Cosmological Perturbation Theory Notes

FRW

FRW Metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Use the time-time component of the EFE to derive the Freidman equation:

$$R_{00} - \frac{1}{2}g_{00}R = 8\pi G T_{00}$$

where

$$T_{00} = \rho.$$

This leads to the Friedman equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}Ga^2\rho - K$$

The FRW can also be put into the form with conformal time τ where $a(\tau)d\tau = dt$ (dt is proper time)

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \gamma_{ij}(\mathbf{x})dx^i dx^j\right]$$

with the maximally symmetric constant curvature 3-metric

$$\gamma_{ij}(\mathbf{x})dx^i dx^j = d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)$$

where the angular radius r depends on curvature

$$r(\chi) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\chi\sqrt{K}), & K > 0 \\ \chi, & K = 0 \\ \frac{1}{\sqrt{-K}}\sinh(\chi\sqrt{-K}), & K < 0 \end{cases}$$

Note: In the $K = 0$ form, FRW metric is conformal to flat (related by conformal factor $a(\tau)^2$).

deSitter

Given a cosmological constant Λ, the EFE are

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

For a perfect fluid, we can absorb the cosmological constant into the energy momentum tensor by defining

$$\tilde{T}_{\mu\nu} = \tilde{p} g_{\mu\nu} + (\tilde{p} + \tilde{\rho}) U_\mu U_\nu$$

where

$$\tilde{p} = p - \frac{\Lambda}{8\pi G}$$
$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}.$$

Making the appropriate substitution $\rho \to \tilde{\rho}$, the Friedman equation for the RW universe becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = a^2 \left(\frac{\Lambda}{3} + \frac{8\pi G}{3}\rho\right) - K.$$

Now if we solve for the vacuum equation in a space with no curvature,

$$\left(\frac{\dot{a}}{a}\right)^2 = a^2 \frac{\Lambda}{3}$$

or

$$\dot{a} = \sqrt{\frac{\Lambda}{3}} a$$

with solution (up to constant of proportionality)

$$a(t) = e^{Ht} = e^{\sqrt{\frac{\Lambda}{3}} t}.$$

In this flat space, the metric goes as

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2).$$

The deSitter metric can also be expressed in other coordinates. In "global" coordinates

$$x_0 = \frac{1}{\sinh \tau}$$
$$x_i = \frac{1}{\omega_i \cosh \tau}$$

where ω_i represent the ith component of the angular 3-sphere metric $d\Omega^2$, i.e. $\omega_1 = \cos\phi$, $\omega_2 = \sin\theta \cos\phi$, the deSitter metric takes the form

$$ds^2 = -d\tau^2 + (\cosh \tau)^2 d\Omega^2.$$

Another coordinate system, conformal coordinates, is related to the prior via

$$\cosh t = \frac{1}{\cosh \tau}.$$

The metric is then

$$ds^2 = \frac{1}{\cosh^2 t} \left(-dt^2 + d\Omega^2\right).$$

FRW scalar perturbations in Newtonian gauage

Restricting ourself to scalar perturbations in the conformal Newtonian gauge with zero curvature ($K = 0$), the perturbed metric is

$$ds^2 = a^2(\tau)\left[-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right].$$

The metric perturbations are related to energy momentum tensor perturbations via EFE

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$$

$$R_{\mu\nu} \equiv R^{\kappa}{}_{\mu\kappa\nu} = \partial_\kappa \Gamma^{\kappa}{}_{\mu\nu} - \partial_\nu \Gamma^{\kappa}{}_{\mu\kappa} + \Gamma^{\kappa}{}_{\alpha\kappa}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\kappa}$$

$$\Gamma^{\kappa}{}_{\mu\nu} = \frac{g^{\kappa\lambda}}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

Start with one time component, $\kappa = 0$ (recall affine connection is symmetric in $\mu\nu$):

$$\begin{aligned}
\Gamma^{0}{}_{\mu\nu} &= \frac{g^{0\lambda}}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)(\partial_\mu g_{0\nu} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu})
\end{aligned}$$

$\mu = \nu = 0$:

$$\begin{aligned}
\Gamma^{0}{}_{00} &= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)\partial_0 g_{00} \\
&= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)\partial_0\left[-a^2(1+2\psi)\right] \\
&= \frac{-1}{a^2}\left(\frac{-1+2\psi}{2}\right)\left[2a\dot{a}(1+2\psi) + 2a^2\partial_0\psi\right] \\
&= \frac{\dot{a}}{a} + \partial_0\psi
\end{aligned}$$

$\mu = 0$, $\nu = i$:

$$\begin{aligned}
\Gamma^{0}{}_{0i} &= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)\partial_i g_{00} \\
&= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)\partial_i\left[-a^2(1+2\psi)\right] \\
&= \left(\frac{1-2\psi}{2}\right)\partial_i(2\psi) \\
&= \partial_i\psi
\end{aligned}$$

$\mu = i$, $\nu = j$:

$$\begin{aligned}
\Gamma^{0}{}_{ij} &= \frac{1}{a^2}\left(\frac{-1+2\psi}{2}\right)(-\partial_0 g_{ij}) \\
&= \frac{1}{a^2}\left(\frac{1-2\psi}{2}\right)\partial_0\left[a^2(1-2\phi)\delta_{ij}\right] \\
&= \delta_{ij}\frac{\dot{a}}{a}[1-2(\psi+\phi)] - \delta_{ij}\partial_0\phi
\end{aligned}$$

Now for the spatial $\kappa = i$

$$\begin{aligned}
\Gamma^i{}_{\mu\nu} &= \frac{g^{i\lambda}}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ij}(\partial_\mu g_{j\nu} + \partial_\nu g_{j\mu} - \partial_j g_{\mu\nu})
\end{aligned}$$

$\mu = \nu = 0$:

$$\begin{aligned}
\Gamma^i{}_{00} &= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ij}(\partial_0 g_{j0} + \partial_0 g_{j0} - \partial_j g_{00}) \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ij}(-\partial_j g_{00}) \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ij}(-\partial_j[-a^2(1+2\psi)] \\
&= \delta^{ij}\partial_j\psi \\
&= \partial_i\psi
\end{aligned}$$

$\mu = j$, $\nu = 0$:

$$\begin{aligned}
\Gamma^i{}_{j0} &= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ik}(\partial_j g_{k0} + \partial_0 g_{kj} - \partial_k g_{j0}) \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{ik}\partial_0 g_{kj} \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta_{ij}\partial_0[a^2(1-2\phi)] \\
&= \delta_{ij}\left(\frac{\dot{a}}{a} - \partial_0\phi\right)
\end{aligned}$$

$\mu = j$, $\nu = k$:

$$\begin{aligned}
\Gamma^i{}_{jk} &= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)\delta^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) \\
&= \frac{1}{a^2}\left(\frac{1+2\phi}{2}\right)(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) \\
&= -(1+2\phi)[\delta_{ik}\partial_j\phi + \delta_{ij}\partial_k\phi - \delta_{jk}\partial_i\phi] \\
&= \delta_{jk}\partial_i\phi - \delta_{ik}\partial_j\phi - \delta_{ij}\partial_k\phi
\end{aligned}$$

Moving to Fourier transformed variables, $\partial_i \to ik_i = ik^i$, we have in total

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{\dot{a}}{a} + \partial_0\psi \\
\Gamma^0{}_{0i} &= ik_i\psi \\
\Gamma^0{}_{ij} &= \delta_{ij}\frac{\dot{a}}{a}[1 - 2(\psi+\phi)] - \delta_{ij}\partial_0\phi \\
\Gamma^i{}_{00} &= ik_i\psi \\
\Gamma^i{}_{j0} &= \delta_{ij}\left(\frac{\dot{a}}{a} - \partial_0\phi\right) \\
\Gamma^i{}_{jk} &= i\phi(\delta_{jk}k_i - \delta_{ik}k_j - \delta_{ij}k_k)
\end{aligned}$$

Now we calculate the Ricci tensor

$$R_{\mu\nu} = \partial_\kappa \Gamma^\kappa{}_{\mu\nu} - \partial_\nu \Gamma^\kappa{}_{\mu\kappa} + \Gamma^\kappa{}_{\alpha\kappa}\Gamma^\alpha{}_{\mu\nu} - \Gamma^\kappa{}_{\alpha\nu}\Gamma^\alpha{}_{\mu\kappa}$$

$\mu = \nu = 0$:

$$\begin{aligned} R_{00} &= \partial_\kappa \Gamma^\kappa{}_{00} - \partial_0 \Gamma^\kappa{}_{0\kappa} + \Gamma^\kappa{}_{\alpha\kappa}\Gamma^\alpha{}_{00} - \Gamma^i{}_{\alpha 0}\Gamma^\alpha{}_{0i} \\ &= \partial_i \Gamma^i{}_{00} - \partial_0 \Gamma^i{}_{0i} + \Gamma^i{}_{\alpha i}\Gamma^\alpha{}_{00} - \Gamma^i{}_{\alpha 0}\Gamma^\alpha{}_{0i} \end{aligned}$$

Looking at individual terms

$$\begin{aligned} \partial_i \Gamma^i{}_{00} &= -k^2\psi \\ \partial_0 \Gamma^i{}_{0i} &= 3\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \partial_0^2\phi\right] \\ \Gamma^i{}_{\alpha i}\Gamma^\alpha{}_{00} &= \Gamma^i{}_{0i}\Gamma^0{}_{00} \\ &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\partial_0(\psi - \phi)\right] \\ \Gamma^i{}_{\alpha 0}\Gamma^\alpha{}_{0i} &= \Gamma^i{}_{j0}\Gamma^j{}_{0i} \\ &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\dot{a}}{a}\partial_0\phi\right]. \end{aligned}$$

Altogether for R_{00}

$$\begin{aligned} R_{00} &= -k^2\psi - 3\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \partial_0^2\phi\right] + 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\partial_0(\psi - \phi)\right] - 3\left[\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\dot{a}}{a}\partial_0\phi\right] \\ &= -k^2\psi - 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + 3\partial_0^2\phi + 3\frac{\dot{a}}{a}\partial_0\left(\psi + \phi\right). \end{aligned}$$

$\mu = i$, $\nu = j$:

$$R_{ij} = \partial_\kappa \Gamma^\kappa{}_{ij} - \partial_j \Gamma^\kappa{}_{i\kappa} + \Gamma^\kappa{}_{\alpha\kappa}\Gamma^\alpha{}_{ij} - \Gamma^\kappa{}_{\alpha j}\Gamma^\alpha{}_{i\kappa}$$

Individual terms,

$$\begin{aligned} \partial_\kappa \Gamma^\kappa{}_{ij} &= \delta_{ij}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 (1 - 2(\psi + \phi)) - 2\frac{\dot{a}}{a}\partial_0(\psi + \phi) - \partial_0^2\phi - k^2\phi\right] + 2\phi k_i k_j \\ \partial_j \Gamma^\kappa{}_{i\kappa} &= -k_i k_j(\psi - 3\phi) \end{aligned}$$

Massless Scalar Field Coupled to Gravity

$$I = \int d^4x(-g)^{1/2}\left[\left(\frac{1}{16\pi G} - \frac{S^2}{12}\right)R^\alpha{}_\alpha + \frac{1}{2}S_\mu S^\mu\right]$$

First we find field equations by varying with respect to metric.

Variation of $g^{\mu\nu}$

$$\begin{aligned}\delta(g^{\mu\alpha}g_{\alpha\beta}) &= 0\\ \delta g^{\mu\alpha}g_{\alpha\beta} &= -\delta g_{\alpha\beta}g^{\mu\alpha}\\ \delta g^{\mu\nu} &= -\delta g_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\end{aligned}$$

Variation of the determinant,

$$\begin{aligned}\delta(-g)^{1/2} &= \frac{1}{2}(-g)^{-1/2}\delta g\\ &= \frac{1}{2}(-g)^{-1/2}(-g)g^{\mu\nu}\delta g_{\mu\nu}\\ &= \frac{1}{2}(-g)^{1/2}g^{\mu\nu}\delta g_{\mu\nu}\\ &= -\frac{1}{2}(-g)^{1/2}g_{\mu\nu}\delta g^{\mu\nu}\end{aligned}$$

Variation of the Ricci tensor,

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \tag{1}$$

From

$$\begin{aligned}\delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\delta g^{\lambda\rho}\delta^\sigma_\rho(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{g^{\lambda\rho}}{2}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu})\\ &= \frac{1}{2}\delta g^{\lambda\rho}g^{\sigma\kappa}g_{\kappa\rho}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{g^{\lambda\rho}}{2}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu})\\ &= \delta g^{\lambda\rho}g_{\rho\kappa}\Gamma^\kappa_{\mu\nu} + \frac{g^{\lambda\rho}}{2}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu})\\ &= \frac{g^{\lambda\rho}}{2}[(\delta g_{\nu\rho})_{;\mu} + (\delta g_{\mu\rho})_{;\nu} - (\delta g_{\mu\nu})_{;\rho}]\\ &= -\frac{1}{2}[g_{\alpha\nu}(\delta g^{\alpha\lambda})_{;\mu} + g_{\alpha\mu}(\delta g^{\alpha\lambda})_{;\nu} - g_{\alpha\mu}g_{\beta\nu}(\delta g^{\alpha\beta})^{;\lambda}],\end{aligned}$$

it follows that,

$$\begin{aligned}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} &= -\frac{1}{2}g_{\alpha\lambda}(\delta g^{\alpha\lambda})^{;\nu}\\ g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} &= -(\delta g^{\alpha\lambda})_{;\alpha} + \frac{1}{2}g_{\alpha\beta}(\delta g^{\alpha\beta})^{;\lambda}\end{aligned}$$

and we may use this in (1) to form

$$g^{\mu\nu}\delta R_{\mu\nu} = (\delta g^{\mu\nu})_{;\mu;\nu} - g_{\mu\nu}(\delta g^{\mu\nu})^{;\alpha}{}_{;\alpha}$$

Putting this altogether, we have for variation of the action

$$\begin{aligned}\delta I = &\int d^4x\ \left[\left(\frac{1}{16\pi G} - \frac{S^2}{12}\right)R^\alpha{}_\alpha + \frac{1}{2}g^{\alpha\beta}S_\alpha S_\beta\right]\delta(-g)^{1/2}\\ &+ \int d^4x\ (-g)^{1/2}\left[\left(\frac{1}{16\pi G} - \frac{S^2}{12}\right)R_{\mu\nu} + \frac{1}{2}S_\mu S_\nu\right]\delta g^{\mu\nu}\end{aligned}$$

$$
\begin{aligned}
&+\int d^4x\,(-g)^{1/2}\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)g^{\mu\nu}\delta R_{\mu\nu}\\
&=-\frac{1}{2}\int d^4x\,(-g)^{1/2}\left[\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)R^\alpha{}_\alpha+\frac{1}{2}g^{\alpha\beta}S_\alpha S_\beta\right]g_{\mu\nu}\delta g^{\mu\nu}\\
&+\int d^4x\,(-g)^{1/2}\left[\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)R_{\mu\nu}+\frac{1}{2}S_\mu S_\nu\right]\delta g^{\mu\nu}\\
&+\int d^4x\,(-g)^{1/2}\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)[(\delta g^{\mu\nu})_{;\mu;\nu}-g_{\mu\nu}(\delta g^{\mu\nu})^{;\alpha}{}_{;\alpha}]
\end{aligned}
$$

For the last integral, we note that by using

$$
V^\mu{}_{;\mu}=\frac{1}{(-g)^{1/2}}\partial_\mu((-g)^{1/2}V^\mu)
$$

we may express the variation in terms of a covariant divergence

$$
\begin{aligned}
\int d^4x\,(-g)^{1/2}\,[(\delta g^{\mu\nu})_{;\mu;\nu}-g_{\mu\nu}(\delta g^{\mu\nu})^{;\alpha}{}_{;\alpha}]&=\int d^4x\,(-g)^{1/2}\left[(\delta g^{\mu\nu})_{;\mu}-g^{\mu\nu}g_{\alpha\beta}(\delta g^{\alpha\beta})_{;\mu}\right]_{;\nu}\\
&=\oint d^3x\,(-g)^{1/2}\left[(\delta g^{\mu\nu})_{;\mu}-g^{\mu\nu}g_{\alpha\beta}(\delta g^{\alpha\beta})_{;\mu}\right]\\
&=\oint d^3x\,(-g)^{1/2}\left[\partial_\mu(\delta g^{\mu\nu})-g^{\mu\nu}g_{\alpha\beta}\partial_\mu(\delta g^{\alpha\beta})\right].
\end{aligned}
$$

The above will vanish if we require not only $\delta g^{\mu\nu}$ but also $\partial_\lambda\delta g^{\mu\nu}$ to vanish on the boundary.

