$3+1$ Decomposition of $G_{\mu\nu} = -\kappa_4^2 T_{\mu\nu}$ in RW

Given the maximal symmetry of the 3-space, it may be that the fluctuation equations can be simplified under a $3+1$ decomposition. We evaluate in a static RW geometry $\Omega(\tau) = 1$. If the equations simplify in this geometry, afterward we can then try to generalize to arbitrary $\Omega(\tau)$.

Separately, we also look at the $4D$ SVT decomposition within the static RW geometry.

1 Background

Given the maximal symmetry of the 3-space,

$$\begin{aligned} ds^2 &= (g^{(0)}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= \left[-dt^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 + h_{\mu\nu}dx^\mu dx^\nu\right] \end{aligned} \tag{1.1}$$

By conditions of homogeneity and isotropy of the background, the background EM tensor must take the form

$$T^{(0)}_{\mu\nu} = [\rho(\tau) + p(\tau)]U^{(0)}_\mu U^{(0)}_\nu + p(\tau) g^{(0)}_{\mu\nu} \tag{1.2}$$

with background four velocity

$$U^{(0)}_\mu = -\delta^0_\mu, \qquad U^\mu_{(0)} = \delta^\mu_0. \tag{1.3}$$

Via the background field equations, we will find that ρ and p are constant in the static geometry, though for the purposes of this document we do not yet explicitly make use of the background equations.

Since the 3-space has maximally symmetry, it may be beneficial to project the fluctuation equations parallel and orthogonal to the background four velocity $g^{(0)}_{\mu\nu}U^\mu_{(0)}U^\nu_{(0)} = -1$, via the projector

$$P_{\mu\nu} = g^{(0)}_{\mu\nu} + U^{(0)}_\mu U^{(0)}_\nu \tag{1.4}$$

with properties

$$U^{(0)}_\mu P^{\mu\nu} = 0, \qquad P_{\mu\nu}P^{\mu\nu} = g_{\mu\nu}P^{\mu\nu} = 3, \qquad P_{\mu\sigma}P^\sigma{}_\nu = P_{\mu\nu}. \tag{1.5}$$

Projecting onto an arbitrary rank 2 tensor $T_{\mu\nu}$ we have

$$T_{\mu\nu} = (\rho + p)U^{(0)}_\mu U^{(0)}_\nu + p g^{(0)}_{\mu\nu} + q_\mu U^{(0)}_\nu + q_\nu U_\mu(0) + \pi_{\mu\nu}, \tag{1.6}$$

where

$$\begin{aligned} \rho &= U^\sigma_{(0)}U^\tau_{(0)}T_{\sigma\tau} = T_{00}, \qquad p = \frac{1}{3}P^{\sigma\tau}T_{\sigma\tau} \qquad q_\mu = -P_\mu{}^\sigma U^\tau_{(0)}T_{\sigma\tau} \\ \pi_{\mu\nu} &= \left[\frac{1}{2}P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2}P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3}P_{\mu\nu}P^{\sigma\tau}\right]T_{\sigma\tau} \end{aligned} \tag{1.7}$$

which obey

$$U^\mu_{(0)}q_\mu = 0, \qquad U^\nu_{(0)}\pi_{\mu\nu} = 0, \qquad \pi_{\mu\nu} = \pi_{\nu\mu}, \qquad g^{\mu\nu}_{(0)}\pi_{\mu\nu} = P^{\mu\nu}\pi_{\mu\nu} = 0 \tag{1.8}$$

With the most general perturbed EM tensor consisting not only of perturbations of the perfect fluid background but also vector and tensor perturbations, we may utilize projector (1.4) to express any perturbed EM tensor as

$$\delta T_{\mu\nu} \quad = \quad (\delta\bar{\rho} + \delta\bar{p})U^{(0)}_\mu U^{(0)}_\nu + \delta\bar{p}g^{(0)}_{\mu\nu} + \bar{q}_\mu U^{(0)}_\nu + \bar{q}_\nu U^{(0)}_\mu + \bar{\pi}_{\mu\nu}. \tag{1.9}$$

Comparison to the background perfect fluid (1.2), we see that $\delta\rho$ and δp represent perturbations of the background, while q_μ and $\pi_{\mu\nu}$ are first order quantities without any respective background component.

