Fourier Transform of SVT

In a flat background of $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, Lie derivatives of $\delta G_{\mu\nu}$ vanish thus making $\delta G_{\mu\nu}$ gauge invariant all on its own. We will decompose $\delta G_{\mu\nu}$ by S.V.T. via the metric

$$ds^2 = -(1+2\phi)d\tau^2 + 2(B_i + \partial_i B)d\tau dx^i + \left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + \partial_i E_j + \partial_j E_i + 2E_{ij}\right] dx^i dx^j.$$

The perturbed Einstein tensor can be expressed in terms of the following gauge invariant quantities:

$$\begin{aligned}
\Psi &= \psi \\
\Phi &= \phi + \dot{B} - \ddot{E} \\
Q_i &= \dot{E}_i - B_i \\
E_{ij} &= E_{ij}.
\end{aligned}$$

The perturbed tensor is then

$$\delta G_{00} = -2\nabla^2\Psi \tag{1}$$

$$\delta G_{0i} = -2\partial_i\dot{\Psi} - \frac{1}{2}\nabla^2 Q_i \tag{2}$$

$$\delta G_{ij} = -2\ddot{\Psi}\delta_{ij} + (\nabla^2\delta_{ij} - \partial_i\partial_j)(\Psi - \Phi) - \frac{1}{2}\left(\partial_i\dot{Q}_j + \partial_j\dot{Q}_i\right) + \Box E_{ij}. \tag{3}$$

The gauge invariant variables are subject to the constraints

$$\partial^i Q_i = 0, \qquad \partial^i E_{ij} = 0, \qquad \delta^{ij} E_{ij} = 0. \tag{4}$$

We may also decompose $\delta T_{\mu\nu}$ in a manner exactly analogous to $\delta g_{\mu\nu}$, where perturbed variables are denoted with bars. Since the zeroth order terms in $T_{\mu\nu}$ vanish, all first order terms are all automatically gauge invariant. In addition to the Einstein equation, we have the conservation of energy

$$\partial^\mu \delta T_{\mu\nu} = 0$$

yielding the two equations

$$-2\dot{\bar{\phi}} - \nabla^2\bar{B} = 0 \tag{5}$$

$$-(\dot{\bar{B}}_i + \partial_i\dot{\bar{B}}) - 2\partial_i\bar{\psi} + 2\partial_i\nabla^2\bar{E} + \nabla^2\bar{E}_i = 0 \tag{6}$$

Let us represent each of the perturbed variables in terms of its Fourier decomposition, i.e.

$$\Psi(x,t) = \int d^3k\ e^{ikx}\hat{\Psi}(k,t) \tag{7}$$

$$\Phi(x,t) = \int d^3k\ e^{ikx}\hat{\Phi}(k,t) \tag{8}$$

$$Q_i(x,t) = \int d^3k\ e^{ikx}\hat{Q}_i(k,t) \tag{9}$$

$$E_{ij}(x,t) = \int d^3k\ e^{ikx}\hat{E}_{ij}(k,t) \tag{10}$$

where the transformed quantities are defined as usual, for example

$$\hat{\Psi}(k,t) = \int d^3x \; e^{-ikx} \Psi(x,t).$$

Now if we substitute (9) and (10) into the constraint equations we have

$$\int d^3k \; e^{ikx} ik^i \hat{Q}_i(k,t) = 0, \qquad \int d^3k \; e^{ikx} ik^i \hat{E}_{ij}(k,t) = 0, \qquad \int d^3k \; e^{ikx} \delta^{ij} \hat{E}_{ij}(k,t) = 0$$

For arbitrary k, it should then follow that the constraints can be expressed as

$$k^i \hat{Q}_i = 0, \qquad k^i \hat{E}_{ij} = 0, \qquad \delta^{ij} \hat{E}_{ij} = 0. \tag{11}$$

Next we will substitute (7-10) into $\delta G_{\mu\nu} = \delta T_{\mu\nu}$. This yields

$$\begin{aligned}
\delta G_{00} - \delta T_{00} &= \int d^3k \; e^{ikx} \left[2k^2 \hat{\Psi} - \delta \hat{T}_{00} \right] = 0 \\
\delta G_{0i} - \delta T_{0i} &= \int d^3k \; e^{ikx} \left[-2ik_i \dot{\hat{\Psi}} + \frac{1}{2} k^2 \hat{Q}_i - \delta \hat{T}_{0i} \right] = 0 \\
\delta G_{ij} - \delta T_{ij} &= \int d^3k \; e^{ikx} \left[-2\ddot{\hat{\Psi}} \delta_{ij} - (k^2 \delta_{ij} - k_i k_j)(\hat{\Psi} - \hat{\Phi}) - \frac{1}{2} \left(ik_i \dot{\hat{Q}}_j + ik_j \dot{\hat{Q}}_i \right) - k^2 \hat{E}_{ij} - \ddot{\hat{E}}_{ij} - \delta \hat{T}_{ij} \right] = 0
\end{aligned}$$

