# Quantum Mechanics III
# HW 3

Matthew Phelps

Due: Feb. 8

2.6 (a) Show that all eigenvalues of a unitary operator have unit modulus.
(b) Show that an operator $U$ is unitary if and only if there is a hermitian operator $A$ such that $U = e^{iA}$.

(a) Unitary operators are normal $[U, U^\dagger] = 0$ and thus have a spectral representation

$$U = \sum_n c_n |n\rangle \langle n| \,.$$

By definition of $UU^\dagger = U^\dagger U = \mathbb{1}$

$$\begin{aligned} \sum_n c_n |n\rangle \langle n| \left( \sum_m c_m^* |m\rangle \langle m| \right) &= \sum_{n,m} c_n c_m^* |n\rangle \langle m| \langle n|m\rangle \\ &= \sum_{n,m} \delta_{nm} c_n c_m^* |n\rangle \langle m| \langle n|m\rangle \\ &= \sum_n |c_n|^2 |n\rangle \langle n| = \mathbb{1} \end{aligned}$$

thus the eigenvalues are of unit modulus $|c_n|^2 = 1$. We can take the adjoint of this expression $(UU^\dagger)^\dagger$ to find the same result.

(b) Using the spectral theorem, any unitary operator $U$ can always be written as

$$\sum_n e^{ia_n} |a_n\rangle \langle a_n| \,.$$

where, of course, the set $\{|a_n\rangle\}$ is an orthonormal basis and $a_n \in \mathbb{R}$. Now define the diagonalized operator

$$A = \sum_n a_n |a_n\rangle \langle a_n| \,.$$

Since the eigenvalues of this diagonalized operator are real, it is hermitian. Then, we have the relation

$$e^{iA} = \sum_m \frac{(iA)^m}{m!} = \sum_m \frac{i^m}{m!} \sum_n \left( a_n^m |a_n\rangle \langle a_n| \right) = \sum_n e^{ia_n} |a_n\rangle \langle a_n| = U$$

To prove that any operator in this form must be unitary, we note

$$UU^\dagger = U^\dagger U = e^{iA} e^{-iA^\dagger} = e^{-iA^\dagger} e^{iA} = e^{-iA} e^{iA} = \mathbb{1}$$

Since *every* unitary operator can be represented by $e^{iA}$ with $A = A^\dagger$, if there does not exist a hermitian operator $A$ such that $U = e^{iA}$, then $U$ cannot be unitary.

2.7 Take a subspace $\mathscr{S}$ of a Hilbert space $\mathscr{H}$. Suppose we have defined an operator $U$ with the property that $(U\psi, U\phi) = (\psi, \phi)$ for all vectors $\phi$ and $\phi$ in the subspace $\mathscr{S}$. Show that the operator $U$ can be extended from $\mathscr{S}$ to the whole Hilbert space $\mathscr{H}$ in such a way that the result is a unitary operator on $\mathscr{H}$. As always in (my version) of QM, you may assume that an arbitrary orthonormal set may be completed to an orthonormal basis.

The preservation of the inner product on a subspace implies, by definition, that the operator $U$ is unitary (on $\mathscr{S}$):

$$\langle \psi_{\mathscr{S}} | U^\dagger U | \phi_{\mathscr{S}} \rangle = \langle \psi_{\mathscr{S}} | \phi_{\mathscr{S}} \rangle \Rightarrow U^\dagger U = \mathbb{1}$$
$$\left( \langle \psi_{\mathscr{S}} | U^\dagger U | \phi_{\mathscr{S}} \rangle \right)^\dagger = \langle \phi_{\mathscr{S}} | U U^\dagger | \psi_{\mathscr{S}} \rangle = \langle \phi_{\mathscr{S}} | \psi_{\mathscr{S}} \rangle \Rightarrow U U^\dagger = \mathbb{1}$$
$$U^\dagger U = U U^\dagger = \mathbb{1}.$$

As such, it is normal and can be spectral decomposed on the subspace $\mathscr{S}$

$$U_{\mathscr{S}} = \sum_{n \in \mathscr{S}} e^{in} |n\rangle \langle n| \,. \tag{1}$$

