$\delta W_{\mu\nu}$ Residual Gauge v3

Residual Gauge for Flat Transverse Traceless $\Box^2 K_{\mu\nu} = 0$

In the transverse gauge $\partial_\nu K^{\mu\nu} = 0$ in the Minkowski background the vacuum equation of motion for the traceless $K_{\mu\nu}$ is

$$\delta W_{\mu\nu} = \eta^{\alpha\beta}\eta^{\sigma\rho}\partial_\alpha\partial_\beta\partial_\sigma\partial_\rho K_{\mu\nu} = 0. \tag{1}$$

The momentum eigenstate solutions take the form

$$K_{\mu\nu} = A_{\mu\nu}e^{ikx} + n_\alpha x^\alpha B_{\mu\nu}e^{ikx} + \text{c.c.} \tag{2}$$

where $n_\alpha = (1,0,0,0)$ and $k^\mu k_\mu = 0$. Following the transverse condition, the solution must obey

$$0 = (ik^\nu A_{\mu\nu} + n^\nu B_{\mu\nu})\, e^{ikx} + (ik^\nu B_{\mu\nu})\, n_\alpha x^\alpha e^{ikx} + \text{c.c.} \tag{3}$$

In addition to the tracelessness condition, to satisfy all x (noting that e^{ikx}, e^{-ikx}, te^{ikx} and te^{-ikx} are linearly independent), we set in (3) each coefficient preceding the space-time dependent function to zero, viz.

$$A^\mu{}_\mu = 0, \qquad B^\mu{}_\mu = 0, \qquad ik^\nu A_{\mu\nu} + n^\nu B_{\mu\nu} = 0, \qquad ik^\nu B_{\mu\nu} = 0. \tag{4}$$

We have a total of 10 conditions upon the 20 total components of $A_{\mu\nu}$ and $B_{\mu\nu}$. It is easy to check that these conditions (and also their implied conjugate expressions) satisfy our choice of transverse coordinate system and retain the tracelessness of $K_{\mu\nu}$. Under infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, $K_{\mu\nu}$ transforms as

$$K'_{\mu\nu} = K_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu + \frac{1}{2}g_{\mu\nu}\partial_\rho\epsilon^\rho. \tag{5}$$

We denote the change in $K_{\mu\nu}$ (Lie derivative) as the tensor

$$\Delta K_{\mu\nu} = -\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu + \frac{1}{2}g_{\mu\nu}\partial_\rho\epsilon^\rho. \tag{6}$$

Noting that $\Delta K_{\mu\nu}$ is manifestly traceless, in order to preserve the tranverse gauge condition $\partial_\mu K^{\mu\nu} = 0$, $\Delta K^{\mu\nu}$ must obey $\partial_\nu \Delta K^{\mu\nu} = 0$, viz.

$$0 = -\partial_\nu\partial^\nu\epsilon^\mu - \frac{1}{2}\partial^\mu\partial_\nu\epsilon^\nu. \tag{7}$$

We take the $\epsilon^\mu(x)$ to be of the plane wave form,

$$\epsilon^\mu(x) = iA^\mu e^{ikx} + iB^\mu n_\alpha x^\alpha e^{ikx} + \text{c.c.}, \tag{8}$$

which obeys the following relations:

$$\partial^\nu\epsilon^\mu = -k^\nu\left(A^\mu e^{ikx} + B^\mu n_\alpha x^\alpha e^{ikx}\right) + in^\nu\left(B^\mu e^{ikx}\right) + \text{c.c.} \tag{9}$$

$$\partial_\nu\partial^\nu\epsilon^\mu = -2k_\nu n^\nu\left(B^\mu e^{ikx}\right) + \text{c.c.}, \tag{10}$$

$$\partial_\mu \partial^\nu \epsilon^\mu = -ik_\mu k^\nu \left(A^\mu e^{ikx} + B^\mu n_\alpha x^\alpha e^{ikx}\right) - \left(k^\nu n_\mu + k_\mu n^\nu\right)\left[B^\mu e^{ikx}\right] + \text{c.c.}, \tag{11}$$

where for reference we also have the relation

$$\partial_\beta \partial^\beta (n_\alpha x^\alpha e^{ikx}) = 2in_\alpha k^\alpha e^{ikx}. \tag{12}$$

The transverse condition per (7) then takes the form

$$0 = 2k_\nu n^\nu \left(B^\mu e^{ikx}\right) + \frac{1}{2} ik_\nu k^\mu \left(A^\nu e^{ikx} + B^\nu n_\alpha x^\alpha e^{ikx}\right) + \frac{1}{2}(k^\mu n_\nu + k_\nu n^\mu)\left[B^\nu e^{ikx}\right] + \text{c.c.}\ . \tag{13}$$

