$$W^{(1)}_{\mu\nu} = 2g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} - 2\left(R^{\alpha}{}_{\alpha}\right)_{;\mu;\nu} - 2R^{\alpha}{}_{\alpha}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^2$$

$$W^{(2)}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_{\mu\beta;\nu}{}^{;\beta} - R_{\nu\beta;\mu}{}^{;\beta} - 2R_{\mu\beta}R_{\nu}{}^{\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left[(\delta g_{\lambda\rho})_{;\mu;\nu} - (\delta g_{\rho\mu})_{;\nu;\lambda} - (\delta g_{\rho\nu})_{;\mu;\lambda} + (\delta g_{\mu\nu})_{;\rho;\lambda}\right]$$

$$\nabla^{\beta}\nabla_{\nu}A_{\mu\beta} - \nabla_{\nu}\nabla^{\beta}A_{\mu\beta} = A^{\sigma\beta}R_{\mu\sigma\nu\beta} - A_{\mu}{}^{\sigma}R_{\sigma\nu}$$

$$\nabla^{\beta}\nabla_{\nu}A_{\mu\beta} = \nabla_{\nu}\nabla^{\beta}A_{\mu\beta} + A^{\sigma\beta}R_{\mu\sigma\nu\beta} - A_{\mu}{}^{\sigma}R_{\sigma\nu}$$

$$W^{(2)new}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_{\mu\beta;}{}^{;\beta}{}_{;\nu} - R_{\nu\beta}{}^{;\beta}{}_{;\mu} - 2R^{\sigma\beta}R_{\mu\sigma\nu\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$

Scalar

$$\begin{aligned}
\delta[\nabla_{\beta}\nabla^{\beta}A] &= \nabla_{\beta}\delta[\nabla^{\beta}A] + \delta\Gamma^{\beta}_{\beta\alpha}\nabla^{\alpha}A \\
&= \nabla_{\beta}(\delta g^{\alpha\beta}\nabla_{\alpha}A + g^{\alpha\beta}\nabla_{\alpha}\delta A) + \delta\Gamma^{\beta}_{\beta\alpha}\nabla^{\alpha}A \\
&= \nabla_{\beta}\nabla^{\beta}\delta A + \delta g^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}A + \nabla_{\beta}\delta g^{\alpha\beta}\nabla_{\alpha}A + \delta\Gamma^{\beta}_{\beta\alpha}\nabla^{\alpha}A \\
&= \nabla_{\beta}\nabla^{\beta}\delta A + \delta g^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}A + \nabla_{\beta}\delta g^{\alpha\beta}\nabla_{\alpha}A + \frac{1}{2}g^{\beta\rho}\left(\nabla_{\beta}\delta g_{\rho\alpha} + \nabla_{\alpha}\delta g_{\rho\beta} - \nabla_{\rho}\delta g_{\alpha\beta}\right)\nabla^{\alpha}A \\
&= \nabla_{\beta}\nabla^{\beta}\delta A + \delta g^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}A - g^{\alpha\gamma}g^{\beta\rho}\nabla_{\rho}\delta g_{\alpha\beta}\nabla_{\gamma}A + \frac{1}{2}g^{\alpha\gamma}g^{\beta\rho}\nabla_{\alpha}\delta g_{\rho\beta}\nabla_{\gamma}A \\
&= \nabla_{\beta}\nabla^{\beta}\delta A - h^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}A - g^{\alpha\gamma}g^{\beta\rho}\nabla_{\rho}h_{\alpha\beta}\nabla_{\gamma}A + \frac{1}{2}g^{\alpha\gamma}g^{\beta\rho}\nabla_{\alpha}h_{\rho\beta}\nabla_{\gamma}A
\end{aligned}$$

