Bach External SVT

Via orthogonal projection to the four velocity U^μ, we may decompose a rank 2 $T_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} + U_\mu q_\nu + U_\nu q_\mu + \pi_{\mu\nu} \tag{1}$$

where

$$U^\mu q_\mu = 0, \qquad U^\nu \pi_{\mu\nu} = 0, \qquad \pi_{\mu\nu} = \pi_{\nu\mu}, \qquad g^{\mu\nu}\pi_{\mu\nu} = U^\mu U^\nu \pi_{\mu\nu} = 0. \tag{2}$$

We evaluate within a Minkowski background $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$.

Given $T_{0i} = -q_i$, let us decompose the q_i into longitudinal and transverse parts by introducing the scalar

$$Q = \int d^3y \; D(x-y)\tilde{\nabla}^i q_i. \tag{3}$$

Now we can form the transverse piece as

$$q_i - \tilde{\nabla}_i Q = Q_i, \tag{4}$$

with it following that $\tilde{\nabla}^i Q_i = 0$. Additionally, we may decompose the 5 component $\pi_{\mu\nu}$ into a transverse traceless π_{ij}, a divergenceless π_i, and a scalar π as

$$\pi_{ij} = -\frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}. \tag{5}$$

Now $T_{\mu\nu}$ can be expressed in the SVT form as

$$\begin{aligned}
&T_{00} = \rho, \\
&T_{0i} = -Q_i - \tilde{\nabla}_i Q, \\
&T_{ij} = \delta_{ij}p - \frac{2}{3}\delta_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\pi + 2\tilde{\nabla}_i\tilde{\nabla}_j\pi + \tilde{\nabla}_i\pi_j + \tilde{\nabla}_j\pi_i + \pi^{T\theta}_{ij}.
\end{aligned} \tag{6}$$

Such a $T_{\mu\nu}$ must be covariantly conserved and thus must obey the four conditions

$$-\partial_t\rho = \tilde{\nabla}_i\tilde{\nabla}^i Q \tag{7}$$

$$0 = \partial_t(Q^i + \tilde{\nabla}^i Q) + \tilde{\nabla}^i p + \frac{4}{3}\tilde{\nabla}^i\tilde{\nabla}^k\tilde{\nabla}_k\pi + \tilde{\nabla}_k\tilde{\nabla}^k\pi^i. \tag{8}$$

In conformal gravity, such an energy momentum tensor must also be traceless and as such must obey

$$\rho = 3p. \tag{9}$$

From the first condition, we may express Q in terms of ρ as

$$Q = -\int d^3y D^3(\mathbf{x} - \mathbf{y})\partial_t\rho. \tag{10}$$

We may extract a scalar condition from the second transverse condition, which takes the form

$$0 = \tilde{\nabla}_a\tilde{\nabla}^a(\partial_t Q + p + \frac{4}{3}\tilde{\nabla}_b\tilde{\nabla}^b\pi). \tag{11}$$

This allows expression of π as

$$\pi = \frac{3}{4}\int d^3y\ D(x-y)\left[\int d^3z\ D(y-z)\partial_t^2\rho - p\right]. \tag{12}$$

Substitution of π back into the transverse condition then yields a vector condition

$$0 = \partial_t Q_i + \tilde{\nabla}_a\tilde{\nabla}^a\pi_i, \tag{13}$$

from which we may solve π_i as

$$\pi_i = -\int d^3y\ D(x-y)\partial_t Q_i. \tag{14}$$

In total, we may express a $\delta T_{\mu\nu}$ in terms of ρ, Q_i and $\pi_{ij}^{T\theta}$, totaling 5 components:

$$\begin{aligned}
\delta T_{00} &= \rho, \\
\delta T_{0i} &= -Q_i + \tilde{\nabla}_i\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t\rho, \\
\delta T_{ij} &= \frac{1}{2}\left(\delta_{ij}\rho - \tilde{\nabla}_i\tilde{\nabla}_j\int d^3y\ D(x-y)\rho\right) - \frac{1}{2}\int d^3y\ D(x-y)\delta_{ij}\partial_t^2\rho \\
&\quad + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j\int d^3y\ D(x-y)\int d^3z\ D(y-z)\partial_t^2\rho - \tilde{\nabla}_i\int d^3y\ D(x-y)\partial_t Q_j \\
&\quad - \tilde{\nabla}_j\int d^3y\ D(x-y)\partial_t Q_i + \pi_{ij}^{T\theta}.
\end{aligned} \tag{15}$$

