Scalar Gauge Invariant RW SVT4 v1

1 Background

$$ds^2 = \Omega^2(\tau)\left(-d\tau^2 + \tilde{g}_{ij}dx^i dx^j\right), \qquad R_{ij} = -2k\tilde{g}_{ij} \tag{1.1}$$

$$R_{\lambda\mu\nu\kappa} = -\tfrac{1}{6}g_{\lambda\nu}g_{\mu\kappa}R + \tfrac{1}{6}g_{\lambda\kappa}g_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R_{\lambda\kappa} + \tfrac{1}{2}g_{\mu\kappa}R_{\lambda\nu} + \tfrac{1}{2}g_{\lambda\nu}R_{\mu\kappa} - \tfrac{1}{2}g_{\lambda\kappa}R_{\mu\nu} \tag{1.2}$$

$$R_{\mu\nu} = (A+B)U_\mu U_\nu + g_{\mu\nu}B, \qquad R = 3B - A \tag{1.3}$$

$$G_{\mu\nu} = \tfrac{1}{2}Ag_{\mu\nu} - \tfrac{1}{2}Bg_{\mu\nu} + AU_\mu U_\nu + BU_\mu U_\nu \tag{1.4}$$

$$g^{\mu\nu}G_{\mu\nu} = A - 3B \tag{1.5}$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \tag{1.6}$$

$$g^{\mu\nu}T_{\mu\nu} = 3p - \rho \tag{1.7}$$

$$\Delta^{(0)}_{\mu\nu} = \tfrac{1}{2}Ag_{\mu\nu} - \tfrac{1}{2}Bg_{\mu\nu} + g_{\mu\nu}p + AU_\mu U_\nu + BU_\mu U_\nu + pU_\mu U_\nu + U_\mu U_\nu \rho \tag{1.8}$$

$$g^{\mu\nu}\Delta^{(0)}_{\mu\nu} = A - 3B + 3p - \rho \tag{1.9}$$

$$\begin{aligned} A &= -\tfrac{1}{2}(3p+\rho) \\ &= -3\dot{\Omega}^2\Omega^{-4} + 3\ddot{\Omega}\Omega^{-3} \end{aligned} \tag{1.10}$$

$$\begin{aligned} B &= \tfrac{1}{2}(p-\rho) \\ &= -\dot{\Omega}^2\Omega^{-4} - \ddot{\Omega}\Omega^{-3} - 2k\Omega^{-2} \end{aligned} \tag{1.11}$$

$$\begin{aligned} \rho &= \tfrac{1}{2}(-A-3B) \\ &= 3\dot{\Omega}^2\Omega^{-4} + 3k\Omega^{-2} \end{aligned} \tag{1.12}$$

$$\begin{aligned} p &= \tfrac{1}{2}(-A+B) \\ &= \dot{\Omega}^2\Omega^{-4} - 2\ddot{\Omega}\Omega^{-3} - k\Omega^{-2} \end{aligned} \tag{1.13}$$

1.1 Identities

$$\nabla_\alpha R = 3\nabla_\alpha B - \nabla_\alpha A \tag{1.14}$$

$$= \quad -2\nabla_\alpha A - (A+B)(U^\beta\nabla_\alpha U_\beta - 2U_\alpha\nabla_\beta U^\beta) \tag{1.15}$$

$$= \quad 2\nabla_\alpha B + 2U_\alpha U^\beta(\nabla_\beta A + \nabla_\beta B) + 2(A+B)U_\alpha\nabla_\beta U^\beta \tag{1.16}$$

2 Trace $g^{\mu\nu}\Delta_{\mu\nu}$

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{2.1}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{2.2}$$

$$g^{\mu\nu}F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad \nabla^\mu F_\mu = 0 \tag{2.3}$$

$$U^\mu \delta U_\mu = \tfrac{1}{2}U^\mu U^\nu h_{\mu\nu}, \qquad U^\mu U_\mu = -1 \tag{2.4}$$

$$Q_\mu \equiv F_\mu + \nabla_\mu F \tag{2.5}$$

$$g^{\mu\nu}\delta G_{\mu\nu} = 6\nabla_\alpha\nabla^\alpha\chi - R\nabla_\alpha Q^\alpha - Q^\alpha\nabla_\alpha R + 2R_{\alpha\beta}\nabla^\beta Q^\alpha + 4F^{\alpha\beta}R_{\alpha\beta} \tag{2.6}$$

$$g^{\mu\nu}\delta T_{\mu\nu} = 3\delta p - \delta\rho - 2R\chi + R\nabla_\alpha Q^\alpha - 2R_{\alpha\beta}\nabla^\beta Q^\alpha - 2F^{\alpha\beta}R_{\alpha\beta} \tag{2.7}$$

$$g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p - \delta\rho - 2R\chi + 6\nabla_\alpha\nabla^\alpha\chi - Q^\alpha\nabla_\alpha R + 2F^{\alpha\beta}R_{\alpha\beta} \tag{2.8}$$

