Covariant Green's Functions v2

1 Summary of Poisson's Curved Space Green's Functions

1.1 Two Point Bitensors

Bitensor fields are tensors of two spacetime points. Primed indices denoted basis vectors with respect to the x' coordinate system, while unprimed indices denoted coordinates with respect to the x coordinate system.

Take $T_{\alpha\beta'}(x, x')$ for example. Such a tensor has transformation rule under $x \to \tilde{x}$

$$T_{\alpha\beta'} \to T_{\tilde{\alpha}\beta'} = \frac{\partial x^{\rho}}{\partial \tilde{x}^{\alpha}} T_{\alpha\beta'} \tag{1}$$

and under $x' \to \hat{x}$,

$$T_{\alpha\beta'} \to T_{\alpha\hat{\beta}'} = \frac{\partial x'^{\rho}}{\partial \hat{x}'^{\beta}} T_{\alpha\beta'}. \tag{2}$$

In Poisson's construction of Green's functions, the bitensors are evaluated on the unique geodesic defined as the set of points x linked to x' that belong within the normal convex neighborhood of x'. (Given smooth manifold with affine connection, the local existence and uniqueness theorem states there exists a unique geodesic connecting two points on the manifold).

1.2 Parallel Propagator

On a geodesic linking x to x' parametrized by λ, introduce the orthonormal basis vectors e^{μ}_{a}, which satisfy

$$g_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b} = \eta_{ab}, \tag{3}$$

and which obey

$$\frac{De^{\mu}_{a}}{d\lambda} = 0. \tag{4}$$

These orthonormal basis vectors which have both coordinate and Lorentz index are equivalent to tetrads (vierbeins).

The parallel propagator which takes a vector at point x and propagates it along the manifold to point x' is defined as

$$g^{\alpha}{}_{\alpha'}(x, x') = e^{\alpha}_{a}(x) e^{a}_{\alpha'}(x'). \tag{5}$$

To motivate this, we may decompose $A^{\mu}(z)$ according to

$$A^{\mu} = A^{a} e^{\mu}_{a}, \tag{6}$$

with components obeying

$$A^{a} = A^{\mu} e^{a}_{\mu} = A^{\mu'} e^{a}_{\mu'}. \tag{7}$$

Under parallel transport, by definition, the components A^a are held constant. Hence we may express

$$\begin{aligned} A^\mu(x) &= (A^{\alpha'}(x')e^a_{\alpha'}(x'))e^\mu_a(x) \\ &= g^\mu{}_{\alpha'}(x,x')A^{\alpha'}(x') \end{aligned} \tag{8}$$

Hence the bitensor $g^\mu{}_{\alpha'}(x,x')$ acts a the parallel propagator transporting A'^μ from x' to x. We can similarly show the inverse relation, which takes A^μ from x to x',

$$A^{\mu'}(x') = g^{\mu'}{}_\alpha(x',x)A^\alpha(x). \tag{9}$$

From (5) we have the identities

$$\begin{aligned} g^\alpha{}_{\alpha'}g^{\alpha'}{}_\beta &= \delta^\alpha{}_\beta, \qquad g^{\alpha'}{}_\alpha g^{\alpha}{}'_\beta = \delta^{\alpha'}{}_{\beta'} \\ g_\alpha{}^{\alpha'}(x,x') &= g^{\alpha'}{}_\alpha(x',x), \qquad g_{\alpha'}{}^\alpha(x',x) = g^\alpha{}_{\alpha'}(x,x') \end{aligned} \tag{10}$$

When evaluated at coincidence (i.e. $\lim_{x\to x'} T(x,x') \equiv [T]$)

$$[g^\alpha{}_{\beta'}] = \delta^{\alpha'}{}_{\beta'}. \tag{11}$$

1.3 Dirac Distribution in Curved Space

Poisson defines the invariant Dirac functional as

$$\int_V f(x)\delta_4(x,x')\sqrt{-g}d^4x = f(x'), \qquad \int_{V'} f(x')\delta_4(x,x')\sqrt{-g'}d^4x' = f(x), \tag{12}$$

with $x \in V$ and $x' \in V'$. This implies the various equivalent forms:

$$\delta_4(x,x') = \frac{\delta_4(x-x')}{(-g)^{1/2}} = \frac{\delta_4(x-x')}{(-g')^{1/2}} = (gg')^{1/4}\delta_4(x-x'), \tag{13}$$

where

$$\delta_4(x-x') = \delta(x_0-x_0')\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3') \tag{14}$$

and

$$(-g)^{1/2} = (-\det[g_{\mu\nu}(x)])^{1/2}, \qquad (-g')^{1/2} = (-\det[g'_{\mu\nu}(x')])^{1/2}. \tag{15}$$

