TT Projection RW

Projections onto the transverse components of Δ_{ij} and Δ_{0i} are applied within the RW geometry

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) = -dt^2 + a(t)^2 g_{ij} dx^i dx^j, \quad (0.1)$$

in order to investigate if it is possible to obtain SVT separation at a lesser derivative order.

Separation into scalar, vector, and tensor sectors is found and given as

$$\begin{aligned}(\nabla^2 - 2k)(\nabla^2 - 3k)\Delta_{ij}^{T\theta} &= -6k^2\ddot{E}_{ij} - 12k^3 E_{ij} - 12k^2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 5k\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}_{ij} + 10k\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} + 16k^2\tilde{\nabla}_a\tilde{\nabla}^a E_{ij} \\ &\quad -\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} - 7k\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a E_{ij} + \tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a E_{ij}\end{aligned} \quad (0.2)$$

$$\begin{aligned}(\nabla^2 - 2k)\Delta_{0i}^{T} &= -2k^2 Q_i + 8k\dot{\Omega}^2 V_i\Omega^{-3} - 4k\ddot{\Omega}V_i\Omega^{-2} + 4k^2 V_i\Omega^{-1} - 4\dot{\Omega}^2\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a V_i + 2\ddot{\Omega}\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a V_i \\ &\quad -2k\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a V_i + \tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a Q_i\end{aligned} \quad (0.3)$$

Combined with Δ_{00} and $g^{ij}\Delta_{ij}$, (0.2) and (0.3) serve as alternative separation equations, totaling 6 independent equations.

1 $h_{\mu\nu}$ General Decomposition

Curvature Tensors:

$$\begin{aligned}R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\ R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\ R &= kD(1-D)\end{aligned} \quad (1.1)$$

Covariant Commutation:

$$\begin{aligned}[\nabla^\sigma\nabla_\nu]W_\sigma &= -R_\nu{}^\sigma W_\sigma = -\frac{R}{D}W_\nu \\ [\nabla^\mu\nabla_\mu, \nabla_\nu]V &= -R_\nu{}^\mu\nabla_\mu V = -\frac{R}{D}\nabla_\nu V \\ [\nabla^2, \nabla_\mu\nabla_\nu]V &= \frac{2g_{\mu\nu}R}{D(D-1)}\nabla^2 V - \frac{2R}{D-1}\nabla_\mu\nabla_\nu V\end{aligned} \quad (1.2)$$

Decomposition:

$$\begin{aligned}h_{\mu\nu} &= h_{\mu\nu}^{T\theta} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\ &\quad + \frac{2-D}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')\nabla^\sigma W_\sigma - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D(x,x')h\end{aligned} \quad (1.3)$$

$$\left(\nabla_\alpha\nabla^\alpha - \frac{R}{D-1}\right)D(x,x') = g^{-1/2}\delta^4(x-x')$$

$$\nabla^\mu h_{\mu\nu} = \left(\nabla_\alpha \nabla^\alpha - \frac{R}{D}\right) W_\nu \tag{1.4}$$

2 $D = 3$ Decomposition

For $D = 3$, we have

$$h_{ij}^{T\theta} = h_{ij} - \nabla_i W_j - \nabla_j W_i + \frac{g_{ij}}{2}(\nabla^k W_k - h) + \frac{1}{2}(\nabla_i \nabla_j + k g_{ij}) \int D(\nabla^k W_k + h) \tag{2.1}$$

where

$$\begin{aligned} (\nabla_a \nabla^a + 3k)\, D(x, x') &= g^{-1/2} \delta^3(x - x') \\ \nabla^\ell h_{k\ell} &= (\nabla_a \nabla^a + 2k)\, W_k. \end{aligned} \tag{2.2}$$

3 $h_{ij}^{T\theta}$ Projection

The idea is to find differential operators that will bring (2.1) into a local form. This entails finding operators that commute through covariant derivatives outside the integrals but still retain the correct form to act on the Green's functions.

