu} = g_{\mu\nu} + U_{\mu\nu}, \tag{4}$$

it follows that

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= (P_\lambda{}^\alpha - U_\lambda{}^\alpha)(P_\mu{}^\beta - U_\mu{}^\beta)(P_\nu{}^\sigma - U_\nu{}^\sigma)(P_\kappa{}^\rho - U_\kappa{}^\rho)R_{\alpha\beta\sigma\rho} \\ &= P_\lambda{}^\alpha P_\mu{}^\beta P_\nu{}^\sigma P_\kappa{}^\rho R_{\alpha\beta\sigma\rho}. \end{aligned} \tag{5}$$

Hence, the Riemann tensor effectively represents the curvature only of the underlying 3-space. Given metric (1), the 3-space is maximally symmetric and so the curvature relations are

$$R_{\lambda\mu\nu\kappa} = k(P_{\mu\nu}P_{\lambda\kappa} - P_{\lambda\nu}P_{\mu\kappa}) \tag{6}$$

$$R_{\mu\nu} = -2kP_{\mu\nu} \tag{7}$$

$$R^\mu{}_\mu = -6k \tag{8}$$

In evaluating $\delta W_{\mu\nu}$ in the metric of (1), we recall

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}(\tfrac{h}{4} g^{(0)}_{\mu\nu}) \tag{9}$$

where

$$\delta W_{\mu\nu}(\tfrac{h}{4} g^{(0)}_{\mu\nu}) = -\frac{h}{4} W^{(0)}_{\mu\nu}. \tag{10}$$

Since (1) can be expressed in a the conformal to flat form (upon taking $a(\tau) = 1$) for arbitrary k, we know the background piece $W^{(0)}_{\mu\nu}$ must vanish. This has also been confirmed by taking $W_{\mu\nu}$ and directly inserting curvature relations (6-8). As such, the fluctuation equations will only depend on $K_{\mu\nu}$.

To calculate $\delta W_{\mu\nu}$ in (1), first take $\delta W_{\mu\nu}$ in gauge ready form

$$
\begin{aligned}
\delta W_{\mu\nu} = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha} - \tfrac{1}{2}K_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta \\
& - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}K_\mu{}^\alpha RR_{\nu\alpha} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \tfrac{1}{2}K_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\zeta}R_{\alpha\gamma\beta\zeta} \\
& - \tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\zeta}R_\mu{}^\gamma{}_\nu{}^\zeta + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\beta K_\mu{}^\beta + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\beta K_\nu{}^\beta \\
& - \tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \tfrac{1}{3}g_{\mu\nu}\nabla_\alpha K^{\gamma\zeta}\nabla^\alpha R_{\gamma\zeta} - 2\nabla_\alpha K^{\gamma\zeta}\nabla^\alpha R_{\mu\gamma\nu\zeta} + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta \\
& - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} \\
& - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \tfrac{2}{3}g_{\mu\nu}R^{\gamma\zeta}\nabla_\beta\nabla^\beta K_{\gamma\zeta} - 2R_{\mu\gamma\nu\zeta}\nabla_\beta\nabla^\beta K^{\gamma\zeta} + \tfrac{1}{3}R\nabla_\beta\nabla^\beta K_{\mu\nu} \\
& + \tfrac{1}{6}g_{\mu\nu}K^{\gamma\zeta}\nabla_\beta\nabla^\beta R_{\gamma\zeta} + \tfrac{1}{2}K_\nu{}^\gamma\nabla_\beta\nabla^\beta R_{\mu\gamma} + \tfrac{1}{2}K_\mu{}^\gamma\nabla_\beta\nabla^\beta R_{\nu\gamma} - K^{\gamma\zeta}\nabla_\beta\nabla^\beta R_{\mu\gamma\nu\zeta} \\
& + \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\zeta\nabla_\gamma K^{\gamma\zeta} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\gamma K_\nu{}^\gamma - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\gamma K_\mu{}^\gamma \\
& - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma + R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} + R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha \\
& + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \tfrac{1}{6}g_{\mu\nu}K_{\alpha\beta}\nabla^\beta\nabla^\alpha R - \tfrac{1}{6}K_{\mu\nu}\nabla^\beta\nabla_\beta R + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} \\
& + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{2}\nabla_\zeta\nabla^\zeta\nabla_\beta\nabla^\beta K_{\mu\nu} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} \\
& - \tfrac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta - \tfrac{1}{6}R^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta} + \tfrac{1}{3}K^{\alpha\beta}\nabla_\mu\nabla_\nu R_{\alpha\beta} \\
& + \tfrac{1}{3}\nabla_\mu\nabla_\nu\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} \\
& + \tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \tfrac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta - \tfrac{1}{2}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta},
\end{aligned} \tag{11}
$$