Since we have a scalar field coupled to the Ricci tensor, we must also find its variation. Setting $S^2\equiv\phi$, note that

$$
[\phi_{;\mu}\delta g^{\mu\nu}-\phi(\delta g^{\mu\nu})_{;\mu}]_{;\nu}=\phi_{;\mu;\nu}\delta g^{\mu\nu}-\phi(\delta g^{\mu\nu})_{;\mu;\nu}
$$

and

$$
[\phi^{;\alpha}g_{\mu\nu}\delta g^{\mu\nu}-\phi g_{\mu\nu}(\delta g^{\mu\nu})^{;\alpha}]_{;\alpha}=\phi^{;\alpha}{}_{;\alpha}g_{\mu\nu}\delta g^{\mu\nu}-\phi g_{\mu\nu}(\delta g^{\mu\nu})^{;\alpha}{}_{;\alpha}.
$$

The divergence terms are integrated on the surface and vanish under the same conditions, i.e. $\delta g^{\mu\nu}=0$ and $\partial_\lambda\delta g^{\mu\nu}=0$.

Gathering all terms together for the entire variation,

$$
\begin{aligned}
\delta I=\int d^4x\,(-g)^{1/2}\delta g^{\mu\nu}\Bigg[&\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^\alpha{}_\alpha\right)\\
&+\frac{1}{2}S_\mu S_\nu-\frac{1}{4}g_{\mu\nu}S^\alpha S_\alpha-\frac{1}{12}(S^2)_{;\mu;\nu}+\frac{1}{12}g_{\mu\nu}(S^2)^{;\alpha}{}_{;\alpha}\Bigg].
\end{aligned}
$$

Since the action must be stationary with respect to arbitrary variation $\delta g^{\mu\nu}$, we have the Einstein field equations

$$
\left(\frac{S^2}{6}-\frac{1}{8\pi G}\right)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^\alpha{}_\alpha\right)=S_\mu S_\nu-\frac{1}{2}g_{\mu\nu}S^\alpha S_\alpha-\frac{1}{6}(S^2)_{;\mu;\nu}+\frac{1}{6}g_{\mu\nu}(S^2)^{;\alpha}{}_{;\alpha}\tag{2}
$$

To find the equations of motion of the scalar field, we vary $\mathcal{L}$ with respect to the scalar field S

$$
\mathcal{L}=(-g)^{1/2}\left[\left(\frac{1}{16\pi G}-\frac{S^2}{12}\right)R^\alpha{}_\alpha+\frac{1}{2}S_\mu S^\mu\right]
$$

$$
\delta\mathcal{L}=(-g)^{1/2}\left[-\frac{1}{6}SR^\alpha{}_\alpha\delta S+\frac{1}{2}\delta(S_\mu S^\mu)\right].
$$

Looking at the last term,

$$\begin{aligned}\frac{1}{2}\delta(S_\mu S^\mu) &= \frac{1}{2}\delta\left(g^{\mu\nu}S_{;\mu}S_{;\nu}\right)\\ &= g^{\mu\nu}S_{;\nu}(\delta S)_{;\mu}\\ &= S^{;\mu}(\delta S)_{;\mu}.\end{aligned}$$

Note the divergence

$$[S^{;\mu}\delta S]_{;\mu} = S^{;\mu}{}_{;\mu}\delta S + S^{;\mu}(\delta S)_{;\mu}.$$

The divergence will vanish on the boundary for arbitrary δS, and so we are left with

$$\delta L = (-g)^{1/2}\delta S\left[-\frac{1}{6}SR^\alpha{}_\alpha - S^{;\mu}{}_{;\mu}\right].$$

Hence the equation of motion is

$$S^{;\mu}{}_{;\mu} = -\frac{1}{6}SR^\alpha{}_\alpha \tag{3}$$

Taking the trace of the Einstein equations (2) yields

$$\begin{aligned}g^{\mu\nu}\left(\frac{S^2}{6}-\frac{1}{8\pi G}\right)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^\alpha{}_\alpha\right) &= g^{\mu\nu}S_\mu S_\nu - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}S_\alpha S^\alpha - \frac{1}{6}g^{\mu\nu}(S^2)_{\mu;\nu} + \frac{1}{6}g^{\mu\nu}g_{\mu\nu}(S^2)^\alpha{}_{;\alpha}\\ \left(\frac{S^2}{6}-\frac{1}{8\pi G}\right)(R^\alpha{}_\alpha - 2R^\alpha{}_\alpha) &= S^\mu S_\mu - 2S^\mu S_\mu - \frac{1}{6}(S^2)^{;\mu}{}_{;\mu} + \frac{4}{6}(S^2)^{;\alpha}{}_{;\alpha}\\ \left(\frac{1}{8\pi G}-\frac{S^2}{6}\right)R^\alpha{}_\alpha &= -S^\alpha S_\alpha + \frac{1}{2}(S^2)^{;\alpha}{}_{;\alpha}\\ \left(\frac{1}{8\pi G}-\frac{S^2}{6}\right)R^\alpha{}_\alpha &= -S^\alpha S_\alpha + \frac{1}{2}[2SS^\alpha]_{;\alpha}\\ \left(\frac{1}{8\pi G}-\frac{S^2}{6}\right)R^\alpha{}_\alpha &= -S^\alpha S_\alpha + S_\alpha S^\alpha + SS^\alpha{}_{;\alpha}\\ \left(\frac{1}{8\pi G}-\frac{S^2}{6}\right)R^\alpha{}_\alpha &= SS^\alpha{}_{;\alpha}.\end{aligned}$$

From here we may insert the equation of motion (3)

$$\left(\frac{1}{8\pi G}-\frac{S^2}{6}\right)R^\alpha{}_\alpha = -\frac{1}{6}S^2R^\alpha{}_\alpha.$$

From inspection, we see that either $G\to\infty$ or $R^\alpha{}_\alpha = 0$.

Before setting the Ricci scalar to zero, it can be shown that in RW the equation of motion (3) can be solved via separation of variables,

$$S(x) = f(t)g(r,\theta,\phi).$$

An important step in showing this involves computing the covariant divergence of S in a RW metric, which simplifies to

$$\begin{aligned}S^\alpha{}_{;\alpha} &= \partial^\mu\partial_\mu S + \Gamma^\mu_{\mu\lambda}S^\lambda\\ &= \partial^\mu\partial_\mu S + \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\rho\mu}S^\lambda\\ &= \partial^\mu\partial_\mu S + S^0\frac{\dot{R}}{R} + \frac{1}{2}S^k\gamma^{ij}\partial_k\gamma_{ij}.\end{aligned}$$

where

$$\gamma^{ij} = R^2 g^{ij}, \qquad \gamma_{ij} = \frac{g_{ij}}{R^2}.$$

Using the relations $(-g)^{1/2} = \gamma^{1/2}R^3$, $f(t) = \frac{f(p)}{R}$, $dp = dt/R$, we may bring the equation of motion for $S(x)$ into the form

$$\frac{1}{f(p)}\left[\frac{d^2 f}{dp^2} + Kf(p)\right] = \frac{1}{g(r,\theta,\phi)}\gamma^{-1/2}\partial_i[\gamma^{1/2}\gamma^{ij}\partial_j g(r,\theta,\phi)]$$

$$S = N_l^m \begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} j_l(\omega r)P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}$$

$$\partial^\alpha S\partial_\alpha S = ...$$

Variation of Conformal Action

The conformally invariant action is

$$I_W = \int d^4x\sqrt{g}\; C^{\lambda\mu\nu\kappa}C_{\lambda\mu\nu\kappa}$$

where

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}R^\alpha{}_\alpha(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})$$

is the Weyl tensor. We will compute its contraction

$$\begin{aligned} C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} = &\left[R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}R^\alpha{}_\alpha(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})\right]\times\\ &\left[R^{\lambda\mu\nu\kappa} - \frac{1}{2}(g^{\lambda\nu}R^{\mu\kappa} - g^{\lambda\kappa}R^{\mu\nu} - g^{\mu\nu}R^{\lambda\kappa} + g^{\mu\kappa}R^{\lambda\nu}) + \frac{1}{6}R^\alpha{}_\alpha(g^{\lambda\nu}g^{\mu\kappa} - g^{\lambda\kappa}g^{\mu\nu})\right]. \end{aligned}$$

Taking individual products,

$$\begin{aligned} R_{\lambda\mu\nu\kappa}\left[-\frac{1}{2}(g^{\lambda\nu}R^{\mu\kappa} - g^{\lambda\kappa}R^{\mu\nu} - g^{\mu\nu}R^{\lambda\kappa} + g^{\mu\kappa}R^{\lambda\nu})\right] &= -\frac{1}{2}\left(R_{\mu\kappa}R^{\mu\kappa} + R^{\mu\nu}g^{\lambda\kappa}R_{\lambda\mu\kappa\nu} + R^{\lambda\kappa}g^{\mu\nu}R_{\mu\lambda\nu\kappa} + R^{\lambda\nu}g^{\mu\kappa}R_{\mu\lambda\kappa\nu}\right)\\ &= -\frac{1}{2}\left(R^{\mu\kappa}R_{\mu\kappa} + R^{\mu\nu}R_{\mu\nu} + R^{\lambda\kappa}R_{\lambda\kappa} + R^{\lambda\nu}R_{\lambda\nu}\right)\\ &= -2R^{\mu\nu}R_{\mu\nu} \end{aligned}$$

$$\begin{aligned} R_{\lambda\mu\nu\kappa}\left[\frac{1}{6}R^\alpha{}_\alpha(g^{\lambda\nu}g^{\mu\kappa} - g^{\lambda\kappa}g^{\mu\nu})\right] &= \frac{1}{6}R^\alpha{}_\alpha\left(g^{\mu\kappa}R_{\mu\kappa} + g^{\lambda\kappa}R_{\lambda\kappa}\right)\\ &= \frac{1}{3}(R^\alpha{}_\alpha)^2 \end{aligned}$$

$$\begin{aligned} &\left[-\frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu})\right]\times\left[-\frac{1}{2}(g^{\lambda\nu}R^{\mu\kappa} - g^{\lambda\kappa}R^{\mu\nu} - g^{\mu\nu}R^{\lambda\kappa} + g^{\mu\kappa}R^{\lambda\nu})\right]\\ &\qquad= \frac{1}{4}[(4R_{\mu\kappa}R^{\mu\kappa} - \delta^\kappa_\nu R_{\mu\kappa}R^{\mu\nu} - \delta^\mu_\lambda R_{\mu\kappa}R^{\lambda\kappa} + (R^\alpha{}_\alpha)^2)\\ &\qquad\quad + (4R_{\mu\kappa}R^{\mu\kappa} - \delta^\nu_\kappa R_{\mu\nu}R^{\mu\kappa} - \delta^\mu_\lambda R_{\mu\nu}R^{\lambda\nu} + (R^\alpha{}_\alpha)^2)\\ &\qquad\quad + (4R_{\mu\kappa}R^{\mu\kappa} - \delta^\kappa_\nu R_{\lambda\kappa}R^{\lambda\nu} - \delta^\lambda_\mu R_{\lambda\kappa}R^{\mu\kappa} + (R^\alpha{}_\alpha)^2)\\ &\qquad\quad + (4R_{\mu\kappa}R^{\mu\kappa} - \delta^\nu_\kappa R_{\lambda\nu}R^{\lambda\kappa} - \delta^\lambda_\mu R_{\lambda\nu}R^{\mu\nu} + (R^\alpha{}_\alpha)^2)]\\ &\qquad= \frac{1}{4}[16R_{\mu\nu}R^{\mu\nu} - 8R_{\mu\nu}R^{\mu\nu} + 4(R^\alpha{}_\alpha)^2]\\ &\qquad= 2R_{\mu\nu}R^{\mu\nu} + (R^\alpha{}_\alpha)^2 \end{aligned}$$

$$
\begin{aligned}
&\left[-\frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa}-g_{\lambda\kappa}R_{\mu\nu}-g_{\mu\nu}R_{\lambda\kappa}+g_{\mu\kappa}R_{\lambda\nu})\right]\left[\frac{1}{6}R^{\alpha}{}_{\alpha}(g^{\lambda\nu}g^{\mu\kappa}-g^{\lambda\kappa}g^{\mu\nu})\right]\\
&\qquad=-\frac{1}{12}R^{\alpha}{}_{\alpha}\left[4R^{\alpha}{}_{\alpha}-R^{\alpha}{}_{\alpha}-R^{\alpha}{}_{\alpha}+4R^{\alpha}{}_{\alpha}-R^{\alpha}{}_{\alpha}+4R^{\alpha}{}_{\alpha}+4R^{\alpha}{}_{\alpha}-R^{\alpha}{}_{\alpha}\right]\\
&\qquad=-\frac{1}{12}R^{\alpha}{}_{\alpha}(12R^{\alpha}{}_{\alpha})\\
&\qquad=-(R^{\alpha}{}_{\alpha})^2
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{6}R^{\alpha}{}_{\alpha}(g_{\lambda\nu}g_{\mu\kappa}-g_{\lambda\kappa}g_{\mu\nu})\right]\left[\frac{1}{6}R^{\alpha}{}_{\alpha}(g^{\lambda\nu}g^{\mu\kappa}-g^{\lambda\kappa}g^{\mu\nu})\right]\\
&\qquad=\frac{1}{36}(R^{\alpha}{}_{\alpha})^2(16-4-4+16)\\
&\qquad=\frac{2}{3}(R^{\alpha}{}_{\alpha})^2.
\end{aligned}
$$

Summing appropriate quantities together, we have

$$
\begin{aligned}
C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}&=R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa}-4R_{\mu\nu}R^{\mu\nu}+\frac{2}{3}(R^{\alpha}{}_{\alpha})^2+2R_{\mu\nu}R^{\mu\nu}+(R^{\alpha}{}_{\alpha})^2-2(R^{\alpha}{}_{\alpha})^2+\frac{2}{3}(R^{\alpha}{}_{\alpha})^2\\
&=R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}(R^{\alpha}{}_{\alpha})^2.
\end{aligned}
$$

