Weyl Tensor Simplifications

$\delta W_{\mu\nu}$ Trace Dependence (General)

In isolating the trace part of the subsitution $h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4} g_{\mu\nu} h$, the perturbed conformal tensor takes the form (after some simplification from Bianchi identity and a substitution like eq. 47 in Cosmology paper, also taking $g_{\mu\nu} \equiv g^{(0)}_{\mu\nu}$ hereonforth)

$$\begin{aligned}\delta W_{\mu\nu}(\tfrac{h}{4} g_{\mu\nu}) = {} & \tfrac{1}{24} g_{\mu\nu} R^2 h - \tfrac{1}{8} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} h - \tfrac{1}{6} R R_{\mu\nu} h + \tfrac{1}{2} R_{\alpha\beta} R_\mu{}^\alpha{}_\nu{}^\beta h + \tfrac{1}{24} g_{\mu\nu} h \nabla_\alpha \nabla^\alpha R \\ & - \tfrac{1}{4} h \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \tfrac{1}{4} \nabla_\alpha h \nabla^\alpha R_{\mu\nu} + \tfrac{1}{4} \nabla_\alpha h \nabla_\beta R_\mu{}^\beta{}_\nu{}^\alpha + \tfrac{1}{4} \nabla_\alpha h \nabla_\nu R_\mu{}^\alpha + \tfrac{1}{12} h \nabla_\nu \nabla_\mu R\end{aligned} \tag{1}$$

Using a once contracted bianchi identity,

$$\begin{aligned}\nabla^\alpha h \nabla_\beta R_\mu{}^\beta{}_{\nu\alpha} &= -\nabla^\alpha h \nabla_\beta R^\beta{}_{\mu\nu\alpha} \\ &= \nabla^\alpha h \nabla_\alpha R_{\mu\nu} - \nabla^\alpha h \nabla_\nu R_{\mu\alpha}.\end{aligned} \tag{2}$$

(1) then becomes

$$\begin{aligned}\delta W_{\mu\nu} = {} & \tfrac{1}{24} g_{\mu\nu} R^2 h - \tfrac{1}{8} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} h - \tfrac{1}{6} R R_{\mu\nu} h + \tfrac{1}{2} R_{\alpha\beta} R_\mu{}^\alpha{}_\nu{}^\beta h \\ & + \tfrac{1}{24} g_{\mu\nu} h \nabla_\alpha \nabla^\alpha R - \tfrac{1}{4} h \nabla_\alpha \nabla^\alpha R_{\mu\nu} + \tfrac{1}{12} h \nabla_\nu \nabla_\mu R\end{aligned} \tag{3}$$

Now note the form of $W_{\mu\nu}$

$$\begin{aligned}W_{\mu\nu} = {} & -\tfrac{1}{6} g_{\mu\nu} R^2 + \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \tfrac{2}{3} R R_{\mu\nu} - 2 R^{\alpha\beta} R_{\mu\alpha\nu\beta} - \tfrac{1}{6} g_{\mu\nu} \nabla_\alpha \nabla^\alpha R \\ & + \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \nabla_\mu \nabla^\alpha R_{\nu\alpha} - \nabla_\nu \nabla^\alpha R_{\mu\alpha} + \tfrac{2}{3} \nabla_\nu \nabla_\mu R\end{aligned} \tag{4}$$

Now use the Bianchi identity on (4) to bring it to

$$W_{\mu\nu} = -\tfrac{1}{6} g_{\mu\nu} R^2 h + \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} h + \tfrac{2}{3} R R_{\mu\nu} h - 2 R^{\alpha\beta} R_{\mu\alpha\nu\beta} h - \tfrac{1}{6} g_{\mu\nu} h \nabla_\alpha \nabla^\alpha R + h \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \tfrac{1}{3} h \nabla_\nu \nabla_\mu R \tag{5}$$

It becomes apparent that

$$\boxed{\delta W_{\mu\nu}(\tfrac{h}{4} g_{\mu\nu}) = -\frac{1}{4} h W_{\mu\nu}.} \tag{6}$$

This result matches eq. 1.13 in *Fluctuations_Summary_Matthew.pdf*, where it was proven in general using the conformal properties of $W_{\mu\nu}$.

$\delta W_{\mu\nu}$ (Flat)

If we perturb $W_{\mu\nu}$ in a flat background we arrive at the form of

$$\delta W_{\mu\nu} = 2 \partial^\lambda \partial^\kappa \delta C_{\mu\lambda\nu\kappa} \tag{7}$$

Weyl quantities in a flat background $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\delta C^L{}_{MNK} = \delta R^L{}_{MNK} + \frac{1}{12} g^{AB} \delta R_{AB} (\delta^L{}_N g_{MK} - \delta^L{}_K g_{MN}) \tag{8}$$
$$- \frac{1}{3} (\delta^L{}_N \delta R_{MK} - \delta^L{}_K \delta R_{MN} - g_{MN} \delta R^L{}_K + g_{MK} \delta R^L{}_N)$$

$$\delta R^L{}_{MNK} = \delta\Gamma^L_{MN;K} - \delta\Gamma^L_{MK;N} \tag{9}$$

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} (\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \tag{10}$$

Upon evaluating $\delta C_{\mu\lambda\nu\kappa}$ as defined above, after simplification the perturbed $W_{\mu\nu}$ reduces to

$$2\partial^\lambda \partial^\kappa \delta C_{\mu\lambda\nu\kappa} = \tfrac{1}{3} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\kappa \partial_\nu \partial_\mu K_{\alpha\lambda} + \tfrac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\alpha K_{\mu\nu} - \tfrac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\mu K_{\nu\alpha} \tag{11}$$
$$- \tfrac{1}{2} \eta^{\alpha\kappa} \eta^{\lambda\beta} \partial_\beta \partial_\lambda \partial_\kappa \partial_\nu K_{\mu\alpha} + \tfrac{1}{6} \eta^{\alpha\kappa} \eta^{\gamma\eta} \eta^{\lambda\beta} \eta_{\mu\nu} \partial_\eta \partial_\gamma \partial_\beta \partial_\kappa K_{\alpha\lambda}.$$

This is equivalent to eq. (50) in Cosmology paper given as

$$\boxed{\delta W_{\mu\nu} = \tfrac{1}{2} \Pi^\rho{}_\mu \Pi^\sigma{}_\nu K_{\rho\sigma} - \tfrac{1}{6} \Pi_{\mu\nu} \Pi^{\rho\sigma} K_{\rho\sigma}} \tag{12}$$

where

$$\Pi_{\mu\nu} = \eta_{\mu\nu} \partial^\alpha \partial_\alpha - \partial_\mu \partial_\nu$$

and where one must keep in mind $\eta^{\alpha\beta} K_{\alpha\beta} = 0$.