then impose tranvserse gauge

$$
\nabla_\mu K^{\mu\nu} = 0, \tag{12}
$$

such that $\delta W_{\mu\nu}$ is now

$$
\begin{aligned}
\delta W_{\mu\nu} = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha} - \tfrac{1}{2}K_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta \\
& - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}K_\mu{}^\alpha RR_{\nu\alpha} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \tfrac{1}{2}K_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\zeta}R_{\alpha\gamma\beta\zeta} \\
& - \tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\zeta}R_\mu{}^\gamma{}_\nu{}^\zeta + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R \\
& - \tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \tfrac{1}{2}K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} \\
& + \tfrac{1}{2}K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} + R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} + R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} \\
& + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \tfrac{1}{6}g_{\mu\nu}K_{\alpha\beta}\nabla^\beta\nabla^\alpha R + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} \\
& - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} \\
& + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} \\
& - \tfrac{1}{6}R^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta} + \tfrac{1}{3}K^{\alpha\beta}\nabla_\mu\nabla_\nu R_{\alpha\beta} - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} \\
& + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \tfrac{1}{2}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta}.
\end{aligned} \tag{13}
$$

Now substitute curvature relations (6-8) and impose

$$
\nabla_\mu K^{\mu\nu} = 0, \qquad \nabla_\mu U_{\alpha\beta} = 0, \qquad U^\alpha{}_\alpha = -1, \qquad U^{\alpha\beta}U_{\alpha\lambda} = -U^\beta{}_\lambda, \qquad K^\alpha{}_\alpha = 0. \tag{14}
$$

This yields

$$
\begin{aligned}
\delta W_{\mu\nu} = & \, 2k^2K_{\mu\nu} - \tfrac{2}{3}k^2g_{\mu\nu}K_{\alpha\beta}U^{\alpha\beta} + 4k^2K_{\nu\alpha}U_\mu{}^\alpha + \tfrac{4}{3}k^2K_{\alpha\beta}U^{\alpha\beta}U_{\mu\nu} - 2k^2K_{\alpha\beta}U^{\beta\alpha}U_{\mu\nu} \\
& + 4k^2K_{\mu\alpha}U_\nu{}^\alpha - 4k^2K_{\alpha\beta}U_\mu{}^\beta U_\nu{}^\alpha + 10k^2K_{\alpha\beta}U_\mu{}^\alpha U_\nu{}^\beta + \tfrac{2}{3}kg^{\alpha\beta}g_{\mu\nu}U^{\gamma\zeta}\nabla_\beta\nabla_\alpha K_{\gamma\zeta} \\
& + 2kg^{\alpha\beta}U^{\gamma\zeta}U_{\mu\nu}\nabla_\beta\nabla_\alpha K_{\gamma\zeta} - 2kg^{\alpha\beta}U_\mu{}^\gamma U_\nu{}^\zeta\nabla_\beta\nabla_\alpha K_{\gamma\zeta} - 2kg^{\alpha\beta}U_\nu{}^\gamma\nabla_\beta\nabla_\alpha K_{\mu\gamma} \\
& - 2kg^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + 2kU^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} - 2kg^{\alpha\beta}U_\mu{}^\gamma\nabla_\beta\nabla_\alpha K_{\nu\gamma} - 2kU^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} \\
& - 2kU^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} + \tfrac{1}{2}g^{\alpha\zeta}g^{\gamma\beta}\nabla_\zeta\nabla_\alpha\nabla_\beta\nabla_\gamma K_{\mu\nu} + \tfrac{1}{3}kU^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta} \\
& + kU^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta}.
\end{aligned} \tag{15}
$$