Again, operating under the assumption that the inverse Fourier transform of zero is zero, we directly evaluate the integrand to zero, yielding the following new set of equations

$$2k^2 \hat{\Psi} = -2\hat{\bar{\phi}} \tag{12}$$

$$-2ik_i \dot{\hat{\Psi}} + \frac{1}{2} k^2 \hat{Q}_i = \hat{\bar{B}}_i + ik_i \hat{\bar{B}} \tag{13}$$

$$-2\ddot{\hat{\Psi}} \delta_{ij} - (k^2 \delta_{ij} - k_i k_j)(\hat{\Psi} - \hat{\Phi}) - \frac{1}{2} \left(ik_i \dot{\hat{Q}}_j + ik_j \dot{\hat{Q}}_i \right) - k^2 \hat{E}_{ij} - \ddot{\hat{E}}_{ij} = -2\hat{\bar{\psi}} \delta_{ij} - 2k_i k_j \hat{\bar{E}} + ik_i \hat{\bar{E}}_j + ik_j \hat{\bar{E}}_i + 2\hat{\bar{E}}_{ij} \tag{14}$$

$$-6\ddot{\hat{\Psi}} - 2k^2(\Psi - \Phi) = -6\bar{\psi} - 2k^2 \bar{E} \tag{15}$$

where we have included the spatial trace as the last equation. We also have the k-space conservation equations

$$-2\dot{\hat{\bar{\phi}}} + k^2 \hat{\bar{B}} = 0 \tag{16}$$

$$-(\dot{\hat{\bar{B}}}_i + ik_i \dot{\hat{\bar{B}}}) - 2ik_i \hat{\bar{\psi}} - 2ik_i k^2 \hat{\bar{E}} - k^2 \hat{\bar{E}}_i = 0. \tag{17}$$

When looking to decompose the equations in k *space* in terms of S.V.T., we first look at δG_{0i} and must assess whether $k_i \hat{\Psi}$ is orthogonal to $\hat{Q}_i$ and $\hat{\bar{B}}_i$. The most straightforward test is to take their scalar product

$$k^i \hat{\Psi} \hat{Q}_i = 0$$

where we have used the constraint eq (11). Clearly then $k_i \hat{\Psi}$ lies along k_i and Q_i is orthogonal to it. Since $\hat{\bar{B}}_i$ follows the same constraint equation as $\hat{Q}_i$, it is also orthogonal to $k_i \hat{\Psi}$. Alternatively, we may choose to apply k^i to eq (13) in which we arrive at the same decomposition. The result is the decomposition of scalar and vector equations:

$$-2\dot{\hat{\Psi}} = \hat{\bar{B}} \tag{18}$$

$$\dot{\hat{\bar{B}}} + 2\hat{\bar{\psi}} + 2k^2 \hat{\bar{E}} = 0 \tag{19}$$

$$\frac{1}{2} k^2 \hat{Q}_i = \hat{\bar{B}}_i \tag{20}$$

$$\dot{\hat{\bar{B}}}_i + k^2 \hat{\bar{E}}_i = 0 \tag{21}$$

Before looking at the spatial piece δG_{ij}, we can try to solve eq (18) and compare it to the solution obtained in Mannheim SVTsolution.pdf. The solution to (18) is

$$-2\hat{\Psi} = \int dt\ \hat{\bar{B}} + \hat{h}(k).$$

Having solved for $\hat{\Psi}$ we can now construct $\Psi(x,t)$ as

$$-2\Psi(x,t) = -2\int d^3k\ e^{ikx}\hat{\Psi} = \int d^3k e^{ikx}\left[\int dt\ \hat{\bar{B}} + \hat{h}(k)\right] = \int dt\ \bar{B} + h(x)$$

thus

$$-2\Psi(x,t) = \int dt\ \bar{B} + h(x). \tag{22}$$

Compare this to the equation calculated in SVTsolution.pdf (recall $\Psi = \psi$)

$$-2\psi(x,t) = \int dt\ \bar{B} + \alpha_j x_j \int dt\ f(t) + \int dt\ g(t) + h(x) \tag{23}$$

where $\nabla^2 h(x) = 0$.

To try to make the discrepancy more transparent, we note that the equation one obtains from solving in position space is

$$-2\nabla^2\dot{\psi} = \nabla^2\bar{B} \tag{24}$$

in which it follows

$$-2\dot{\psi} = \bar{B} + A(x,t)$$

where $\nabla^2 A(x,t) = 0$. The solution is then

$$-2\psi(x,t) = \int dt\ \bar{B} + \int dt\ A(x,t) + h(x). \tag{25}$$

However, if we transform (24) into Fourier components we get

$$-2k^2\hat{\Psi} = k^2\hat{\bar{B}}$$

which reduces to

$$-2\hat{\Psi} = \hat{\bar{B}}$$

with the solution given in eq. (22).

Related to this problem is that there seems to reside an ambiguity when we consider a vector that is both longitudinal and transverse at the same time, as in the vector $\partial_i A$ where

$$\nabla^2 A = 0.$$

Decomposing A into its Fourier transform we see

$$\nabla^2\int d^3k\ e^{ikx}\hat{A}(k,t) = -\int d^3k\ e^{ikx}k^2\hat{A}(k,t) = 0$$

and hence

$$k^2\hat{A} = 0$$

which for arbitrary k implies that $\hat{A} = 0$. The problem then is that if we try to construct $A(x,t)$ via

$$A(x,t) = \int d^3k\ e^{ikx}\hat{A}(k,t)$$

we find that $A(x,t) = 0$ which we know is not the general solution of Laplace's equation.