$\delta W_{\mu\nu}$ (General)

No Riemann (56 terms)

$$\begin{aligned}
\delta W_{\mu\nu} = &-\tfrac{1}{6} K_{\mu\nu} R^2 + \tfrac{1}{3} g_{\mu\nu} K^{\alpha\beta} R R_{\alpha\beta} + \tfrac{1}{2} K_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - g_{\mu\nu} K^{\alpha\beta} R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3} K^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} + 2K^{\alpha\beta} R_{\mu\alpha} R_{\nu\beta} \\
&+ \tfrac{1}{3} R \nabla_\alpha \nabla^\alpha K_{\mu\nu} - \tfrac{1}{6} K_{\mu\nu} \nabla_\alpha \nabla^\alpha R - \tfrac{1}{3} R \nabla_\alpha \nabla_\mu K_\nu{}^\alpha - \tfrac{1}{2} \nabla_\alpha \nabla_\mu \nabla_\beta \nabla^\beta K_\nu{}^\alpha - \tfrac{1}{3} R \nabla_\alpha \nabla_\nu K_\mu{}^\alpha - \tfrac{1}{2} \nabla_\alpha \nabla_\nu \nabla_\beta \nabla^\beta K_\mu{}^\alpha \\
&- \tfrac{1}{6} \nabla_\alpha K_{\mu\nu} \nabla^\alpha R + \tfrac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta} \nabla_\beta R_{\mu\nu} + \tfrac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3} R_{\mu\nu} \nabla_\beta \nabla_\alpha K^{\alpha\beta} + R_\nu{}^\alpha \nabla_\beta \nabla_\alpha K_\mu{}^\beta \\
&- R^{\alpha\beta} \nabla_\beta \nabla_\alpha K_{\mu\nu} + R_\mu{}^\alpha \nabla_\beta \nabla_\alpha K_\nu{}^\beta + \tfrac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\beta \nabla_\alpha R - K^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} - R_\nu{}^\alpha \nabla_\beta \nabla^\beta K_{\mu\alpha} - R_\mu{}^\alpha \nabla_\beta \nabla^\beta K_{\nu\alpha} \\
&+ \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha K_{\mu\nu} - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\mu K_\nu{}^\alpha - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\nu K_\mu{}^\alpha + R_\nu{}^\alpha \nabla_\beta \nabla_\mu K_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\mu K_{\nu\alpha} + K^{\alpha\beta} \nabla_\beta \nabla_\mu R_{\nu\alpha} \\
&+ \tfrac{1}{2} \nabla_\beta \nabla_\mu \nabla_\alpha \nabla^\beta K_\nu{}^\alpha + \tfrac{1}{2} \nabla_\beta \nabla_\mu \nabla_\alpha \nabla_\nu K^{\alpha\beta} + R_\mu{}^\alpha \nabla_\beta \nabla_\nu K_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\nu K_{\mu\alpha} + K^{\alpha\beta} \nabla_\beta \nabla_\nu R_{\mu\alpha} + \tfrac{1}{2} \nabla_\beta \nabla_\nu \nabla_\alpha \nabla^\beta K_\mu{}^\alpha \\
&+ \tfrac{1}{2} \nabla_\beta \nabla_\nu \nabla_\alpha \nabla_\mu K^{\alpha\beta} + \nabla_\alpha R_{\nu\beta} \nabla^\beta K_\mu{}^\alpha - \nabla_\beta R_{\nu\alpha} \nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta} \nabla^\beta K_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha} \nabla^\beta K_\nu{}^\alpha - g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla_\beta K_\alpha{}^\gamma \\
&+ \tfrac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma K_{\alpha\beta} + \tfrac{1}{6} g_{\mu\nu} K^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} + \tfrac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha K^{\alpha\beta} + \tfrac{1}{3} g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma K^{\alpha\beta} + \tfrac{1}{6} \nabla^\alpha R \nabla_\mu K_{\nu\alpha} \\
&+ \nabla_\alpha K^{\alpha\beta} \nabla_\mu R_{\nu\beta} - \tfrac{1}{6} \nabla_\mu R_{\alpha\beta} \nabla_\nu K^{\alpha\beta} + \tfrac{1}{6} \nabla^\alpha R \nabla_\nu K_{\mu\alpha} - \tfrac{1}{6} \nabla_\mu K^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + \nabla_\alpha K^{\alpha\beta} \nabla_\nu R_{\mu\beta} - \tfrac{2}{3} R^{\alpha\beta} \nabla_\nu \nabla_\mu K_{\alpha\beta} \\
&- \tfrac{2}{3} K^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} - \tfrac{2}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha K^{\alpha\beta} - \tfrac{1}{4} h W_{\mu\nu}(g^{(0)}_{\mu\nu})
\end{aligned}$$

With Riemann (57 terms)

$$
\begin{aligned}
\delta W_{\mu\nu} = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}K_\nu{}^\alpha RR_{\mu\alpha} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}K_\mu{}^\alpha RR_{\nu\alpha} - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\
& - \tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} - K_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta - K_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} \\
& - \tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \tfrac{1}{6}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\
& + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha K_\mu{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha K_\nu{}^\beta + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} + \tfrac{1}{2}K_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} \\
& + \tfrac{1}{2}K_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha K_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha K_\mu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma K_\alpha{}^\gamma \\
& - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu K_\alpha{}^\beta - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu K_\alpha{}^\beta + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} \\
& + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma K^{\alpha\beta} \\
& + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha K^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} - \tfrac{1}{3}R\nabla_\mu\nabla_\alpha K_\nu{}^\alpha + R^{\alpha\beta}\nabla_\mu\nabla_\beta K_{\nu\alpha} \\
& - \nabla_\beta R_{\mu\alpha}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \tfrac{1}{3}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \tfrac{1}{3}R\nabla_\nu\nabla_\alpha K_\mu{}^\alpha + R^{\alpha\beta}\nabla_\nu\nabla_\beta K_{\mu\alpha} \\
& - \tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} + \tfrac{1}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta} - \tfrac{1}{4}hW_{\mu\nu}(g^{(0)}_{\mu\nu})
\end{aligned}
$$

$\delta C_{\lambda\mu\nu\kappa}$ and Trace Dependence (General)

