

Coordinate Transformations RW $k < 0$ v6

For $K < 1$ FRW cosmology with $L^2 a^2 = t^2 + d^2$, the line element takes the form

$$\begin{aligned} ds^2 &= dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \\ &= d^2 \left[du^2 - (1 + u^2) \left(\frac{dv^2}{1 + v^2} + v^2 d\Omega^2 \right) \right], \end{aligned} \quad (1)$$

where we have introduced

$$u = \frac{t}{d}, \quad v = \frac{r}{L}. \quad (2)$$

Original Coordinates for K_{33}

Transformations and Asymptotics:

$$p' = \frac{u}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}}, \quad r' = \frac{v}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}} \quad (3)$$

$$u^2 = \frac{4p'^2}{(1 - (p' + r')^2)(1 - (p' - r')^2)}, \quad v = \left(\frac{r'}{p'} \right) u \quad (4)$$

$$\Omega^2(p', r') = \frac{4L^2 a^2}{(1 - (p' + r')^2)(1 - (p' - r')^2)} = d^2(1 + u^2) \left[(1 + u^2)^{1/2} + (1 + v^2)^{1/2} \right]^2 \quad (5)$$

Null Trajectory

In the u, v geometry, the condition for null separation (at fixed angle) is $u = v$. Inspection of coordinate transformation (3-5) shows the leading order ($u \gg 1$) contributions for null separation:

$$p' \sim 1, \quad r' \sim 1, \quad \Omega^2 \sim u^4. \quad (6)$$

The leading behavior for the full $K_{33}^{(cm)}$ then behaves as

$$\boxed{K_{33}^{(cm)} = K_{33}^{(P)} \approx \Omega^2(p', r') r'^2 p' B_{\mu\nu} e^{ik(z' - p')} \sim u^4.} \quad (7)$$

This result coincides with the null configuration given in PRD 2012.

Timelike Trajectory

For coordinate separations which are timelike we take $u \gg v$. In order to find the leading contribution in u , we will effectively take v to be finite on the order $\mathcal{O}(1)$, and take $u \gg 1$. These results yield a leading behavior of:

$$p' \sim 1, \quad r' \sim \frac{1}{u}, \quad \Omega^2 \sim u^4. \quad (8)$$

The leading behavior for the full $K_{33}^{(cm)}$ then behaves as

$$\boxed{K_{33}^{(cm)} = K_{33}^{(P)} \approx \Omega^2(p', r') r'^2 p' B_{\mu\nu} e^{ik(z' - p')} \sim u^2.} \quad (9)$$

New Coordinates for K_{33}

Transformations:

$$T = \left[u + (1 + u^2)^{1/2} \right] (1 + v^2)^{1/2}, \quad R = \left[u + (1 + u^2)^{1/2} \right] v \quad (10)$$

$$\Omega^2(T, R) = \frac{L^2 a^2}{T^2 - R^2} = d^2 \frac{(1 + u^2)}{(u + (1 + u^2)^{1/2})^2} \quad (11)$$

Null Trajectory

In the comoving u, v geometry, the condition for null separation (at fixed angle) is $u = v$. Inspection of coordinate transformation (10-11) shows the leading order ($u \gg 1$) contributions for null separation:

$$T \sim u^2, \quad R \sim u^2, \quad \Omega^2 \sim 1. \quad (12)$$

The leading behavior for the full $K_{33}^{(cm)}$ then behaves as

$$\boxed{K_{33}^{(cm)} = K_{33}^{(P)} \approx \Omega^2(T, R) R^2 T B_{\mu\nu} e^{ik(Z-T)} \sim u^6.} \quad (13)$$

Timelike Trajectory

For coordinate separations which are timelike we take $u \gg v$. In order to find the leading contribution in u , we will effectively take v to be finite on the order $\mathcal{O}(1)$, and take $u \gg 1$. These results yield a leading behavior of:

$$T \sim u, \quad R \sim u, \quad \Omega^2 \sim 1. \quad (14)$$

The leading behavior for the full $K_{33}^{(cm)}$ then behaves as

$$\boxed{K_{33}^{(cm)} = K_{33}^{(P)} \approx \Omega^2(T, R) R^2 T B_{\mu\nu} e^{ik(Z-T)} \sim u^3.} \quad (15)$$

PRD 2012

Employing both the transverse gauge $\nabla^\nu K_{\mu\nu} = 0$ and the synchronous conditions $K_{0\mu} = 0$, within the geometry of

$$ds^2 = dp'^2 - dr'^2 - r'^2 d\Omega^2, \quad (16)$$

solutions to the equation

$$\nabla_\alpha \nabla^\alpha \nabla_\beta \nabla^\beta K_{\mu\nu} = 0 \quad (17)$$

were found as

$$K_{33} = -(A_{33} + B_{33} p') r' e^{ik(r' - p')}. \quad (18)$$

Upon conformal transformation, the full $K_{33}^{(cm)}$ is then calculated as

$$K_{33}^{(cm)} = \Omega^2(p', r') (A_{33} + B_{33} p') r' e^{ik(r' - p')}. \quad (19)$$

In comparison to APM2018, we note the difference in overall factor of r' .

