# Special Gauge v5 Matthew

## Tranverse Gauge

Under the perturbation $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, the Einstein tensor is

$$\begin{aligned}\delta G_{\mu\nu} = &-\tfrac{1}{2}h_{\mu\nu}R + \tfrac{1}{2}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{2}h_\nu{}^\alpha R_{\mu\alpha} + \tfrac{1}{2}h_\mu{}^\alpha R_{\nu\alpha} - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} \\ &- \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\mu h. \end{aligned} \tag{1}$$

If we now substitute the covariant tranverse gauge, i.e

$$\nabla^\mu h_{\mu\nu} = \frac{1}{2}\nabla_\nu h \tag{2}$$

then the perturbation equation becomes

$$\delta G_{\mu\nu} = -\tfrac{1}{2}h_{\mu\nu}R + \tfrac{1}{2}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{2}h_\nu{}^\alpha R_{\mu\alpha} + \tfrac{1}{2}h_\mu{}^\alpha R_{\nu\alpha} - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h. \tag{3}$$

Now set the background metric as $g^{(0)}_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, such that (3) becomes

$$\begin{aligned}\delta G_{\mu\nu} = &-\eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-4}\partial_\alpha\Omega\partial_\beta\Omega \\ &+ 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-4}\partial_\alpha\Omega\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\Omega^{-2}\partial_\beta\partial_\alpha h_{\mu\nu} \\ &- \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h - 3\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-3}\partial_\beta\partial_\alpha\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\Omega^{-3}\partial_\beta\partial_\mu\Omega \\ &+ 2\eta^{\alpha\beta}h_{\mu\alpha}\Omega^{-3}\partial_\beta\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - 6\eta^{\alpha\beta}h_{\nu\beta}\Omega^{-4}\partial_\alpha\Omega\partial_\mu\Omega \\ &- \eta^{\alpha\beta}\Omega^{-3}\partial_\beta h_{\nu\alpha}\partial_\mu\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - 6\eta^{\alpha\beta}h_{\mu\beta}\Omega^{-4}\partial_\alpha\Omega\partial_\nu\Omega \\ &- \eta^{\alpha\beta}\Omega^{-3}\partial_\beta h_{\mu\alpha}\partial_\nu\Omega + 3h\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega - h\Omega^{-1}\partial_\nu\partial_\mu\Omega. \end{aligned} \tag{4}$$

where $h = g^{\mu\nu}_{(0)}h_{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}h_{\mu\nu}$. Note that the covariant box term $\frac{1}{2}\nabla_\alpha\nabla^\alpha$ yields another gauge condition to be substituted from the $-\eta^{\alpha\beta}\Omega^{-3}\partial_\beta h_{\nu\alpha}\partial_\mu\Omega$ and $-\eta^{\alpha\beta}\Omega^{-3}\partial_\beta h_{\mu\alpha}\partial_\nu\Omega$ terms. As such, we will need an expression for the transverse gauge with respect to $g^{(0)}_{\mu\nu} = \Omega^2\eta_{\mu\nu}$. This is

$$\eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\Omega + \tfrac{1}{2}\Omega^2\partial_\nu h + \Omega h\partial_\nu\Omega. \tag{5}$$

Within the above gauge, the Einstein tensor becomes

$$\begin{aligned}\delta G_{\mu\nu} = &-\eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-4}\partial_\alpha\Omega\partial_\beta\Omega \\ &+ 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-4}\partial_\alpha\Omega\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\Omega^{-2}\partial_\beta\partial_\alpha h_{\mu\nu} \\ &- \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h - 3\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-3}\partial_\beta\partial_\alpha\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\Omega^{-3}\partial_\beta\partial_\mu\Omega \\ &+ 2\eta^{\alpha\beta}h_{\mu\alpha}\Omega^{-3}\partial_\beta\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - 4\eta^{\alpha\beta}h_{\nu\beta}\Omega^{-4}\partial_\alpha\Omega\partial_\mu\Omega \\ &+ \eta^{\alpha\beta}\Omega^{-3}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \tfrac{1}{2}\Omega^{-1}\partial_\mu\Omega\partial_\nu h - 4\eta^{\alpha\beta}h_{\mu\beta}\Omega^{-4}\partial_\alpha\Omega\partial_\nu\Omega - \tfrac{1}{2}\Omega^{-1}\partial_\mu h\partial_\nu\Omega \\ &+ h\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega - h\Omega^{-1}\partial_\nu\partial_\mu\Omega. \end{aligned} \tag{6}$$