2.1 L^{-2} Scalars

Let S_2 be a linear superposition of all scalars with dimension L^{-2},

$$S_2 = A_1\nabla^\alpha\nabla^\beta h_{\alpha\beta} + A_2\nabla_\alpha\nabla^\alpha h + A_3 Rh + A_4 R^{\alpha\beta}h_{\alpha\beta} \tag{2.9}$$

$$\begin{aligned} = \; & -2(4A_3 + A_4)R\chi - 2(A_1 + 4A_2)\nabla_\alpha\nabla^\alpha\chi + 2A_3 R\nabla_\alpha Q^\alpha - A_1 Q^\alpha\nabla_\alpha R \\ & +2(A_1 + A_2)\nabla_\beta\nabla^\beta\nabla_\alpha Q^\alpha + 2(-A_1 + A_4)R_{\alpha\beta}\nabla^\beta Q^\alpha + 2A_4 F^{\alpha\beta}R_{\alpha\beta}. \end{aligned} \tag{2.10}$$

Setting $\{A_1, A_2, A_3, A_4\} = \{1, -1, 0, 1\}$ we have

$$\begin{aligned} S_2^{\{1,-1,0,1\}} &= h_{\alpha\beta}R^{\alpha\beta} - \nabla_\alpha\nabla^\alpha h + \nabla^\alpha\nabla^\beta h_{\alpha\beta} \\ &= 6\nabla_\alpha\nabla^\alpha\chi - 2R\chi - Q^\alpha\nabla_\alpha R + 2F^{\alpha\beta}R_{\alpha\beta}. \end{aligned} \tag{2.11}$$

This combination was chosen because in a maximally symmetric space, it follows that

$$S_2^{\{1,-1,0,1\}} = 6\nabla_\alpha\nabla^\alpha\chi - 2R\chi, \tag{2.12}$$

which aligns with the gauge invariant scalar found in *TT_Projection_Curved_v4*. However, if our background does not possess maximal symmetry, the combination $h_{\alpha\beta}R^{\alpha\beta} - \nabla_\alpha\nabla^\alpha h + \nabla^\alpha\nabla^\beta h_{\alpha\beta}$ is not gauge invariant. Specifically, it transforms as

$$(h_{\alpha\beta}R^{\alpha\beta} - \nabla_\alpha\nabla^\alpha h + \nabla^\alpha\nabla^\beta h_{\alpha\beta}) \to (h_{\alpha\beta}R^{\alpha\beta} - \nabla_\alpha\nabla^\alpha h + \nabla^\alpha\nabla^\beta h_{\alpha\beta}) - \epsilon^\alpha\nabla_\alpha R. \tag{2.13}$$

Now, since we also note that

$$g^{\mu\nu}\Delta_{\mu\nu} - (3\delta p - \delta\rho) = -2R\chi + 6\nabla_\alpha\nabla^\alpha\chi - Q^\alpha\nabla_\alpha R + 2F^{\alpha\beta}R_{\alpha\beta} \tag{2.14}$$

$$= \quad h_{\alpha\beta}R^{\alpha\beta} - \nabla_\alpha\nabla^\alpha h + \nabla^\alpha\nabla^\beta h_{\alpha\beta}, \tag{2.15}$$

we conclude from the gauge invariance of $\Delta_{\mu\nu}$ that

$$(3\delta p - \delta\rho) \to (3\delta p - \delta\rho) + \epsilon^\alpha\nabla_\alpha R. \tag{2.16}$$

By analyzing the gauge transformation of the general S_2, we conclude that is there does not exist a scalar of order L^{-2} that is gauge invariant.

