Statistical Mechanics
HW 5

Matthew Phelps

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5.1 Show that any density operator defined as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

for any set of normalized (but not necessarily orthogonal) states $\{\left|\psi_i\right\rangle\}$ can also be written as

$$\rho = \sum_n p'_n \left|n\right\rangle \left\langle n\right|$$

where $\{\left|n\right\rangle\}$ is an orthonormal basis, and the p'_n are also probabilities.

As a hermitian operator, the eigenvalues of ρ are real and the eigenvectors form an orthogonal basis. Let's denote these normalized eigenvectors as

$$\rho \left|n\right\rangle = \lambda \left|n\right\rangle$$

Using the identity operator twice, the matrix representation of ρ is

$$\begin{aligned}
\rho &= \sum_i p_i \sum_{n,m} \left|n\right\rangle \left\langle n|\psi_i\right\rangle \left\langle\psi_i|m\right\rangle \left\langle m\right| \\
&= \sum_{n,m} \left|n\right\rangle \left\langle m\right| \sum_i \left\langle n|p_i|\psi_i\right\rangle \left\langle\psi_i|m\right\rangle \\
&= \sum_{n,m} \left\langle n|\rho|m\right\rangle \left|n\right\rangle \left\langle m\right| .
\end{aligned}$$

Since the basis is the diagonal basis, this becomes

$$\rho = \sum_n \lambda_n \left\langle n|\rho|n\right\rangle \left|n\right\rangle \left\langle n\right| .$$

The expectation value of the density operator is real due to hermiticity and is positive definite by

$$\left\langle n|\rho|n\right\rangle = \sum_i p_i |\left\langle\psi|n\right\rangle|^2 = \lambda_n \geq 0.$$

Therefore we may write ρ in an orthogonal basis with probabilities $p'_n = \lambda_n^2$

$$\rho = \sum_n p'_n \left\langle n\right| \left|n\right\rangle .$$

5.2 Given an entangled state of $|\phi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ of three qubits, what is the state of one of the individual subsystems?

As an entangled state, it cannot be written as a tensor product of three pure state. *However we may write it as a density matrix, which suggests the individual subsystems are actually ensembles?*

Start with the entangled pure state

$$\rho = |\phi\rangle \langle\phi| \,.$$

To find the ensemble state of say subsystem A, we must trace out the other two subsystem B and C:

$$\sum_{b,c} \langle b|_B \langle c|_C |\phi\rangle \langle\phi| |c\rangle_C |b\rangle_B = \sum_{b,c} \langle bc|\phi\rangle \langle\phi|bc\rangle \,.$$

Now applied to the given system,

$$\langle\phi|cb\rangle = \frac{1}{\sqrt{2}}\delta_{c,b} \langle b|$$

thus

$$\langle bc|\phi\rangle = \frac{1}{\sqrt{2}}\delta_{c,b} |b\rangle$$

so

$$\sum_{b,c} \langle bc|\phi\rangle \langle\phi|bc\rangle = \sum_{b,c} \frac{1}{2}\delta_{c,b} |b\rangle \langle b| = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|).$$

Therefore, the state of each individual subsystem i may be written as

$$\rho_i = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|).$$

5.3 Take a qubit with the two states $|\pm\rangle$. Write down the density operators in matrix form corresponding to the superposition state $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and to the 50/50 mixture of the states $|+\rangle$ and $|-\rangle$.

The superposition state is a pure state

$$\rho = |\psi\rangle \langle\psi| = \sum_{n,m} \langle n|\rho|m\rangle |n\rangle \langle m| = \sum_{n,m} \langle n|\psi\rangle \langle\psi|m\rangle |n\rangle \langle m| \,.$$

The elements are easily worked out

$$\langle\pm|\psi\rangle = \frac{1}{\sqrt{2}}$$

and our matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

For the ensemble, our density operator is

$$\rho = \frac{1}{2}|+\rangle \langle +| + \frac{1}{2}|-\rangle \langle -|$$

so our matrix is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5.4 Show that the relation that we used in class is correct:

$$p_m = \text{tr}\left(P_m \rho P_m\right),$$

where p_m is the probability of being in eigenstate $|m\rangle$ and P_m is the projector onto this state.

Forming the trace of the density operator $\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$ in the basis of $|m\rangle$ (which should be orthogonal considering they are eigenstates)

$$\begin{aligned} \text{tr}\left(P_m \rho P_m\right) &= \sum_i p_i \sum_n \langle n|m\rangle \langle m|\psi_i\rangle \langle \psi_i|m\rangle \langle m|n\rangle \\ &= \sum_i p_i |\langle m|\psi_i\rangle|^2 \\ &= p_m \end{aligned}$$

If the density operator is a pure state, then $\rho = |\psi\rangle \langle\psi|$ and the trace reduces to

$$\text{tr}\left(P_m \rho P_m\right) = |\langle m|\psi\rangle|^2$$

which is the familiar probability of obtaining eigenstate $|m\rangle$ for a known pure state.