TT Projection Curved Space v3

1 Curved Space TT

1.1 Summary

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \tag{1.1}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{1.2}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)} h \tag{1.3}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x') h \tag{1.4}$$

$$\begin{aligned} F_\mu &= W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \\ 2F_{\mu\nu} &= 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \end{aligned} \tag{1.5}$$

Conditions upon W_μ and Ψ:

$$\Psi = \int g^{1/2} D(x,x') h \tag{1.6}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right] W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.7}$$

$$\nabla_\alpha\nabla^\alpha W_\nu + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla^\alpha W_\alpha - R_{\nu\alpha}W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.8}$$

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.9}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{1.10}$$

1.2 TT Decomposition

Assume $h_{\mu\nu}$ to be of the form:

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \underbrace{\left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D} g_{\mu\nu}\nabla^\alpha W_\alpha\right)}_{W_{\mu\nu}} + \underbrace{\frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi}_{S_{\mu\nu}} \tag{1.11}$$

Taking the trace of (1.11), we find the vector sector $W_{\mu\nu}$ is decoupled from the trace and Ψ can easily be inverted,

$$g^{\mu\nu}W_{\mu\nu} = 0 \tag{1.12}$$

$$g^{\mu\nu}S_{\mu\nu} = \nabla_\alpha\nabla^\alpha\Psi = h \qquad \to \Psi = \int g^{1/2}D(x,x')h \tag{1.13}$$

Taking the divergence of (1.11), we have

$$\nabla^\mu h_{\mu\nu} = \nabla^\mu W_{\mu\nu} + \nabla^\mu S_{\mu\nu}(h) \tag{1.14}$$

By substituting (1.13), the above serves to define an equation for W_μ in terms of h and $h_{\mu\nu}$, namely

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h \tag{1.15}$$

Commuting derivatives, (1.15) can be expressed in the equivalent forms,

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \nabla_\alpha\nabla_\nu - \frac{2}{D}\nabla_\nu\nabla_\alpha\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\int g^{1/2}D(x,x')h \tag{1.16}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}R_{\nu\alpha}\nabla^\alpha\int g^{1/2}D(x,x')h. \tag{1.17}$$

Similar to (2.4), the requisite Green's function that solves W_α is a bi-tensor defined as

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]D^{\alpha\gamma'} = g^{\alpha\gamma'}g^{-1/2}\delta^{(D)}(x,x'). \tag{1.18}$$

Hence, W_μ takes the form

$$W_\mu = \int g^{1/2}D_\mu{}^{\sigma'}\left[\nabla^{\rho'}h_{\sigma'\rho'} - \frac{1}{D-1}R_{\sigma'\rho'}\nabla^{\rho'}\int g^{1/2}D(x',x'')h\right]. \tag{1.19}$$

1.3 SVTD Decomposition

Starting with

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi, \tag{1.20}$$

we decompose W_μ into transverse and longitudinal components viz.

$$W_\mu = \underbrace{W_\mu - \nabla_\mu\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{F_\mu} + \nabla_\mu\underbrace{\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{H}. \tag{1.21}$$

Setting $h^{T\theta}_{\mu\nu} = 2F_{\mu\nu}$, (1.44) becomes

$$h_{\mu\nu} = 2F_{\mu\nu} + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2\nabla_\mu\nabla_\nu H - \frac{2}{D}g_{\mu\nu}\nabla_\alpha\nabla^\alpha H + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi. \tag{1.22}$$

Upon further defining

$$F = H - \frac{1}{2(D-1)}\Psi \tag{1.23}$$

$$\chi = \frac{1}{D}\nabla_\alpha\nabla^\alpha H - \frac{1}{2(D-1)}\nabla_\alpha\nabla^\alpha\Psi, \tag{1.24}$$

we may express (1.44) as the desired SVTD form:

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{1.25}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{1.26}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x')h \tag{1.27}$$

$$F_\mu = W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \tag{1.28}$$

$$2F_{\mu\nu} = 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \tag{1.29}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.30}$$

$$\frac{2(D-1)}{D}\nabla_\alpha\nabla^\alpha\nabla^\sigma W_\sigma - \nabla^\alpha R W_\alpha - 2R^{\alpha\beta}\nabla_\alpha W_\beta = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{1}{(D-1)}\left[\tfrac{1}{2}\nabla^\alpha R\nabla_\alpha + R^{\alpha\beta}\nabla_\alpha\nabla_\beta\right]\Psi \tag{1.31}$$