Likewise we may express a general $\delta W_{\mu\nu}$ in terms of the barred quantities

$$\begin{aligned}
\delta W_{00} &= \bar{\rho}, \\
\delta W_{0i} &= -\bar{Q}_i + \tilde{\nabla}_i\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t\bar{\rho}, \\
\delta W_{ij} &= \frac{1}{2}\left(\delta_{ij}\bar{\rho} - \tilde{\nabla}_i\tilde{\nabla}_j\int d^3y\ D(x-y)\bar{\rho}\right) - \frac{1}{2}\int d^3y\ D(x-y)\delta_{ij}\partial_t^2\bar{\rho} \\
&\quad + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j\int d^3y\ D(x-y)\int d^3z\ D(y-z)\partial_t^2\bar{\rho} - \tilde{\nabla}_i\int d^3y\ D(x-y)\partial_t\bar{Q}_j \\
&\quad - \tilde{\nabla}_j\int d^3y\ D(x-y)\partial_t\bar{Q}_i + \bar{\pi}_{ij}^{T\theta}.
\end{aligned} \tag{16}$$

Solving for $\delta W_{00} = \delta T_{00}$ fixes ρ, and $\delta W_{0i} = \delta T_{0i}$ fixes Q_i viz.

$$\bar{\rho} = \rho, \qquad \bar{Q}_i = Q_i. \tag{17}$$

It then follows that these terms mutually cancel within $\delta W_{ij} = \delta T_{ij}$, leaving the remaing expression

$$\bar{\pi}_{ij}^{T\theta} = \pi_{ij}^{T\theta}. \tag{18}$$

Thus we can express the entire $\delta W_{\mu\nu} = \delta T_{\mu\nu}$ field equation in terms of irreducible SO(3) equations as

$$\begin{aligned}
\bar{\rho} &= \rho \\
\bar{Q}_i &= Q_i \\
\bar{\pi}_{ij}^{T\theta} &= \pi_{ij}^{T\theta}.
\end{aligned} \tag{19}$$

We can try to express the above SVT relations in terms of the actual tensor components. Recall the flat 3+1 projector

$$P_{\mu\nu} = \eta_{\mu\nu} + U_\mu U_\nu, \qquad U_\mu = -\delta^0_\mu, \qquad U^\mu = \delta^\mu_0. \tag{20}$$

In terms of the the flat space projectors, the splitting of the 3+1 components goes as

$$\rho = U^\sigma U^\tau T_{\sigma\tau} = T_{00}, \qquad q_i = -P_i{}^\sigma U^\tau T_{\sigma\tau} = -T_{0i}, \qquad \pi_{\mu\nu} = \left[\frac{1}{2} P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2} P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3} P_{\mu\nu} P^{\sigma\tau}\right] T_{\sigma\tau}, \tag{21}$$

in which it follows

$$\pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} T_{kl}. \tag{22}$$

We recall the definition of Q_i as

$$Q_i = q_i - \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^i q_i. \tag{23}$$

This may be alternatively expressed as

$$Q_i = -T_{0i} + \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^j T_{0j} \tag{24}$$

Noting that π_{ij} is already traceless by construction, we may project out its transverse part and define $\pi^{T\theta}_{ij}$ as

$$\begin{aligned} \pi^{T\theta}_{ij} = \pi_{ij} &- \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^k \pi_{jk} - \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}^k \pi_{ik} \\ &+ \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}_k \int d^3z\ D(y-z) \tilde{\nabla}_l \pi^{kl}. \end{aligned} \tag{25}$$

Substituting in $\pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}$, we have

$$\begin{aligned} \pi^{T\theta}_{ij} = &\left(T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}\right) - \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^k \left(T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}T_{mn}\right) \\ &- \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}^k \left(T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}T_{mn}\right) \\ &+ \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}_k \int d^3z\ D(y-z) \tilde{\nabla}_l \left(T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}T_{mn}\right). \end{aligned} \tag{26}$$