Now perform 3+1 split, where Latin indices are spatial and

$$K'_{\mu\nu} = \frac{\partial K_{\mu\nu}}{\partial \tau}, \qquad \nabla^a\nabla_a = g^{ab}\nabla_a\nabla_b, \qquad U^{\alpha\beta} = u^\alpha u^\beta, \qquad u^\alpha = (1,0,0,0), \qquad u_\alpha = (-1,0,0,0) \tag{16}$$

The result is

$$\delta W_{00} = \tfrac{1}{2}K''''_{00} - 2kK''_{00} - \nabla_a\nabla^a K''_{00} + \tfrac{4}{3}k\nabla_a\nabla^a K_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a K_{00} \tag{17}$$

$$\delta W_{0i} = \tfrac{1}{2}K''''_{0i} - 2k^2 K_{0i} - \nabla_a\nabla^a K''_{0i} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a K_{0i} - \tfrac{2}{3}k\nabla_i K'_{00} \tag{18}$$

$$\begin{aligned}\delta W_{ij} = {} & \tfrac{1}{2}K''''_{ij} + 4kK''_{ij} - \tfrac{2}{3}kK''_{00}g_{ij} + 2k^2 K_{ij} - \tfrac{2}{3}k^2 g_{ij}K_{00} - \nabla_a\nabla^a K''_{ij} - 2k\nabla_a\nabla^a K_{ij} \\ & + \tfrac{2}{3}kg_{ij}\nabla_a\nabla^a K_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a K_{ij} - 2k\nabla_i K'_{0j} + \tfrac{4}{3}k\nabla_i\nabla_j K_{00} - 2k\nabla_j K'_{0i}.\end{aligned} \tag{19}$$

In the limit $k \to 0$, this reduces to

$$\delta W_{\mu\nu} = \frac{1}{2}\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta K_{\mu\nu}. \tag{20}$$

As a step towards the comoving coordinates, we evaluate (17-19) in the metric

$$-ds^2 = -\frac{dt^2}{a^2(t)} + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{21}$$

To do this we must transform the coordinates as

$$d\tau = \frac{dt}{a(t)}. \tag{22}$$

Under this coordinate change, $K_{\mu\nu}$ transforms as

$$K_{\mu\nu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\bar{K}_{\alpha\beta} \tag{23}$$

where $\bar{x}^\mu$ denotes the new coordinate system in (t, r, θ, ϕ) of (21). Only the time components are affected:

$$K_{00} = \frac{\partial \bar{x}^\alpha}{\partial x^0}\frac{\partial \bar{x}^\beta}{\partial x^0}\bar{K}_{\alpha\beta} = \delta^\alpha_0\delta^\beta_0\left(\frac{dt}{d\tau}\right)^2\bar{K}_{\alpha\beta} = a^2(t)\bar{K}_{00} \tag{24}$$

$$K_{0i} = \frac{\partial \bar{x}^\alpha}{\partial x^0}\frac{\partial \bar{x}^\beta}{\partial x^i}\bar{K}_{\alpha\beta} = \delta^\alpha_0\delta^\beta_i\frac{dt}{d\tau}\bar{K}_{\alpha\beta} = a(t)\bar{K}_{0i} \tag{25}$$

$$K_{ij} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial \bar{x}^\beta}{\partial x^j}\bar{K}_{\alpha\beta} = \delta^\alpha_i\delta^\beta_j\bar{K}_{\alpha\beta} = \bar{K}_{ij}. \tag{26}$$

In addition, only one Christoffel term changes, the one with all time components

$$\bar{\Gamma}^0_{00} = -\frac{\dot{a}}{a}. \tag{27}$$

This is useful because any covariant derivative with respect to a spatial index then retains its form in the new coordinate system. Time derivatives onto $K_{\mu\nu}$ must be transformed as

$$\begin{aligned}K'_{\mu\nu} &= a\dot{K}_{\mu\nu} \\ K''_{\mu\nu} &= a^2\ddot{K}_{\mu\nu} + a\dot{a}\dot{K}_{\mu\nu} \\ K'''_{\mu\nu} &= a^3\dddot{K}_{\mu\nu} + 3a^2\dot{a}\ddot{K}_{\mu\nu} + a\dot{a}^2\dot{K}_{\mu\nu} + a^2\ddot{a}\dot{K}_{\mu\nu} \\ K''''_{\mu\nu} &= a^4\ddddot{K}_{\mu\nu} + 6a^3\dot{a}\dddot{K}_{\mu\nu} + 7a^2\dot{a}^2\ddot{K}_{\mu\nu} + 4a^3\ddot{a}\ddot{K}_{\mu\nu} + a\dot{a}^3\dot{K}_{\mu\nu} + 4a^2\dot{a}\ddot{a}\dot{K}_{\mu\nu} + a^3\dddot{a}\dot{K}_{\mu\nu}.\end{aligned} \tag{28}$$

