4D SVT dS$_4$ Einstein v2

1 $h_{\mu\nu}$ Decomposition

Curvature Tensors:

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D) \end{aligned} \tag{1.1}$$

Covariant Commutation:

$$\begin{aligned} [\nabla^\sigma \nabla_\nu] W_\sigma &= -R_\nu{}^\sigma W_\sigma = -\frac{R}{D}W_\nu \\ [\nabla^\mu \nabla_\mu, \nabla_\nu]V &= -R_\nu{}^\mu \nabla_\mu V = -\frac{R}{D}\nabla_\nu V \\ [\nabla^2, \nabla_\mu\nabla_\nu]V &= \frac{2g_{\mu\nu}R}{D(D-1)}\nabla^2 V - \frac{2R}{D-1}\nabla_\mu\nabla_\nu V \end{aligned} \tag{1.2}$$

Decomposition:

$$\begin{aligned} h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\ &+ \frac{2-D}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\nabla^\sigma W_\sigma - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')h \end{aligned} \tag{1.3}$$

$$\begin{aligned} \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)D(x,x') &= g^{-1/2}\delta^4(x-x') \\ \nabla^\mu h_{\mu\nu} &= \left(\nabla_\alpha\nabla^\alpha - \frac{R}{D}\right)W_\nu \end{aligned} \tag{1.4}$$

With the box-like operator mixing indices of W_ν, the particular integral solution for W_ν involves a bi-tensor Green's function $F_{\sigma\rho'}$ which obeys

$$\left(\nabla^\alpha\nabla_\alpha - \frac{R}{D}\right)F_{\sigma\rho'}(x,x') = g_{\sigma\rho'}g^{-1/2}\delta^4(x-x'). \tag{1.5}$$

Here $g_{\sigma\rho'}$ represents a parallel propagator, defined in terms of Vierbeins e^a_μ:

$$g^{\alpha'}{}_\beta(x,x') = e^{\alpha'}_a(x')e^a_\beta(x), \qquad g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu. \tag{1.6}$$

In terms of (1.5), W_ν has particular solution

$$W_\nu = \int F_\nu{}^{\rho'}(x,x')\nabla^{\sigma'}h_{\rho'\sigma'}. \tag{1.7}$$

To construct a transverse vector E_μ, split W_μ

$$\begin{aligned} W_\mu &= \underbrace{W_\mu - \nabla_\mu \int A(x,x')\nabla^\sigma W_\sigma}_{E_\mu} + \nabla_\mu \int A(x,x')\nabla^\sigma W_\sigma \\ \nabla_\alpha \nabla^\alpha A(x,x') &= g^{-1/2}\delta^4(x-x') \end{aligned} \tag{1.8}$$

With $h^{T\theta}_{\mu\nu} = 2E_{\mu\nu}$, (1.3) may be expressed as

$$\begin{aligned} h_{\mu\nu} &= 2E^{T\theta}_{\mu\nu} + \nabla_\mu E_\nu + \nabla_\nu E_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) + 2\nabla_\mu\nabla_\nu \int A(x,x')\nabla^\sigma W_\sigma \\ &+ \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\left[(2-D)\nabla^\sigma W_\sigma - h\right]. \end{aligned} \tag{1.9}$$

We may simplify this to

$$\begin{aligned} h_{\mu\nu} &= 2E^{T\theta}_{\mu\nu} + \nabla_\mu E_\nu + \nabla_\nu E_\mu \\ &+ 2\nabla_\mu\nabla_\nu\left(\int A(x,x')\nabla^\sigma W_\sigma + \frac{1}{2(D-1)}\int D(x,x')[(2-D)\nabla^\sigma W_\sigma - h]\right) \\ &- \frac{2g_{\mu\nu}}{2(D-1)}\left(\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)}\int D(x,x')[(2-D)\nabla^\sigma W_\sigma - h]\right). \end{aligned} \tag{1.10}$$

SVT Definitions:

$$\begin{aligned} 2E^{T\theta}_{\mu\nu} &= h^{T\theta}_{\mu\nu} \\ E_\mu &= W_\mu - \nabla_\mu \int A(x,x')\nabla^\sigma W_\sigma \\ E &= \int A(x,x')\nabla^\sigma W_\sigma + \frac{1}{2(D-1)}\int D(x,x')[(2-D)\nabla^\sigma W_\sigma - h] \\ \psi &= \frac{1}{2(D-1)}\left(\nabla^\sigma W_\sigma - h + \frac{R}{D(D-1)}\int D(x,x')[(2-D)\nabla^\sigma W_\sigma - h]\right) \end{aligned} \tag{1.11}$$

In the flat space limit, $A(x,x') = D(x,x')$ and we have

$$\begin{aligned} 2E^{T\theta}_{\mu\nu} &= h^{T\theta}_{\mu\nu} \\ E_\mu &= W_\mu - \nabla_\mu \int D(x,x')\nabla^\sigma W_\sigma \\ E &= \frac{1}{2(D-1)}\int D(x,x')[D\nabla^\sigma W_\sigma - h] \\ \psi &= \frac{1}{2(D-1)}\left(\nabla^\sigma W_\sigma - h\right), \end{aligned} \tag{1.12}$$

a form that coincides with *Localization_Condition_Matthew* (2.1).

