General Relativity
HW 5

Matthew Phelps

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1. (a) As we know, we may reduce the problem of two orbiting masses to an effective one body problem with reduced mass

$$\mu = \frac{mM}{m+M}$$

with radial separation r and coordinates relative to the center of mass

$$r_1 = \frac{m}{M+m}r, \qquad r_2 = \frac{M}{m+M}r.$$

The potential is then a function of the separation r only, $V(r)$. Due to this spherical symmetry, the total angular momentum vector is conserved and it ensues that motion takes place in a plane. The lagrangian for said system is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Here I will set $\mu = m$ until the end of the problem. We recall the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

Here θ is cyclic and thus the orbital angular momentum is conserved

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}mr^2\dot{\theta} = 0$$

$$\Rightarrow mr^2\dot{\theta} = l.$$

In terms of a force $f(r) = -\frac{\partial V(r)}{\partial r}$ the radial Euler-Lagrange equation yields

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r).$$

At this point, we may use our relation for the angular momentum to change from time derivatives to those with respect to θ

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}.$$

Performing a substitution of $u = 1/r$, the radial equation is now

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

or

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right).$$

Integrating this equation for θ we have

$$\theta = \theta_0 \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}.$$

For the newtonian potential $V = -\frac{k}{r}$ this is

$$\theta = \theta' \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}.$$

Evaluating the integral,

$$\theta = \theta' - \cos^{-1}\left[\frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}\right].$$

Going back to $r = 1/u$ this gives the equation of orbit

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \theta')\right).$$

The general equation for an ellipse takes the form of

$$\frac{1}{r} = C[1 + e\cos(\theta - \theta')]$$

with eccentricity e. Thus we immediately identify the eccentricity of the orbit equation as

$$\boxed{e = \sqrt{1 + \frac{2El^2}{mk^2}}}.$$

The closest distance of approach is given by minimum r, and thus by the semi-minor axis. For an ellipse, the semi-minor b is related to the semi-major a by

$$b = a\sqrt{1 - e^2}.$$

Thus we seek to solve for a. This can be accomplished by looking at the total energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} - \frac{k}{r}.$$

The turning points are points at which the radial velocity is zero, i.e. $\dot{r} = 0$

$$E - \frac{l^2}{2mr^2} + \frac{k}{r} = 0.$$

There are two roots to this equation r_1 and r_2. The average of these then gives the semi major axis a

$$a = \frac{|r_1 + r_2|}{2} = \frac{k}{2E}.$$

Now using this we may solve for the distance of closest approach $l_0 = b$

$$\boxed{l_0 = \frac{k}{2E}\sqrt{1 - e^2} = \sqrt{\frac{l^2}{2mE}}}.$$

To find the period, we may use some of the properties of an ellipse. From conservation of angular momentum, the area $dA = \frac{1}{2}r^2 d\theta$ is conserved over time

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}$$

The total area of the ellipse $A = \pi ab.$ is given by the time τ to complete one orbit

$$\int_0^\tau \frac{dA}{dt} dt = A = \frac{l\tau}{2m}.$$

Recalling our relation earlier representing the semi-minor in terms of the eccentricity and semi-major, this leads us to

$$\tau = \frac{2m}{l} \pi a^2 \sqrt{1 - e^2} = 2\pi a^{3/2} \sqrt{\frac{m}{k}}.$$

Thus the orbital period is

$$\boxed{\tau = \pi k^2 \sqrt{\frac{m}{2E^3}}}$$

where $k = GmM$ and $\mu \equiv m$.

(b) Recall the quadropole moment defined as

$$D_{ij} = \int d^3x \; x^i x^j T^{00}.$$

In our binary system, we have

$$T^{00} = M\delta(\vec{x}_1 - \vec{x}) + m\delta(\vec{x}_2 - \vec{x})$$

with coordinates

$$x_1 = r_1 \cos\theta, \quad y_1 = r_1 \sin\theta$$

$$x_2 = -r_2 \cos\theta, \quad y_2 = -r_2 \sin\theta.$$

The distances r_1 and r_2 are relative to the center of mass, and can therefore be expressed in terms of the distance between the two masses r

$$r_1 = \frac{m}{M+m} r, \qquad r_2 = \frac{M}{m+M} r$$

with reduced mass

$$\mu = \frac{mM}{m+M}.$$

Substituting these in for r_1 and r_2, we may now evaluate the quadropole moment $D_{ij} = M(x_1^i x_1^j) + m(x_2^i x_2^j)$:

$$D_{xx} = \mu r^2 \cos^2\theta$$

$$D_{yy} = \mu r^2 \sin^2\theta$$

$$D_{xy} = \mu r^2 \sin\theta\cos\theta.$$

All other quadropole terms vanish.