Evaluating for $\Omega(\tau)$ yields

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-8\Omega^{-6}\dot{\Omega}^2 + 4\Omega^{-5}\ddot{\Omega})h_{00} + (-2\Omega^{-4}\dot{\Omega}^2 + \Omega^{-3}\ddot{\Omega} - \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)h \tag{7}$$

$$\delta G_{00} = (5\Omega^{-4}\dot{\Omega}^2 - \Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-3}\dot{\Omega}\partial_0)h_{00} + (\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 - \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{3}{2}\Omega^{-1}\dot{\Omega}\partial_0)h. \tag{8}$$

$$\delta G_{0i} = -\Omega^{-3}\dot{\Omega}\partial_i h_{00} + (2\Omega^{-4}\dot{\Omega}^2 + \Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu)h_{0i} - \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_i h. \tag{9}$$

$$\delta G_{11} = \Omega^{-4}\dot{\Omega}^2 h_{00} - 2\Omega^{-3}\dot{\Omega}\partial_1 h_{01} + (-2\Omega^{-4}\dot{\Omega}^2 + 3\Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)h_{11} + (\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2 - \tfrac{1}{2}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_0)h. \tag{10}$$

$$\delta G_{12} = -\Omega^{-3}\dot{\Omega}\partial_2 h_{01} - \Omega^{-3}\dot{\Omega}\partial_1 h_{02} + (-2\Omega^{-4}\dot{\Omega}^2 + 3\Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)h_{12}. \tag{11}$$

In a deSitter background $\Omega(\tau) = \frac{1}{H\tau}$, $\delta G_{\mu\nu}$ evaluates to

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-\tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h \tag{12}$$

$$\delta G_{00} = (3H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2\tau\partial_0)h_{00} + (-\tfrac{1}{2}\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{3}{2}\tau^{-1}\partial_0)h \tag{13}$$

$$\delta G_{0i} = H^2\tau\partial_i h_{00} + (4H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu)h_{0i} + \tfrac{1}{2}\tau^{-1}\partial_i h \tag{14}$$

$$\delta G_{11} = H^2 h_{00} + 2H^2\tau\partial_1 h_{01} + (4H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h_{11} + (-\tfrac{1}{2}\tau^{-2} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}\tau^{-1}\partial_0)h \tag{15}$$

$$\delta G_{12} = H^2\tau\partial_2 h_{01} + H^2\tau\partial_1 h_{02} + (4H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h_{12}. \tag{16}$$

# Special $K_{\mu\nu}$ Gauge

The perturbed Einstein tensor $\delta G_{\mu\nu}(h_{\mu\nu})$ evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{17}$$

is calculated as

$$\begin{aligned}\delta G_{\mu\nu} = {} & \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{18}$$

Now we split $h_{\mu\nu}$ into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \tag{19}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. With this substitution, (2) takes the form

$$\begin{aligned}\delta G_{\mu\nu} = &-2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\alpha\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta K_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ &+ \eta^{\alpha\beta}K_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha K_{\mu\nu} + 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ &- 2\eta^{\alpha\beta}K_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta K_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta K_{\alpha\gamma} \\ &- \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\mu\partial_\beta K_{\nu\alpha} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} + 2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega \\ &- \tfrac{1}{2}\eta^{\alpha\beta}\partial_\nu\partial_\beta K_{\mu\alpha} + \tfrac{3}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ &- \tfrac{1}{8}\partial_\mu\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu h\partial_\nu\Omega + \tfrac{3}{8}\partial_\nu\partial_\mu h. \end{aligned} \tag{20}$$

