Scalar Gauge Invariant RW SVT4 v2

1 Background and Fluctuations

$$ds^2 = \Omega^2(\tau)(\tilde{g}_{\mu\nu} + f_{\mu\nu})dx^\mu dx^\nu = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{1.1}$$

$$\tilde{g}_{\mu\nu} = \mathrm{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right), \qquad \tilde{\Gamma}^\lambda_{\alpha\beta} = \delta^\lambda_i \delta^j_\alpha \delta^k_\beta \tilde{\Gamma}^i_{jk} \tag{1.2}$$

$$\begin{aligned} f_{\mu\nu} &= -2\tilde{g}_{\mu\nu}\chi + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu F + \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu} \\ \tilde{g}^{\mu\nu} f_{\mu\nu} &\equiv f = -8\chi + 2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha F \end{aligned} \tag{1.3}$$

$$\begin{aligned} f_{00} &= -2\phi \\ f_{0i} &= B_i + \tilde{\nabla}_i B \\ f_{ij} &= -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \\ \tilde{g}^{ij} f_{ij} &= -6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E \\ \tilde{g}^{\mu\nu} f_{\mu\nu} &= 2\phi - 6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E \end{aligned} \tag{1.4}$$

2 $SVT3 \to SVT4$

Gauge Invariant Scalar:

$$\begin{aligned} (\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\psi - \dot{\Omega}\Omega^{-1}(B - \dot{E})] &= -\dot{\Omega}\Omega^{-1}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}^b f_{0b} + \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 3\dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} \\ &\quad -\tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 2k + \dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}^{cd} f_{cd}) \end{aligned} \tag{2.1}$$

Using U_μ and $P_{\mu\nu}$ we may construct the covariant (with respect to $\tilde{g}_{\mu\nu}$) $4D$ extension of the RHS of (2.1).

$$U_\mu = -\delta^0_\mu, \qquad U^\mu = \delta^\mu_0, \qquad P_{\mu\nu} = (\tilde{g}_{\mu\nu} + U_\mu U_\nu) \tag{2.2}$$

$$\begin{aligned} &-\dot{\Omega}\Omega^{-1}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}^b f_{0b} + \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 3\dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} \\ &-\tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 2k + \dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}^{cd} f_{cd}) \\ &= \\ &-\tfrac{1}{2}k\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f - \tfrac{1}{4}\dot{\Omega}U^\alpha\Omega^{-1}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta f + \tfrac{3}{4}\dot{\Omega}U^\alpha\Omega^{-1}\tilde{\nabla}_\alpha\tilde{\nabla}_\gamma\tilde{\nabla}_\beta f^{\beta\gamma} - 3k\dot{\Omega}U^\alpha\Omega^{-1}\tilde{\nabla}_\beta f_\alpha{}^\beta \\ &-\tfrac{1}{2}kU^\alpha U^\beta\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f - \tfrac{1}{4}U^\alpha U^\beta\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma f + \tfrac{1}{4}U^\alpha U^\beta\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma f^{\gamma\zeta} - \tfrac{1}{4}\tilde{\nabla}_\beta\tilde{\nabla}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f \\ &-3k\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\tilde{\nabla}_\gamma f_{\alpha\beta} - \tfrac{1}{4}\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\tilde{\nabla}_\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f + \tfrac{1}{2}\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\tilde{\nabla}_\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\zeta f_\alpha{}^\zeta \\ &-\tfrac{1}{2}kU^\alpha U^\beta\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma f_{\alpha\beta} - \dot{\Omega}U^\alpha\Omega^{-1}\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma\tilde{\nabla}_\beta f_\alpha{}^\beta + \tfrac{1}{4}\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f^{\alpha\beta} - \tfrac{1}{4}U^\alpha U^\beta\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f \\ &-\tfrac{1}{4}\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\tilde{\nabla}_\gamma\tilde{\nabla}_\zeta\tilde{\nabla}^\zeta f_{\alpha\beta} - \tfrac{1}{2}kU^\alpha U^\beta U^\gamma U^\zeta\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma f_{\alpha\beta} - \tfrac{1}{4}U^\alpha U^\beta U^\gamma U^\zeta\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f \\ &+\tfrac{1}{2}U^\alpha U^\beta U^\gamma U^\zeta\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma\tilde{\nabla}_\beta\tilde{\nabla}_\eta f_\alpha{}^\eta - \tfrac{1}{4}U^\alpha U^\beta U^\gamma U^\zeta\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma\tilde{\nabla}_\eta\tilde{\nabla}^\eta f_{\alpha\beta} + \tfrac{1}{2}U^\alpha U^\beta\tilde{\nabla}_\zeta\tilde{\nabla}^\zeta\tilde{\nabla}_\beta\tilde{\nabla}_\gamma f_\alpha{}^\gamma \\ &-\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\tilde{\nabla}_\zeta\tilde{\nabla}^\zeta\tilde{\nabla}_\gamma f_{\alpha\beta} - \tfrac{1}{4}U^\alpha U^\beta\tilde{\nabla}_\zeta\tilde{\nabla}^\zeta\tilde{\nabla}_\gamma\tilde{\nabla}^\gamma f_{\alpha\beta} - \tfrac{1}{2}\dot{\Omega}U^\alpha U^\beta U^\gamma U^\zeta U^\eta\Omega^{-1}\tilde{\nabla}_\eta\tilde{\nabla}_\zeta\tilde{\nabla}_\gamma f_{\alpha\beta} \end{aligned} \tag{2.3}$$

