Quantum Mechanics III
HW 4

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Due: Feb. 15

3.5 In some three-dimensional matrix representation, a density operator reads

$$\rho = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Is this a pure or a mixed state?

Take the square of ρ

$$\rho^2 = \begin{pmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \rho$$

Hence $\rho^2 = \rho \Rightarrow$ pure state.

3.8 The *no-cloning theorem.* In a quantum cloner, one starts with the system S in an arbitrary and possibly unknown state $|\psi\rangle$, prepares another identical system S in a reference state $|r\rangle$, and applies a unitary transformation U in such a way that the copy of the system also ends up in the state $|\psi\rangle : U\left(|\psi\rangle \otimes |r\rangle\right) = |\psi\rangle \otimes |\psi\rangle$. But there is a problem: Show that, if the dimension of the Hilbert space of the system S is at least two, such a unitary transformation does not exist. This problem presents all sorts of subtleties. To avoid these, assume that all states you consider are normalized to unity.

Assuming there exists a unitary operator $U : U\left(|\psi\rangle \otimes |r\rangle\right) = |\psi\rangle \otimes |\psi\rangle$ for all normalized states $|\psi\rangle, |r\rangle \in \mathscr{H}$, let us take two arbitrary states $|\psi\rangle, |\phi\rangle \in \mathscr{H}$ and form two joint states $|\psi\rangle|r\rangle, |\phi\rangle|r\rangle \in \mathscr{H} \otimes \mathscr{H}$. Now take the inner products of these joint states

$$\begin{aligned} \langle r|\langle\psi||\phi\rangle|r\rangle &= \langle\psi|\phi\rangle \\ &= \langle r|\langle\psi|UU^\dagger|\phi\rangle|r\rangle \\ &= \langle\psi|\langle\psi||\phi\rangle|\phi\rangle \\ &= \langle\psi|\phi\rangle^2 \end{aligned}$$

So for two arbitrary states we find

$$\langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2$$

and taking the magnitudes ($|z^n| = |z|^n$)

$$|\langle\psi|\phi\rangle| = |\langle\psi|\phi\rangle|^2$$

This equality holds for $|\langle\psi|\phi\rangle| = 0$ or $|\langle\psi|\phi\rangle| = 1$, which implies they are orthogonal or the same state, respectively. If the dimension of the Hilbert space is at least $N = 2$, then the inner product of any two *arbitrary* states $|\psi\rangle$ and $|\phi\rangle$ cannot be restricted to values of zero or unity, i.e. $0 \leq |\langle\psi|\phi\rangle| \leq 1$. Thus, there cannot exist a unitary transformation that can copy an arbitrary system.

3.11 Given N orthonormal vectors $\{|n\rangle\}$, let us compose of an equal mixture of them according to $\rho = \frac{1}{N}\sum_n |n\rangle\langle n|$. Likewise define a density operator as a mixture of some other orthonormal vectors $\{|\alpha\rangle\}$, $\rho_\alpha = \sum_\alpha p_\alpha |\alpha\rangle\langle\alpha|$, with $p_\alpha > 0$. The question is, when is $\rho = \rho_\alpha$?

(a) Show that if $\rho = \rho_\alpha$ is to hold true, the vectors $\{|n\rangle\}$ and $\{|\alpha\rangle\}$ must span the same subspace. There are therefore equally many of them. From now on, consider only this subspace as if it were the entire Hilbert space.

(b) Suppose $\{|n\rangle\}$ is a given orthonormal basis. We know that $\sum_n |n\rangle\langle n| = 1$ is a possible "resolution" of the unit operator. Conversely, show that this is the only way to represent the unit operator as an expansion of the dyads $|n\rangle\langle m|$ made of the vectors $|n\rangle$.

(c) Characterize completely the mixtures ρ_α that reproduce the density operator ρ.

(a) Denote the space spanned by $\{|n\rangle\}$ as $\mathscr{S}$ and assume $\rho = \rho_\alpha$. Take an arbitrary vector from the orthogonal compliment space $|\psi\rangle \in \mathscr{S}_\perp$ and form the expectation value

$$\langle\psi|\rho|\psi\rangle = \frac{1}{N}\sum_n |\langle\psi|n\rangle|^2 = \sum_\alpha p_\alpha |\langle\psi|\alpha\rangle|^2$$

From orthogonality, $\langle\psi|n\rangle = 0$ for all n and thus

$$\sum_\alpha p_\alpha |\langle\psi|\alpha\rangle|^2 = 0.$$

Since $|\psi\rangle$ is an arbitrary vector (orthogonal to $|\alpha\rangle$), each α term must vanish independently

$$\langle\psi|\alpha\rangle = 0.$$

This can only be true if $\{|\alpha\rangle\}$ belongs to the orthogonal compliment of $\mathscr{S}_\perp$, i.e. belongs to the space (or subspace) of $\mathscr{S}$. This implies that

$$\dim(\{|\alpha\rangle\}) \leq \dim(\{|n\rangle\}).$$

It remains to show that the spaces of $\{|n\rangle\}$ and $\{|\alpha\rangle\}$ are of the same dimensionality.

