Quantum Mechanics II
HW 10

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1. Dirac matrices and Helicity

   (a) Verify that the Dirac matrices really do satisfy the anti-commutation relations:
   $$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

   (b) Verify that the free Dirac hamiltonian commutes with the "helicity" operator (the projection of the spin along the direction of the momentum):
   $$\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}$$

   (a) Let's separate the space and time indices. With $\mu = \nu = 0$ we have
   $$\{\gamma_0, \gamma_0\} = 2\gamma_0^2 = 2\mathbb{1}.$$
   For $\nu = 0$ we note that
   $$\gamma_i\gamma_0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} = -\gamma_0\gamma_i$$
   Therefore
   $$\{\gamma_i, \gamma_0\} = 2\delta_{i0}\mathbb{1}.$$
   Now for the space indices i, j,
   $$\gamma_i\gamma_j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix}$$
   hence
   $$\{\gamma_i, \gamma_j\} = -\begin{pmatrix} \{\sigma_i, \sigma_j\} & 0 \\ 0 & \{\sigma_i, \sigma_j\} \end{pmatrix}.$$
   Using the anti-commutation property of the Pauli matrices,
   $$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$$
   we have
   $$\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbb{1}.$$
   With our results,
   $$\{\gamma_i, \gamma_0\} = 2\delta_{i0}\mathbb{1}; \qquad \{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbb{1},$$
   we see that we get back our Minkowski metric with a factor of 2
   $$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}.$$

(b) The free Dirac Hamiltonian is

$$H = c\gamma^0\gamma^i p_i + \gamma^0 mc^2$$

or

$$H = c\vec{\alpha}\cdot\vec{p} + \vec{\beta}mc^2$$

Spin operators commute with position/momentum operators (identity in tensor product space). Commuting with the Hamiltonian

$$\left[H, \frac{S_i p_i}{\sqrt{p^j p_j}}\right] = c\left[\alpha^i p_i, \frac{S^i p_i}{\sqrt{p^j p_j}}\right] + mc^2\left[\gamma^0, \frac{S^i p_i}{\sqrt{p^j p_j}}\right]$$

First we note that with $[p_\mu, p_\nu] = 0$ we have

$$\left[p_i, \frac{1}{\sqrt{p^j p_j}}\right] \sim [p_i, p^j p_j] = 0$$

Thus we can pull out the factor of $\frac{1}{|\vec{p}|}$

$$c\left[\alpha^i p_i, \frac{S^i p_i}{\sqrt{p^j p_j}}\right] + mc^2\left[\gamma^0, \frac{S^i p_i}{\sqrt{p^j p_j}}\right] = \frac{1}{|\vec{p}|}\left(c[\alpha^i p_i, S^j p_j] + mc^2[\gamma^0, S^j p_j]\right)$$

Looking at the first commutator

$$\begin{aligned}
[\alpha^i p_i, S^j p_j] &= [\alpha^i p_i, S^j]p_j - S^j[\alpha^i p_i, p_j] \\
&= [\alpha^i p_i, S^j]p_j \\
&= p_i p_j[\alpha^i, S^j] \qquad (1)
\end{aligned}$$

The components of the spin operator are

$$S_i = \frac{\hbar}{2}\sigma_i \to \frac{\hbar}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & \sigma_i\end{pmatrix}.$$

Using this we find the commutator of α_i with S_j

$$\begin{aligned}
[\alpha_i, S_j] &= \frac{\hbar}{2}\left[\begin{pmatrix}0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}, \begin{pmatrix}\sigma_j & 0 \\ 0 & \sigma_j\end{pmatrix}\right] \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0\end{pmatrix} \\
&= i\hbar\begin{pmatrix}0 & \epsilon^{ijk}\sigma_k \\ \epsilon^{ijk}\sigma_k & 0\end{pmatrix} \qquad (2)
\end{aligned}$$

Using (1) and (2) we can construct the operator over all components of $\vec{\alpha}$ and $\vec{S}$

$$\begin{aligned}
[\alpha^i p_i, S^j p_j] &= p_i p_j[\alpha^i, S^j] \\
&= i\hbar\epsilon^{ijk}\, p_i p_j \sigma_k \begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}. \qquad (3)
\end{aligned}$$

This sum is antisymmetric with respect to ij, and thus equates to zero when summed over all permutations. Hence all that remains is

$$[\gamma^0, S^i p_i] = p_i[\gamma^0, S^i] = p_i\left(\gamma^0 S^i - S^i\gamma^0\right) = p_i\left(\gamma^0 S^i - \gamma^0 S^i\right) = 0. \qquad (4)$$