$$
\begin{aligned}
\delta C_{\lambda\mu\nu\kappa} = & -\tfrac{1}{6}g_{\mu\nu}K_{\kappa\lambda}R + \tfrac{1}{6}g_{\lambda\nu}K_{\kappa\mu}R + \tfrac{1}{6}g_{\kappa\mu}K_{\lambda\nu}R - \tfrac{1}{6}g_{\kappa\lambda}K_{\mu\nu}R - \tfrac{1}{6}g_{\kappa\mu}g_{\lambda\nu}K^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{6}g_{\kappa\lambda}g_{\mu\nu}K^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\kappa\lambda} \\
& - \tfrac{1}{2}K_{\lambda\nu}R_{\kappa\mu} - \tfrac{1}{2}K_{\kappa\mu}R_{\lambda\nu} + \tfrac{1}{2}K_{\kappa\lambda}R_{\mu\nu} + K_\lambda{}^\alpha R_{\kappa\nu\mu\alpha} + \tfrac{1}{4}g_{\mu\nu}\nabla_\alpha\nabla^\alpha K_{\kappa\lambda} - \tfrac{1}{4}g_{\lambda\nu}\nabla_\alpha\nabla^\alpha K_{\kappa\mu} - \tfrac{1}{4}g_{\kappa\mu}\nabla_\alpha\nabla^\alpha K_{\lambda\nu} \\
& + \tfrac{1}{4}g_{\kappa\lambda}\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\nabla_\alpha\nabla_\kappa K_\lambda{}^\alpha + \tfrac{1}{4}g_{\lambda\nu}\nabla_\alpha\nabla_\kappa K_\mu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}\nabla_\alpha\nabla_\lambda K_\kappa{}^\alpha + \tfrac{1}{4}g_{\kappa\mu}\nabla_\alpha\nabla_\lambda K_\nu{}^\alpha + \tfrac{1}{4}g_{\lambda\nu}\nabla_\alpha\nabla_\mu K_\kappa{}^\alpha \\
& - \tfrac{1}{4}g_{\kappa\lambda}\nabla_\alpha\nabla_\mu K_\nu{}^\alpha + \tfrac{1}{4}g_{\kappa\mu}\nabla_\alpha\nabla_\nu K_\lambda{}^\alpha - \tfrac{1}{4}g_{\kappa\lambda}\nabla_\alpha\nabla_\nu K_\mu{}^\alpha - \tfrac{1}{6}g_{\kappa\mu}g_{\lambda\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{6}g_{\kappa\lambda}g_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\
& - \tfrac{1}{2}\nabla_\kappa\nabla_\lambda K_{\mu\nu} + \tfrac{1}{2}\nabla_\kappa\nabla_\mu K_{\lambda\nu} + \tfrac{1}{2}\nabla_\kappa\nabla_\nu K_{\lambda\mu} - \tfrac{1}{2}\nabla_\nu\nabla_\kappa K_{\lambda\mu} + \tfrac{1}{2}\nabla_\nu\nabla_\lambda K_{\kappa\mu} - \tfrac{1}{2}\nabla_\nu\nabla_\mu K_{\kappa\lambda} + \tfrac{1}{4}hC_{\lambda\mu\nu\kappa}
\end{aligned}
$$

where the trace dependent terms are

$$
\begin{aligned}
\delta C_{\lambda\mu\nu\kappa}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) &= \tfrac{1}{24}g_{\kappa\mu}g_{\lambda\nu}Rh - \tfrac{1}{24}g_{\kappa\lambda}g_{\mu\nu}Rh + \tfrac{1}{8}g_{\mu\nu}R_{\kappa\lambda}h - \tfrac{1}{8}g_{\lambda\nu}R_{\kappa\mu}h - \tfrac{1}{8}g_{\kappa\mu}R_{\lambda\nu}h + \tfrac{1}{8}g_{\kappa\lambda}R_{\mu\nu}h - \tfrac{1}{4}R_{\kappa\nu\lambda\mu}h \\
&= \tfrac{1}{4}hC_{\lambda\mu\nu\kappa}.
\end{aligned}
\tag{13}
$$

Note that this is opposite in sign compared to the trace dependence of $\delta W_{\mu\nu}$. To see this, under conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$$

we know the Weyl tensor is invariant

$$C^\lambda{}_{\mu\nu\kappa} \to C^\lambda{}_{\mu\nu\kappa}.$$

This is equivalent to

$$g^{\lambda\alpha}C_{\alpha\mu\nu\kappa} \to \Omega^{-2}(x)g^{\lambda\alpha}\tilde{C}_{\alpha\mu\nu\kappa},$$

and thus for the quantity to remain invariant, the covariant Weyl tensor must transform as

$$C_{\lambda\mu\nu\kappa} \to \Omega^2(x)C_{\lambda\mu\nu\kappa}.$$

The conformal symmetry applies to the trace as the following:

$$
\begin{aligned}
C_{\lambda\mu\nu\kappa}\left((1+\tfrac{h}{4})g^{(0)}_{\mu\nu}\right) &= \left(1+\tfrac{h}{4}\right)C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) \\
C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) + \delta C_{\lambda\mu\nu\kappa}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) &= C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu}) + \tfrac{h}{4}C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu})
\end{aligned}
$$

and therefore

$$\boxed{\delta C_{\lambda\mu\nu\kappa}\left(\tfrac{h}{4}g^{(0)}_{\mu\nu}\right) = \tfrac{h}{4}C_{\lambda\mu\nu\kappa}(g^{(0)}_{\mu\nu})}. \tag{14}$$

Latest Format

$$W^{(1)}_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R^2 - 2 R R_{\mu\nu} + 2 g_{\mu\nu} \nabla_\alpha \nabla^\alpha R - 2 \nabla_\nu \nabla_\mu R. \tag{15}$$

$$W^{(2)}_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2 R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \tfrac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla^\alpha R + \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \nabla_\mu \nabla^\alpha R_{\nu\alpha} - \nabla_\nu \nabla^\alpha R_{\mu\alpha}. \tag{16}$$

$$\begin{aligned} W_{\mu\nu} = & -\tfrac{1}{6} g_{\mu\nu} R^2 + \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \tfrac{2}{3} R R_{\mu\nu} - 2 R^{\alpha\beta} R_{\mu\alpha\nu\beta} - \tfrac{1}{6} g_{\mu\nu} \nabla_\alpha \nabla^\alpha R + \nabla_\alpha \nabla^\alpha R_{\mu\nu} \\ & - \tfrac{1}{3} \nabla_\nu \nabla_\mu R. \end{aligned} \tag{17}$$

where we use the perturbed quantities

$$\begin{aligned} \delta R_{\lambda\mu\nu\kappa} = & \, h^\alpha{}_\lambda R_{\alpha\mu\nu\kappa} - \tfrac{1}{2} \nabla_\kappa \nabla_\lambda h_{\mu\nu} + \tfrac{1}{2} \nabla_\kappa \nabla_\mu h_{\nu\lambda} + \tfrac{1}{2} \nabla_\kappa \nabla_\nu h_{\mu\lambda} - \tfrac{1}{2} \nabla_\nu \nabla_\kappa h_{\mu\lambda} + \tfrac{1}{2} \nabla_\nu \nabla_\lambda h_{\kappa\mu} \\ & - \tfrac{1}{2} \nabla_\nu \nabla_\mu h_{\kappa\lambda} \end{aligned} \tag{18}$$

$$\delta R_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta} (\nabla_\alpha \nabla_\beta h_{\mu\nu} - \nabla_\alpha \nabla_\mu h_{\beta\nu} - \nabla_\alpha \nabla_\nu h_{\beta\mu} + \nabla_\nu \nabla_\mu h_{\alpha\beta}). \tag{19}$$