$$\begin{aligned}
\delta[\nabla_{\nu}\nabla_{\mu}A] &= \nabla_{\nu}\nabla_{\mu}\delta A - \delta\Gamma^{\lambda}_{\nu\mu}\nabla_{\lambda}A \\
&= \nabla_{\nu}\nabla_{\mu}\delta A - \frac{1}{2}g^{\lambda\rho}\left(\nabla_{\nu}h_{\mu\rho} + \nabla_{\mu}h_{\nu\rho} - \nabla_{\rho}h_{\mu\nu}\right)\nabla_{\lambda}A
\end{aligned}$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}[\nabla_{\nu}h_{\rho\mu} + \nabla_{\mu}h_{\rho\nu} - \nabla_{\rho}h_{\mu\nu}]$$

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left[(\delta g_{\lambda\rho})_{;\mu;\nu} - (\delta g_{\rho\mu})_{;\nu;\lambda} - (\delta g_{\rho\nu})_{;\mu;\lambda} + (\delta g_{\mu\nu})_{;\rho;\lambda}\right]$$

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_{\kappa}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$$

$$\begin{aligned}
\delta R^{\lambda}{}_{\mu\nu\kappa} &= \partial_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) - \partial_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa}) + \delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} + \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\kappa\eta} - \delta\Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\kappa}\delta\Gamma^{\lambda}_{\nu\eta} \\
&= \nabla_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa})
\end{aligned}$$

$$\nabla_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) = \partial_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) + \Gamma^{\lambda}_{\kappa\eta}\delta\Gamma^{\eta}_{\mu\nu} - \Gamma^{\eta}_{\mu\kappa}\delta\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\nu\kappa}\delta\Gamma^{\lambda}_{\mu\eta}$$

$$\nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa}) = \partial_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa}) + \Gamma^{\lambda}_{\nu\eta}\delta\Gamma^{\eta}_{\mu\kappa} - \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\nu\kappa}\delta\Gamma^{\lambda}_{\mu\eta}$$

$$\delta R^{\lambda}{}_{\mu\nu\kappa} = \nabla_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa})$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}[\nabla_{\nu}h_{\rho\mu} + \nabla_{\mu}h_{\rho\nu} - \nabla_{\rho}h_{\mu\nu}]$$

$$\delta\Gamma^{\lambda}_{\mu\kappa} = \frac{1}{2}g^{\lambda\rho}[\nabla_{\kappa}h_{\rho\mu} + \nabla_{\mu}h_{\rho\kappa} - \nabla_{\rho}h_{\mu\kappa}]$$

$$\begin{aligned}\delta R^{\lambda}{}_{\mu\nu\kappa} &= \frac{1}{2}g^{\lambda\rho}[\nabla_\kappa\nabla_\nu h_{\rho\mu} + \nabla_\kappa\nabla_\mu h_{\rho\nu} - \nabla_\kappa\nabla_\rho h_{\mu\nu}] \\ &\quad - \frac{1}{2}g^{\lambda\rho}[\nabla_\nu\nabla_\kappa h_{\rho\mu} + \nabla_\nu\nabla_\mu h_{\rho\kappa} - \nabla_\nu\nabla_\rho h_{\mu\kappa}] \\ &= \nabla_\kappa\nabla_\nu h^{\lambda}{}_{\mu} + \nabla_\kappa\nabla_\mu h^{\lambda}{}_{\nu} - \nabla_\kappa\nabla^{\lambda} h_{\mu\nu} - \nabla_\nu\nabla_\kappa h^{\lambda}{}_{\mu} - \nabla_\nu\nabla_\mu h^{\lambda}{}_{\kappa} + \nabla_\nu\nabla^{\lambda} h_{\mu\kappa}\end{aligned}$$

