Quantum Mechanics III
HW 11

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7.9 Let us study a single mode a of an electromagnetic field under the usual cavity damping, so that the master equation reads

$$\dot{\rho} = \gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a).$$

Ignore the Hamiltonian evolution, which just unnecessarily complicates the argument.

(a) Show that the normally ordered moments of the electromagnetic field $M_{p,q}(t) = \mathrm{Tr}[\rho(t)a^{p\dagger}a^q]$, with p and q being nonnegative integers, evolve in time according to $M_{p,q}(t) = e^{-(p+q)\gamma t}M_{p,q}(0)$.

(b) According to a famous theorem by Glauber (paraphrased), the one and only state of the electromagnetic field such that $M_{p,q} = \alpha^{p*}\alpha^q$ for some complex number α and all nonnegative integers p, q is the coherent state $|\alpha\rangle$. Use this theorem to demonstrate that evolution according to the master equation (7.189) preserves a coherent state; i.e., a coherent state at one time, a coherent state at all others.

(a) We can express $M_{p,q}(t)$ as a first order ODE by taking its time derivative. This should lead to the form desired

$$\begin{aligned}
\dot{M}_{p,q}(t) &= \mathrm{Tr}[\dot{\rho}(t)a^{p\dagger}a^q] \\
&= \gamma\mathrm{Tr}[2a\rho a^\dagger a^{p\dagger}a^q - a^\dagger a\rho a^{p\dagger}a^q - \rho a^\dagger a a^{p\dagger}a^q] \\
&= \gamma\mathrm{Tr}[2a\rho a^\dagger a^{p\dagger}a^q - a\rho a^{p\dagger}a^q a^\dagger - \rho a^\dagger a a^{p\dagger}a^q] \\
&= \gamma\mathrm{Tr}[2a\rho a^\dagger a^{p\dagger}a^q - \rho a^\dagger[a, a^{p\dagger}]a^q - a\rho a^{p\dagger}[a^q, a^\dagger] - 2a\rho a^\dagger a^{p\dagger}a^q] \\
&= \gamma\mathrm{Tr}[-q\rho a^{p\dagger}a^q - p\rho a^{p\dagger}a^q] \\
&= -\gamma(p+q)M_{p,q}(t)
\end{aligned}$$

Now we may solve the differential equation for $M_{p,q}(t)$ to find

$$M_{p,q}(t) = e^{-(p+q)\gamma t}M_{p,q}(0)$$

(b) According to Glauber's theorem, we have

$$\dot{M}_{p,q}(t) = 0 \quad \text{for} \quad \rho = |\alpha\rangle\langle\alpha|.$$

$$\Rightarrow 0 = -\gamma(p+q)\alpha^{*p}\alpha^q.$$

Now for integers $p, q > 0$ and $\alpha \neq 0$, this implies $\gamma = 0$, and thus $\dot{\rho} = 0$ for a coherent state. In the case that $\alpha = 0$, each term cancels in the master equation and still $\dot{\rho} = 0$. Coherent state is persevered.

7.12 In parametric downconversion photons from an incoming field are split in two. A stylistic but serviceable Hamiltonian for downconversion to a photon mode with the annihilation operator a reads

$$H = \frac{\hbar\xi}{2}(aa + a^\dagger a^\dagger)$$

where the parameter ξ proportional to the amplitude of the driving field is real.

(a) Write down the Heisenberg equations of motion for the operators $a(t)$ and $a^\dagger(t)$.

(b) Solve these equations with the ansatz $a(t) = f(t)a(0) + g^*(t)a^\dagger(0)$.

(c) Suppose the photon mode starts at $t = 0$ in the vacuum state. Show that all time the photon is in a squeezed vacuum state, an eigenstate of an operator of the form (7.120) with eigenvalue 0.