Therefore, using (3) and (4) we may conclude

$$\left[H, \frac{S_i p_i}{\sqrt{p^j p_j}}\right] = 0.$$

2. Lorentz transformation of the Dirac current

   (a) Verify that with the transformation $\psi \to \mathcal{M}\psi$ derived in class, for an infinitesimal Lorentz transformation, the Dirac current density $c\bar{\psi}\gamma^\mu\psi$ transforms as a vector under the Lorentz transformation: $j^\mu \to \Lambda^\mu_\nu j^\nu$.

I am going to re-derive the infinitesimal transformation matrix $\mathcal{M}$ in $\psi' = \mathcal{M}\psi$.
We require that the Dirac equation take the same form in all inertial frames of reference. Measurable quantities in the unprimed frame relate to those in a primed frame via a Lorentz transformation:

$$p_\mu = \Lambda^\nu_\mu p'_\nu$$

$$\psi = \mathcal{M}^{-1}\psi'.$$

Forming the Dirac equation

$$\begin{aligned} \gamma^\mu p_\mu \psi = mc\psi \quad &\Rightarrow \quad \gamma^\mu \Lambda^\nu_\mu p'_\nu \mathcal{M}^{-1}\psi' = mc\mathcal{M}^{-1}\psi' \\ &\Rightarrow \quad \mathcal{M}\gamma^\mu \Lambda^\nu_\mu p'_\nu \mathcal{M}^{-1}\psi' = mc\psi'. \end{aligned}$$

Now, for the Dirac equation to be Lorentz invariant we require

$$\mathcal{M}\gamma^\mu \Lambda^\nu_\mu \mathcal{M}^{-1} = \gamma^\nu$$

or

$$\gamma^\mu \Lambda^\nu_\mu = \mathcal{M}^{-1}\gamma^\nu \mathcal{M}. \tag{5}$$

From here on we work only to first order. The Lorentz transformation is then the identity plus a first order (infinitesimal) change

$$\Lambda^\nu_\mu = g^\nu_\mu + \Delta\omega^\nu_\mu$$

If we take an infinitesimal Lorentz transformation along with its inverse $(\Lambda^\rho_\mu)^T = (\Lambda^\rho_\mu)^{-1}$ we expect to obtain the identity

$$\Lambda^\mu_\nu \Lambda^\rho_\mu = g^\rho_\nu$$

hence

$$(g^\mu_\nu + \Delta\omega^\mu_\nu)(g^\rho_\mu + \Delta\omega^\rho_\mu) = g^\rho_\nu$$

From this, we deduce that the infinitesimal parameters of the Lorentz transformation are antisymmetric

$$\Delta\omega^{\rho\nu} + \Delta\omega^{\nu\rho} = 0,$$

and thus given by $(n+1)n/2$ independent elements in $n = 4$ dimensions. We now use these elements to construct the transformation matrix $\mathcal{M}$. The 4×4 matrix $\mathcal{M}$ can be viewed a function of the Lorentz parameters, and thus may be written to first order as

$$\mathcal{M} = \mathbb{1} - \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}.$$

The factor of i ensures unitarity and in the limit that $\omega^{\alpha\beta} \to 0$ we obtain the identity.

Having $\mathcal{M}$, we go back and form (5)

$$\gamma^\mu(g^\nu_\mu + \Delta\omega^\nu_\mu) = \left(\mathbb{1} - \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right)\left(\mathbb{1} + \frac{i}{4}\Delta\omega^{\alpha\beta}\sigma_{\alpha\beta}\right).$$

To first order then

$$\gamma^\mu \Delta\omega^\nu_\mu = \frac{i}{4}\Delta\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\nu]. \tag{6}$$

Using antisymmetry, we may re-express the left hand side of (6)

$$\gamma^\mu \Delta\omega^\nu_\mu = g^{\mu\beta}\gamma_\beta \Delta\omega^\nu_\mu = \Delta\omega^{\nu\beta}\gamma_\beta = \Delta\omega^{\alpha\beta}g^\nu_\alpha\gamma_\beta = \frac{1}{2}\Delta\omega^{\alpha\beta}\left(g^\nu_\alpha\gamma_\beta - g^\nu_\beta\gamma_\alpha\right).$$