Hence, in the geometry of (1.1), the background and fluctuation field equations obey

$$\begin{aligned} G^{(0)}_{\mu\nu} &= -\kappa_4^2 T^{(0)}_{\mu\nu} \\ R^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(0)} &= -\kappa_4^2\left[(\rho(\tau) + p(\tau))U^{(0)}_\mu U^{(0)}_\nu + p(\tau)g^{(0)}_{\mu\nu}\right] \end{aligned} \tag{1.10}$$

$$\begin{aligned} \delta G_{\mu\nu} &= -\kappa_4^2 \delta T_{\mu\nu} \\ \delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\delta R_{\alpha\beta} - \frac{1}{2}h_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} &= -\kappa_4^2\left[(\delta\rho + \delta p)U^{(0)}_\mu U^{(0)}_\nu + \delta p g^{(0)}_{\mu\nu} + q_\mu U^{(0)}_\nu + q_\nu U^{(0)}_\mu + \pi_{\mu\nu}\right] \end{aligned} \tag{1.11}$$

2 $3+1$

We decompose $h_{\mu\nu}$ according to

$$h_{\mu\nu} \quad = \quad (\delta\rho + \delta p)U^{(0)}_\mu U^{(0)}_\nu + \delta p g^{(0)}_{\mu\nu} + q_\mu U^{(0)}_\nu + q_\nu U^{(0)}_\mu + \pi_{\mu\nu}. \tag{2.1}$$

We will evaluate each projected component of $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$. From the gauge invariance of $\delta G_{\mu\nu} = -\kappa_4^2\delta T_{\mu\nu}$, the $3+1$ projected equations will be gauge invariant.

First we substitute the symmetric 3-space forms of the curvature terms into (1.11), via

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= (P_{\mu\nu}P_{\lambda\kappa} - P_{\lambda\nu}P_{\mu\kappa})k, \\ R_{\mu\nu} &= -2P_{\mu\nu}k, \\ R &= -6k. \end{aligned} \tag{2.2}$$

Now we compose each projected sector:

$$U^\sigma_{(0)}U^\tau_{(0)}\delta G_{\sigma\tau} \quad = \quad -\kappa_4^2 U^\sigma_{(0)}U^\tau_{(0)}\delta T_{\sigma\tau} \tag{2.3}$$

$$3k\delta p + 3k\delta\rho + \nabla_\alpha\nabla^\alpha\delta p - \tfrac{1}{2}\nabla_\beta\nabla_\alpha\pi^{\alpha\beta} + U^\alpha U^\beta\nabla_\beta\nabla_\alpha\delta p \quad = \quad -\kappa_4^2\bar{\delta}\rho \tag{2.4}$$

$$\frac{1}{3}P^{\sigma\tau}\delta G_{\sigma\tau} \quad = \quad -\kappa_4^2\frac{1}{3}P^{\sigma\tau}\delta T_{\sigma\tau} \tag{2.5}$$

$$-\tfrac{1}{3}\nabla_\alpha\nabla^\alpha\delta p + \tfrac{1}{3}\nabla_\alpha\nabla^\alpha\delta\rho + \tfrac{2}{3}U^\alpha\nabla_\alpha\nabla_\beta q^\beta + \tfrac{1}{6}\nabla_\beta\nabla_\alpha\pi^{\alpha\beta} + \tfrac{2}{3}U^\alpha U^\beta\nabla_\beta\nabla_\alpha\delta p + \tfrac{1}{3}U^\alpha U^\beta\nabla_\beta\nabla_\alpha\delta\rho \quad = \quad -\kappa_4^2\delta\bar{p} \tag{2.6}$$

