Gravitational Invariants

1 Summary

In a Minkowski background, the two gravitational gauge invariants are δR and $\delta R_{\mu\nu}$. With the Bianchi identities, this yields 6 independent gauge invariants, which are taken as δG and $\delta G^{T\theta}_{\mu\nu}$.

In a dS_4, δR, $\delta R_{\mu\nu}$, and $\delta G_{\mu\nu}$ are not gauge invariant. However, we may construct a gravitational gauge invariant $\Delta_{\mu\nu} = \delta G_{\mu\nu} - 3kh_{\mu\nu}$. Being conserved, the 6 components are analogously Δ and $\Delta^{T\theta}_{\mu\nu}$.

By virtue of $\delta G_{\mu\nu} = \delta T_{\mu\nu}$, Einstein gravity does not impose any equation of motion upon the gravitational gauge invariants - it merely equates gravitational gauge invariants to matter gauge invariants.

In conformal gravity, the gravitational invariants are dynamic. In a Minkowski background, the gravitational invariant obeys

$$\begin{aligned} \delta W_{\mu\nu} &= \nabla^2 \delta G^{T\theta}_{\mu\nu} \\ \rightarrow \nabla^2 \delta G^{T\theta}_{\mu\nu} &= \delta T_{\mu\nu} \end{aligned} \tag{1.1}$$

In a dS_4 background, we have determined

$$\begin{aligned} \delta W_{\mu\nu} &= (\nabla^2 - 4k)\Delta^{T\theta}_{\mu\nu} \\ \rightarrow (\nabla^2 - 4k)\Delta^{T\theta}_{\mu\nu} &= \delta T_{\mu\nu} \end{aligned} \tag{1.2}$$

2 Minkowski

$$
\begin{aligned}
ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
\delta W_{\mu\nu} &= \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\
&\quad -\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
\delta G_{\mu\nu} &= \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\mu h \\
\delta G &= \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h \\
\delta G^{T\theta}_{\mu\nu} &= \delta G_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\delta G + \frac{1}{3}\nabla_\mu\nabla_\nu\int D\delta G \\
\nabla^2\delta G^{T\theta}_{\mu\nu} &= \nabla^2\delta G_{\mu\nu} + \frac{1}{3}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2\right]\delta G \\
\nabla^2\delta G^{T\theta}_{\mu\nu} &= \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha \\
&\quad -\tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&= \delta W_{\mu\nu}
\end{aligned} \tag{2.1}
$$

2.1 Gauge Transformation

Under $x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x)$,

$$
\begin{aligned}
\delta\bar{W}_{\mu\nu} &= \delta W_{\mu\nu} + W^{(0)}_{\rho\mu}g^{\lambda\rho}\nabla_\nu\epsilon_\lambda + W^{(0)}_{\rho\nu}g^{\lambda\rho}\nabla_\mu\epsilon_\lambda + \epsilon^\lambda\nabla_\lambda W^{(0)}_{\mu\nu} \\
&= 0 \\
\delta\bar{G}_{\mu\nu} &= \delta G_{\mu\nu} + G^{(0)}_{\rho\mu}g^{\lambda\rho}\nabla_\nu\epsilon_\lambda + G^{(0)}_{\rho\nu}g^{\lambda\rho}\nabla_\mu\epsilon_\lambda + \epsilon^\lambda\nabla_\lambda G^{(0)}_{\mu\nu} \\
&= 0
\end{aligned} \tag{2.2}
$$