Previous Work (v4,v5)

Fluctuation Equation Solution

If we take the conformal gauge

$$\nabla_\nu K^{\mu\nu} - \frac{1}{2} K^{\mu\nu} \Omega^{-2} g_{(0)}^{\alpha\beta} \partial_\nu g_{\alpha\beta}^{(0)} = 0, \quad (20)$$

within the geometry of

$$ds^2 = -\Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu = -(g_{\mu\nu}^{(0)} + K_{\mu\nu})dx^\mu dx^\nu, \quad (21)$$

then as outlined in APM (55), the conformal gauge condition may be expressed in the covariant form

$$\nabla_\nu K^{\mu\nu} = 4\Omega^{-1} K^{\mu\nu} \partial_\nu \Omega. \quad (22)$$

The above form transforms conformally only if the background is Minkowski, and is also equivalent to

$$\partial_\nu k^{\mu\nu} = 0. \quad (23)$$

Evaluating $\delta W_{\mu\nu}$ in the geometry of (9) using covariant gauge (10) then yields

$$\delta W_{\mu\nu} = \frac{1}{2} \Omega^{-2} \eta^{\sigma\rho} \eta^{\alpha\beta} \partial_\sigma \partial_\rho \partial_\alpha \partial_\beta (\Omega^{-2} K_{\mu\nu}) = \frac{1}{2} \Omega^{-2} \eta^{\sigma\rho} \eta^{\alpha\beta} \partial_\sigma \partial_\rho \partial_\alpha \partial_\beta k_{\mu\nu}. \quad (24)$$

The momentum eigenstate solution for $\square^2 k_{\mu\nu} = 0$ is then

$$k_{\mu\nu} = A_{\mu\nu} e^{ikx} + B_{\mu\nu} n_\alpha x^\alpha e^{ikx}. \quad (25)$$

As our gauge condition is covariant, if it is satisfied in r', p' coordinates, then it must also be satisfied in T, R coordinates.

Summary

In the radiation dominated early universe with scale factor $L^2 a^2(t) = (d^2 + t^2)$, the leading order large time behavior for $K_{\mu\nu}$ as evaluated in the comoving $k < 0$ R.W. background takes the form:

$$\begin{aligned} K_{00}^{(cm)} &\sim 1 \\ K_{01}^{(cm)} &\sim \frac{d}{L}(u) \\ K_{02}^{(cm)} &\sim d(u) \\ K_{03}^{(cm)} &\sim d(u) \\ K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^2) \\ K_{22}^{(cm)} &\sim d^2(u^2) \\ K_{33}^{(cm)} &\sim d^2(u^2) \\ K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\ K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\ K_{23}^{(cm)} &\sim d^2(u^2), \end{aligned} \quad (26)$$

where $u = t/d$. This result differs from APM3 perturbations by a u^{-1} suppression for each angular index. This is due to the Cartesian to polar coordinate transformation, where factors of $r'(t, r)$ or $R(t, r)$ in the transformation have non-negligible u dependence. The large time behavior for the new coordinate system of (T, R) was found to only match that of the old coordinate system of (p', r') when integrating the z-direction plane wave over the full solid angle.

It remains to look into the necessity (or non-necessity) of spatial averaging.

Notation

From the original form of the scale factor

$$a^2(t) = \frac{2AL^2}{S_0^2} + \frac{t^2}{L^2} \quad (27)$$

we see that for setting up a definition for large t , we should take

$$\frac{t^2}{L^2} \gg \frac{2AL^2}{S_0^2}. \quad (28)$$

This is equivalent to requiring $t \gg d$. If the scale behaves such that $2AL^2/S_0^2 \ll 1$, then $t \gg d$ does not necessarily imply $t \gg L$. Noting in addition the R.W. comoving geometry distance r/L , we introduce two scales of comparison

$$u \equiv \frac{t}{d}, \quad v \equiv \frac{r}{L}. \quad (29)$$

Thus we define large t behavior as taking $u \gg 1$, holding v finite.

In terms of u and v , the scale factor takes the form

$$a^2(u) = \frac{d^2}{L^2}(1 + u^2) \quad (30)$$

comoving R.W. metric takes the form

$$\begin{aligned} ds^2 &= dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \\ &= d^2 \left[du^2 - (1 + u^2) \left(\frac{dv^2}{1 + v^2} + v^2 d\Omega^2 \right) \right] \end{aligned} \quad (31)$$

