Boundary Conditions

Under infinitesimal coordinate transformation $x^\mu \to \bar{x}^\mu = x^\mu + \epsilon^\mu(x)$ where

$$\epsilon^0 = T, \qquad \epsilon^i = \tilde{\nabla}^i L + L^i, \qquad \tilde{\nabla}^i L_i = 0,$$

it follows that h_{0i} transforms as

$$\bar{h}_{0i} = h_{0i} - (\tilde{\nabla}_i \dot{L} + L_i) + \partial_i T \tag{1}$$

which evaluates to

$$\tilde{\nabla}_i \bar{B} + \bar{B}_i = \tilde{\nabla}_i B + B_i - \tilde{\nabla}_i \dot{L} - \dot{L}_i + \tilde{\nabla}_i T. \tag{2}$$

or

$$\tilde{\nabla}_i \bar{B} + \bar{B}_i = \tilde{\nabla}_i (B - \dot{L} + T) + B_i. \tag{3}$$

Since an arbitrary gradient of a scalar such as $\tilde{\nabla}_i T$ could in fact be transverse, we cannot immediately separate scalars to scalars and vectors to vectors. If we take the divergence, we arrive at

$$\tilde{\nabla}_a \tilde{\nabla}^a \bar{B} = \tilde{\nabla}_a \tilde{\nabla}^a (B - \dot{L} + T), \tag{4}$$

in which we may define $\bar{B}$ as

$$\begin{aligned}
\bar{B} &= \int d^3y\ D^3(\mathbf{x} - \mathbf{y}) \tilde{\nabla}^y_a \tilde{\nabla}^a_y (B - \dot{L} + T) \\
&= \int d^3y\ \tilde{\nabla}^y_a \tilde{\nabla}^a_y \left[D^3(\mathbf{x} - \mathbf{y})(B - \dot{L} + T) \right] - \int d^3y\ \tilde{\nabla}^y_a \tilde{\nabla}^a_y D^3(\mathbf{x} - \mathbf{y})(B - \dot{L} + T) \\
&= B - \dot{L} + T + \int dS_a\ \tilde{\nabla}^a_y \left[D^3(\mathbf{x} - \mathbf{y})(B - \dot{L} + T) \right] \\
&= B - \dot{L} + T + \chi.
\end{aligned} \tag{5}$$

The surface term takes the form

$$\chi = \int dS_a\ \tilde{\nabla}^a_y D^3(\mathbf{x} - \mathbf{y})(B - \dot{L} + T) + \int dS_a\ D^3(\mathbf{x} - \mathbf{y}) \tilde{\nabla}^a_y (B - \dot{L} + T). \tag{6}$$

The discussion in Jackson Electrodynamics pg. 39 suggests that a given Green's function $D(\mathbf{x}, \mathbf{y})$, may be defined up to an arbitrary function $F(\mathbf{x}, \mathbf{y})$ which satisfies $\nabla^2 F(\mathbf{x}, \mathbf{y}) = 0$. It is then suggested that the freedom in $F(\mathbf{x}, \mathbf{y})$ may be used to formulate the solution for $\bar{B}$ in terms of either Dirichlet or Neumann boundary conditions by finding an $F(\mathbf{x}, \mathbf{y})$ such that

$$D(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for} \quad \mathbf{x} \text{ on } S, \qquad \text{or} \qquad \tilde{\nabla}_a D(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for} \quad \mathbf{x} \text{ on } S. \tag{7}$$

Let us assume we were able to find an $F(\mathbf{x}, \mathbf{y})$ that allows for Dirichlet boundary conditions, i.e.

$$D(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for} \quad \mathbf{x} \text{ on } S, \tag{8}$$

then in order to arrive at the desired equation of

$$\bar{B} = B - \dot{L} + T \tag{9}$$

we must require that

$$B - \dot{L} + T = 0 \quad \text{for} \quad \mathbf{x} \text{ on } S, \tag{10}$$

with S being the asymptotic boundary surface at infinity. Imposing such a boundary condition would seem to allow better constraints when expanding the perturbation functions in momentum space viz.

$$B(t,x) = \int d^3k \; e^{ikx} \tilde{B}(t,k). \tag{11}$$

For example, an equation such as

$$\tilde{\nabla}_a \tilde{\nabla}^a (B - E) = 0, \tag{12}$$

leads to

$$\int d^3k \; e^{ikx} k^2 [-\tilde{B}(t,k) + \tilde{E}(t,k)] = 0. \tag{13}$$

Without boundary conditions, either $\tilde{B}(t,k) = \tilde{E}(t,k)$ or $\tilde{B}(t,k) = \tilde{E}(t,k) + \delta(k)$ (or perhaps $k^n \delta(k)$ for $n > -2$). However, the requirement that $B(t,x)$ and $E(t,x)$ vanish at spatial infinity excludes the possible $\delta(k)$ solutions and thus yields $\tilde{B}(t,k) = \tilde{E}(t,k)$ and consequently $B(t,x) = E(t,x)$.

