Vector, Tensor RW PDE's

1 Vector $\sinh\chi$ RW

$$\tilde{g}_{ij} = \mathrm{Diag}\left(1, \sinh^2\chi, \sinh^2\chi \sin^2\theta\right) \tag{1.1}$$

$$A^i = \tilde{\nabla}_a \tilde{\nabla}^a B^i \tag{1.2}$$

$$\tilde{\nabla}_a B^a = \frac{B_2\cos\theta}{\sin\theta\sinh^2\chi} + \frac{2B_1\cosh\chi}{\sinh\chi} + \partial_1 B_1 + \frac{\partial_2 B_2}{\sinh^2\chi} + \frac{\partial_3 B_3}{\sin^2\theta\sinh^2\chi} = 0 \tag{1.3}$$

$$\begin{aligned} A^1 = {} & B_1\left(-2 - \frac{2}{\sinh^2\chi}\right) - \frac{2B_2\cos\theta\cosh\chi}{\sin\theta\sinh^3\chi} + \frac{2\cosh\chi\partial_1 B_1}{\sinh\chi} + \partial_1\partial_1 B_1 + \frac{\cos\theta\partial_2 B_1}{\sin\theta\sinh^2\chi} \\ & - \frac{2\cosh\chi\partial_2 B_2}{\sinh^3\chi} + \frac{\partial_2\partial_2 B_1}{\sinh^2\chi} - \frac{2\cosh\chi\partial_3 B_3}{\sin^2\theta\sinh^3\chi} + \frac{\partial_3\partial_3 B_1}{\sin^2\theta\sinh^2\chi} \end{aligned} \tag{1.4}$$

$$\begin{aligned} A^2 = {} & B_2\left(-\frac{1}{\sin^2\theta\sinh^4\chi} - \frac{2}{\sinh^2\chi}\right) + \frac{\partial_1\partial_1 B_2}{\sinh^2\chi} + \frac{2\cosh\chi\partial_2 B_1}{\sinh^3\chi} + \frac{\cos\theta\partial_2 B_2}{\sin\theta\sinh^4\chi} + \frac{\partial_2\partial_2 B_2}{\sinh^4\chi} \\ & - \frac{2\cos\theta\partial_3 B_3}{\sin^3\theta\sinh^4\chi} + \frac{\partial_3\partial_3 B_2}{\sin^2\theta\sinh^4\chi} \end{aligned} \tag{1.5}$$

$$\begin{aligned} A^3 = {} & -\frac{2B_3}{\sin^2\theta\sinh^2\chi} + \frac{\partial_1\partial_1 B_3}{\sin^2\theta\sinh^2\chi} - \frac{\cos\theta\partial_2 B_3}{\sin^3\theta\sinh^4\chi} + \frac{\partial_2\partial_2 B_3}{\sin^2\theta\sinh^4\chi} + \frac{2\cosh\chi\partial_3 B_1}{\sin^2\theta\sinh^3\chi} \\ & + \frac{2\cos\theta\partial_3 B_2}{\sin^3\theta\sinh^4\chi} + \frac{\partial_3\partial_3 B_3}{\sin^4\theta\sinh^4\chi} \end{aligned} \tag{1.6}$$

With imposition of trace and transverse conditions:

$$A^1 = B_1\left(2 + \frac{2}{\sinh^2\chi}\right) + \frac{4\cosh\chi\partial_1 B_1}{\sinh\chi} + \partial_1\partial_1 B_1 + \frac{\cos\theta\partial_2 B_1}{\sin\theta\sinh^2\chi} + \frac{\partial_2\partial_2 B_1}{\sinh^2\chi} + \frac{\partial_3\partial_3 B_1}{\sin^2\theta\sinh^2\chi} \tag{1.7}$$

$$\begin{aligned} A^2 = {} & B_2\left(-\frac{2}{\sinh^4\chi} + \frac{1}{\sin^2\theta\sinh^4\chi} - \frac{2}{\sinh^2\chi}\right) + \frac{4B_1\cos\theta\cosh\chi}{\sin\theta\sinh^3\chi} + \frac{2\cos\theta\partial_1 B_1}{\sin\theta\sinh^2\chi} + \frac{\partial_1\partial_1 B_2}{\sinh^2\chi} \\ & + \frac{2\cosh\chi\partial_2 B_1}{\sinh^3\chi} + \frac{3\cos\theta\partial_2 B_2}{\sin\theta\sinh^4\chi} + \frac{\partial_2\partial_2 B_2}{\sinh^4\chi} + \frac{\partial_3\partial_3 B_2}{\sin^2\theta\sinh^4\chi} \end{aligned} \tag{1.8}$$

$$\begin{aligned} A^3 = {} & -\frac{2B_3}{\sin^2\theta\sinh^2\chi} + \frac{\partial_1\partial_1 B_3}{\sin^2\theta\sinh^2\chi} - \frac{\cos\theta\partial_2 B_3}{\sin^3\theta\sinh^4\chi} + \frac{\partial_2\partial_2 B_3}{\sin^2\theta\sinh^4\chi} + \frac{2\cosh\chi\partial_3 B_1}{\sin^2\theta\sinh^3\chi} \\ & + \frac{2\cos\theta\partial_3 B_2}{\sin^3\theta\sinh^4\chi} + \frac{\partial_3\partial_3 B_3}{\sin^4\theta\sinh^4\chi} \end{aligned} \tag{1.9}$$