General, no Bianchi, no explicit covariant derivative commutation

80 Terms

$$\begin{aligned} \delta W_{\mu\nu}(h_{\mu\nu}) = & -\tfrac{1}{6} h_{\mu\nu} R^2 + \tfrac{1}{3} g_{\mu\nu} h^{\alpha\beta} R R_{\alpha\beta} + \tfrac{1}{2} h_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - g_{\mu\nu} h^{\alpha\beta} R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3} h^{\alpha\beta} R_{\alpha\beta} R_{\mu\nu} \\ & + 2 h^{\alpha\beta} R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} - \tfrac{1}{6} h_{\mu\nu} \nabla_\alpha \nabla^\alpha R + \tfrac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_\beta \nabla_\alpha R - h^{\alpha\beta} \nabla_\beta \nabla_\alpha R_{\mu\nu} \\ & + \tfrac{1}{6} g_{\mu\nu} h^{\alpha\beta} \nabla_\gamma \nabla^\gamma R_{\alpha\beta} + h^{\alpha\beta} \nabla_\mu \nabla_\beta R_{\nu\alpha} + h^{\alpha\beta} \nabla_\nu \nabla_\beta R_{\mu\alpha} - \tfrac{2}{3} h^{\alpha\beta} \nabla_\nu \nabla_\mu R_{\alpha\beta} \end{aligned} \tag{20}$$

$$\begin{aligned} & + \tfrac{1}{3} R \nabla_\alpha \nabla^\alpha h_{\mu\nu} + R_{\mu\beta\nu\gamma} \nabla_\alpha \nabla^\gamma h^{\alpha\beta} + R_{\mu\gamma\nu\beta} \nabla_\alpha \nabla^\gamma h^{\alpha\beta} - \tfrac{1}{3} R \nabla_\alpha \nabla_\mu h_\nu{}^\alpha - \tfrac{1}{3} R \nabla_\alpha \nabla_\nu h_\mu{}^\alpha \\ & + \tfrac{1}{3} \nabla_\alpha h_{\mu\nu} \nabla^\alpha R + \tfrac{1}{6} g_{\mu\nu} \nabla^\alpha R \nabla_\beta h_\alpha{}^\beta - \nabla^\alpha h_{\mu\nu} \nabla_\beta R_\alpha{}^\beta - \nabla_\alpha h^{\alpha\beta} \nabla_\beta R_{\mu\nu} \\ & + \tfrac{1}{3} g_{\mu\nu} R \nabla_\beta \nabla_\alpha h^{\alpha\beta} - \tfrac{2}{3} R_{\mu\nu} \nabla_\beta \nabla_\alpha h^{\alpha\beta} + \tfrac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\alpha h_\mu{}^\beta - R^{\alpha\beta} \nabla_\beta \nabla_\alpha h_{\mu\nu} \\ & + \tfrac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\alpha h_\nu{}^\beta - \tfrac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla^\beta h_{\mu\alpha} - \tfrac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla^\beta h_{\nu\alpha} + \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha h_{\mu\nu} \\ & - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\mu h_\nu{}^\alpha - \tfrac{1}{2} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla_\nu h_\mu{}^\alpha - \tfrac{1}{2} R_\nu{}^\alpha \nabla_\beta \nabla_\mu h_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\mu h_{\nu\alpha} \\ & - \tfrac{1}{2} R_\mu{}^\alpha \nabla_\beta \nabla_\nu h_\alpha{}^\beta + R^{\alpha\beta} \nabla_\beta \nabla_\nu h_{\mu\alpha} + \nabla_\alpha R_{\nu\beta} \nabla^\beta h_\mu{}^\alpha - \nabla_\beta R_{\nu\alpha} \nabla^\beta h_\mu{}^\alpha \\ & + \nabla_\alpha R_{\mu\beta} \nabla^\beta h_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha} \nabla^\beta h_\nu{}^\alpha - g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla_\beta h_\alpha{}^\gamma + \tfrac{2}{3} g_{\mu\nu} R^{\alpha\beta} \nabla_\gamma \nabla^\gamma h_{\alpha\beta} \\ & - R_{\mu\alpha\nu\beta} \nabla_\gamma \nabla^\gamma h^{\alpha\beta} + \tfrac{1}{6} g_{\mu\nu} \nabla_\gamma \nabla^\gamma \nabla_\beta \nabla_\alpha h^{\alpha\beta} + \tfrac{1}{3} g_{\mu\nu} \nabla_\gamma R_{\alpha\beta} \nabla^\gamma h^{\alpha\beta} - \tfrac{1}{3} \nabla^\alpha R \nabla_\mu h_{\nu\alpha} \\ & + \nabla_\beta R_\alpha{}^\beta \nabla_\mu h_\nu{}^\alpha + \nabla_\alpha h^{\alpha\beta} \nabla_\mu R_{\nu\beta} - \tfrac{1}{2} \nabla_\mu \nabla_\alpha \nabla_\beta \nabla^\beta h_\nu{}^\alpha + R_\nu{}^\alpha \nabla_\mu \nabla_\beta h_\alpha{}^\beta \\ & + \tfrac{1}{2} \nabla_\mu \nabla_\beta \nabla_\alpha \nabla^\beta h_\nu{}^\alpha + \tfrac{1}{2} \nabla_\mu \nabla_\beta \nabla_\alpha \nabla_\nu h^{\alpha\beta} + \tfrac{1}{2} R^{\alpha\beta} \nabla_\mu \nabla_\nu h_{\alpha\beta} - \tfrac{1}{6} \nabla_\mu R_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \\ & - \tfrac{1}{3} \nabla^\alpha R \nabla_\nu h_{\mu\alpha} + \nabla_\beta R_\alpha{}^\beta \nabla_\nu h_\mu{}^\alpha - \tfrac{1}{6} \nabla_\mu h^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + \nabla_\alpha h^{\alpha\beta} \nabla_\nu R_{\mu\beta} \\ & - \tfrac{1}{2} \nabla_\nu \nabla_\alpha \nabla_\beta \nabla^\beta h_\mu{}^\alpha + R_\mu{}^\alpha \nabla_\nu \nabla_\beta h_\alpha{}^\beta + \tfrac{1}{2} \nabla_\nu \nabla_\beta \nabla_\alpha \nabla^\beta h_\mu{}^\alpha + \tfrac{1}{2} \nabla_\nu \nabla_\beta \nabla_\alpha \nabla_\mu h^{\alpha\beta} \\ & - \tfrac{7}{6} R^{\alpha\beta} \nabla_\nu \nabla_\mu h_{\alpha\beta} - \tfrac{2}{3} \nabla_\nu \nabla_\mu \nabla_\beta \nabla_\alpha h^{\alpha\beta} \end{aligned} \tag{21}$$

$$\begin{aligned} & - \tfrac{1}{3} g_{\mu\nu} R \nabla_\alpha \nabla^\alpha h + \tfrac{2}{3} R_{\mu\nu} \nabla_\alpha \nabla^\alpha h + \tfrac{1}{2} \nabla_\alpha \nabla^\alpha \nabla_\nu \nabla_\mu h - \tfrac{1}{12} g_{\mu\nu} \nabla_\alpha h \nabla^\alpha R \\ & + \tfrac{1}{2} \nabla_\alpha R_{\mu\nu} \nabla^\alpha h + \tfrac{1}{2} g_{\mu\nu} R^{\alpha\beta} \nabla_\beta \nabla_\alpha h - \tfrac{1}{6} g_{\mu\nu} \nabla_\beta \nabla^\beta \nabla_\alpha \nabla^\alpha h - R_{\mu\alpha\nu\beta} \nabla^\beta \nabla^\alpha h \\ & - \tfrac{1}{2} \nabla^\alpha h \nabla_\mu R_{\nu\alpha} - \tfrac{1}{2} R_\nu{}^\alpha \nabla_\mu \nabla_\alpha h - \tfrac{1}{2} \nabla_\mu \nabla_\alpha \nabla^\alpha \nabla_\nu h - \tfrac{1}{2} \nabla^\alpha h \nabla_\nu R_{\mu\alpha} - \tfrac{1}{2} R_\mu{}^\alpha \nabla_\nu \nabla_\alpha h \\ & - \tfrac{1}{2} \nabla_\nu \nabla_\alpha \nabla^\alpha \nabla_\mu h + \tfrac{1}{3} R \nabla_\nu \nabla_\mu h + \tfrac{2}{3} \nabla_\nu \nabla_\mu \nabla_\alpha \nabla^\alpha h. \end{aligned} \tag{22}$$

