SVT3 dS$_4$ Conformal Einstein

The SVT separation of field equations $\delta G_{\mu\nu} = -\kappa_4^2 \delta T_{\mu\nu}$ in dS$_4$ are computed in three cases:
i.) general solutions, ii.) particular solutions, iii). trivial solutions

1 Background and Fluctuations

$$G^{(0)}_{\mu\nu} = 3\alpha^2 g_{\mu\nu} \tag{1.1}$$

$$R^{(0)}_{\lambda\mu\nu\kappa} = \alpha^2(g_{\mu\nu}g_{\lambda\kappa} - g_{\lambda\nu}g_{\mu\kappa}), \qquad R^{(0)}_{\mu\kappa} = -3\alpha^2 g_{\mu\kappa}, \qquad R^{(0)} = -12\alpha^2, \tag{1.2}$$

$$ds^2 = \Omega^2(\tau)[\tilde{g}_{\mu\nu} + f_{\mu\nu}]dx^\mu dx^\nu, \qquad \Omega^2(\tau) = \frac{1}{(\alpha\tau)^2} \tag{1.3}$$

$$\tilde{g}_{\mu\nu} = \mathrm{diag}(-1,1,1,1) \quad \text{or} \quad \mathrm{diag}(-1,1,r^2,r^2\sin^2\theta) \tag{1.4}$$

$$f_{00} = -2\phi, \qquad f_{0i} = \tilde{\nabla}_i B + B_i, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{1.5}$$

$$\delta G_{00} = -6\dot{\psi}\tau^{-1} - 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B + 2\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi \tag{1.6}$$

$$\delta G_{0i} = 3\tau^{-2}\tilde{\nabla}_i B - 2\tilde{\nabla}_i\dot{\psi} + 2\tau^{-1}\tilde{\nabla}_i\phi + 3B_i\tau^{-2} + \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i \tag{1.7}$$

$$\begin{aligned}
\delta G_{ij} = {} & -2\ddot{\psi}\tilde{g}_{ij} + 2\dot{\phi}\tilde{g}_{ij}\tau^{-1} + 4\dot{\psi}\tilde{g}_{ij}\tau^{-1} - 6\tilde{g}_{ij}\tau^{-2}\phi - 6\tilde{g}_{ij}\tau^{-2}\psi + 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B \\
& -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} - 2\tilde{g}_{ij}\tau^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi \\
& -2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} + 2\tau^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} + 6\tau^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + \tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi \\
& -\tau^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j + \tau^{-1}\tilde{\nabla}_i\dot{E}_j + 3\tau^{-2}\tilde{\nabla}_i E_j - \tau^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i \\
& -\tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i + \tau^{-1}\tilde{\nabla}_j\dot{E}_i + 3\tau^{-2}\tilde{\nabla}_j E_i - \ddot{E}_{ij} + 6E_{ij}\tau^{-2} + 2\dot{E}_{ij}\tau^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}
\end{aligned} \tag{1.8}$$

$$\begin{aligned}
\delta G = {} & \Omega^{-2}(-\delta G_{00} + \tilde{g}^{ab}\delta G_{ab}) \\
= {} & \alpha^2(6\dot{\phi}\tau + 18\dot{\psi}\tau - 6\ddot{\psi}\tau^2 - 18\phi - 18\psi + 6\tau\tilde{\nabla}_a\tilde{\nabla}^a B - 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} \\
& -6\tau\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} + 6\tilde{\nabla}_a\tilde{\nabla}^a E - 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\psi)
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
\Omega^{-2}\tilde{g}^{ab}\delta G_{ab} = {} & \alpha^2\big(6\dot{\phi}\tau + 12\dot{\psi}\tau - 6\ddot{\psi}\tau^2 - 18\phi - 18\psi + 4\tau\tilde{\nabla}_a\tilde{\nabla}^a B - 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} \\
& -4\tau\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} + 6\tilde{\nabla}_a\tilde{\nabla}^a E - 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\phi + 2\tau^2\tilde{\nabla}_a\tilde{\nabla}^a\psi\big)
\end{aligned} \tag{1.10}$$