1.1 $B_i = g_i(\chi)$

$$\tilde{\nabla}_a B^a = g_1' + \frac{g_2\cos\theta}{\sin\theta\sinh^2\chi} + \frac{2g_1\cosh\chi}{\sinh\chi} = 0 \tag{1.10}$$

$$A^1 = g_1'' + g_1\Big(-2 - \frac{2}{\sinh^2\chi}\Big) - \frac{2g_2\cos\theta\cosh\chi}{\sin\theta\sinh^3\chi} + \frac{2\cosh\chi g_1'}{\sinh\chi} \tag{1.11}$$

Substituting g_2 (1.12)

$$\begin{aligned} A^1 &= \cos\theta g_1'' + g_1\Big[\frac{\sin^2\theta(-8 - \frac{8}{\sinh^2\chi})}{\cos\theta} + \cos\theta\Big(-2 - \frac{4}{\sinh^2\chi}\Big) + \frac{4 + \frac{4}{\sinh^2\chi}}{\cos\theta}\Big] \\ &\quad + g_1'\Big(\frac{2\cosh\chi}{\cos\theta\sinh\chi} + \frac{2\cos\theta\cosh\chi}{\sinh\chi} - \frac{4\cosh\chi\sin^2\theta}{\cos\theta\sinh\chi}\Big) \end{aligned} \tag{1.13}$$

$$A^2 = g_2\Big(-\frac{1}{\sin^2\theta\sinh^4\chi} - \frac{2}{\sinh^2\chi}\Big) + \frac{g_2''}{\sinh^2\chi} \tag{1.14}$$

$$A^3 = -\frac{2g_3}{\sin^2\theta\sinh^2\chi} + \frac{g_3''}{\sin^2\theta\sinh^2\chi} \tag{1.15}$$

1.2 $B_i = h_i(\chi)\cos\theta$

$$\tilde{\nabla}_a B^a = \cos\theta h_1' + h_2\left(\frac{1}{\sin\theta\sinh^2\chi} - \frac{2\sin\theta}{\sinh^2\chi}\right) + \frac{2h_1\cos\theta\cosh\chi}{\sinh\chi} = 0 \tag{1.16}$$

$$A^1 = \cos\theta h_1'' + h_2\left(-\frac{2\cosh\chi}{\sin\theta\sinh^3\chi} + \frac{4\cosh\chi\sin\theta}{\sinh^3\chi}\right) + h_1\cos\theta\left(-2 - \frac{4}{\sinh^2\chi}\right) + \frac{2\cos\theta\cosh\chi h_1'}{\sinh\chi} \tag{1.17}$$

Substituting h_2 (1.18)

$$\begin{aligned} A^1 &= \cos\theta h_1'' \\ &\quad + h_1\Big(-2\cos\theta - \frac{4\cos\theta}{-1+2\sin^2\theta} + \frac{8\cos\theta\sin^2\theta}{-1+2\sin^2\theta} - \frac{4\cos\theta}{\sinh^2\chi} - \frac{4\cos\theta}{(-1+2\sin^2\theta)\sinh^2\chi} + \frac{8\cos\theta\sin^2\theta}{(-1+2\sin^2\theta)\sinh^2\chi}\Big) \\ &\quad + h_1'\Big(\frac{2\cos\theta\cosh\chi}{\sinh\chi} - \frac{2\cos\theta\cosh\chi}{(-1+2\sin^2\theta)\sinh\chi} + \frac{4\cos\theta\cosh\chi\sin^2\theta}{(-1+2\sin^2\theta)\sinh\chi}\Big) \end{aligned} \tag{1.19}$$

$$A^2 = h_2\Big[\cos\theta\big(-\frac{2}{\sinh^4\chi} - \frac{2}{\sinh^2\chi}\big) - \frac{\cos\theta}{\sin^2\theta\sinh^4\chi}\Big] - \frac{2h_1\cosh\chi\sin\theta}{\sinh^3\chi} + \frac{\cos\theta h_2''}{\sinh^2\chi} \tag{1.20}$$

$$A^3 = -\frac{2h_3\cos\theta}{\sin^2\theta\sinh^2\chi} + \frac{\cos\theta h_3''}{\sin^2\theta\sinh^2\chi} \tag{1.21}$$

2 Tensor $\sinh\chi$ RW

$$\tilde{g}_{ij} = \mathrm{Diag}\left(1, \sinh^2\chi, \sinh^2\chi\sin^2\theta\right) \tag{2.1}$$

$$A^{ij} = \tilde{\nabla}_a\tilde{\nabla}^a E^{ij} \tag{2.2}$$

$$\tilde{g}^{ab}E_{ab} = E_{11} + \frac{E_{22}}{\sinh^2\chi} + \frac{E_{33}}{\sin^2\theta\sinh^2\chi} = 0 \tag{2.3}$$

$$\begin{aligned}\tilde{\nabla}_a E^{a1} = & -\frac{\cosh\chi E_{22}}{\sinh^3\chi} - \frac{\cosh\chi E_{33}}{\sin^2\theta\sinh^3\chi} + \frac{\cos\theta E_{12}}{\sin\theta\sinh^2\chi} + \frac{2\cosh\chi E_{11}}{\sinh\chi} + \partial_1 E_{11} + \frac{\partial_2 E_{12}}{\sinh^2\chi} \\ & + \frac{\partial_3 E_{13}}{\sin^2\theta\sinh^2\chi} = 0\end{aligned} \tag{2.4}$$

$$\tilde{\nabla}_a E^{a2} = -\frac{\cos\theta E_{33}}{\sin^3\theta\sinh^4\chi} + \frac{\cos\theta E_{22}}{\sin\theta\sinh^4\chi} + \frac{2\cosh\chi E_{12}}{\sinh^3\chi} + \frac{\partial_1 E_{12}}{\sinh^2\chi} + \frac{\partial_2 E_{22}}{\sinh^4\chi} + \frac{\partial_3 E_{23}}{\sin^2\theta\sinh^4\chi} \tag{2.5}$$