2 $\delta T_{\mu\nu}$ Decomposition

For a conserved $\delta T_{\mu\nu}$ we take $W_\mu = 0$.

$$
\begin{aligned}
\delta T_{\mu\nu} &= \delta T^{T\theta}_{\mu\nu} + \frac{g_{\mu\nu}}{D-1}\delta T - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\delta T \\
6\bar{\chi} &= \int D(x,x')\delta T \\
2\bar{E}_{\mu\nu} &= \delta T^{T\theta}_{\mu\nu} \\
6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} &= \delta T \\
\delta T_{\mu\nu} &= 2\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3k g_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\bar{\chi} + 2\bar{E}_{\mu\nu} \qquad (2.1)
\end{aligned}
$$

3 dS$_4$ Background and Fluctuations

$$
\begin{aligned}
G^{(0)}_{\mu\nu} &= 3kg_{\mu\nu} \\
R^{(0)}_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\
R^{(0)}_{\mu\kappa} &= -3kg_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\
R^{(0)} &= -12k \\
\\
ds^2 &= (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
\\
\delta G_{\mu\nu} &= 2kh_{\mu\nu} - \tfrac{1}{2}kg_{\mu\nu}h + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha \\
&\quad -\tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\mu h \\
\\
\delta G &= \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h \\
\\
h_{\mu\nu} &= -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu E_\nu + \nabla_\nu E_\mu + 2E_{\mu\nu} \\
\\
\delta G_{\mu\nu} &= 4kE_{\mu\nu} + \nabla_\alpha\nabla^\alpha E_{\mu\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + 3k\nabla_\mu E_\nu + 3k\nabla_\nu E_\mu + 6k\nabla_\nu\nabla_\mu F - 2\nabla_\nu\nabla_\mu\chi \\
\\
\nabla^\mu\delta G_{\mu\nu} &= 3k\nabla^\mu h_{\mu\nu} \\
&= -6k\nabla_\nu\chi + 18k^2\nabla_\nu F + 6k\nabla_\nu\nabla_\alpha\nabla^\alpha F + 9k^2E_\nu + 3k\nabla_\alpha\nabla^\alpha E_\nu \\
\\
-\kappa_4^2T^{(0)}_{\mu\nu} &= 3kg_{\mu\nu} \\
\\
-\kappa_4^2\delta T^{(b)}_{\mu\nu} &= 3kh_{\mu\nu} \\
&= -6kg_{\mu\nu}\chi + 6k\nabla_\mu\nabla_\nu F + 3k\nabla_\mu E_\nu + 3k\nabla_\nu E_\mu + 6kE_{\mu\nu} \\
\\
\nabla^\mu(\delta G_{\mu\nu} + \kappa_4^2\delta T^{(b)}_{\mu\nu}) &= 0 \\
\\
\delta T_{\mu\nu} &= \delta T^{(b)}_{\mu\nu} + \delta T^{(s)}_{\mu\nu} \\
\\
-\kappa_4^2\delta T^{(s)}_{\mu\nu} &= 2\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\bar{\chi} + 2\bar{E}_{\mu\nu}, \qquad -\kappa_4^2\nabla^\mu\delta T^{(s)}_{\mu\nu} = 0 \qquad (3.1)
\end{aligned}
$$

4 SVT Separation

$$
\begin{aligned}
\delta G_{\mu\nu} + \kappa_4^2\delta T^{(b)}_{\mu\nu} &= -\kappa_4^2\delta T^{(s)}_{\mu\nu} \\
2\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\chi + (\nabla_\alpha\nabla^\alpha - 2k)E_{\mu\nu} &= 2\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\bar{\chi} + 2\bar{E}_{\mu\nu} \qquad (4.1)
\end{aligned}
$$

Trace (4.1):

$$
6(\nabla_\alpha\nabla^\alpha + 4k)\chi = 6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} \qquad (4.2)
$$

For $\chi = \bar{\chi}$, (4.1) becomes

$$
(\nabla_\alpha\nabla^\alpha - 2k)E_{\mu\nu} = 2\bar{E}_{\mu\nu} \qquad (4.3)
$$