2.2 L^{-4} Scalars

From Asanka's *SVT_in_RW_full*, we see that we must go to fourth order derivatives to find the SVT3 scalar gauge invariants. If such a gauge invariant is fourth order, i.e. L^{-4}, then it must reduce to a factorized expression of $6\nabla_\alpha\nabla^\alpha\chi - 2R\chi$ within the maximally symmetric limit. Analogous to S_2, we find that there are 18 possible scalars with dimension L^{-4}.

$$\begin{array}{lll}
A_1 & h\nabla_\alpha\nabla^\alpha R & 2\nabla_\alpha\epsilon^\alpha\nabla_\beta\nabla^\beta R \\
A_2 & R\nabla_\alpha\nabla^\alpha h & 2R\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha \\
A_3 & h^{\sigma\rho}\nabla_\alpha\nabla^\alpha R_{\sigma\rho} & 2\nabla^\beta\epsilon^\alpha\nabla_\gamma\nabla^\gamma R_{\alpha\beta} \\
A_4 & R_{\sigma\rho}\nabla_\alpha\nabla^\alpha h^{\sigma\rho} & (-\frac{2}{3}R^2 + 2R_{\beta\gamma}R^{\beta\gamma})\nabla_\alpha\epsilon^\alpha + (\frac{8}{3}RR_{\alpha\beta} - 6R_\alpha{}^\gamma R_{\beta\gamma})\nabla^\beta\epsilon^\alpha \\
 & & +\frac{1}{6}\epsilon^\alpha\big(R\nabla_\alpha R - R_{\alpha\beta}\nabla^\beta R + 6R^{\beta\gamma}(\nabla_\alpha R_{\beta\gamma} - \nabla_\gamma R_{\alpha\beta})\big) + 2R_{\alpha\gamma}\nabla^\gamma\nabla_\beta\nabla^\beta\epsilon^\alpha \\
A_5 & \nabla_\sigma h^{\alpha\beta}\nabla^\sigma R_{\alpha\beta} & 2\nabla_\gamma R_{\alpha\beta}\nabla^\gamma\nabla^\beta\epsilon^\alpha \\
A_6 & \nabla_\alpha h\nabla^\alpha R & 2\nabla_\alpha\nabla_\beta\epsilon^\beta\nabla^\alpha R \\
A_7 & \nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta h & 2\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha \\
A_8 & \nabla_\alpha\nabla^\alpha\nabla_\sigma\nabla_\rho h^{\sigma\rho} & \frac{2}{3}(R^2 - 3R_{\beta\gamma}R^{\beta\gamma})\nabla_\alpha\epsilon^\alpha - \nabla^\alpha R\nabla_\beta\nabla^\beta\epsilon_\alpha \\
 & & +\epsilon^\alpha\big(-\frac{1}{6}R\nabla_\alpha R - \nabla_\alpha\nabla_\beta\nabla^\beta R + \frac{7}{6}R_{\alpha\beta}\nabla^\beta R + R^{\beta\gamma}(-\nabla_\alpha R_{\beta\gamma} + \nabla_\gamma R_{\alpha\beta})\big) \\
 & & +\nabla^\beta\epsilon^\alpha\big(-\frac{8}{3}RR_{\alpha\beta} + 6R_\alpha{}^\gamma R_{\beta\gamma} - 2(\nabla_\beta\nabla_\alpha R + \nabla_\gamma\nabla^\gamma R_{\alpha\beta})\big) + 2\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha \\
 & & -2R_{\alpha\gamma}\nabla^\gamma\nabla_\beta\nabla^\beta\epsilon^\alpha - 4\nabla_\gamma R_{\alpha\beta}\nabla^\gamma\nabla^\beta\epsilon^\alpha \\
A_9 & hR^2 & 2R^2\nabla_\alpha\epsilon^\alpha \\
A_{10} & h^{\alpha\beta}RR_{\alpha\beta} & 2RR_{\alpha\beta}\nabla^\beta\epsilon^\alpha \\
A_{11} & R\nabla_\alpha\nabla_\beta h^{\alpha\beta} & -\epsilon^\alpha R\nabla_\alpha R + 2R\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha - 2RR_{\alpha\beta}\nabla^\beta\epsilon^\alpha \\
A_{12} & hR_{\alpha\beta}R^{\alpha\beta} & 2R_{\beta\gamma}R^{\beta\gamma}\nabla_\alpha\epsilon^\alpha \\
A_{13} & R^{\alpha\beta}\nabla_\beta\nabla_\alpha h & 2R^{\alpha\beta}\nabla_\beta\nabla_\alpha\nabla_\gamma\epsilon^\gamma \\
A_{14} & h^{\alpha\beta}R_{\alpha\gamma}R_\beta{}^\gamma & 2R_\alpha{}^\gamma R_{\beta\gamma}\nabla^\beta\epsilon^\alpha \\
A_{15} & \nabla_\alpha R\nabla_\beta h^{\alpha\beta} & \nabla_\alpha\nabla_\beta\epsilon^\beta\nabla^\alpha R + \nabla^\alpha R\nabla_\beta\nabla^\beta\epsilon_\alpha - \epsilon^\alpha R_{\alpha\beta}\nabla^\beta R \\
A_{16} & h^{\alpha\beta}\nabla_\beta\nabla_\alpha R & 2\nabla_\beta\nabla_\alpha R\nabla^\beta\epsilon^\alpha \\
A_{17} & R^{\alpha\beta}\nabla_\beta\nabla_\gamma h_\alpha{}^\gamma & R^{\alpha\beta}\nabla_\beta\nabla_\alpha\nabla_\gamma\epsilon^\gamma - R_\alpha{}^\gamma R_{\beta\gamma}\nabla^\beta\epsilon^\alpha - \epsilon^\alpha R^{\beta\gamma}\nabla_\gamma R_{\alpha\beta} + R_{\alpha\gamma}\nabla^\gamma\nabla_\beta\nabla^\beta\epsilon^\alpha \\
A_{18} & \nabla_\beta R_{\alpha\gamma}\nabla^\gamma h^{\alpha\beta} & (\nabla_\alpha R_{\beta\gamma} + \nabla_\beta R_{\alpha\gamma})\nabla^\gamma\nabla^\beta\epsilon^\alpha
\end{array} \tag{2.17}$$