$$\tilde{\nabla}_a E^{a3} = \frac{\cos\theta E_{23}}{\sin^3\theta\sinh^4\chi} + \frac{2\cosh\chi E_{13}}{\sin^2\theta\sinh^3\chi} + \frac{\partial_1 E_{13}}{\sin^2\theta\sinh^2\chi} + \frac{\partial_2 E_{23}}{\sin^2\theta\sinh^4\chi} + \frac{\partial_3 E_{33}}{\sin^4\theta\sinh^4\chi} \tag{2.6}$$

$$\begin{aligned}A^{11} = & E_{11}\left(-4 - \frac{4}{\sinh^2\chi}\right) + E_{22}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) + \frac{E_{33}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} \\ & - \frac{4\cos\theta\cosh\chi E_{12}}{\sin\theta\sinh^3\chi} + \frac{2\cosh\chi\partial_1 E_{11}}{\sinh\chi} + \partial_1\partial_1 E_{11} + \frac{\cos\theta\partial_2 E_{11}}{\sin\theta\sinh^2\chi} - \frac{4\cosh\chi\partial_2 E_{12}}{\sinh^3\chi} + \frac{\partial_2\partial_2 E_{11}}{\sinh^2\chi} \\ & - \frac{4\cosh\chi\partial_3 E_{13}}{\sin^2\theta\sinh^3\chi} + \frac{\partial_3\partial_3 E_{11}}{\sin^2\theta\sinh^2\chi}\end{aligned} \tag{2.7}$$

$$\begin{aligned}A^{22} = & E_{33}\left(\frac{2}{\sin^4\theta\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi}\right) + E_{22}\left(\frac{2}{\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi} - \frac{2}{\sinh^4\chi}\right) \\ & + E_{11}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) - \frac{2\cosh\chi\partial_1 E_{22}}{\sinh^5\chi} + \frac{\partial_1\partial_1 E_{22}}{\sinh^4\chi} + \frac{4\cosh\chi\partial_2 E_{12}}{\sinh^5\chi} + \frac{\cos\theta\partial_2 E_{22}}{\sin\theta\sinh^6\chi} \\ & + \frac{\partial_2\partial_2 E_{22}}{\sinh^6\chi} - \frac{4\cos\theta\partial_3 E_{23}}{\sin^3\theta\sinh^6\chi} + \frac{\partial_3\partial_3 E_{22}}{\sin^2\theta\sinh^6\chi}\end{aligned} \tag{2.8}$$

$$\begin{aligned}A^{33} = & E_{22}\left(\frac{2}{\sin^4\theta\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi}\right) + E_{33}\left(\frac{2}{\sin^6\theta\sinh^6\chi} - \frac{2}{\sin^4\theta\sinh^4\chi}\right) \\ & + \frac{E_{11}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} + \frac{4\cos\theta\cosh\chi E_{12}}{\sin^3\theta\sinh^5\chi} - \frac{2\cosh\chi\partial_1 E_{33}}{\sin^4\theta\sinh^5\chi} + \frac{\partial_1\partial_1 E_{33}}{\sin^4\theta\sinh^4\chi} \\ & - \frac{3\cos\theta\partial_2 E_{33}}{\sin^5\theta\sinh^6\chi} + \frac{\partial_2\partial_2 E_{33}}{\sin^4\theta\sinh^6\chi} + \frac{4\cosh\chi\partial_3 E_{13}}{\sin^4\theta\sinh^5\chi} + \frac{4\cos\theta\partial_3 E_{23}}{\sin^5\theta\sinh^6\chi} + \frac{\partial_3\partial_3 E_{33}}{\sin^6\theta\sinh^6\chi}\end{aligned} \tag{2.9}$$

$$\begin{aligned}A^{12} = & E_{12}\left(-\frac{4}{\sinh^4\chi} - \frac{1}{\sin^2\theta\sinh^4\chi} - \frac{6}{\sinh^2\chi}\right) + \frac{2\cos\theta\cosh\chi E_{33}}{\sin^3\theta\sinh^5\chi} - \frac{2\cos\theta\cosh\chi E_{22}}{\sin\theta\sinh^5\chi} \\ & + \frac{\partial_1\partial_1 E_{12}}{\sinh^2\chi} + \frac{2\cosh\chi\partial_2 E_{11}}{\sinh^3\chi} + \frac{\cos\theta\partial_2 E_{12}}{\sin\theta\sinh^4\chi} - \frac{2\cosh\chi\partial_2 E_{22}}{\sinh^5\chi} + \frac{\partial_2\partial_2 E_{12}}{\sinh^4\chi} - \frac{2\cos\theta\partial_3 E_{13}}{\sin^3\theta\sinh^4\chi} \\ & - \frac{2\cosh\chi\partial_3 E_{23}}{\sin^2\theta\sinh^5\chi} + \frac{\partial_3\partial_3 E_{12}}{\sin^2\theta\sinh^4\chi}\end{aligned} \tag{2.10}$$

$$A^{13} = \frac{E_{13}\left(-\frac{4}{\sinh^4\chi} - \frac{6}{\sinh^2\chi}\right)}{\sin^2\theta} - \frac{2\cos\theta\cosh\chi E_{23}}{\sin^3\theta\sinh^5\chi} + \frac{\partial_1\partial_1 E_{13}}{\sin^2\theta\sinh^2\chi} - \frac{\cos\theta\partial_2 E_{13}}{\sin^3\theta\sinh^4\chi}$$