Assume $\{|\alpha\rangle\} \in \mathscr{S}_1$ where $\mathscr{S}_1$ is a subspace of $\mathscr{S}$ such that

$$\dim \mathscr{S}_1 < \dim \mathscr{S}$$

Now take a vector $|\phi\rangle \in \mathscr{S}$ that also lies in the orthogonal complement to $\mathscr{S}_1$ and form the inner product

$$\langle\phi|\rho|\phi\rangle = \frac{1}{N}\sum_n |\langle\phi|n\rangle|^2 = \langle\phi|\rho_\alpha|\phi\rangle = 0.$$

Since $\sum_n |\langle\phi|n\rangle|^2 > 0$ for $|\phi\rangle \in \mathscr{S} \cap \mathscr{S}_1^\perp$ and since $|\phi\rangle$ is orthogonal to all $|\alpha\rangle$, we have a contradictory result. Hence we must have

$$\dim \mathscr{S}_1 = \dim \mathscr{S},$$

and because both ρ and ρ_α are composed of orthonormal vectors, they must span the entire space so

$$\mathscr{S}_1 = \mathscr{S}.$$

(b) The most general expansion of dyads in a space $\mathscr{H}$ spanned by finite orthonormal basis $\{|n\rangle\}$ is

$$A = \sum_{n.m} c_{nm} |n\rangle \langle m| .$$

If this operator were to act as the identity, it must have the property of $AA = A$:

$$\begin{aligned} A^2 &= \sum_{n,m} c_{nm} |n\rangle \langle m| \left(\sum_{n',m'} c_{n'm'} |n'\rangle \langle m'| \right) \\ &= \sum_{n,m,n',m'} c_{nm} c_{n'm'} |n\rangle \langle m'| \delta_{m,n'} \\ &= \sum_{n,m,m'} c_{nm} c_{mm'} |n\rangle \langle m'| \\ &\stackrel{!}{=} \sum_{n.m} c_{nm} |n\rangle \langle m| \end{aligned}$$

The last equality can only be satisfied if

$$c_{mm'} = \delta_{m,m'} \Rightarrow c_{nm} = \delta_{n,m}.$$

Substituting this coefficient relation into A, we have

$$A = \sum_{n,m} \delta_{n,m} |n\rangle \langle m| = \sum_{n} |n\rangle \langle n| .$$

We may confirm that the result is indeed the unit operator: $A|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle \in \mathscr{H}$.

(c) Given that $\{|n\rangle_i\}$ and $\{|\alpha\rangle_i\}$ are two (equal dimension) orthonormal sets that span $\mathscr{H}$, we may use the identity operator to expand one in terms of the other

$$|n\rangle_i = \left(\sum_j |\alpha_j\rangle \langle \alpha_j| \right) |n_i\rangle = \sum_j \langle \alpha_j | n_i \rangle |\alpha_j\rangle = \sum_j c_{ij} |\alpha_j\rangle .$$

The coefficients c_{ij} form a matrix. To see its form, let's take the inner product

$$\langle n_{i'} | n_i \rangle = \sum_j c^*_{ji'} \langle \alpha_j| \left(\sum_k c_{ik} |\alpha_k\rangle \right) = \sum_{j,k} c^*_{ji'} c_{ik} \delta_{j,k} = \sum_j c^*_{ji'} c_{ij}$$

thus we find

$$\sum_j c_{ij} c^*_{ji'} = \delta_{i,i'}. \tag{1}$$

If we denote the matrix $(U)_{ij} = c_{ij}$ then (1) represents

$$U^\dagger U = 1.$$

Together, with the adjoint of (1), we verify that U is unitary

$$UU^\dagger = U^\dagger U = 1.$$

Equating the two density operators

$$\rho = \frac{1}{N} \sum_i^N |n_i\rangle \langle n_i| = \frac{1}{N} \sum_{i,j,j'}^N u_{ij} u^*_{j'i} |\alpha_j\rangle \langle \alpha_{j'}| = \frac{1}{N} \sum_i^N |\alpha_i\rangle \langle \alpha_i| \stackrel{!}{=} \sum_i^N p_i |\alpha_i\rangle \langle \alpha_i| = \rho_\alpha.$$

In summary, we see that the two density operator expansions are related by unitary matrix

$$|n_i\rangle = U|\alpha_i\rangle = \sum_j (U)_{ij}\,|\alpha_j\rangle = \sum_j \langle\alpha_j|n_i\rangle\,|\alpha_j\rangle$$

and that probabilities are evenly dispersed

$$p_i = \frac{1}{N}.$$