We now vary the (equivalent) action

$$
I_W=-2\alpha_s\int d^4x\sqrt{g}\ \left[R_{\mu\kappa}R^{\mu\kappa}-\frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right].
$$

Recalling that

$$
\delta\sqrt{g}=\frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu},
$$

we have the variation

$$
-\left(\frac{1}{2\alpha_s\sqrt{g}}\right)\frac{\delta I_W}{\delta g_{\mu\nu}}=\frac{1}{2}g^{\mu\nu}\left[R_{\mu\kappa}R^{\mu\kappa}-\frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right]+\frac{\delta}{\delta g_{\mu\nu}}\left[R_{\mu\kappa}R^{\mu\kappa}-\frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right].
$$

Looking the first individual term,

$$
\begin{aligned}
\delta(R_{\mu\nu}R^{\mu\nu})&=\delta(R_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta})\\
&=2\delta(R_{\mu\nu})R^{\mu\nu}+\delta(g^{\mu\alpha})g^{\nu\beta}R_{\alpha\beta}R_{\mu\nu}+\delta(g^{\nu\beta})g^{\mu\alpha}R_{\alpha\beta}R_{\mu\nu}\\
&=2\delta(R_{\mu\nu})R^{\mu\nu}-2\delta g_{\mu\nu}R_{\rho\sigma}(g^{\mu\rho}R^{\nu\sigma}+g^{\nu\sigma}R^{\rho\nu})\\
&=2\delta(R_{\mu\nu})R^{\mu\nu}-2\delta g_{\mu\nu}R^{\mu}{}_{\alpha}R^{\nu\alpha}.
\end{aligned}
$$

From earlier we have

$$
\delta R_{\mu\nu}=(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}-(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}
$$

thus it follows

$$
\begin{aligned}
2\delta(R_{\mu\nu})R^{\mu\nu}&=2R^{\mu\nu}\left[(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}-(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}\right]\\
&=2\left\{\left[(R^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}-R^{\mu\nu}{}_{;\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right]-\left[(R^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}-R^{\mu\nu}{}_{;\lambda}\delta\Gamma^{\lambda}_{\mu\nu}\right]\right\}.
\end{aligned}
$$

Since the equations of motion are fourth order, the variation and its derivative vanish on the boundary. We see from

$$
\delta\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\rho}\left[(\delta g_{\mu\rho})_{;\nu}+(\delta g_{\rho\nu})_{;\mu}-(\delta g_{\mu\nu})_{;\rho}\right]
$$

that $\delta\Gamma$ will vanish on the boundary. Hence we are left with

$$
\begin{aligned}
2\delta(R_{\mu\nu})R^{\mu\nu} &= 2\left[R^{\mu\nu}{}_{;\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - R^{\mu\nu}{}_{;\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right] \\
&= R^{\mu\nu}{}_{;\lambda}\left\{g^{\lambda\rho}\left[(\delta g_{\mu\rho})_{;\nu} + (\delta g_{\rho\nu})_{;\mu} - (\delta g_{\mu\nu})_{;\rho}\right]\right\} \\
&\quad - R^{\mu\nu}{}_{;\nu}\left\{g^{\lambda\rho}\left[(\delta g_{\mu\rho})_{;\lambda} + (\delta g_{\rho\lambda})_{;\mu} - (\delta g_{\mu\lambda})_{;\rho}\right]\right\}
\end{aligned}
$$

As each is a product of derivatives, we can use

$$f'g' = (f'g)' - f''g$$

for each term. Thus each $f'g$ term vanishes on the boundary and we effectively commute derivatives with a change in sign:

$$
\begin{aligned}
2\delta(R_{\mu\nu})R^{\mu\nu} &= g^{\lambda\rho}\left[R^{\mu\nu}{}_{;\nu;\lambda}\delta g_{\mu\rho} + R^{\mu\nu}{}_{;\nu;\mu}\delta g_{\rho\lambda} - R^{\mu\nu}{}_{;\nu;\rho}\delta g_{\mu\lambda} - R^{\mu\nu}{}_{;\lambda;\nu}\delta g_{\mu\rho} - R^{\mu\nu}{}_{;\lambda;\mu}\delta g_{\rho\nu} + R^{\mu\nu}{}_{;\lambda;\rho}\delta g_{\mu\nu}\right] \\
&= \delta g_{\mu\nu}\left[R^{\mu\alpha}{}_{;\alpha}{}^{;\nu} + g^{\mu\nu}R^{\alpha\beta}{}_{;\alpha;\beta} - R^{\mu\alpha}{}_{;\alpha}{}^{;\nu} - R^{\mu\alpha;\nu}{}_{;\alpha} - R^{\alpha\nu;\mu}{}_{;\alpha} + R^{\mu\nu;\alpha}{}_{;\alpha}\right] \\
&= \delta g_{\mu\nu}\left(g^{\mu\nu}R^{\alpha\beta}{}_{;\alpha;\beta} - R^{\mu\alpha;\nu}{}_{;\alpha} - R^{\alpha\nu;\mu}{}_{;\alpha} + R^{\mu\nu;\alpha}{}_{;\alpha}\right).
\end{aligned}
$$

The first term may be re-expressed, by using the last of the three following identities (DeWitt 1964 pg 720):

$$R^{\mu\sigma\nu\tau}{}_{;\sigma;\tau} = R^{\mu\nu}{}_{;\sigma}{}^{;\sigma} - R^{\mu\sigma;\nu}{}_{;\sigma}$$

$$R^{\mu\sigma;\nu}{}_{;\sigma} = \frac{1}{2}R^{;\mu;\nu} + R^{\mu\sigma\nu\tau}R_{\sigma\tau} - R^{\mu}{}_{\sigma}R^{\nu\sigma}$$

$$R^{\mu\nu}{}_{;\mu;\nu} = \frac{1}{2}R_{;\mu}{}^{;\mu}.$$

The above follow from the Bianchi identity and are also given in Weinberg. Putting this altogether we recover the $W^{\mu\nu}_{(2)}$ term.

As for $W^{\mu\nu}_{(1)}$,

$$
\begin{aligned}
\delta[(R^{\alpha}{}_{\alpha})^2] &= 2R^{\alpha}{}_{\alpha}[\delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu})] \\
&= 2R^{\alpha}{}_{\alpha}[-\delta g_{\mu\nu}R^{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu})
\end{aligned}
$$

$$
\begin{aligned}
2R^{\alpha}{}_{\alpha}g^{\mu\nu}\delta R_{\mu\nu} &= 2g^{\mu\nu}\left[R^{\alpha}{}_{\alpha}(\delta\Gamma^{\lambda}_{\lambda\mu})_{;\nu} - R^{\alpha}{}_{\alpha}(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}\right] \\
&= 2g^{\mu\nu}\left\{\left[\left(R^{\alpha}{}_{\alpha}\delta\Gamma^{\lambda}_{\lambda\mu}\right)_{;\nu} - (R^{\alpha}{}_{\alpha})_{;\nu}\delta\Gamma^{\lambda}_{\lambda\mu}\right] - \left[\left(R^{\alpha}{}_{\alpha}\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} - (R^{\alpha}{}_{\alpha})_{;\lambda}\delta\Gamma^{\lambda}_{\mu\nu}\right]\right\} \\
&\Rightarrow 2g^{\mu\nu}\left[(R^{\alpha}{}_{\alpha})_{;\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - (R^{\alpha}{}_{\alpha})_{;\nu}\delta\Gamma^{\lambda}_{\lambda\mu}\right] \\
&= g^{\mu\nu}(R^{\alpha}{}_{\alpha})_{;\lambda}\left[g^{\lambda\rho}\left((\delta g_{\mu\rho})_{;\nu} + (\delta g_{\nu\rho})_{;\mu} - (\delta g_{\mu\nu})_{;\rho}\right)\right] \\
&\quad - g^{\mu\nu}(R^{\alpha}{}_{\alpha})_{;\nu}\left[g^{\lambda\rho}\left((\delta g_{\mu\rho})_{;\lambda} + (\delta g_{\lambda\rho})_{;\mu} - (\delta g_{\mu\lambda})_{;\rho}\right)\right] \\
&\Rightarrow \delta g_{\mu\nu}\big[-g^{\mu\beta}g^{\lambda\nu}(R^{\alpha}{}_{\alpha})_{;\lambda;\beta} - g^{\beta\mu}g^{\lambda\nu}(R^{\alpha}{}_{\alpha})_{;\lambda;\beta} + g^{\mu\nu}g^{\lambda\rho}(R^{\alpha}{}_{\alpha})_{;\lambda;\rho} \\
&\quad + g^{\mu\beta}g^{\lambda\nu}(R^{\alpha}{}_{\alpha})_{;\beta;\lambda} - g^{\mu\nu}g^{\lambda\rho}(R^{\alpha}{}_{\alpha})_{;\lambda;\rho} + g^{\mu\beta}g^{\nu\rho}(R^{\alpha}{}_{\alpha})_{;\beta;\rho}\big] \\
&= \delta g_{\mu\nu}\left[-2(R^{\alpha}{}_{\alpha})^{;\mu;\nu} + 2g^{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta}\right].
\end{aligned}
$$

Forming all the pieces above together, we recover the $W^{\mu\nu}_{(1)}$ term.

Alternatively, another method for $\delta(R^\alpha{}_\alpha)^2$ would be

$$\begin{aligned}2R^\alpha{}_\alpha g^{\mu\nu}\delta R_{\mu\nu} &= R^\alpha{}_\alpha g^{\mu\nu}g^{\lambda\rho}\left[(\delta g_{\lambda\rho})_{;\mu;\nu} - (\delta g_{\rho\mu})_{;\nu;\lambda} - (\delta g_{\rho\nu})_{;\mu;\lambda} + (\delta g_{\mu\nu})_{;\rho;\lambda}\right] \\ &= R^\alpha{}_\alpha(\delta g_{\mu\nu})_{;\lambda;\rho}(g^{\lambda\rho}g^{\mu\nu} - 2g^{\lambda\nu}g^{\rho\mu} + g^{\mu\nu}g^{\rho\lambda}) \\ &= 2R^\alpha{}_\alpha(\delta g_{\mu\nu})_{;\lambda;\rho}(g^{\lambda\rho}g^{\mu\nu} - g^{\lambda\nu}g^{\rho\mu}) \\ &= 2R^\alpha{}_\alpha[g^{\mu\nu}(\delta g_{\mu\nu})^{;\lambda}{}_{;\lambda} - (\delta g_{\mu\nu})^{;\mu;\nu}]\end{aligned}$$

$$(R^\alpha{}_\alpha)(\delta g_{\mu\nu})^{;\lambda}{}_{;\lambda} = \left[R^\alpha{}_\alpha(\delta g_{\mu\nu})^{;\lambda} - (R^\alpha{}_\alpha)^{;\lambda}\delta g_{\mu\nu}\right]_{;\lambda} + (R^\alpha{}_\alpha)^{;\lambda}{}_{;\lambda}\delta g_{\mu\nu}$$

$$R^\alpha{}_\alpha(\delta g_{\mu\nu})^{;\mu;\nu} = [R^\alpha{}_\alpha(\delta g_{\mu\nu})^{;\mu} - (R^\alpha{}_\alpha)^{;\mu}\delta g_{\mu\nu}]^{;\nu} + (R^\alpha{}_\alpha)^{;\mu;\nu}\delta g_{\mu\nu}.$$

Einstein-Hilbert Action

$$\begin{aligned}I &= \int d^4x\sqrt{g}\ R \\ \delta g &= gg^{\mu\nu}\delta g_{\mu\nu} \\ \delta(\sqrt{g}) &= \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \\ \delta g^{\mu\nu} &= -\delta g_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\end{aligned}$$

$$\begin{aligned}\delta\Gamma^\lambda_{\mu\nu} &= -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left[\partial_\nu\delta g_{\rho\mu} + \partial_\mu\delta g_{\rho\mu} - \partial_\rho\delta g_{\mu\nu}\right] \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\delta g_{\rho\mu})_{;\nu} + (\delta g_{\rho\nu})_{;\mu} - (\delta_{\mu\nu})_{;\rho}\right]\end{aligned}$$

$$\begin{aligned}\delta R_{\mu\nu} &= (\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\delta g_{\lambda\rho})_{;\mu;\nu} - (\delta g_{\rho\mu})_{;\nu;\lambda} - (\delta g_{\rho\nu})_{;\mu;\lambda} + (\delta_{\mu\nu})_{;\rho;\lambda}\right]\end{aligned}$$

$$\begin{aligned}\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{g}\left[(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})_{;\lambda}\right] \equiv \sqrt{g}(V^\nu{}_{;\nu} - W^\lambda{}_{;\lambda}) \\ &= \partial_\nu\sqrt{g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\sqrt{g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} \\ &= \partial_\nu[\sqrt{g}(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} - g^{\alpha\beta}\delta\Gamma^\nu_{\alpha\beta})]\end{aligned}$$

$$\begin{aligned}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} - g^{\alpha\beta}\delta\Gamma^\nu_{\alpha\beta} &= g^{\mu\nu}g^{\lambda\rho}[(\delta g_{\rho\lambda})_{;\mu} - (\delta g_{\mu\lambda})_{;\rho}] \\ &= g^{\mu\nu}g^{\lambda\rho}\left[\Gamma^\alpha_{\rho\lambda}\delta g_{\mu\alpha} - \Gamma^\alpha_{\mu\lambda}\delta g_{\rho\alpha} + \partial_\mu(\delta g_{\rho\lambda}) - \partial_\rho(\delta g_{\mu\lambda})\right]\end{aligned}$$

$$\delta I = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) + \int_{\partial V} d^3x\sqrt{\gamma}\ n_\nu\left\{g^{\mu\nu}g^{\lambda\rho}\left[\Gamma^\alpha_{\rho\lambda}\delta g_{\mu\alpha} - \Gamma^\alpha_{\mu\lambda}\delta g_{\rho\alpha} + \partial_\mu(\delta g_{\rho\lambda}) - \partial_\rho(\delta g_{\mu\lambda})\right]\right\}$$

$$\Rightarrow \delta I = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) + \int_{\partial V} d^3x\sqrt{\gamma}\ n_\nu\left\{g^{\mu\nu}g^{\lambda\rho}\left[\partial_\mu(\delta g_{\rho\lambda}) - \partial_\rho(\delta g_{\mu\lambda})\right]\right\}$$

$$\begin{aligned}\delta Q(p) &= Q(x^\alpha(p)) - Q^{(0)}(x^\alpha(p)) \\ \delta Q'(p) &= Q'(x'^\alpha(p)) - Q^{(0)}(x'^\alpha(p))\end{aligned}$$

All evaluated at p,

$$\begin{aligned}\delta Q' - \delta Q &= Q'(x'^\alpha) - Q^{(0)}(x'^\alpha) - Q(x^\alpha) + Q^{(0)}(x^\alpha) \\ &= Q'(x^\alpha) + \xi^\lambda \partial_\lambda Q'(x^\alpha) - Q^{(0)}(x'^\alpha) - Q(x^\alpha) + Q^{(0)}(x^\alpha) \\ ! &= \xi^\lambda \partial_\lambda Q'(x^\alpha)\end{aligned}$$

if the following holds

$$Q'(x^\alpha) = Q(x^\alpha), \qquad Q^{(0)}(x'^\alpha) = Q^{(0)}(x^\alpha)$$

Under infinitesimal transformation $x^\alpha \to x'^\alpha = x^\alpha - \xi^\alpha$

$$h_{ij}(x) \to h'_{ij}(x') = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} h_{kl}(x)$$

$$\begin{aligned}h'_{ij}(x') &= h_{ij}(x) + \xi^\lambda \partial_\lambda h_{ij}(x) + h_{i\lambda}(x)\partial_j \xi^\lambda + h_{\lambda j}(x)\partial_i \xi^\lambda \\ &= h_{ij}(x) + \xi^\lambda h_{ij;\lambda} + h_{i\lambda}\xi^\lambda{}_{;j} + h_{\lambda j}\xi^\lambda{}_{;i} \\ &= h_{ij}(x) + \mathscr{L}_\xi[h_{ij}(x)]\end{aligned}$$

FRW Fluctuations in Synchronous Gauge

Metric:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \left(\frac{1}{1-Kr^2}\right)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

$$g_{00} = -a^2, \quad g_{rr} = \frac{a^2}{1-Kr^2}, \quad g_{\theta\theta} = a^2 r^2, \quad g_{\phi\phi} = a^2 r^2 \sin^2\theta$$

Connection Terms:

$$\Gamma^0_{00} = \frac{\dot{a}}{a}, \quad \Gamma^0_{ii} = \frac{\dot{a}}{a}(a^{-2}g_{ii})$$

$$\Gamma^r_{r0} = \frac{\dot{a}}{a}, \quad \Gamma^r_{rr} = \frac{Kr}{1-Kr^2}, \quad \Gamma^r_{\theta\theta} = -r(1-Kr^2), \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta(1-Kr^2),$$

$$\Gamma^\theta_{\theta 0} = \frac{\dot{a}}{a}, \quad \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi 0} = \frac{\dot{a}}{a}, \quad \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = \cot\theta$$

All other terms vanish.

