4D SVT Thoughts

Green's Identity

For any function ϕ, we may always represent it as

$$\phi = \int D\nabla^2\phi + \int \nabla^\rho[\nabla_\rho D\phi - D\nabla_\rho\phi]. \tag{1}$$

Given $\nabla^2\phi = \rho$, this leads us to the fundamental solution to Laplace's equation,

$$\phi = \int D\rho + \int dS^\nu[\nabla_\rho D\phi - D\nabla_\rho\phi]. \tag{2}$$

In flat space, the same identity holds for vectors, i.e.

$$A_\mu = \int D\nabla^2 A_\mu + \int \nabla^\rho[\nabla_\rho DA_\mu - D\nabla_\rho A_\mu]. \tag{3}$$

Given $\nabla^2 A_\mu = J_\mu$, the fundamental solution is then

$$A_\mu = \int DJ_\mu + \int \nabla^\rho[\nabla_\rho DA_\mu - D\nabla_\rho A_\mu]. \tag{4}$$

It is of interest to ask what happens when we take the divergence:

$$\begin{aligned}
\nabla^\mu A_\mu &= \nabla^\mu \int DJ_\mu + \nabla^\mu \int \nabla^\rho[\nabla_\rho DA_\mu - D\nabla_\rho A_\mu] \\
&= \int D\nabla^\mu J_\mu - \int \nabla^\mu(DJ_\mu) + \int \nabla^\rho[\nabla^\mu D\nabla_\rho A_\mu - \nabla^\mu\nabla_\rho DA_\mu] \\
&= \int D\nabla^\mu J_\mu + \int \nabla^\rho[\nabla^\mu D\nabla_\rho A_\mu - \nabla^\mu\nabla_\rho DA_\mu - D\nabla^2 A_\rho]
\end{aligned} \tag{5}$$

Using the delta function relation

$$\int \nabla^2 D\nabla^\mu A_\mu = -\int \nabla^\mu\nabla^2 DA_\mu \tag{6}$$

we may express (5) as

$$\nabla^\mu A_\mu = \int D\nabla^\mu J_\mu + \int \nabla^\rho[\nabla_\rho D\nabla^\mu A_\mu - D\nabla_\rho\nabla^\mu A_\mu], \tag{7}$$

which we may recognize as the fundamental solution to $\nabla^2(\nabla^\mu A_\mu) = \nabla^\mu J_\mu$, where we treat $\nabla^\mu A_\mu$ as a scalar with no apriori assumptions on the form of A_μ.

Hence we have shown that the divergence of A_μ as defined by (4) is consistent with the fundamental solution of $\nabla^2\phi = \rho$ where $\phi = \nabla^\mu A_\mu$ and $\rho = \nabla^\mu J_\mu$. In order to construct an A_μ that obeys $\nabla^\mu A_\mu = \int D\nabla^\mu J_\mu$ we require $\nabla^\mu A_\mu$ and D to vanish on the surface.

4D SVT

In reference to your email, according to decomposition (C), W_μ is fixed by condition (D)

$$\nabla^2 W_\mu = \nabla^\alpha h_{\alpha\mu}. \tag{8}$$

The fundamental solution to (D) is

$$W_\mu = \int D\nabla^\alpha h_{\alpha\mu} + \int \nabla^\alpha[\nabla_\alpha D W_\mu - D\nabla_\alpha W_\mu]. \tag{9}$$

Taking the divergence, we have from (7)

$$\nabla^\alpha W_\alpha \quad = \quad \int D\nabla^\alpha\nabla^\beta h_{\alpha\beta} + \int \nabla^\rho[\nabla_\rho D\nabla^\alpha W_\alpha - D\nabla_\rho\nabla^\alpha W_\alpha] \tag{10}$$

From (D) we may construct (G) as

$$\nabla^2[\nabla^\alpha W_\alpha - h] = \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla^2 h, \tag{11}$$

which is equivalent to

$$\nabla^2\nabla^\alpha W_\alpha = \nabla^\alpha\nabla^\beta h_{\alpha\beta}. \tag{12}$$

Based on the Green's identities, we have shown that fundamental solution to the above $\nabla^2\nabla^\alpha W_\alpha = \nabla^\alpha\nabla^\beta h_{\alpha\beta}$ is again just (10). Decomposing h into its harmonic and non-harmonic components via

$$h = \int D\nabla^2 h + \int \nabla^\alpha[\nabla_\alpha D h - D\nabla_\alpha h], \tag{13}$$

the most general combination $\nabla^\alpha W_\alpha - h$ that satisfies the condition $\nabla^2 W_\mu = \nabla^\alpha h_{\mu\alpha}$ must be

$$\nabla^\alpha W_\alpha - h \quad = \quad \int D[\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla^2 h] + \underbrace{\oint dS^\rho[\nabla_\rho D(\nabla^\alpha W_\alpha - h) - D\nabla_\rho(\nabla^\alpha W_\alpha - h)]}_{A}. \tag{14}$$

The harmonic surface term A may vanish given that D, h, and $\nabla^\alpha W_\alpha$ vanish on the surface. Such constraints would appear to correspond to the freedom to perform integration by parts. Hence it would appear we cannot construct a ψ or $\nabla^2 E_{\mu\nu}$ that is gauge invariant under large spatial gauge transformations.

Nonetheless, in defining

$$\psi \quad = \quad \nabla^\alpha W_\alpha - h, \tag{15}$$

it holds that $\nabla^2\psi$, $\nabla^2 E_{\mu\nu} + (D-2)\nabla_\mu\nabla_\nu\psi$ and $\nabla^4 E_{\mu\nu}$ are gauge invariant.

Though we can reframe the decomposition entirely in terms of a W_μ which must obey $\nabla^2 W_\mu = \nabla^\alpha h_{\mu\alpha}$, the most general decomposition that is gauge invariant under all transformations would appear to require a W_μ with the form given in (9).