To hold for arbitrary x, we have the two separate conditions,

$$2k_\nu n^\nu B^\mu + \frac{1}{2} ik_\nu k^\mu A^\nu + \frac{1}{2}(k^\mu n_\nu + k_\nu n^\mu) B^\nu = 0, \qquad \frac{1}{2} ik_\nu k^\mu B^\nu = 0. \tag{14}$$

For arbitrary k^μ, the second condition in 14 implies $k_\nu B^\nu = 0$. As such, the remaining condition is

$$2k_\nu n^\nu B^\mu + \frac{1}{2} k^\mu n_\nu B^\nu + \frac{1}{2} ik_\nu k^\mu A^\nu = 0. \tag{15}$$

Let us now take a wave propagating in the z direction, with wavevector

$$k^\mu = (k, 0, 0, k), \qquad k_\mu = (-k, 0, 0, k). \tag{16}$$

The transverse condition $\partial^\mu \Delta K_{\mu\nu}$ then entails

$$B_0 = -B_3, \qquad B_0 = \frac{i}{5} k(A_0 + A_3), \qquad B_1 = B_2 = 0. \tag{17}$$

For the tensor polarizations $A_{\mu\nu}$ and $B_{\mu\nu}$ the transverse relations take the form

$$B^\mu{}_\mu = A^\mu{}_\mu = 0, \qquad B_{0\mu} = -B_{3\mu}, \qquad ik(A_{\mu 0} + A_{\mu 3}) = B_{0\mu}. \tag{18}$$

Although this would appear to be 10 total constraints, the condition $B_{00} = -B_{30}$ reduces the equation

$$ik(A_{\mu 0} + A_{\mu 3}) = B_{0\mu}, \tag{19}$$

from 4 to 3 conditions, namely

$$ik(A_{10} + A_{13}) = B_{01}, \qquad ik(A_{20} + A_{23}) = B_{02}, \qquad A_{00} + 2A_{03} + A_{33} = 0. \tag{20}$$

The form for the transformation (Lie derivative) onto $K_{\mu\nu}$ is

$$\begin{aligned} \Delta K_{\mu\nu} = & \left[k_\nu A_\mu + k_\mu A_\nu - i\left(n_\nu B_\mu + n_\mu B_\nu\right) - \frac{1}{2} g_{\mu\nu} A^\alpha k_\alpha + \frac{i}{2} g_{\mu\nu} n_\alpha B^\alpha\right] e^{ikx} \\ & + \left[k_\nu B_\mu + k_\mu B_\nu\right] n_\alpha x^\alpha e^{ikx}. \end{aligned} \tag{21}$$

It will be useful to evaluate this for different components:

$$\begin{aligned} \Delta K_{00} &= \left[-2kA_0 + \frac{1}{2} k(A_0 + A_3) - \frac{3i}{2} B_0\right] e^{ikx} - \left[2kB_0\right] n_\alpha x^\alpha e^{ikx} \\ \Delta K_{01} &= -kA_1 e^{ikx}, \qquad \Delta K_{02} = -kA_2 e^{ikx} \\ \Delta K_{03} &= \left[-kA_3 + kA_0 - iB_3\right] e^{ikx} - \left[2kB_3\right] n_\alpha x^\alpha e^{ikx} \\ \Delta K_{11} &= \Delta K_{22} = \left[-\frac{1}{2} k(A_0 + A_3) - \frac{i}{2} B_0\right] e^{ikx}, \qquad \Delta K_{12} = 0 \\ \Delta K_{13} &= [kA_1] e^{ikx}, \qquad \Delta K_{23} = [kA_2] e^{ikx} \\ \Delta K_{33} &= \left[2kA_3 - \frac{1}{2} k(A_0 + A_3) - \frac{i}{2} B_0\right] e^{ikx} + \left[2kB_3\right] n_\alpha x^\alpha e^{ikx}. \end{aligned} \tag{22}$$