Ricci Tensor:

$$R_{\mu\nu} = \partial_\kappa \Gamma^\kappa_{\mu\nu} - \partial_\nu \Gamma^\kappa_{\mu\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{\mu\nu} - \Gamma^\kappa_{\alpha\nu}\Gamma^\alpha_{\mu\kappa}$$

$$\begin{aligned}
R_{00} &= \partial_\kappa \Gamma^\kappa_{00} - \partial_0 \Gamma^\kappa_{0\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{00} - \Gamma^\kappa_{\alpha 0}\Gamma^\alpha_{0\kappa} \\
&= \partial_0 \Gamma^0_{00} - (\partial_0 \Gamma^0_{00} + \partial_0 \Gamma^i_{0i}) + \Gamma^\kappa_{0\kappa}\Gamma^0_{00} - (\Gamma^0_{00}\Gamma^0_{00} + \Gamma^i_{0i}\Gamma^i_{0i}) \\
&= -\partial_0 \Gamma^i_{0i} \\
&= -3\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right]
\end{aligned}$$

$$\begin{aligned}
R_{0i} &= \partial_\kappa \Gamma^\kappa_{0i} - \partial_i \Gamma^\kappa_{0\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{0i} - \Gamma^\kappa_{\alpha i}\Gamma^\alpha_{0\kappa} \\
&= \Gamma^\kappa_{i\kappa}\Gamma^i_{0i} - (\Gamma^\kappa_{0i}\Gamma^0_{0\kappa} + \Gamma^\kappa_{ji}\Gamma^j_{0\kappa}) \\
&= 0
\end{aligned}$$

$i \neq j$

$$\begin{aligned}
R_{ij} &= \partial_\kappa \Gamma^\kappa_{ij} - \partial_j \Gamma^\kappa_{i\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{ij} - \Gamma^\kappa_{\alpha j}\Gamma^\alpha_{i\kappa} \\
&= 0
\end{aligned}$$

Have to step through this one for different values of i and j. One simplificiation is that any derivatives with respect to ϕ are zero, and there are only three nonvanishing Γ^k_{ij}'s for $i \neq j$.

$$\begin{aligned}
R_{rr} &= \partial_\kappa \Gamma^\kappa_{rr} - \partial_r \Gamma^\kappa_{r\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{rr} - \Gamma^\kappa_{\alpha r}\Gamma^\alpha_{r\kappa} \\
&= \partial_0 \Gamma^0_{rr} - \partial_r(\Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}) + \Gamma^\kappa_{r\kappa}\Gamma^r_{rr} + \Gamma^\kappa_{0\kappa}\Gamma^0_{rr} - [2\Gamma^0_{rr}\Gamma^r_{r0} + (\Gamma^r_{rr})^2 + (\Gamma^\theta_{\theta r})^2 + (\Gamma^\phi_{\phi r})^2] \\
&= \partial_0 \Gamma^0_{rr} - \partial_r(\Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}) + \Gamma^r_{rr}(\Gamma^\theta_{r\theta} + \Gamma^\phi_{\phi r}) + \Gamma^0_{rr}(\Gamma^\theta_{\theta 0} + \Gamma^\phi_{\phi 0}) - [\Gamma^0_{rr}\Gamma^r_{r0} + (\Gamma^\theta_{\theta r})^2 + (\Gamma^\phi_{\phi r})^2] \\
&= \frac{1}{1-Kr^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{2K}{1-Kr^2} \\
&= \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2K\right)\frac{1}{1-Kr^2}
\end{aligned}$$

$$R_{\theta\theta} = \partial_\kappa \Gamma^\kappa_{\theta\theta} - \partial_\theta \Gamma^\kappa_{\theta\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{\theta\theta} - \Gamma^\kappa_{\alpha\theta}\Gamma^\alpha_{\theta\kappa}$$

$$
\begin{aligned}
&= \partial_0 \Gamma^0_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^\kappa_{0\kappa}\Gamma^0_{\theta\theta} + \Gamma^\kappa_{r\kappa}\Gamma^r_{\theta\theta} - (\Gamma^\phi_{\phi\theta}\Gamma^\phi_{\phi\theta} + \Gamma^0_{\theta\theta}\Gamma^\theta_{\theta 0}) \\
&= \partial_0 \Gamma^0_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^0_{\theta\theta}(\Gamma^r_{0r} + \Gamma^\theta_{0\theta} + \Gamma^\phi_{0\phi}) + \Gamma^r_{\theta\theta}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - (\Gamma^\phi_{\phi\theta}\Gamma^\phi_{\phi\theta} + \Gamma^0_{\theta\theta}\Gamma^\theta_{\theta 0}) \\
&= \partial_0 \Gamma^0_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^0_{\theta\theta}(\Gamma^r_{0r} + \Gamma^\phi_{0\phi}) + \Gamma^r_{\theta\theta}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - \Gamma^\phi_{\phi\theta}\Gamma^\phi_{\phi\theta} \\
&= \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2K\right) r^2
\end{aligned}
$$

$$
\begin{aligned}
R_{\phi\phi} &= \partial_\kappa \Gamma^\kappa_{\phi\phi} - \partial_\phi \Gamma^\kappa_{\phi\kappa} + \Gamma^\kappa_{\alpha\kappa}\Gamma^\alpha_{\phi\phi} - \Gamma^\kappa_{\alpha\phi}\Gamma^\alpha_{\phi\kappa} \\
&= \partial_0 \Gamma^0_{\phi\phi} + \partial_r \Gamma^r_{\phi\phi} + \Gamma^0_{\phi\phi}(\Gamma^r_{0r} + \Gamma^\theta_{0\theta} + \Gamma^\phi_{0\phi}) + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} + \Gamma^r_{\phi\phi}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) \\
&= \partial_0 \Gamma^0_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^0_{\theta\theta}(\Gamma^r_{0r} + \Gamma^\theta_{0\theta} + \Gamma^\phi_{0\phi}) + \Gamma^r_{\theta\theta}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) \\
&= \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2K\right) r^2 \sin^2\theta
\end{aligned}
$$

Fluctuations in synchronous gauge:

$$
\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\left(\delta g_{\mu\nu} R - g_{\mu\nu}\delta R\right)
$$

Using $\delta g_{\mu\nu} = h_{\mu\nu}$,

$$
\begin{aligned}
\delta R_{\mu\nu} &= (\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \\
&= \frac{1}{2} g^{\lambda\rho}\left(h_{\rho\mu;\nu} + h_{\rho\nu;\mu} - h_{\mu\nu;\rho}\right)
\end{aligned}
$$

Variation of affine connection:

$$
\delta\Gamma^\lambda_{\mu\nu} = -g^{\lambda\rho} h_{\rho\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2} g^{\lambda\rho}\left(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}\right)
$$

$$
\delta\Gamma^0_{0\mu} = \delta\Gamma^i_{00} = 0
$$

$$
\delta\Gamma^0_{ij} = \frac{a^{-2}}{2}\partial_0 h_{ij}
$$

$$
\delta\Gamma^i_{0j} = \frac{a^{-2}}{2}\gamma^{ik}\left(\partial_0 h_{kj} - 2\frac{\dot{a}}{a} h_{kj}\right)
$$

$$
\delta\Gamma^i_{jk} = -g^{il} h_{lm}\Gamma^m_{jk} + \frac{1}{2} g^{il}\left(\partial_k h_{lj} + \partial_j h_{lk} - \partial_l h_{jk}\right)
$$

$$
\begin{aligned}
\delta\Gamma^i_{i\mu} &= -g^{il} h_{lm}\Gamma^m_{i\mu} + \frac{1}{2} g^{il}\left(\partial_\mu h_{li} + \partial_i h_{l\mu} - \partial_l h_{i\mu}\right) \\
&= -g^{ii} h_{im}\Gamma^m_{i\mu} + \frac{1}{2} g^{ii}\left(\partial_\mu h_{ii}\right)
\end{aligned}
$$

$$
\begin{aligned}
\delta\Gamma^i_{i0} &= -g^{ii} h_{im}\Gamma^m_{i0} \\
&= -a^{-2}\gamma^{ii} h_{ii}\Gamma^i_{i0} \\
&= -a^{-2}\left(\frac{\dot{a}}{a}\right)\gamma^{ii} h_{ii}
\end{aligned}
$$

$\delta\Gamma^i_{ir}$:

$$
\delta\Gamma^i_{ir} = -g^{ii} h_{ii}\Gamma^i_{ir} + \frac{1}{2} g^{ii}\partial_r h_{ii}
$$

$$\delta\Gamma^r_{rr} = -g^{rr}h_{rr}\Gamma^r_{rr} + \frac{1}{2}a^{-2}(1-Kr^2)\partial_r h_{rr}$$
$$= -\frac{Kr}{a^2}h_{rr} + \frac{1}{2}a^{-2}(1-Kr^2)\partial_r h_{rr}$$

$$\delta\Gamma^\theta_{\theta r} = -g^{\theta\theta}h_{\theta\theta}\Gamma^\theta_{\theta r} + \frac{1}{2}g^{\theta\theta}\partial_r h_{\theta\theta}$$
$$= -a^{-2}r^{-3}h_{\theta\theta} + \frac{1}{2}a^2r^2\partial_r h_{\theta\theta}$$

$$\delta\Gamma^\phi_{\phi r} = -g^{\phi\phi}h_{\phi\phi}\Gamma^\phi_{\phi r} + \frac{1}{2}g^{\phi\phi}\partial_r h_{\phi\phi}$$
$$= -a^{-2}r^{-3}\sin^{-2}\theta h_{\phi\phi} + \frac{1}{2}a^2r^2\sin^2\theta\partial_r h_{\phi\phi}$$

$\delta\Gamma^i_{i\theta}$:

$$\delta\Gamma^i_{i\theta} = -g^{ii}h_{ii}\Gamma^i_{i\theta} + \frac{1}{2}g^{ii}\partial_\theta h_{ii}$$

$$\delta\Gamma^r_{r\theta} = \frac{1}{2}g^{rr}\partial_\theta h_{rr}$$
$$= \frac{1}{2}\left(\frac{1-Kr^2}{a^2}\right)\partial_\theta h_{rr}$$

$$\delta\Gamma^\theta_{\theta\theta} = \frac{1}{2}r^{-2}a^{-2}\partial_\theta h_{\theta\theta}$$

$$\delta\Gamma^\phi_{\phi\theta} = -a^{-2}r^{-2}\sin^{-2}\theta h_{\phi\phi}\cot\theta + \frac{1}{2}a...$$

SVT Decomposition

The most general form we may take for our perturbation to the FRW metric is

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \gamma_{ij}dx^i dx^j + h_{\mu\nu}dx^\mu dx^\nu\right].$$

Seperating out $h_{\mu\nu}$ into space and time components,

$$ds^2 = a^2(\tau)\left[-(1+\psi)d\tau^2 + w_i dx^i d\tau + (\phi\gamma_{ij} + S_{ij})\right].$$

Here we have taken the trace of h_{ij} and placed it in ϕ so that S_{ij} is a symmetric traceless tensor. Given the 3 vector w_i, we may decompose it into its scalar (curl-free) and vector (divergence-free) components

$$w_i = w_i^{(S)} + w_i^{(V)}$$

where

$$w_i^{(S)} = \nabla_i w$$
$$\nabla^i w_i^{(V)} = 0.$$

Our symmetric traceless tensor S_{ij} may also be broken up accordingly

$$S_{ij} = S_{ij}^{(S)} + S_{ij}^{(V)} + S_{ij}^{(T)}$$

where

$$S_{ij}^{(S)} = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S$$

$$S_{ij}^{(V)} = \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right), \qquad \nabla^i S_i = 0$$
$$\nabla^i S_{ij}^{(T)} = 0.$$

Note that the vector S_i is divergence-less, which is as we expect for a true vector, and that S_{ij}^T is transverse. Additionally, the actual scalar, vector, and tensor functions are not unique. If we put all these together, our metric takes the form

$$ds^2 = a(\tau^2)\left\{-(1+\psi)d\tau^2 + (\nabla_i w + w_i)dx^i d\tau + \left[\phi\gamma_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S + \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right) + S_{ij}^T\right]\right\}.$$

Mode superscripts have been dropped, so we make take a vector like w_i to be a true vector $\nabla^i w_i = 0$. Counting the degrees of freedom, we have 4 scalar fields (ψ, w, S, ϕ), 2 two-component vectors (w_i, S_i), and one traceless transverse symmetric tensor S_{ij}^T, $4+4+2=10$.

$$\begin{aligned}
h_{00} &= -\psi \\
h_{0i} &= \nabla_i w + w_i \\
h_{ij} &= \phi\gamma_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)S + \frac{1}{2}\left(\nabla_i S_j + \nabla_j S_i\right) + S_{ij}^T
\end{aligned}$$

In flat space $h = \psi + 3\phi$ and

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\nu\lambda} - \partial_\nu\partial^\lambda h_{\mu\lambda} + \partial_\mu\partial_\nu h^\lambda{}_\lambda\right)$$

$$\begin{aligned}
\delta R_{00} &= \frac{1}{2}\left(\partial^\lambda\partial_\lambda h_{00} - 2\partial_0\partial^\lambda h_{0\lambda} + \ddot{h}\right) \\
&= \frac{1}{2}\left[-\partial_i\partial^i\psi - 2\partial_0(\nabla^2 w + \partial^i w_i) + 2\ddot{\phi}\right]
\end{aligned}$$

Under the transverse or synchronous gauge, we see that δR_{00} relates scalars.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\left(h_{\alpha\beta}R^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)$$
$$\delta G^\mu_\nu = \delta(g^{\mu\lambda}G_{\lambda\nu}) = h^{\mu\lambda}G_{\lambda\nu} + g^{\mu\lambda}\delta G_{\lambda\nu}$$

Synchronous:

$$\delta G^0{}_0 = g^{00}\delta G_{00} = -a^{-2}(\tau)\delta G_{00}$$
$$\delta G_{00} = \delta R_{00} - \frac{1}{2}h_{00}R + \frac{1}{2}a^2(\tau)\left(h_{\alpha\beta}R^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)$$

$h_{\mu\nu}$ is not trace-free. In conformal gravity, we may do the standard perscription with perturbations involving $h_{\mu\nu}$ and we will find that the fluctation equations do not depend on the trace, h. Equivalently, we may instead work with traceless "gauge" $K_{\mu\nu}$. What we cannot do within the code is remove the trace terms h, and then use the full gauge $h_{\mu\nu}$. However, we can do the substitution $h_{\mu\nu} \to K_{\mu\nu}$ without problems.

xAct Note: Noting the difference in notation between Wald and Weinberg, we must replace R, $R_{\mu\nu}$ and $R_{\lambda\mu\nu\kappa}$ with a negative. Consequently the Weyl tensor also changes sign. Varying the action of $C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$ yielded conformal invariant equations, however they were not equal to $2C^{\lambda\mu\nu\kappa}{}_{;\mu;\kappa} - C^{\lambda\mu\nu\kappa}R_{\mu\kappa}$ nor $W_2 - 1/3W_1$, nor any apparant variation of the minus signs on the curvature tensors. Varying the same action, except this time in its contracted form (in terms of Riemann and Ricci tensors) did not yield conformal invariant equations. The Lanczos identity is not computed correctly either. We did, however get the right expression for $W_2 - 1/3W_1|_{negativecurvatureterms}$ by varying the action $R_{\mu\nu}R^{\mu\nu} - 1/3R^2$.

