SVT Gauge Transformation

S.V.T. decomposition:

$$
\begin{aligned}
ds^2 &= -(g^{(0)}_{\mu\nu} + h_{\mu\nu}) \\
&= \Omega^2(x)\{(1+2\phi)dt^2 - 2(\tilde{\nabla}_i B + B_i)dtdx^i - [(1-2\psi\gamma_{ij}) + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\}. \qquad (1)
\end{aligned}
$$

For reference,

$$g_{00} = -\Omega^2 \qquad h_{00} = \Omega^2(-2\phi) \qquad (2)$$

$$g_{0i} = 0 \qquad h_{0i} = \Omega^2(\tilde{\nabla}_i B + B_i) \qquad (3)$$

$$g_{ij} = \Omega^2\gamma_{ij} \qquad h_{ij} = \Omega^2(-2\psi\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}) \qquad (4)$$

Under coordinate transformation $x^\mu \to \bar{x}^\mu = x^\mu + \epsilon^\mu$, the metric perturbation transforms as

$$\bar{h}_{\mu\nu}(x) = h_{\mu\nu}(x) - \nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu. \qquad (5)$$

Using $\epsilon_\alpha = g_{\alpha\beta}\epsilon^\beta$, we rewrite this as

$$
\begin{aligned}
\bar{h}_{\mu\nu}(x) &= h_{\mu\nu}(x) - g_{\alpha\nu}\nabla_\mu\epsilon^\alpha - g_{\alpha\mu}\nabla_\nu\epsilon^\alpha & (6)\\
&= h_{\mu\nu} - [g_{\alpha\nu}\partial_\mu\epsilon^\alpha + \tfrac{1}{2}(\partial_\mu g_{\nu\beta} + \partial_\beta g_{\mu\nu} - \partial_\nu g_{\mu\beta})\epsilon^\beta] - [g_{\alpha\mu}\partial_\nu\epsilon^\alpha + \tfrac{1}{2}(\partial_\nu g_{\mu\beta} + \partial_\beta g_{\mu\nu} - \partial_\mu g_{\nu\beta})\epsilon^\beta] & (7)\\
&= h_{\mu\nu} - g_{\alpha\nu}\partial_\mu\epsilon^\alpha - g_{\alpha\mu}\partial_\nu\epsilon^\alpha - \epsilon^\beta\partial_\beta g_{\mu\nu} & (8)
\end{aligned}
$$

To facilitate the S.V.T. decomposition, we decompose the coordinate transformation ϵ^μ as

$$\epsilon^0 = T, \qquad \epsilon^i = \tilde{\nabla}^i L + L^i, \qquad \tilde{\nabla}^i L_i = 0$$

where $\tilde{\nabla}$ denotes the covariant derivative with respect to the 3-space metric γ_{ij}. The transformations go as:

$$
\begin{aligned}
\bar{h}_{00} &= h_{00} + 2\Omega^2\dot{T} + 2\Omega\epsilon^\alpha\nabla_\alpha\Omega & (9)\\
-2\bar{\phi} &= -2\phi + 2\dot{T} + 2\Omega^{-1}\epsilon^\alpha\nabla_\alpha\Omega & (10)\\
\bar{\phi} &= \phi - \dot{T} - \Omega^{-1}\epsilon^\alpha\nabla_\alpha\Omega & (11)
\end{aligned}
$$

$$
\begin{aligned}
\bar{h}_{0i} &= h_{0i} - \Omega^2\gamma_{ij}\partial_0(\tilde{\nabla}^j L + L^j) + \Omega^2\partial_i T & (12)\\
\tilde{\nabla}_i\bar{B} + \bar{B}_i &= \tilde{\nabla}_i B + B_i - \tilde{\nabla}_i\dot{L} - \dot{L}_i + \tilde{\nabla}_i T & (13)\\
\bar{B} &= B - \dot{L} + T & (14)\\
\bar{B}_i &= B_i - \dot{L}_i & (15)
\end{aligned}
$$

$$
\begin{aligned}
\bar{h}_{ij} &= h_{ij} - \Omega^2\gamma_{jk}\partial_i(\tilde{\nabla}^k L + L^k) - \Omega^2\gamma_{ik}\partial_j(\tilde{\nabla}^k L + L^k) - \Omega^2(\tilde{\nabla}^k L + L^k)\partial_k\gamma_{ij} - 2\Omega\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega & (16)\\
\Omega^{-2}\bar{h}_{ij} &= \Omega^{-2}h_{ij} - \gamma_{jk}\partial_i(\gamma^{kl}\tilde{\nabla}_l L + \gamma^{kl}L^l) - \gamma_{ik}\partial_j(\gamma^{kl}\tilde{\nabla}_l L + \gamma^{kl}L^l) - (\gamma^{kl}\tilde{\nabla}_l L + \gamma^{kl}L_l)\partial_k\gamma_{ij} \\
&\quad - 2\Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega & (17)\\
&= \Omega^{-2}h_{ij} - 2\partial_i\partial_j L - \partial_i L_j - \partial_j L_i - \gamma_{jk}(\partial_i\gamma^{kl})\partial_l L - \gamma_{ik}(\partial_j\gamma^{kl})\partial_l L - \gamma^{kl}(\partial_k\gamma_{ij})\partial_l L \\
&\quad - \gamma_{jk}(\partial_i\gamma^{kl})L_l - \gamma_{ik}(\partial_j\gamma^{kl})L_l - 2\Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega - \gamma^{kl}(\partial_k\gamma_{ij})L_l & (18)
\end{aligned}
$$