$$
-\frac{2\cosh\chi\partial_2 E_{23}}{\sin^2\theta\sinh^5\chi}+\frac{\partial_2\partial_2 E_{13}}{\sin^2\theta\sinh^4\chi}+\frac{2\cosh\chi\partial_3 E_{11}}{\sin^2\theta\sinh^3\chi}+\frac{2\cos\theta\partial_3 E_{12}}{\sin^3\theta\sinh^4\chi}-\frac{2\cosh\chi\partial_3 E_{33}}{\sin^4\theta\sinh^5\chi}
+\frac{\partial_3\partial_3 E_{13}}{\sin^4\theta\sinh^4\chi} \tag{2.11}
$$

$$
\begin{aligned}
A^{23} &= E_{23}\left(\frac{\frac{4}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}}{\sin^2\theta}-\frac{3}{\sin^4\theta\sinh^6\chi}\right)-\frac{4\cos\theta\cosh\chi E_{13}}{\sin^3\theta\sinh^5\chi}-\frac{2\cosh\chi\partial_1 E_{23}}{\sin^2\theta\sinh^5\chi}\\
&+\frac{\partial_1\partial_1 E_{23}}{\sin^2\theta\sinh^4\chi}+\frac{2\cosh\chi\partial_2 E_{13}}{\sin^2\theta\sinh^5\chi}-\frac{\cos\theta\partial_2 E_{23}}{\sin^3\theta\sinh^6\chi}+\frac{\partial_2\partial_2 E_{23}}{\sin^2\theta\sinh^6\chi}+\frac{2\cosh\chi\partial_3 E_{12}}{\sin^2\theta\sinh^5\chi}\\
&+\frac{2\cos\theta\partial_3 E_{22}}{\sin^3\theta\sinh^6\chi}-\frac{2\cos\theta\partial_3 E_{33}}{\sin^5\theta\sinh^6\chi}+\frac{\partial_3\partial_3 E_{23}}{\sin^4\theta\sinh^6\chi}
\end{aligned} \tag{2.12}
$$

With imposition of trace and transverse conditions:

$$
A^{11} = E_{11}\left(6+\frac{6}{\sinh^2\chi}\right)+\frac{6\cosh\chi\partial_1 E_{11}}{\sinh\chi}+\partial_1\partial_1 E_{11}+\frac{\cos\theta\partial_2 E_{11}}{\sin\theta\sinh^2\chi}+\frac{\partial_2\partial_2 E_{11}}{\sinh^2\chi}+\frac{\partial_3\partial_3 E_{11}}{\sin^2\theta\sinh^2\chi} \tag{2.13}
$$

$$
\begin{aligned}
A^{22} &= \frac{4E_{22}}{\sinh^6\chi}-\frac{4E_{22}}{\sin^2\theta\sinh^6\chi}+\frac{4E_{11}}{\sinh^4\chi}-\frac{2E_{22}}{\sinh^4\chi}-\frac{2E_{11}}{\sin^2\theta\sinh^4\chi}+\frac{2E_{11}}{\sinh^2\chi}-\frac{2\cosh\chi\partial_1 E_{22}}{\sinh^5\chi}\\
&+\frac{\partial_1\partial_1 E_{22}}{\sinh^4\chi}+\frac{4\cosh\chi\partial_2 E_{12}}{\sinh^5\chi}+\frac{\cos\theta\partial_2 E_{22}}{\sin\theta\sinh^6\chi}+\frac{\partial_2\partial_2 E_{22}}{\sinh^6\chi}-\frac{4\cos\theta\partial_3 E_{23}}{\sin^3\theta\sinh^6\chi}+\frac{\partial_3\partial_3 E_{22}}{\sin^2\theta\sinh^6\chi}
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
A^{33} &= \frac{E_{33}\left(\frac{2}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}\right)}{\sin^4\theta}+E_{11}\left(\frac{2}{\sin^4\theta\sinh^4\chi}+\frac{2}{\sin^2\theta\sinh^2\chi}\right)-\frac{4\cos\theta\cosh\chi E_{12}}{\sin^3\theta\sinh^5\chi}\\
&-\frac{4\cos\theta\partial_1 E_{12}}{\sin^3\theta\sinh^4\chi}-\frac{2\cosh\chi\partial_1 E_{33}}{\sin^4\theta\sinh^5\chi}+\frac{\partial_1\partial_1 E_{33}}{\sin^4\theta\sinh^4\chi}+\frac{4\cos\theta\partial_2 E_{11}}{\sin^3\theta\sinh^4\chi}+\frac{\cos\theta\partial_2 E_{33}}{\sin^5\theta\sinh^6\chi}\\
&+\frac{\partial_2\partial_2 E_{33}}{\sin^4\theta\sinh^6\chi}+\frac{4\cosh\chi\partial_3 E_{13}}{\sin^4\theta\sinh^5\chi}+\frac{\partial_3\partial_3 E_{33}}{\sin^6\theta\sinh^6\chi}
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
A^{12} &= E_{12}\left(-\frac{1}{\sin^2\theta\sinh^4\chi}-\frac{2}{\sinh^2\chi}\right)+\frac{2\cosh\chi\partial_1 E_{12}}{\sinh^3\chi}+\frac{\partial_1\partial_1 E_{12}}{\sinh^2\chi}+\frac{2\cosh\chi\partial_2 E_{11}}{\sinh^3\chi}\\
&+\frac{\cos\theta\partial_2 E_{12}}{\sin\theta\sinh^4\chi}+\frac{\partial_2\partial_2 E_{12}}{\sinh^4\chi}-\frac{2\cos\theta\partial_3 E_{13}}{\sin^3\theta\sinh^4\chi}+\frac{\partial_3\partial_3 E_{12}}{\sin^2\theta\sinh^4\chi}
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
A^{13} &= -\frac{2E_{13}}{\sin^2\theta\sinh^2\chi}+\frac{2\cosh\chi\partial_1 E_{13}}{\sin^2\theta\sinh^3\chi}+\frac{\partial_1\partial_1 E_{13}}{\sin^2\theta\sinh^2\chi}-\frac{\cos\theta\partial_2 E_{13}}{\sin^3\theta\sinh^4\chi}+\frac{\partial_2\partial_2 E_{13}}{\sin^2\theta\sinh^4\chi}\\
&+\frac{2\cosh\chi\partial_3 E_{11}}{\sin^2\theta\sinh^3\chi}+\frac{2\cos\theta\partial_3 E_{12}}{\sin^3\theta\sinh^4\chi}+\frac{\partial_3\partial_3 E_{13}}{\sin^4\theta\sinh^4\chi}
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
A^{23} &= E_{23}\left(\frac{\frac{2}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}}{\sin^2\theta}-\frac{1}{\sin^4\theta\sinh^6\chi}\right)+\frac{2\cos\theta\partial_1 E_{13}}{\sin^3\theta\sinh^4\chi}-\frac{2\cosh\chi\partial_1 E_{23}}{\sin^2\theta\sinh^5\chi}+\frac{\partial_1\partial_1 E_{23}}{\sin^2\theta\sinh^4\chi}\\
&+\frac{2\cosh\chi\partial_2 E_{13}}{\sin^2\theta\sinh^5\chi}+\frac{\cos\theta\partial_2 E_{23}}{\sin^3\theta\sinh^6\chi}+\frac{\partial_2\partial_2 E_{23}}{\sin^2\theta\sinh^6\chi}+\frac{2\cosh\chi\partial_3 E_{12}}{\sin^2\theta\sinh^5\chi}+\frac{2\cos\theta\partial_3 E_{22}}{\sin^3\theta\sinh^6\chi}\\
&+\frac{\partial_3\partial_3 E_{23}}{\sin^4\theta\sinh^6\chi}
\end{aligned} \tag{2.18}
$$