(c) From general coordinate invariance, it always possible to choose a gauge such that the perturbed metric (and thus the polarization tensor) is tranverse to a chosen direction, for example the z-axis. Within such a gauge, to leading order $1/r$, the following conditions hold ($G = 1$ units):

$$h_{z\mu} = 0$$

$$h_{xx} = -h_{yy} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} (D_{xx} - D_{yy})$$

$$h_{xy} = -2\omega^2 \frac{e^{i\omega(r-t)}}{r} D_{xy}.$$

Now if we look for radiation along the z axis, using coordinates with mass separation R, we see that

$$h_{xx} = -h_{yy} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} \mu R^2 (\cos^2\theta - \sin^2\theta)$$

$$h_{xy} = -2\omega^2 \frac{e^{i\omega(r-t)}}{r} \mu R^2 \sin\theta\cos\theta.$$

Now if we are to look at radiation along the x-axis, we must rearrange our gauge conditions such that only components transerve to x are non-zero. This is implemented by a change in the gauge conditions of $z \to x$, $y \to z$, $x \to y$:

$$h_{z0} \to h_{x0} = 0$$

$$(h_{xx} = -h_{yy}) \to (h_{yy} = -h_{zz}) = -\omega^2 \frac{e^{i\omega(r-t)}}{r} (D_{yy} - D_{zz})$$

$$h_{xy} \to h_{yz} = -2\omega^2 \frac{e^{i\omega(r-t)}}{r} D_{yz}.$$

This leaves us with

$$h_{yy} = -h_{zz} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} \mu R^2 \sin^2\theta$$

$$h_{yz} = 0.$$

Now, as we take $M \to m$, we have $\mu \to m$ and $r_1 = r_2$. Given appropriate initial conditions, the distance between masses may remain fixed and we have circular motion, $\theta = \omega t$. Looking for radiation along the z axis, we may express the components as

$$h_{z\mu} = 0$$

$$h_{xx} = -h_{yy} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} mR^2 [\cos(2\omega t) + const]$$

$$h_{xy} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} mR^2 \sin(2\omega t).$$

Note that μR^2 becomes constant here. In terms of the spin 2 polarization vector

$$e_\pm = e_{11} \mp i e_{12}$$

we have

$$e_+ = 2e_{xx}, \qquad e_- = 0.$$

Thus the radition is circularly polarized (traveling in +1 helicity direction, i.e. counter clockwise along the z-axis) when measured in the z-direction.

Now measuing along the x-direction, we have

$$h_{yy} = -h_{zz} = -\omega^2 \frac{e^{i\omega(r-t)}}{r} mR^2 \sin^2\theta$$

$$h_{yz} = 0.$$

In this case we have $e_+ = e_- = h_{yy}$. We know that an equally weighted linear combination of two circularly polarized vectors gives rise to linear polarization, and thus for measurements along the x-direction we have linearly polarized radiation. This conforms to the results given in class.

2. The static spherically symmetric metric is

$$g_{tt} = -B(r), \quad g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta,$$

and its inverse is

$$g^{tt} = -B^{-1}(r), \quad g^{rr} = A^{-1}(r), \quad g^{\theta\theta} = 1/r^2, \quad g^{\phi\phi} = 1/r^2 \sin^2\theta.$$

For a perfect fluid, the EM tensor is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}$$

where we have normalization

$$g_{\mu\nu} U^\mu U^\nu = -1.$$

For fluid at rest, $U^r = U^\theta = U^\phi = 0$, and from the above we find

$$U^t = B^{-1/2}.$$

Now we evaluate $T^{\mu\nu}$:

$$T^{tt} = \rho B^{-1}, \quad T^{rr} = pA^{-1}, \quad T^{\theta\theta} = pr^{-2}, \quad T^{\phi\phi} = pr^{-2}\sin^{-2}\theta.$$

