Special Gauge Matthew v12

Setup

Metric decomposed to first order:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + f_{\mu\nu}). \tag{1}$$

We then split $f_{\mu\nu}$ into its traceless and trace components, i.e.

$$f_{\mu\nu} = k_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu} f \tag{2}$$

where $f = \eta^{\mu\nu} f_{\mu\nu}$. The fluctuations then take the form

$$\begin{aligned}\delta G_{\mu\nu} = {}& \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta k_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha f\partial_\beta\Omega - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}k_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \eta^{\alpha\beta}k_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha k_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha f + 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}k_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ & - 2\eta^{\alpha\beta}k_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu k_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu k_{\mu\alpha} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta k_{\beta\gamma} \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta k_{\alpha\gamma} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu k_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu k_{\mu\beta} - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu f \\ & - \tfrac{1}{4}\Omega^{-1}\partial_\mu f\partial_\nu\Omega + \tfrac{1}{4}\partial_\nu\partial_\mu f.\end{aligned} \tag{3}$$

We impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha k_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}k_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu f + R\Omega^{-1}f\partial_\nu\Omega. \tag{4}$$

and take

$$J = -4, \qquad R = 2P - \frac{3}{2}. \tag{5}$$

$\Omega(\tau) : k_{\mu\nu},\ f$

$$\begin{aligned}\eta^{\mu\nu}\delta G_{\mu\nu} &= (-8\Omega^{-2}\dot\Omega^2 + 4\Omega^{-1}\ddot\Omega)k_{00} + (\tfrac{3}{2}\Omega^{-2}\dot\Omega^2 - 2P\Omega^{-2}\dot\Omega^2 + \tfrac{3}{2}\Omega^{-1}\ddot\Omega - 2P\Omega^{-1}\ddot\Omega - \tfrac{3}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu \\ &\quad + P\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\Omega^{-1}\dot\Omega\partial_0 - 4P\Omega^{-1}\dot\Omega\partial_0)f. \\ &= (-8\Omega^{-2}\dot\Omega^2 + 4\Omega^{-1}\ddot\Omega)k_{00} + (P - \tfrac{3}{4})\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 f)\end{aligned} \tag{6}$$

$$\begin{aligned}\delta G_{00} = {}& (2\Omega^{-2}\dot\Omega^2 - 2\Omega^{-1}\ddot\Omega + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 3\Omega^{-1}\dot\Omega\partial_0)k_{00} + (-\tfrac{3}{4}\Omega^{-2}\dot\Omega^2 + P\Omega^{-2}\dot\Omega^2 + \tfrac{3}{4}\Omega^{-1}\ddot\Omega \\ & - P\Omega^{-1}\ddot\Omega + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - P\Omega^{-1}\dot\Omega\partial_0 + \tfrac{1}{4}\partial_0\partial_0 - P\partial_0\partial_0)f.\end{aligned} \tag{7}$$

$$\begin{aligned}\delta G_{0i} = {}& -\Omega^{-1}\dot\Omega\partial_i k_{00} + (\Omega^{-2}\dot\Omega^2 + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\Omega^{-1}\dot\Omega\partial_0)k_{0i} + (\tfrac{1}{2}\Omega^{-1}\dot\Omega\partial_i - P\Omega^{-1}\dot\Omega\partial_i \\ & + \tfrac{1}{4}\partial_i\partial_0 - P\partial_i\partial_0)f.\end{aligned} \tag{8}$$

$$\delta G_{ij} = \delta_{ij}\Omega^{-2}\dot{\Omega}^2 k_{00} - \Omega^{-1}\dot{\Omega}\partial_j k_{0i} - \Omega^{-1}\dot{\Omega}\partial_i k_{0j} + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)k_{ij}$$
$$+ \delta_{ij}(-\tfrac{3}{4}\Omega^{-2}\dot{\Omega}^2 + P\Omega^{-2}\dot{\Omega}^2 + \tfrac{3}{4}\Omega^{-1}\ddot{\Omega} - P\Omega^{-1}\ddot{\Omega} - \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-1}\dot{\Omega}\partial_0$$
$$- P\Omega^{-1}\dot{\Omega}\partial_0)f + (\tfrac{1}{4}\partial_i\partial_j - P\partial_i\partial_j)f. \tag{9}$$

$\Omega(\tau) = \frac{1}{H\tau}$: $k_{\mu\nu},\ f$

Now set $\Omega(\tau) = \frac{1}{H\tau}$, with the fluctuations being evaluated as

$$\eta^{\mu\nu}\delta G_{\mu\nu} = (P - \tfrac{3}{4})\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta(\Omega^2 f) \tag{10}$$

$$\delta G_{00} = (-2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 3\tau^{-1}\partial_0)k_{00} + (\tfrac{3}{4}\tau^{-2} - P\tau^{-2} + \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu$$
$$+ P\tau^{-1}\partial_0 + \tfrac{1}{4}\partial_0\partial_0 - P\partial_0\partial_0)f. \tag{11}$$