2.1 $E_{ij} = g_{ij}(\chi)$

$$
\tilde{g}^{ab}E_{ab} = g_{11}+\frac{g_{22}}{\sinh^2\chi}+\frac{g_{33}}{\sin^2\theta\sinh^2\chi}=0 \tag{2.19}
$$

$$\tilde{\nabla}_a E^{a1} = g'_{11} - \frac{\cosh\chi g_{22}}{\sinh^3\chi} - \frac{\cosh\chi g_{33}}{\sin^2\theta\sinh^3\chi} + \frac{\cos\theta g_{12}}{\sin\theta\sinh^2\chi} + \frac{2\cosh\chi g_{11}}{\sinh\chi} \tag{2.20}$$

$$\tilde{\nabla}_a E^{a2} = -\frac{\cos\theta g_{33}}{\sin^3\theta\sinh^4\chi} + \frac{\cos\theta g_{22}}{\sin\theta\sinh^4\chi} + \frac{2\cosh\chi g_{12}}{\sinh^3\chi} + \frac{g'_{12}}{\sinh^2\chi} \tag{2.21}$$

$$\tilde{\nabla}_a E^{a3} = \frac{\cos\theta g_{23}}{\sin^3\theta\sinh^4\chi} + \frac{2\cosh\chi g_{13}}{\sin^2\theta\sinh^3\chi} + \frac{g'_{13}}{\sin^2\theta\sinh^2\chi} \tag{2.22}$$

Without imposing trace or transverse conditions:

$$\begin{aligned} A^{11} &= g''_{11} + g_{11}\left(-4 - \frac{4}{\sinh^2\chi}\right) + g_{22}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) + \frac{g_{33}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} \\ &\quad - \frac{4\cos\theta\cosh\chi g_{12}}{\sin\theta\sinh^3\chi} + \frac{2\cosh\chi g'_{11}}{\sinh\chi} \end{aligned} \tag{2.23}$$

$$\begin{aligned} A^{22} &= g_{33}\left(\frac{2}{\sin^4\theta\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi}\right) + g_{22}\left(\frac{2}{\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi} - \frac{2}{\sinh^4\chi}\right) \\ &\quad + g_{11}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) - \frac{2\cosh\chi g'_{22}}{\sinh^5\chi} + \frac{g''_{22}}{\sinh^4\chi} \end{aligned} \tag{2.24}$$

$$\begin{aligned} A^{33} &= g_{22}\left(\frac{2}{\sin^4\theta\sinh^6\chi} - \frac{2}{\sin^2\theta\sinh^6\chi}\right) + g_{33}\left(\frac{2}{\sin^6\theta\sinh^6\chi} - \frac{2}{\sin^4\theta\sinh^4\chi}\right) \\ &\quad + \frac{g_{11}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} - \frac{2\cosh\chi g'_{33}}{\sin^4\theta\sinh^5\chi} + \frac{4\cos\theta\cosh\chi g_{12}}{\sin^3\theta\sinh^5\chi} + \frac{g''_{33}}{\sin^4\theta\sinh^4\chi} \end{aligned} \tag{2.25}$$

$$A^{12} = g_{12}\left(-\frac{4}{\sinh^4\chi} - \frac{1}{\sin^2\theta\sinh^4\chi} - \frac{6}{\sinh^2\chi}\right) + \frac{2\cos\theta\cosh\chi g_{33}}{\sin^3\theta\sinh^5\chi} - \frac{2\cos\theta\cosh\chi g_{22}}{\sin\theta\sinh^5\chi} + \frac{g''_{12}}{\sinh^2\chi} \tag{2.26}$$

