General Gauge:

$$\eta^{\alpha\beta}\ \partial_\alpha h_{\beta\nu} == \frac{J\ \eta^{\alpha\beta}\ \ h_{\nu\alpha}\ \ \partial_\beta\Omega}{\Omega} + P\,\Omega^2\,\partial_\nu h + R\,h\,\Omega\,\partial_\nu\Omega$$

Ricci Tensor, Ω = 1/Ht

No gauge usage. Has $h_{0\,i}$ and other terms present

$\eta^{\mu\nu}\delta R_{\mu\nu}$ =

$$\begin{aligned}
&-2H^2\ h_{11} - 2H^2\ h_{22} - 2H^2\ h_{33} + \frac{2\,\partial_0 h}{t} - \frac{\partial_0\partial_0 h}{2} + \frac{\partial_1\partial_1 h}{2} + \frac{\partial_2\partial_2 h}{2} + \\
&\ t\left(3H^2\,\partial_0 h_{00} - H^2\,\partial_0 h_{11} - H^2\,\partial_0 h_{22} - H^2\,\partial_0 h_{33} - 2H^2\,\partial_1 h_{01} - 2H^2\,\partial_2 h_{02} - 2H^2\,\partial_3 h_{03}\right) + \\
&\ t^2\left(-\frac{1}{2}H^2\,\partial_0\partial_0 h_{00} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{11} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{22} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{33} + H^2\,\partial_0\partial_1 h_{01} + \right.\\
&\quad H^2\,\partial_0\partial_2 h_{02} + H^2\,\partial_0\partial_3 h_{03} + H^2\,\partial_1\partial_0 h_{01} - \frac{1}{2}H^2\,\partial_1\partial_1 h_{00} - \frac{1}{2}H^2\,\partial_1\partial_1 h_{11} + \frac{1}{2}H^2\,\partial_1\partial_1 h_{22} + \\
&\quad \frac{1}{2}H^2\,\partial_1\partial_1 h_{33} - H^2\,\partial_1\partial_2 h_{12} - H^2\,\partial_1\partial_3 h_{13} + H^2\,\partial_2\partial_0 h_{02} - H^2\,\partial_2\partial_1 h_{12} - \frac{1}{2}H^2\,\partial_2\partial_2 h_{00} + \\
&\quad \frac{1}{2}H^2\,\partial_2\partial_2 h_{11} - \frac{1}{2}H^2\,\partial_2\partial_2 h_{22} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{33} - H^2\,\partial_2\partial_3 h_{23} + H^2\,\partial_3\partial_0 h_{03} - H^2\,\partial_3\partial_1 h_{13} - \\
&\quad \left. H^2\,\partial_3\partial_2 h_{23} - \frac{1}{2}H^2\,\partial_3\partial_3 h_{00} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{11} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{22} - \frac{1}{2}H^2\,\partial_3\partial_3 h_{33}\right) + \frac{\partial_3\partial_3 h}{2}
\end{aligned}$$

General Gauge

$\eta^{\mu\nu}\delta R_{\mu\nu}$ =

$$\begin{aligned}
&H^2\,J\ h_{00} - 2H^2\ h_{11} - 2H^2\ h_{22} - 2H^2\ h_{33} + \frac{4\,J\,h + 2\,R\,h}{t^2} + \frac{2\,\partial_0 h - P\,\partial_0 h - R\,\partial_0 h}{t} - \\
&\ \frac{\partial_0\partial_0 h}{2} + P\,\partial_0\partial_0 h + \frac{\partial_1\partial_1 h}{2} - P\,\partial_1\partial_1 h + \frac{\partial_2\partial_2 h}{2} - P\,\partial_2\partial_2 h + t\left(H^2\,\partial_0 h_{00} + H^2\,J\,\partial_0 h_{00} - \right.\\
&\quad \left. H^2\,\partial_0 h_{11} - H^2\,\partial_0 h_{22} - H^2\,\partial_0 h_{33} - H^2\,J\,\partial_1 h_{01} - H^2\,J\,\partial_2 h_{02} - H^2\,J\,\partial_3 h_{03}\right) + \\
&\ t^2\left(\frac{1}{2}H^2\,\partial_0\partial_0 h_{00} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{11} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{22} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{33} - \frac{1}{2}H^2\,\partial_1\partial_1 h_{00} + \frac{1}{2}H^2\,\partial_1\partial_1 h_{11} + \right.\\
&\quad \frac{1}{2}H^2\,\partial_1\partial_1 h_{22} + \frac{1}{2}H^2\,\partial_1\partial_1 h_{33} - \frac{1}{2}H^2\,\partial_2\partial_2 h_{00} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{11} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{22} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{33} - \\
&\quad \left. \frac{1}{2}H^2\,\partial_3\partial_3 h_{00} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{11} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{22} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{33}\right) + \frac{\partial_3\partial_3 h}{2} - P\,\partial_3\partial_3 h
\end{aligned}$$

We see that we need J=0, so then we have

$\eta^{\mu\nu}\delta R_{\mu\nu}$ =

$$\begin{aligned}
&-2H^2\ h_{11} - 2H^2\ h_{22} - 2H^2\ h_{33} + \frac{2\,R\,h}{t^2} + t\left(H^2\,\partial_0 h_{00} - H^2\,\partial_0 h_{11} - H^2\,\partial_0 h_{22} - H^2\,\partial_0 h_{33}\right) + \\
&\ \frac{2\,\partial_0 h - P\,\partial_0 h - R\,\partial_0 h}{t} - \frac{\partial_0\partial_0 h}{2} + P\,\partial_0\partial_0 h + \frac{\partial_1\partial_1 h}{2} - P\,\partial_1\partial_1 h + \frac{\partial_2\partial_2 h}{2} - P\,\partial_2\partial_2 h + \\
&\ t^2\left(\frac{1}{2}H^2\,\partial_0\partial_0 h_{00} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{11} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{22} - \frac{1}{2}H^2\,\partial_0\partial_0 h_{33} - \frac{1}{2}H^2\,\partial_1\partial_1 h_{00} + \frac{1}{2}H^2\,\partial_1\partial_1 h_{11} + \right.\\
&\quad \frac{1}{2}H^2\,\partial_1\partial_1 h_{22} + \frac{1}{2}H^2\,\partial_1\partial_1 h_{33} - \frac{1}{2}H^2\,\partial_2\partial_2 h_{00} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{11} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{22} + \frac{1}{2}H^2\,\partial_2\partial_2 h_{33} - \\
&\quad \left. \frac{1}{2}H^2\,\partial_3\partial_3 h_{00} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{11} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{22} + \frac{1}{2}H^2\,\partial_3\partial_3 h_{33}\right) + \frac{\partial_3\partial_3 h}{2} - P\,\partial_3\partial_3 h
\end{aligned}$$

This is equivalent to

$\eta^{\mu\nu}\delta R_{\mu\nu}$ =

$$-2\,H^2\;h_{11}-2\,H^2\;h_{22}-2\,H^2\;h_{33}+\left(\frac{1}{2}-P\right)\Box h+\frac{1}{2}\,H^2\,\Box h\,t^2+\frac{2\,R\,h}{t^2}+H^2\,t\,\partial_0 h+\frac{2\,\partial_0 h-P\,\partial_0 h-R\,\partial_0 h}{t}$$

No choice of P or R can yield $\eta^{\mu\nu}\,\delta R_{\mu\nu}$ in terms of h entirely.