Lecture 10

02/24/2016

Physical interpretation:

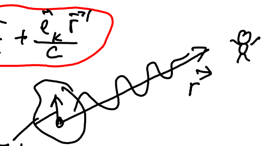
$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|} d^3r' \cong \frac{\mu_0}{4\pi}\frac{1}{r}\int \vec{j}\left(\vec{r}\,', t - \frac{r}{c} + \frac{\vec{e}_r \vec{r}\,'}{c}\right) d^3r'}$$

$$t' = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

For the radiation zone:

$$r \gg r' \Rightarrow |\vec{r}-\vec{r}\,'| \cong r - \vec{e}_r\vec{r}\,' \text{ and } \left(t - \frac{r}{c} + \frac{\vec{e}_k\vec{r}\,'}{c}\right)$$

$$\vec{e}_r \equiv \vec{e}_k$$



Term "$\frac{\vec{e}_k\vec{r}\,'}{c}$" could be considered as a small term with respect to the variable $t' = t - r/c$:

$$\vec{A} = \frac{\mu_0}{4\pi r}\int \vec{j}\left(\vec{r}\,', \underbrace{t - \frac{r}{c}}_{t'}\right) d^3r' + \frac{\mu_0}{4\pi r}\int \frac{\partial \vec{j}}{\partial t'}\left(\frac{\vec{e}_k\vec{r}\,'}{c}\right) d^3r' + \dots$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\left(\sum_i q_i\vec{v}_i\right) + \frac{\mu_0}{4\pi r c}\sum_i \frac{\partial}{\partial t'} q_i\vec{v}_i(\vec{e}_k\vec{r}_i) + \dots$$

$$\vec{j} = \sum_i q_i\vec{v}_i\,\delta(\vec{r}\,' - \vec{r}_i) \rightarrow \int \vec{j}\, d^3r' = \sum_i q_i\vec{v}_i$$

$$\vec{r}\,' = \vec{r}_i$$

$$\vec{A} = \frac{\mu_0}{4\pi r}\frac{\partial}{\partial t'}\left(\sum_i q_i\vec{r}_i\right) + \frac{\mu_0}{4\pi r c}\frac{\partial}{\partial t'}\left(\sum_i \frac{\partial \vec{r}_i}{\partial t'}(\vec{e}_k\vec{r}_i)\right) + \dots$$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt}$$

electric dipole

electric quadrupole and magnetic dipole

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi r}\dot{\vec{p}}(t') + \frac{\mu_0}{4\pi r c}\frac{\partial}{\partial t'}\left(\sum_i \frac{q_i}{2}\frac{\partial}{\partial t'}\left(\vec{r}_i(\vec{e}_k\vec{r}_i)\right) + \frac{q_i}{2}\left(\vec{r}_i\times\frac{\partial \vec{r}_i}{\partial t'}\right)\times\vec{e}_k\right)}$$

$$\boxed{t' = t - \frac{r}{c}}$$

$$\left(\vec{p} = \sum_i q_i\vec{r}_i\right)$$

$$\frac{d\vec{r}_i}{dt}(\vec{e}_k\vec{r}_i) = \frac{1}{2}\frac{d}{dt}\left(\vec{r}_i(\vec{e}_k\vec{r}_i)\right) + \frac{1}{2}\left(\vec{r}_i\times\frac{d\vec{r}_i}{dt}\right)\times\vec{e}_k$$

Magnetic moment: $\boxed{\vec{m} = \sum_i \frac{(\vec{r}_i\times q_i\vec{v}_i)}{2}}$

$$\vec{A} = \frac{\mu_0}{4\pi r}\overbrace{\dot{\vec{p}}(t')}^{\text{electric dipole}} + \frac{\mu_0}{4\pi r c}\frac{1}{6}\frac{\partial^2}{\partial t^2}\underbrace{\left(\sum_i q_i\left(3\vec{r}_i(\vec{e}_k\vec{r}_i) - \vec{e}_k r_i^2\right)\right)}_{\text{This is the product:}} + \frac{\mu_0}{4\pi r c}\underbrace{\dot{\vec{m}}\times\vec{e}_k}_{\text{magnetic dipole}}$$

$$\left(\vec{p}(t) \int \frac{d\omega}{2\pi} e^{-i\omega t}\vec{p}_\omega\right)$$

electric dipole

$$(\vec{e}_k)_\beta Q_{\alpha\beta} \equiv (\vec{Q}_k)_\alpha \qquad Q_{\alpha\beta} = \sum_i q_i(3x_\alpha x_\beta - \delta_{\alpha\beta}r^2)$$

quadrupole moment

$$\vec{A} \simeq \frac{\mu_0}{4\pi r}\dot{\vec{p}}(t') + \frac{\mu_0}{24\pi r c}\ddot{\vec{Q}}_k + \frac{\mu_0}{4\pi r c}\dot{\vec{m}}\times\vec{e}_k$$