Using the expression

$$\gamma_{jk}\partial_i\gamma^{kl} = -\gamma^{kl}\partial_i\gamma_{jk}$$

$\bar{h}_{ij}$ can be expressed as

$$\Omega^{-2}\bar{h}_{ij} = \Omega^{-2}h_{ij} - 2\partial_i\partial_j L - \partial_i L_j - \partial_j L_i + \gamma^{kl}(\partial_i\gamma_{jk})\partial_l L + \gamma^{kl}(\partial_j\gamma_{ik})\partial_l L - \gamma^{kl}(\partial_k\gamma_{ij})\partial_l L \tag{19}$$

$$+ \gamma^{kl}(\partial_i\gamma_{jk})L_l + \gamma^{kl}(\partial_j\gamma_{ik})L_l - \gamma^{kl}(\partial_k\gamma_{ij})L_l - 2\Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega. \tag{20}$$

Noting the covariant derivative relation,

$$\tilde{\nabla}_i A_j = \partial_i A_j - \tfrac{1}{2}\gamma^{kl}(\partial_i\gamma_{jk} + \partial_j\gamma_{ik} - \partial_k\gamma_{ij})A_l \tag{21}$$

$\bar{h}_{ij}$ becomes

$$\Omega^{-2}\bar{h}_{ij} = \Omega^{-2}h_{ij} - 2\tilde{\nabla}_i\tilde{\nabla}_j L - \tilde{\nabla}_i L_j - \tilde{\nabla}_j L_i - 2\Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega. \tag{22}$$

Equating the scalar pieces:

$$-2\bar{\psi}\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j\bar{E} = -2\psi\gamma_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E - 2\tilde{\nabla}_i\tilde{\nabla}_j L - 2\Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega \tag{23}$$

$$\bar{\psi} = \psi + \Omega^{-1}\gamma_{ij}\epsilon^\alpha\nabla_\alpha\Omega \tag{24}$$

$$\bar{E} = E - L. \tag{25}$$

Equating the vector pieces:

$$\tilde{\nabla}_i\bar{E}_j + \tilde{\nabla}_j\bar{E}_i = \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i - \tilde{\nabla}_i L_j - \tilde{\nabla}_j L_i \tag{26}$$

$$\bar{E}_i = E_i - L_i. \tag{27}$$

Equating the tensor pieces:

$$\bar{E}_{ij} = E_{ij}. \tag{28}$$

Altogether:

$$\bar{\phi} = \phi - \dot{T} - \Omega^{-1}\epsilon^\alpha\nabla_\alpha\Omega \tag{29}$$

$$\bar{\psi} = \psi + \Omega^{-1}\epsilon^\alpha\nabla_\alpha\Omega \tag{30}$$

$$\bar{B} = B - \dot{L} + T \tag{31}$$

$$\bar{B}_i = B_i - \dot{L}_i \tag{32}$$

$$\bar{E} = E - L \tag{33}$$

$$\bar{E}_i = E_i - L_i \tag{34}$$

$$\bar{E}_{ij} = E_{ij} \tag{35}$$

In order to have arrived at the following equations, it was necessary that $\bar{\gamma}_{ij} = \gamma_{ij}$. Since γ_{ij} is the zeroth order background, it must be gauge invariant on its own. This can also be seen from

$$\bar{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu \tag{36}$$

$$\bar{g}^{(0)}_{\mu\nu}(x) + \bar{h}_{\mu\nu} = g^{(0)}_{\mu\nu}(x) + h_{\mu\nu} - \nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu \tag{37}$$

and hence to zeroth order

$$\bar{g}^{(0)}_{\mu\nu} = g^{(0)}_{\mu\nu}. \tag{38}$$

Lastly, we note that the gauge transformations as calculated above make no imposition upon the form of γ_{ij} or $\Omega(x)$.

$-h_{\mu\nu}dx^\mu dx^\nu$, we have

$\Gamma^0_{00} = \Omega^{-1}\partial_0\Omega$, $\Gamma^i_{00} = \Omega^{-1}\delta^{ij}\partial_j\Omega$.

Thus under gauge $h_{00} \to h_{00} + 2\partial_0\epsilon_0 - 2\Gamma^0_{00}\epsilon_0 - 2\Gamma^i_{00}\epsilon_i$

i.e. $h_{00} \to h_{00} + 2\partial_0\epsilon_0 - 2\epsilon_0\Omega^{-1}\partial_0\Omega - 2\epsilon_i\Omega^{-1}\delta^{ij}\partial_j\Omega$

Thus define $\epsilon_0 = -\Omega^2 T$, $\epsilon_i = \Omega^2(L_i + \partial_i L)$, $\delta^{ij}\partial_j L_i = 0$

and we get

$\phi = \Omega^{-2}h_{00}/2 \to \phi - \partial_0 T - 2T\Omega^{-1}\partial_0\Omega + T\Omega^{-1}\partial_0\Omega - (L_i + \partial_i L)\Omega^{-1}\delta^{ij}\partial_j\Omega$

i.e.

$\phi \to \phi - \partial_0 T - T\Omega^{-1}\partial_0\Omega - (L_i + \partial_i L)\Omega^{-1}\delta^{ij}\partial_j\Omega$

We now raise with the full $ds^2 = \Omega^2[dt^2 - \delta_{ij}dx^i dx^j] = -g_{\mu\nu}dx^\mu dx^\nu$ and define

$\epsilon^0 = g^{0\mu}\epsilon_\mu = T, \quad \epsilon^i = g^{i\mu}\epsilon_\mu = \delta^{ij}(L_i + \partial_i L)$

and can thus write

$\phi \to \phi - \partial_0 T - \Omega^{-1}\epsilon^\mu\partial_\mu\Omega$

Philip