Coordinate Transformations

Cartesian to Polar

In going from the geometry of

$$ds^2 = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu \quad (32)$$

to

$$ds^2 = \Omega^2(dt^2 - dr^2 - r^2 d\Omega^2 + k_{\mu\nu}^{(P)} dx^\mu dx^\nu), \quad (33)$$

we must perform the appropriate coordinate transformation (given in the Appendix). Denoting the polar coordinate system as $x^{(P)}$, we find, after imposing the transverse and residual relations, the following:

$$\begin{aligned} k_{00}^{(P)} &= 0 \\ k_{01}^{(P)} &= k_{01} \sin(\theta) \cos(\phi) + k_{02} \sin(\theta) \sin(\phi) \\ k_{02}^{(P)} &= k_{01} r \cos(\theta) \cos(\phi) + k_{02} r \cos(\theta) \sin(\phi) \\ k_{03}^{(P)} &= -k_{01} r \sin(\theta) \sin(\phi) + k_{02} r \sin(\theta) \cos(\phi) \\ k_{11}^{(P)} &= k_{11} \sin^2(\theta) \cos(2\phi) + k_{12} \sin^2(\theta) \sin(2\phi) \\ k_{22}^{(P)} &= k_{11} r^2 \cos^2(\theta) \cos(2\phi) + k_{12} r^2 \cos^2(\theta) \sin(2\phi) \\ k_{33}^{(P)} &= -k_{11} r^2 \sin^2(\theta) \cos(2\phi) - 2k_{12} r^2 \sin^2(\theta) \sin(\phi) \cos(\phi) \\ k_{12}^{(P)} &= \frac{1}{2} k_{11} r \sin(2\theta) \cos(2\phi) + k_{12} r \sin(\theta) \cos(\theta) \sin(2\phi) \\ k_{13}^{(P)} &= -2k_{11} r \sin^2(\theta) \sin(\phi) \cos(\phi) + k_{12} r \sin^2(\theta) \cos(2\phi) \\ k_{23}^{(P)} &= -2k_{11} r^2 \sin(\theta) \cos(\theta) \sin(\phi) \cos(\phi) + k_{12} r^2 \sin(\theta) \cos(\theta) \cos(2\phi) \end{aligned} \quad (34)$$

Since the $\square^2 k_{\mu\nu} = 0$ is only valid in a conformal to Minkowski background, upon transforming the solution for $k_{\mu\nu}$ to polar coordinates, we must account for the factors of $R(t, r)$ and $r'(t, r)$ in regards to the asymptotic time behavior. As a rule, every angular index gets a power of r .

Original Coordinates

Performing coordinate transformations

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \quad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \quad (35)$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(p', r')(dp'^2 - dr'^2 - r'^2 d\Omega^2) \quad (36)$$

with conformal factor

$$\Omega^2(p', r') = \frac{4L^2 a^2}{(1 - (p' + r')^2)(1 - (p' - r')^2)} = d^2(1 + u^2) \left[(1 + u^2)^{1/2} + (1 + v^2)^{1/2} \right]^2. \quad (37)$$

We will soon make use of the coordinate relations

$$\begin{aligned} \frac{\partial p'}{\partial t} &= \frac{1}{d} \frac{\partial p'}{\partial u} = \left(\frac{1}{d} \right) \frac{1 + (1 + u^2)^{1/2}(1 + v^2)^{1/2}}{(1 + u^2)^{1/2}((1 + u^2)^{1/2} + (1 + v^2)^{1/2})^2} \\ \frac{\partial p'}{\partial r} &= \frac{1}{L} \frac{\partial p'}{\partial v} = - \left(\frac{1}{L} \right) \frac{uv}{(1 + v^2)^{1/2}((1 + u^2)^{1/2} + (1 + v^2)^{1/2})^2} \\ \frac{\partial r'}{\partial t} &= \frac{1}{d} \frac{\partial r'}{\partial u} = - \left(\frac{1}{d} \right) \frac{uv}{(1 + u^2)^{1/2}((1 + u^2)^{1/2} + (1 + v^2)^{1/2})^2} \\ \frac{\partial r'}{\partial r} &= \frac{1}{L} \frac{\partial r'}{\partial v} = \left(\frac{1}{L} \right) \frac{1 + (1 + u^2)^{1/2}(1 + v^2)^{1/2}}{(1 + v^2)^{1/2}((1 + u^2)^{1/2} + (1 + v^2)^{1/2})^2} \end{aligned} \quad (38)$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned} k_{00}^{(cm)} &= 2 \frac{\partial p'}{\partial t} \frac{\partial r'}{\partial t} k_{01}^{(P)} + \left(\frac{\partial r'}{\partial t} \right)^2 k_{11}^{(P)} \\ k_{01}^{(cm)} &= \frac{\partial p'}{\partial t} \frac{\partial r'}{\partial r} k_{01}^{(P)} + \frac{\partial r'}{\partial t} \frac{\partial p'}{\partial r} k_{01}^{(P)} + \frac{\partial r'}{\partial t} \frac{\partial r'}{\partial r} k_{11}^{(P)} \\ k_{02}^{(cm)} &= \frac{\partial p'}{\partial t} k_{02}^{(P)} + \frac{\partial r'}{\partial t} k_{12}^{(P)} \\ k_{03}^{(cm)} &= \frac{\partial p'}{\partial t} k_{03}^{(P)} + \frac{\partial r'}{\partial t} k_{13}^{(P)} \\ k_{11}^{(cm)} &= 2 \frac{\partial p'}{\partial r} \frac{\partial r'}{\partial r} k_{01}^{(P)} + \left(\frac{\partial r'}{\partial r} \right)^2 k_{11}^{(P)} \\ k_{22}^{(cm)} &= k_{22}^{(P)} \\ k_{33}^{(cm)} &= k_{33}^{(P)} \\ k_{12}^{(cm)} &= \frac{\partial p'}{\partial r} k_{02}^{(P)} + \frac{\partial r'}{\partial r} k_{12}^{(P)} \\ k_{13}^{(cm)} &= \frac{\partial p'}{\partial r} k_{03}^{(P)} + \frac{\partial r'}{\partial r} k_{13}^{(P)} \\ k_{23}^{(cm)} &= k_{23}^{(P)} \end{aligned} \quad (39)$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the z' axis is