Given the orthonormal subset $\{|n\rangle \in \mathscr{S}\}$ defined in (1), let us complete this set so that it forms an orthonormal basis on the Hilbert space $\mathscr{H}$. Likewise, let's extend the unitary operator in the same form as

$$U = \sum_{n \in \mathscr{H}} e^{in} |n\rangle \langle n| \,.$$

To clarify, the eigenvalues and eigenvectors of the subset $|n\rangle \in \mathscr{S}$ remain the same. Now the extended operator $U$ is also unitary on the entire Hilbert space. Take two arbitrary vectors $|\psi\rangle, |\phi\rangle \in \mathscr{H}$

$$\begin{aligned} \langle \psi | U^\dagger U | \phi \rangle &= \langle \psi | \sum_n e^{in} |n\rangle \langle n| \left( \sum_m e^{-im} |m\rangle \langle m| \right) |\phi\rangle \\ &= \langle \psi | \sum_n |n\rangle \langle n| \, |\phi\rangle \\ &= \langle \psi | \phi \rangle \,. \end{aligned}$$

Take the adjoint of this argument as we see that

$$U^\dagger U = U U^\dagger = \mathbb{1}.$$

2.9 Consider a unitary transformation of quantum mechanics defined by the unitary operator $U$ that may in the general case depend explicitly on time.

(a) Show that the time evolution of the transformed state vector is generated by the modified or "effective" Hamiltonian
$$H_E = U H U^\dagger + i\hbar \frac{dU}{dt} U^\dagger.$$

(b) Starting from the preceding result, show that in the Heisenberg picture, the effective Hamiltonian equals the zero operator.

(c) Suppose the Hamiltonian is of the form $H = H_0 + H'$. The transformation generated by $U = e^{iH_0 t/\hbar}$ leads to what is commonly referred to as the interaction picture. What is the effective Hamiltonian?

(a) Under the unitary transformation $U$, states transform as

$$|\psi\rangle \to |\tilde{\psi}\rangle = U|\psi\rangle$$

so

$$|\psi\rangle = U^\dagger |\tilde{\psi}\rangle .$$

Taking the time evolution equation

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(U^\dagger|\tilde{\psi}\rangle\right) &= HU^\dagger|\tilde{\psi}\rangle \\
i\hbar U^\dagger \frac{\partial}{\partial t}|\tilde{\psi}\rangle &= -i\hbar\frac{\partial U^\dagger}{\partial t}|\tilde{\psi}\rangle + HU^\dagger|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle &= -i\hbar U\frac{\partial U^\dagger}{\partial t}|\tilde{\psi}\rangle + UHU^\dagger|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle &= \left(-i\hbar U\frac{\partial U^\dagger}{\partial t} + UHU^\dagger\right)|\tilde{\psi}\rangle
\end{aligned}$$

hence

$$H_E = -i\hbar U\frac{\partial U^\dagger}{\partial t} + UHU^\dagger .$$

As all unitary (and possibly time dependent) operators may be written as

$$U = e^{iA(t)} \qquad \text{with} \qquad A(t) = A^\dagger(t)$$

then

$$H_E = -\hbar UU^\dagger\frac{\partial A(t)}{\partial t} + UHU^\dagger = -\hbar U^\dagger U\frac{\partial A(t)}{\partial t} + UHU^\dagger = H_E^\dagger .$$

Thus we may alternatively express the effective Hamiltonian as

$$H_E = i\hbar\frac{dU}{dt}U^\dagger + UHU^\dagger .$$

(b) Starting with

$$i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle = H_E|\tilde{\psi}\rangle$$

we can choose to represent the time evolution through a unitary operator as:

$$|\tilde{\psi}(t+\delta t)\rangle = \left(1 - \frac{i\delta t H_E}{\hbar}\right)|\tilde{\psi}(t)\rangle$$

and apply $n$ infinitesimal steps

$$|\tilde{\psi}(t+n\delta t)\rangle = \left(1 - \frac{i\delta t H_E}{\hbar}\right)^n|\tilde{\psi}(t)\rangle .$$

Now start from initial time $t_0 = 0$ take $n$ steps $\delta t = t/n$ and take the limit

$$|\tilde{\psi}(t)\rangle = \lim_{n\to\infty}\left(1 - \frac{iH_E t}{\hbar n}\right)^n|\tilde{\psi}(0)\rangle = e^{-\frac{iH_E t}{\hbar}}|\tilde{\psi}(0)\rangle .$$

Thus the unitary time evolution operator is

$$U(t) = e^{-\frac{iH_E t}{\hbar}}; \qquad |\tilde{\psi}(t)\rangle = U(t)|\tilde{\psi}(0)\rangle .$$