We can also show that the determinant of the parallel propagator obeys

$$\det[g^\alpha{}_{\alpha'}] = \frac{(-g')^{1/2}}{(-g)^{1/2}} \tag{16}$$

2 TT Curved Space Decomposition

From (14.41) in MAP decomposition paper, the equation governing W_μ must be

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]W^\alpha = \nabla^\alpha h_{\alpha\nu} - \frac{1}{D-1}R_{\nu\alpha}\nabla^\alpha\int g^{1/2}D(x,x')h. \tag{17}$$

We see that the requisite Green's function that solves W_α is a bi-tensor $D^{\alpha\gamma'}$, defined according to

$$\left[g_{\nu\alpha}\nabla_\beta\nabla^\beta + \left(\frac{D-2}{D}\right)\nabla_\nu\nabla_\alpha - R_{\nu\alpha}\right]D^{\alpha\gamma'}(x,x') = g_\nu{}^{\gamma'}(x,x')(-g')^{-1/2}\delta^{(D)}(x-x'), \tag{18}$$

where

$$g_\nu{}^{\gamma'} = e^a_\nu(x)e^{\gamma'}_a(x), \qquad (-g)^{1/2} = (-\det[g_{\mu\nu}(x)])^{1/2}, \qquad (-g')^{1/2} = (-\det[g'_{\mu\nu}(x')])^{1/2}. \tag{19}$$

Hence, W_ν takes the form

$$W_\nu = \int d^4x'(-g')^{1/2} D_\nu{}^{\sigma'} \left[\nabla^{\rho'} h_{\sigma'\rho'} - \frac{1}{D-1} R_{\sigma'\rho'} \nabla^{\rho'} \int d^4x''(-g'')^{1/2} D(x',x'')h\right]. \tag{20}$$

Schematically, we can see that given an arbitrary differential operator $\mathcal{L}$ acting on a vector,

$$\mathcal{L}_{\nu\alpha}(x) W^\alpha(x) = V_\nu(x), \tag{21}$$

and a bitensor Greens function $D^{\alpha\gamma'}$ that obeys

$$\mathcal{L}_{\nu\alpha}(x) D^{\alpha\gamma'}(x,x') = g_\nu{}^{\gamma'}(x,x')(-g')^{-1/2}\delta^{(D)}(x-x'), \tag{22}$$

then the action of $\mathcal{L}$ upon the integral solution takes the form

$$\begin{aligned} \mathcal{L}_{\nu\alpha}(x) W^\alpha &= \mathcal{L}_{\nu\alpha}(x) \int d^4x'(-g')^{1/2} D^{\alpha\gamma'}(x,x') V_{\gamma'}(x') \\ &= \int d^4x' \delta^{(D)}(x-x') \qquad \underbrace{\left(g_\nu{}^{\gamma'}(x,x') V_{\gamma'}(x')\right)}_{\text{Parallel propagation of } V_{\gamma'}(x') \text{ to } V_\nu(x)} \\ &= V_\nu(x). \end{aligned} \tag{23}$$

References

Poisson, Eric, Adam Pound, and Ian Vega. "The motion of point particles in curved spacetime." Living Reviews in Relativity 14.1 (2011): 7.

DeWitt, Bryce S., and Robert W. Brehme. "Radiation damping in a gravitational field." Annals of Physics 9.2 (1960): 220-259.

Allen, Bruce, and Michael Turyn. "An evaluation of the graviton propagator in de Sitter space." Nuclear Physics B 292 (1987): 813-852.

Fröb, Markus B., and Mojtaba Taslimi Tehrani. "Green's functions and Hadamard parametrices for vector and tensor fields in general linear covariant gauges." Physical Review D 97.2 (2018): 025022.

Hobbs, J. M. "A vierbein formalism of radiation damping." Annals of Physics 47.1 (1968): 141-165.