$$\nabla_\nu\nabla_\kappa h^{\lambda}{}_{\mu}$$

$$\begin{aligned}\delta R_{\rho\mu\nu\kappa} &= [\nabla_\kappa\nabla_\nu h_{\rho\mu} + \nabla_\kappa\nabla_\mu h_{\rho\nu} - \nabla_\kappa\nabla_\rho h_{\mu\nu}] \\ &\quad - [\nabla_\nu\nabla_\kappa h_{\rho\mu} + \nabla_\nu\nabla_\mu h_{\rho\kappa} - \nabla_\nu\nabla_\rho h_{\mu\kappa}]\end{aligned}$$

$$\delta(R_{\alpha\mu\beta\nu}) = \delta(g_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu}) = h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + g_{\alpha\lambda}\delta R^{\lambda}{}_{\mu\beta\nu}$$

$$h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} = h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} - \nabla_\nu\nabla_\beta h_{\alpha\nu} + \nabla_\beta\nabla_\nu h_{\alpha\mu}$$

$$\delta R_{\alpha\mu\beta\nu} = \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\nu h_{\alpha\mu} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu}$$

$$\begin{aligned}\delta R_{\alpha\mu\beta\nu} &= [\nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta}] \\ &\quad - [\nabla_\beta\nabla_\nu h_{\alpha\mu} + \nabla_\beta\nabla_\mu h_{\alpha\nu} - \nabla_\beta\nabla_\alpha h_{\mu\nu}] \\ &= \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\nu h_{\alpha\mu} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu}\end{aligned}$$

$$h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} = h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} - \nabla_\nu\nabla_\beta h_{\alpha\nu} + \nabla_\beta\nabla_\nu h_{\alpha\mu}$$

$$R^{\mu\kappa}\delta R_{\rho\mu\nu\kappa} = R^{\mu\kappa}\left(\nabla_\kappa\nabla_\nu h_{\rho\mu} + \nabla_\kappa\nabla_\mu h_{\rho\nu} - \nabla_\kappa\nabla_\rho h_{\mu\nu} - \nabla_\nu\nabla_\kappa h_{\rho\mu} - \nabla_\nu\nabla_\mu h_{\rho\kappa} + \nabla_\nu\nabla_\rho h_{\mu\kappa}\right)$$

$$\delta(R_{\alpha\mu\beta\nu}) = \delta(g_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu}) = h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + g_{\alpha\lambda}\delta R^{\lambda}{}_{\mu\beta\nu}$$

$$h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} = h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} - \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\beta\nabla_\nu h_{\alpha\mu}$$

$$\begin{aligned}g_{\alpha\lambda}\delta R^{\lambda}{}_{\mu\beta\nu} &= \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\nu h_{\alpha\mu} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu} \\ &= \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\mu\nabla_\beta h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu} + h^{\gamma}{}_{\nu}R_{\alpha\gamma\beta\mu} + h^{\gamma}{}_{\mu}R_{\alpha\gamma\beta\mu} + h^{\gamma}{}_{\alpha}R_{\mu\gamma\beta\nu} + h^{\gamma}{}_{\alpha}R_{\nu\gamma\beta\mu}\end{aligned}$$

$$\begin{aligned}h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + g_{\alpha\lambda}\delta R^{\lambda}{}_{\mu\beta\nu} = &\quad h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} - \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\beta\nabla_\nu h_{\alpha\mu} \\ &+ \nabla_\nu\nabla_\beta h_{\alpha\mu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\nu h_{\alpha\mu} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu} \\ &= h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} + \nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu}\end{aligned}$$

$$R^{\mu\nu}\delta R_{\alpha\mu\beta\nu} = h^{\lambda}{}_{\mu}R_{\alpha\lambda\beta\nu} + \frac{1}{2}\left(\nabla_\nu\nabla_\mu h_{\alpha\beta} - \nabla_\nu\nabla_\alpha h_{\mu\beta} - \nabla_\beta\nabla_\mu h_{\alpha\nu} + \nabla_\beta\nabla_\alpha h_{\mu\nu}\right)$$

$$W^{(1)}_{\mu\nu} = 2g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} - 2\left(R^{\alpha}{}_{\alpha}\right)_{;\mu;\nu} - 2R^{\alpha}{}_{\alpha}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^2$$