3 $\mathbf{dS}_4$

$$
\begin{aligned}
G^{(0)}_{\mu\nu} &= 3kg_{\mu\nu} \\
R^{(0)}_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\
R^{(0)}_{\mu\kappa} &= -3kg_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\
R^{(0)} &= -12k \\
\\
ds^2 &= (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\
\\
\delta W_{\mu\nu} &= 4k^2 h_{\mu\nu} - k^2 g_{\mu\nu}h - 3k\nabla_\alpha\nabla^\alpha h_{\mu\nu} + \tfrac{1}{2}kg_{\mu\nu}\nabla_\alpha\nabla^\alpha h + kg_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&\quad + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}k\nabla_\mu\nabla_\alpha h_\nu{}^\alpha \\
&\quad - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}k\nabla_\nu\nabla_\alpha h_\mu{}^\alpha - \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + k\nabla_\nu\nabla_\mu h \\
&\quad + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta}. \\
\\
\delta G_{\mu\nu} &= 2kh_{\mu\nu} - \tfrac{1}{2}kg_{\mu\nu}h + \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h + \tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha \\
&\quad - \tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha + \tfrac{1}{2}\nabla_\nu\nabla_\mu h \\
\\
\delta G &= \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla_\alpha\nabla^\alpha h \\
\\
\Delta_{\mu\nu} &= \delta G_{\mu\nu} - 3kh_{\mu\nu} \\
\\
\Delta &= \delta G - 3kh \\
\\
\Delta^{T\theta}_{\mu\nu} &= \Delta_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\Delta + \frac{1}{3}\left(\nabla_\mu\nabla_\nu + kg_{\mu\nu}\right)\int D\Delta \\
(\nabla^2 - 4k)\Delta^{T\theta}_{\mu\nu} &= (\nabla^2 - 4k)\Delta_{\mu\nu} + \frac{1}{3}\left[\nabla_\mu\nabla_\nu + 3kg_{\mu\nu} - g_{\mu\nu}\nabla^2\right]\Delta \\
(\nabla^2 - 4k)\Delta^{T\theta}_{\mu\nu} &= 4k^2 h_{\mu\nu} - k^2 g_{\mu\nu}h - 3k\nabla_\alpha\nabla^\alpha h_{\mu\nu} + \tfrac{1}{2}kg_{\mu\nu}\nabla_\alpha\nabla^\alpha h + kg_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&\quad + \tfrac{1}{2}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_\beta\nabla^\beta\nabla_\alpha\nabla^\alpha h + \tfrac{1}{6}g_{\mu\nu}\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla_\alpha h^{\alpha\beta} - \tfrac{1}{2}k\nabla_\mu\nabla_\alpha h_\nu{}^\alpha \\
&\quad - \tfrac{1}{2}\nabla_\mu\nabla_\beta\nabla^\beta\nabla_\alpha h_\nu{}^\alpha - \tfrac{1}{2}k\nabla_\nu\nabla_\alpha h_\mu{}^\alpha - \tfrac{1}{2}\nabla_\nu\nabla_\beta\nabla^\beta\nabla_\alpha h_\mu{}^\alpha + k\nabla_\nu\nabla_\mu h \\
&\quad + \tfrac{1}{6}\nabla_\nu\nabla_\mu\nabla_\alpha\nabla^\alpha h + \tfrac{1}{3}\nabla_\nu\nabla_\mu\nabla_\beta\nabla_\alpha h^{\alpha\beta} \\
&= \delta W_{\mu\nu} \qquad (3.1)
\end{aligned}
$$

3.1 Gauge Transformation

Background:

$$G^{(0)}_{\mu\nu} = 3kg_{\mu\nu} \qquad (3.2)$$

Gravitational Invariant:

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} - 3kh_{\mu\nu} \qquad (3.3)$$