$$-\kappa_4^2\delta T_{\mu\nu} = 3\alpha^2\Omega^2 f_{\mu\nu} \tag{1.11}$$

$$-\kappa_4^2\delta T_{00} = -6\tau^{-2}\phi, \tag{1.12}$$

$$-\kappa_4^2\delta T_{0i} = 3\tau^{-2}(\tilde\nabla_i B + B_i) \tag{1.13}$$

$$-\kappa_4^2\delta T_{ij} = \tau^{-2}\big(-6\tilde g_{ij}\psi + 6\tilde\nabla_i\tilde\nabla_j E + 3\tilde\nabla_i E_j + 3\tilde\nabla_j E_i + 6E_{ij}\big) \tag{1.14}$$

$$-\kappa_4^2\delta T = \alpha^2\big(6\phi - 18\psi + 6\tilde\nabla_a\tilde\nabla^a E\big) \tag{1.15}$$

$$-\kappa_4^2\Omega^{-2}\tilde g^{ab}\delta T_{ab} = \alpha^2\big(-18\psi + 6\tilde\nabla_a\tilde\nabla^a E\big) \tag{1.16}$$

2 Field Equations (Mathematica)

$$\eta = \phi + \frac{\dot\Omega}{\Omega}(B-\dot E) + (\dot B - \ddot E) = \phi - \tau^{-1}(B-\dot E) + (\dot B - \ddot E) \tag{2.1}$$

$$\xi = \psi - \frac{\dot\Omega}{\Omega}(B-\dot E) = \psi + \tau^{-1}(B-\dot E) \tag{2.2}$$

$$\Delta_{\mu\nu} \equiv \delta G_{\mu\nu} + \kappa_4^2\delta T_{\mu\nu} = 0 \tag{2.3}$$

$$\Delta_{00} = 6\eta\tau^{-2} - 6\dot\xi\tau^{-1} - 2\tilde\nabla_a\tilde\nabla^a\xi \tag{2.4}$$

$$\Delta_{0i} = 2\tau^{-1}\tilde\nabla_i\eta - 2\tilde\nabla_i\dot\xi + \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a B_i - \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a\dot E_i \tag{2.5}$$

$$\begin{aligned}\Delta_{ij} = {} & -2\ddot\xi\tilde g_{ij} - 6\eta\tilde g_{ij}\tau^{-2} + 2\dot\eta\tilde g_{ij}\tau^{-1} + 4\dot\xi\tilde g_{ij}\tau^{-1} - \tilde g_{ij}\tilde\nabla_a\tilde\nabla^a\eta + \tilde g_{ij}\tilde\nabla_a\tilde\nabla^a\xi + \tilde\nabla_j\tilde\nabla_i\eta \\ & -\tilde\nabla_j\tilde\nabla_i\xi - \tau^{-1}\tilde\nabla_i B_j + \tfrac{1}{2}\tilde\nabla_i\dot B_j - \tfrac{1}{2}\tilde\nabla_i\ddot E_j + \tau^{-1}\tilde\nabla_i\dot E_j - \tau^{-1}\tilde\nabla_j B_i + \tfrac{1}{2}\tilde\nabla_j\dot B_i \\ & -\tfrac{1}{2}\tilde\nabla_j\ddot E_i + \tau^{-1}\tilde\nabla_j\dot E_i - \ddot E_{ij} + 2\dot E_{ij}\tau^{-1} + \tilde\nabla_a\tilde\nabla^a E_{ij}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\Delta &= \Omega^{-2}(-\Delta_{00} + \tilde g^{ab}\Delta_{ab}) \\ &= -24\eta + 6\dot\eta\tau + 18\dot\xi\tau - 6\ddot\xi\tau^2 - 2\tau^2\tilde\nabla_a\tilde\nabla^a\eta + 4\tau^2\tilde\nabla_a\tilde\nabla^a\xi\end{aligned} \tag{2.7}$$