$$A^{13} = \frac{g_{13}\left(-\frac{4}{\sinh^4\chi} - \frac{6}{\sinh^2\chi}\right)}{\sin^2\theta} - \frac{2\cos\theta\cosh\chi g_{23}}{\sin^3\theta\sinh^5\chi} + \frac{g''_{13}}{\sin^2\theta\sinh^2\chi} \tag{2.27}$$

$$A^{23} = g_{23}\left(\frac{\frac{4}{\sinh^6\chi} - \frac{2}{\sinh^4\chi}}{\sin^2\theta} - \frac{3}{\sin^4\theta\sinh^6\chi}\right) - \frac{4\cos\theta\cosh\chi g_{13}}{\sin^3\theta\sinh^5\chi} - \frac{2\cosh\chi g'_{23}}{\sin^2\theta\sinh^5\chi} + \frac{g''_{23}}{\sin^2\theta\sinh^4\chi} \tag{2.28}$$

With imposition of trace and transverse conditions:

$$A^{11} = g''_{11} + g_{11}(6 + \frac{6}{\sinh^2\chi}) + \frac{6\cosh\chi g'_{11}}{\sinh\chi} \tag{2.29}$$

$$\begin{aligned} A^{22} &= g_{22}(\frac{4}{\sinh^6\chi} - \frac{4}{\sin^2\theta\sinh^6\chi} - \frac{2}{\sinh^4\chi}) + g_{11}(\frac{4}{\sinh^4\chi} - \frac{2}{\sin^2\theta\sinh^4\chi} + \frac{2}{\sinh^2\chi}) \\ &\quad - \frac{2\cosh\chi g'_{22}}{\sinh^5\chi} + \frac{g''_{22}}{\sinh^4\chi} \end{aligned} \tag{2.30}$$

$$\begin{aligned} A^{33} &= \frac{g_{33}(\frac{2}{\sinh^6\chi} - \frac{2}{\sinh^4\chi})}{\sin^4\theta} + g_{11}(-\frac{\frac{8}{\sinh^4\chi} + \frac{10}{\sinh^2\chi}}{\sin^2\theta} - \frac{2}{\sin^4\theta\sinh^4\chi}) - \frac{2\cosh\chi g'_{33}}{\sin^4\theta\sinh^5\chi} \\ &\quad + \frac{g''_{33}}{\sin^4\theta\sinh^4\chi} - \frac{4\cosh\chi g'_{11}}{\sin^2\theta\sinh^3\chi} \end{aligned} \tag{2.31}$$

$$A^{12} = g_{12}(-\frac{1}{\sin^2\theta\sinh^4\chi} - \frac{2}{\sinh^2\chi}) + \frac{2\cosh\chi g'_{12}}{\sinh^3\chi} + \frac{g''_{12}}{\sinh^2\chi} \tag{2.32}$$

$$A^{13} = \frac{2\cosh\chi g'_{13}}{\sin^2\theta\sinh^3\chi} + \frac{g''_{13}}{\sin^2\theta\sinh^2\chi} - \frac{2g_{13}}{\sin^2\theta\sinh^2\chi} \tag{2.33}$$

$$A^{23} = g_{23}\left(\frac{\frac{4}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}}{\sin^2\theta} - \frac{3}{\sin^4\theta\sinh^6\chi}\right) - \frac{4\cos\theta\cosh\chi g_{13}}{\sin^3\theta\sinh^5\chi} - \frac{2\cosh\chi g'_{23}}{\sin^2\theta\sinh^5\chi} + \frac{g''_{23}}{\sin^2\theta\sinh^4\chi} \tag{2.34}$$

2.2 $E_{ij} = h_{ij}(\chi)\cos\theta$

$$\tilde{g}^{ab}E_{ab} = h_{11} + \frac{h_{22}}{\sinh^2\chi} + \frac{h_{33}}{\sin^2\theta\sinh^2\chi} = 0 \tag{2.35}$$

$$\begin{aligned}\tilde{\nabla}_a E^{a1} &= \cos\theta h'_{11} + h_{12}\left(\frac{1}{\sin\theta\sinh^2\chi} - \frac{2\sin\theta}{\sinh^2\chi}\right) - \frac{\cos\theta\cosh\chi h_{22}}{\sinh^3\chi} - \frac{\cos\theta\cosh\chi h_{33}}{\sin^2\theta\sinh^3\chi}\\ &\quad + \frac{2\cos\theta\cosh\chi h_{11}}{\sinh\chi}\end{aligned} \tag{2.36}$$

$$\begin{aligned}\tilde{\nabla}_a E^{a2} &= h_{33}\left(-\frac{1}{\sin^3\theta\sinh^4\chi} + \frac{1}{\sin\theta\sinh^4\chi}\right) + h_{22}\left(\frac{1}{\sin\theta\sinh^4\chi} - \frac{2\sin\theta}{\sinh^4\chi}\right) + \frac{2\cos\theta\cosh\chi h_{12}}{\sinh^3\chi}\\ &\quad + \frac{\cos\theta h'_{12}}{\sinh^2\chi}\end{aligned} \tag{2.37}$$

$$\tilde{\nabla}_a E^{a3} = h_{23}\left(\frac{1}{\sin^3\theta\sinh^4\chi} - \frac{2}{\sin\theta\sinh^4\chi}\right) + \frac{2\cos\theta\cosh\chi h_{13}}{\sin^2\theta\sinh^3\chi} + \frac{\cos\theta h'_{13}}{\sin^2\theta\sinh^2\chi} \tag{2.38}$$

