Covariant Green's Functions

1 Flat Space $D = 2$

1.1 Scalar

Restricting to $D = 2$ we apply the Laplacian to the scalar f

$$\nabla_j \nabla^j f = \left(\frac{\partial^2}{\partial_x^2} + \frac{\partial^2}{\partial_y^2} \right) f. \tag{1}$$

Take coordinate transformation $x^i \to x'^i$ affected by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}. \tag{2}$$

As a scalar $f(x,y) \to f(r,\theta)$. The partial derivatives transform as

$$\frac{\partial}{\partial x} \to \frac{dr}{dx}\frac{\partial}{\partial r} + \frac{d\theta}{dx}\frac{\partial}{\partial\theta}. \tag{3}$$

To determine the Jacobian of transformation, we calculate the total variations

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r}dr & \frac{\partial x}{\partial\theta}d\theta \\ \frac{\partial y}{\partial r}dr & \frac{\partial y}{\partial\theta}d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta dr & -r\sin\theta d\theta \\ \sin\theta dr & r\cos\theta d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta. \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} \tag{4}$$

With the Jacobian J as,

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5}$$

we recall the inverse of a 2×2 matrix is

$$J^{-1} = \frac{1}{\mathrm{Det}(J)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}, \tag{6}$$

and hence we calculate dx'^i

$$\begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \tag{7}$$

With partial derivatives in hand, the polar Laplacian then takes then form

$$\nabla_j \nabla^j f(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \right) f(r,\theta). \tag{8}$$

Regarding the line element, we have in Cartesian coordinates

$$ds^2 = \delta_{ij} dx^i dx^j, \tag{9}$$

and polar coordinates,

$$ds^2 = g_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2. \tag{10}$$

1.2 Vector

Covariant Transformation

Taking A_i as a 2 dimensional vector, under coordinate transformation $x^i \to x'^i$ it transforms as

$$A_i \to A'_i = \frac{\partial x^j}{\partial x'^i} A_j. \tag{11}$$

Defining B_{ij} as

$$B_{ij} = \frac{\partial}{\partial x^i} A_j \tag{12}$$

under coordinate transformation $x^i \to x'^i$ it follows

$$B_{ij} \to B'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} B_{kl} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \frac{\partial}{\partial x^k} A_l \tag{13}$$

To express B'_{ij} in terms of A'_i, we make use of the identity

$$A_i = \delta_i^j A_j = \frac{\partial x^j}{\partial x^i} A_j = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^l} A_j = \frac{\partial x'^l}{\partial x^i} A'_l. \tag{14}$$

Inserting the identity into B'_i and expressing derivatives in terms of the x' coordinate system, we have We would like to express the partial derivatives in terms of the x' coordinate system and as such use the chain rule

$$\begin{aligned}
B'_{ij} &= \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \frac{\partial}{\partial x^k} A_l \\
&= \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \frac{\partial}{\partial x^k} \left(\frac{\partial x'^p}{\partial x^l} A'_p \right) \\
&= \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \frac{\partial x'^q}{\partial x^k} \frac{\partial}{\partial x'^q} \left(\frac{\partial x'^p}{\partial x^l} A'_p \right) \\
&= \frac{\partial x^l}{\partial x'^j} \frac{\partial}{\partial x'^i} \left(\frac{\partial x'^p}{\partial x^l} A'_p \right) \\
&= \frac{\partial}{\partial x'^i} A'_j + \frac{\partial x^l}{\partial x'^j} \left(\frac{\partial}{\partial x'^i} \frac{\partial x'^p}{\partial x^l} \right) A'_p \\
&= \frac{\partial}{\partial x'^i} A'_j - \frac{\partial x'^p}{\partial x^l} \frac{\partial^2 x^l}{\partial x'^i \partial x'^j} A'_p.
\end{aligned}$$

From

$$g_{ij} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} \delta_{mn}, \tag{15}$$

we recall that the affine connection may be expressed as

$$\Gamma^k_{ij} = \frac{\partial x'^k}{\partial x^l} \frac{\partial^2 x^l}{\partial x'^i \partial x'^j}. \tag{16}$$