The total transformation on each polarization tensor, for $A_{\mu\nu} \to A'_{\mu\nu}$ and $B_{\mu\nu} \to B'_{\mu\nu}$, is

$$
\begin{aligned}
A'_{00} &= A_{00} - 2kA_0 - 4iB_0 & B'_{00} &= B_{00} - 2kB_0 \\
A'_{01} &= A_{01} - kA_1 & B'_{01} &= B_{01} \\
A'_{02} &= A_{02} - kA_2 & B'_{02} &= B_{02} \\
A'_{03} &= A_{03} + 2kA_0 + 6iB_0 & B'_{03} &= B_{03} + 2kB_0 \\
A'_{11} &= A_{11} + 2iB_0 & B'_{11} &= B_{11} \\
A'_{22} &= A_{22} + 2iB_0 & B'_{22} &= B_{22} \\
A'_{33} &= A_{33} - 2kA_0 - 8iB_0 & B'_{33} &= B_{33} - 2kB_0 \\
A'_{12} &= A_{12} & B'_{12} &= B_{12} \\
A'_{13} &= A_{13} + kA_1 & B'_{13} &= B_{13} \\
A'_{23} &= A_{23} + kA_2 & B'_{23} &= B_{23}.
\end{aligned} \tag{23}
$$

Neither the polarizations nor the gauge terms A_μ and B_μ are all independent. Their dependencies are:

$$
\begin{aligned}
-A_{00} + A_{11} + A_{22} + A_{33} &= 0 & A_3 &= -A_0 - \frac{5i}{k}B_0 \\
ik(A_{\mu 0} + A_{\mu 3}) &= B_{0\mu} & B_3 &= -B_0 \\
B_{0\mu} &= -B_{3\mu} & B_1 &= B_2 = 0 \\
-B_{00} + B_{11} + B_{22} + B_{33} &= 0
\end{aligned} \tag{24}
$$

Looking more closely at these dependencies amongst $B_{\mu\nu}$, we note:

$$
B_{33} = -B_{03} = B_{00}, \qquad B_{23} = -B_{02}, \qquad B_{13} = -B_{01}, \qquad B_{22} = -B_{11}. \tag{25}
$$

This leaves $B_{\mu\nu}$ with 5 total independent components, chosen as: B_{33}, B_{12}, B_{11}, B_{01} and B_{02}.

As for the $A_{\mu\nu}$, we note:

$$
A_{13} = -\frac{i}{k}B_{01} - A_{01}, \quad A_{23} = -\frac{i}{k}B_{02} - A_{02}, \quad A_{22} = A_{00} - A_{11} - A_{33}, \quad A_{03} = -\frac{1}{2}(A_{00} + A_{33}). \tag{26}
$$

$A_{\mu\nu}$ thus has a total of 6 independent components, here chosen as: A_{00}, A_{01}, A_{02}, A_{11}, A_{33}, and A_{12}. Regarding the gauge invariance, we may choose to set

$$
B_0 = \frac{B_{33}}{2k} \tag{27}
$$

which eliminates B'_{33}, B'_{00}, and B'_{03}. This leaves $B_{\mu\nu}$ with four total independent gauge invariant quantities that cannot be eliminated: B_{12}, B_{11}, B_{01} and B_{02}.

As for $A_{\mu\nu}$, we first set

$$
A_1 = \frac{A_{01}}{k}, \quad A_2 = \frac{A_{02}}{k}. \tag{28}
$$

This eliminates A'_{01} and A'_{02}. Through some various manipulation of the dependencies, we also see that if we set

$$
A_0 = \frac{2A_{00} - A_{33}}{2k}, \tag{29}
$$

this will eliminate A_{00} and A_{33}. This leaves $A_{\mu\nu}$ with two total independent gauge invariant quantities which cannot be eliminated: A_{12} and A_{11}.

In summary we are left with gauge invariant wave solutions of the form

$$
K_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \begin{pmatrix} 0 & B_{01} & B_{02} & 0 \\ B_{01} & B_{11} & B_{12} & 0 \\ B_{02} & B_{12} & -B_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} n_\alpha x^\alpha e^{ikx} \tag{30}
$$

Gauge Invariant $\delta W_{\mu\nu} = \delta T_{\mu\nu}$

Via the 3+1 projection followed by a helicity decomposition, we may express an arbitrary traceless, transverse, symmetric rank 2 tensor as

$$
\begin{aligned}
&\delta T_{00} = \rho, \\
&\delta T_{0i} = -Q_i + \tilde{\nabla}_i \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t \rho, \\
&\delta T_{ij} = \frac{1}{2}\delta_{ij}\rho - \frac{1}{2}\delta_{ij}\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t^2\rho + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j \int d^3y D^3(\mathbf{x}-\mathbf{y})\left(\int d^3z D^3(\mathbf{y}-\mathbf{z})\partial_t^2\rho - \frac{1}{3}\rho\right) \\
&\qquad - \tilde{\nabla}_i \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_0 Q_j - \tilde{\nabla}_j \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_0 Q_i + \pi^{T\theta}_{ij}.
\end{aligned} \tag{31}
$$