SVT Decomposition from Orthogonal Projections

Fundamental observers are locally at rest with respect to the matter fluid. Motivated by these "preferred" frames, we seek to split a given rank 2 tensor T_{ab} into components parallel and orthogonal to a velocity vector u_μ. The rest frames locally define surfaces of constant t. The induced metric for the surfaces of simultaneity is

$$h_{ab} = g_{ab} + u_a u_b.$$

That this acts like a 3-space metric can verified by

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^c{}_c = 3, \qquad h^a{}_b u^b = 0.$$

Note that the last relation uses $h^a{}_b$ to project the components orthogonal to u^a. Likewise $U^a{}_b \equiv -u^a u_b$ projects components parallel to u_a.

We can use these projectors to decompose a tensor into components parallel and orthogonal to the local velocity. Take arbitrary symmetric rank 2 tensor T_{ab}

$$\begin{aligned}
T_{ab} &= g_a{}^c g_b{}^d T_{cd} \\
&= (h_a{}^c + U_a{}^c)(h_b{}^d + U_b{}^d) T_{cd} \\
&= h_a{}^c h_b{}^d T_{cd} - u_a(u^c h_b{}^d T_{cd}) - u_b(u^d h_a{}^c T_{cd}) + u_a u_b (u^c u^d T_{cd}).
\end{aligned}$$

This can be expressed in terms of symmetric trace free quantities

$$\begin{aligned}
T_{ab} = \frac{1}{3} h_{ab} h^{cd} T_{cd} + \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h_{ab} h^{cd} \right] T_{cd} + h^a{}_b h^c{}_d T_{[cd]} \\
- u_a(h_b{}^d T_{cd} u^c) - u_b(h_a{}^c T_{cd} u^d) + u_a u_b (u^c u^d T_{cd}).
\end{aligned}$$

where

$$T_{<ab>} \equiv \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h_{ab} h^{cd} \right] T_{cd}$$

is the symmetric trace free projection orthogonal to u_a. For a vector, we have orthogonal projection

$$V_{<a>} \equiv h_a{}^b V_a.$$

Let us take T_{ab} to be symmetric and relabel the following quantities:

$$\begin{aligned}
\rho &= u^a u^b T_{ab} \\
p &= \frac{1}{3} h^{ab} T_{ab} \\
q_a &= q_{<a>} = -h_a{}^b T_{bc} u^c = -T_{<a>b} u^b \\
\pi_{ab} &= \pi_{<ab>} = T_{<ab>}.
\end{aligned}$$

Now the energy momentum tensor may be expressed as

$$\begin{aligned}
T_{ab} &= u_a u_b \rho + h_{ab} p + u_a q_b + u_b q_a + \pi_{ab} \\
&= (\rho + p) u_a u_b + p g_{ab} + u_a q_b + u_b q_a + \pi_{ab}
\end{aligned}$$

Note that $u^a q_a = 0$, and that π_{ab} (projected symmetric traceless) has 5 components. Thus 2 scalars, 3 vector components, and 5 from the tensor give us 10 in total.

In comoving coordinates in FLRW space, the velocity vector is

$$u^a = \delta^a_0, \qquad u_a = -\delta^0_a$$

and the only non-zero components of π_{ab} are π_{ij} (spatial). According to York (1973) we may decompose a symmetric tensor on a positive definite Riemannian space as

$$\pi_{ab} = \pi_{ab}^{TT} + \pi_{ab}^{L} + \pi_{ab}^{Tr}$$

where

$$\pi_{ab}^{Tr} = \frac{1}{3} g_{ab} g^{cd} \pi_{cd}$$

and

$$\pi_{ab}^{L} = \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c.$$

By construction

$$g^{ab} \pi_{ab}^{TT} = 0.$$

The transverse requirement leads to an equation for the vector W_a

$$-\nabla_b \pi^{ab(L)} = -\nabla_b (\pi^{ab} - \frac{1}{3} g^{ab} g_{cd} \pi^{cd}).$$

York shows that such a vector W_a must exists and is unique, up to conformal Killing vectors. Moreover, he also shows that decomposition actually holds its form under conformal transformation on the metric.

Going back to the symmetric traceless tensor π_{ab}, we may write this as

$$\begin{aligned}\pi_{ab} &= \pi_{ab}^{TT} + \pi_{ab}^{L} \\ &= \pi_{ab}^{TT} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c.\end{aligned}$$

Going back to the general symmetric tensor, if we instead make the substitutions

$$\begin{aligned}\rho &= -2\phi \\ p &= -2\left(\psi' - \frac{1}{3}\nabla^2 E\right) = \psi \\ q_a &= -(B_a + \nabla_a B); \qquad \nabla_a B^a = 0 \\ W_a &= (E_a + \nabla_a E); \qquad \nabla_a E^a = 0\end{aligned}$$

and

$$\begin{aligned}\pi_{ab} &= \pi_{ab}^{TT} + \nabla_a W_b + \nabla_b W_a - \frac{2}{3} g_{ab} \nabla_c W^c \\ &= E_{ab} + \nabla_a E_b + \nabla_b E_a + 2\nabla_a \nabla_b E - \frac{2}{3} h_{ab} \nabla^2 E\end{aligned}$$

we then end up with the same form of the perturbation metric as given in the standard SVT decomposition:

$$T_{ab} = -2\phi u_a u_b - (B_b + \nabla_b B) u_a - (B_a + \nabla_a B) u_b - 2\gamma_{ab}\psi + \nabla_a E_b + \nabla_b E_a + +2E_{ab}.$$

In flat space the spacetime interval is

$$ds^2 = -(1+2\phi)dt^2 + 2(B_i + \nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i \nabla_j E + 2E_{ij}\right] dx^i dx^j.$$

Thus the SVT decomposition can be achieved first by orthogonal decomposition of a symmetric tensor relative to the four velocity, and then decomposing the projected symmetric trace-free portion into transverse and longitudinal components.

Harmonic Flat Space Conformal Transformation

Harmonic condition:

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\mu{}_\mu = 0$$

$$\nabla_\mu h^\mu{}_\nu - \frac{1}{2}\nabla_\nu h^\mu{}_\mu = 0.$$

Conformal transformation:

$$\Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$

$$\Omega^{-2} g^{\mu\nu} = \bar{g}^{\mu\nu}$$

$$h^\mu{}_\nu = g^{(0)}_{\rho\nu} h^{\mu\rho} = (\Omega^{-2}\bar{g}^{(0)}_{\rho\nu})(\Omega^2 \bar{h}^{\mu\rho}) = \bar{h}^\mu{}_\nu$$

The following will be useful within our gauge transformation:

$$\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} - \Omega^{-1}\left(\delta^\lambda_\nu \partial_\mu \Omega + \delta^\lambda_\mu \partial_\nu \Omega - n_{\mu\nu} n^{\lambda\rho}\partial_\rho \Omega\right)$$

$$\begin{aligned}
\nabla_\mu h^\mu{}_\nu - \frac{1}{2}\nabla_\nu h^\mu{}_\mu &= \partial_\mu h^\mu{}_\nu + \Gamma^\mu_{\mu\rho} h^\rho{}_\nu - \Gamma^\rho_{\mu\nu} h^\mu{}_\rho - \frac{1}{2}\partial_\nu h^\mu{}_\mu \\
&= \partial_\mu \bar{h}^\mu{}_\nu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu} h^\mu{}_\rho - \frac{1}{2}\partial_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho \Omega + \Omega^{-1}\bar{h}^\mu{}_\rho\left(\delta^\rho_\nu \partial_\mu \Omega + \delta^\rho_\mu \partial_\nu \Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha \Omega\right) \\
&= \bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho \Omega + \Omega^{-1}\bar{h}^\mu{}_\rho\left(\delta^\rho_\nu \partial_\mu \Omega + \delta^\rho_\mu \partial_\nu \Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha \Omega\right) \\
&= \& \quad \bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho \Omega + \Omega^{-1}\bar{h}^\mu{}_\mu \partial_\nu \Omega \quad \&.
\end{aligned}$$

In a conformal to flat space, we need to work in the gauge

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu = 4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho \Omega - \Omega^{-1} h^\mu{}_\mu \partial_\nu \Omega$$

Einstein Perturbation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\left(h_{\alpha\beta}R^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)$$

Weinberg (10.9.3)

$$\begin{aligned}
\delta R_{\mu\nu} &= (\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \\
&= \frac{1}{2}g^{\lambda\rho}\left[(h_{\lambda\rho})_{;\mu;\nu} - (h_{\rho\mu})_{;\nu;\lambda} - (h_{\rho\nu})_{;\mu;\lambda} + (h_{\mu\nu})_{;\rho;\lambda}\right] \\
&= \frac{1}{2}\left(\nabla_\nu \nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda \nabla_\nu h^\lambda{}_\mu - \nabla_\lambda \nabla_\mu h^\lambda{}_\nu + \nabla_\lambda \nabla^\lambda h_{\mu\nu}\right)
\end{aligned}$$

$$\begin{aligned}
\delta G_{\mu\nu} = &\frac{1}{2}\left(\nabla_\nu \nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda \nabla_\nu h^\lambda{}_\mu - \nabla_\lambda \nabla_\mu h^\lambda{}_\nu + \nabla_\lambda \nabla^\lambda h_{\mu\nu}\right) - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta} \\
&- \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\left(\nabla_\beta \nabla_\alpha h^\lambda{}_\lambda - \nabla_\lambda \nabla_\beta h^\lambda{}_\alpha - \nabla_\lambda \nabla_\alpha h^\lambda{}_\beta + \nabla_\lambda \nabla^\lambda h_{\alpha\beta}\right) \\
= &\frac{1}{2}\left(\nabla_\nu \nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda \nabla_\nu h^\lambda{}_\mu - \nabla_\lambda \nabla_\mu h^\lambda{}_\nu + \nabla_\lambda \nabla^\lambda h_{\mu\nu}\right) - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta} \\
&- \frac{1}{4}g_{\mu\nu}\left(2\nabla_\beta \nabla^\beta h^\lambda{}_\lambda - 2\nabla^\alpha \nabla_\lambda h^\lambda{}_\alpha\right) \\
= &\frac{1}{2}\left(\nabla_\nu \nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda \nabla_\nu h^\lambda{}_\mu - \nabla_\lambda \nabla_\mu h^\lambda{}_\nu + \nabla_\lambda \nabla^\lambda h_{\mu\nu}\right) - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta} \\
&- \frac{1}{2}g_{\mu\nu}\left(\nabla_\beta \nabla^\beta h^\lambda{}_\lambda - \nabla^\alpha \nabla_\lambda h^\lambda{}_\alpha\right)
\end{aligned}$$

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \\
G^{\mu}{}_{\mu} &= R - 2R = -R = T^{\mu}{}_{\mu} \\
R_{\mu\nu} &= T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda} \equiv S_{\mu\nu} \\
\delta R_{\mu\nu} &= \delta S_{\mu\nu} \\
\delta R_{\mu\nu} &= \frac{1}{2} \left(\nabla_\nu \nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\lambda \nabla_\nu h^{\lambda}{}_{\mu} - \nabla_\lambda \nabla_\mu h^{\lambda}{}_{\nu} + \nabla_\lambda \nabla^\lambda h_{\mu\nu} \right) \\
\bar{\nabla}_\mu \bar{h}^{\mu}{}_{\nu} - \frac{1}{2} \bar{\nabla}_\nu \bar{h}^{\mu}{}_{\mu} &= 4\Omega^{-1} \bar{h}^{\rho}{}_{\nu} \partial_\rho \Omega - \Omega^{-1} h^{\mu}{}_{\mu} \partial_\nu \Omega
\end{aligned}
$$

Referring to Mannheim (35), we may use the covariant interchange identity to express the Ricci variation as

$$
\delta R_{\mu\nu} = \frac{1}{2} \left(\nabla_\nu \nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\nu \nabla_\lambda h^{\lambda}{}_{\mu} - \nabla_\mu \nabla_\lambda h^{\lambda}{}_{\nu} + \nabla_\lambda \nabla^\lambda h_{\mu\nu} \right) + \frac{1}{2} g^{\lambda\rho} \left(h^{\sigma}{}_{\rho} R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho} R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu} R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu} R_{\rho\sigma\mu\lambda} \right).
$$

Substituting our gauge choice for the middle two covariant derivative terms

$$
\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2} \left(\nabla_\nu \nabla_\mu h^{\lambda}{}_{\lambda} - \nabla_\nu \nabla_\lambda h^{\lambda}{}_{\mu} - \nabla_\mu \nabla_\lambda h^{\lambda}{}_{\nu} + \nabla_\lambda \nabla^\lambda h_{\mu\nu} \right) + \frac{1}{2} g^{\lambda\rho} \left(h^{\sigma}{}_{\rho} R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho} R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu} R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu} R_{\rho\sigma\mu\lambda} \right) \\
&= \frac{1}{2} \left(\nabla_\nu \nabla_\mu h^{\lambda}{}_{\lambda} - \frac{1}{2} \nabla_\nu \nabla_\mu h^{\lambda}{}_{\lambda} - \frac{1}{2} \nabla_\mu \nabla_\nu h^{\lambda}{}_{\lambda} + \nabla_\lambda \nabla^\lambda h_{\mu\nu} \right) + \frac{1}{2} g^{\lambda\rho} \left(h^{\sigma}{}_{\rho} R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho} R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu} R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu} R_{\rho\sigma\mu\lambda} \right) \\
&\quad - \nabla_\nu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\mu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\mu \Omega \right) - \nabla_\mu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\nu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\nu \Omega \right) \\
&= \frac{1}{2} \nabla_\lambda \nabla^\lambda h_{\mu\nu} + \frac{1}{2} g^{\lambda\rho} \left(h^{\sigma}{}_{\rho} R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho} R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu} R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu} R_{\rho\sigma\mu\lambda} \right) \\
&\quad - \frac{1}{2} \nabla_\nu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\mu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\mu \Omega \right) - \frac{1}{2} \nabla_\mu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\nu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\nu \Omega \right)
\end{aligned}
$$