Without imposing trace or transverse conditions:

$$\begin{aligned}A^{11} &= \cos\theta h''_{11} + h_{12}\left(-\frac{4\cosh\chi}{\sin\theta\sinh^3\chi} + \frac{8\cosh\chi\sin\theta}{\sinh^3\chi}\right) + \cos\theta h_{11}\left(-4 - \frac{6}{\sinh^2\chi}\right)\\ &\quad + \cos\theta h_{22}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) + \frac{\cos\theta h_{33}\left(\frac{2}{\sinh^4\chi}+\frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} + \frac{2\cos\theta\cosh\chi h'_{11}}{\sinh\chi}\end{aligned} \tag{2.39}$$

$$\begin{aligned}A^{22} &= h_{33}\left(\frac{2\cos\theta}{\sin^4\theta\sinh^6\chi} - \frac{2\cos\theta}{\sin^2\theta\sinh^6\chi}\right) + h_{22}\left(-\frac{2\cos\theta}{\sin^2\theta\sinh^6\chi} - \frac{2\cos\theta}{\sinh^4\chi}\right)\\ &\quad + \cos\theta h_{11}\left(\frac{2}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}\right) - \frac{2\cos\theta\cosh\chi h'_{22}}{\sinh^5\chi} - \frac{4\cosh\chi h_{12}\sin\theta}{\sinh^5\chi} + \frac{\cos\theta h''_{22}}{\sinh^4\chi}\end{aligned} \tag{2.40}$$

$$\begin{aligned}A^{33} &= h_{33}\left(\frac{\cos\theta\left(\frac{2}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}\right)}{\sin^4\theta} + \frac{2\cos\theta}{\sin^6\theta\sinh^6\chi}\right) + h_{12}\left(\frac{4\cosh\chi}{\sin^3\theta\sinh^5\chi} - \frac{4\cosh\chi}{\sin\theta\sinh^5\chi}\right)\\ &\quad + \frac{\cos\theta h_{11}\left(\frac{2}{\sinh^4\chi}+\frac{2}{\sinh^2\chi}\right)}{\sin^2\theta} + \frac{2\cos^3\theta h_{22}}{\sin^4\theta\sinh^6\chi} - \frac{2\cos\theta\cosh\chi h'_{33}}{\sin^4\theta\sinh^5\chi} + \frac{\cos\theta h''_{33}}{\sin^4\theta\sinh^4\chi}\end{aligned} \tag{2.41}$$

$$\begin{aligned}A^{12} &= h_{33}\left(\frac{2\cosh\chi}{\sin^3\theta\sinh^5\chi} - \frac{2\cosh\chi}{\sin\theta\sinh^5\chi}\right) + h_{22}\left(-\frac{2\cosh\chi}{\sin\theta\sinh^5\chi} + \frac{4\cosh\chi\sin\theta}{\sinh^5\chi}\right)\\ &\quad + h_{12}\Big(\cos\theta\left(-\frac{6}{\sinh^4\chi} - \frac{6}{\sinh^2\chi}\right) - \frac{\cos\theta}{\sin^2\theta\sinh^4\chi}\Big) - \frac{2\cosh\chi h_{11}\sin\theta}{\sinh^3\chi} + \frac{\cos\theta h''_{12}}{\sinh^2\chi}\end{aligned} \tag{2.42}$$

$$A^{13} = h_{23}\left(-\frac{2\cosh\chi}{\sin^3\theta\sinh^5\chi} + \frac{4\cosh\chi}{\sin\theta\sinh^5\chi}\right) + \frac{\cos\theta h_{13}\left(-\frac{4}{\sinh^4\chi}-\frac{6}{\sinh^2\chi}\right)}{\sin^2\theta} + \frac{\cos\theta h''_{13}}{\sin^2\theta\sinh^2\chi} \tag{2.43}$$

$$\begin{aligned}A^{23} &= h_{23}\left(\frac{\cos\theta\left(\frac{4}{\sinh^6\chi}-\frac{2}{\sinh^4\chi}\right)}{\sin^2\theta} - \frac{3\cos\theta}{\sin^4\theta\sinh^6\chi}\right) + h_{13}\left(-\frac{4\cosh\chi}{\sin^3\theta\sinh^5\chi} + \frac{2\cosh\chi}{\sin\theta\sinh^5\chi}\right)\\ &\quad - \frac{2\cos\theta\cosh\chi h'_{23}}{\sin^2\theta\sinh^5\chi} + \frac{\cos\theta h''_{23}}{\sin^2\theta\sinh^4\chi}\end{aligned} \tag{2.44}$$

With imposition of trace and transverse conditions:

$$\begin{aligned}
A^{11} &= \cos\theta h_{11}'' \\
&+h_{11}\big(-6\cos\theta - \frac{12\cos\theta}{-1+2\sin^2\theta} + \frac{24\cos\theta\sin^2\theta}{-1+2\sin^2\theta} - \frac{8\cos\theta}{\sinh^2\chi} - \frac{12\cos\theta}{(-1+2\sin^2\theta)\sinh^2\chi} + \frac{24\cos\theta\sin^2\theta}{(-1+2\sin^2\theta)\sinh^2\chi}\big) \\
&+h_{11}'\big(\frac{2\cos\theta\cosh\chi}{\sinh\chi} - \frac{4\cos\theta\cosh\chi}{(-1+2\sin^2\theta)\sinh\chi} + \frac{8\cos\theta\cosh\chi\sin^2\theta}{(-1+2\sin^2\theta)\sinh\chi}\big)
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
A^{22} &= h_{22}\big(\cos\theta(\frac{2}{\sinh^6\chi} - \frac{2}{\sinh^4\chi}) - \frac{4\cos\theta}{\sin^2\theta\sinh^6\chi}\big) \\
&+h_{11}\big(\cos\theta(\frac{4}{\sinh^4\chi} + \frac{2}{\sinh^2\chi}) - \frac{2\cos\theta}{\sin^2\theta\sinh^4\chi}\big) - \frac{2\cos\theta\cosh\chi h_{22}'}{\sinh^5\chi} - \frac{4\cosh\chi h_{12}\sin\theta}{\sinh^5\chi} \\
&+\frac{\cos\theta h_{22}''}{\sinh^4\chi}
\end{aligned} \tag{2.46}$$