1.4 Curved TT in Max. Symmetric Space (Incomplete)

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi \tag{1.32}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu}. \tag{1.33}$$

$$\chi = \frac{1}{D}\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}h \tag{1.34}$$

$$F = \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma - \frac{1}{2(D-1)}\int g^{1/2} D(x,x')h \tag{1.35}$$

$$\begin{aligned} F_\mu &= W_\mu - \nabla_\mu \int g^{1/2} D(x,x')\nabla^\sigma W_\sigma \\ 2F_{\mu\nu} &= 2g_{\mu\nu}\chi - 2\nabla_\mu\nabla_\nu F - \nabla_\mu F_\nu - \nabla_\nu F_\mu - h_{\mu\nu} \end{aligned} \tag{1.36}$$

In a space of maximal symmetry defined by

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D), \end{aligned} \tag{1.37}$$

the conditions upon W_μ and Ψ reduce to

$$\Psi = \int g^{1/2} D(x,x')h \tag{1.38}$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)W_\nu + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{R}{D(D-1)}\nabla_\nu\Psi \tag{1.39}$$

$$\nabla_\alpha\nabla^\alpha W_\nu + \nabla^\alpha\nabla_\nu W_\alpha - \frac{2}{D}\nabla_\nu\nabla^\alpha W_\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}\left(\nabla_\nu\nabla_\alpha\nabla^\alpha - \nabla_\alpha\nabla^\alpha\nabla_\nu\right)\Psi \tag{1.40}$$

$$\frac{2(D-1)}{D}\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\nabla^\sigma W_\sigma = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{R}{D(D-1)}\nabla_\alpha\nabla^\alpha\Psi \tag{1.41}$$

From (1.41), we may determine χ and F as

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)\chi = \frac{1}{2(D-1)}\left(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{R}{D(D-1)}h\right) \tag{1.42}$$

$$\nabla_\alpha\nabla^\alpha F = \frac{D}{2(D-1)}\left(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \frac{R}{D(D-1)}h - \frac{D-1}{D(D-1)}h\right) \tag{1.43}$$

1.5 Curved TT in Minkowski

$$h_{\mu\nu} = h^{T\theta}_{\mu\nu} + \left(\nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{2}{D}g_{\mu\nu}\nabla^\alpha W_\alpha\right) + \frac{1}{D-1}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right)\Psi, \tag{1.44}$$

In a Minkowski geometry, the defining equation for W_μ reduces to

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} \tag{1.45}$$

$$W_\mu = \int g^{1/2} D_\mu{}^{\sigma'}\nabla^{\rho'} h_{\sigma'\rho'} \tag{1.46}$$

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha\right]D^{\alpha\gamma'} = g^{\alpha\gamma'}\delta^{(D)}(x,x') \tag{1.47}$$

$$\nabla_\alpha\nabla^\alpha\Psi = h \tag{1.48}$$

$$\Psi = \int g^{1/2}D(x,x')h \tag{1.49}$$

$$\nabla_\alpha\nabla^\alpha D(x,x') = g^{-1/2}\delta^{(D)}(x-x') \tag{1.50}$$

Decompose W_μ into transverse and longitudinal components viz.

$$W_\mu = \underbrace{W_\mu - \nabla_\mu\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{F_\mu} + \nabla_\mu\underbrace{\int g^{1/2}D(x,x')\nabla^\sigma W_\sigma}_{H}. \tag{1.51}$$

2 Maximally Symmetric Space TT

$$\begin{aligned} h_{\mu\nu} = {} & h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\ & + \frac{2-D}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\nabla^\sigma W_\sigma - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')h \end{aligned} \tag{2.1}$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)D(x,x') = g^{-1/2}\delta^{(D)}(x-x') \tag{2.2}$$

$$\nabla^\mu h_{\mu\nu} = \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)W_\nu \tag{2.3}$$

With the covariant operator $(\nabla^2 - R/D)$ mixing indices of W_ν, the particular integral solution for W_ν involves a bi-tensor Green's function $D_{\sigma\rho'}$ which obeys

$$\left(\nabla^\alpha\nabla_\alpha - \frac{R}{D}\right)D_{\sigma\rho'}(x,x') = g_{\sigma\rho'}g^{-1/2}\delta^4(x-x'). \tag{2.4}$$

Here $g_{\sigma\rho'}$ represents a parallel propagator, defined in terms of Vierbeins e^a_μ:

$$g^{\alpha'}{}_{\beta}(x,x') = e^{\alpha'}_a(x')e^a_\beta(x), \qquad g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu. \tag{2.5}$$

In terms of (2.4), W_ν has particular solution

$$W_\nu = g^{1/2}\int D_\nu{}^{\rho'}(x,x')\nabla^{\sigma'}h_{\rho'\sigma'}. \tag{2.6}$$