Lecture 3

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Greens Function for the Wave Equation

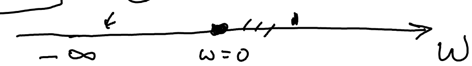
(Jackson, Chapter 6 ≡ J6)

source

some physical value "$\Psi(\vec r, t)$" $\Rightarrow$ $\boxed{\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec r, t)}$ (1)

$$\boxed{\Psi(\vec r, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_\omega(\vec r)e^{-i\omega t}\,d\omega}$$

$e^{-i|\omega|t}$ $\quad$ $e^{i|\omega|t}$



Fourier transformation

Fourier amplitude

$\pm\omega$ complex conjugated values

General solution of Eq.1:

$$\boxed{\Psi(\vec r, t) = \Psi_{in}(\vec r, t) + \int dt'\int d^3r'\, G(\vec r, t, \vec r', t')f(\vec r', t')} \quad (2)$$

where $G(\vec r, t, \vec r', t')$ is the Green Function (propagator): (3)

$$\boxed{\nabla_r^2 G(\vec r, t, \vec r', t') - \frac{1}{c^2}\frac{\partial^2 G(\vec r, t, \vec r', t')}{\partial t^2} = -4\pi\delta(\vec r - \vec r')\delta(t-t')}$$

We will show, that Eq.2 provides a general solution of the wave equation:

$$\int_{-\infty}^{+\infty} dt'\int d^3r'\, f(\vec r', t')\left\{\nabla_r^2 G(\vec r, t, \vec r', t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\vec r, t, \vec r', t')\right\} =$$

$$= -4\pi\int_{-\infty}^{+\infty} dt'\, d^3r'\,\delta(\vec r - \vec r')\delta(t-t')f(\vec r', t')$$

$$\nabla_r^2\Big(\underbrace{\int dt'\int d^3r'\, G(\vec r, t, \vec r', t')f(\vec r', t')}_{\Psi(\vec r, t)}\Big) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(\underbrace{\int dt'\int d^3r'\, G(\vec r, t, \vec r', t')f(\vec r', t')}_{\Psi(\vec r, t)}\Big) =$$

$$= -4\pi f(\vec r, t)$$

We obtained the Ψ-solution of Eq.1

Retarded Green Function:

We will compute the retarded Green Function,

$$\boxed{G(\vec r, t, \vec r', t') = \frac{\delta\left(t - t' - \frac{|\vec r - \vec r'|}{c}\right)}{|\vec r - \vec r'|}}$$

which yields the retarded solution of Eq.1

$$\boxed{\Psi(\vec r, t) = \Psi_{in}(\vec r, t) + \int d^3r'\,\frac{f\left(\vec r', t - \frac{|\vec r - \vec r'|}{c}\right)}{|\vec r - \vec r'|}}$$

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Retarded Green Function.

$$\vec{\nabla}_r^{\,2} G(\vec{r},t,\vec{r}\,',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G(\vec{r},t,\vec{r}\,',t') = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

New variable: $\boxed{\tau = t - t'}$ (Our solutions should be valid for the uniform space and time).

$$G(\vec{r},t,\vec{r}\,',t') = G(\vec{r},\vec{r}\,',\tau)$$

for our time-uniform system

Fourier transformation:

$$G(\vec{r},\vec{r}\,',\tau) = \int_{-\infty}^{+\infty} G_\omega(\vec{r},\vec{r}\,')\, e^{-i\omega\tau}\frac{d\omega}{(2\pi)}$$

The equation for the Fourier amplitude $G_\omega(\vec{r},\vec{r}\,')$:

$$\boxed{\int \frac{d\omega}{(2\pi)} e^{-i\omega\tau}\left(\vec{\nabla}_r^{\,2} G_\omega(\vec{r},\vec{r}\,') - \frac{1}{c^2}(-i\omega)^2 G_\omega(\vec{r},\vec{r}\,')\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,\delta(\tau)} \quad (4)$$

$$\frac{\partial^2}{\partial t^2} \equiv \frac{\partial^2}{\partial \tau^2}$$

$$\frac{\partial G}{\partial \tau} = -i\omega\, G$$

$$\frac{\partial^2 G}{\partial \tau^2} = (-i\omega)^2 G$$

$$\frac{\omega}{c} = k \equiv \frac{2\pi}{\lambda}$$

$$\underbrace{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi} e^{-i\omega\tau}}_{\delta(\tau)}\underbrace{\left(\vec{\nabla}_r^{\,2} G_\omega(\vec{r},\vec{r}\,') + k^2 G_\omega(\vec{r},\vec{r}\,')\right)} = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,\delta(\tau)$$

If this part does not depend on "ω" or "$k=\omega/c$", we could integrate over "ω"! $\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi} e^{-i\omega\tau} = \delta(\tau)$

$$\cancel{\delta(\tau)}\left\{\vec{\nabla}_r^{\,2} G_\omega(\vec{r},\vec{r}\,') + k^2 G_\omega(\vec{r},\vec{r}\,')\right\} = -4\pi\,\delta(\vec{r}-\vec{r}\,')\cdot\cancel{\delta(\tau)}$$

$$\boxed{\vec{\nabla}_r^{\,2} G_\omega(\vec{r},\vec{r}\,') + k^2 G_\omega(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')} \quad (5)$$

Yes, this part of the equation is the constant, $-4\pi\,\delta(\vec{r}-\vec{r}\,')$, which does not depend on "ω" (or "k").