From here we would like to evaluate the Riemann tensor for a conformal to flat metric. From Weinberg (6.1.5) we have

$$
\begin{aligned}
\Omega^2 g_{\mu\nu} &= \bar{g}_{\mu\nu} \\
\Omega^{-2} g^{\mu\nu} &= \bar{g}^{\mu\nu}
\end{aligned}
$$

$$
R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta}.
$$

We will need an expression for the Christoffel symbol:

$$
\Gamma^{\lambda}_{\mu\nu} = \Omega^{-1} \left(\delta^{\lambda}_{\nu} \partial_\mu \Omega + \delta^{\lambda}_{\mu} \partial_\nu \Omega - n_{\mu\nu} n^{\lambda\rho} \partial_\rho \Omega \right).
$$

Now form the Riemann tensor

$$
\begin{aligned}
R_{\lambda\mu\nu\kappa} &= g_{\lambda\rho} (\partial_\kappa \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\rho}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\rho}_{\nu\eta}) \\
&= \Omega \left(\eta_{\lambda\nu} \partial_\mu \partial_\kappa \Omega + \eta_{\kappa\mu} \partial_\nu \partial_\lambda \Omega - \eta_{\mu\nu} \partial_\lambda \partial_\kappa \Omega - \eta_{\kappa\lambda} \partial_\mu \partial_\nu \Omega \right) + \eta_{\mu\kappa} \eta_{\lambda\nu} \partial_\alpha \Omega \partial^\alpha \Omega - \eta_{\kappa\lambda} \eta_{\mu\nu} \partial_\alpha \Omega \partial^\alpha \Omega \\
&\quad + 2\eta_{\mu\nu} \partial_\kappa \Omega \partial_\lambda \Omega - 2\eta_{\lambda\nu} \partial_\kappa \Omega \partial_\mu \Omega - 2\eta_{\kappa\mu} \partial_\lambda \Omega \partial_\nu \Omega + 2\eta_{\kappa\lambda} \partial_\mu \Omega \partial_\nu \Omega
\end{aligned}
$$

$$
\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2} \nabla_\lambda \nabla^\lambda h_{\mu\nu} + \frac{1}{2} g^{\lambda\rho} \left(h^{\sigma}{}_{\rho} R_{\sigma\nu\mu\lambda} + h^{\sigma}{}_{\rho} R_{\sigma\mu\nu\lambda} - h^{\sigma}{}_{\mu} R_{\rho\sigma\nu\lambda} - h^{\sigma}{}_{\nu} R_{\rho\sigma\mu\lambda} \right) \\
&\quad - \nabla_\nu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\mu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\mu \Omega \right) - \nabla_\mu \left(4\Omega^{-1} \bar{h}^{\rho}{}_{\nu} \partial_\rho \Omega - \Omega^{-1} h^{\lambda}{}_{\lambda} \partial_\nu \Omega \right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega-\Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) &= 4\Omega^{-1}\left(\nabla_\nu\bar{h}^\rho{}_\mu\partial_\rho\Omega+\bar{h}^\rho{}_\mu\nabla_\nu\nabla_\rho\Omega-\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\nu\Omega\partial_\rho\Omega\right)\\
&\quad-\Omega^{-1}\left(\partial_\nu\bar{h}^\lambda{}_\lambda\partial_\mu\Omega+\bar{h}^\lambda{}_\lambda\nabla_\nu\nabla_\mu\Omega-\Omega^{-1}\bar{h}^\lambda{}_\lambda\partial_\nu\Omega\partial_\mu\Omega\right)\\
&= 4\Omega^{-3}\eta^{\rho\kappa}\left(\nabla_\nu\bar{h}_{\kappa\mu}\partial_\rho\Omega+\bar{h}_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega-\Omega^{-1}\bar{h}_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega\right)\\
&\quad-\Omega^{-3}\eta^{\lambda\kappa}\left(\partial_\nu\bar{h}_{\kappa\lambda}\partial_\mu\Omega+\bar{h}_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega-\Omega^{-1}\bar{h}_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right)\\
&= \Omega^{-3}\left(4\eta^{\rho\kappa}\nabla_\nu h_{\kappa\mu}\partial_\rho\Omega+4\eta^{\rho\kappa}h_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega-\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega-\eta^{\lambda\kappa}h_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega\right)\\
&\quad\ \Omega^{-4}\left(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega+\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu h_{\kappa\mu} &= \partial_\nu h_{\kappa\mu}+\Omega^{-1}\left(\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega+\eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega-h_{\mu\nu}\partial_\kappa\Omega-h_{\kappa\nu}\partial_\mu\Omega-2h_{\kappa\mu}\partial_\nu\Omega\right)\\
\nabla_\nu\nabla_\rho\Omega &= \partial_\rho\partial_\nu\Omega+\Omega^{-1}(\eta^{\alpha\beta}\eta_{\nu\rho}\partial_\alpha\Omega\partial_\beta\Omega-2\partial_\nu\Omega\partial_\rho\Omega)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega-\Omega^{-1}\bar{h}^\lambda{}_\lambda\partial_\mu\Omega\right) &=\\
&\Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega+4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega-\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega-\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right)\\
&+\Omega^{-4}(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega+\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega+4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega\\
&\quad+4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega\partial_\rho\Omega-4\eta^{\rho\kappa}h_{\mu\nu}\partial_\kappa\Omega\partial_\rho\Omega-4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega\\
&\quad-8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega+4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\nu\rho}h_{\kappa\mu}\partial_\alpha\Omega\partial_\beta\Omega-8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega\\
&\quad-\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega+2\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega)\\
&= \quad\Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega+4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega-\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega-\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right)\\
&\quad+\Omega^{-4}(3\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega+4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega-4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega\\
&\quad-16\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega-\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega)
\end{aligned}
$$

As long as all expressions involving $h_{\mu\nu}$ are converted to down indices, we need to take the conformal transformations of $\Omega^{-2}h_{\mu\nu}$. There is a problem in the program in the conformal weighting. With the above replacement, all terms $h_{\mu\nu}$ can be intrepreted at $\bar{h}_{\mu\nu}$ within the code.

$$\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda \nabla^\lambda h_{\mu\nu} + \frac{1}{2} g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)$$
$$- \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu \partial_\rho \Omega - \Omega^{-1} h^\lambda{}_\lambda \partial_\mu \Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu \partial_\rho \Omega - \Omega^{-1} h^\lambda{}_\lambda \partial_\nu \Omega\right)$$

$$\begin{aligned}
\frac{1}{2}\nabla_\lambda \nabla^\lambda h_{\mu\nu} = \Omega^{-2}&\left(\frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu}\right) \\
&+ \Omega^{-3}\Big(- \eta^{\alpha\beta} h_{\mu\nu}\partial_\beta\partial_\alpha\Omega - \eta^{\alpha\gamma}\partial_\alpha h_{\mu\nu}\partial_\gamma\Omega + \eta^{\alpha\eta}\partial_\mu h_{\nu\alpha}\partial_\eta\Omega - \eta^{\alpha\beta}\partial_\beta h_{\nu\alpha}\partial_\mu\Omega \\
&\qquad + \eta^{\alpha\lambda}\partial_\nu h_{\mu\alpha}\partial_\lambda\Omega - \eta^{\alpha\beta}\partial_\beta h_{\mu\alpha}\partial_\nu\Omega\Big) \\
&+ \Omega^{-4}\Big(\eta^{\alpha\eta}\eta^{\beta\rho}\eta_{\mu\nu}h_{\alpha\beta}\partial_\eta\Omega\partial_\rho\Omega - 2\eta^{\alpha\kappa}h_{\nu\alpha}\partial_\kappa\Omega\partial_\mu\Omega + \eta^{\alpha\beta}h_{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega - 2\eta^{\alpha\rho}h_{\mu\alpha}\partial_\nu\Omega\partial_\rho\Omega\Big)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \\
\Omega^{-3}&\Big(\eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\partial_\mu\Omega + 2\eta^{\alpha\beta}h_{\mu\alpha}\partial_\beta\partial_\nu\Omega \\
&\qquad - \eta^{\alpha\beta}\eta^{\eta\gamma}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\eta\partial_\beta\Omega - \eta^{\alpha\beta}h_{\alpha\beta}\partial_\nu\partial_\mu\Omega\Big) \\
+ \Omega^{-4}&\Big(- \eta^{\alpha\eta}\eta^{\gamma\beta}\eta_{\mu\nu}h_{\beta\gamma}\partial_\alpha\Omega\partial_\eta\Omega + 2\eta^{\alpha\kappa}h_{\mu\nu}\partial_\alpha\Omega\partial_\kappa\Omega \\
&\qquad - 4\eta^{\alpha\rho}h_{\nu\alpha}\partial_\rho\Omega\partial_\mu\Omega - 4\eta^{\alpha\eta}h_{\mu\alpha}\partial_\eta\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}h_{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega \\
&\qquad + 2\eta^{\alpha\lambda}\eta^{\beta\rho}\eta_{\mu\nu}h_{\alpha\beta}\partial_\lambda\Omega\partial_\rho\Omega\Big)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right) = \\
\Omega^{-3}&\left(2\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 2\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \frac{1}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right) \\
+ \Omega^{-4}&\Big(\frac{3}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 2\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 2\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\
&\qquad - 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\Big) \\
+ (\mu &\leftrightarrow \nu)
\end{aligned}$$

$$\frac{1}{2}\Box\left(\Omega^{-2}\bar{h}_{\mu\nu}\right) = \frac{1}{2}\Omega^{-2}\Box\bar{h}_{\mu\nu} - \Omega^{-3}\bar{h}_{\mu\nu}\Box\Omega$$

Under conformal transformation

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$

an arbitrary tensor $T_{\mu\nu}(g_{\mu\nu})$ transforms as

$$T_{\mu\nu}(g_{\mu\nu}) \to T_{\mu\nu}(\Omega^2 g_{\mu\nu}) = \bar{T}_{\mu\nu}(g_{\mu\nu}).$$

The tensor as evaluated in the conformal geometry $\bar{g}_{\mu\nu}$, that is $\bar{T}_{\mu\nu}(\Omega^2 g_{\mu\nu})$ must again be calculated and is in general not proportional to $T_{\mu\nu}(g_{\mu\nu})$. In C^2 theory, however, we have

$$W_{\mu\nu} \to W_{\mu\nu}(\bar{g}_{\mu\nu}) = \bar{W}_{\mu\nu}(g_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(g_{\mu\nu})$$

and thus

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(\Omega^{-2}\bar{g}_{\mu\nu})$$

asdfasdfasdfasdf

$$\begin{aligned} W_{\mu\nu}(\bar{g}_{\mu\nu}) &= \Omega^{-2} W_{\mu\nu}(g_{\mu\nu}) \\ &\equiv \bar{W}_{\mu\nu}(g_{\mu\nu}) \\ &\to \bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2} W_{\mu\nu}(\Omega^{-2}\bar{g}_{\mu\nu}) \end{aligned}$$

$$\begin{aligned} R_{\mu\nu}(\bar{g}_{\mu\nu}) &= R_{\mu\nu}(g_{\mu\nu}) + S_{\mu\nu}(g_{\mu\nu}) \\ &\equiv \bar{R}_{\mu\nu}(g_{\mu\nu}) \end{aligned}$$

$$\begin{aligned} \bar{R}_{\mu\nu}(g_{\mu\nu}) &= R_{\mu\nu}(g_{\mu\nu}) + S_{\mu\nu}(g_{\mu\nu}) \\ &\to \bar{R}_{\mu\nu}(\bar{g}_{\mu\nu}) = R_{\mu\nu}(\Omega^{-2}\bar{g}_{\mu\nu}) + S_{\mu\nu}(\Omega^{-2} g_{\mu\nu}) \end{aligned}$$

hmasdf

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

$$\begin{aligned} W_{\mu\nu}(g_{\mu\nu}) &= W_{\mu\nu}(g^{(0)}_{\mu\nu} + h_{\mu\nu}) \\ &= W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}(h_{\mu\nu}) \end{aligned}$$

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}\left(K_{\mu\nu} + \frac{h}{4} g^{(0)}_{\mu\nu}\right) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4} g^{(0)}_{\mu\nu}\right)$$

$$\begin{aligned} W_{\mu\nu}\left(g^{(0)}_{\mu\nu} + \frac{h}{4} g^{(0)}_{\mu\nu}\right) &= W_{\mu\nu}\left[\left(1 + \frac{h}{4}\right) g^{(0)}_{\mu\nu}\right] \\ \left(1 - \frac{h}{4}\right) W_{\mu\nu}(g^{(0)}_{\mu\nu}) &= W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4} g^{(0)}_{\mu\nu}\right) \\ -\frac{h}{4} W_{\mu\nu}(g^{(0)}_{\mu\nu}) &= \delta W_{\mu\nu}\left(\frac{h}{4} g^{(0)}_{\mu\nu}\right) \end{aligned}$$

asdfasdfa

$$g^{\mu\nu} W_{\mu\nu}(g_{\mu\nu}) = \left(g^{(0)\mu\nu} - h^{\mu\nu}\right)\left(W^{(0)}_{\mu\nu} + \delta W_{\mu\nu}\right) = 0$$

To first order,

$$-h^{\mu\nu} W^{(0)}_{\mu\nu} + g^{(0)\mu\nu} \delta W_{\mu\nu} = 0$$

and thus

$$g^{(0)\mu\nu} \delta W_{\mu\nu}(h_{\mu\nu}) = h^{\mu\nu} W_{\mu\nu}(g^{(0)}_{\mu\nu})$$

Supplementary

I would like to verify the sign in Weinberg 10.9.2, since this is commonly referenced opposite in literature.

