de Sitter Geometries

de Sitter space can be described as a submanifold embedded in a higher dimension Minskowski space. Working in $D = 4$, take the $D + 1$ Minkowski space defined as

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{1}$$

Now let us constrain our coordinates to a hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \alpha^2. \tag{2}$$

Taking the differential of (2), we may relate dx_4 to the remaining coordinates

$$\begin{aligned} f(x_0, x_1, x_2, x_3, x_4) &= \alpha^2 \\ df &= \frac{\partial f}{\partial x_\mu} dx^\mu = 0 \\ \Rightarrow \quad x_4 dx_4 &= x_0 dx_0 - \mathbf{x} \cdot d\mathbf{x} \\ \Rightarrow \quad dx_4^2 &= \frac{(x_0 dx_0 - \mathbf{x} \cdot d\mathbf{x})^2}{x_4^2} \\ \Rightarrow \quad dx_4^2 &= \frac{(x_0 dx_0 - \mathbf{x} \cdot d\mathbf{x})^2}{\alpha^2 + x_0^2 - \mathbf{x}^2}. \end{aligned} \tag{3}$$

Hence we may express (1) in terms of four coordinates

$$\begin{aligned} ds^2 &= -dx_0^2 + (d\mathbf{x})^2 + \frac{x_0^2 dx_0^2 + (\mathbf{x} \cdot d\mathbf{x})^2 - 2x_0 dx_0 (\mathbf{x} \cdot d\mathbf{x})}{\alpha^2 + x_0^2 - \mathbf{x}^2} \\ &= \frac{1}{\alpha^2 + x_0^2 - \mathbf{x}^2} \left[-dx_0^2(\alpha^2 - \mathbf{x}^2) + dx_1^2(\alpha^2 + x_0^2 + x_1^2 - \mathbf{x}^2) + ... - 2x_0 dx_0 (\mathbf{x} \cdot d\mathbf{x}) \right]. \end{aligned} \tag{4}$$

Before proceeding, it is also worth noting how the curvature tensors are related to α^2 in a $D = 4$ maximally symmetric space

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= \frac{1}{\alpha^2}(g_{\mu\nu} g_{\lambda\kappa} - g_{\lambda\nu} g_{\mu\kappa}) \\ R_{\mu\kappa} &= -3/\alpha^2 g_{\mu\kappa} \\ R &= -12/\alpha^2 \end{aligned} \tag{5}$$

Going back to the rather complicated line element, we may choose coordinates

$$\begin{aligned} x_0 &= \alpha \sinh(t/\alpha) \\ x_1 &= \alpha \cosh(t/\alpha) \cos\chi \\ x_2 &= \alpha \cosh(t/\alpha) \sin\chi \cos\theta \\ x_3 &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi, \end{aligned} \tag{6}$$

which brings (4) to

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha) d\chi^2 + \alpha^2 \cosh^2(t/\alpha) \sin^2\chi (d\theta^2 + \sin^2 d\phi^2). \tag{7}$$

We may bring the de Sitter line element into a conformal flat form by first choosing coordinates

$$\begin{aligned} x_0 &= \alpha \sinh(t/\alpha) + e^{t/\alpha}\mathbf{x}\cdot\mathbf{x}/2\alpha \\ x_1 &= \alpha \cosh(t/\alpha) - e^{t/\alpha}\mathbf{x}\cdot\mathbf{x}/2\alpha \\ x_2 &= e^{t/\alpha} X_1 \\ x_3 &= e^{t/\alpha} X_2, \end{aligned} \tag{8}$$

in which the line element becomes

$$ds^2 = -dt^2 + e^{2t/\alpha}(d\mathbf{X})^2. \tag{9}$$

We see that this looks like $k = 0$ RW with $a(t) = e^{2t/\alpha}$. To bring to conformal form, take

$$\begin{aligned} dt &= e^{t/\alpha} d\tau \\ d\tau &= \int dt e^{-t/\alpha} \\ \tau &= -\alpha e^{-t/\alpha} + C \\ \tau &= -\alpha e^{-t/\alpha} + \tau_\infty \\ \frac{\alpha^2}{(\tau - \tau_\infty)^2} &= e^{2t/\alpha}. \end{aligned} \tag{10}$$

If time in both the τ and t coordinates has the same infinitesimal direction (i.e positive $dt \Rightarrow +d\tau$) then coordinate time τ will be negative for $t > 0$.

From (10), we may express (9) as

$$ds^2 = \frac{\alpha^2}{(\tau - \tau_\infty)^2}\left[-d\tau^2 + d\mathbf{X}^2\right]. \tag{11}$$

If we further define time coordinate $p = \tau - \tau_\infty$, then we may write the line element more conveniently as

$$ds^2 = \frac{\alpha^2}{p^2}\left[-dp^2 + d\mathbf{X}^2\right]. \tag{12}$$

Using curvature relations (5), we may express the Einstein tensor as

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= \frac{3}{\alpha^2} g_{\mu\nu}. \end{aligned} \tag{13}$$

Hence if we define a background $T_{\mu\nu}$ according to $-\kappa_4^2 T_{\mu\nu} = \frac{3}{\alpha^2} g_{\mu\nu}$, then it follows that the perturbation of the background $T_{\mu\nu}$ yields

$$-\kappa_4^2 \delta T_{\mu\nu} = \frac{3}{\alpha^2} h_{\mu\nu} \tag{14}$$

$$\begin{aligned} -3(\dot\beta - \alpha) &= \tau \nabla^2 \beta \\ -3\tau(\ddot\beta - \dot\alpha) + 12(\dot\beta - \alpha) &= \tau^2 \nabla^2 \alpha - 3\tau \nabla^2 \beta \end{aligned} \tag{15}$$