Lecture 5

02/03/2016

recap: harmonic solutions

$$\boxed{\begin{aligned}\vec{A}(\vec{r},t) &= e^{-i\omega t}\,\vec{A}_\omega(\vec{r})\\ \vec{A}_\omega(\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}\,')\,e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3r'\end{aligned}}$$

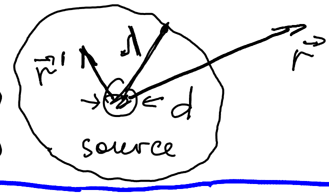
$\vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r})\,e^{-i\omega t}$

$\rho(\vec{r},t) = \rho_\omega(\vec{r})\,e^{-i\omega t}$

$k = \omega/c$

a) static zone $\quad r \ll \lambda \quad$ or $\quad kr \ll 1$

$k|\vec{r}-\vec{r}\,'| \sim kr \ll 1$; $\quad e^{ik|\vec{r}-\vec{r}\,'|} \simeq 1$



and $\boxed{\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}}$ $\leftarrow$ static field (Biot-Savart)

multipole expansion

$$\boxed{\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\int d\vec{r}\,'\cdot r'^{\,l}\,\vec{J}_\omega(\vec{r}\,')\,Y^*_{lm}(\theta',\phi')\,d^3r'}$$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l,m}\frac{4\pi}{2l+1}\,Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta',\phi')\,\frac{r'^{\,l}}{r^{l+1}} \quad \text{(expansion for } r'<r)}$$

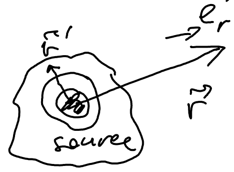
The radiation zone:

$r \gg \lambda \;\Rightarrow\; k|\vec{r}-\vec{r}\,'| \gg 1 \quad (d \ll \lambda)$

$r \gg d \sim r' \;\Rightarrow\; |\vec{r}-\vec{r}\,'| = r - \vec{r}\,'\cdot\vec{\nabla}(r) + \ldots = r - \vec{r}\,'\,\frac{\vec{r}}{r} = r - \vec{r}\,'\cdot\vec{e}_r$

$f(\vec{r}-\vec{r}\,') = e^{-\vec{r}\,'\vec{\nabla}} f(\vec{r})$
$r' \ll r$

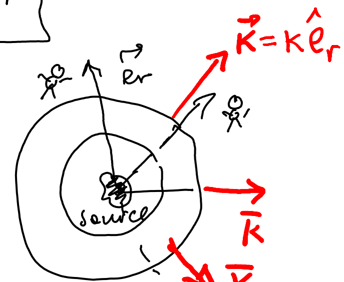
Unit vector $\vec{e}_r = \frac{\vec{r}}{r}$



$$\boxed{\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int \vec{J}_\omega(\vec{r}\,')\,e^{-ik\vec{e}_r\cdot\vec{r}\,'}\,d^3r'}$$

Outgoing spherical waves:

$$\boxed{\begin{aligned}\vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\,\vec{F}(\vec{k})\\ \vec{F}(\vec{k}) &= \int \vec{J}_\omega(\vec{r}\,')\,e^{-i\vec{k}\vec{r}\,'}\,d^3r'\end{aligned}}$$



The function $\vec{F}(\vec{k})$ depends on the $\vec{k}$-vector

$\boxed{\vec{k} = \vec{e}_r\cdot k}$ wave vector

Spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

The expansion with the spherical Bessel function

Math. $\rightarrow$ $$\boxed{e^{i\vec{k}\vec{r}} = 4\pi \sum_{l,m} i^l \, Y^*_{lm}(\hat{k}) \, Y_{lm}(\hat{r}) \, j_l(kr)}$$

The spherical Bessel function $j_l(kr)$:

equation for the $j_l(kr)$
$$\left\{ \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + k^2 - \frac{l(l+1)}{r^2} \right\} j_l(kr) = 0$$

$$j_l(x) = (-1)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}; \qquad j_0(x) = \sin x / x$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x};$$

$\ldots\ldots$

General solution of the wave equation in the radiation zone:

$$\vec{A}(\vec{r},t) = e^{-i\omega t} \vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int d^3r' \, \vec{j}_\omega(\vec{r}') e^{-i\vec{k}\vec{r}'} =$$

$$= \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \sum_{l,m} 4\pi \, i^l \, Y^*_{lm}(-\hat{k}) \int d^3r' \, \vec{j}_\omega(\vec{r}') \, Y_{lm}(\hat{r}') \, j_l(kr')$$

Parity: $Y^*_{lm}(-\hat{k}) = (-1)^l \, Y^*_{lm}(\hat{k})$
$(\vec{k} \to -\vec{k};\ \theta \to \pi - \theta)$

$$\boxed{\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{l,m} (-i)^l \, Y^*_{lm}(\hat{k}) \int Y_{lm}(\hat{r}') \, j_l(kr') \, \vec{j}_\omega(\vec{r}') \, d^3r'}$$

Simplest case: $\quad kr' = 2\pi \frac{r'}{\lambda} \ll 1 \qquad$ radiation zone: $r \gg \lambda$

For small values of kr':
$$j_l(kr') \simeq \frac{(kr')^l}{1\cdot 3\cdot 5 \cdot \ldots (2l+1)} = \frac{(kr')^l}{(2l+1)!!} \qquad (kr' \ll 1)$$

$$\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr - i\omega t}}{r} \cdot \sum_{l,m} (-i)^l \frac{Y^*_{lm}(\hat{k})}{(2l+1)!!} \int Y_{lm}(\hat{r}') (kr')^l \, \vec{j}_\omega(\vec{r}') \, d^3r'$$

The leading term $l=0$:
$Y_{00} = 1;\ j_{l=0} \simeq 1 \ (kr' \ll 1)$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int \vec{j}_\omega(\vec{r}') \, d^3r'}$$

The same result can be obtained, if we neglect an exponential factor:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int \vec{j}_\omega(\vec{r}') e^{-i\vec{k}\vec{r}'} d^3r' \simeq \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \cdot \int \vec{j}_\omega(\vec{r}') \, d^3r'$$

$$e^{-i\vec{k}\vec{r}'} = 1 - ikr' + \ldots \simeq 1$$