Now make subsitution

$$h_{\mu\nu} = K_{\mu\nu} + \tfrac{1}{4} h g^{(0)}_{\mu\nu} \tag{23}$$

in which it follows

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu} + \tfrac{1}{4}hg^{(0)}_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}(\tfrac{1}{4}hg^{(0)}_{\mu\nu}). \tag{24}$$

64 Terms

$$\begin{aligned}\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_\alpha{}^\gamma R_{\beta\gamma} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\
& + 2K^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{6}K_{\mu\nu}\nabla_\alpha\nabla^\alpha R + R_{\mu\beta\nu\gamma}\nabla_\alpha\nabla^\gamma K^{\alpha\beta} \\
& + R_{\mu\gamma\nu\beta}\nabla_\alpha\nabla^\gamma K^{\alpha\beta} - \tfrac{1}{3}R\nabla_\alpha\nabla_\mu K_\nu{}^\alpha - \tfrac{1}{3}R\nabla_\alpha\nabla_\nu K_\mu{}^\alpha + \tfrac{1}{3}\nabla_\alpha K_{\mu\nu}\nabla^\alpha R \\
& + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta K_\alpha{}^\beta - \nabla^\alpha K_{\mu\nu}\nabla_\beta R_\alpha{}^\beta - \nabla_\alpha K^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha K^{\alpha\beta} \\
& - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\alpha K_\mu{}^\beta - R^{\alpha\beta}\nabla_\beta\nabla_\alpha K_{\mu\nu} + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\alpha K_\nu{}^\beta \\
& + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\beta\nabla_\alpha R - K^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla^\beta K_{\mu\alpha} - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla^\beta K_{\nu\alpha} \\
& + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha K_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\mu K_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla_\nu K_\mu{}^\alpha - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\beta\nabla_\mu K_\alpha{}^\beta \\
& + R^{\alpha\beta}\nabla_\beta\nabla_\mu K_{\nu\alpha} - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\beta\nabla_\nu K_\alpha{}^\beta + R^{\alpha\beta}\nabla_\beta\nabla_\nu K_{\mu\alpha} + \nabla_\alpha R_{\nu\beta}\nabla^\beta K_\mu{}^\alpha \\
& - \nabla_\beta R_{\nu\alpha}\nabla^\beta K_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta K_\nu{}^\alpha - \nabla_\beta R_{\mu\alpha}\nabla^\beta K_\nu{}^\alpha - g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla_\beta K_\alpha{}^\gamma \\
& + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma K^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
& + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma K^{\alpha\beta} - \tfrac{1}{3}\nabla^\alpha R\nabla_\mu K_{\nu\alpha} + \nabla_\beta R_\alpha{}^\beta\nabla_\mu K_\nu{}^\alpha \\
& + \nabla_\alpha K^{\alpha\beta}\nabla_\mu R_{\nu\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha\nabla_\beta\nabla^\beta K_\nu{}^\alpha + R_\nu{}^\alpha\nabla_\mu\nabla_\beta K_\alpha{}^\beta + K^{\alpha\beta}\nabla_\mu\nabla_\beta R_{\nu\alpha} \\
& + \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla^\beta K_\nu{}^\alpha + \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla_\alpha\nabla_\nu K^{\alpha\beta} + \tfrac{1}{2}R^{\alpha\beta}\nabla_\mu\nabla_\nu K_{\alpha\beta} - \tfrac{1}{6}\nabla_\mu R_{\alpha\beta}\nabla_\nu K^{\alpha\beta} \\
& - \tfrac{1}{3}\nabla^\alpha R\nabla_\nu K_{\mu\alpha} + \nabla_\beta R_\alpha{}^\beta\nabla_\nu K_\mu{}^\alpha - \tfrac{1}{6}\nabla_\mu K^{\alpha\beta}\nabla_\nu R_{\alpha\beta} + \nabla_\alpha K^{\alpha\beta}\nabla_\nu R_{\mu\beta} \\
& - \tfrac{1}{2}\nabla_\nu\nabla_\alpha\nabla_\beta\nabla^\beta K_\mu{}^\alpha + R_\mu{}^\alpha\nabla_\nu\nabla_\beta K_\alpha{}^\beta + K^{\alpha\beta}\nabla_\nu\nabla_\beta R_{\mu\alpha} + \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla^\beta K_\mu{}^\alpha \\
& + \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla_\alpha\nabla_\mu K^{\alpha\beta} - \tfrac{7}{6}R^{\alpha\beta}\nabla_\nu\nabla_\mu K_{\alpha\beta} - \tfrac{2}{3}K^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} - \tfrac{2}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha K^{\alpha\beta}.\end{aligned} \tag{25}$$

21 Terms

$$\begin{aligned}\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = & \tfrac{1}{24}g_{\mu\nu}R^2h - \tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \tfrac{1}{6}RR_{\mu\nu}h + \tfrac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h + \tfrac{1}{24}g_{\mu\nu}h\nabla_\alpha\nabla^\alpha R \\
& - \tfrac{1}{4}h\nabla_\alpha\nabla^\alpha R_{\mu\nu} + \tfrac{1}{4}h\nabla_\mu\nabla_\alpha R_\nu{}^\alpha + \tfrac{1}{4}h\nabla_\nu\nabla_\alpha R_\mu{}^\alpha - \tfrac{1}{6}h\nabla_\nu\nabla_\mu R \\
& + \tfrac{1}{4}\nabla_\alpha\nabla^\alpha\nabla_\nu\nabla_\mu h + \tfrac{1}{8}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R - \tfrac{1}{4}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h - \tfrac{1}{4}g_{\mu\nu}\nabla^\alpha h\nabla_\beta R_\alpha{}^\beta \\
& - \tfrac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\beta\nabla^\alpha h + \tfrac{1}{4}\nabla_\alpha R_\nu{}^\alpha\nabla_\mu h - \tfrac{1}{4}\nabla_\mu\nabla_\alpha\nabla^\alpha\nabla_\nu h - \tfrac{1}{8}\nabla_\mu h\nabla_\nu R + \tfrac{1}{4}\nabla_\alpha R_\mu{}^\alpha\nabla_\nu h \\
& - \tfrac{1}{8}\nabla_\mu R\nabla_\nu h - \tfrac{1}{4}\nabla_\nu\nabla_\alpha\nabla^\alpha\nabla_\mu h + \tfrac{1}{4}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h.\end{aligned} \tag{26}$$