Hence B'_{ij} may be expressed as

$$\begin{aligned}
B'_{ij} &= \frac{\partial}{\partial x'^i} A'_j - \frac{\partial x'^p}{\partial x^l} \frac{\partial^2 x^l}{\partial x'^i \partial x'^j} A'_p \\
&= \frac{\partial}{\partial x'^i} A'_j - \Gamma^p_{ij} A'_p \\
&= \nabla_i A'_j.
\end{aligned}$$

While the above example was performed for one covariant derivative, the same process applies any general tensor composed of derivatives. The procedure simply entails re-expressing all Cartesian derivatives ∂_i in terms of covariant derivatives ∇_i.

As a result, we may determine the transformation of the Laplacian of a vector under $x^i \to x'^i$ as

$$J_i = \partial_j \partial^j A_i \to J'_i = \nabla_j \nabla^j A'_i, \tag{17}$$

with covariant derivatives understood as being defined relative to metric g_{ij} belonging to the x' coordinate system.

Cartesian Green's Function

Given the equation

$$\partial_j \partial^j A_i = J_i, \tag{18}$$

we may form a solution for A_i as

$$A_i = \int d^2x' \; G(\mathbf{x}, \mathbf{x}') J_i(\mathbf{x}'), \tag{19}$$

where

$$\partial_j \partial^j G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \tag{20}$$

With the differential operator $\mathcal{L} = \partial_j \partial^j$ being translation invariant, and with $\delta(\mathbf{x} - \mathbf{x}')$ being translation invariant under $\mathbf{x} \to \mathbf{x} + \mathbf{a}$, $x' \to \mathbf{x}' + \mathbf{a}$, it follows that $G(\mathbf{x}, \mathbf{x}')$ must take the same form, i.e. be a function only of the difference $\mathbf{x} - \mathbf{x}'$. If the above argument does not hold, we then assume $G(\mathbf{x}, \mathbf{x}')$ to be of the form $G(|\mathbf{x} - \mathbf{x}'|)$.

To determine the form of $D = 2$ Green's function of the Laplacian, integrate (**??**) over a region of radius a centered at x',

$$\begin{aligned}
&\int_{r<a} d^2x \; \partial_j \partial^j G(|\mathbf{x} - \mathbf{x}'|) = 1 \\
&\oint_{r=a} dS_j \; \partial^j G(|\mathbf{x} - \mathbf{x}'|) = 1 \\
&\oint_{x^2+y^2=a} (-dy\hat{\mathbf{x}} + dx\hat{\mathbf{y}}) \cdot \left(\frac{\partial G(|\mathbf{x} - \mathbf{x}'|)}{\partial x} \hat{\mathbf{x}} + \frac{G(|\mathbf{x} - \mathbf{x}'|)}{\partial y} \hat{\mathbf{y}} \right) = 1.
\end{aligned} \tag{21}$$

Noting that

$$\frac{\partial f(|\mathbf{r} - \mathbf{r}'|)}{\partial r} = \frac{\partial f(|\mathbf{r} - \mathbf{r}'|)}{\partial (r - r')}, \tag{22}$$

it will be convenient to solve (**??**) in polar coordinates,

$$\begin{aligned}
&\oint_{r=a} d\theta \; r \sin\theta \; \hat{\mathbf{r}} \cdot \left(\frac{\partial G(|\mathbf{r} - \mathbf{r}'|)}{\partial (r - r')} \hat{\mathbf{r}} + \frac{G(|\mathbf{r} - \mathbf{r}'|)}{\partial \theta} \hat{\theta} \right) = 1 \\
&a \oint_{r=a} d\theta \; \sin\theta \frac{\partial G(|\mathbf{r} - \mathbf{r}'|)}{\partial (r - r')} = 1 \\
&2\pi a \frac{\partial G(\tilde{r})}{\partial \tilde{r}} \bigg|_{\tilde{r}=a} = 2\pi a \frac{\partial G(|\mathbf{r} - \mathbf{r}'|)}{\partial (r - r')} \bigg|_{|r-r'|=a} = 1.
\end{aligned}$$