$$K_{\mu\nu} = \Omega^2(p', r') p' [C_{\mu\nu} \cos(k(r' \cos \theta - p')) + D_{\mu\nu} \sin(k(r' \cos \theta - p'))] \quad (40)$$

where $k_\mu = (-k, 0, 0, k)$, $z' = r' \cos \theta$, $C_{\mu\nu} = B_{\mu\nu} + B_{\mu\nu}^*$, and $D_{\mu\nu} = i(B_{\mu\nu} - B_{\mu\nu}^*)$.

Up to leading order in u , we have:

$$p' \sim 1, \quad r' \sim \frac{1}{u}, \quad \Omega^2(p', r') \sim d^2 u^4. \quad (41)$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{d} \left(\frac{1}{u^2} \right), \quad \frac{\partial p'}{\partial r} \sim -\frac{1}{L} \left(\frac{1}{u} \right), \quad \frac{\partial r'}{\partial t} \sim -\frac{1}{d} \left(\frac{1}{u^2} \right), \quad \frac{\partial r'}{\partial r} \sim \frac{1}{L} \left(\frac{1}{u} \right). \quad (42)$$

For the plane wave $\sin(k(z' - p'))$, the phase equates to

$$z' - p' = \frac{v \cos \theta - u}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}}. \quad (43)$$

For $u \rightarrow \infty$, the above converges and has asymptotic expansion

$$z' - p' \approx -1 + \frac{1}{u} (v + (1 + v^2)^{1/2}) \cos \theta - \frac{1}{u^2} \left(\frac{1}{2} + v^2 + v(1 + v^2)^{1/2} \cos \theta \right) + O\left(\frac{1}{u^3}\right). \quad (44)$$

Hence, to second leading order, the (p', z') plane wave behave asymptotically as

$$\begin{aligned} \sin(k(z' - p')) &\approx -\sin(k) + \frac{k \cos(k)}{u} (v \cos \theta + (1 + v^2)^{1/2}) \\ \cos(k(z' - p')) &\approx \cos(k) + \frac{k \sin(k)}{u} (v \cos \theta + (1 + v^2)^{1/2}) \end{aligned} \quad (45)$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim r'$, the leading large u behavior of $B_{\mu\nu}^{(cm)}$ is calculated as:

$$\begin{aligned} B_{00}^{(cm)} &\sim \frac{1}{d^2} \left(\frac{1}{u^4} \right), \quad B_{01}^{(cm)} \sim \frac{1}{dL} \left(\frac{1}{u^3} \right), \quad B_{02}^{(cm)} \sim \frac{1}{d} \left(\frac{1}{u^3} \right), \quad B_{03}^{(cm)} \sim \frac{1}{d} \left(\frac{1}{u^3} \right) \\ B_{11}^{(cm)} &\sim \frac{1}{L^2} \left(\frac{1}{u^2} \right), \quad B_{22}^{(cm)} \sim \frac{1}{u^2}, \quad B_{33}^{(cm)} \sim \frac{1}{u^2}, \quad B_{12}^{(cm)} \sim \frac{1}{L} \left(\frac{1}{u^2} \right), \quad B_{13}^{(cm)} \sim \frac{1}{L} \left(\frac{1}{u^2} \right), \quad B_{23}^{(cm)} \sim \frac{1}{u^2} \end{aligned} \quad (46)$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K_{\mu\nu}^{(cm)}$, which follows

$$K_{\mu\nu}^{(cm)} = \Omega^2(p', r') B_{\mu\nu}^{(cm)} p' \sin(k(z' - p')) \sim d^2 u^4 B_{\mu\nu}^{(cm)}. \quad (47)$$

$$\begin{aligned} K_{00}^{(cm)} &\sim 1 \\ K_{01}^{(cm)} &\sim \frac{d}{L}(u) \\ K_{02}^{(cm)} &\sim d(u) \\ K_{03}^{(cm)} &\sim d(u) \\ K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^2) \\ K_{22}^{(cm)} &\sim d^2(u^2) \\ K_{33}^{(cm)} &\sim d^2(u^2) \\ K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\ K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\ K_{23}^{(cm)} &\sim d^2(u^2) \end{aligned} \quad (48)$$