General, with Bianchi, no explicit covariant derivative commutation

52 Terms

$$\begin{aligned}\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - g_{\mu\nu}K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\beta\gamma} - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} \\ & + 2K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\mu\gamma\nu\beta} - \tfrac{1}{6}K_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R - K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} \\ & + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} + \tfrac{1}{2}K^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R_{\alpha\beta} - \tfrac{1}{6}K^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}R_{\alpha\beta}\end{aligned} \tag{27}$$

$$\begin{aligned} & + \tfrac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} + R_{\mu\beta\nu\gamma}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla_{\alpha}\nabla^{\gamma}K^{\alpha\beta} - \tfrac{1}{3}R\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} \\ & - \tfrac{1}{3}R\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \tfrac{1}{6}\nabla_{\alpha}K_{\mu\nu}\nabla^{\alpha}R + \tfrac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}K_{\alpha}{}^{\beta} - \nabla_{\alpha}K^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} \\ & + \tfrac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\mu}{}^{\beta} - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}K_{\mu\nu} \\ & + \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\alpha}K_{\nu}{}^{\beta} - \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\mu\alpha} - \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}K_{\nu\alpha} + \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} \\ & - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\mu}K_{\nu}{}^{\alpha} - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla_{\nu}K_{\mu}{}^{\alpha} - \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\beta}\nabla_{\mu}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\mu}K_{\nu\alpha} \\ & - \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\beta}\nabla_{\nu}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\beta}\nabla_{\nu}K_{\mu\alpha} + \nabla_{\alpha}R_{\nu\beta}\nabla^{\beta}K_{\mu}{}^{\alpha} - \nabla_{\beta}R_{\nu\alpha}\nabla^{\beta}K_{\mu}{}^{\alpha} \\ & + \nabla_{\alpha}R_{\mu\beta}\nabla^{\beta}K_{\nu}{}^{\alpha} - \nabla_{\beta}R_{\mu\alpha}\nabla^{\beta}K_{\nu}{}^{\alpha} - g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla_{\beta}K_{\alpha}{}^{\gamma} + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}K_{\alpha\beta} \\ & - R_{\mu\alpha\nu\beta}\nabla_{\gamma}\nabla^{\gamma}K^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_{\gamma}R_{\alpha\beta}\nabla^{\gamma}K^{\alpha\beta} - \nabla_{\beta}R_{\nu\alpha}\nabla_{\mu}K^{\alpha\beta} \\ & + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\mu}K_{\nu\alpha} + \tfrac{1}{2}R^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}K_{\alpha\beta} - \nabla_{\beta}R_{\mu\alpha}\nabla_{\nu}K^{\alpha\beta} + \tfrac{1}{3}\nabla_{\mu}R_{\alpha\beta}\nabla_{\nu}K^{\alpha\beta} \\ & + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\nu}K_{\mu\alpha} + \tfrac{1}{3}\nabla_{\mu}K^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta} - \tfrac{7}{6}R^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}K_{\alpha\beta} + \tfrac{1}{3}\nabla_{\nu}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta}.\end{aligned} \tag{28}$$

15 Terms

$$\begin{aligned}\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = & \tfrac{1}{24}g_{\mu\nu}R^2h - \tfrac{1}{8}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}h - \tfrac{1}{6}RR_{\mu\nu}h + \tfrac{1}{2}R^{\alpha\beta}R_{\mu\alpha\nu\beta}h + \tfrac{1}{24}g_{\mu\nu}h\nabla_{\alpha}\nabla^{\alpha}R \\ & - \tfrac{1}{4}h\nabla_{\alpha}\nabla^{\alpha}R_{\mu\nu} + \tfrac{1}{12}h\nabla_{\nu}\nabla_{\mu}R + \tfrac{1}{4}\nabla_{\alpha}\nabla^{\alpha}\nabla_{\nu}\nabla_{\mu}h - \tfrac{1}{4}\nabla_{\alpha}R_{\mu\nu}\nabla^{\alpha}h - \tfrac{1}{2}R_{\mu\alpha\nu\beta}\nabla^{\beta}\nabla^{\alpha}h \\ & + \tfrac{1}{4}\nabla^{\alpha}h\nabla_{\mu}R_{\nu\alpha} + \tfrac{1}{4}R_{\nu}{}^{\alpha}\nabla_{\mu}\nabla_{\alpha}h + \tfrac{1}{4}\nabla^{\alpha}h\nabla_{\nu}R_{\mu\alpha} + \tfrac{1}{4}R_{\mu}{}^{\alpha}\nabla_{\nu}\nabla_{\alpha}h - \tfrac{1}{4}\nabla_{\nu}\nabla_{\mu}\nabla_{\alpha}\nabla^{\alpha}h.\end{aligned} \tag{29}$$