For the rest of the discussion denote the transformed state of part (a) as $|\psi\rangle$. In the Heisenberg picture, we effectively apply a unitary transformation that is the adjoint of the time evolution operator: $U'(t) = e^{\frac{iH_E t}{\hbar}}$. With that,

$$|\psi\rangle \to |\tilde{\psi}\rangle = U'(t)\,|\psi\rangle\,; \qquad |\psi\rangle = U'^{\dagger}\,|\tilde{\psi}\rangle\,.$$

Now we utilize the time evolution equation for the Hamiltonian $H_E$ of part (a)

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(U'^{\dagger}\,|\tilde{\psi}\rangle\right) &= H_E U'^{\dagger}\,|\tilde{\psi}\rangle \\
i\hbar U'^{\dagger}\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= \left(-H_E U'^{\dagger} + H_E U'^{\dagger}\right)|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= \left(-U' H_E U'^{\dagger} + U' H_E U'^{\dagger}\right)|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= 0 = H_{E'}\,|\tilde{\psi}\rangle
\end{aligned}$$

Hence in the Heisenberg picture, the effective Hamiltonian is the zero operator.

(c) Define the transformation $U = e^{\frac{iH_0 t}{\hbar}}$ with total Hamiltonian $H = H_0 + H'$

$$|\psi\rangle \to |\tilde{\psi}\rangle = U\,|\psi\rangle\,; \qquad |\psi\rangle = U^{\dagger}\,|\tilde{\psi}\rangle\,.$$

Then we have the time evolution

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(U^{\dagger}\,|\tilde{\psi}\rangle\right) &= (H_0 + H')U^{\dagger}\,|\tilde{\psi}\rangle \\
i\hbar U^{\dagger}\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= \left(-i\hbar\frac{\partial U^{\dagger}}{\partial t} + (H_0 + H')U^{\dagger}\right)|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= \left(-U H_0 U^{\dagger} + U(H_0 + H')U^{\dagger}\right)|\tilde{\psi}\rangle \\
i\hbar\frac{\partial}{\partial t}\,|\tilde{\psi}\rangle &= U H' U^{\dagger}\,|\tilde{\psi}\rangle\,.
\end{aligned}$$

Thus in the interaction picture, the unitary transform of the perturbation is the effective Hamiltonian

$$H_E = U H' U^{\dagger}.$$

3.3 (a) Take two distinct quantum systems 1 and 2 evolving completely independently of one another with the Hamiltonians $H_1$ and $H_2$. Show that time evolution operator for the joint system is the tensor product of the time evolution operators of individual systems.

(b) Use the result of part (a) to argue that the Hamiltonian for the joint system is $H = H_1 + H_2$.

(c) Now apply the above observations to two spins 1/2. Suppose the spin are initially in the entangled state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_1\,|\downarrow_z\rangle_2 - |\downarrow_z\rangle_1\,|\uparrow_z\rangle_2\right)$$

and that each spin evolves in a magnetic field that points in the $x$ direction,

$$H = \hbar\omega(S_{1,x} + S_{2,x}).$$

What is the state at an arbitrary time $t$?

(a) The time evolution of a state in system 1 and system 2 is

$$|\psi(t)\rangle_1 = U_1(t,t_0)\,|\psi(t_0)\rangle_1\,; \quad |\psi(t)\rangle_2 = U_2(t,t_0)\,|\psi(t_0)\rangle_2\,.$$

Taking $t_0 = 0$ for convenience, the combined two state system is given as a tensor product of the two states

$$\begin{aligned}|\psi(t)\rangle_1\,|\psi(t)\rangle_2 &= |\psi(t)\rangle_1 \otimes |\psi(t)\rangle_2\\ &= (U_1(t)\,|\psi(t_0)\rangle_1) \otimes (U_2(t)\,|\psi(t_0)\rangle_2\,|\psi(t)\rangle_2)\\ &= U_1(t)\otimes U_2(t)\,|\psi(0)\rangle_1\,|\psi(0)\rangle_2\,.\end{aligned}$$

We may also express this as

$$\begin{aligned}|\psi(t)\rangle_1\,|\psi(t)\rangle_2 &= \left(e^{-\frac{iH_1 t}{\hbar}}\right)\otimes\left(e^{-\frac{iH_2 t}{\hbar}}\right)|\psi(0)\rangle_1\,|\psi(0)\rangle_2\\ &= \left(e^{-\frac{iH_1 t}{\hbar}}\right)\left(e^{-\frac{iH_2 t}{\hbar}}\right)|\psi(0)\rangle_1\,|\psi(0)\rangle_2\end{aligned}$$

where it understood that $H_1$ only acts on system 1, i.e. $H_1 = (H_1 \otimes \mathbb{1})$ and likewise for $H_2$.