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\begin{aligned}
S_4 \quad = \quad & (A_{15}+2A_6)\nabla_\alpha\nabla_\beta\epsilon^\beta\nabla^\alpha R + (2A_{13}+A_{17})R^{\alpha\beta}\nabla_\beta\nabla_\alpha\nabla_\gamma\epsilon^\gamma + (A_{15}-A_8)\nabla^\alpha R\nabla_\beta\nabla^\beta\epsilon_\alpha \\
& +\nabla_\alpha\epsilon^\alpha\big(\tfrac{2}{3}(-A_4+A_8+3A_9)R^2 + 2(A_{12}+A_4-A_8)R_{\beta\gamma}R^{\beta\gamma} + 2A_1\nabla_\beta\nabla^\beta R\big) \\
& +2(A_{11}+A_2)R\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha \\
& +\tfrac{1}{6}\epsilon^\alpha\Big((-6A_{11}+A_4-A_8)R\nabla_\alpha R - 6A_8\nabla_\alpha\nabla_\beta\nabla^\beta R - (6A_{15}+A_4-7A_8)R_{\alpha\beta}\nabla^\beta R \\
& +6R^{\beta\gamma}\big((A_4-A_8)\nabla_\alpha R_{\beta\gamma} - (A_{17}+A_4-A_8)\nabla_\gamma R_{\alpha\beta}\big)\Big) \\
& +\nabla^\beta\epsilon^\alpha\big(\tfrac{2}{3}(3A_{10}-3A_{11}+4A_4-4A_8)RR_{\alpha\beta} + (2A_{14}-A_{17}-6A_4+6A_8)R_\alpha{}^\gamma R_{\beta\gamma} + 2(A_{16}-A_8)\nabla_\beta\nabla_\alpha R \\
& +2(A_3-A_8)\nabla_\gamma\nabla^\gamma R_{\alpha\beta}\big) \\
& +2(A_7+A_8)\nabla_\gamma\nabla^\gamma\nabla_\beta\nabla^\beta\nabla_\alpha\epsilon^\alpha + (A_{17}+2A_4-2A_8)R_{\alpha\gamma}\nabla^\gamma\nabla_\beta\nabla^\beta\epsilon^\alpha \\
& +\big(A_{18}(\nabla_\alpha R_{\beta\gamma}+\nabla_\beta R_{\alpha\gamma}) + 2(A_5-2A_8)\nabla_\gamma R_{\alpha\beta}\big)\nabla^\gamma\nabla^\beta\epsilon^\alpha \qquad (2.18)
\end{aligned}
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