In total, we may express relations (19) explicitly in terms of the components of the tensors as the following:

$$\bar{\rho} - \rho = \delta W_{00} - \delta T_{00} \tag{27}$$

$$\bar{Q}_i - Q_i = -(\delta W_{0i} - \delta T_{0i}) + \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^j (\delta W_{0j} - \delta T_{0j}) \tag{28}$$

$$\begin{aligned} \bar{\pi}^{T\theta}_{ij} - \pi^{T\theta}_{ij} = &\left[\delta W_{ij} - \delta T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\left(\delta W_{kl} - \delta T_{kl}\right)\right] \\ &- \tilde{\nabla}_i \int d^3y\ D(x-y) \tilde{\nabla}^k \left[\delta W_{jk} - \delta T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right] \\ &- \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}^k \left[\delta W_{ij} - \delta T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right] \\ &+ \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y\ D(x-y) \tilde{\nabla}_k \int d^3z\ D(y-z) \tilde{\nabla}_l \left[\delta W_{kl} - \delta T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}\left(\delta W_{mn} - \delta T_{mn}\right)\right]. \end{aligned} \tag{29}$$

Now we explicitly evaluate $\delta W_{\mu\nu}$ in terms of its SVT metric components (see appendix for reference). The scalar portion ρ takes the form

$$\rho = -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(\phi + \psi + \dot{B} - \ddot{E}). \tag{30}$$

The two component tranverse vector Q_i evaluates to

$$\begin{aligned} Q_i &= -\delta W_{0i} + \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^j\delta W_{0j} \\ &= \frac{2}{3}\tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a\partial_t(\phi + \psi + \dot{B} - \ddot{E}) - \frac{1}{2}\left[\tilde{\nabla}_a\tilde{\nabla}^a\left(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_t^2\right)(B_i - \dot{E}_i)\right] \\ &\quad - \tilde{\nabla}_i \int d^3y\ D(x-y)\left[\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b\partial_t(\phi + \psi + \dot{B} - \ddot{E})\right]. \end{aligned} \tag{31}$$

Let us denote the scalar quantity

$$\Psi = \tilde{\nabla}_a\tilde{\nabla}^a(\phi + \psi + \dot{B} - \ddot{E}), \tag{32}$$

then we may rewrite Q_i in the simpler form

$$Q_i = -\frac{1}{2}\left[\tilde{\nabla}_a\tilde{\nabla}^a\left(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_t^2\right)(B_i - \dot{E}_i)\right] + \frac{2}{3}\tilde{\nabla}_i\partial_t\left(\Psi - \int d^3y\ D(x-y)\tilde{\nabla}_a\tilde{\nabla}^a\Psi\right). \tag{33}$$

Looking at the form for Ψ, we recall that we can decompose any scalar into longitudinal and transverse components as

$$\begin{aligned} \phi(x) &= \int d^3y\ D(x-y)\tilde{\nabla}_a\tilde{\nabla}^a\phi(y) + \int dS_a\left[\phi(y)\tilde{\nabla}^a D(x-y) - D(x-y)\tilde{\nabla}^a\phi(y)\right] \\ &= \phi^L(x) + \phi^T(x). \end{aligned} \tag{34}$$

Through the above decomposition, the only $\tilde{\nabla}_a\tilde{\nabla}^a\phi^L$ that vanishes is one for which ϕ^L itself vanish. Additionally, the transverse ϕ identically obeys $\tilde{\nabla}_a\tilde{\nabla}^a\phi^T = 0$.

Upon analyzing Q_i, we see that in fact it is only Ψ^T that contributes. We can show that Ψ^T can be expressed solely as a surface integral (as given in (34)) if we perform an integration by parts.

Moreover, looking back at our equation for ρ, we note that this may be expressed as

$$\rho = -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\Psi^L. \tag{35}$$

Hence taking $\Psi = \tilde{\nabla}_a\tilde{\nabla}^a(\phi + \psi + \dot{B} - \ddot{E})$, we may express the scalar and vector equations as

$$\rho = -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\Psi^L \tag{36}$$

$$Q_i = -\frac{1}{2}\left[\tilde{\nabla}_a\tilde{\nabla}^a\left(\tilde{\nabla}_b\tilde{\nabla}^b - \partial_t^2\right)(B_i - \dot{E}_i)\right] + \frac{2}{3}\tilde{\nabla}_i\partial_t\Psi^T \tag{37}$$

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Incomplete (Tensor Sector)

