Weyl Tensor Simplifications

$\delta W_{\mu\nu}$ Trace Dependence (General)

In isolating the trace part of the subsitution $h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}g_{\mu\nu}h$, the perturbed conformal tensor takes the form (after some simplification from Bianchi identity and a substitution like eq. 47 in Cosmology paper, also taking $g_{\mu\nu} \equiv g^{(0)}_{\mu\nu}$ hereonforth)

$$\begin{aligned}\delta W_{\mu\nu}(\tfrac{h}{4}g_{\mu\nu}) = {}& \tfrac{1}{24}g_{\mu\nu}R^2h - \tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \tfrac{1}{6}RR_{\mu\nu}h + \tfrac{1}{2}R_{\alpha\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}h + \tfrac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R \\ & - \tfrac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} - \tfrac{1}{4}\nabla_\alpha h\nabla^\alpha R_{\mu\nu} + \tfrac{1}{4}\nabla_\alpha h\nabla_\beta R_{\mu}{}^{\beta}{}_{\nu}{}^{\alpha} + \tfrac{1}{4}\nabla_\alpha h\nabla_\nu R_{\mu}{}^{\alpha} + \tfrac{1}{12}h\nabla_\nu\nabla_\mu R\end{aligned} \tag{1}$$

Using a once contracted bianchi identity,

$$\begin{aligned}\nabla^\alpha h\nabla_\beta R_{\mu}{}^{\beta}{}_{\nu\alpha} &= -\nabla^\alpha h\nabla_\beta R^{\beta}{}_{\mu\nu\alpha} \\ &= \nabla^\alpha h\nabla_\alpha R_{\mu\nu} - \nabla^\alpha h\nabla_\nu R_{\mu\alpha}.\end{aligned} \tag{2}$$

(1) then becomes

$$\begin{aligned}\delta W_{\mu\nu} = {}& \tfrac{1}{24}g_{\mu\nu}R^2h - \tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \tfrac{1}{6}RR_{\mu\nu}h + \tfrac{1}{2}R_{\alpha\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}h \\ & + \tfrac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R - \tfrac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} + \tfrac{1}{12}h\nabla_\nu\nabla_\mu R\end{aligned} \tag{3}$$

Now note the form of $W_{\mu\nu}$

$$\begin{aligned}W_{\mu\nu} = {}& -\tfrac{1}{6}g_{\mu\nu}R^2 + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R \\ & + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\mu\nabla^\alpha R_{\nu\alpha} - \nabla_\nu\nabla^\alpha R_{\mu\alpha} + \tfrac{2}{3}\nabla_\nu\nabla_\mu R\end{aligned} \tag{4}$$

Now use the Bianchi identity on (4) to bring it to

$$W_{\mu\nu} = -\tfrac{1}{6}g_{\mu\nu}R^2h + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h + \tfrac{2}{3}RR_{\mu\nu}h - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta}h - \tfrac{1}{6}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R + h\nabla_\alpha\nabla^\alpha R_{\mu\nu} - \tfrac{1}{3}h\nabla_\nu\nabla_\mu R \tag{5}$$

It becomes apparent that

$$\boxed{\delta W_{\mu\nu}(\tfrac{h}{4}g_{\mu\nu}) = -\frac{1}{4}hW_{\mu\nu}.} \tag{6}$$

This result matches eq. 1.13 in *Fluctuations_Summary_Matthew.pdf*, where it was proven in general using the conformal properties of $W_{\mu\nu}$.

$\delta W_{\mu\nu}$ (Flat)

If we perturb $W_{\mu\nu}$ in a flat background we arrive at the form of

$$\delta W_{\mu\nu} = 2\partial^\lambda\partial^\kappa\delta C_{\mu\lambda\nu\kappa} \tag{7}$$

Weyl quantities in a flat background $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\delta C^{L}{}_{MNK} = \delta R^{L}{}_{MNK} + \frac{1}{12}g^{AB}\delta R_{AB}(\delta^{L}{}_{N}g_{MK} - \delta^{L}{}_{K}g_{MN}) \tag{8}$$

$$
-\frac{1}{3}(\delta^L{}_N \delta R_{MK} - \delta^L{}_K \delta R_{MN} - g_{MN} \delta R^L{}_K + g_{MK} \delta R^L{}_N)
$$

$$
\delta R^L{}_{MNK} = \delta\Gamma^L_{MN;K} - \delta\Gamma^L_{MK;N} \tag{9}
$$

$$
\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \tag{10}
$$

Upon evaluating $\delta C_{\mu\lambda\nu\kappa}$ as defined above, after simplification the perturbed $W_{\mu\nu}$ reduces to