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}$$

$$\begin{aligned}
\delta R_{\mu\nu} &= \delta R^{\lambda}{}_{\mu\lambda\nu} \\
&= \delta\left(\partial_\nu\Gamma^{\lambda}_{\mu\lambda} - \partial_\lambda\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\lambda\alpha}\right) \\
&= \partial_\nu(\delta\Gamma^{\lambda}_{\mu\lambda}) - \partial_\lambda(\delta\Gamma^{\lambda}_{\mu\nu}) + (\delta\Gamma^{\alpha}_{\mu\lambda})\Gamma^{\lambda}_{\nu\alpha} + \Gamma^{\alpha}_{\mu\lambda}\delta(\Gamma^{\lambda}_{\nu\alpha}) - \delta(\Gamma^{\alpha}_{\mu\nu})\Gamma^{\lambda}_{\lambda\alpha} - \Gamma^{\alpha}_{\mu\nu}\delta(\Gamma^{\lambda}_{\lambda\alpha}) \\
&= \left\{\partial_\nu(\delta\Gamma^{\lambda}_{\mu\lambda}) - \Gamma^{\alpha}_{\mu\nu}\delta(\Gamma^{\lambda}_{\lambda\alpha})\right\} - \left\{\partial_\lambda(\delta\Gamma^{\lambda}_{\mu\nu}) - \Gamma^{\lambda}_{\nu\alpha}(\delta\Gamma^{\alpha}_{\mu\lambda}) - \Gamma^{\alpha}_{\mu\lambda}\delta(\Gamma^{\lambda}_{\nu\alpha}) + \Gamma^{\lambda}_{\lambda\alpha}\delta(\Gamma^{\alpha}_{\mu\nu})\right\} \\
&= (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}
\end{aligned}$$

Verify that $R_{\mu\nu} = R_{\nu\mu}$:

$$R_{\mu\nu} = g^{\lambda\kappa}R_{\lambda\mu\kappa\nu} = g^{\lambda\kappa}R_{\kappa\nu\lambda\mu} = R_{\nu\mu}.$$

Explicitly evaluating $R_{\mu\nu}$ and looking for symmetry is actually quite difficult. Best to look directly at curvature tensor and its associated symmetry, then contract.

Verify under the transformation

$$\begin{aligned}
g_{\mu\nu}(x) &\to e^{2\alpha(x)}g_{\mu\nu}(x), \\
C^{\lambda}{}_{\mu\nu\kappa}(x) &\to C^{\lambda}{}_{\mu\nu\kappa}(x).
\end{aligned}$$

Weyl tensor:

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) + \frac{1}{6}R^{\alpha}{}_{\alpha}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right)$$

Since

$$C^{\lambda}{}_{\mu\nu\kappa} = g^{\lambda\rho}C_{\rho\mu\nu\kappa} \to \left(g^{\lambda\rho}C_{\rho\mu\nu\kappa}\right)' = e^{-2\alpha(x)}g^{\lambda\rho}C'_{\rho\mu\nu\kappa}$$

where we have used

$$g^{\mu\nu} \to e^{-2\alpha(x)}g^{\mu\nu}.$$

Hence we seek to show that under the conformal transformation given above,

$$C_{\lambda\mu\nu\kappa} \to C'_{\lambda\mu\nu\kappa} = e^{2\alpha(x)}C_{\lambda\mu\nu\kappa}.$$

Begin with the Riemann curvature tensor

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\,\partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\,\partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\,\partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\,\partial x^\lambda}\right] + g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right].$$

Take a look at $\Gamma^{\eta}_{\nu\lambda}$

$$\Gamma^{\eta}_{\nu\lambda} = \frac{1}{2}g^{\eta\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}\right).$$

We can decompose this into two terms, one from the derivaties on $g_{\mu\nu}$ and the other from derivatives on $e^{2\alpha(x)}$. Since we have the inverse metric out front, the conformal contributions cancel for the first derivative section, i.e.

$$\Gamma^{\eta}_{\nu\lambda} \to \Gamma^{\eta}_{\nu\lambda} + \delta\Gamma^{\eta}_{\nu\lambda}$$

From this fact, we can actually see that the whole $R_{\lambda\mu\nu\kappa}$ will transform as

$$R_{\lambda\mu\nu\kappa} \to e^{2\alpha(x)}R_{\lambda\mu\nu\kappa} + \text{Derivatives of } \alpha(x).$$

Lets figure out what the function is for the connection terms. The following will be useful

$$\partial_\mu e^{2\alpha(x)} = e^{2\alpha}\partial_\mu(2\alpha)$$

$$\begin{aligned}
\partial_\nu\partial_\mu e^{2\alpha} &= e^{2\alpha}\partial_\nu\partial_\mu(2\alpha) + e^{2\alpha}\partial_\nu(2\alpha)\partial_\mu(2\alpha)\\
&= e^{2\alpha}\left(\partial_\nu\partial_\mu(2\alpha) + \partial_\nu(2\alpha)\partial_\mu(2\alpha)\right).
\end{aligned}$$

Back to $\delta\Gamma$

$$\begin{aligned}
\delta\Gamma^\eta_{\nu\lambda} &= \frac{1}{2}g^{\eta\rho}\left[\partial_\nu(2\alpha)g_{\rho\lambda} + \partial_\lambda(2\alpha)g_{\rho\nu} - \partial_\rho(2\alpha)g_{\nu\lambda}\right]\\
&= \delta^\eta_\lambda\partial_\nu\alpha + \delta^\eta_\nu\partial_\lambda\alpha - g_{\nu\lambda}g^{\eta\rho}\partial_\rho\alpha.
\end{aligned}$$

$$g_{\eta\sigma}\delta\Gamma^\eta_{\nu\lambda} = g_{\sigma\lambda}\partial_\nu\alpha + g_{\sigma\nu}\partial_\lambda\alpha - g_{\nu\lambda}\partial_\sigma\alpha$$

Note that the variation of the connection is a tensor! Now lets take a product

$$\begin{aligned}
\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} &\to (\Gamma^\eta_{\nu\lambda} + \delta\Gamma^\eta_{\nu\lambda})(\Gamma^\sigma_{\mu\kappa} + \delta\Gamma^\sigma_{\mu\kappa})\\
&= \Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} + \delta(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa})
\end{aligned}$$

$$\begin{aligned}
g_{\eta\sigma}\delta(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}) &= g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\delta\Gamma^\sigma_{\mu\kappa} + \Gamma^\sigma_{\mu\kappa}\delta\Gamma^\eta_{\nu\lambda} + \delta\Gamma^\eta_{\nu\lambda}\delta\Gamma^\sigma_{\mu\kappa}\right)\\
&= \left\{\frac{1}{2}\partial_\mu\alpha\left(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\kappa\nu} - \partial_\kappa g_{\nu\lambda}\right) + \frac{1}{2}\partial_\kappa\alpha\left(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda}\right) - (\partial_\sigma\alpha)g_{\mu\kappa}\Gamma^\sigma_{\nu\lambda}\right\}\\
&\quad + \left\{\frac{1}{2}\partial_\nu\alpha\left(\partial_\mu g_{\lambda\kappa} + \partial_\kappa g_{\lambda\mu} - \partial_\lambda g_{\mu\kappa}\right) + \frac{1}{2}\partial_\lambda\alpha\left(\partial_\mu g_{\nu\kappa} + \partial_\kappa g_{\nu\mu} - \partial_\nu g_{\mu\kappa}\right) - (\partial_\sigma\alpha)g_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}\right\}\\
&\quad - \left\{g_{\mu\kappa}\left(2\partial_\nu\alpha\partial_\lambda\alpha - g_{\nu\lambda}\partial^\sigma\alpha\partial_\sigma\alpha\right)\right\}
\end{aligned}$$

From here we note that when we subtract the second pair of Γ terms in $R_{\lambda\mu\nu\kappa}$, any term that is symmetric under $\nu \Leftrightarrow \kappa$ vanishes. We collect the remaining anti-symmetric terms

$$\begin{aligned}
g_{\eta\sigma}\delta(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}) - g_{\eta\sigma}\delta(\Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}) &= \partial_\mu\alpha\left(\partial_\nu g_{\kappa\lambda} - \partial_\kappa g_{\nu\lambda}\right) + \partial_\lambda\alpha\left(\partial_\kappa g_{\nu\lambda} - \partial_\nu g_{\mu\kappa}\right)\\
&\quad + \partial_\kappa\alpha\left(\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda}\right) + \partial_\nu\alpha\left(\partial_\mu g_{\lambda\kappa} - \partial_\lambda g_{\mu\kappa}\right)\\
&\quad - (\partial_\sigma\alpha)\left(g_{\mu\kappa}\Gamma^\sigma_{\nu\lambda} + g_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - g_{\mu\nu}\Gamma^\sigma_{\kappa\lambda} - g_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right)\\
&\quad - g_{\mu\kappa}\left(2\partial_\nu\alpha\partial_\lambda\alpha - g_{\nu\lambda}\partial^\sigma\alpha\partial_\sigma\alpha\right) + g_{\mu\nu}\left(2\partial_\kappa\alpha\partial_\lambda\alpha - g_{\kappa\lambda}\partial^\sigma\alpha\partial_\sigma\alpha\right)
\end{aligned}$$

$$\partial_\kappa\partial_\mu(e^{2\alpha}g_{\lambda\nu}) = e^{2\alpha}\left[g_{\lambda\nu}\partial_\kappa(2\alpha)\partial_\mu(2\alpha) + g_{\lambda\nu}\partial_\kappa\partial_\mu(2\alpha) + \partial_\kappa g_{\lambda\nu}\partial_\mu(2\alpha) + \partial_\mu g_{\lambda\nu}\partial_\kappa(2\alpha) + \partial_\kappa\partial_\mu g_{\lambda\nu}\right]$$

$$
\begin{aligned}
\delta(\partial_\kappa\partial_\mu g_{\lambda\nu}) &= e^{2\alpha}\left[g_{\lambda\nu}\partial_\kappa(2\alpha)\partial_\mu(2\alpha) + g_{\lambda\nu}\partial_\kappa\partial_\mu(2\alpha) + \partial_\kappa g_{\lambda\nu}\partial_\mu(2\alpha) + \partial_\mu g_{\lambda\nu}\partial_\kappa(2\alpha)\right] \\
-\delta(\partial_\kappa\partial_\lambda g_{\mu\nu}) &= e^{2\alpha}\left[g_{\mu\nu}\partial_\kappa(2\alpha)\partial_\lambda(2\alpha) + g_{\mu\nu}\partial_\kappa\partial_\lambda(2\alpha) + \partial_\kappa g_{\mu\nu}\partial_\lambda(2\alpha) + \partial_\lambda g_{\mu\nu}\partial_\kappa(2\alpha)\right] \\
-\delta(\partial_\nu\partial_\mu g_{\lambda\kappa}) &= e^{2\alpha}\left[g_{\lambda\kappa}\partial_\nu(2\alpha)\partial_\mu(2\alpha) + g_{\lambda\kappa}\partial_\nu\partial_\mu(2\alpha) + \partial_\nu g_{\lambda\kappa}\partial_\mu(2\alpha) + \partial_\mu g_{\lambda\kappa}\partial_\nu(2\alpha)\right] \\
\delta(\partial_\nu\partial_\lambda g_{\mu\kappa}) &= e^{2\alpha}\left[g_{\mu\kappa}\partial_\nu(2\alpha)\partial_\lambda(2\alpha) + g_{\mu\kappa}\partial_\nu\partial_\lambda(2\alpha) + \partial_\nu g_{\mu\kappa}\partial_\lambda(2\alpha) + \partial_\lambda g_{\mu\kappa}\partial_\nu(2\alpha)\right]
\end{aligned}
$$

$$R^{\lambda}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}.$$

Under the conformal transformation $g_{\mu\nu} \to e^{2\alpha(x)} g_{\mu\nu}$ we have the connection change

$$\Gamma^{\eta}_{\nu\lambda} \to \Gamma^{\eta}_{\nu\lambda} + \delta\Gamma^{\eta}_{\nu\lambda}$$

where

$$\begin{aligned}\delta\Gamma^{\eta}_{\nu\lambda} &= \frac{1}{2} g^{\eta\rho} \left[\partial_\nu (2\alpha) g_{\rho\lambda} + \partial_\lambda (2\alpha) g_{\rho\nu} - \partial_\rho (2\alpha) g_{\nu\lambda}\right] \\ &= \delta^{\eta}_{\lambda}\partial_\nu \alpha + \delta^{\eta}_{\nu}\partial_\lambda \alpha - g_{\nu\lambda} g^{\eta\rho}\partial_\rho \alpha.\end{aligned}$$

Thus

$$\delta(\partial_\kappa \Gamma^{\lambda}_{\mu\nu}) = \delta^{\lambda}_{\nu}\partial_\kappa\partial_\mu\alpha + \delta^{\lambda}_{\mu}\partial_\kappa\partial_\nu\alpha - \partial_\kappa\left(g_{\mu\nu}\partial^\lambda\alpha\right)$$
$$\delta(\partial_\nu \Gamma^{\lambda}_{\mu\kappa}) = \delta^{\lambda}_{\kappa}\partial_\nu\partial_\mu\alpha + \delta^{\lambda}_{\mu}\partial_\nu\partial_\kappa\alpha - \partial_\nu\left(g_{\mu\kappa}\partial^\lambda\alpha\right)$$

$$\delta(\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}) = \delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} + \delta\Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu} + \delta\Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\kappa\eta}$$

$$\begin{aligned}\delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} &= (\delta^{\eta}_{\mu}\partial_\nu\alpha + \delta^{\eta}_{\nu}\partial_\mu\alpha - g_{\mu\nu}\partial^\eta\alpha)\Gamma^{\lambda}_{\kappa\eta} \\ &= \Gamma^{\lambda}_{\kappa\mu}\partial_\nu\alpha + \Gamma^{\lambda}_{\kappa\nu}\partial_\mu\alpha - g_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}\partial^\eta\alpha\end{aligned}$$

$$\delta\Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu} = (\delta^{\lambda}_{\kappa}\partial_\eta\alpha + \delta^{\lambda}_{\eta}\partial_\kappa\alpha - g_{\kappa\eta}\partial^\lambda\alpha)\Gamma^{\eta}_{\mu\nu}$$

$$\begin{aligned}\delta\Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\kappa\eta} &= (\delta^{\eta}_{\mu}\partial_\nu\alpha + \delta^{\eta}_{\nu}\partial_\mu\alpha - g_{\mu\nu}\partial^\eta\alpha)(\delta^{\lambda}_{\kappa}\partial_\eta\alpha + \delta^{\lambda}_{\eta}\partial_\kappa\alpha - g_{\kappa\eta}\partial^\lambda\alpha) \\ &= \partial_\nu\alpha(\delta^{\lambda}_{\kappa}\partial_\mu\alpha + \delta^{\lambda}_{\mu}\partial_\kappa\alpha - g_{\kappa\mu}\partial^\lambda\alpha) \\ &\quad + \partial_\mu\alpha(\delta^{\lambda}_{\kappa}\partial_\nu\alpha + \delta^{\lambda}_{\nu}\partial_\kappa\alpha - g_{\kappa\nu}\partial^\lambda\alpha) \\ &\quad - \delta^{\lambda}_{\kappa} g_{\mu\nu}\partial^\eta\alpha\partial_\eta\alpha - g_{\mu\nu}\partial^\lambda\alpha\partial_\kappa\alpha + g_{\mu\nu}\partial_\kappa\alpha\partial^\lambda\alpha \\ &= 2\delta^{\lambda}_{\kappa}\partial_\mu\alpha\partial_\nu\alpha + \delta^{\lambda}_{\mu}\partial_\nu\alpha\partial_\kappa\alpha + \delta^{\lambda}_{\nu}\partial_\mu\alpha\partial_\kappa\alpha - g_{\kappa\mu}\partial^\lambda\alpha\partial_\nu\alpha - g_{\kappa\nu}\partial^\lambda\alpha\partial_\mu\alpha - \delta^{\lambda}_{\kappa} g_{\mu\nu}\partial^\eta\alpha\partial_\eta\alpha\end{aligned}$$

$$(\delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}) + (\delta\Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu}) = \left[\Gamma^{\lambda}_{\kappa\mu}\partial_\nu\alpha + \Gamma^{\lambda}_{\kappa\nu}\partial_\mu\alpha - g_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}\partial^\eta\alpha\right] + \left[(\delta^{\lambda}_{\kappa}\partial_\eta\alpha + \delta^{\lambda}_{\eta}\partial_\kappa\alpha - g_{\kappa\eta}\partial^\lambda\alpha)\Gamma^{\eta}_{\mu\nu}\right]$$

$$(\delta\Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}) + (\delta\Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\kappa}) = \left[\Gamma^{\lambda}_{\nu\mu}\partial_\kappa\alpha + \Gamma^{\lambda}_{\kappa\nu}\partial_\mu\alpha - g_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}\partial^\eta\alpha\right] + \left[(\delta^{\lambda}_{\nu}\partial_\eta\alpha + \delta^{\lambda}_{\eta}\partial_\nu\alpha - g_{\nu\eta}\partial^\lambda\alpha)\Gamma^{\eta}_{\mu\kappa}\right]$$