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$$\begin{aligned} \pi_{ij}^{T\theta} &= \left(\delta W_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}\delta W_{kl}\right) - \tilde{\nabla}_i \int d^3y\ D(x-y)\tilde{\nabla}^k\left(\delta W_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}\delta W_{mn}\right) \\ &\quad - \tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}^k\left(\delta W_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}\delta W_{mn}\right) \\ &\quad + \tilde{\nabla}_i\tilde{\nabla}_j \int d^3y\ D(x-y)\tilde{\nabla}_k \int d^3z\ D(y-z)\tilde{\nabla}_l\left(\delta W^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}\delta W_{mn}\right). \end{aligned} \tag{38}$$

$$\delta W_{ij}^{(S)} = \frac{1}{3}\left[\delta_{ij}\left(\partial_t^2 - \tilde{\nabla}_a\tilde{\nabla}^a\right)\Psi + \tilde{\nabla}_i\tilde{\nabla}_j\left(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_t^2\right)\tilde{\nabla}^{-2}\Psi\right] \tag{39}$$

For the transverse tensor contribution, the trace is

$$\delta^{ij}\delta W_{ij} = -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\Psi. \tag{40}$$

$$\left(\delta W_{ij}^{(S)} - \frac{1}{3}\delta_{ij}\delta^{kl}\delta W_{kl}^{(S)}\right) = \frac{1}{3}\left[\delta_{ij}\left(\partial_t^2 - \frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a\right)\Psi + \tilde{\nabla}_i\tilde{\nabla}_j\left(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_t^2\right)\tilde{\nabla}^{-2}\Psi\right] \tag{41}$$

$$\tilde{\nabla}^i\left(\delta W_{ij}^{(S)} - \frac{1}{3}\delta_{ij}\delta^{kl}\delta W_{kl}^{(S)}\right) = \frac{2}{3}\left(\frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a - \partial_t^2\right)\tilde{\nabla}_j\Psi \tag{42}$$

$$\begin{aligned}\pi_{ij}^{T\theta} = \delta W_{ij}^{(V)} + \delta W_{ij}^{(T)} + \frac{1}{3}&\left[\delta_{ij}\left(\partial_t^2 - \frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a\right)\Psi + \tilde{\nabla}_i\tilde{\nabla}_j\left(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_t^2\right)\tilde{\nabla}^{-2}\Psi\right]\\ &- \frac{2}{3}\tilde{\nabla}_i\int d^3y\ D(x-y)\left(\frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a - \partial_t^2\right)\tilde{\nabla}_j\Psi\\ &- \frac{2}{3}\tilde{\nabla}_j\int d^3y\ D(x-y)\left(\frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a - \partial_t^2\right)\tilde{\nabla}_i\Psi\\ &+ \frac{2}{3}\tilde{\nabla}_i\tilde{\nabla}_j\int d^3y\ D(x-y)\tilde{\nabla}_k\ \int d^3z\ D(y-z)\left(\frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a - \partial_t^2\right)\tilde{\nabla}^k\Psi\end{aligned} \tag{43}$$

Nonzero vector contributions also present in integrals.

Appendix

$$
\begin{aligned}
\delta W_{00} &= -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(\phi+\psi+\dot{B}-\ddot{E}), \\
\delta W_{0i} &= -\frac{2}{3}\tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a\partial_t(\phi+\psi+\dot{B}-\ddot{E})+\frac{1}{2}\left[\tilde{\nabla}_a\tilde{\nabla}^a\left(\tilde{\nabla}_b\tilde{\nabla}^b-\partial_t^2\right)(B_i-\dot{E}_i)\right], \\
\delta W_{ij} &= \frac{1}{3}\left[\delta_{ij}\left(\partial_t^2-\tilde{\nabla}_a\tilde{\nabla}^a\right)\tilde{\nabla}_b\tilde{\nabla}^b(\phi+\psi+\dot{B}-\ddot{E})+\tilde{\nabla}_i\tilde{\nabla}_j\left(\tilde{\nabla}_a\tilde{\nabla}^a-3\partial_t^2\right)(\phi+\psi+\dot{B}-\ddot{E})\right] \\
&+\frac{1}{2}\left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_i\partial_t(B_j-\dot{E}_j)+\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_j\partial_t(B_i-\dot{E}_i)-\tilde{\nabla}_i\partial_t^3(B_j-\dot{E}_j)-\tilde{\nabla}_j\partial_t^3(B_i-\dot{E}_i)\right] \\
&+\left[\tilde{\nabla}_a\tilde{\nabla}^a-\partial_t^2\right]^2 E_{ij}.
\end{aligned}
\tag{44}
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