In the last step, we have shifted the origin to $\mathbf{r}_2$, in which $|\mathbf{r} - \mathbf{r}'| = \tilde{r}$. The solution to

$$\frac{\partial G(|\mathbf{r} - \mathbf{r}'|)}{\partial (r - r')} \bigg|_{|r-r'|=a} = \frac{1}{2\pi a} \tag{23}$$

for arbitrary a may be found by integration to be

$$G(|\mathbf{r}-\mathbf{r}'|) = \frac{1}{2\pi}\ln(|\mathbf{r}-\mathbf{r}'|). \tag{24}$$

Accordingly, the solution A_i is

$$A_i(x) = \frac{1}{2\pi}\int d^2y \ \ln(|x-y|)J_i(y). \tag{25}$$

1.3 Coordinate Transformation

As a vector, A_i, under coordinate transformation $x \to x'$, must obey

$$A_i \to A'_i = \frac{\partial x^j}{\partial x'^i}A_j. \tag{26}$$

To compare a Cartesian and polar system specifically, we note that in Cartesian coordinates, the Green's solution to A_i is

$$A_x = \frac{1}{2\pi}\int d^2z \ \ln(|x-z|)J_x(z)$$
$$A_y = \frac{1}{2\pi}\int d^2z \ \ln(|x-z|)J_y(z),$$

where we integrate over z for clarity. According to transformation (**??**), the components in the polar basis are related via

$$\begin{aligned} A_r(x) &= \frac{\partial x^j}{\partial r}A_j(x) = \frac{\partial x}{\partial r}A_x + \frac{\partial y}{\partial r}A_y = \cos\theta A_x + \sin\theta A_y \\ &= \frac{1}{2\pi}\cos\theta\int d^2z \ \ln(|x-z|)J_x(z) + \frac{1}{2\pi}\sin\theta\int d^2z \ \ln(|x-z|)J_y(z) \end{aligned} \tag{27}$$

$$\begin{aligned} A_\theta(x) &= \frac{\partial x^j}{\partial \theta}A_j(x) = \frac{\partial x}{\partial \theta}A_x + \frac{\partial y}{\partial \theta}A_y = -r\sin\theta A_x + r\cos\theta A_y \\ &= -\frac{1}{2\pi}r\sin\theta\int d^2z \ \ln(|x-z|)J_x(z) + \frac{1}{2\pi}r\cos\theta\int d^2z \ \ln(|x-z|)J_y(z) \end{aligned} \tag{28}$$

We may also express J_x and J_y in the polar basis via

$$J_i = \frac{\partial x'^j}{\partial x^i}J'_j$$

to obtain

$$J_x = \frac{\partial r}{\partial x}J_r + \frac{\partial \theta}{\partial x}J_\theta = \cos\theta J_r - \frac{\sin\theta}{r}J_\theta \tag{29}$$

$$J_y = \frac{\partial r}{\partial y}J_r + \frac{\partial \theta}{\partial y}J_\theta = \sin\theta J_r + \frac{\cos\theta}{r}J_\theta. \tag{30}$$

Inserting these back into (**??**) and (**??**) yields

$$A_r = \frac{1}{2\pi}\cos^2\theta\int d^2z \ \ln(|x-z|)J_r(z) - \frac{1}{2\pi}\left(\frac{\sin\theta\cos\theta}{r}\right)\int d^2z \ \ln(|x-z|)J_\theta(z) \tag{31}$$

$$+\frac{1}{2\pi}\sin^2\theta\int d^2z \ \ln(|x-z|)J_r(z) + \frac{1}{2\pi}\left(\frac{\sin\theta\cos\theta}{r}\right)\int d^2z \ \ln(|x-z|)J_\theta(z) \tag{32}$$

$$= \frac{1}{2\pi}\int d^2z \ \ln(|x-z|)J_r(z) \tag{33}$$

$$A_\theta = -\frac{1}{2\pi} r \sin\theta\cos\theta \int d^2z\ \ln(|x-z|) J_r(z) + \frac{1}{2\pi}\sin^2\theta \int d^2z\ \ln(|x-z|) J_\theta(z) \tag{34}$$