Angular Average Over Plane Wave

$$\int \sin(k(r \cos \theta - t)) d\Omega = -4\pi \frac{\sin(kt) \sin(kr)}{kr} \quad (49)$$

In terms of the respective coordinates, this is

$$\langle \sin(k(z' - p')) \rangle = -4\pi \frac{\sin(kp') \sin(kr')}{kr'}. \quad (50)$$

Asymptotically, for large u , this behaves as

$$\langle \sin(k(z' - p')) \rangle \sim -\sin(k). \quad (51)$$

This in fact agrees with our asymptotic expansion of $\sin(k(z' - p'))$ and thus presents no change to the overall behavior.

New Coordinates

Performing coordinate transformations

$$T = \left[u + (1 + u^2)^{1/2} \right] (1 + v^2)^{1/2}, \quad R = \left[u + (1 + u^2)^{1/2} \right] v, \quad X^2 = T^2 - R^2, \quad (52)$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(T, R)(dT^2 - dR^2 - R^2 d\Omega^2) \quad (53)$$

with conformal factor

$$\Omega^2(T, R) = \frac{L^2 a^2}{T^2 - R^2} = d^2(1 + u^2)((1 + u^2)^{1/2} - u)^2. \quad (54)$$

We will soon make use of the coordinate relations

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{1}{d} \frac{\partial T}{\partial u} = \left(\frac{1}{d} \right) \frac{(u + (1 + u^2)^{1/2})(1 + v^2)^{1/2}}{(1 + u^2)^{1/2}} \\ \frac{\partial T}{\partial r} &= \frac{1}{L} \frac{\partial T}{\partial v} = \left(\frac{1}{L} \right) \frac{(u + (1 + u^2)^{1/2})v}{(1 + v^2)^{1/2}} \\ \frac{\partial R}{\partial t} &= \frac{1}{d} \frac{\partial R}{\partial u} = \left(\frac{1}{d} \right) \frac{(u + (1 + u^2)^{1/2})v}{(1 + u^2)^{1/2}} \\ \frac{\partial R}{\partial r} &= \frac{1}{L} \frac{\partial R}{\partial v} = \left(\frac{1}{L} \right) (u + (1 + u^2)^{1/2}) \end{aligned} \quad (55)$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned}
k_{00}^{(cm)} &= 2 \frac{\partial T}{\partial t} \frac{\partial R}{\partial t} k_{01}^{(P)} + \left(\frac{\partial R}{\partial t} \right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial T}{\partial t} \frac{\partial R}{\partial r} k_{01}^{(P)} + \frac{\partial R}{\partial t} \frac{\partial T}{\partial r} k_{01}^{(P)} + \frac{\partial R}{\partial t} \frac{\partial R}{\partial r} k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial T}{\partial t} k_{02}^{(P)} + \frac{\partial R}{\partial t} k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial T}{\partial t} k_{03}^{(P)} + \frac{\partial R}{\partial t} k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2 \frac{\partial T}{\partial r} \frac{\partial R}{\partial r} k_{01}^{(P)} + \left(\frac{\partial R}{\partial r} \right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial T}{\partial r} k_{02}^{(P)} + \frac{\partial R}{\partial r} k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial T}{\partial r} k_{03}^{(P)} + \frac{\partial R}{\partial r} k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{56}$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the Z axis is

$$K_{\mu\nu} = \Omega^2(T, R) T [C_{\mu\nu} \cos(k(R \cos \theta - T)) + D_{\mu\nu} \sin(k(R \cos \theta - T))] \tag{57}$$

where $k_\mu = (-k, 0, 0, k)$, $Z = R \cos \theta$, $C_{\mu\nu} = B_{\mu\nu} + B_{\mu\nu}^*$, and $D_{\mu\nu} = i(B_{\mu\nu} - B_{\mu\nu}^*)$.

Up to leading order in u , we have:

$$T \sim u, \quad R \sim u, \quad \Omega^2(T, R) \sim d^2 \tag{58}$$

$$\frac{\partial T}{\partial t} \sim \frac{1}{d}, \quad \frac{\partial T}{\partial r} \sim \frac{u}{L}, \quad \frac{\partial R}{\partial t} \sim \frac{1}{d}, \quad \frac{\partial R}{\partial r} \sim \frac{u}{L} \tag{59}$$

For the plane wave $\sin(k(Z - T))$, the phase equates to

$$Z - T = \left[u + (1 + u^2)^{1/2} \right] \left[v \cos \theta - (1 + v^2)^{1/2} \right] \tag{60}$$