As an aside, we take the Laplacian of the boundary term χ, which evaluates to

$$\begin{aligned} \tilde{\nabla}^x_b \tilde{\nabla}^b_x \chi &= \int dS_a \; \tilde{\nabla}^a_y \delta^3(\mathbf{x}-\mathbf{y})(B - \dot{L} + T) + \int dS_a \; \delta^3(\mathbf{x}-\mathbf{y}) \tilde{\nabla}^a_y (B - \dot{L} + T) \\ &= -\tilde{\nabla}^a_x \int dS_a \; \delta^3(\mathbf{x}-\mathbf{y})(B - \dot{L} + T) + \int dS_a \; \delta^3(\mathbf{x}-\mathbf{y}) \tilde{\nabla}^a_y (B - \dot{L} + T) \end{aligned} \tag{14}$$

The quantity $\nabla^2 \chi$ is only supported asymptotically, but even if $\mathbf{x}$ is evaluated at a point on the infinite surface, the two surface terms will mutually cancel. Therfore, for all $\mathbf{x}$ such a χ obeys

$$\tilde{\nabla}_a \tilde{\nabla}^a \chi = 0. \tag{15}$$

Introduce the scalar propagator $D(x - x')$, which obeys

$$\partial_\nu \partial^\nu D(x - x') = \delta(x - x'). \tag{16}$$

Take the mathematical identity

$$\phi(x') \partial_\nu \partial^\nu D(x - x') = D(x - x') \partial_\nu \partial^\nu \phi(x') + \partial_\nu \left[\phi(x') \partial^\nu D(x - x') - D(x - x') \partial^\nu \phi(x') \right], \tag{17}$$

where here $\partial_\nu = \frac{\partial}{\partial x'^\nu}$. Now integrate over a region S,

$$\begin{aligned} \int d^4x' \; \phi(x') \partial_\nu \partial^\nu D(x - x') &= \int d^4x' \; D(x - x') \partial_\nu \partial^\nu \phi(x') + \int dS_\nu \left[\phi(x') \partial^\nu D(x - x') - D(x - x') \partial^\nu \phi(x') \right] \\ \phi(x) &= \int d^4x' \; D(x - x') \partial_\nu \partial^\nu \phi(x') + \int dS_\nu \left[\phi(x') \partial^\nu D(x - x') - D(x - x') \partial^\nu \phi(x') \right]. \end{aligned} \tag{18}$$

Thus we have separated ϕ into two parts

$$\phi(x) = \int d^4x' \; D(x - x') \partial_\nu \partial^\nu \phi(x') + \int dS_\nu \left[\phi(x') \partial^\nu D(x - x') - D(x - x') \partial^\nu \phi(x') \right]. \tag{19}$$

$$\phi = \phi^L + \phi^T \tag{20}$$

where

$$\phi^L = \int d^3x' \ D(x - x')\nabla_a \nabla^a \phi(x'), \qquad \phi^T = \int dS_a \left[\phi(x')\nabla^a D(x - x') - D(x - x')\nabla^a \phi(x')\right]. \tag{21}$$

Taking the Laplacian

$$\nabla_a \nabla^a \phi^L = \nabla_a \nabla^a \phi, \qquad \nabla_a \nabla^a \phi^T = \int dS_a \left[\phi(x')\nabla^a \delta(x - x') - \delta(x - x')\nabla^a \phi(x')\right]. \tag{22}$$

We see that $\nabla_a \nabla^a \phi^L = 0$ only if $\nabla_a \nabla^a \phi = 0$, but this then entails by definition of ϕ^L that $\phi^L = 0$. Therefore, only a completely transverse $\nabla_a \nabla^a \phi$ entails $\nabla_a \nabla^a \phi^L = 0$.