The covariant conservation equation is of the form

$$T^{\mu\nu}{}_{;\mu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0.$$

Only $\nu = r$ is non-vanishing,

$$T^{\mu r}{}_{;\mu} = \partial_r T^{rr} + \Gamma^\mu_{\mu r} T^{rr} + \Gamma^r_{\mu\lambda} T^{\mu\lambda}.$$

The first connection term is

$$\begin{aligned}
\Gamma^\mu_{\mu r} &= \frac{1}{2} g^{\mu\rho} \left[\partial_r g_{\rho\mu} + \partial_\mu g_{\rho r} - \partial_\rho g_{\mu r}\right] = \frac{1}{2} g^{\mu\rho} \partial_r g_{\mu\rho} \\
&= \frac{1}{2}\left(B^{-1}B' + A^{-1}A' + 4r^{-1}\right).
\end{aligned}$$

The other is

$$\begin{aligned}
\Gamma^r_{\mu\lambda} &= \frac{1}{2} g^{r\rho} \left[\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda}\right] \\
&= \delta^r_\lambda \delta^r_\mu (g^{rr}\partial_r g_{rr}) - \frac{1}{2} g^{rr} \partial_r g_{\mu\lambda} \\
&= A^{-1}A' \delta^r_\lambda \delta^r_\mu - \frac{1}{2} A^{-1} \partial_r g_{\mu\lambda}.
\end{aligned}$$

Contracting this with $T^{\mu\lambda}$ gives

$$\Gamma^r_{\mu\lambda} T^{\mu\lambda} = pA^{-2}A' - \frac{1}{2} A^{-1}\left(-\rho B^{-1}B' + pA^{-1}A' + 4pr^{-1}\right).$$

Now putting all of this together

$$\begin{aligned}
T^{\mu r}{}_{;\mu} &= \partial_r T^{rr} + \Gamma^\mu_{\mu r} T^{rr} + \Gamma^r_{\mu\lambda} T^{\mu\lambda} \\
&= p'A^{-1} - pA^{-2}A' + \frac{1}{2} pA^{-1}\left(B^{-1}B' + A^{-1}A' + 4r^{-1}\right) + pA^{-2}A' - \frac{1}{2}\left(-\rho A^{-1}B^{-1}B' + pA^{-2}A' + 4pA^{-1}r^{-1}\right) \\
&= p'A^{-1} + \frac{1}{2} A^{-1}B^{-1}B'(\rho + p) \\
&= A^{-1}\left[p' + \frac{1}{2} B^{-1}B'(\rho + p)\right] = 0
\end{aligned}$$

which yields

$$(\rho + p)B' = -2p'B.$$

3. The static spherically symmetric metric for our star is given as

$$g_{tt} = -B(r), \quad g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta.$$

We may first find a differential equation for $A(r)$ by taking the following combination of Ricci tensor components:

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2B} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = -8\pi G\rho$$

which simplifies to

$$\left(\frac{r}{A}\right)' = 1 - 8\pi G\rho r^2.$$

If, for conveinence, we define the function

$$m(r) = \frac{1}{2}r(1 - A(r))$$

then the differential equation for A above becomes

$$m' = 4\pi r^2 \rho. \tag{1}$$

In addition to this equation, we will need the T.O.V. equation, which is supplied by taking the $G_{rr} = 8\pi G T_{rr}$ component of the field equation and then subtituting in the result from the last problem ($(\rho + p)B' = -2p'B$). This yields the following equation for m in terms of the pressure and density

$$p' = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}.$$

Typically then, one may use the last two equations, along with an equation of state, to solve for ρ, p, and m. However, in our case, we will not use an equation of state and instead can use $\rho = const$ to express m in terms of a critical $p = p_c$.