$$W^{(2)}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_{\mu\beta;\nu}{}^{;\beta} - R_{\nu\beta;\mu}{}^{;\beta} - 2R_{\mu\beta}R_{\nu}{}^{\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$

$$W^{(2)new}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(R^{\alpha}{}_{\alpha}\right)^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - \left(R^{\alpha}{}_{\alpha}\right)_{;\mu;\nu} - 2R^{\sigma\beta}R_{\mu\sigma\nu\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$

$$\delta R^{\lambda}{}_{\mu\nu\kappa} = \nabla_{\kappa}(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\kappa})$$

$$\begin{aligned}
\delta(R_{\alpha\mu\beta\nu}) = \delta(g_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu}) &= h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + g_{\alpha\lambda}\delta R^{\lambda}{}_{\mu\beta\nu} \\
&= h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + \frac{1}{2}\left(\nabla_{\nu}\nabla_{\beta}h_{\alpha\mu} + \nabla_{\nu}\nabla_{\mu}h_{\alpha\beta} - \nabla_{\nu}\nabla_{\alpha}h_{\mu\beta} - \nabla_{\beta}\nabla_{\nu}h_{\alpha\mu} - \nabla_{\beta}\nabla_{\mu}h_{\alpha\nu} + \nabla_{\beta}\nabla_{\alpha}h_{\mu\nu}\right) \\
&= h_{\alpha\lambda}R^{\lambda}{}_{\mu\beta\nu} + \frac{1}{2}[\nabla_{\nu}\nabla_{\mu}h_{\alpha\beta} - \nabla_{\nu}\nabla_{\alpha}h_{\mu\beta} - \nabla_{\mu}\nabla_{\beta}h_{\alpha\nu} + \nabla_{\beta}\nabla_{\alpha}h_{\mu\nu} \\
&\qquad - h_{\mu\lambda}R^{\lambda}{}_{\alpha\beta\nu} - h_{\nu\lambda}R^{\lambda}{}_{\alpha\beta\mu} - h_{\alpha\lambda}(R^{\lambda}{}_{\nu\beta\mu} + R^{\lambda}{}_{\mu\beta\nu})] \\
&= \frac{1}{2}[\nabla_{\nu}\nabla_{\mu}h_{\alpha\beta} - \nabla_{\nu}\nabla_{\alpha}h_{\mu\beta} - \nabla_{\mu}\nabla_{\beta}h_{\alpha\nu} + \nabla_{\beta}\nabla_{\alpha}h_{\mu\nu} \\
&\qquad - h_{\mu\lambda}R^{\lambda}{}_{\alpha\beta\nu} - h_{\nu\lambda}R^{\lambda}{}_{\alpha\beta\mu} - h_{\alpha\lambda}(R^{\lambda}{}_{\nu\beta\mu} - R^{\lambda}{}_{\mu\beta\nu})]
\end{aligned}$$

$$\begin{aligned}
\nabla_{\beta}\nabla_{\alpha}\nabla_{\mu}\nabla_{\nu}V - \nabla_{\mu}\nabla_{\nu}\nabla_{\alpha}\nabla_{\beta}V = {} & R_{\nu\sigma\mu\alpha}\nabla_{\beta}\nabla^{\sigma}V - R_{\sigma\beta\mu\alpha}\nabla^{\sigma}\nabla_{\nu}V + \nabla_{\alpha}(R_{\nu\sigma\mu\beta}\nabla^{\sigma}V) \\
& + R_{\nu\sigma\alpha\beta}\nabla_{\mu}\nabla^{\sigma}V - R_{\sigma\mu\alpha\beta}\nabla^{\sigma}\nabla_{\nu}V + \nabla_{\mu}(R_{\sigma\beta\alpha\nu}\nabla^{\sigma}V)
\end{aligned}$$