$$\Omega^{-2}\tilde g^{ab}\Delta_{ab} = -18\eta + 6\dot\eta\tau + 12\dot\xi\tau - 6\ddot\xi\tau^2 - 2\tau^2\tilde\nabla_a\tilde\nabla^a\eta + 2\tau^2\tilde\nabla_a\tilde\nabla^a\xi \tag{2.8}$$

3 Field Equations (Simplified)

$$\Delta_{00} = \frac{6}{\tau}\left(\frac{\eta}{\tau} - \dot\xi\right) - 2\tilde\nabla_a\tilde\nabla^a\xi \tag{3.1}$$

$$\Delta_{0i} = 2\tilde\nabla_i\left(\frac{\eta}{\tau} - \dot\xi\right) + \tfrac{1}{2}\tilde\nabla_a\tilde\nabla^a(B_i - \dot E_i) \tag{3.2}$$

$$\begin{aligned}\Delta_{ij} = {} & g_{ij}\left[2\frac{d}{d\tau}\left(\frac{\eta}{\tau} - \dot\xi\right) - \frac{4}{\tau}\left(\frac{\eta}{\tau} - \dot\xi\right) - \tilde\nabla_a\tilde\nabla^a(\eta - \xi)\right] + \tilde\nabla_i\tilde\nabla_j(\eta - \xi) \\ & -\frac{1}{\tau}\tilde\nabla_i(B_j - \dot E_j) - \frac{1}{\tau}\tilde\nabla_j(B_i - \dot E_i) + \frac{1}{2}\tilde\nabla_i(\dot B_j - \ddot E_j) + \frac{1}{2}\tilde\nabla_j(\dot B_i - \ddot E_j) \\ & -\ddot E_{ij} + \frac{2}{\tau}\dot E_{ij} + \tilde\nabla_a\tilde\nabla^a E_{ij}\end{aligned} \tag{3.3}$$

$$\Delta = 6\tau^2 \frac{d}{d\tau}\left(\frac{\eta}{\tau} - \dot{\xi}\right) - 18\tau\left(\frac{\eta}{\tau} - \dot{\xi}\right) - 2\tau^2 \tilde{\nabla}_a \tilde{\nabla}^a (\eta - 2\xi) \tag{3.4}$$

$$\Omega^{-2}\tilde{g}^{ab}\Delta_{ab} = 6\tau^2 \frac{d}{d\tau}\left(\frac{\eta}{\tau} - \dot{\xi}\right) - 12\tau\left(\frac{\eta}{\tau} - \dot{\xi}\right) - 2\tau^2 \tilde{\nabla}_a \tilde{\nabla}^a (\eta - \xi) \tag{3.5}$$

$$\tilde{g}^{ab}\Delta_{ab} = 6\frac{d}{d\tau}\left(\frac{\eta}{\tau} - \dot{\xi}\right) - \frac{12}{\tau}\left(\frac{\eta}{\tau} - \dot{\xi}\right) - 2\tilde{\nabla}_a \tilde{\nabla}^a (\eta - \xi) \tag{3.6}$$

$$\tilde{\nabla}^i \tilde{\nabla}^j \Delta_{ij} = 2\tilde{\nabla}_a \tilde{\nabla}^a \left[\frac{d}{d\tau}\left(\frac{\eta}{\tau} - \dot{\xi}\right)\right] - \frac{4}{\tau}\tilde{\nabla}_a \tilde{\nabla}^a \left(\frac{\eta}{\tau} - \dot{\xi}\right) \tag{3.7}$$

4 SVT Separation (General)

4.1 Scalar, Vector

$$\left(\frac{\eta}{\tau} - \dot{\xi}\right) = \frac{\tau}{3}\tilde{\nabla}_a \tilde{\nabla}^a \xi \tag{4.1}$$

$$\tilde{\nabla}_a \tilde{\nabla}^a \left(\frac{\eta}{\tau} - \dot{\xi}\right) = 0 \tag{4.2}$$

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^a (B_i - \dot{E}_i) = 0 \tag{4.3}$$

The scalar equations from the 3-trace and $\tilde{\nabla}^i \tilde{\nabla}^j \Delta_{ij}$ are formed by linear combinations of (4.1) and (4.2).

