SVT Decomposition of $\delta W_{\mu\nu}$

Under conformal transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $W_{\mu\nu}$ transforms as

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(g_{\mu\nu}).$$

Perturbing the metric,

$$\bar{g}_{\mu\nu} = \bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu} = \Omega^2 g^{(0)}_{\mu\nu} + \Omega^2 h_{\mu\nu}$$

it follows that to first order

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2} \delta W_{\mu\nu}(h_{\mu\nu}). \tag{1}$$

Under an infinitesimal oordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, the perturbed tensor $\delta W_{\mu\nu}$ transforms as

$$\delta W_{\mu\nu}(h_{\mu\nu}) \to \delta W'_{\mu\nu}(h'_{\mu\nu}) = \delta W_{\mu\nu}(h_{\mu\nu}) - \delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu})$$

At the same time, we also consider the transformation of the entire $W_{\mu\nu}$ under the infinitesimal coordinate transformation

$$W_{\mu\nu} \to W'_{\mu\nu} = W_{\mu\nu} + \mathcal{L}_e(W_{\mu\nu}) \tag{2}$$

where the Lie derivative $\mathcal{L}_e$ for the rank 2 tensor is

$$\mathcal{L}_e(W_{\mu\nu}) = W^\lambda{}_\mu \epsilon_{\lambda;\nu} + W^\lambda{}_\nu \epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda.$$

Defining $\delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}) \equiv \delta W_{\mu\nu}(\epsilon)$, if we expand eq (2) to first order, we conclude that

$$\delta W_{\mu\nu}(\epsilon) = W^\lambda{}_\mu \epsilon_{\lambda;\nu} + W^\lambda_\nu \epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda.$$

Hence, in any background that is conformal to flat, the Lie derivative vanishes and thus $\delta W_{\mu\nu}$ must be gauge invariant. As such, it must always be possible to express $\delta W_{\mu\nu}$ in terms of 5 gauge invariant quantities (10 symmetric components - 4 coordinate transformation - 1 traceless condition = 5). This is shown below. Alternatively, we may also fix the gauge, as we have done to make $\delta W_{\mu\nu}$ diagonal in its indicies.

Now decomposing $h_{\mu\nu}$ according to

$$ds^2 = \Omega^2 \left\{ -(1+2\phi)d\tau^2 + (\partial_i B - B_i)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}]dx^i dx^j \right\}$$

(same as Ellis when tensor mode E_{ij} is doubled), we have in flat space $\delta W_{\mu\nu}(h_{\mu\nu})$:

00	$-\frac{2}{3}\nabla^4(\psi+\phi+\partial_0 B-\partial_0\partial_0 E)$
11	$-\frac{1}{3}[\Box^2+\Box(\partial_0\partial_0-\partial_1\partial_1)+2\partial_1\partial_1\partial_0\partial_0](\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \Box\partial_1(\partial_0 B_1+\partial_0\partial_0 E_1)\ +\ \Box^2 E_{11}$
22	$-\frac{1}{3}[\Box^2+\Box(\partial_0\partial_0-\partial_2\partial_2)+2\partial_2\partial_2\partial_0\partial_0](\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \Box\partial_2(\partial_0 B_2+\partial_0\partial_0 E_2)\ +\ \Box^2 E_{22}$
33	$-\frac{1}{3}[\Box^2+\Box(\partial_0\partial_0-\partial_3\partial_3)+2\partial_3\partial_3\partial_0\partial_0](\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \Box\partial_3(\partial_0 B_3+\partial_0\partial_0 E_3)\ +\ \Box^2 E_{33}$
01	$-\frac{2}{3}\nabla^2\partial_1(\partial_0\psi+\partial_0\phi+\partial_0\partial_0 B-\partial_0\partial_0\partial_0 E)-\frac{1}{2}(\nabla^4-\nabla^2\partial_0\partial_0)(B_1+\partial_0 E_1)$
02	$-\frac{2}{3}\nabla^2\partial_2(\partial_0\psi+\partial_0\phi+\partial_0\partial_0 B-\partial_0\partial_0\partial_0 E)-\frac{1}{2}(\nabla^4-\nabla^2\partial_0\partial_0)(B_2+\partial_0 E_2)$
03	$-\frac{2}{3}\nabla^2\partial_3(\partial_0\psi+\partial_0\phi+\partial_0\partial_0 B-\partial_0\partial_0\partial_0 E)-\frac{1}{2}(\nabla^4-\nabla^2\partial_0\partial_0)(B_3+\partial_0 E_3)$
12	$\frac{1}{3}(\Box-2\partial_0\partial_0)\partial_1\partial_2(\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \frac{1}{2}\Box\partial_1\partial_0(B_2+\partial_0 E_2)\ -\ \frac{1}{2}\Box\partial_2\partial_0(B_1+\partial_0 E_1)\ +\ \Box^2 E_{12}$
13	$\frac{1}{3}(\Box-2\partial_0\partial_0)\partial_1\partial_3(\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \frac{1}{2}\Box\partial_1\partial_0(B_3+\partial_0 E_3)\ -\ \frac{1}{2}\Box\partial_3\partial_0(B_1+\partial_0 E_1)\ +\ \Box^2 E_{13}$
23	$\frac{1}{3}(\Box-2\partial_0\partial_0)\partial_2\partial_3(\psi+\phi+\partial_0 B-\partial_0\partial_0 E)\ -\ \frac{1}{2}\Box\partial_2\partial_0(B_3+\partial_0 E_3)\ -\ \frac{1}{2}\Box\partial_3\partial_0(B_2+\partial_0 E_2)\ +\ \Box^2 E_{23}$

According to eq. (1), we may find $\delta W_{\mu\nu}$ based on a conformal to flat background by simply multiplying the above by a factor of Ω^{-2}.

The gauge invariant SVT quantities in the RW $K=0$ space are

$$\bar{\phi}=\phi-\frac{\dot{\Omega}}{\Omega}(\dot{E}-B)-(\ddot{E}-\dot{B}) \tag{3}$$

$$\bar{\psi}=\psi+\frac{\dot{\Omega}}{\Omega}(\dot{E}-B) \tag{4}$$

$$F_i=\dot{E}_i+B_i \tag{5}$$

$$E_{ij}=E_{ij}. \tag{6}$$

In the flat space all $\dot{\Omega}$ gauge quantities vanish and we see immediately $\delta W_{\mu\nu}$ can be expressed solely in terms of $\bar{\phi},\bar{\psi},F_i$, and E_{ij}. In fact, we see that ψ and ϕ are on equal footing everywhere, and thus we may combine the two invariant scalars

$$\bar{\xi}=\bar{\phi}+\bar{\psi}=\phi+\psi-(\ddot{E}-\dot{B}).$$

so that we are now exactly at 5 independent degrees of freedom. Now, in the conformal to flat background, we must use the gauge invariant quantities eq. (3-6). However we note that the quantity $\bar{\xi}$ remains unchanged, as the $\dot{\Omega}$ terms cancel. We see now that even in a conformal to flat background, we are able to retain the same gauge invariant quantities whilst preseving the conformal symmetry, i.e.

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu})=\Omega^{-2}\delta W_{\mu\nu}(h_{\mu\nu}).$$