Making substitutions (24-28), $\delta W_{\mu\nu}$ can be expressed in the metric of (21) as

$$\begin{aligned}\delta W_{00} = &\tfrac{1}{2}a^6\ddddot{K}_{00} + 7a^5\dot{a}\dddot{K}_{00} - 2ka^4\ddot{K}_{00} + \tfrac{55}{2}a^4\dot{a}^2\ddot{K}_{00} + 8a^5\ddot{a}\ddot{K}_{00} - 10ka^3\dot{a}\dot{K}_{00} + \tfrac{65}{2}a^3\dot{a}^3\dot{K}_{00} \\ &+ 40a^4\dot{a}\ddot{a}\dot{K}_{00} + \tfrac{9}{2}a^5\dddot{a}\dot{K}_{00} - 8ka^2\dot{a}^2K_{00} + 8a^2\dot{a}^4K_{00} - 4ka^3\ddot{a}K_{00} + 33a^3\dot{a}^2\ddot{a}K_{00} \\ &+ 7a^4\ddot{a}^2K_{00} + 11a^4\dot{a}\dddot{a}K_{00} + a^5\ddddot{a}K_{00} - a^4\nabla_a\nabla^a\ddot{K}_{00} - 5a^3\dot{a}\nabla_a\nabla^a\dot{K}_{00} \\ &+ \tfrac{4}{3}ka^2\nabla_a\nabla^aK_{00} - 4a^2\dot{a}^2\nabla_a\nabla^aK_{00} - 2a^3\ddot{a}\nabla_a\nabla^aK_{00} + \tfrac{1}{2}a^2\nabla_b\nabla^b\nabla_a\nabla^aK_{00}.\end{aligned} \tag{29}$$

$$\begin{aligned}\delta W_{0i} = &\tfrac{1}{2}a^5\ddddot{K}_{i0} + 5a^4\dot{a}\dddot{K}_{i0} + \tfrac{25}{2}a^3\dot{a}^2\ddot{K}_{i0} + 5a^4\ddot{a}\ddot{K}_{i0} + \tfrac{15}{2}a^2\dot{a}^3\dot{K}_{i0} + 15a^3\dot{a}\ddot{a}\dot{K}_{i0} + \tfrac{5}{2}a^4\dddot{a}\dot{K}_{i0} \\ &- 2k^2aK_{i0} + \tfrac{1}{2}a\dot{a}^4K_{i0} + \tfrac{11}{2}a^2\dot{a}^2\ddot{a}K_{i0} + 2a^3\ddot{a}^2K_{i0} + \tfrac{7}{2}a^3\dot{a}\dddot{a}K_{i0} + \tfrac{1}{2}a^4\ddddot{a}K_{i0} \\ &- a^3\nabla_a\nabla^a\ddot{K}_{i0} - 3a^2\dot{a}\nabla_a\nabla^a\dot{K}_{i0} - a\dot{a}^2\nabla_a\nabla^aK_{i0} - a^2\ddot{a}\nabla_a\nabla^aK_{i0} + \tfrac{1}{2}a\nabla_b\nabla^b\nabla_a\nabla^aK_{i0} \\ &- \tfrac{2}{3}ka^3\nabla_i\dot{K}_{00} - \tfrac{4}{3}ka^2\dot{a}\nabla_iK_{00}.\end{aligned} \tag{30}$$