4.2 Tensor

To isolate E_{ij}, we evaluate $\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^b \Delta^{T\theta}_{ij}$. To shorten notation, denote $\tilde{\nabla}^2 = \tilde{\nabla}_a \tilde{\nabla}^a$ and $\tilde{\nabla}^4 = \tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^b$.

$$\begin{aligned}\tilde{\nabla}^4 \Delta^{T\theta}_{ij} = {} & \tilde{\nabla}^4 \Delta_{ij} - \tilde{\nabla}^2(\tilde{\nabla}_i \tilde{\nabla}^k \Delta_{kj} + \tilde{\nabla}_j \tilde{\nabla}^k \Delta_{ki}) + \frac{1}{2} g_{ij}(\tilde{\nabla}^4 \tilde{g}^{kl}\Delta_{kl} - \tilde{\nabla}^2 \tilde{\nabla}^k \tilde{\nabla}^l \Delta_{kl}) \\ & + \frac{1}{2}\tilde{\nabla}_i \tilde{\nabla}_j (\tilde{\nabla}^2 \tilde{g}^{kl}\Delta_{kl} + \tilde{\nabla}^k \tilde{\nabla}^l \Delta_{kl}).\end{aligned} \tag{4.4}$$

The result is:

$$\tilde{\nabla}^4 \Delta^{T\theta}_{ij} = \tilde{\nabla}^4 \left(\tilde{\nabla}^2 E_{ij} + \frac{2}{\tau}\tilde{\nabla}^2 \dot{E}_{ij} - \ddot{E}_{ij}\right) \tag{4.5}$$

4.3 Fluctuation Equations

$$\left(\frac{\eta}{\tau} - \dot{\xi}\right) - \frac{\tau}{3}\tilde{\nabla}_a \tilde{\nabla}^a \xi = 0 \tag{4.6}$$

$$\tilde{\nabla}_a \tilde{\nabla}^a \left(\frac{\eta}{\tau} - \dot{\xi}\right) = 0 \tag{4.7}$$

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\nabla}_b \tilde{\nabla}^a (B_i - \dot{E}_i) = 0 \tag{4.8}$$

$$\tilde{\nabla}^4 \left(\tilde{\nabla}^2 E_{ij} + \frac{2}{\tau}\tilde{\nabla}^2 \dot{E}_{ij} - \ddot{E}_{ij}\right) = 0 \tag{4.9}$$

5 2nd Order SVT Separation (Particular)

To obtain a separation for E_{ij} that is second order without coupling B_i and E_i to scalars η and ξ, we first require

$$\tilde{\nabla}_i (B_j - \dot{E}_j) = 0. \tag{5.1}$$

From (3.2) it follows

$$\tilde{\nabla}_i \left(\frac{\eta}{\tau} - \dot{\xi}\right) = 0. \tag{5.2}$$

Using (4.1) and (5.2) this brings Δ_{ij} to the form

$$\Delta_{ij} = \frac{1}{3}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a(\xi - \tau\dot{\xi}) + \tilde{\nabla}_i\tilde{\nabla}_j(\xi + \tau\dot{\xi}) - \ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij}. \tag{5.3}$$