5 SVT4 Covariant vs. Conformal

Perturbing on the background source, we have in SV4 covariant (barred quantities):

$$\Delta = 6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} \tag{5.1}$$

$$\Delta_{\mu\nu} = 2\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu} + 3kg_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\bar{\chi} + (\nabla_\alpha\nabla^\alpha - 2k)\bar{E}_{\mu\nu} \tag{5.2}$$

This may be compared to SVT4 conformal (unbarred quantities):

$$\begin{aligned}\Delta_{\mu\nu} = {}& \tau^{-2}\big(6\ddot{F}\tilde{g}_{\mu\nu} + 2\dot{\chi}\tilde{g}_{\mu\nu}\tau + 6\tilde{g}_{\mu\nu}\chi + 2\tilde{g}_{\mu\nu}\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 2\tilde{g}_{\mu\nu}\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi + 2\tau U_\nu\tilde{\nabla}_\mu\chi + 2\tau U_\mu\tilde{\nabla}_\nu\chi \\ & -2\tau\tilde{\nabla}_\nu\tilde{\nabla}_\mu\dot{F} - 2\tau^2\tilde{\nabla}_\nu\tilde{\nabla}_\mu\chi + 6\dot{E}^\alpha\tilde{g}_{\mu\nu}U_\alpha + 2\tilde{g}_{\mu\nu}\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha - 2\tau U^\alpha\tilde{\nabla}_\nu\tilde{\nabla}_\mu E_\alpha \\ & +2\dot{E}_{\mu\nu}\tau + 6E_{\alpha\beta}\tilde{g}_{\mu\nu}U^\alpha U^\beta + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha E_{\mu\nu} - 2\tau U^\alpha\tilde{\nabla}_\mu E_{\nu\alpha} - 2\tau U^\alpha\tilde{\nabla}_\nu E_{\mu\alpha}\big)\end{aligned} \tag{5.3}$$

$$\Delta = k(24\ddot{F} + 12\dot{\chi}\tau + 24\chi + 6\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi + 24\dot{E}^\alpha U_\alpha + 6\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha + 24E_{\alpha\beta}U^\alpha U^\beta) \tag{5.4}$$

Evaluating (5.4) in conformal flat coordinates

$$6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} = 6k(2\dot{\bar{\chi}}\tau + 4\bar{\chi} + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi}) \tag{5.5}$$

Equating (5.1) to (5.4)

$$\begin{aligned}12\dot{\bar{\chi}}\tau + 24\bar{\chi} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi} = {}& 24\ddot{F} + 12\dot{\chi}\tau + 24\chi + 6\tau\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\dot{F} + 6\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\chi \\ & +24\dot{E}^\alpha U_\alpha + 6\tau U^\alpha\tilde{\nabla}_\beta\tilde{\nabla}^\beta E_\alpha + 24E_{\alpha\beta}U^\alpha U^\beta\end{aligned} \tag{5.6}$$

$$6(\nabla_\alpha\nabla^\alpha + 4k)\bar{\chi} = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - (\nabla_\alpha\nabla^\alpha + 3k)h \tag{5.7}$$

Various Relations:

$$\bar{\chi} = \frac{1}{6}\int g^{1/2}D(x,x')\left[\nabla^\alpha\nabla^\beta h_{\alpha\beta} - (\nabla_\alpha\nabla^\alpha + 3k)h\right] \tag{5.8}$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)D(x,x') = g^{-1/2}\delta^4(x-x') \tag{5.9}$$

$$\nabla_\alpha\nabla^\alpha\bar{\chi} = k(2\dot{\chi}\tau + \tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\bar{\chi}) \tag{5.10}$$

$$\begin{aligned}\nabla^\alpha\nabla^\beta h_{\alpha\beta} - (\nabla_\alpha\nabla^\alpha + 3k)h = {}& -\Omega^{-2}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f - f\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Omega - 3\Omega^{-3}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha f + 2f\Omega^{-4}\tilde{\nabla}_\alpha\Omega\tilde{\nabla}^\alpha\Omega \\ & +6\Omega^{-3}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}_\beta f_\alpha{}^\beta + \Omega^{-2}\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f^{\alpha\beta} + 4f^{\alpha\beta}\Omega^{-3}\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\Omega + 4f_{\alpha\beta}\Omega^{-4}\tilde{\nabla}^\alpha\Omega\tilde{\nabla}^\beta\Omega \\ = {}& -3k\dot{f}\tau + 12kf_{\alpha\beta}U^\alpha U^\beta - k\tau^2\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha f \\ & +6k\tau U^\alpha\tilde{\nabla}_\beta f_\alpha{}^\beta + k\tau^2\tilde{\nabla}_\beta\tilde{\nabla}_\alpha f^{\alpha\beta}\end{aligned} \tag{5.11}$$