Hence (6) becomes

$$g^{\nu}_{\alpha}\gamma_{\beta} - g^{\nu}_{\beta}\gamma_{\alpha} = \frac{i}{2}[\sigma_{\alpha\beta}, \gamma^{\nu}]$$

or

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}].$$

We finally have an expression to first order for the spinor transformation matrix $\mathcal{M}$ in terms of the gamma matrices and Lorentz parameters

$$\mathcal{M} = \mathbb{1} + \frac{1}{8}\Delta\omega^{\alpha\beta}[\gamma_{\alpha}, \gamma_{\beta}]. \tag{7}$$

The current density is given as

$$j^{\mu} = c\bar{\psi}\gamma^{\mu}\psi$$

with

$$\bar{\psi} = \psi^{\dagger}\gamma^{0}.$$

Under a Lorentz transformation $j^{\mu} \to j'^{\mu}$ and so

$$\begin{aligned} j'^{\mu} &= \bar{\psi}'\gamma^{\mu}\psi' \\ &= \psi^{\dagger}\mathcal{M}^{\dagger}\gamma^{0}\gamma^{\mu}\mathcal{M}\psi \\ &= \bar{\psi}\gamma^{0}\mathcal{M}^{\dagger}\gamma^{0}\gamma^{\mu}\mathcal{M}\psi. \end{aligned}$$

Using

$$\mathcal{M}^{\dagger} = \gamma^{0}\mathcal{M}^{-1}\gamma^{0}$$

the current becomes

$$j'^{\mu} = \bar{\psi}\mathcal{M}^{-1}\gamma^{\mu}\mathcal{M}\psi.$$

Recalling eq. (5) ($\gamma^{\mu}\Lambda^{\nu}_{\mu} = \mathcal{M}^{-1}\gamma^{\nu}\mathcal{M}$) we finally have

$$j^{\mu} \to j'^{\mu} = \Lambda^{\mu}_{\nu}j^{\nu} = \Lambda^{\mu}_{\nu}(\bar{\psi}\gamma^{\nu}\psi).$$

(b) Verify that $\partial_{\alpha}F^{\alpha\mu}$ also transforms the same way. Why is this important?

Under a Lorentz transformation, a tensor transforms as

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta}.$$

The EM current density

$$j^{\nu} = \partial_{\mu}F^{\mu\nu} = (-i\hbar)p_{\mu}F^{\mu\nu}$$

transforms to

$$\begin{aligned} j'^{\nu} to &= (-i\hbar)p'_{\mu}F'^{\mu\nu} \\ &= (-i\hbar)\Lambda^{\rho}_{\mu}p_{\rho}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta} \\ &= (-i\hbar)\Lambda^{\rho}_{\mu}\Lambda^{\mu}_{\alpha}p_{\rho}\Lambda^{\nu}_{\beta}F^{\alpha\beta} \\ &= (-i\hbar)g^{\rho}_{\alpha}p_{\rho}\Lambda^{\nu}_{\beta}F^{\alpha\beta} \\ &= (-i\hbar)p_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta} \\ &= \Lambda^{\nu}_{\beta}(\partial_{\alpha}F^{\alpha\beta}) \\ &= \Lambda^{\nu}_{\beta}j^{\beta}. \end{aligned}$$

Hence the electromagnetic current density also transforms as a vector under a Lorentz transformation. This is important because we expect Maxwells equations $\partial_{\mu}F^{\mu\nu} = 0$ to be the same in differential inertial frames of reference, i.e. Lorentz invariant. Likewise, charge is conserved under a Lorentz transformation.

3. Spin Matrices

(a) Verify that the "spin" matrices $\Sigma_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ satisfy the relation needed in the proof of Lorentz covariance of the Dirac equation:

$$\left(g^\nu_\alpha \gamma_\beta - g^\nu_\beta \gamma_\alpha\right) = \frac{i}{4}[\Sigma_{\alpha\beta}, \gamma^\nu]$$