Subtracting the above two from each other

$$\begin{aligned}&\left[(\delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}) + (\delta\Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu})\right] - \left[(\delta\Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}) + (\delta\Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\kappa})\right] \\ &\qquad = -g_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}\partial^\eta\alpha + \delta^{\lambda}_{\kappa}\Gamma^{\eta}_{\mu\nu}\partial_\eta\alpha - g_{\kappa\eta}\Gamma^{\eta}_{\mu\nu}\partial^\lambda\alpha + g_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}\partial^\eta\alpha - \delta^{\lambda}_{\nu}\Gamma^{\eta}_{\mu\kappa}\partial_\eta\alpha + g_{\nu\eta}\Gamma^{\eta}_{\mu\kappa}\partial^\lambda\alpha\end{aligned}$$

Lets compare the like derivative terms

$$\begin{aligned}\partial^\lambda\alpha\left(g_{\nu\eta}\Gamma^{\eta}_{\mu\kappa} - g_{\kappa\eta}\Gamma^{\eta}_{\mu\nu}\right) a &= \partial^\lambda\alpha\left[(\partial_\mu g_{\nu\kappa} + \partial_\kappa g_{\nu\mu} - \partial_\nu g_{\mu\kappa}) - (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu})\right] \\ &= 2\partial^\lambda\alpha(\partial_\kappa g_{\mu\nu} - \partial_\nu g_{\mu\kappa})\end{aligned}$$

$$\partial_\eta\alpha\left(\delta^{\lambda}_{\kappa}\Gamma^{\eta}_{\mu\nu} - \delta^{\lambda}_{\nu}\Gamma^{\eta}_{\mu\kappa}\right) =$$

$$\partial^\eta\alpha\left(g_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} - g_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}\right) =$$

$$
\begin{aligned}
\nabla^\beta\nabla_\nu[\delta R_{\mu\beta}+3H^2h_{\mu\beta}] &= \nabla_\nu\nabla^\beta[\delta R_{\mu\beta}+3H^2h_{\mu\beta}]+4H^2[\delta R_{\mu\beta}+3H^2h_{\mu\beta}]-H^2g_{\mu\nu}g^{\alpha\beta}[\delta R_{\alpha\beta}+3H^2h_{\alpha\beta}] \\
&= \nabla_\nu\nabla^\beta[\delta R_{\mu\beta}+3H^2h_{\mu\beta}]+4H^2\delta R_{\mu\nu}+12H^4h_{\mu\nu}-H^2g_{\mu\nu}g^{\alpha\beta}\delta R_{\alpha\beta}-3H^4g_{\mu\nu}h
\end{aligned}
$$

$$
\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}[-(h^{\alpha\sigma}g^{\beta\tau}+g^{\alpha\sigma}h^{\beta\tau})R_{\sigma\tau}]=-9H^4g_{\mu\nu}h
$$

$$
\begin{aligned}
2h^{\alpha\beta}R_{\mu\alpha}R_{\nu\beta}-2g^{\alpha\beta}\delta R_{\mu\alpha}R_{\nu\beta}-2g^{\alpha\beta}R_{\mu\alpha}\delta R_{\nu\beta}+\frac{1}{2}h_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} &= 18H^4h_{\mu\nu}+6H^2\delta R_{\mu\nu}+6H^2\delta R_{\mu\nu}+18H^4h_{\mu\nu} \\
&= 36H^4h_{\mu\nu}+12H^2\delta R_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
\nabla^\beta\nabla_\nu A^\mu{}_\beta-\nabla_\nu\nabla^\beta A^\mu{}_\beta &= g^{\beta\kappa}[A^\sigma{}_\beta R^\mu{}_{\sigma\nu\kappa}-A^\mu{}_\sigma R^\sigma{}_{\beta\nu\kappa}] \\
\nabla^\beta\nabla_\nu A_{\alpha\beta}-\nabla_\nu\nabla^\beta A_{\alpha\beta} &= g^{\beta\kappa}[A^\sigma{}_\beta R_{\alpha\sigma\nu\kappa}-A_{\alpha\sigma}R^\sigma{}_{\beta\nu\kappa}]
\end{aligned}
$$

$$\nabla_P A^P{}_M = \partial_P A^P{}_M + \Gamma^P_{PQ} A^Q{}_M - \Gamma^Q_{PM} A^P{}_Q \tag{4}$$

With

$$\Gamma^Q_{PM} = \Omega^{-1}\left[\delta^Q_M \partial_P + \delta^Q_P \partial_M - \eta^{QR}\eta_{PM}\partial_R\right]\Omega$$

it follows that

$$\begin{aligned}\Gamma^P_{PQ} &= \Omega^{-1}\left[\delta^P_Q \partial_P + \delta^P_P \partial_Q - \eta^{PR}\eta_{PQ}\partial_R\right]\Omega \\ &= \Omega^{-1}\left[\partial_Q + 4\partial_Q - \delta^R_Q \partial_R\right]\Omega \\ &= 4\Omega^{-1}\partial_Q.\end{aligned}$$

Now evaluating the covariant derivative (4)

$$\begin{aligned}\nabla_P A^P{}_M &= \Omega^{-2}\eta^{PQ}\partial_P A_{QM} + 4\Omega^{-3}\eta^{QR}A_{RM}\partial_Q\Omega - \Omega^{-3}\eta^{PL}A_{LQ}\left[\delta^Q_M\partial_P + \delta^Q_P\partial_M - \eta^{QR}\eta_{PM}\partial_R\right]\Omega \\ &= \Omega^{-2}\eta^{PQ}\partial_P A_{QM} + 4\Omega^{-3}\eta^{QR}A_{RM}\partial_Q\Omega - \Omega^{-3}\eta^{PL}A_{LM}\partial_P\Omega - \Omega^{-3}\eta^{PL}A_{LP}\partial_M\Omega + \Omega^{-3}\eta^{QR}A_{MQ}\partial_R\Omega \\ &= \Omega^{-2}\eta^{PQ}\partial_P A_{QM} + 4\Omega^{-3}\eta^{QR}A_{RM}\partial_Q\Omega - \Omega^{-3}\eta^{PL}A_{LP}\partial_M\Omega\end{aligned}$$

$$\begin{aligned}-\Omega^{-1}\partial_N\Omega\nabla_L A^L{}_M &= -\Omega^{-3}\eta^{LR}\partial_N\Omega\partial_L A_{RM} - 4\Omega^{-4}\eta^{TX}\partial_X\Omega\partial_N\Omega A_{TM} + \Omega^{-4}\eta^{TX}\partial_N\Omega\partial_M\Omega A_{TX} \\ -\Omega^{-1}\partial_M\Omega\nabla_L A^L{}_N &= -\Omega^{-3}\eta^{LR}\partial_M\Omega\partial_L A_{RN} - 4\Omega^{-4}\eta^{TX}\partial_X\Omega\partial_M\Omega A_{TN} + \Omega^{-4}\eta^{TX}\partial_N\Omega\partial_M\Omega A_{TX}\end{aligned}$$

For $S = S_0$, the equation of motion is

$$\frac{1}{6} S_0 R - 4\lambda S_0^3 + h\bar{\psi}\psi = 0$$

or

$$-hS_0\bar{\psi}\psi = \frac{1}{6} S_0^2 R - 4\lambda S_0^4.$$

Varying the matter action

$$\frac{1}{2}\frac{\delta I_m}{\delta g^{\mu\nu}} = (-g)^{1/2}\left\{\left[i\bar{\psi}\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS_0\bar{\psi}\psi - \frac{1}{12}S_0^2 R + \lambda S_0^4\right] g_{\mu\nu} + \frac{1}{6}S_0^2\left(\frac{1}{2}Rg_{\mu\nu} - R_{\mu\nu}\right)\right\}.$$

Inserting the equation of motion, this becomes

$$\frac{1}{2}\frac{\delta I_m}{\delta g^{\mu\nu}} = (-g)^{1/2}\left\{\left[i\bar{\psi}\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi + \frac{1}{6}S_0^2 R - 3\lambda S_0^4\right] g_{\mu\nu} - \frac{1}{6}S_0^2 R_{\mu\nu}\right\}$$

Curvature tensors under conformal transformations:

$$R^\lambda_{\mu\nu\kappa}(g_{\mu\nu}) \to \bar{R}^\lambda_{\mu\nu\kappa}(\bar{g}_{\mu\nu}) = R^\lambda_{\mu\nu\kappa}(g_{\mu\nu}) + \delta R^\lambda_{\mu\nu\kappa}(g_{\mu\nu})$$

where

$$\begin{aligned}\delta R^\lambda_{\mu\nu\kappa}(g_{\mu\nu}) = \Omega^{-2}&\left(\delta^\lambda_\nu g_{\kappa\mu}\nabla_\alpha\Omega\nabla^\alpha\Omega - \delta^\lambda_\kappa g_{\mu\nu}\nabla_\alpha\Omega\nabla^\alpha\Omega + 2g_{\mu\nu}\nabla_\kappa\Omega\nabla^\lambda\Omega - 2\delta^\lambda_\nu\nabla_\kappa\Omega\nabla_\mu\Omega - 2g_{\mu\kappa}\nabla^\lambda\Omega\nabla_\nu\Omega + 2\delta^\lambda_\kappa\nabla_\mu\Omega\nabla_\nu\Omega\right)\\ &+ \Omega^{-1}\left(g_{\mu\kappa}\nabla_\nu\nabla^\lambda\Omega + \delta^\lambda_\nu\nabla_\mu\nabla_\kappa\Omega - g_{\mu\nu}\nabla^\lambda\nabla_\kappa\Omega - \delta^\lambda_\kappa\nabla_\nu\nabla_\mu\Omega\right)\end{aligned}$$

$$R_{\mu\nu}(g_{\mu\nu}) \to \bar{R}_{\mu\nu}(\bar{g}_{\mu\nu}) = R_{\mu\nu}(g_{\mu\nu}) + \delta R_{\mu\nu}(g_{\mu\nu})$$

where

$$\delta R_{\mu\nu}(g_{\mu\nu}) = \Omega^{-2}\left(g_{\mu\nu}\nabla_\alpha\Omega\nabla^\alpha\Omega - 4\nabla_\mu\Omega\nabla_\nu\Omega\right) + \Omega^{-1}\left(g_{\mu\nu}\nabla_\alpha\Omega\nabla^\alpha\Omega + 2\nabla_\mu\nabla_\nu\Omega\right)$$

$$R^\lambda{}_\lambda(g_{\mu\nu}) \to \bar{R}^\lambda{}_\lambda(\bar{g}_{\mu\nu}) = \Omega^2 R^\lambda{}_\lambda(g_{\mu\nu}) + \delta R^\lambda{}_\lambda(g_{\mu\nu})$$

where

$$\delta R^\lambda{}_\lambda(g_{\mu\nu}) = 6\Omega\nabla_\alpha\nabla^\alpha\Omega$$

$$S_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\delta R_{\alpha\beta}$$

Under conformal transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} \to \Gamma^\lambda_{\mu\nu} + \bar{\Gamma}^\lambda_{\mu\nu} \tag{5}$$

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Omega^{-1}\left(\delta^\lambda_\mu\nabla_\nu + \delta^\lambda_\nu\nabla_\mu - g_{\mu\nu}g^{\lambda\rho}\nabla_\rho\right)\Omega \tag{6}$$

Evaluating gauge condition in conformal background

$$\bar{\nabla}^\nu K_{\mu\nu} = g^{\alpha\nu}(\partial_\alpha K_{\mu\nu} - \Gamma^\lambda_{\mu\alpha}K_{\lambda\nu} - \Gamma^\lambda_{\nu\alpha}K_{\lambda\mu}) \tag{7}$$

In the conformal background, the connection and metric changes as indicated above, so that we then have

$$\bar{\nabla}^\nu K_{\mu\nu} \to \Omega^{-2}\nabla^\nu K_{\mu\nu} - \Omega^{-2}g^{\alpha\nu}(\tilde{\Gamma}^\lambda_{\mu\alpha}K_{\lambda\nu} + \tilde{\Gamma}^\lambda_{\nu\alpha}K_{\lambda\mu})$$

$$= \Omega^{-2}\nabla^{\nu}K_{\mu\nu} - \Omega^{-3}g^{\alpha\nu}(\tag{8}$$

$$= \Omega^{-2}\nabla^{\mu}K_{\mu\nu} + 2\Omega^{-3}K_{\mu\nu}\nabla^{\nu}\Omega \tag{9}$$

Under conformal transformation,

$$g_{\mu\nu} \to e^{\alpha(x)}g_{\mu\nu} \tag{10}$$

according to Mannheim

$$A^{\mu}(x) \to A^{\mu}(x). \tag{11}$$

That this must be true can be seen by starting with the action,

$$S = \int d^4x\sqrt{-g}F_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}. \tag{12}$$

Then, we know the following transformations

$$d^4x\sqrt{-g} \to \Omega^4 d^4x\sqrt{-g}, \qquad g^{\mu\alpha}g^{\nu\beta} \to \Omega^{-4}g^{\mu\alpha}g^{\nu\beta}. \tag{13}$$

Thus, in order for the Lagrangian to be conformal invariant, then

$$F_{\mu\nu} \to F_{\mu\nu}. \tag{14}$$

Note that the EM tensor can be expressed as

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} \tag{15}$$

$$= (\partial_{\mu}A_{\nu} - \Gamma^{\lambda}_{\mu\nu}A_{\lambda}) - (\partial_{\nu}A_{\nu} - \Gamma^{\lambda}_{\mu\nu}A_{\lambda}) \tag{16}$$

$$= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{17}$$

From this, we can see that if $A_{\mu}(x) \to A_{\mu}(x)$ then $F_{\mu\nu} \to F_{\mu\nu}$.

In a flat geometry (not necessarily Minkowski), the perturbed Bach tensor can be related to the perturbed Einstein tensor via

$$\delta W_{\mu\nu} = \nabla_{\alpha}\nabla^{\alpha}\delta G_{\mu\nu} + \frac{1}{3}(\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha})\delta G^{\lambda}{}_{\lambda} \tag{18}$$

$$= \nabla_{\alpha}\nabla^{\alpha}\delta G^{Tr\theta}_{\mu\nu} + \frac{1}{3}(\nabla_{\mu}\nabla_{\nu} - \tfrac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha})\delta G^{\lambda}{}_{\lambda} \tag{19}$$

where we define the traceless

$$\delta G^{Tr\theta}_{\mu\nu} = \delta(R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R). \tag{20}$$