$$+ \frac{1}{2\pi} r \sin\theta\cos\theta \int d^2z\ \ln(|x-z|) J_r(z) + \frac{1}{2\pi}\cos^2\theta \int d^2z\ \ln(|x-z|) J_\theta(z) \tag{35}$$

$$= \frac{1}{2\pi}\int d^2z\ \ln(|x-z|) J_\theta(z). \tag{36}$$

From these coordinate transformation, we find that in a polar coordinate system the solution to A_i is given as

$$A_r(r) = \frac{1}{2\pi}\int d^2z\ \ln(|r-z|) J_r(z), \qquad A_\theta(r) = \frac{1}{2\pi}\int d^2z\ \ln(|r-z|) J_\theta(z). \tag{37}$$

Since these solutions are merely coordinate transformations from the known Cartesian solutions, it must be that they satisfy the covariant differential equation

$$\nabla_j \nabla^j A_i(r) = J_i(r), \tag{38}$$

since this equation is itself derived via a coordinate transformation from Cartesian to polar.

To verify the solution, we note the covariant box acting on a vector may be decomposed into Christoffels as

$$\begin{aligned} g^{jk}\nabla_j\nabla_k A_i = &\left(g^{jk}\partial_j\partial_k - \Gamma^m{}_{jk} g^{jk}\partial_m\right) A_i + A_m \Gamma^m{}_{kn}\Gamma^n{}_{ji} g^{jk} + A_m \Gamma^m{}_{ni}\Gamma^n{}_{jk} g^{jk} \\ &- A_m g^{jk}\partial_j \Gamma^m{}_{ki} - 2\Gamma^m{}_{ji} g^{jk}\partial_k A_m. \end{aligned} \tag{39}$$

The Christoffels evaluate to

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^r_{rr} = \Gamma^\theta_{rr} = \Gamma^r_{r\theta} = \Gamma^\theta_{\theta\theta} = 0. \tag{40}$$

We may then evaluate the components of $J_i(r)$ as

$$\begin{aligned} J_r &= -A_r r^{-2} + r^{-1}\frac{\partial}{\partial r}A_r + \frac{\partial^2}{\partial r^2}A_r - 2r^{-3}\frac{\partial}{\partial\theta}A_\theta + r^{-2}\frac{\partial^2}{\partial\theta^2}A_r \\ &= (\nabla_j\nabla^j)A_r - r^{-2}A_r - 2r^{-3}\frac{\partial}{\partial\theta}A_\theta \end{aligned} \tag{41}$$

$$\begin{aligned} J_\theta &= -r^{-1}\frac{\partial}{\partial r}A_\theta + \frac{\partial^2}{\partial r^2}A_\theta + 2r^{-1}\frac{\partial}{\partial\theta}A_r + r^{-2}\frac{\partial^2}{\partial\theta^2}A_\theta \\ &= (\nabla_j\nabla^j)A_\theta - 2r^{-1}\frac{\partial}{\partial r}A_\theta + 2r^{-1}\frac{\partial}{\partial\theta}A_r \end{aligned} \tag{42}$$

where $(\nabla_j\nabla^j)$ denotes the scalar covariant box in polar coordinates (i.e. compute $\nabla_j\nabla^j A$, then set $A = A_i$).

However, here we note J_r is coupled to A_r and A_θ (and likewise for J_θ). Substitution of the A_i given in (37) would appear to yield an inconsistent equation.

2 Summary of Poisson's Curved Space Green's Functions

2.1 Two Point Bitensors

Bitensors are tensors of two spacetime points. Primed indices denoted coordinates with respect to x', while unprimed indices denoted coordinates with respect to x. For example

$$T_{\alpha\beta'}(x, x'). \tag{43}$$

Such a tensor has transformation rule under $x \to \tilde{x}$

$$T_{\alpha\beta'} \to T_{\tilde{\alpha}\beta'} = \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} T_{\alpha\beta'} \tag{44}$$

and under $x' \to \hat{x}$,

$$T_{\alpha\beta'} \to T_{\alpha\hat{\beta}'} = \frac{\partial x'^\rho}{\partial \hat{x}'^\beta} T_{\alpha\beta'}. \tag{45}$$

In the Poison's construction of Green's functions, the bitensors are evaluated on the unique geodesic defined as the set of points x linked to x' that belong within the normal convex neighborhood of x'.