Noting that

$$\begin{aligned}[\gamma_\mu, \gamma_\nu] &= 2\gamma_\mu\gamma_\nu - \{\gamma_\mu, \gamma_\nu\} \\ &= 2(\gamma_\mu\gamma_\nu - g_{\mu\nu})\end{aligned}$$

and that

$$\{\gamma^\mu, \gamma_\nu\} = \{g^{\mu\alpha}\gamma_\alpha, \gamma_\nu\} = g^{\mu\alpha}\{\gamma_\alpha, \gamma_\nu\} = 2g^{\mu\alpha}g_{\alpha\nu} = 2g^\mu_\nu$$

we form the commutator of the Sigma matrix,

$$\begin{aligned}\frac{i}{4}[\Sigma_{\alpha\beta}, \gamma^\nu] &= -\frac{1}{16}[[\gamma_\alpha, \gamma_\beta], \gamma^\nu] \\ &= \frac{1}{8}[\gamma^\nu, \gamma_\alpha\gamma_\beta + g_{\alpha\beta}] \\ &= \frac{1}{8}\left([\gamma^\nu, \gamma_\alpha\gamma_\beta] + [\gamma^\nu, g_{\alpha\beta}]\right) \\ &= \frac{1}{8}\left(\gamma_\alpha[\gamma^\nu, \gamma_\beta] + [\gamma^\nu, \gamma_\alpha]\gamma_\beta + 0\right) \\ &= \frac{1}{8}\left(\{\gamma_\alpha, \gamma^\nu\}\gamma_\beta - \gamma_\alpha\{\gamma_\beta, \gamma^\nu\}\right) \\ &= \frac{1}{4}\left(g^\nu_\alpha\gamma_\beta - g^\nu_\beta\gamma_\alpha\right).\end{aligned}$$

If the Sigma matrix is normalized as $\Sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]$, then we obtain the relation

$$\frac{i}{4}[\Sigma_{\alpha\beta}, \gamma^\nu] = g^\nu_\alpha\gamma_\beta - g^\nu_\beta\gamma_\alpha.$$

(b) Express Σ_{0i} and Σ_{ij} in terms of their 2×2 sub-blocks using the Pauli matrices.

For one time index,

$$\Sigma_{0i} = \frac{i}{4}[\gamma_0, \gamma_i] = \frac{i}{4}\left[\begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

For Σ_{ij},

$$\Sigma_{ij} = \frac{i}{4}\left[-\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} + \begin{pmatrix} \sigma_j\sigma_i & 0 \\ 0 & \sigma_j\sigma_i \end{pmatrix}\right] = -\frac{i}{4}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix}.$$

WIth the Pauli commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

this becomes

$$\Sigma_{ij} = \frac{1}{2}\epsilon^k_{ij}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}.$$

(c) Define $K_i = \Sigma_{0i}$ and $J_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk}$, and compute the commutators:

$$[K_i, K_j]; \qquad [K_i, J_j]; \qquad [J_i, J_j]$$

For K_i we have

$$K_i = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

For J_i we have

$$J_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk} = \frac{1}{4}\epsilon_{ijk}\epsilon^{jkl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}.$$

Using

$$\epsilon_{ijk}\epsilon^{jkl} = \epsilon_{ijk}\epsilon^{ljk} = 2\delta_i^l$$

we then have

$$J_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

Commutators between the vector elements of $\vec{J}$ are

$$[J_i, J_j] = \frac{1}{4}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix} = \frac{i}{2}\epsilon_{ij}^k\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = i\Sigma_{ij} = i\epsilon_{ij}^k J_k.$$

Commutators between the vector elements of $\vec{K}$ are

$$[K_i, K_j] = -\frac{1}{4}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix} = -\frac{i}{2}\epsilon_{ij}^k\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = \epsilon_{ij}^k\Sigma_{ij} = -i\epsilon_{ij}^k J_k.$$

Commutators between vector differing elements

$$[K_i, J_j] = -\frac{i}{4}\begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix} = \frac{1}{2}\epsilon_{ij}^k\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = -i\epsilon_{ij}^k\Sigma_{0k} = -i\epsilon_{ij}^k K_k.$$

All together then

$$[K_i, K_j] = -i\epsilon_{ij}^k J_k; \qquad [J_i, J_j] = i\epsilon_{ij}^k J_k; \qquad [K_i, J_j] = -i\epsilon_{ij}^k K_k$$

(d) Comment on the results of the previous part.

We have 6 independent elements (two vectors) that are closed under commutation and thus form some sort of group. The vector $\vec{J}$ appears to transform just like angular momentum. If we take $\vec{J}$ as the generators of angular momentum and $\vec{K}$ as generators of boosts, we form a representation of the Lorentz group in terms of Dirac matrices. Two successive rotations relate to a rotation about the third axis, two successive boosts relate to a rotation (interesting), and a boost followed by rotation relates to a boost about the untouched axis. I'm sure there is even much more that could be said about these pairs of commutators from a group theory perspective.