As a verification, applying the $3+1$ splitting to the RHS of (2.3) yields the LHS. Evaluating $f_{\mu\nu}$ in terms of SVT4 quantities, (2.3) can be expressed in terms of χ, $Q_\alpha = \tilde{\nabla}_\alpha F + F_\alpha$, and $F_{\mu\nu}$:

$$
\begin{aligned}
&(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\psi - \dot{\Omega}\Omega^{-1}(B-\dot{E})] = \\
&3k\nabla_\alpha\nabla^\alpha\chi + 3kU^\alpha U^\beta\nabla_\beta\nabla_\alpha\chi + \nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha\chi + 2U^\alpha U^\beta\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha\chi \\
&+U^\alpha U^\beta U^\gamma U^\zeta\nabla_\zeta\nabla_\gamma\nabla_\beta\nabla_\alpha\chi - 3k\dot{\Omega}U^\alpha\Omega^{-1}\nabla_\beta\nabla^\beta Q_\alpha - 3k\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\nabla_\gamma\nabla_\beta Q_\alpha \\
&-\dot{\Omega}U^\alpha\Omega^{-1}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta Q_\alpha - 2\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\nabla_\zeta\nabla^\zeta\nabla_\gamma\nabla_\beta Q_\alpha \\
&-\dot{\Omega}U^\alpha U^\beta U^\gamma U^\zeta U^\eta\Omega^{-1}\nabla_\eta\nabla_\zeta\nabla_\gamma\nabla_\beta Q_\alpha - 6k\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\nabla_\gamma F_{\alpha\beta} - kU^\alpha U^\beta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \\
&-kU^\alpha U^\beta U^\gamma U^\zeta\nabla_\zeta\nabla_\gamma F_{\alpha\beta} - \tfrac{5}{2}\dot{\Omega}U^\alpha U^\beta U^\gamma\Omega^{-1}\nabla_\zeta\nabla^\zeta\nabla_\gamma F_{\alpha\beta} - \tfrac{1}{2}U^\alpha U^\beta\nabla_\zeta\nabla^\zeta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \\
&-\dot{\Omega}U^\alpha U^\beta U^\gamma U^\zeta U^\eta\Omega^{-1}\nabla_\eta\nabla_\zeta\nabla_\gamma F_{\alpha\beta} - \tfrac{1}{2}U^\alpha U^\beta U^\gamma U^\zeta\nabla_\eta\nabla^\eta\nabla_\zeta\nabla_\gamma F_{\alpha\beta}.
\end{aligned} \tag{2.4}
$$

Finally, performing a $3+1$ split upon (2.4) and setting $U^\alpha Q_\alpha = Q_0$, we have

$$
\begin{aligned}
& (\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\psi - \dot{\Omega}\Omega^{-1}(B-\dot{E})] \\
= \;& 3k\tilde{\nabla}_a\tilde{\nabla}^a\chi + \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\chi - 3k\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b Q_0 - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b Q_0 + \tfrac{3}{2}\dddot{F}_{00}\dot{\Omega}\Omega^{-1} \\
& -6k\dot{F}_{00}\dot{\Omega}\Omega^{-1} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{F}_{00} - \tfrac{5}{2}\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F}_{00} - k\tilde{\nabla}_a\tilde{\nabla}^a F_{00} - \tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a F_{00}
\end{aligned} \tag{2.5}
$$

Factorizing (2.5) yields

$$
\begin{aligned}
&(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\psi - \dot{\Omega}\Omega^{-1}(B-\dot{E})] = \\
&(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b(\chi - \dot{\Omega}\Omega^{-1}Q_0) - \tfrac{1}{2}\left(\tilde{\nabla}_a\tilde{\nabla}^a + 2k - \partial_0^2\right)\tilde{\nabla}_b\tilde{\nabla}^b F_{00} - \tfrac{1}{2}\dot{\Omega}\Omega^{-1}\left(5\tilde{\nabla}_a\tilde{\nabla}^a\partial_0 + 12k\partial_0 - 3\partial_0^3\right)F_{00}
\end{aligned} \tag{2.6}
$$

Appendix A Gauge Transformations

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x) \quad \Longrightarrow \quad h'_{\mu\nu} = h_{\mu\nu} + \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} \tag{A.1}$$

$$f_{\mu} = \Omega^2 \epsilon_{\mu}, \qquad f^{\mu} = \epsilon^{\mu} \tag{A.2}$$

$$\Delta_{\epsilon}\left[f_{\mu\nu}\right] = \tilde{\nabla}_{\alpha} f_{\beta} + \tilde{\nabla}_{\beta} f_{\alpha} + 2 f^{\gamma} \tilde{g}_{\alpha\beta} \Omega^{-1} \tilde{\nabla}_{\gamma}\Omega \tag{A.3}$$

$$\Delta_{\epsilon}\left[\tilde{g}^{\mu\nu} f_{\mu\nu}\right] = 2\tilde{\nabla}_{\alpha} f^{\alpha} + 8 f^{\alpha} \Omega^{-1} \tilde{\nabla}_{\alpha}\Omega \tag{A.4}$$

$$\Delta_{\epsilon}\left[\tilde{f}_{00}\right] = 2\dot{f}_0 + 2 f_0 \Omega^{-1}\dot{\Omega} \tag{A.5}$$

$$\Delta_{\epsilon}\left[\tilde{f}_{0i}\right] = \dot{f}_i + \tilde{\nabla}_i f_0 \tag{A.6}$$

$$\Delta_{\epsilon}\left[\tilde{f}_{ij}\right] = \tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i - 2\tilde{g}_{ij} f_0 \Omega^{-1}\dot{\Omega} \tag{A.7}$$

$$\Delta_{\epsilon}\left[\tilde{f}\right] = -2\dot{f}_0 + 2\tilde{\nabla}^k f_k - 8 f_0 \Omega^{-1}\dot{\Omega} \tag{A.8}$$