(a)

$$\dot{a} = -\frac{i}{\hbar}[a, H] = -\frac{i\xi}{2}[a, a^\dagger a^\dagger] = -\frac{i\xi}{2}(2a^\dagger) = -i\xi a^\dagger$$
$$\Rightarrow \dot{a}^\dagger = i\xi a$$

(b) The equation from (a) may be decoupled as

$$\frac{d^2 a}{dt^2} = \xi^2 a, \qquad \frac{d^2 a^\dagger}{dt^2} = \xi^2 a^\dagger.$$

Using the ansatz we have

$$\frac{d^2 f(t)}{dt^2}a(0) + \frac{d^2 g^*(t)}{dt^2}a^\dagger(0) = \xi^2(f(t)a(0) + g^*(t)a^\dagger(0)).$$

Viewing differentiation as a linear operator, this implies

$$\Rightarrow \frac{d^2 f(t)}{dt^2} = \xi^2 f(t)$$

and likewise for $g^*(t)$. The general solution is

$$f(t) = Ae^{\xi t} + Be^{-\xi t}$$

and likewise

$$g^*(t) = Ce^{\xi t} + De^{-\xi t}.$$

The coefficients from $f(t)$ may be related to those of $g(t)$ by the first derivative relations of the ansatz

$$\frac{df(t)}{dt} = -i\xi g(t), \qquad \frac{dg^*(t)}{dt} = -i\xi f^*(t).$$
$$\Rightarrow A = -iC^* \qquad B = -iD^*.$$
$$\Rightarrow C = -iA^* \qquad D = -iB^*.$$

Initial conditions

$$f(0) = 1, \qquad g(0) = 0.$$

Now our solution is

$$f(t) = Ae^{\xi t} + Be^{-\xi t}, \qquad g^*(t) = i(A^* e^{\xi t} + B^* e^{-\xi t}).$$

We may also take the commutator

$$\begin{aligned}[a, a^\dagger] &= [f(t)a(0) + g^*(t)a^\dagger(0), f(t)^* a^\dagger(0) + g(t)a(0)] \\ &= |f(t)|^2 - |g(t)|^2\end{aligned}$$

$$=$$

$$a(t) = (Ae^{\xi t} + Be^{-\xi t})a(0) + i(A^* e^{\xi t} + B^* e^{-\xi t})a^\dagger(0)$$

..... The final solution is

$$a(t) = a(0)\cosh(\xi t) - ia^\dagger(0)\sinh(\xi t).$$

(c) The time evolution of the vacuum state is given as

$$|\psi(t)\rangle = e^{-iHt/\hbar}|0\rangle$$

and in the Schrodinger picture we have

$$a = e^{-iHt/\hbar}a(t)e^{iHt/\hbar}.$$

Unitary transformations preserve eigen-equations

Using this, we act upon the time dependent state with the annihilation operator. If it annihilates, the state remained in the vacuum state.

$$\begin{aligned} a|\psi(t)\rangle &= e^{-iHt/\hbar}a(t)e^{iHt/\hbar}e^{-iHt/\hbar}|0\rangle \\ &= e^{-iHt/\hbar}a(t)|0\rangle \\ &= 0 \end{aligned}$$

Thus the squeezed vacuum state

$$|\psi(t)\rangle = e^{-iHt/\hbar}|0\rangle = |0, \xi t\rangle$$

($A = 0$ squeezed vacuum)

is an eigenstate of our a with zero eigenvalue and matches the form of 7.120.

7.14 (a) Take an operator F that satisfied $[F, F^\dagger] = -C$. Show that the expectation value of the operator $F^\dagger F$ in any state satisfies $\langle F^\dagger F\rangle \geq \langle C\rangle$.

In a Heisenberg-like picture, a phase-insensitive linear amplifier of a boson mode (say, single mode of light) is represented by the equation $a_0 = Ga_i + F$, where a_i and a_0 are the input and output mode operators, the amplitude gain satisfied $G > 1$, and F is an operator having to do with the internal degrees of freedom of the amplifier. This being a Heisenberg-like picture, the state of the system $|\psi\rangle$, for the input mode and the internal degrees of freedom, is constant. Let us assume that the input and the amplifier are initially uncorrelated, so that $|\psi\rangle$ is a direct product of the state of the input and of the amplifier. Finally, in order that the internal degrees of freedom do not generate a false signal at the output, let us assume that $\langle F\rangle = 0$.

(b) Show that, given a coherent input state $|\alpha\rangle$, the expectation value of the output is $G\alpha$. This is, of course, as it should be for an amplifier with gain G.

(c) Now, the output operator a_0 must be a boson operator. Show that a non-trivial "noise operator" F *must* be present, and satisfies $[F, F^\dagger] = 1 - G^2$.

(d) Show that, for any input, the boson number at the output satisfies $\langle a_0^\dagger a_0\rangle \geq G^2\langle a_i^\dagger a_i\rangle + G^2 - 1$.