$$\begin{aligned}\delta W_{ij} = &\tfrac{1}{2}a^4\ddddot{K}_{ij} + 3a^3\dot{a}\dddot{K}_{ij} + 4ka^2\ddot{K}_{ij} + \tfrac{7}{2}a^2\dot{a}^2\ddot{K}_{ij} + 2a^3\ddot{a}\ddot{K}_{ij} + 4ka\dot{a}\dot{K}_{ij} + \tfrac{1}{2}a\dot{a}^3\dot{K}_{ij} \\ &+ 2a^2\dot{a}\ddot{a}\dot{K}_{ij} + \tfrac{1}{2}a^3\dddot{a}\dot{K}_{ij} - \tfrac{2}{3}ka^4\ddot{K}_{00}g_{ij} - \tfrac{10}{3}ka^3\dot{a}\dot{K}_{00}g_{ij} + 2k^2K_{ij} - \tfrac{2}{3}k^2a^2g_{ij}K_{00} \\ &- \tfrac{8}{3}ka^2\dot{a}^2g_{ij}K_{00} - \tfrac{4}{3}ka^3\ddot{a}g_{ij}K_{00} - a^2\nabla_a\nabla^a\ddot{K}_{ij} - a\dot{a}\nabla_a\nabla^a\dot{K}_{ij} - 2k\nabla_a\nabla^aK_{ij} \\ &+ \tfrac{2}{3}ka^2g_{ij}\nabla_a\nabla^aK_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^aK_{ij} - 2ka^2\nabla_i\dot{K}_{j0} - 2ka\dot{a}\nabla_iK_{j0} + \tfrac{4}{3}ka^2\nabla_i\nabla_jK_{00} \\ &- 2ka^2\nabla_j\dot{K}_{i0} - 2ka\dot{a}\nabla_jK_{i0}.\end{aligned} \tag{31}$$

Lastly, within the conformal metric $\bar{g}^{(0)}_{\mu\nu} = \Omega^2 g^{(0)}_{\mu\nu}$ we recall that the Bach tensor transforms as

$$\bar{\delta}W_{\mu\nu}(\bar{K}_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(K_{\mu\nu}). \tag{32}$$

As such, we may express $\delta W_{\mu\nu}$ in the RW metric (in conformal time or comoving coordinates) by multiplying by the appropriate conformal factor.

Summary

Within the RW conformal time metric

$$-ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right], \tag{33}$$

$\delta W_{\mu\nu}$ is evaluated as

$$\delta W_{00} = a^{-2}(\tau)\left[\tfrac{1}{2}K''''_{00} - 2kK''_{00} - \nabla_a\nabla^aK''_{00} + \tfrac{4}{3}k\nabla_a\nabla^aK_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^aK_{00}\right] \tag{34}$$

$$\delta W_{0i} = a^{-2}(\tau)\left[\tfrac{1}{2}K''''_{0i} - 2k^2K_{0i} - \nabla_a\nabla^aK''_{0i} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^aK_{0i} - \tfrac{2}{3}k\nabla_iK'_{00}\right] \tag{35}$$

$$\begin{aligned}\delta W_{ij} = a^{-2}(\tau)\big[&\tfrac{1}{2}K''''_{ij} + 4kK''_{ij} - \tfrac{2}{3}kK''_{00}g_{ij} + 2k^2K_{ij} - \tfrac{2}{3}k^2g_{ij}K_{00} - \nabla_a\nabla^aK''_{ij} - 2k\nabla_a\nabla^aK_{ij} \\ &+ \tfrac{2}{3}kg_{ij}\nabla_a\nabla^aK_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^aK_{ij} - 2k\nabla_iK'_{0j} + \tfrac{4}{3}k\nabla_i\nabla_jK_{00} - 2k\nabla_jK'_{0i}\big].\end{aligned} \tag{36}$$

In the RW comoving metric

$$-ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right], \tag{37}$$

$\delta W_{\mu\nu}$ is evaluated as

$$\begin{aligned}\delta W_{00} = {} & \tfrac{1}{2}a^4\ddddot{K}_{00} + 7a^3\dot{a}\dddot{K}_{00} - 2ka^2\ddot{K}_{00} + \tfrac{55}{2}a^2\dot{a}^2\ddot{K}_{00} + 8a^3\ddot{a}\ddot{K}_{00} - 10ka\dot{a}\dot{K}_{00} + \tfrac{65}{2}a\dot{a}^3\dot{K}_{00} \\ & + 40a^2\dot{a}\ddot{a}\dot{K}_{00} + \tfrac{9}{2}a^3\dddot{a}\dot{K}_{00} - 8k\dot{a}^2K_{00} + 8\dot{a}^4K_{00} - 4ka\ddot{a}K_{00} + 33a\dot{a}^2\ddot{a}K_{00} + 7a^2\ddot{a}^2K_{00} \\ & + 11a^2\dot{a}\dddot{a}K_{00} + a^3\ddddot{a}K_{00} - a^2\nabla_a\nabla^a\ddot{K}_{00} - 5a\dot{a}\nabla_a\nabla^a\dot{K}_{00} + \tfrac{4}{3}k\nabla_a\nabla^aK_{00} \\ & - 4\dot{a}^2\nabla_a\nabla^aK_{00} - 2a\ddot{a}\nabla_a\nabla^aK_{00} + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^aK_{00}.\end{aligned} \tag{38}$$