Alternative method to obtain Eq. 5 from Eq. 4:

From Eq. 4: (*)

$$\boxed{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left\{\vec{\nabla}_r^2 G_\omega(\vec{r},\vec{r}\,')+k^2 G_\omega(\vec{r},\vec{r}\,')\right\}=-4\pi\delta(\vec{r}-\vec{r}\,')\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}}$$

Equation (*) can be written as

We used the δ-function representation $\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}=\delta(\tau)$

(**) $$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\underbrace{\left\{\vec{\nabla}_r^2 G_\omega(\vec{r},\vec{r}\,')+k^2 G_\omega(\vec{r},\vec{r}\,')+4\pi\,\delta(\vec{r}-\vec{r}\,')\right\}}_{\text{"0"}}=0$$

The left part of Eq. (**) has to be zero at any values of papameters τ, $\vec{r}$ and $\vec{r}\,'$.

It is possible only ~~for~~

(***) $$\boxed{\vec{\nabla}_r^2 G_\omega(\vec{r},\vec{r}\,')+k^2 G_\omega(\vec{r},\vec{r}\,')+4\pi\,\delta(\vec{r}-\vec{r}\,')=0}$$

Equation (5) or (***) can be solved for uniform space:

$$G_\omega(\vec{r},\vec{r}\,')=G_\omega(\vec{r}-\vec{r}\,')\equiv G_\omega(R),\quad\text{where } R=|\vec{r}-\vec{r}\,'|$$

$$\vec{\nabla}_r^2 G_\omega=\vec{\nabla}_R^2 G_\omega=\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}G_\omega\right)+\text{angular part}$$

"0" for a uniform space

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dG_\omega}{dR}\right)+k^2G_\omega=-4\pi\delta(\vec{R})\qquad(\vec{R}=\vec{r}-\vec{r}\,')$$

We introduce a new function: $\boxed{\chi=G_\omega(R)\cdot R}$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\frac{\chi}{R}\right)=\frac{1}{R^2}\frac{d}{dR}\left(R\frac{d\chi}{dR}-\chi\right)=\frac{1}{R}\frac{d^2\chi}{dR^2}+\cancel{\frac{1}{R^2}\frac{d\chi}{dR}}-\cancel{\frac{1}{R^2}\frac{d\chi}{dR}}=$$

$$=\frac{1}{R}\frac{d^2\chi}{dR^2}$$

$$\frac{1}{R}\frac{d^2\chi}{dR^2}+k^2\frac{\chi}{R}=-4\pi\delta(R)\Rightarrow\boxed{\frac{d^2\chi}{dR^2}+k^2\chi=-4\pi\delta(R)\cdot R}$$

For $R\neq 0$: $\frac{d^2\chi}{dR^2}+k^2\chi=0$; $\chi=A\cdot e^{ikr}+Be^{-ikr}\Rightarrow$

$$\Rightarrow\quad G_\omega(R)=A\frac{e^{ikR}}{R}+B\frac{e^{-ikR}}{R}\qquad\text{(A and B are constants)}$$

$$\vec{\nabla}_R^2 G\underset{R\to 0}{=}A\vec{\nabla}^2\frac{1}{R}+B\vec{\nabla}^2\frac{1}{R}=(A+B)\vec{\nabla}^2\frac{1}{R}=(A+B)*(-4\pi)\delta(R)$$

$$\vec{\nabla}^2\left(\frac{1}{R}\right)=-4\pi\delta(R)$$

$\Rightarrow$ $\boxed{A+B=1}$

Simplest case: $A=1,\ B=0 \Rightarrow$ Retarded Green Function

$A=0,\ B=1 \Rightarrow$ Advanced Green Function

Retarded Green Function:

$$G_\omega(R) = G_\omega(\vec r, \vec r\,') = \frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|}$$

$R = |\vec r - \vec r\,'|$

$$G(\vec r, \vec r\,', \tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)} e^{-i\omega\tau} \cdot \frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} = \frac{1}{2\pi|\vec r - \vec r\,'|} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega\tau + i\frac{\omega}{c}|\vec r - \vec r\,'|};$$

$k = \omega/c$

$$G(\vec r, \vec r\,', \tau) = \frac{1}{2\pi|\vec r - \vec r\,'|} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega\left(\tau - \frac{|\vec r - \vec r\,'|}{c}\right)} = \frac{\cancel{2}\pi\, \delta\left(\tau - \frac{|\vec r - \vec r\,'|}{c}\right)}{\cancel{2}\pi\, |\vec r - \vec r\,'|}$$

$\tau = t - t'$

The last equation provides an analytical formula for the retarded Green Function:

$$\boxed{G(\vec r, t, \vec r\,', t') = \frac{\delta\left(t - t' - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}}$$

Solution of the wave equation is:

$$\boxed{\Psi(\vec r, t) = \Psi_{in}(\vec r, t) + \iint dt'\, d^3 r'\, G(\vec r, t, \vec r\,', t')\, f(\vec r\,', t')}$$

solution of the homogeneous wave equation

$$\vec\nabla^2 \Psi_{in} - \frac{1}{c^2}\frac{\partial^2 \Psi_{in}}{\partial^2 t^2} = 0$$

Integration over "t" yields the retarded solution of the inhomogeneous equation:

$$\boxed{\Psi(\vec r, t) = \Psi_{in}(\vec r, t) + \int d^3 r'\, \frac{f\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}}$$