With Covariant Derivative Commutation and Bianchi

71 Terms

$$\begin{aligned}
\delta W_{\mu\nu}(h_{\mu\nu}) = & -\tfrac{1}{6}h_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}h^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}h_\nu{}^\alpha RR_{\mu\alpha} - \tfrac{1}{2}h_\nu{}^\alpha R_{\alpha\beta}R_\mu{}^\beta \\
& - \tfrac{2}{3}h^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}h_\mu{}^\alpha RR_{\nu\alpha} + h^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \tfrac{1}{2}h_\mu{}^\alpha R_{\alpha\beta}R_\nu{}^\beta - g_{\mu\nu}h^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\
& - \tfrac{2}{3}h^{\alpha\beta}RR_{\mu\alpha\nu\beta} - h_\nu{}^\alpha R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2h^{\alpha\beta}R_\alpha{}^\gamma R_{\mu\gamma\nu\beta} + 2h^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_\mu{}^\gamma{}_\nu{}^\eta \\
& - h_\mu{}^\alpha R^{\beta\gamma}R_{\nu\beta\alpha\gamma} - \tfrac{1}{6}h_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\beta\nabla_\alpha R - h^{\alpha\beta}\nabla_\beta\nabla_\alpha R_{\mu\nu} \\
& + \tfrac{1}{2}h_\nu{}^\alpha\nabla_\beta\nabla^\beta R_{\mu\alpha} + \tfrac{1}{2}h_\mu{}^\alpha\nabla_\beta\nabla^\beta R_{\nu\alpha} + \tfrac{1}{6}g_{\mu\nu}h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\alpha\beta} - h^{\alpha\beta}\nabla_\gamma\nabla^\gamma R_{\mu\alpha\nu\beta} \\
& + \tfrac{1}{3}h^{\alpha\beta}\nabla_\nu\nabla_\mu R_{\alpha\beta} \\
& \\
& + \tfrac{1}{3}R\nabla_\alpha\nabla^\alpha h_{\mu\nu} + \tfrac{1}{2}R_\nu{}^\alpha\nabla_\alpha\nabla_\beta h_\mu{}^\beta + \tfrac{1}{2}R_\mu{}^\alpha\nabla_\alpha\nabla_\beta h_\nu{}^\beta - \tfrac{1}{6}\nabla_\alpha h_{\mu\nu}\nabla^\alpha R \\
& + \tfrac{1}{6}g_{\mu\nu}\nabla^\alpha R\nabla_\beta h_\alpha{}^\beta - \nabla_\alpha h^{\alpha\beta}\nabla_\beta R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& - R^{\alpha\beta}\nabla_\beta\nabla_\alpha h_{\mu\nu} + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\nu\nabla_\alpha h_\mu{}^\alpha \\
& - g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\gamma h_\alpha{}^\gamma + \nabla_\alpha R_{\nu\beta}\nabla^\beta h_\mu{}^\alpha + \nabla_\alpha R_{\mu\beta}\nabla^\beta h_\nu{}^\alpha + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_\gamma\nabla^\gamma h_{\alpha\beta} \\
& - 2R_{\mu\alpha\nu\beta}\nabla_\gamma\nabla^\gamma h^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla^\gamma h^{\alpha\beta} \\
& - 2\nabla_\gamma R_{\mu\alpha\nu\beta}\nabla^\gamma h^{\alpha\beta} + R_{\mu\beta\nu\gamma}\nabla^\gamma\nabla_\alpha h^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^\gamma\nabla_\alpha h^{\alpha\beta} - \nabla_\beta R_{\nu\alpha}\nabla_\mu h^{\alpha\beta} \\
& + \tfrac{1}{6}\nabla^\alpha R\nabla_\mu h_{\nu\alpha} - \tfrac{1}{3}R\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\beta h_\alpha{}^\beta + R^{\alpha\beta}\nabla_\mu\nabla_\beta h_{\nu\alpha} - \nabla_\beta R_{\mu\alpha}\nabla_\nu h^{\alpha\beta} \\
& + \tfrac{1}{3}\nabla_\mu R_{\alpha\beta}\nabla_\nu h^{\alpha\beta} + \tfrac{1}{6}\nabla^\alpha R\nabla_\nu h_{\mu\alpha} + \tfrac{1}{3}\nabla_\mu h^{\alpha\beta}\nabla_\nu R_{\alpha\beta} - \tfrac{1}{3}R\nabla_\nu\nabla_\alpha h_\mu{}^\alpha \\
& - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\beta h_\alpha{}^\beta + R^{\alpha\beta}\nabla_\nu\nabla_\beta h_{\mu\alpha} - \tfrac{2}{3}R^{\alpha\beta}\nabla_\nu\nabla_\mu h_{\alpha\beta} + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
& \\
& - \tfrac{1}{3}g_{\mu\nu}R\nabla_\alpha\nabla^\alpha h + \tfrac{2}{3}R_{\mu\nu}\nabla_\alpha\nabla^\alpha h - \tfrac{1}{12}g_{\mu\nu}\nabla_\alpha h\nabla^\alpha R + \tfrac{1}{2}\nabla_\alpha R_{\mu\nu}\nabla^\alpha h \\
& + \tfrac{1}{2}\nabla^\alpha h\nabla_\beta R_\mu{}^\beta{}_{\nu\alpha} + \tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}\nabla_\beta\nabla_\alpha h - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h - \tfrac{1}{2}\nabla^\alpha h\nabla_\mu R_{\nu\alpha} \\
& - \tfrac{1}{2}R_\nu{}^\alpha\nabla_\mu\nabla_\alpha h - \tfrac{1}{2}R_\mu{}^\alpha\nabla_\nu\nabla_\alpha h + \tfrac{1}{3}R\nabla_\nu\nabla_\mu h + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h.
\end{aligned} \tag{30}$$

59 Terms

$$
\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) = & -\tfrac{1}{6}K_{\mu\nu}R^2 + \tfrac{1}{3}g_{\mu\nu}K^{\alpha\beta}RR_{\alpha\beta} + \tfrac{1}{2}K_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}K_{\nu}{}^{\alpha}RR_{\mu\alpha} - \tfrac{1}{2}K_{\nu}{}^{\alpha}R_{\alpha\beta}R_{\mu}{}^{\beta} \\
& - \tfrac{2}{3}K^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu} + \tfrac{1}{3}K_{\mu}{}^{\alpha}RR_{\nu\alpha} + K^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta} - \tfrac{1}{2}K_{\mu}{}^{\alpha}R_{\alpha\beta}R_{\nu}{}^{\beta} - g_{\mu\nu}K^{\alpha\beta}R^{\gamma\eta}R_{\alpha\gamma\beta\eta} \\
& - \tfrac{2}{3}K^{\alpha\beta}RR_{\mu\alpha\nu\beta} - K_{\nu}{}^{\alpha}R^{\beta\gamma}R_{\mu\beta\alpha\gamma} + 2K^{\alpha\beta}R_{\alpha}{}^{\gamma}R_{\mu\gamma\nu\beta} + 2K^{\alpha\beta}R_{\alpha\gamma\beta\eta}R_{\mu}{}^{\gamma}{}_{\nu}{}^{\eta} \\
& - K_{\mu}{}^{\alpha}R^{\beta\gamma}R_{\nu\beta\alpha\gamma} - \tfrac{1}{6}K_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}R + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R - K^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}R_{\mu\nu} \\
& + \tfrac{1}{2}K_{\nu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}R_{\mu\alpha} + \tfrac{1}{2}K_{\mu}{}^{\alpha}\nabla_{\beta}\nabla^{\beta}R_{\nu\alpha} + \tfrac{1}{6}g_{\mu\nu}K^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\alpha\beta} - K^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}R_{\mu\alpha\nu\beta} \\
& + \tfrac{1}{3}K^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}R_{\alpha\beta} \\
& \\
& + \tfrac{1}{3}R\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} + \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\alpha}\nabla_{\beta}K_{\mu}{}^{\beta} + \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\alpha}\nabla_{\beta}K_{\nu}{}^{\beta} - \tfrac{1}{6}\nabla_{\alpha}K_{\mu\nu}\nabla^{\alpha}R \\
& + \tfrac{1}{6}g_{\mu\nu}\nabla^{\alpha}R\nabla_{\beta}K_{\alpha}{}^{\beta} - \nabla_{\alpha}K^{\alpha\beta}\nabla_{\beta}R_{\mu\nu} + \tfrac{1}{3}g_{\mu\nu}R\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} - \tfrac{2}{3}R_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} \\
& - R^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}K_{\mu\nu} + \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\alpha}\nabla^{\alpha}K_{\mu\nu} - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\mu}\nabla_{\alpha}K_{\nu}{}^{\alpha} - \tfrac{1}{2}\nabla_{\beta}\nabla^{\beta}\nabla_{\nu}\nabla_{\alpha}K_{\mu}{}^{\alpha} \\
& - g_{\mu\nu}R^{\alpha\beta}\nabla_{\beta}\nabla_{\gamma}K_{\alpha}{}^{\gamma} + \nabla_{\alpha}R_{\nu\beta}\nabla^{\beta}K_{\mu}{}^{\alpha} + \nabla_{\alpha}R_{\mu\beta}\nabla^{\beta}K_{\nu}{}^{\alpha} + \tfrac{2}{3}g_{\mu\nu}R^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}K_{\alpha\beta} \\
& - 2R_{\mu\alpha\nu\beta}\nabla_{\gamma}\nabla^{\gamma}K^{\alpha\beta} + \tfrac{1}{6}g_{\mu\nu}\nabla_{\gamma}\nabla^{\gamma}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta} + \tfrac{1}{3}g_{\mu\nu}\nabla_{\gamma}R_{\alpha\beta}\nabla^{\gamma}K^{\alpha\beta} \\
& - 2\nabla_{\gamma}R_{\mu\alpha\nu\beta}\nabla^{\gamma}K^{\alpha\beta} + R_{\mu\beta\nu\gamma}\nabla^{\gamma}\nabla_{\alpha}K^{\alpha\beta} + R_{\mu\gamma\nu\beta}\nabla^{\gamma}\nabla_{\alpha}K^{\alpha\beta} - \nabla_{\beta}R_{\nu\alpha}\nabla_{\mu}K^{\alpha\beta} \\
& + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\mu}K_{\nu\alpha} - \tfrac{1}{3}R\nabla_{\mu}\nabla_{\alpha}K_{\nu}{}^{\alpha} - \tfrac{1}{2}R_{\nu}{}^{\alpha}\nabla_{\mu}\nabla_{\beta}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\mu}\nabla_{\beta}K_{\nu\alpha} \\
& - \nabla_{\beta}R_{\mu\alpha}\nabla_{\nu}K^{\alpha\beta} + \tfrac{1}{3}\nabla_{\mu}R_{\alpha\beta}\nabla_{\nu}K^{\alpha\beta} + \tfrac{1}{6}\nabla^{\alpha}R\nabla_{\nu}K_{\mu\alpha} + \tfrac{1}{3}\nabla_{\mu}K^{\alpha\beta}\nabla_{\nu}R_{\alpha\beta} \\
& - \tfrac{1}{3}R\nabla_{\nu}\nabla_{\alpha}K_{\mu}{}^{\alpha} - \tfrac{1}{2}R_{\mu}{}^{\alpha}\nabla_{\nu}\nabla_{\beta}K_{\alpha}{}^{\beta} + R^{\alpha\beta}\nabla_{\nu}\nabla_{\beta}K_{\mu\alpha} - \tfrac{2}{3}R^{\alpha\beta}\nabla_{\nu}\nabla_{\mu}K_{\alpha\beta} \\
& + \tfrac{1}{3}\nabla_{\nu}\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}K^{\alpha\beta}. \qquad (31)
\end{aligned}
$$