2.2 Parallel Propogator

On a geodesic linking x to x', introduce the orthonormal basis e^μ_a, which satisfy

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}, \tag{46}$$

and which obey

$$\frac{De^\mu_a}{d\lambda} = 0. \tag{47}$$

The parallel propogator which takes a vector at point x and propogates along the manifold to point x' is defined as

$$g^\alpha{}_{\alpha'}(x, x') = e^\alpha_a(x) e^a_{\alpha'}(x'). \tag{48}$$

For example:

$$A^{\alpha'}(x') = g^{\alpha'}{}_\alpha(x', x) A^\alpha. \tag{49}$$

These orthonormal basis vectors which have both coordinate and Lorentz index are equivalent to tetrads (vierbeins).

2.3 Dirac Distribution in Curved Space

Poisson defines the invariant Dirac functional as

$$\int_V f(x)\delta_4(x, x')\sqrt{-g}d^4x = f(x'), \quad \int_{V'} f(x')\delta_4(x, x')\sqrt{-g'}d^4x' = f(x). \tag{50}$$

This implies the various equivalent forms:

$$\delta_4(x, x') = \frac{\delta_4(x - x')}{\sqrt{-g}} = \frac{\delta_4(x - x')}{\sqrt{-g'}} = (gg')^{1/4}\delta_4(x - x'). \tag{51}$$

When acted upon by a covariant derivative, it obeys

$$\nabla_\alpha(g^\alpha{}_{\alpha'}(x, x')\delta_4(x, x')) = -\partial_{\alpha'}\delta_4(x, x') \tag{52}$$

$$\nabla_{\alpha'}(g^{\alpha'}{}_\alpha(x, x')\delta_4(x, x')) = -\partial_\alpha\delta_4(x, x') \tag{53}$$

2.4 Flat Space Greens Functions

2.5 Curved Space Greens Functions

Scalar

$$(\nabla_\alpha\nabla^\alpha - \xi R)\Phi(x) = -4\pi\mu(x) \tag{54}$$

$$\Phi(x) = \int G(x, x')u(x')\sqrt{-g'}d^4x' \tag{55}$$

Vector

See Poisson Paper.

3 Potential Application

3.0.1 SVT Basis

In terms of the SVT decomposition

$$\begin{aligned}
ds^2 &= -(g^{(0)}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
&= (1+2\phi)d\tau^2 - 2(\nabla_i B + B_i)dtdx^i - [(1-2\psi\gamma_{ij}) + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}]dx^i dx^j,
\end{aligned} \tag{56}$$

$\delta W_{\mu\nu}$ takes the form

$$\begin{aligned}
\delta W_{00} =& -2k\nabla_a\nabla^a\dot{B} + 2k\nabla_a\nabla^a\ddot{E} + \tfrac{8}{3}k^2\nabla_a\nabla^a E - 2k\nabla_a\nabla^a\phi - 2k\nabla_a\nabla^a\psi - \tfrac{2}{3}k\nabla_b\nabla_a\nabla^b\nabla^a E \\
& - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\dot{B} + \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\ddot{E} + \tfrac{2}{3}k\nabla_b\nabla^b\nabla_a\nabla^a E - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\phi \\
& - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\psi,
\end{aligned} \tag{57}$$

$$\begin{aligned}
\delta W_{0i} =& \tfrac{4}{3}k\nabla_a\nabla^a\nabla_i\dot{E} - 2k\nabla_i\ddot{B} + 2k\nabla_i\dddot{E} - 4k^2\nabla_i\dot{E} - 2k\nabla_i\dot{\phi} - 2k\nabla_i\dot{\psi} - \tfrac{2}{3}\nabla_i\nabla_a\nabla^a\ddot{B} \\
& + \tfrac{2}{3}\nabla_i\nabla_a\nabla^a\dddot{E} - \tfrac{4}{3}k\nabla_i\nabla_a\nabla^a\dot{E} - \tfrac{2}{3}\nabla_i\nabla_a\nabla^a\dot{\phi} - \tfrac{2}{3}\nabla_i\nabla_a\nabla^a\dot{\psi} \\
& - 2k^2 B_i - k\ddot{B}_i + k\dddot{E}_i + 2k^2\dot{E}_i - \tfrac{1}{2}\nabla_a\nabla^a\ddot{B}_i + \tfrac{1}{2}\nabla_a\nabla^a\dddot{E}_i + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a B_i \\
& - \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a\dot{E}_i,
\end{aligned} \tag{58}$$