For $u \rightarrow \infty$, the above diverges and has asymptotic expansion

$$Z - T \approx 2u \left(v \cos \theta - (1 + v^2)^{1/2} \right) + \frac{1}{2u} \left(v \cos \theta - (1 + v^2)^{1/2} \right) + O\left(\frac{1}{u^3}\right) \tag{61}$$

Hence, in the (T, Z) coordinate system, plane waves remain at least periodic with asymptotic form

$$\begin{aligned}
\sin(k(Z - T)) &\approx \sin \left[2ku \left(v \cos \theta - (1 + v^2)^{1/2} \right) \right] \\
\cos(k(Z - T)) &\approx \cos \left[2ku \left(v \cos \theta - (1 + v^2)^{1/2} \right) \right]
\end{aligned} \tag{62}$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim R$, the leading large u behavior of $B_{\mu\nu}^{(cm)}$ is calculated as:

$$\begin{aligned}
B_{00}^{(cm)} &\sim \frac{1}{d^2}, & B_{01}^{(cm)} &\sim \frac{u}{dL}, & B_{02}^{(cm)} &\sim \frac{u}{d}, & B_{03}^{(cm)} &\sim \frac{u}{d}, & B_{11}^{(cm)} &\sim \frac{u^2}{L^2} \\
B_{22}^{(cm)} &\sim u^2, & B_{33}^{(cm)} &\sim u^2, & B_{12}^{(cm)} &\sim \frac{u^2}{L}, & B_{13}^{(cm)} &\sim \frac{u^2}{L}, & B_{23}^{(cm)} &\sim u^2
\end{aligned} \tag{63}$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K_{\mu\nu}^{(cm)}$, which follows

$$K_{\mu\nu}^{(cm)} = \Omega^2(T, R) B_{\mu\nu}^{(cm)} T \sin(k(Z - T)) \sim d^2 u B_{\mu\nu}^{(cm)}. \quad (64)$$

$$\begin{aligned} K_{00}^{(cm)} &\sim u \\ K_{01}^{(cm)} &\sim \frac{d}{L} u^2 \\ K_{02}^{(cm)} &\sim d(u^2) \\ K_{03}^{(cm)} &\sim d(u^2) \\ K_{11}^{(cm)} &\sim \frac{d^2}{L^2} (u^3) \\ K_{22}^{(cm)} &\sim d^2(u^3) \\ K_{33}^{(cm)} &\sim d^2(u^3) \\ K_{12}^{(cm)} &\sim \frac{d^2}{L} (u^3) \\ K_{13}^{(cm)} &\sim \frac{d^2}{L} (u^3) \\ K_{23}^{(cm)} &\sim d^2(u^3) \end{aligned} \quad (65)$$

Angular Average Over Plane Wave

$$\int \sin(k(r \cos \theta - t)) d\Omega = -4\pi \frac{\sin(kt) \sin(kr)}{kr} \quad (66)$$

In terms of the respective coordinates, this is

$$\langle \sin(k(Z - T)) \rangle = -4\pi \frac{\sin(kT) \sin(kR)}{kR}. \quad (67)$$

Asymptotically, for large u , this behaves as

$$\langle \sin(k(Z - T)) \rangle \sim \frac{1}{u} \sin(kT) \sin(kR). \quad (68)$$

The averaged plane wave angular behavior will thus reduce the large time behavior by an overall u^{-1} . As a result, we have the angular averaged asymptotic behavior

$$\begin{aligned} K_{00}^{(cm)} &\sim 1 \\ K_{01}^{(cm)} &\sim \frac{d}{L} (u) \\ K_{02}^{(cm)} &\sim d(u) \\ K_{03}^{(cm)} &\sim d(u) \\ K_{11}^{(cm)} &\sim \frac{d^2}{L^2} (u^2) \\ K_{22}^{(cm)} &\sim d^2(u^2) \\ K_{33}^{(cm)} &\sim d^2(u^2) \\ K_{12}^{(cm)} &\sim \frac{d^2}{L} (u^2) \\ K_{13}^{(cm)} &\sim \frac{d^2}{L} (u^2) \\ K_{23}^{(cm)} &\sim d^2(u^2) \end{aligned} \quad (69)$$

Appendix

Early Universe Setup

Given the geometry

$$ds^2 = (g_{\mu\nu} + K_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu, \quad (70)$$

upon imposing the conformal gauge condition $\nabla_\nu K^{\mu\nu} - \frac{1}{2}K^{\mu\nu}g_{(0)}^{\alpha\beta}\partial_\nu g_{\alpha\beta}^{(0)} = 0$, solutions to the first order source free Bach tensor $\delta W_{\mu\nu} = 0$ are found to obey

$$\frac{1}{2}\Omega^{-2}\square^2 k_{\mu\nu} = 0 \quad (71)$$

After performing residual gauge transformations to eliminate gauge degrees of freedom, the general momentum eigenstate solution to (46) for a given k -mode is

$$k_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \begin{pmatrix} 0 & B_{01} & B_{02} & 0 \\ B_{01} & B_{11} & B_{12} & 0 \\ B_{02} & B_{12} & -B_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} n_\alpha x^\alpha e^{ikx} \quad (72)$$

with timelike $n_\alpha = (1, 0, 0, 0)$. The full solution for $K_{\mu\nu}$ is then given as

$$K_{\mu\nu} = \Omega^2 k_{\mu\nu}. \quad (73)$$

The $k < 0$ R.W. line element is given in comoving coordinates as

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1+r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \quad (74)$$

where $k = -1/L^2$ (with $k < 0$). By coordinate transformation, the hyperbolic R.W. background geometry may be expressed in the form of $g_{\mu\nu}^{(0)} = \Omega^2 \eta_{\mu\nu}$, with the general conformal factor Ω having time and spatial dependence in the Minkowski coordinates.