$$
\delta W_{\mu\nu}(\tfrac{h}{4}g^{(0)}_{\mu\nu}) = -\frac{1}{4}hW_{\mu\nu}(g^{(0)}_{\mu\nu}) \qquad (32)
$$

Weyl Tensor Flat

$$
\begin{aligned}
\delta C_{\lambda\mu\nu\kappa} = & \tfrac{1}{4}\eta_{\mu\nu}\partial_{\alpha}\partial^{\alpha}K_{\kappa\lambda} - \tfrac{1}{4}\eta_{\lambda\nu}\partial_{\alpha}\partial^{\alpha}K_{\kappa\mu} - \tfrac{1}{4}\eta_{\kappa\mu}\partial_{\alpha}\partial^{\alpha}K_{\lambda\nu} + \tfrac{1}{4}\eta_{\kappa\lambda}\partial_{\alpha}\partial^{\alpha}K_{\mu\nu} \\
& - \tfrac{1}{6}\eta_{\kappa\mu}\eta_{\lambda\nu}\partial_{\beta}\partial_{\alpha}K^{\alpha\beta} + \tfrac{1}{6}\eta_{\kappa\lambda}\eta_{\mu\nu}\partial_{\beta}\partial_{\alpha}K^{\alpha\beta} - \tfrac{1}{4}\eta_{\mu\nu}\partial_{\kappa}\partial_{\alpha}K_{\lambda}{}^{\alpha} + \tfrac{1}{4}\eta_{\lambda\nu}\partial_{\kappa}\partial_{\alpha}K_{\mu}{}^{\alpha} \\
& + \tfrac{1}{2}\partial_{\kappa}\partial_{\mu}K_{\lambda\nu} - \tfrac{1}{4}\eta_{\mu\nu}\partial_{\lambda}\partial_{\alpha}K_{\kappa}{}^{\alpha} + \tfrac{1}{4}\eta_{\kappa\mu}\partial_{\lambda}\partial_{\alpha}K_{\nu}{}^{\alpha} - \tfrac{1}{2}\partial_{\lambda}\partial_{\kappa}K_{\mu\nu} + \tfrac{1}{2}\partial_{\lambda}\partial_{\nu}K_{\kappa\mu} \\
& + \tfrac{1}{4}\eta_{\lambda\nu}\partial_{\mu}\partial_{\alpha}K_{\kappa}{}^{\alpha} - \tfrac{1}{4}\eta_{\kappa\lambda}\partial_{\mu}\partial_{\alpha}K_{\nu}{}^{\alpha} - \tfrac{1}{2}\partial_{\mu}\partial_{\nu}K_{\kappa\lambda} + \tfrac{1}{4}\eta_{\kappa\mu}\partial_{\nu}\partial_{\alpha}K_{\lambda}{}^{\alpha} - \tfrac{1}{4}\eta_{\kappa\lambda}\partial_{\nu}\partial_{\alpha}K_{\mu}{}^{\alpha}. \qquad (33)
\end{aligned}
$$

Applying Gauge

Now we apply the gauge condition

$$
\nabla_{\nu}K^{\mu\nu} = 4\Omega^{-1}K^{\mu\nu}\partial_{\nu}\Omega \qquad (34)
$$

or the equivalent gauge covariant in $K_{\mu\nu}$

$$
\eta^{\alpha\beta}\partial_{\alpha}K_{\mu\beta} = 2\Omega^{-1}\eta^{\alpha\beta}K_{\mu\beta}\partial_{\alpha}\Omega. \qquad (35)
$$

and $\delta W_{\mu\nu}$ reduces to

$$
\begin{aligned}
\delta W_{\mu\nu}(K_{\mu\nu}) &= -48\Omega^{-7}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma K_{\mu\nu} + 24\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\rho\partial_\beta\Omega\partial_\sigma K_{\mu\nu} \\
&+ 60\Omega^{-8}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\rho\Omega\partial_\sigma\Omega - 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta K_{\mu\nu} \\
&+ 6\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\rho\partial_\alpha\Omega\partial_\sigma\partial_\beta\Omega + 12\Omega^{-6}\eta^{\alpha\rho}\eta^{\beta\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho K_{\mu\nu} \\
&+ 6\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho K_{\mu\nu} - 2\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho K_{\mu\nu} \\
&+ 12\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\beta K_{\mu\nu}\partial_\sigma\partial_\rho\Omega - 48\Omega^{-7}\eta^{\alpha\rho}\eta^{\beta\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega \\
&- 24\Omega^{-7}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega\partial_\sigma\partial_\rho\Omega + 3\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\beta\partial_\alpha\Omega\partial_\sigma\partial_\rho\Omega \\
&- 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta K_{\mu\nu} - 4\Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\alpha K_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\Omega \\
&+ 12\Omega^{-6}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\alpha\Omega\partial_\sigma\partial_\rho\partial_\beta\Omega + \tfrac{1}{2}\Omega^{-4}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha K_{\mu\nu} \\
&- \Omega^{-5}\eta^{\alpha\beta}\eta^{\rho\sigma}K_{\mu\nu}\partial_\sigma\partial_\rho\partial_\beta\partial_\alpha\Omega \\
&= \frac{1}{2}\Omega^{-2}\eta^{\sigma\rho}\eta^{\alpha\beta}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta(\Omega^{-2}K_{\mu\nu}) \qquad (36)
\end{aligned}
$$