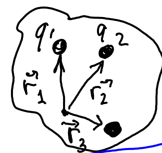
Lecture 11

02/29/2016

recap: General expression for the multipole emission terms for the total emisson power:

$$P = P^{(p)} + P^{(m)} + P^{(Q)}$$

Electric dipole: $$P^{(p)} = \frac{2}{3}\frac{|\ddot{\vec{p}}|^2}{c^3} = \frac{2}{3}\frac{|\sum q_i \ddot{\vec{r}}_i(t)|^2}{c^3}$$



electric dipole

$$\vec{p}(t) = \sum_i q_i \vec{r}_i(t)$$

Electric quadrupole: $$P^{(Q)} = \frac{1}{180c^5}\left(\sum_{\alpha,\beta} \dddot{Q}_{\alpha\beta}\right)^2$$

electric quadrupole

$$Q_{\alpha\beta} = \sum_{i,\alpha,\beta} q_i\left(3x_{i\alpha}x_{i\beta} - \delta_{\alpha\beta} r_i^2\right)$$

Magnetic dipole: $$P^{(m)} = \frac{2}{3c^3}|\ddot{\vec{m}}|^2$$

magnetic dipole

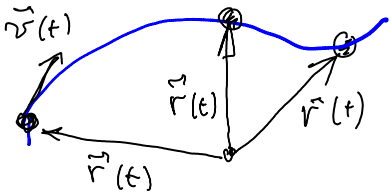
$$\vec{m} = \frac{1}{2c}\sum_i q_i(\vec{r}_i \times \vec{v}_i)$$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt}$$

These formulas are written in Gaussian System of Units

Radiation induced by accelerated charge

For a single charge: $\boxed{\vec{p} = q\vec{r}(t)}$ where $\vec{r}(t)$ is the charge radius-vector

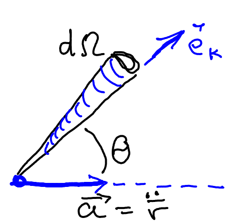


$$P^{(p)} = \frac{2}{3}\frac{q^2}{c^3}|\ddot{\vec{r}}|^2 = \frac{2}{3}\frac{q^2}{c^3}|\vec{a}|^2$$

acceleration of the charged particle q

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Angular distribution of emission induced by a point charge particle q:



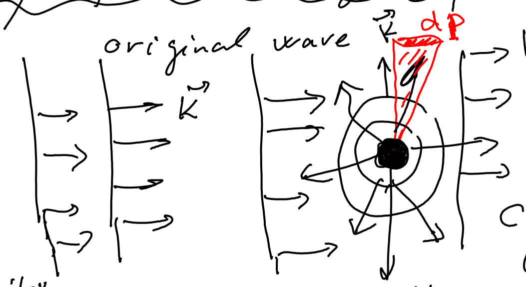
$\vec{e}_K = \frac{\vec{r}}{r}$ – direction of observation of EM waves

$$\boxed{I(\theta) = \frac{dP(\theta)}{d\Omega} = \frac{1}{4\pi}\frac{q^2}{c^3}|\ddot{\vec{r}}|^2 \cdot \sin^2\theta}$$

where θ is the angle between the vector of acceleration $\vec{a} = \ddot{\vec{r}}$ and direction of observation $\vec{e}_K = \frac{\vec{r}}{r}$.

Scattering of EM waves by free charged particles

Poynting vector $\vec{S}_0$ of incoming EM waves



$$\vec{S}_0 = \frac{\vec{E}\times\vec{B}}{\mu_0} \equiv \frac{c}{4\pi}(\vec{E}\times\vec{B})$$

"SI" "Gaussian Units"

Definition: Differential cross-sections

$$d\sigma(\vec{e}_k) = \frac{dP}{S_0} = I(\theta)\, d\Omega / S_0$$

elemental area

$[d\sigma] = cm^2$

$I = I(\theta)$ – intensity of the EM radiation

Gaussian Units

$$\vec{S}_0 = \frac{c}{4\pi}(\vec{E}\times\vec{B})$$

$$w_E = \frac{E^2}{8\pi};\ w_B = \frac{B^2}{8\pi}$$

in a vacuum $B = H$, $E = D$

differential cross section

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{I(\theta)}{S_0}}$$

← differential cross sections of EM scattering

The total cross section

$$\sigma_{total} = \int d\sigma \equiv \int_{(4\pi)} \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{\int dP}{S_0} = \frac{P}{S_0}$$

$$\boxed{\sigma_{total} = \frac{P}{S_0}}$$

Total cross section

Density of the energy flux in incoming wave:

$$\vec{S}_0 = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{c}{4\pi}E^2\,\vec{e}_k =$$

$(B \equiv E)$ Gaussian Units

$$= \frac{c}{4\pi}E_0^2 \cos^2(kr - \omega t)\,\vec{e}_k$$

amplitude

For the plane waves: $\vec{E} = \vec{E}_0 \cos(\bar{k}\bar{r} - \omega t)$ $\quad k = \frac{\omega}{c}$

Electron's motion: (Newton's equation) $\quad \ddot{\vec{x}} = \frac{e}{m_e}\vec{E} \qquad \ddot{\bar{r}} = \frac{e}{m_e}\vec{E}_0 \cos(\bar{k}\bar{r} - \omega t)$

Dipole vector: $\vec{p}(t) = e\vec{x}_e(t) = \frac{e^2}{m_e}\vec{E}_0\cos(kr-\omega t)$; $\boxed{\vec{p} \uparrow\uparrow E_0}$

Emission power:

$\frac{da}{r^2} = d\Omega$

$$dP = \frac{1}{4\pi c^3 r^2}(\ddot{\vec{p}} \times \vec{n})\, da = \frac{|\ddot{\vec{p}}|^2 \cdot \sin^2\theta}{4\pi c^3}\, d\Omega$$

"θ" - the angle between $\vec{e}$ and $\vec{p}$ or $\vec{E}_0$

Differential cross section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\frac{dP}{d\Omega}}{4\pi c^3} \Big/ S_0\right) = \frac{1}{4\pi c^3}\frac{\left(\frac{e^2}{m_e}\right)^2 \cancel{E_0^2}}{\frac{c}{4\pi}\cancel{E_0^2}}\cdot \sin^2\theta$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{m_e c^2}\right)^2 \sin^2\theta}$$

Thomson's Formula

$$\boxed{\sigma_{total} = \int_{(4\pi)}\left(\frac{d\sigma}{d\Omega}\right)\cdot d\Omega = \left(\frac{e^2}{m_e c^2}\right)^2 \int_0^\pi 2\pi \sin\theta\, d\theta \cdot \sin^2\theta = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2}$$

$\boxed{r_0 = \frac{e^2}{m_e c^2}}$ ← classical electron radius

$r_0 = 2.82 * 10^{-13}$ cm

Thomson's cross section: $\boxed{\sigma = \frac{8\pi}{3} r_0^2}$

$$\frac{r_0}{a_B} = \frac{e^2}{\cancel{m_e} c^2 \frac{\hbar^2}{\cancel{m_e} e^2}} = \left(\frac{e^2}{\hbar c}\right)^2 << 1 ; \quad \frac{r_0}{a_B} = \alpha^2$$

(Bohr-radius)

$a_B = \frac{\hbar^2}{m_e e^2}$

$\alpha = \frac{e^2}{\hbar c}$ ↙ the fine structure constant

($\alpha \sim 1/137$)