The formulas, derived for a system of point charges, are identical to equations of "harmonic oscillating" charge distributions $\rho = \rho_\omega e^{-i\omega t}$ and $\vec{J} = \vec{J}_\omega \cdot e^{-i\omega t}$.

$$\boxed{\vec{A} = \vec{A}^{(p)}(\vec{r},t) + \vec{A}^{(Q)}(\vec{r},t) + \vec{A}^{(m)}(\vec{r},t)}$$

$t' = t - \frac{r}{c}$

Electric dipole

$$\vec{A}^{(\vec{p})}(\vec{r},t) = \frac{\mu_0}{4\pi r}\dot{\vec{p}}(t) \quad \Big/\!\!\Big/ \quad \vec{p}(t') = \vec{p}_\omega e^{-i\omega t'} = \vec{p}_\omega e^{-i\omega t + i\frac{\omega r}{c}}$$

$$t' = t - \frac{r}{c}$$

$$\boxed{\vec{A}^{(p)} = -i\omega \frac{\mu_0 e^{ikr - i\omega t}}{4\pi r}\vec{p}_\omega}$$

$$\omega = ck$$

Quadrupole:

$$\boxed{\vec{Q} = \vec{Q}_\omega \cdot e^{-i\omega t'}} \quad (t' = t - \frac{r}{c})$$

$$\vec{A}^{(Q)} = -\frac{\mu_0 \omega^2}{24\pi c}\vec{Q}_\omega \frac{e^{ikr - i\omega t}}{r}$$

$$\vec{Q} = \{Q_\alpha\}$$
$$Q_\alpha = D_{\alpha\beta}(\vec{e}_k)_\beta$$
$$D_{\alpha\beta} = \int 3(x_\alpha x_\beta - \delta_{\alpha\beta} r^2)\rho(\vec{x})\, d\vec{x}^3$$

Magnetic moment

$$\vec{m} = \vec{m}_\omega e^{-i\omega t'}$$

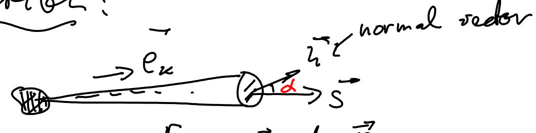
$$\vec{A}^{(m)} = \frac{\mu_0}{4\pi r \cdot c}\dot{\vec{m}} \times \vec{e}_k = -\frac{i\mu_0 k}{4\pi}\frac{e^{ikr - i\omega t}}{r}(\vec{m}_\omega \times \vec{e}_k)$$

$\vec{A} \equiv \vec{A}(r, t - \frac{r}{c})$

B-calculations: $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\times\vec{e}_k \quad (\vec{\nabla} \to -\frac{\vec{e}_k}{c}\frac{\partial}{\partial t}\ldots)$

Power of induced radiation:

Poynting vector: $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$; $\begin{cases}\vec{B} = \vec{\nabla}\times\vec{A} \equiv i\vec{k}\times\vec{A} \\ \vec{E} = c(\vec{B}\times\vec{e}_k)\end{cases}$



$d\vec{a} = da \cdot \vec{n}$

$\vec{e}_k = \frac{\vec{r}}{r} \equiv \frac{\vec{k}}{k}$

We may calculate $\vec{B}$ and $\vec{E}$ and Poynting vector for $\vec{A}^p$, $\vec{A}^Q$ and $\vec{A}^m$.

Power of the EM emission:

$$\boxed{dP = \vec{S}\, d\vec{a} = S\, da \cos\alpha = S \cdot r^2\, d\Omega}$$

$I = \frac{dP}{d\Omega} \leftarrow$ intensity (power per unit of solid angle)

Total power (Gaussian Units)

$$\boxed{P = \frac{2}{3c^3}\ddot{\vec{p}}^2 + \frac{1}{180c^3}\left(\sum_{\alpha,\beta}\dddot{Q}_{\alpha\beta}\right)^2 + \frac{2}{3c^3}\ddot{\vec{m}}^2}$$

electric dipole — quadrupole — magnetic dipole

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