The trace of the above yields $\tilde{\nabla}_a\tilde{\nabla}^a\xi = 0$, which from (4.1) implies $\eta/\tau - \dot{\xi} = 0$. Hence $\tilde{\nabla}_i(B_j - \dot{E}_j) = 0 \implies (\eta/\tau - \dot{\xi}) = 0$. The separation requirement is

$$\tilde{\nabla}_i\tilde{\nabla}_j(\xi + \tau\dot{\xi}) = 0, \tag{5.4}$$

for an ξ that obeys

$$\tilde{\nabla}_a\tilde{\nabla}^a\xi = 0. \tag{5.5}$$

Taking ξ to be separable in time and space, $\xi = f(t)g(\mathbf{r})$ the requirement is

$$(f + \tau\dot{f})\tilde{\nabla}_i\tilde{\nabla}_j g(\mathbf{r}) = 0, \qquad \tilde{\nabla}_a\tilde{\nabla}^a g(\mathbf{r}) = 0. \tag{5.6}$$

Two possibilities:

$$\begin{aligned} f + \tau\dot{f} = 0 \quad &\implies \quad f(\tau) = \tau \\ \tilde{\nabla}_i\tilde{\nabla}_j g(\mathbf{r}) = 0 \quad \text{and} \quad \tilde{\nabla}_a\tilde{\nabla}^a g(\mathbf{r}) = 0 \quad &\implies \quad g(\mathbf{r}) = x + y + z. \end{aligned} \tag{5.7}$$

5.1 Fluctuation Equations

Solution 1:

$$\begin{aligned} \xi = \eta &= \frac{\tau}{r} \\ \tilde{\nabla}_i(B_j - \dot{E}_j) &= 0 \\ -\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= 0. \end{aligned} \tag{5.8}$$

Solution 2:

$$\begin{aligned} \xi = f(\tau)g(\mathbf{r}), \qquad \eta &= h(\tau)k(\mathbf{r}) \\ \frac{h(\tau)}{\tau} = \dot{f}(\tau), \qquad k(\mathbf{r}) = h(\mathbf{r}) &= x + y + z \\ \tilde{\nabla}_i(B_j - \dot{E}_j) &= 0 \\ -\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= 0. \end{aligned} \tag{5.9}$$

6 Trivial Separation

6.1 Asymptotically Vanishing

Restricting to solutions that vanish on the boundary entails $\phi = \int D\nabla^2\phi$. From $\tilde{\nabla}^i\Delta_{0i}$ we find $\eta/\tau = \dot{\xi}$, whereby from Δ_{00} it must follow that

$$\tilde{\nabla}_a\tilde{\nabla}^a\xi = 0. \tag{6.1}$$

By restricting to asymptotically vanishing solutions, this means

$$\begin{aligned} \eta = \xi &= 0 \\ B_i - \dot{E}_i &= 0 \\ -\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} &= 0. \end{aligned} \tag{6.2}$$

6.2 No Asymptotic Constraints

Let us instead impose the trivial constraints

$$\frac{\eta}{\tau} - \dot{\xi} = 0, \qquad \eta - \xi = 0, \qquad B_i - \dot{E}_i = 0. \tag{6.3}$$

Taking ξ to be separable in time and space, $\xi = f(t)g(\mathbf{r})$, the above constraints imply $f(\tau) = \tau$,

$$\xi = \tau g(\mathbf{r}). \tag{6.4}$$

From Δ_{00}, such a $g(\mathbf{r})$ must obey

$$\tilde{\nabla}_a \tilde{\nabla}^a g(\mathbf{r}) = 0. \tag{6.5}$$

For a solution that is well behaved asymptotically (but not at the origin) we may take $\xi = \tau/r$. Hence the trivial solutions are the same as Solution 1 of the previous particular separation.

6.2.1 Fluctuation Equations

$$\begin{aligned} \xi = \eta &= \frac{\tau}{r} \\ B_j - \dot{E}_j &= 0 \\ -\ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}_a \tilde{\nabla}^a E_{ij} &= 0. \end{aligned} \tag{6.6}$$