$$\delta G_{0i} = \tau^{-1}\partial_i k_{00} + (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + 2\tau^{-1}\partial_0)k_{0i} + (-\tfrac{1}{2}\tau^{-1}\partial_i + P\tau^{-1}\partial_i + \tfrac{1}{4}\partial_i\partial_0$$
$$- P\partial_i\partial_0)f. \tag{12}$$

$$\delta G_{ij} = \delta_{ij}\tau^{-2}k_{00} + \tau^{-1}\partial_j k_{0i} + \tau^{-1}\partial_i k_{0j} + (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)k_{ij} + \delta_{ij}(\tfrac{3}{4}\tau^{-2} - P\tau^{-2}$$
$$- \tfrac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\eta^{\mu\nu}\partial_\mu\partial_\nu - \tau^{-1}\partial_0 + P\tau^{-1}\partial_0)f + (\tfrac{1}{4}\partial_i\partial_j - P\partial_i\partial_j)f. \tag{13}$$

Now we will further express our results in terms of

$$K_{\mu\nu} = \Omega^2 k_{\mu\nu} \qquad \text{and} \qquad h = \Omega^2 f \tag{14}$$

$\Omega(\tau)$: $K_{\mu\nu},\ h$

Working with a time dependent conformal factor, $\Omega(\tau)$, the fluctuations are evaluated as

$$\eta^{\mu\nu}\delta G_{\mu\nu} = (-8\Omega^{-4}\dot{\Omega}^2 + 4\Omega^{-3}\ddot{\Omega})K_{00} + (-\tfrac{3}{4}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu)h$$
$$= \Omega^{-2}(-8\Omega^{-2}\dot{\Omega}^2 + 4\Omega^{-1}\ddot{\Omega})K_{00} + (P - \tfrac{3}{4})\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h \tag{15}$$

$$\delta G_{00} = (5\Omega^{-4}\dot{\Omega}^2 - \Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-3}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{3}{4}\Omega^{-4}\dot{\Omega}^2 + \tfrac{3}{4}\Omega^{-3}\ddot{\Omega}$$
$$+ \tfrac{1}{4}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}P\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\Omega^{-3}\dot{\Omega}\partial_0 + \tfrac{1}{4}\Omega^{-2}\partial_0\partial_0 - P\Omega^{-2}\partial_0\partial_0)h. \tag{16}$$

$$\delta G_{0i} = -\Omega^{-3}\dot{\Omega}\partial_i K_{00} + (2\Omega^{-4}\dot{\Omega}^2 + \Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu)K_{0i} + (P\Omega^{-3}\dot{\Omega}\partial_i + \tfrac{1}{4}\Omega^{-2}\partial_i\partial_0$$
$$- P\Omega^{-2}\partial_i\partial_0)h. \tag{17}$$

$$\delta G_{ij} = \delta_{ij}\Omega^{-4}\dot{\Omega}^2 K_{00} - \Omega^{-3}\dot{\Omega}\partial_j K_{0i} - \Omega^{-3}\dot{\Omega}\partial_i K_{0j} + (-2\Omega^{-4}\dot{\Omega}^2 + 3\Omega^{-3}\ddot{\Omega} + \tfrac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)K_{ij}$$
$$+ \delta_{ij}(-\tfrac{5}{4}\Omega^{-4}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-3}\ddot{\Omega} - \tfrac{1}{4}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}P\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + P\Omega^{-3}\dot{\Omega}\partial_0)h + (\tfrac{1}{4}\Omega^{-2}\partial_i\partial_j$$
$$- P\Omega^{-2}\partial_i\partial_j)h. \tag{18}$$

$\Omega(\tau) = \frac{1}{H\tau}$: $K_{\mu\nu}$, h

Now set $\Omega(\tau) = \frac{1}{H\tau}$, with the fluctuations being evaluated as

$$\begin{aligned}\eta^{\mu\nu}\delta G_{\mu\nu} =&(-\tfrac{3}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu)h\\ =&(P-\tfrac{3}{4})\Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h\end{aligned} \tag{19}$$

$$\begin{aligned}\delta G_{00} = &(3H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2\tau\partial_0)K_{00} + (\tfrac{3}{4}H^2 + \tfrac{1}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{2}H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu\\ &- H^2P\tau\partial_0 + \tfrac{1}{4}H^2\tau^2\partial_0\partial_0 - H^2P\tau^2\partial_0\partial_0)h.\end{aligned} \tag{20}$$

$$\begin{aligned}\delta G_{0i} = &H^2\tau\partial_i K_{00} + (4H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu)K_{0i} + (-H^2P\tau\partial_i + \tfrac{1}{4}H^2\tau^2\partial_i\partial_0\\ &- H^2P\tau^2\partial_i\partial_0)h.\end{aligned} \tag{21}$$

$$\begin{aligned}\delta G_{ij} = &\delta_{ij}H^2K_{00} + H^2\tau\partial_j K_{0i} + H^2\tau\partial_i K_{0j} + (4H^2 + \tfrac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)K_{ij}\\ &+ \delta_{ij}(-\tfrac{3}{4}H^2 - \tfrac{1}{4}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{2}H^2P\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2P\tau\partial_0)h + (\tfrac{1}{4}H^2\tau^2\partial_i\partial_j - H^2P\tau^2\partial_i\partial_j)h.\end{aligned} \tag{22}$$