$$\begin{aligned}
\delta W_{ij} =& -\tfrac{2}{3}kg_{ij}\nabla_a\nabla^a\dot{B} + \tfrac{1}{3}g_{ij}\nabla_a\nabla^a\dddot{B} - \tfrac{1}{3}g_{ij}\nabla_a\nabla^a\ddddot{E} + \tfrac{2}{3}kg_{ij}\nabla_a\nabla^a\ddot{E} + \tfrac{1}{3}g_{ij}\nabla_a\nabla^a\ddot{\phi} \\
& + \tfrac{1}{3}g_{ij}\nabla_a\nabla^a\ddot{\psi} + \tfrac{20}{3}k^2 g_{ij}\nabla_a\nabla^a E - \tfrac{2}{3}kg_{ij}\nabla_a\nabla^a\phi - \tfrac{2}{3}kg_{ij}\nabla_a\nabla^a\psi + \tfrac{4}{3}k\nabla_a\nabla_j\nabla^a\nabla_i E \\
& - \tfrac{4}{3}kg_{ij}\nabla_b\nabla_a\nabla^b\nabla^a E - \tfrac{1}{3}g_{ij}\nabla_b\nabla^b\nabla_a\nabla^a\dot{B} + \tfrac{1}{3}g_{ij}\nabla_b\nabla^b\nabla_a\nabla^a\ddot{E} + \tfrac{4}{3}kg_{ij}\nabla_b\nabla^b\nabla_a\nabla^a E \\
& - \tfrac{1}{3}g_{ij}\nabla_b\nabla^b\nabla_a\nabla^a\phi - \tfrac{1}{3}g_{ij}\nabla_b\nabla^b\nabla_a\nabla^a\psi + 2k\nabla_j\nabla_a\nabla^a\nabla_i E - \nabla_j\nabla_i\dddot{B} + \nabla_j\nabla_i\ddddot{E} \\
& - \nabla_j\nabla_i\ddot{\phi} - \nabla_j\nabla_i\ddot{\psi} - \tfrac{40}{3}k^2\nabla_j\nabla_i E + \tfrac{1}{3}\nabla_j\nabla_i\nabla_a\nabla^a\dot{B} - \tfrac{1}{3}\nabla_j\nabla_i\nabla_a\nabla^a\ddot{E} \\
& - \tfrac{10}{3}k\nabla_j\nabla_i\nabla_a\nabla^a E + \tfrac{1}{3}\nabla_j\nabla_i\nabla_a\nabla^a\phi + \tfrac{1}{3}\nabla_j\nabla_i\nabla_a\nabla^a\psi \\
& + k\nabla_a\nabla^a\nabla_i E_j + k\nabla_a\nabla^a\nabla_j E_i - k\nabla_i\dot{B}_j - \tfrac{1}{2}\nabla_i\dddot{B}_j + \tfrac{1}{2}\nabla_i\ddddot{E}_j + k\nabla_i\ddot{E}_j - 4k^2\nabla_i E_j \\
& + \tfrac{1}{2}\nabla_i\nabla_a\nabla^a\dot{B}_j - \tfrac{1}{2}\nabla_i\nabla_a\nabla^a\ddot{E}_j - k\nabla_i\nabla_a\nabla^a E_j - k\nabla_j\dot{B}_i - \tfrac{1}{2}\nabla_j\dddot{B}_i + \tfrac{1}{2}\nabla_j\ddddot{E}_i \\
& + k\nabla_j\ddot{E}_i - 4k^2\nabla_j E_i + \tfrac{1}{2}\nabla_j\nabla_a\nabla^a\dot{B}_i - \tfrac{1}{2}\nabla_j\nabla_a\nabla^a\ddot{E}_i - k\nabla_j\nabla_a\nabla^a E_i \\
& + \ddddot{E}_{ij} + 8k\ddot{E}_{ij} + 4k^2 E_{ij} - 2\nabla_a\nabla^a\ddot{E}_{ij} - 4k\nabla_a\nabla^a E_{ij} + \nabla_b\nabla^b\nabla_a\nabla^a E_{ij}.
\end{aligned} \tag{59}$$