$$-P_\mu{}^\sigma U^\tau_{(0)}\delta G_{\sigma\tau} \quad = \quad -\kappa_4^2 P_\mu{}^\sigma U^\tau_{(0)}\delta T_{\sigma\tau} \tag{2.7}$$

$$\begin{aligned} &2kq_\mu + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha q_\mu + \tfrac{1}{2}U^\alpha\nabla_\alpha\nabla_\beta\pi_\mu{}^\beta - \tfrac{1}{2}U^\alpha U_\mu\nabla_\alpha\nabla_\beta q^\beta - U^\alpha U^\beta U_\mu\nabla_\beta\nabla_\alpha\delta p \\ &+\tfrac{1}{2}U^\alpha U^\beta\nabla_\beta\nabla_\alpha q_\mu - U^\alpha\nabla_\mu\nabla_\alpha\delta p - \tfrac{1}{2}\nabla_\mu\nabla_\alpha q^\alpha = \bar{q}_\mu \end{aligned} \tag{2.8}$$

$$\left[\frac{1}{2}P_{\mu}{}^{\sigma}P_{\nu}{}^{\tau}+\frac{1}{2}P_{\nu}{}^{\sigma}P_{\mu}{}^{\tau}-\frac{1}{3}P_{\mu\nu}P^{\sigma\tau}\right]\delta G_{\sigma\tau} = -\kappa_4^2\left[\frac{1}{2}P_{\mu}{}^{\sigma}P_{\nu}{}^{\tau}+\frac{1}{2}P_{\nu}{}^{\sigma}P_{\mu}{}^{\tau}-\frac{1}{3}P_{\mu\nu}P^{\sigma\tau}\right]\delta T_{\sigma\tau} \tag{2.9}$$

$$\begin{aligned}
&\tfrac{1}{2}\nabla_{\alpha}\nabla^{\alpha}\pi_{\mu\nu}-\tfrac{1}{6}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\delta p-\tfrac{1}{6}U_{\mu}U_{\nu}\nabla_{\alpha}\nabla^{\alpha}\delta p+\tfrac{1}{6}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\delta\rho+\tfrac{1}{6}U_{\mu}U_{\nu}\nabla_{\alpha}\nabla^{\alpha}\delta\rho\\
&+\tfrac{1}{3}g_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}\pi^{\alpha\beta}+\tfrac{1}{3}U_{\mu}U_{\nu}\nabla_{\beta}\nabla_{\alpha}\pi^{\alpha\beta}-\tfrac{1}{2}U^{\alpha}U_{\nu}\nabla_{\beta}\nabla_{\alpha}\pi_{\mu}{}^{\beta}-\tfrac{1}{2}U^{\alpha}U_{\mu}\nabla_{\beta}\nabla_{\alpha}\pi_{\nu}{}^{\beta}\\
&-\tfrac{1}{6}g_{\mu\nu}U^{\alpha}U^{\beta}\nabla_{\beta}\nabla_{\alpha}\delta p+\tfrac{1}{3}U^{\alpha}U^{\beta}U_{\mu}U_{\nu}\nabla_{\beta}\nabla_{\alpha}\delta p+\tfrac{1}{3}g_{\mu\nu}U^{\alpha}\nabla_{\beta}\nabla_{\alpha}q^{\beta}\\
&+\tfrac{1}{3}U^{\alpha}U_{\mu}U_{\nu}\nabla_{\beta}\nabla_{\alpha}q^{\beta}-\tfrac{1}{2}U^{\alpha}U^{\beta}U_{\nu}\nabla_{\beta}\nabla_{\alpha}q_{\mu}-\tfrac{1}{2}U^{\alpha}U^{\beta}U_{\mu}\nabla_{\beta}\nabla_{\alpha}q_{\nu}\\
&+\tfrac{1}{6}g_{\mu\nu}U^{\alpha}U^{\beta}\nabla_{\beta}\nabla_{\alpha}\delta\rho-\tfrac{1}{3}U^{\alpha}U^{\beta}U_{\mu}U_{\nu}\nabla_{\beta}\nabla_{\alpha}\delta\rho-\tfrac{1}{2}\nabla_{\mu}\nabla_{\alpha}\pi_{\nu}{}^{\alpha}+\tfrac{1}{2}U^{\alpha}U_{\nu}\nabla_{\mu}\nabla_{\alpha}\delta p\\
&-\tfrac{1}{2}U^{\alpha}\nabla_{\mu}\nabla_{\alpha}q_{\nu}-\tfrac{1}{2}U^{\alpha}U_{\nu}\nabla_{\mu}\nabla_{\alpha}\delta\rho-\tfrac{1}{2}\nabla_{\nu}\nabla_{\alpha}\pi_{\mu}{}^{\alpha}+\tfrac{1}{2}U^{\alpha}U_{\mu}\nabla_{\nu}\nabla_{\alpha}\delta p-\tfrac{1}{2}U^{\alpha}\nabla_{\nu}\nabla_{\alpha}q_{\mu}\\
&-\tfrac{1}{2}U^{\alpha}U_{\mu}\nabla_{\nu}\nabla_{\alpha}\delta\rho+\tfrac{1}{2}\nabla_{\nu}\nabla_{\mu}\delta p-\tfrac{1}{2}\nabla_{\nu}\nabla_{\mu}\delta\rho=-\kappa_4^2\bar{\pi}_{\mu\nu}
\end{aligned} \tag{2.10}$$