Under the present conditions (basically, no correlation between the input and the initial state of the amplifier), the output is not just an amplified copy of the input, but the process of amplification invariably adds at least $G^2 - 1$ uncorrelated bosons to worsen the signal to noise ratio.

(a) Take the expectation value in an arbitrary normalized state $|\psi\rangle$

$$\begin{aligned}\langle\psi|[F,F^\dagger]|\psi\rangle &= -\langle\psi|C|\psi\rangle = -\langle C\rangle \\ \Rightarrow \langle F^\dagger F\rangle &= \langle C\rangle + \langle FF^\dagger\rangle \end{aligned} \tag{1}$$

By virtue of the hermiticity of $[F,F^\dagger]$, $\langle C\rangle \in \mathbb{R}$. Likewise $FF^\dagger$ is hermitian so $\langle FF^\dagger\rangle \in \mathbb{R}$. Moreover, $FF^\dagger$ is a positive operator by

$$(\psi, FF^\dagger\psi) = (F^\dagger\psi, F^\dagger\psi) \equiv (\phi,\phi) \geq 0.$$

Thus from (1) we have

$$\langle F^\dagger F\rangle \geq \langle C\rangle .$$

(b) The state $|\psi\rangle$ with coherent input $|\alpha\rangle$ may be written as $|\alpha\rangle|\phi\rangle$ where $|\phi\rangle$ represents the (normalized and uncorrelated) internal degrees of freedom of the state. Taking the expectation value

$$\begin{aligned}\langle\phi|\langle\alpha|\,a_0\,|\alpha\rangle|\phi\rangle &= \langle\phi|\langle\alpha|\,(Ga_i+F)\,|\alpha\rangle|\phi\rangle \\ &= G\langle\alpha|a_i|\alpha\rangle + \langle\phi|F|\phi\rangle = G\alpha.\end{aligned}$$

(c) As a boson operator, a_0 must satisfy commutation relation

$$\begin{aligned}[a_0, a_0^\dagger] &= 1 \\ &= [Ga_i + F, Ga_i^\dagger + F^\dagger] \\ &= G^2 + [F,F^\dagger].\end{aligned}$$

Without the noise operator F present in the definition of a_0, the output boson commutation relation cannot hold for $G \neq 1$. Also note that photon operators and internal operators F commute. Thus we have

$$[F,F^\dagger] = 1 - G^2.$$

(d) Given an arbitrary input, construct the general expectation value

$$\begin{aligned}\langle a_0^\dagger a_0\rangle &= \langle (Ga_i^\dagger + F^\dagger)(Ga_i+F)\rangle \\ &= G^2\langle a_i^\dagger a_i\rangle + G\langle a_i^\dagger F\rangle + G\langle a_i F^\dagger\rangle + \langle F^\dagger F\rangle \\ &= G^2\langle a_i^\dagger a_i\rangle + G\langle a_i^\dagger F\rangle + G\langle a_i F^\dagger\rangle + \langle FF^\dagger - [F,F^\dagger]\rangle \\ &= G^2\langle a_i^\dagger a_i\rangle + G\langle a_i^\dagger F\rangle + G\langle a_i F^\dagger\rangle + \langle FF^\dagger\rangle + G^2 - 1.\end{aligned}$$

As the input is not correlated to the initial state of the amplifier, we may utilize the assumption $\langle F\rangle = 0$ to allow the cross terms to vanish

$$\langle F\rangle = 0 = \langle F^\dagger\rangle^* \Rightarrow \langle F^\dagger\rangle = 0$$

$$\Rightarrow \langle a_i F^\dagger\rangle = 0, \qquad \langle a_i^\dagger F\rangle = 0.$$

Just to be clear, because of no correlation, the expectation is between a product of states, which could be defined as arbitrary initial boson state $|\xi\rangle$ and internal state $|\phi\rangle$, so

$$\langle a_i F^\dagger \rangle = \langle \xi | a_i | \xi \rangle \langle \phi | F^\dagger | \phi \rangle = 0.$$

Now eliminating the vanishing products in the expectation of the output number

$$\begin{aligned} \langle a_0^\dagger a_0 \rangle &= G^2 \langle a_i^\dagger a_i \rangle + \langle F F^\dagger \rangle + G^2 - 1 \\ &\geq G^2 \langle a_i^\dagger a_i \rangle + G^2 - 1. \end{aligned}$$

by virtue of $FF^\dagger$ as a positive operator.