Now we impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega. \tag{21}$$

Within this gauge, (4) is evaluated as

$$\begin{aligned}\delta G_{\mu\nu} = &-2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\alpha\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta K_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega \\ &+ 2P\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \tfrac{1}{2}JP\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \tfrac{1}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega \\ &- \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{3}{2}J\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ &+ \tfrac{1}{2}J^2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}K_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{3}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ &+ \tfrac{1}{2}JR\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha K_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h + \tfrac{1}{2}P\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ &+ 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega + \tfrac{1}{2}J\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - 2\eta^{\alpha\beta}K_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ &+ \tfrac{1}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} - \tfrac{1}{2}J\eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} \\ &+ \tfrac{1}{2}J\eta^{\alpha\beta}K_{\nu\beta}\Omega^{-2}\partial_\alpha\Omega\partial_\mu\Omega - \tfrac{1}{2}J\eta^{\alpha\beta}K_{\nu\beta}\Omega^{-1}\partial_\mu\partial_\alpha\Omega - \tfrac{1}{8}\partial_\mu\partial_\nu h - \tfrac{1}{2}P\partial_\mu\partial_\nu h \\ &- \tfrac{1}{2}Rh\Omega^{-1}\partial_\mu\partial_\nu\Omega - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} - \tfrac{1}{2}J\eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu h \\ &- \tfrac{1}{2}R\Omega^{-1}\partial_\mu\Omega\partial_\nu h + \tfrac{1}{2}J\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-2}\partial_\alpha\Omega\partial_\nu\Omega - \tfrac{1}{4}\Omega^{-1}\partial_\mu h\partial_\nu\Omega - \tfrac{1}{2}R\Omega^{-1}\partial_\mu h\partial_\nu\Omega \\ &+ Rh\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega - \tfrac{1}{2}J\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega + \tfrac{3}{8}\partial_\nu\partial_\mu h \\ &- \tfrac{1}{2}P\partial_\nu\partial_\mu h - \tfrac{1}{2}Rh\Omega^{-1}\partial_\nu\partial_\mu\Omega. \end{aligned} \tag{22}$$

Upon taking $J = -2$, $P = \frac{1}{2}$, and $R = 0$, viz.

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + \tfrac{1}{2}\partial_\nu h, \tag{23}$$

for a strictly time dependent conformal factor $\Omega(\tau)$, we find the flucuations take the form

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-10\Omega^{-4}\dot{\Omega}^2 + 6\Omega^{-3}\ddot{\Omega})K_{00} + (-\tfrac{1}{4}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}\Omega^{-3}\dot{\Omega}\partial_0)h \tag{24}$$

$$\delta G_{00} = (3\Omega^{-2}\dot{\Omega}^2 - \Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0 - \tfrac{1}{4}\partial_0\partial_0)h. \tag{25}$$

$$\delta G_{0i} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{0i} + (-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_i - \tfrac{1}{4}\partial_i\partial_0)h. \tag{26}$$

$$\begin{aligned}\delta G_{11} = &(-2\Omega^{-2}\dot{\Omega}^2 + \Omega^{-1}\ddot{\Omega})K_{00} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{11} \\ &+ (-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0 - \tfrac{1}{4}\partial_1\partial_1)h. \end{aligned} \tag{27}$$

$$\delta G_{12} = (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{12} - \tfrac{1}{4}\partial_2\partial_1 h. \tag{28}$$

In the deSitter background, we take $\Omega(\tau) = \frac{1}{H\tau}$, in which $\delta G_{\mu\nu}$ reduces to

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = 2H^2K_{00} + (-\tfrac{1}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}H^2\tau\partial_0)h \tag{29}$$

$$\delta G_{00} = (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{00} + (\tfrac{1}{4}\tau^{-1}\partial_0 - \tfrac{1}{4}\partial_0\partial_0)h \tag{30}$$

$$\delta G_{0i} = (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{0i} + (\tfrac{1}{4}\tau^{-1}\partial_i - \tfrac{1}{4}\partial_i\partial_0)h \tag{31}$$

$$\delta G_{11} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\tfrac{1}{4}\tau^{-1}\partial_0 - \tfrac{1}{4}\partial_1\partial_1)h. \tag{32}$$

$$\delta G_{12} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} - \tfrac{1}{4}\partial_2\partial_1 h. \tag{33}$$