Under $x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x)$,

$$
\begin{aligned}
\delta\bar{W}_{\mu\nu} &= \delta W_{\mu\nu} + W^{(0)}_{\rho\mu} g^{\lambda\rho} \nabla_\nu \epsilon_\lambda + W^{(0)}_{\rho\nu} g^{\lambda\rho} \nabla_\mu \epsilon_\lambda + \epsilon^\lambda \nabla_\lambda W^{(0)}_{\mu\nu} \\
&= 0 \\
\\
\delta\bar{G}_{\mu\nu} &= \delta G_{\mu\nu} + G^{(0)}_{\rho\mu} g^{\lambda\rho} \nabla_\nu \epsilon_\lambda + G^{(0)}_{\rho\nu} g^{\lambda\rho} \nabla_\mu \epsilon_\lambda + \epsilon^\lambda \nabla_\lambda G^{(0)}_{\mu\nu} \\
&= \delta G_{\mu\nu} + 3k(\nabla_\nu \epsilon_\mu + \nabla_\mu \epsilon_\nu) \\
\\
\bar{\Delta}_{\mu\nu} &= \delta G_{\mu\nu} + 3k(\nabla_\nu \epsilon_\mu + \nabla_\mu \epsilon_\nu) - 3k h_{\mu\nu} - 3k(\nabla_\nu \epsilon_\mu + \nabla_\mu \epsilon_\nu) \\
&= \Delta_{\mu\nu}
\end{aligned}
\tag{3.4}
$$

4 Conformal to Flat

$$
\begin{aligned}
ds^2 &= (\tilde{g}_{\mu\nu} + \delta\tilde{g}_{\mu\nu}) dx^\mu dx^\nu \\
&= \Omega^2(x)(\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \\
\\
\delta\tilde{W}_{\mu\nu} &= \Omega^{-2} \delta W_{\mu\nu} \\
&= \Omega^{-2} \nabla^2 \delta G^{T\theta}_{\mu\nu}
\end{aligned}
\tag{4.1}
$$

To be continued.

Appendix A $h^{T\theta}_{\mu\nu}$

A.1 Minkowski

$$
\begin{aligned}
h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\
&+ \frac{2-D}{D-1}\nabla_\mu\nabla_\nu \int D\nabla^\sigma W_\sigma - \frac{1}{D-1}\nabla_\mu\nabla_\nu\int Dh
\end{aligned} \tag{A.1}
$$

with scalar Green's function

$$
\nabla^\sigma\nabla_\sigma D(x,x') = \delta^4(x-x'). \tag{A.2}
$$

Taking the trace of (A.1), we find

$$
h = h. \tag{A.3}
$$

As for the transverse component we find a condition upon vector W_ν

$$
\nabla^\sigma h_{\nu\sigma} = \nabla^\sigma\nabla_\sigma W_\nu. \tag{A.4}
$$

The particular integral solution for W_ν is

$$
W_\nu = \int D\nabla^\sigma h_{\mu\sigma}. \tag{A.5}
$$

If decompose a $T_{\mu\nu}$ that is apriori transverse, then with $W_\mu = 0$ the decomposition reduces to

$$
T^{T\theta}_{\mu\nu} = T_{\mu\nu} - \frac{g_{\mu\nu}}{D-1}T + \frac{1}{D-1}\nabla_\mu\nabla_\nu\int DT \tag{A.6}
$$

To bring into a local form, we apply the box operator

$$
\nabla^2 T^{T\theta}_{\mu\nu} = \nabla^2 T_{\mu\nu} + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2\right]T \tag{A.7}
$$

A.2 Maximally Symmetric

Curvature Tensors:

$$
\begin{aligned}
R_{\lambda\mu\nu\kappa} &= k(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}) \\
R_{\mu\kappa} &= k(1-D)g_{\mu\kappa} = \frac{R}{D}g_{\mu\kappa} \\
R &= kD(1-D)
\end{aligned} \tag{A.8}
$$

Covariant Commutation:

$$
\begin{aligned}
[\nabla^\sigma\nabla_\nu]W_\sigma &= -R_\nu{}^\sigma W_\sigma = -\frac{R}{D}W_\nu \\
[\nabla^\mu\nabla_\mu, \nabla_\nu]V &= -R_\nu{}^\mu\nabla_\mu V = -\frac{R}{D}\nabla_\nu V \\
[\nabla^2, \nabla_\mu\nabla_\nu]V &= \frac{2g_{\mu\nu}R}{D(D-1)}\nabla^2 V - \frac{2R}{D-1}\nabla_\mu\nabla_\nu V
\end{aligned} \tag{A.9}
$$