We may note that the scalar equations for $\delta\bar{\rho}$ and $\delta\bar{p}$ involve only the transverse components of q_{μ} and $\pi_{\mu\nu}$. In the $3+1$ splitting, we have a total of 10 projected equations. There is a redundancy in these equations due to the Bianchi and conservation laws. Thus, among these 10 equations, we need to reduce to six independent equations.

3 4D SVT

Within the geometry

$$ds^2 = \left[-dt^2+\frac{dr^2}{1-kr^2}+r^2d\Omega^2+h_{\mu\nu}dx^{\mu}dx^{\nu}\right] \tag{3.1}$$

we take $h_{\mu\nu}$ as the 4D SVT decomposition

$$h_{\mu\nu} = -2g_{\mu\nu}\psi+2\nabla_{\mu}\nabla_{\nu}E+\nabla_{\mu}E_{\nu}+\nabla_{\nu}E_{\mu}+2E_{\mu\nu}. \tag{3.2}$$

Inserting the above into (1.11), we have for the perturbed Einstein tensor

$$\begin{aligned}
\delta G_{\mu\nu} ={}& -E_{\mu\nu}R+E^{\alpha\beta}g_{\mu\nu}R_{\alpha\beta}+E_{\nu}{}^{\alpha}R_{\mu\alpha}+E_{\mu}{}^{\alpha}R_{\nu\alpha}-2E^{\alpha\beta}R_{\mu\alpha\nu\beta}-\tfrac{1}{2}E^{\alpha}g_{\mu\nu}\nabla_{\alpha}R\\
&+\nabla_{\alpha}\nabla^{\alpha}E_{\mu\nu}+2g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\psi-\tfrac{1}{2}g_{\mu\nu}\nabla_{\alpha}R\nabla^{\alpha}E+\tfrac{1}{2}E^{\alpha}\nabla_{\beta}R_{\mu\alpha\nu}{}^{\beta}+\tfrac{1}{2}E^{\alpha}\nabla_{\beta}R_{\mu}{}^{\beta}{}_{\nu\alpha}\\
&+\nabla^{\alpha}E\nabla_{\beta}R_{\mu}{}^{\beta}{}_{\nu\alpha}+R_{\mu\alpha\nu\beta}\nabla^{\beta}\nabla^{\alpha}E-R_{\mu\beta\nu\alpha}\nabla^{\beta}\nabla^{\alpha}E+R_{\nu\alpha}\nabla_{\mu}E^{\alpha}-\tfrac{1}{2}R\nabla_{\mu}E_{\nu}\\
&+\tfrac{1}{2}E^{\alpha}\nabla_{\mu}R_{\nu\alpha}+R_{\nu\alpha}\nabla_{\mu}\nabla^{\alpha}E+R_{\mu\alpha}\nabla_{\nu}E^{\alpha}-\tfrac{1}{2}R\nabla_{\nu}E_{\mu}+\tfrac{1}{2}E^{\alpha}\nabla_{\nu}R_{\mu\alpha}+\nabla^{\alpha}E\nabla_{\nu}R_{\mu\alpha}\\
&+R_{\mu\alpha}\nabla_{\nu}\nabla^{\alpha}E-R\nabla_{\nu}\nabla_{\mu}E-2\nabla_{\nu}\nabla_{\mu}\psi
\end{aligned} \tag{3.3}$$

