Perturbed Curvature Quantities

$$\delta R_{\lambda\mu\nu\kappa} = h^{\alpha}{}_{\lambda} R_{\alpha\mu\nu\kappa} - \tfrac{1}{2}\nabla_\kappa\nabla_\lambda h_{\mu\nu} + \tfrac{1}{2}\nabla_\kappa\nabla_\mu h_{\nu\lambda} + \tfrac{1}{2}\nabla_\kappa\nabla_\nu h_{\mu\lambda} - \tfrac{1}{2}\nabla_\nu\nabla_\kappa h_{\mu\lambda} + \tfrac{1}{2}\nabla_\nu\nabla_\lambda h_{\kappa\mu} - \tfrac{1}{2}\nabla_\nu\nabla_\mu h_{\kappa\lambda}. \quad (1)$$

$$\delta R_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(\nabla_\alpha\nabla_\beta h_{\mu\nu} - \nabla_\alpha\nabla_\mu h_{\beta\nu} - \nabla_\alpha\nabla_\nu h_{\beta\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta}) \quad (2)$$

$$W^1_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R^2 - 2RR_{\mu\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - 2\nabla_\nu\nabla_\mu R. \quad (3)$$

$$W^2_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \tfrac{1}{2} g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\mu\nabla^\alpha R_{\nu\alpha} - \nabla_\nu\nabla^\alpha R_{\mu\alpha}. \quad (4)$$

No Bianchi, No Explicit Commutations

29 Terms

$$\begin{aligned}
\delta W^1_{\mu\nu} = {} & \tfrac{1}{2} h_{\mu\nu} R^2 - g_{\mu\nu} h^{\alpha\beta} R R_{\alpha\beta} + 2h^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} - R\nabla_\alpha\nabla^\alpha h_{\mu\nu} + 2h_{\mu\nu}\nabla_\alpha\nabla^\alpha R \\
& + g_{\mu\nu} R\nabla_\alpha\nabla^\alpha h - 2R_{\mu\nu}\nabla_\alpha\nabla^\alpha h + R\nabla_\alpha\nabla_\mu h_\nu{}^\alpha + R\nabla_\alpha\nabla_\nu h_\mu{}^\alpha - \nabla_\alpha h_{\mu\nu}\nabla^\alpha R \\
& + g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R - 2g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta - g_{\mu\nu} R\nabla_\beta\nabla_\alpha h^{\alpha\beta} + 2R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& - 2g_{\mu\nu} h^{\alpha\beta}\nabla_\beta\nabla_\alpha R + 2g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h - 2g_{\mu\nu} R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} - 2g_{\mu\nu} h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
& - 2g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - 4g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} + \nabla^\alpha R\nabla_\mu h_{\nu\alpha} + 2\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} \\
& + \nabla^\alpha R\nabla_\nu h_{\mu\alpha} + 2\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + 2R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} + 2h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} - R\nabla_\nu\nabla_\mu h \\
& - 2\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + 2\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}. \quad (5)
\end{aligned}$$

62 Terms

$$\begin{aligned}
\delta W^2_{\mu\nu} = {} & \tfrac{1}{2} h_{\mu\nu} R^2 - g_{\mu\nu} h^{\alpha\beta} R R_{\alpha\beta} + 2h^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} - R\nabla_\alpha\nabla^\alpha h_{\mu\nu} + 2h_{\mu\nu}\nabla_\alpha\nabla^\alpha R \\
& + g_{\mu\nu} R\nabla_\alpha\nabla^\alpha h - 2R_{\mu\nu}\nabla_\alpha\nabla^\alpha h + R\nabla_\alpha\nabla_\mu h_\nu{}^\alpha + R\nabla_\alpha\nabla_\nu h_\mu{}^\alpha - \nabla_\alpha h_{\mu\nu}\nabla^\alpha R \\
& + g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R - 2g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta - g_{\mu\nu} R\nabla_\beta\nabla_\alpha h^{\alpha\beta} + 2R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& - 2g_{\mu\nu} h^{\alpha\beta}\nabla_\beta\nabla_\alpha R + 2g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h - 2g_{\mu\nu} R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} - 2g_{\mu\nu} h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
& - 2g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - 4g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} + \nabla^\alpha R\nabla_\mu h_{\nu\alpha} + 2\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} \\
& + \nabla^\alpha R\nabla_\nu h_{\mu\alpha} + 2\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + 2R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} + 2h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} - R\nabla_\nu\nabla_\mu h \\
& - 2\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + 2\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}. \quad (6)
\end{aligned}$$

$$\nabla_\kappa\nabla_\nu T_{\lambda\mu} = \nabla_\nu\nabla_\kappa T_{\lambda\mu} + R_{\lambda\sigma\nu\kappa} T^\sigma{}_\mu - R_{\sigma\mu\nu\kappa} T_\lambda{}^\sigma \quad (7)$$

obeyed by any rank two tensor, so that we can write $W^{\mu\nu}$ as

$$\begin{aligned}
W^{\mu\nu} = {} & -\frac{1}{6} g^{\mu\nu}\nabla_\beta\nabla^\beta R^\alpha{}_\alpha + \nabla_\beta\nabla^\beta R^{\mu\nu} - \frac{1}{3}\nabla_\mu\nabla_\nu R^\alpha{}_\alpha - R^\beta{}_\sigma R^\sigma{}_{\mu\beta\nu} \\
& - R^\beta{}_\sigma R^\sigma{}_{\nu\beta\mu} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{3} R^\alpha{}_\alpha R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} (R^\alpha{}_\alpha)^2 \quad (8)
\end{aligned}$$

Perturbing $W^{\mu\nu}$ about metric $g_{\mu\nu} + h_{\mu\nu}$ with background metric $g_{\mu\nu}$ and fluctuation $h_{\mu\nu}$ gives (following a machine calculation)