Decomposition:

$$
\begin{aligned}
h_{\mu\nu} &= h^{T\theta}_{\mu\nu} + \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{g_{\mu\nu}}{D-1}(\nabla^\sigma W_\sigma - h) \\
&+ \frac{2-D}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int D\nabla^\sigma W_\sigma - \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int Dh
\end{aligned} \tag{A.10}
$$

with scalar Green's function

$$\left(\nabla^\sigma\nabla_\sigma - \frac{R}{D-1}\right) D(x,x') \quad = \quad g^{-1/2}\delta^4(x-x'). \tag{A.11}$$

Taking the trace of (A.10), we find

$$h \quad = \quad h. \tag{A.12}$$

As for the transverse component we find, upon applying covariant commutations (A.9), a condition upon vector W_ν

$$\nabla^\sigma h_{\nu\sigma} \quad = \quad \nabla^\sigma\nabla_\sigma W_\nu. \tag{A.13}$$

With the box operator mixing indices of W_ν, the particular integral solution for W_ν involves a bi-tensor Green's function $F_{\sigma\rho'}$ which obeys

$$\nabla^\alpha\nabla_\alpha F_{\sigma\rho'}(x,x') \quad = \quad g_{\sigma\rho'}g^{-1/2}\delta^4(x-x') \tag{A.14}$$

$$W_\nu \quad = \quad \int F_\nu{}^{\rho'}\nabla^{\sigma'}h_{\rho'\sigma'}. \tag{A.15}$$

If a tensor $T_{\mu\nu}$ is apriori transverse, then we again may set $W_\mu = 0$ to find for a conserved tensor, the decomposition

$$T^{T\theta}_{\mu\nu} \quad = \quad T_{\mu\nu} - \frac{g_{\mu\nu}}{D-1}T + \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu - \frac{g_{\mu\nu}R}{D(D-1)}\right)\int DT. \tag{A.16}$$

We see that to retain transversality, we cannot simply just extract the trace in a trivial way.

To form a second order equation for $T^{T\theta}_{\mu\nu}$ that is absent of the non-local integral, we need to apply a specific box operator. Acting upon a scalar V, the desired operator is given below with commutation relation

$$\left(\nabla^2 + \frac{R}{D-1}\right)\left(\nabla_\mu\nabla_\nu - \frac{Rg_{\mu\nu}}{D(D-1)}\right)V \quad = \quad \left(\nabla_\mu\nabla_\nu + \frac{Rg_{\mu\nu}}{D(D-1)}\right)\left(\nabla^2 - \frac{R}{D-1}\right)V, \tag{A.17}$$

which may be verified using (A.9).

Now applying this operator to (A.16), we find

$$\begin{aligned}\left(\nabla^2 + \frac{R}{D-1}\right)T^{T\theta}_{\mu\nu} \quad &= \quad \left(\nabla^2 + \frac{R}{D-1}\right)T_{\mu\nu} - \frac{g_{\mu\nu}}{D-1}\left(\nabla^2 + \frac{R}{D-1}\right)T \\ &\quad + \frac{1}{D-1}\left(\nabla_\mu\nabla_\nu + \frac{g_{\mu\nu}R}{D(D-1)}\right)T.\end{aligned} \tag{A.18}$$

Expressed in terms of curvature constant $R = -kD(D-1)$, the above becomes

$$(\nabla^2 - Dk)T^{T\theta}_{\mu\nu} \quad = \quad (\nabla^2 - Dk)T_{\mu\nu} + \frac{1}{D-1}\left[\nabla_\mu\nabla_\nu + (D-1)kg_{\mu\nu} - g_{\mu\nu}\nabla^2\right]T. \tag{A.19}$$