Note that the trace h vanishes as expected, since our metric is of RW form with $\Omega(x) = 1$, which we know may be expressed in conformal to flat form (with $W^{(0)}_{\mu\nu}$ thereby vanishing).

3.0.2 Conformal Transformation

Under general conformal transformation $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$, the perturbed Bach tensor transforms as

$$\delta W_{\mu\nu} \to \Omega^{-2}(x)\delta W_{\mu\nu}. \tag{60}$$

Hence, we can express $\delta W_{\mu\nu}$ in the proper RW form with metric

$$ds^2 = -\Omega(\tau)^2\left[-(1+h_{00})d\tau^2 + (g_{ij} + h_{ij})dx^i dx^j\right], \tag{61}$$

by multiplying the net results above by $\Omega^{-2}(\tau)$.

3.0.3 Projected Components

Based on the Energy Momentum Tensor section, we can simplify the equation $\delta W_{\mu\nu} = \delta T_{\mu\nu}$ by looking at each SO(3) sector viz.

$$\begin{aligned}
\bar{\rho} &= \rho \\
\bar{Q}_i &= Q_i \\
\bar{\pi}^{T\theta}_{ij} &= \pi^{T\theta}_{ij}.
\end{aligned} \tag{62}$$

where

$$
\begin{aligned}
\rho &= \delta W_{00} \\
Q_i &= -\delta W_{0i} - \nabla_i \int d^3x' \sqrt{g}\ A(x,x') \nabla^j \delta W_{0j} \\
\pi^{T\theta}_{ij} &= \delta W^{T\theta}_{ij}.
\end{aligned} \tag{63}
$$

Again, we set to zero the surfrace terms generated by integration by parts. Reading off the scalar, vector, and tensor components from (ref 100) according to (ref 101) yields for $\delta T_{\mu\nu} = \delta W_{\mu\nu}$:

$$
\begin{aligned}
\bar{\rho} =\ & -2k\nabla_a\nabla^a\dot{B} + 2k\nabla_a\nabla^a\ddot{E} + \tfrac{8}{3}k^2\nabla_a\nabla^a E - 2k\nabla_a\nabla^a\phi - 2k\nabla_a\nabla^a\psi - \tfrac{2}{3}k\nabla_b\nabla_a\nabla^b\nabla^a E \\
& - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\dot{B} + \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\ddot{E} + \tfrac{2}{3}k\nabla_b\nabla^b\nabla_a\nabla^a E - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\phi \\
& - \tfrac{2}{3}\nabla_b\nabla^b\nabla_a\nabla^a\psi, \\
\bar{Q}_i =\ & -2k^2B_i - k\ddot{B}_i + k\dddot{E}_i + 2k^2\dot{E}_i - \tfrac{1}{2}\nabla_a\nabla^a\ddot{B}_i + \tfrac{1}{2}\nabla_a\nabla^a\dddot{E}_i + \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a B_i \\
& - \tfrac{1}{2}\nabla_b\nabla^b\nabla_a\nabla^a\dot{E}_i, \\
\bar{\pi}^{T\theta}_{ij} =\ & \ddddot{E}_{ij} + 8k\ddot{E}_{ij} + 4k^2E_{ij} - 2\nabla_a\nabla^a\ddot{E}_{ij} - 4k\nabla_a\nabla^a E_{ij} + \nabla_b\nabla^b\nabla_a\nabla^a E_{ij}.
\end{aligned} \tag{64}
$$

Under conformal transformation each SO(3) section simply scales as $\Omega^{-2}(\tau)$.

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