Within the early universe radiation era, the perfect fluid energy momentum tensor obeys $\rho = 3p$, $\rho = A/a^4(t)$, $A > 0$, with $a(t)$ following the evolution equation

$$\begin{aligned} \dot{a}^2 - \frac{1}{L^2} &= \alpha a^2 - \frac{2A}{S_0^2 a^2} \\ &= -2 \frac{a^2}{S_0^2} \left(\lambda_S S_0^4 + \frac{A}{a^4} \right) \end{aligned} \quad (75)$$

With the radiation dominating over the cosmological constant in the early universe (since $a(t)$ is small), i.e.

$$\frac{A}{a^4} \gg \lambda_S S_0^4, \quad (76)$$

the evolution equation can then be brought to the form

$$L^2 \dot{a}^2 = 1 - \frac{d^2}{L^2} \left(\frac{1}{a^2} \right), \quad (77)$$

in which the solution $a(t)$ is

$$a^2(t) = \frac{1}{L^2} (d^2 + t^2) \quad (78)$$

where we have defined

$$d^2 \equiv \frac{2AL^4}{S_0^2}. \quad (79)$$

(With $A \sim [L]^{-4}$ and $S_0 \sim [L]^{-1}$ fixed early on, the relevant quantities to compare in the radiation dominated era should be the dimensionless $a(t)$ and λ_S).

Cartesian to Polar

Transformation Matrices

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \quad (80)$$

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \frac{\cos \theta \cos \phi}{r} & \frac{\cos \theta \sin \phi}{r} & -\frac{\sin \theta}{r} \\ -\frac{\sin \phi}{r \sin \theta} & \frac{\cos \phi}{r \sin \theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \quad (81)$$

Time-Time

$$K'_{00} = K_{00} \quad (82)$$

Time-Space

$$K'_{0i} = \frac{\partial x^j}{\partial x'^i} K_{0j} \quad (83)$$

$$\begin{pmatrix} K'_{01} \\ K'_{02} \\ K'_{03} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^2}{\partial x'^3} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{01} \\ K_{02} \\ K_{03} \end{pmatrix} \quad (84)$$

$$K'_{01} = K_{01} \sin(\theta) \cos(\phi) + K_{02} \sin(\theta) \sin(\phi) + K_{03} \cos(\theta) \quad (85)$$

$$K'_{02} = K_{01} r \cos(\theta) \cos(\phi) + K_{02} r \cos(\theta) \sin(\phi) - K_{03} r \sin(\theta) \quad (86)$$

$$K'_{03} = -K_{01} r \sin(\theta) \sin(\phi) + K_{02} r \sin(\theta) \cos(\phi) \quad (87)$$

Space-Space

$$K'_{ij} = \frac{\partial x^k}{\partial x'^i} K_{kl} \frac{\partial x^l}{\partial x'^j} \quad (88)$$

$$\begin{pmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{21} & K'_{22} & K'_{23} \\ K'_{31} & K'_{32} & K'_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^2}{\partial x'^3} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^2}{\partial x'^3} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}^T \quad (89)$$

$$K'_{11} = K_{11} \sin^2(\theta) \cos^2(\phi) + K_{12} \sin^2(\theta) \sin(2\phi) + K_{13} \sin(2\theta) \cos(\phi) + K_{22} \sin^2(\theta) \sin^2(\phi) + K_{23} \sin(2\theta) \sin(\phi) + K_{33} \cos^2(\theta) \quad (90)$$

$$K'_{22} = K_{11} r^2 \cos^2(\theta) \cos^2(\phi) + K_{12} r^2 \cos^2(\theta) \sin(2\phi) - K_{13} r^2 \sin(2\theta) \cos(\phi) + K_{22} r^2 \cos^2(\theta) \sin^2(\phi) - K_{23} r^2 \sin(2\theta) \sin(\phi) + K_{33} r^2 \sin^2(\theta) \quad (91)$$

$$K'_{33} = K_{11} r^2 \sin^2(\theta) \sin^2(\phi) - 2K_{12} r^2 \sin^2(\theta) \sin(\phi) \cos(\phi) + K_{22} r^2 \sin^2(\theta) \cos^2(\phi) \quad (92)$$