Commutators:

$$[\nabla^2 \nabla_i, \nabla_i \nabla^2] A_j = 2k(\nabla_i A_j + \nabla_j A_i - g_{ij} \nabla^k A_k) \tag{3.1}$$

$$[\nabla^2 \nabla_i, \nabla^i \nabla^2] A_i = -2k \nabla^k A_k \tag{3.2}$$

$$[\nabla^2 \nabla_i \nabla_j, \nabla_i \nabla_j \nabla^2] S = 2k(3 \nabla_i \nabla_j S - g_{ij} \nabla^2 S) \tag{3.3}$$

Useful relations:

$$(\nabla^2 - 3k)(\nabla_i \nabla_j + k g_{ij}) = (\nabla_i \nabla_j - k g_{ij})(\nabla^2 + 3k) \tag{3.4}$$

$$\begin{aligned} (\nabla^2 - 2k)(\nabla^2 - 3k)(\nabla_i W_j + \nabla_j W_i) &= -4k g_{ij}(2\nabla^2 + k)\nabla^k W_k + \nabla_j \nabla^2 (\nabla^2 + 2k) W_i + k \nabla_j (\nabla^2 + 2k) W_i \\ &\quad + \nabla_i \nabla^2 (\nabla^2 + 2k) W_j + k \nabla_i (\nabla^2 + 2k) W_j \end{aligned} \tag{3.5}$$

$$\begin{aligned} \nabla_i \nabla^2 (\nabla^2 + 2k) W_j &= \nabla^2 \nabla_i (\nabla^2 + 2k) W_j - 2k \nabla_j (\nabla^2 + 2k) W_i - 2k \nabla_i (\nabla^2 + 2k) W_j \\ &\quad + 2k g_{ij} (\nabla^2 + 4k) \nabla^k W_k \end{aligned} \tag{3.6}$$

$$\begin{aligned} (\nabla^2 - 2k)(\nabla^2 - 3k)(\nabla_i W_j + \nabla_j W_i) &= \nabla^2 \nabla_i (\nabla^2 + 2k) W_j + \nabla^2 \nabla_j (\nabla^2 + 2k) W_i - 3k \nabla_j (\nabla^2 + 2k) W_i - 3k \nabla_i (\nabla^2 + 2k) W_j \\ &\quad - 4k g_{ij} \nabla^2 \nabla^k W_k + 12 k^2 g_{ij} \nabla^k W_k \end{aligned} \tag{3.7}$$

$$(\nabla^2 + 4k) \nabla^k W_k = \nabla^k \nabla^l h_{kl} \tag{3.8}$$

$$(\nabla^2 - 2k)(\nabla^2 - 3k) \left[\frac{g_{ij}}{2}(\nabla^k W_k - h) + \frac{1}{2}(\nabla_i \nabla_j + k g_{ij}) \int D(\nabla^k W_k + h) \right]$$

$$
\begin{aligned}
= \quad & \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)\nabla^k W_k + \tfrac{1}{2}g_{ij}\nabla^2(\nabla^2+4k)\nabla^k W_k \\
& -6kg_{ij}\nabla^2\nabla^k W_k + 4k^2 g_{ij}\nabla^k W_k + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h - \tfrac{1}{2}g_{ij}\nabla^4 h + kg_{ij}\nabla^2 h - 2k^2 g_{ij}h
\end{aligned}
\tag{3.9}
$$

Result:

$$
\begin{aligned}
(\nabla^2-2k)(\nabla^2-3k)h^{T\theta}_{ij} \quad = \quad & (\nabla^2-2k)(\nabla^2-3k)h_{ij} - \nabla^2\nabla_i(\nabla^2+2k)W_j - \nabla^2\nabla_j(\nabla^2+2k)W_i + 3k\nabla_j(\nabla^2+2k)W_i \\
& +3k\nabla_i(\nabla^2+2k)W_j + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)\nabla^k W_k + \tfrac{1}{2}g_{ij}\nabla^2(\nabla^2+4k)\nabla^k W_k \\
& -2kg_{ij}(\nabla^2+4k)\nabla^k W_k + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h - \tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)h - \tfrac{1}{2}k(\nabla^2+4k)h
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
= \quad & (\nabla^2-2k)(\nabla^2-3k)h_{ij} - \nabla^2\nabla_i\nabla^l h_{jl} - \nabla^2\nabla_j\nabla^l h_{il} + 3k\nabla_j\nabla^l h_{il} + 3k\nabla_i\nabla^l h_{jl} \\
& +\tfrac{1}{2}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl} + \tfrac{1}{2}g_{ij}\nabla^2\nabla^k\nabla^l h_{kl} - 2kg_{ij}\nabla^l\nabla^k h_{kl} + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)h \\
& -\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)h - \tfrac{1}{2}g_{ij}k(\nabla^2+4k)h
\end{aligned}
\tag{3.11}
$$

4 $\Delta^{T\theta}_{ij}$ Projection

Now we use (3.11) as applied to the particular tensor Δ_{ij}, i.e.:

$$
\begin{aligned}
(\nabla^2-2k)(\nabla^2-3k)\Delta^{T\theta}_{ij} \quad = \quad & (\nabla^2-2k)(\nabla^2-3k)\Delta_{ij} - \nabla^2\nabla_i\nabla^l\Delta_{jl} - \nabla^2\nabla_j\nabla^l\Delta_{il} + 3k\nabla_j\nabla^l\Delta_{il} + 3k\nabla_i\nabla^l\Delta_{jl} \\
& +\tfrac{1}{2}\nabla_i\nabla_j\nabla^k\nabla^l\Delta_{kl} + \tfrac{1}{2}g_{ij}\nabla^2\nabla^k\nabla^l\Delta_{kl} - 2kg_{ij}\nabla^l\nabla^k\Delta_{kl} + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)\Delta \\
& -\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)\Delta - \tfrac{1}{2}g_{ij}k(\nabla^2+4k)\Delta,
\end{aligned}
\tag{4.1}
$$

where Δ is the 3-trace $\Delta = g^{ab}\Delta_{ab}$.

Inputting the explicit form of Δ_{ij}, we find

$$
\begin{aligned}
(\nabla^2-2k)(\nabla^2-3k)\Delta^{T\theta}_{ij} \quad = \quad & -6k^2\ddot{E}_{ij} - 12k^3E_{ij} - 12k^2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 5k\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}_{ij} + 10k\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} + 16k^2\tilde{\nabla}_a\tilde{\nabla}^a E_{ij} \\
& -\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_{ij} - 7k\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a E_{ij} + \tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^a E_{ij}.
\end{aligned}
\tag{4.2}
$$

5 Δ^T_{0i} Projection

For a vector, the transverse component may be expressed as

$$
\Delta^T_{0i} \quad = \quad \Delta_{0i} - \nabla_i\int A\nabla^k\Delta_{0k}
\tag{5.1}
$$

where

$$
\nabla_a\nabla^a A(x,x') = g^{-1/2}\delta(x-x').
\tag{5.2}
$$

To bring this into a local form, we apply

$$
(\nabla^2-2k)\Delta^T_{0i} \quad = \quad (\nabla^2-2k)\Delta_{0i} - \nabla_i\nabla^k\Delta_{0k}.
\tag{5.3}
$$

Inputting the explicit form of Δ_{0i}, we find

$$
\begin{aligned}
(\nabla^2-2k)\Delta^T_{0i} \quad = \quad & -2k^2Q_i + 8k\dot{\Omega}^2V_i\Omega^{-3} - 4k\ddot{\Omega}V_i\Omega^{-2} + 4k^2V_i\Omega^{-1} - 4\dot{\Omega}^2\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^aV_i + 2\ddot{\Omega}\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^aV_i \\
& -2k\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^aV_i + \tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a\tilde{\nabla}^aQ_i.
\end{aligned}
\tag{5.4}
$$