We may equivalently express $\delta W_{\mu\nu}$ in terms of the analogous barred perturbation quantities ($\bar{\rho}$, $\bar{Q}_i$, $\bar{E}_{ij}$) as

$$
\begin{aligned}
&\delta W_{00} = \bar{\rho}, \\
&\delta W_{0i} = -\bar{Q}_i + \tilde{\nabla}_i \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t \bar{\rho}, \\
&\delta W_{ij} = \frac{1}{2}\delta_{ij}\bar{\rho} - \frac{1}{2}\delta_{ij}\int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_t^2\bar{\rho} + \frac{3}{2}\tilde{\nabla}_i\tilde{\nabla}_j \int d^3y D^3(\mathbf{x}-\mathbf{y})\left(\int d^3z D^3(\mathbf{y}-\mathbf{z})\partial_t^2\bar{\rho} - \frac{1}{3}\bar{\rho}\right) \\
&\qquad - \tilde{\nabla}_i \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_0 \bar{Q}_j - \tilde{\nabla}_j \int d^3y D^3(\mathbf{x}-\mathbf{y})\partial_0 \bar{Q}_i + \bar{\pi}^{T\theta}_{ij}.
\end{aligned} \tag{32}
$$

Then, the fluctuation equation $\delta W_{\mu\nu} = \delta T_{\mu\nu}$ then entails

$$
\begin{aligned}
\bar{\rho} &= \rho \\
\bar{Q}_i &= Q_i \\
\bar{\pi}^{T\theta}_{ij} &= \pi^{T\theta}_{ij}.
\end{aligned} \tag{33}
$$

The $\delta W_{00} = \delta T_{00}$ fixes ρ, allowing $\delta W_{0i} = \delta T_{0i}$ to fix Q_i, thereby leading to $\bar{\pi}^{T\theta}_{ij} = \pi^{T\theta}_{ij}$ without having to apply transverse projections or deal with additional homogenous solutions such as $\tilde{\nabla}_i\tilde{\nabla}_j\tilde{\nabla}_a\tilde{\nabla}^a\chi = 0$. This is also why the fluctuations equations have been expressed in terms of Q_i rather than π_i, as the equation of π_i necessarily leads to

$$
\tilde{\nabla}_a\tilde{\nabla}^a\bar{\pi}_i = \tilde{\nabla}_a\tilde{\nabla}^a\pi_i, \tag{34}
$$

which only permits equivalence of $\bar{\pi}_i = \pi_i$ under assumptions upon the boundary conditions of the perturbations (see helicity_decomposition_v1.pdf).

Upon carrying through the same analogous helicity decomposition on $K_{\mu\nu}$, we find that the helicity components of $\delta W_{\mu\nu}$ take the form

$$
\begin{aligned}
\bar{\rho} &= -\frac{2}{3}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(\phi + \psi + \partial_0 B - \partial_0^2 E) \\
\bar{Q}_i &= -\frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\left(-\partial_0^2 + \tilde{\nabla}_b\tilde{\nabla}^b\right)(B_i - \partial_0 E_i) \\
\bar{\pi}^{T\theta}_{ij} &= \left(-\partial_0^2 + \tilde{\nabla}_a\tilde{\nabla}^a\right)^2 E_{ij}.
\end{aligned} \tag{35}
$$

According to these solutions, we see that the gauge invariant quantity $\phi + \psi + \dot{B} - \ddot{E}$ represents a static potential, whereas $B_i - \dot{E}_i$ admit second order wave solutions, and E_{ij} admit fourth order wave solutions. That is, in terms of momentum eigenstate solutions

$$
(B_i - \dot{E}_i) = A_i e^{ikx} + \text{c.c.} \tag{36}
$$

$$
E_{ij} = A_{ij}e^{ikx} + n_\alpha x^\alpha B_{ij}e^{ikx} + \text{c.c.} \tag{37}
$$

Requiring E_i to be transverse entails A_i to be transverse also, leaving only A_1 and A_2 as independent polarizations. Under spatial rotation, these modes transform as helicity ± 1 components.

Requiring E_{ij} to be transverse and traceless entails

$$k^i A_{ij} = 0, \quad k^i B_{ij} = 0, \quad A^i{}_i = 0, \quad B^i{}_i = 0, \tag{38}$$

leaving only A_{12}, $A_{11} = -A_{22}$, B_{12}, and $B_{11} = -B_{22}$ as independent polarizations. As expected from E_{ij}, these are helicity ± 2 components.