(b) Denote the total Hamiltonian of the joint system as

$$\mathbf{H} = \mathbf{H_1} + \mathbf{H_2} = (H_1 \otimes \mathbb{1}) + (\mathbb{1} \otimes H_2).$$

Now form the unitary operator

$$\begin{aligned}e^{-\frac{i\mathbf{H}t}{\hbar}} &= e^{-\frac{it}{\hbar}[(H_1\otimes\mathbb{1})+(\mathbb{1}\otimes H_2)]}\\ &= e^{-\frac{it}{\hbar}(H_1\otimes\mathbb{1})}e^{-\frac{it}{\hbar}(H_2\otimes\mathbb{1})}e^{\frac{it}{2\hbar}[(H_1\otimes\mathbb{1}),(H_2\otimes\mathbb{1})]}\\ &= e^{-\frac{it}{\hbar}(H_1\otimes\mathbb{1})}e^{-\frac{it}{\hbar}(H_2\otimes\mathbb{1})}\\ &= e^{-\frac{i\mathbf{H_1}t}{\hbar}}e^{-\frac{i\mathbf{H_2}t}{\hbar}}.\end{aligned}$$

Comparison to part (a) verifies that we may express the time evolution of the joint system by a single unitary operator with total hamiltonian

$$\mathbf{H} = \mathbf{H_1} + \mathbf{H_2}.$$

(c) First, let's determine the evolution of a single spin 1/2 particle in a magnetic system as described earlier:

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right); \quad |\downarrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - |\downarrow_z\rangle\right)$$

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right); \quad |\downarrow_z\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right)$$

$$S_x\,|\uparrow_z\rangle = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}\right)\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right) = \frac{\hbar}{2}\,|\downarrow\rangle_z$$

$$S_x\,|\downarrow_z\rangle = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}\right)\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right) = \frac{\hbar}{2}\,|\uparrow\rangle_z$$

Now we apply the unitary operator with Hamiltonian $H = \omega(S_{1,x} + S_{2,x})$ on the entangled system

$$U(t)\,|\psi(0)\rangle = e^{-\frac{iHt}{\hbar}}\,|\psi\rangle$$

$$
\begin{aligned}
&= \left(e^{-\frac{i\omega S_{1x}}{\hbar}} e^{-\frac{i\omega S_{2x}}{\hbar}}\right) \frac{1}{\sqrt{2}} \left(\left|\uparrow_z\right\rangle_1 \left|\downarrow_z\right\rangle_2 - \left|\downarrow_z\right\rangle_1 \left|\uparrow_z\right\rangle_2\right) \\
&= \left(e^{-\frac{i\omega S_{1x}}{\hbar}} e^{-\frac{i\omega S_{2x}}{\hbar}}\right) \frac{1}{2\sqrt{2}} \left[\left(\left|\uparrow_x\right\rangle_1 + \left|\downarrow_x\right\rangle_1\right)\left(\left|\uparrow_x\right\rangle_2 - \left|\downarrow_x\right\rangle_2\right) - \left(\left|\uparrow_x\right\rangle_1 - \left|\downarrow_x\right\rangle_1\right)\left(\left|\uparrow_x\right\rangle_2 + \left|\downarrow_x\right\rangle_2\right)\right] \\
&= \left(e^{-\frac{i\omega S_{1x}}{\hbar}} e^{-\frac{i\omega S_{2x}}{\hbar}}\right) \frac{1}{\sqrt{2}} \left(\left|\downarrow_x\right\rangle_1 \left|\uparrow_x\right\rangle_2 - \left|\uparrow_x\right\rangle_1 \left|\downarrow_x\right\rangle_2\right) \\
&= \frac{1}{\sqrt{2}} \left(\left|\downarrow_x\right\rangle_1 \left|\uparrow_x\right\rangle_2 - \left|\uparrow_x\right\rangle_1 \left|\downarrow_x\right\rangle_2\right) \\
&= \left|\psi(0)\right\rangle
\end{aligned}
$$