with trace

$$\begin{aligned}
g^{\alpha\beta}\delta G_{\alpha\beta} ={}& 4E^{\alpha\beta}R_{\alpha\beta}-E^{\alpha}\nabla_{\alpha}R-R\nabla_{\alpha}\nabla^{\alpha}E+6\nabla_{\alpha}\nabla^{\alpha}\psi-\nabla_{\alpha}R\nabla^{\alpha}E+2R_{\alpha\beta}\nabla^{\beta}E^{\alpha}\\
&+2R_{\alpha\beta}\nabla^{\beta}\nabla^{\alpha}E.
\end{aligned} \tag{3.4}$$

If we substitute the symmetric 3-space forms of the curvature terms into the above, via

$$\begin{aligned}
R_{\lambda\mu\nu\kappa} &= (P_{\mu\nu}P_{\lambda\kappa}-P_{\lambda\nu}P_{\mu\kappa})k,\\
R_{\mu\nu} &= -2P_{\mu\nu}k,\\
R &= -6k
\end{aligned} \tag{3.5}$$

we are left with

$$\begin{aligned}
\delta G_{\mu\nu} ={}& -4kE_{\nu\alpha}U^{\alpha}U_{\mu}-4kE_{\mu\alpha}U^{\alpha}U_{\nu}+\nabla_{\alpha}\nabla^{\alpha}E_{\mu\nu}+2g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\psi-2kU^{\alpha}U_{\nu}\nabla_{\mu}E_{\alpha}\\
&+k\nabla_{\mu}E_{\nu}-2kU^{\alpha}U_{\nu}\nabla_{\mu}\nabla_{\alpha}E-2kU^{\alpha}U_{\mu}\nabla_{\nu}E_{\alpha}+k\nabla_{\nu}E_{\mu}-2kU^{\alpha}U_{\mu}\nabla_{\nu}\nabla_{\alpha}E\\
&+2k\nabla_{\nu}\nabla_{\mu}E-2\nabla_{\nu}\nabla_{\mu}\psi
\end{aligned} \tag{3.6}$$

and trace

$$g^{\alpha\beta}\delta G_{\alpha\beta} \quad = \quad -8kE_{\alpha\beta}U^{\alpha}U^{\beta} + 2k\nabla_{\alpha}\nabla^{\alpha}E + 6\nabla_{\alpha}\nabla^{\alpha}\psi - 4kU^{\alpha}U^{\beta}\nabla_{\beta}E_{\alpha} - 4kU^{\alpha}U^{\beta}\nabla_{\beta}\nabla_{\alpha}E. \tag{3.7}$$

For a $\delta T_{\mu\nu}$ that is similarly decomposed we have

$$\delta T_{\mu\nu} \quad = \quad -2g_{\mu\nu}\bar{\psi} + 2\nabla_{\mu}\nabla_{\nu}\bar{E} + \nabla_{\mu}\bar{E}_{\nu} + \nabla_{\nu}\bar{E}_{\mu} + 2\bar{E}_{\mu\nu}, \tag{3.8}$$

with trace

$$g^{\alpha\beta}\delta T_{\alpha\beta} \quad = \quad -8\bar{\psi} + 2\nabla^2\bar{E}. \tag{3.9}$$

Though in flat space the trace allowed us to simplify the remaining equations, it remains to see if one can do so in the static RW geometry.