$$\begin{aligned}\delta W_{0i} = {} & \tfrac{1}{2}a^3\ddddot{K}_{i0} + 5a^2\dot{a}\dddot{K}_{i0} + \tfrac{25}{2}a\dot{a}^2\ddot{K}_{i0} + 5a^2\ddot{a}\ddot{K}_{i0} + \tfrac{15}{2}\dot{a}^3\dot{K}_{i0} + 15a\dot{a}\ddot{a}\dot{K}_{i0} + \tfrac{5}{2}a^2\dddot{a}\dot{K}_{i0} \\ & - 2k^2a^{-1}K_{i0} + \tfrac{1}{2}a^{-1}\dot{a}^4K_{i0} + \tfrac{11}{2}\dot{a}^2\ddot{a}K_{i0} + 2a\ddot{a}^2K_{i0} + \tfrac{7}{2}a\dot{a}\dddot{a}K_{i0} + \tfrac{1}{2}a^2\ddddot{a}K_{i0} \\ & - a\nabla_a\nabla^a\ddot{K}_{i0} - 3\dot{a}\nabla_a\nabla^a\dot{K}_{i0} - a^{-1}\dot{a}^2\nabla_a\nabla^aK_{i0} - \ddot{a}\nabla_a\nabla^aK_{i0} + \tfrac{1}{2}a^{-1}\nabla_b\nabla^b\nabla_a\nabla^aK_{i0} \\ & - \tfrac{2}{3}ka\nabla_i\dot{K}_{00} - \tfrac{4}{3}k\dot{a}\nabla_iK_{00}.\end{aligned} \tag{39}$$

$$\begin{aligned}\delta W_{ij} = {} & \tfrac{1}{2}a^2\ddddot{K}_{ij} + 3a\dot{a}\dddot{K}_{ij} + 4k\ddot{K}_{ij} + \tfrac{7}{2}\dot{a}^2\ddot{K}_{ij} + 2a\ddot{a}\ddot{K}_{ij} + 4ka^{-1}\dot{a}\dot{K}_{ij} + \tfrac{1}{2}a^{-1}\dot{a}^3\dot{K}_{ij} + 2\dot{a}\ddot{a}\dot{K}_{ij} \\ & + \tfrac{1}{2}a\dddot{a}\dot{K}_{ij} - \tfrac{2}{3}ka^2\ddot{K}_{00}g_{ij} - \tfrac{10}{3}ka\dot{a}\dot{K}_{00}g_{ij} + 2k^2a^{-2}K_{ij} - \tfrac{2}{3}k^2g_{ij}K_{00} - \tfrac{8}{3}k\dot{a}^2g_{ij}K_{00} \\ & - \tfrac{4}{3}ka\ddot{a}g_{ij}K_{00} - \nabla_a\nabla^a\ddot{K}_{ij} - a^{-1}\dot{a}\nabla_a\nabla^a\dot{K}_{ij} - 2ka^{-2}\nabla_a\nabla^aK_{ij} + \tfrac{2}{3}kg_{ij}\nabla_a\nabla^aK_{00} \\ & + \tfrac{1}{2}a^{-2}\nabla_b\nabla^b\nabla_a\nabla^aK_{ij} - 2k\nabla_i\dot{K}_{j0} - 2ka^{-1}\dot{a}\nabla_iK_{j0} + \tfrac{4}{3}k\nabla_i\nabla_jK_{00} - 2k\nabla_j\dot{K}_{i0} \\ & - 2ka^{-1}\dot{a}\nabla_jK_{i0}.\end{aligned} \tag{40}$$