Firstly, we solve for m in (1) by integration

$$m(r) = \begin{cases} \frac{4}{3}\pi\rho r^3 & r \leq R \\ \frac{4}{3}\pi\rho R^3 & r \geq R \end{cases} \tag{2}$$

The exterior solution is defined in terms of the Scharwzchild mass parameter $M = \frac{4}{3}\pi\rho R^3$. Note $m(r)$ is continuous at the boundary. Going back to the interior solution, we may solve for the T.O.V. equation by substituting in the appropriate form for $m(r)$ - this yields

$$p' = -\frac{4}{3}\pi r \frac{(\rho + p)(\rho + 3p)}{1 - 8\pi r^2 \rho/3}.$$

Now integrate from 0 to arbitrary r, defining $p(0) = p_c$,

$$\frac{1}{2\rho} \ln\left[\frac{(\rho + p_c)(3p_\rho)}{(3p_c + \rho)(p + \rho)}\right] = \frac{1}{4\rho} \ln\left[1 - \frac{8}{3}\pi r^2 \rho\right].$$

Again making use of our definition of $m(r)$, we can express this as

$$\frac{\rho + 3p}{\rho + p} = \left(1 - 2\frac{m}{r}\right)^{1/2} \frac{\rho + 3p_c}{\rho + p_c}.$$

For p to be continuous, it must be zero at the boundary $r = R$. Thus we may evaluate the above at $r = R$ to find an expression for the radius in terms of the critical density

$$1 = \left(1 - \frac{8}{3}\pi\rho R^2\right)^{1/2} \frac{\rho + 3p_c}{\rho + p_c}$$

and solving for R

$$\boxed{R = \left[\frac{3}{8\pi\rho}\left(1 - \frac{(\rho+p_c)^2}{(\rho+3p_c)^2}\right)\right]^{1/2}}.$$

We may also express the mass of the star in terms of the critical pressure and radius R by subsituting

$$M = \frac{4\pi\rho R^3}{3}$$

into the above and solving for M. This yields

$$\boxed{M = \frac{2Rp_c(2p_c+\rho)}{(3p_c+\rho)^2}}.$$

Recalling our definition of $m(r)$ in terms of $A(r)$, we may easily solve for $A(r)$ as

$$\boxed{A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1}}$$

where $m(r)$ for any r is given in (2). To solve for $B(r)$ we go back to the result

$$(\rho + p)B' = -2p'B.$$

Before we solve for $B(r)$, first we note that from the rr component of the Einstein equation, we have

$$-\frac{1}{r^2}B(1-B^{-1}) + \frac{2}{r}\frac{B'}{B} = pA = p\left(1 - \frac{2m(r)}{r}\right)^{-1}.$$

This simplifies to

$$\frac{B'}{B} = \frac{m(r) + 4\pi r^3 p}{r(r-2m(r))}.$$

Outside the star, $p = \rho = 0$ and $m(r) = const = M$, and solving the above equation gives the usual Scharzchild solution

$$B(r) = 1 - \frac{2M}{r} \qquad r \geq R.$$

where the initial condition $B(\infty) = 1$ has been applied for asymptotic limit. Now solving for B interior, we start with

$$\frac{B'}{B} = \frac{m(r) + 4\pi r^3 p}{r(r-2m(r))}$$

and transform it to

$$B = C \exp\left[-\int \frac{2p'}{\rho + p}\right].$$

C here is an integration constant. Making use of

$$\frac{\rho + 3p}{\rho + p} = \left(1 - 2\frac{m}{r}\right)^{1/2} \frac{\rho + 3p_c}{\rho + p_c}.$$

we may first solve for $p(r)$ as well as its derivative in terms of r, ρ, p_c and m. Then, using

$$\rho = \frac{3M}{4\pi R^3}$$

and

$$p_c = \rho\frac{(1-2Mr^2/R^3)^{1/2} - (1-2M/R)^{1/2}}{3(1-2M/R)^{1/2} - (1-2Mr^2/R^3)^{1/2}}$$

we may express the entire integral in terms of M, R and r. Lastly, we use the boundary condition to solve for the arbitrary integration constant

$$B(R) = \frac{4}{3}\pi\rho R^3.$$

This is a long process and easier to keep track of things in Mathematica, but the end result yields an expression for $B(r)$ in the interior of the star

$$\boxed{B(r) = \left[\frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}\right]^2} \qquad r \leq R$$