$$\begin{aligned}
A^{33} &= h_{33}\big(\frac{\frac{2\cos\theta}{\sinh^6\chi} - \frac{2\cos^3\theta}{\sinh^6\chi}}{\sin^6\theta} + \frac{\cos\theta(\frac{2}{\sinh^6\chi} - \frac{2}{\sinh^4\chi})}{\sin^4\theta}\big) \\
&+h_{11}'\big(-\frac{4\cos\theta\cosh\chi}{(-1+2\sin^2\theta)\sinh^3\chi} + \frac{4\cos\theta\cosh\chi}{\sin^2\theta(-1+2\sin^2\theta)\sinh^3\chi}\big) \\
&+h_{11}\big(-\frac{2\cos^3\theta}{\sin^4\theta\sinh^4\chi} + \frac{2\cos\theta}{\sin^2\theta\sinh^4\chi} - \frac{12\cos\theta}{(-1+2\sin^2\theta)\sinh^4\chi} \\
&+\frac{12\cos\theta}{\sin^2\theta(-1+2\sin^2\theta)\sinh^4\chi} + \frac{2\cos\theta}{\sin^2\theta\sinh^2\chi} - \frac{12\cos\theta}{(-1+2\sin^2\theta)\sinh^2\chi} + \frac{12\cos\theta}{\sin^2\theta(-1+2\sin^2\theta)\sinh^2\chi}\big) \\
&-\frac{2\cos\theta\cosh\chi h_{33}'}{\sin^4\theta\sinh^5\chi} + \frac{\cos\theta h_{33}''}{\sin^4\theta\sinh^4\chi}
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
A^{12} &= h_{33}\big(\frac{2\cosh\chi}{\sin^3\theta\sinh^5\chi} + \frac{2\cosh\chi}{\sin^3\theta(-1+2\sin^2\theta)\sinh^5\chi} - \frac{8\cosh\chi}{\sin\theta(-1+2\sin^2\theta)\sinh^5\chi} + \frac{6\cosh\chi\sin\theta}{(-1+2\sin^2\theta)\sinh^5\chi}\big) \\
&+h_{12}'\big(-\frac{2\cos\theta\cosh\chi}{(-1+2\sin^2\theta)\sinh^3\chi} + \frac{6\cos\theta\cosh\chi\sin^2\theta}{(-1+2\sin^2\theta)\sinh^3\chi}\big) \\
&+h_{12}\big(-\frac{6\cos\theta}{\sinh^4\chi} - \frac{\cos\theta}{\sin^2\theta\sinh^4\chi} - \frac{4\cos\theta}{(-1+2\sin^2\theta)\sinh^4\chi} \\
&+\frac{12\cos\theta\sin^2\theta}{(-1+2\sin^2\theta)\sinh^4\chi} - \frac{6\cos\theta}{\sinh^2\chi} - \frac{4\cos\theta}{(-1+2\sin^2\theta)\sinh^2\chi} + \frac{12\cos\theta\sin^2\theta}{(-1+2\sin^2\theta)\sinh^2\chi}\big) \\
&+\frac{\cos\theta h_{12}''}{\sinh^2\chi}
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
A^{13} &= h_{13}'\big(\frac{4\cos\theta\cosh\chi}{(-1+2\sin^2\theta)\sinh^3\chi} - \frac{2\cos\theta\cosh\chi}{\sin^2\theta(-1+2\sin^2\theta)\sinh^3\chi}\big) \\
&+h_{13}\big(-\frac{4\cos\theta}{\sin^2\theta\sinh^4\chi} + \frac{8\cos\theta}{(-1+2\sin^2\theta)\sinh^4\chi} - \frac{4\cos\theta}{\sin^2\theta(-1+2\sin^2\theta)\sinh^4\chi} \\
&-\frac{6\cos\theta}{\sin^2\theta\sinh^2\chi} + \frac{8\cos\theta}{(-1+2\sin^2\theta)\sinh^2\chi} - \frac{4\cos\theta}{\sin^2\theta(-1+2\sin^2\theta)\sinh^2\chi}\big) \\
&+\frac{\cos\theta h_{13}''}{\sin^2\theta\sinh^2\chi}
\end{aligned} \tag{2.49}$$

$$A^{23} = h_{23}\big(\frac{\cos\theta(\frac{4}{\sinh^6\chi} - \frac{2}{\sinh^4\chi})}{\sin^2\theta} - \frac{3\cos\theta}{\sin^4\theta\sinh^6\chi}\big) + h_{13}\big(-\frac{4\cosh\chi}{\sin^3\theta\sinh^5\chi} + \frac{2\cosh\chi}{\sin\theta\sinh^5\chi}\big)$$

$$-\frac{2\cos\theta\cosh\chi h_{23}'}{\sin^2\theta\sinh^5\chi}+\frac{\cos\theta h_{23}''}{\sin^2\theta\sinh^4\chi} \tag{2.50}$$