$$K'_{12} = K_{11}r \sin(\theta) \cos(\theta) \cos^2(\phi) + K_{12}r \sin(\theta) \cos(\theta) \sin(2\phi) + K_{13}r \cos(2\theta) \cos(\phi) + K_{22}r \sin(\theta) \cos(\theta) \sin^2(\phi) + K_{23}r \cos(2\theta) \sin(\phi) - K_{33}r \sin(\theta) \cos(\theta) \quad (93)$$

$$K'_{13} = -K_{11}r \sin^2(\theta) \sin(\phi) \cos(\phi) + K_{12}r \sin^2(\theta) \cos(2\phi) - K_{13}r \sin(\theta) \cos(\theta) \sin(\phi) + K_{22}r \sin^2(\theta) \sin(\phi) \cos(\phi) + K_{23}r \sin(\theta) \cos(\theta) \cos(\phi) \quad (94)$$

$$K'_{23} = -K_{11}r^2 \sin(\theta) \cos(\theta) \sin(\phi) \cos(\phi) + K_{12}r^2 \sin(\theta) \cos(\theta) \cos(2\phi) + K_{13}r^2 \sin^2(\theta) \sin(\phi) + K_{22}r^2 \sin(\theta) \cos(\theta) \sin(\phi) \cos(\phi) - K_{23}r^2 \sin^2(\theta) \cos(\phi) \quad (95)$$

Polar to Comoving

$$K'_{\mu\nu}(t, r, \theta, \phi) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} k_{\alpha\beta}(T, R, \theta, \phi) \quad (96)$$

$$J_{\mu\nu} = \frac{\partial x^\nu}{\partial x'^\mu}, \quad \text{where } x(T, R, \theta, \phi) \quad x'(t, r, \theta, \phi) \quad (97)$$

$$J_{\mu\nu} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (98)$$

$$k_{\mu\nu}^{(cm)} = \frac{\partial x_{(P)}^k}{\partial x_{(cm)}^i} k_{kl}^{(P)} \frac{\partial x_{(P)}^l}{\partial x_{(cm)}^j} \quad (99)$$

$$\begin{pmatrix} k_{00}^{(cm)} & k_{01}^{(cm)} & k_{02}^{(cm)} & k_{03}^{(cm)} \\ k_{10}^{(cm)} & k_{11}^{(cm)} & k_{12}^{(cm)} & k_{13}^{(cm)} \\ k_{20}^{(cm)} & k_{21}^{(cm)} & k_{22}^{(cm)} & k_{23}^{(cm)} \\ k_{30}^{(cm)} & k_{31}^{(cm)} & k_{32}^{(cm)} & k_{33}^{(cm)} \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{00}^{(P)} & k_{01}^{(P)} & k_{02}^{(P)} & k_{03}^{(P)} \\ k_{10}^{(P)} & k_{11}^{(P)} & k_{12}^{(P)} & k_{13}^{(P)} \\ k_{20}^{(P)} & k_{21}^{(P)} & k_{22}^{(P)} & k_{23}^{(P)} \\ k_{30}^{(P)} & k_{31}^{(P)} & k_{32}^{(P)} & k_{33}^{(P)} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \quad (100)$$

$$\begin{aligned} k_{00}^{(cm)} &= 2 \frac{\partial T}{\partial t} \frac{\partial R}{\partial t} k_{01}^{(P)} + \left(\frac{\partial R}{\partial t} \right)^2 k_{11}^{(P)} \\ k_{01}^{(cm)} &= \frac{\partial T}{\partial t} \frac{\partial R}{\partial r} k_{01}^{(P)} + \frac{\partial R}{\partial t} \frac{\partial T}{\partial r} k_{01}^{(P)} + \frac{\partial R}{\partial t} \frac{\partial R}{\partial r} k_{11}^{(P)} \\ k_{02}^{(cm)} &= \frac{\partial T}{\partial t} k_{02}^{(P)} + \frac{\partial R}{\partial t} k_{12}^{(P)} \\ k_{03}^{(cm)} &= \frac{\partial T}{\partial t} k_{03}^{(P)} + \frac{\partial R}{\partial t} k_{13}^{(P)} \\ k_{11}^{(cm)} &= 2 \frac{\partial T}{\partial r} \frac{\partial R}{\partial r} k_{01}^{(P)} + \left(\frac{\partial R}{\partial r} \right)^2 k_{11}^{(P)} \\ k_{22}^{(cm)} &= k_{22}^{(P)} \\ k_{33}^{(cm)} &= k_{33}^{(P)} \\ k_{12}^{(cm)} &= \frac{\partial T}{\partial r} k_{02}^{(P)} + \frac{\partial R}{\partial r} k_{12}^{(P)} \\ k_{13}^{(cm)} &= \frac{\partial T}{\partial r} k_{03}^{(P)} + \frac{\partial R}{\partial r} k_{13}^{(P)} \\ k_{23}^{(cm)} &= k_{23}^{(P)} \end{aligned} \quad (101)$$