$$
\begin{aligned}
2\partial^\lambda\partial^\kappa \delta C_{\mu\lambda\nu\kappa} = {} & \tfrac{1}{3}\eta^{\alpha\kappa}\eta^{\lambda\beta}\partial_\beta\partial_\kappa\partial_\nu\partial_\mu K_{\alpha\lambda} + \tfrac{1}{2}\eta^{\alpha\kappa}\eta^{\lambda\beta}\partial_\beta\partial_\lambda\partial_\kappa\partial_\alpha K_{\mu\nu} - \tfrac{1}{2}\eta^{\alpha\kappa}\eta^{\lambda\beta}\partial_\beta\partial_\lambda\partial_\kappa\partial_\mu K_{\nu\alpha} \\
& - \tfrac{1}{2}\eta^{\alpha\kappa}\eta^{\lambda\beta}\partial_\beta\partial_\lambda\partial_\kappa\partial_\nu K_{\mu\alpha} + \tfrac{1}{6}\eta^{\alpha\kappa}\eta^{\gamma\eta}\eta^{\lambda\beta}\eta_{\mu\nu}\partial_\eta\partial_\gamma\partial_\beta\partial_\kappa K_{\alpha\lambda}.
\end{aligned} \tag{11}
$$

This is equivalent to eq. (50) in Cosmology paper given as

$$
\boxed{\delta W_{\mu\nu} = \tfrac{1}{2}\Pi^\rho{}_\mu \Pi^\sigma{}_\nu K_{\rho\sigma} - \tfrac{1}{6}\Pi_{\mu\nu}\Pi^{\rho\sigma}K_{\rho\sigma}} \tag{12}
$$

where

$$
\Pi_{\mu\nu} = \eta_{\mu\nu}\partial^\alpha\partial_\alpha - \partial_\mu\partial_\nu
$$

and where one must keep in mind $\eta^{\alpha\beta}K_{\alpha\beta} = 0$.

$\delta W_{\mu\nu}$ (General)

No Riemann (56 terms)

$$
\begin{aligned}
\delta W_{\mu\nu} = {} & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + 2K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} \\
& + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \tfrac{1}{3}R\nabla_\alpha\nabla_\mu K_\nu{}^\alpha - \tfrac{1}{2}\nabla_\alpha\nabla_\mu\nabla_\beta\nabla^\beta K_\nu{}^\alpha - \tfrac{1}{3}R\nabla_\alpha\nabla_\nu K_\mu{}^\alpha - \tfrac{1}{2}\nabla_\alpha\nabla_\nu\nabla_\beta\nabla^\beta K_\mu{}^\alpha \\
& - \tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} + R_\nu{}^\alpha\nabla_\beta\nabla_\alpha K_\mu{}^\beta \\
& - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + R_\mu{}^\alpha\nabla_\beta\nabla_\alpha K_\nu{}^\beta + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} - R_\nu{}^\alpha\nabla_\beta\nabla^\beta K_{\mu\alpha} - R_\mu{}^\alpha\nabla_\beta\nabla^\beta K_{\nu\alpha} \\
& + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu K_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu K_\mu{}^\alpha + R_\nu{}^\alpha\nabla_\beta\nabla_\mu K_\alpha{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} + K^{\alpha\beta}\nabla_\beta\nabla_\mu R_{\nu\alpha} \\
& + \tfrac{1}{2}\nabla_\beta\nabla_\mu\nabla_\alpha\nabla^\beta K_\nu{}^\alpha + \tfrac{1}{2}\nabla_\beta\nabla_\mu\nabla_\alpha\nabla_\nu K^{\alpha\beta} + R_\mu{}^\alpha\nabla_\beta\nabla_\nu K_\alpha{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} + K^{\alpha\beta}\nabla_\beta\nabla_\nu R_{\mu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla_\nu\nabla_\alpha\nabla^\beta K_\mu{}^\alpha \\
& + \tfrac{1}{2}\nabla_\beta\nabla_\nu\nabla_\alpha\nabla_\mu K^{\alpha\beta} + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha - \nabla_\beta R_{\nu\alpha}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha}\nabla^\beta K_\nu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta K_\alpha{}^\gamma \\
& + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} \\
& + \nabla_\alpha K^{\alpha\beta}\nabla_\mu R_{\nu\beta} - \tfrac{1}{6}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} - \tfrac{1}{6}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + \nabla_\alpha K^{\alpha\beta}\nabla_\nu R_{\mu\beta} - \tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} \\
& - \tfrac{2}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} - \tfrac{2}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{1}{4}hW_{\mu\nu}(g^{(0)}_{\mu\nu})
\end{aligned}
$$

With Riemann (57 terms)

$$
\begin{aligned}
\delta W_{\mu\nu} = {} & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}K_\mu{}^\alpha RR_{\nu\alpha} - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\
& - \tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} - K_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta - K_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} \\
& - \tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\
& + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha K_\mu{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha K_\nu{}^\beta + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \tfrac{1}{2}K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} \\
& + \tfrac{1}{2}K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha K_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma \\
& - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu K_\alpha{}^\beta - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu K_\alpha{}^\beta + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} \\
& + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta} \\
& + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} - \tfrac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha + R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} \\
& - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \tfrac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha + R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha} \\
& - \tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} + \tfrac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{1}{4}hW_{\mu\nu}(g^{(0)}_{\mu\nu})
\end{aligned}
$$

$\delta C_{\lambda\mu\nu\kappa}$ and Trace Dependence (General)

$$\begin{aligned}\delta C_{\lambda\mu\nu\kappa} = &-\tfrac{1}{6}g_{\mu\nu}K_{\kappa\lambda}R + \tfrac{1}{6}g_{\lambda\nu}K_{\kappa\mu}R + \tfrac{1}{6}g_{\kappa\mu}K_{\lambda\nu}R - \tfrac{1}{6}g_{\kappa\lambda}K_{\mu\nu}R - \tfrac{1}{6}g_{\kappa\mu}g_{\lambda\nu}K^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{6}g_{\kappa\lambda}g_{\mu\nu}K^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\kappa\lambda} \\ &- \tfrac{1}{2}K_{\lambda\nu}R_{\kappa\mu} - \tfrac{1}{2}K_{\kappa\mu}R_{\lambda\nu} + \tfrac{1}{2}K_{\kappa\lambda}R_{\mu\nu} + K_{\lambda}{}^{\alpha}R_{\kappa\nu\mu\alpha} + \tfrac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}K_{\kappa\lambda} - \tfrac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla^{\alpha}K_{\kappa\mu} - \tfrac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla^{\alpha}K_{\lambda\nu} \\ &+ \tfrac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla_{\kappa}K_{\lambda}{}^{\alpha} + \tfrac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla_{\kappa}K_{\mu}{}^{\alpha} - \tfrac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla_{\lambda}K_{\kappa}{}^{\alpha} + \tfrac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla_{\lambda}K_{\nu}{}^{\alpha} + \tfrac{1}{4}g_{\lambda\nu}\nabla_{\alpha}\nabla_{\mu}K_{\kappa}{}^{\alpha} \\ &- \tfrac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} + \tfrac{1}{4}g_{\kappa\mu}\nabla_{\alpha}\nabla_{\nu}K_{\lambda}{}^{\alpha} - \tfrac{1}{4}g_{\kappa\lambda}\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \tfrac{1}{6}g_{\kappa\mu}g_{\lambda\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{6}g_{\kappa\lambda}g_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} \\ &- \tfrac{1}{2}\nabla_{\kappa}\nabla_{\lambda}K_{\mu\nu} + \tfrac{1}{2}\nabla_{\kappa}\nabla_{\mu}K_{\lambda\nu} + \tfrac{1}{2}\nabla_{\kappa}\nabla_{\nu}K_{\lambda\mu} - \tfrac{1}{2}\nabla_{\nu}\nabla_{\kappa}K_{\lambda\mu} + \tfrac{1}{2}\nabla_{\nu}\nabla_{\lambda}K_{\kappa\mu} - \tfrac{1}{2}\nabla_{\nu}\nabla_{\mu}K_{\kappa\lambda} + \tfrac{1}{4}hC_{\lambda\mu\nu\kappa}\end{aligned}$$

where the trace dependent terms are

$$\begin{aligned}\delta C_{\lambda\mu\nu\kappa}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) &= \tfrac{1}{24}g_{\kappa\mu}g_{\lambda\nu}Rh - \tfrac{1}{24}g_{\kappa\lambda}g_{\mu\nu}Rh + \tfrac{1}{8}g_{\mu\nu}R_{\kappa\lambda}h - \tfrac{1}{8}g_{\lambda\nu}R_{\kappa\mu}h - \tfrac{1}{8}g_{\kappa\mu}R_{\lambda\nu}h + \tfrac{1}{8}g_{\kappa\lambda}R_{\mu\nu}h - \tfrac{1}{4}R_{\kappa\nu\lambda\mu}h \\ &= \tfrac{1}{4}hC_{\lambda\mu\nu\kappa}.\end{aligned} \tag{13}$$

Note that this is opposite in sign compared to the trace dependence of $\delta W_{\mu\nu}$. To see this, under conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$$

we know the Weyl tensor is invariant

$$C^{\lambda}{}_{\mu\nu\kappa} \to C^{\lambda}{}_{\mu\nu\kappa}.$$

This is equivalent to

$$g^{\lambda\alpha}C_{\alpha\mu\nu\kappa} \to \Omega^{-2}(x)g^{\lambda\alpha}\tilde{C}_{\alpha\mu\nu\kappa},$$

and thus for the quantity to remain invariant, the covariant Weyl tensor must transform as

$$C_{\lambda\mu\nu\kappa} \to \Omega^2(x)C_{\lambda\mu\nu\kappa}.$$

The conformal symmetry applies to the trace as the following:

$$\begin{aligned}C_{\lambda\mu\nu\kappa}\left((1+\tfrac{h}{4})g^{(0)}_{\mu\nu}\right) &= \left(1+\tfrac{h}{4}\right)C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) \\ C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) + \delta C_{\lambda\mu\nu\kappa}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) &= C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) + \tfrac{h}{4}C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu})\end{aligned}$$

and therefore

$$\boxed{\delta C_{\lambda\mu\nu\kappa}\left(\tfrac{h}{4}g^{(0)}_{\mu\nu}\right) = \tfrac{h}{4}C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu})}. \tag{14}$$