$$g^{\mu\nu}\delta W^{(1)}_{\mu\nu}$$
 v3

1 Generalized RW Geometry

We use the following geometry which facilitates easy computation by either taking k = 0 or $\Omega(x) = \Omega(\tau)$ depending, on the desired coordinates (conformal flat or conformal comoving respectively).

$$ds^{2} = (g_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu} = \Omega^{2}(x)(\tilde{g}_{\mu\nu} + f_{\mu\nu})dx^{\mu}dx^{\nu}$$
(1.1)

$$\tilde{g}_{\mu\nu} = \operatorname{diag}\left(-1, \frac{1}{1 - kr^2}, r^2, r^2 \sin^2 \theta\right) \qquad \tilde{\Gamma}^{\lambda}_{\alpha\beta} = \delta^{\lambda}_i \delta^j_{\alpha} \delta^k_{\beta} \tilde{\Gamma}^i_{jk}$$

$$(1.2)$$

1.1 W_1

As evaluated in the background geometry of (1.1):

$$\begin{split} g^{\mu\nu}W^{(1)}_{\mu\nu} &=& 6\nabla_{\alpha}\nabla^{\alpha}R \\ &=& 6\Omega^{-4}\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\alpha}\tilde{R} - 12\tilde{R}\Omega^{-5}\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\alpha}\Omega - 12\Omega^{-5}\tilde{\nabla}_{\alpha}\tilde{R}\tilde{\nabla}^{\alpha}\Omega + 12\tilde{R}\Omega^{-6}\tilde{\nabla}_{\alpha}\Omega\tilde{\nabla}^{\alpha}\Omega - 144\Omega^{-6}\tilde{\nabla}_{\alpha}\tilde{\nabla}_{\beta}\tilde{\nabla}^{\beta}\Omega\tilde{\nabla}^{\alpha}\Omega \\ &- 108\Omega^{-6}\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\alpha}\Omega\tilde{\nabla}_{\beta}\tilde{\nabla}^{\beta}\Omega + 216\Omega^{-7}\tilde{\nabla}_{\alpha}\Omega\tilde{\nabla}^{\alpha}\Omega\tilde{\nabla}_{\beta}\tilde{\nabla}^{\beta}\Omega + 36\Omega^{-5}\tilde{\nabla}_{\beta}\tilde{\nabla}^{\beta}\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\alpha}\Omega \\ &=& 36\Omega^{-5}\bigg[\ddot{\Omega} - 2k\ddot{\Omega} + 6\ddot{\Omega}\dot{\Omega}^{2}\Omega^{-2} - 3\ddot{\Omega}^{2}\Omega^{-1} - 4\ddot{\Omega}\dot{\Omega}\Omega^{-1} + 2k\dot{\Omega}^{2}\Omega^{-1} - 2\tilde{\nabla}_{a}\tilde{\nabla}^{a}\tilde{\Omega} + 4\dot{\Omega}\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\dot{\Omega} \\ &- 6\dot{\Omega}^{2}\Omega^{-2}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega + 6\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega + 4\Omega^{-1}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\tilde{\Omega} - 6\ddot{\Omega}\Omega^{-2}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega + 6k\Omega^{-1}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega \\ &+ \tilde{\nabla}_{b}\tilde{\nabla}_{a}\tilde{\nabla}^{b}\tilde{\nabla}^{a}\Omega - 3\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega + 6\Omega^{-2}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega - 4\Omega^{-1}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\tilde{\nabla}_{a}\Omega\bigg] \end{split} \tag{1.3}$$

1.2 $g^{\mu\nu}\delta W^{(1)}_{\mu\nu}$

We set k=0 and substitute the gauge invariants and null trace condition $(g^{\mu\nu}W^{(1)}_{\mu\nu}=0)$,

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \qquad \gamma = \psi - \Omega^{-1} [(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a \Omega],$$

$$Q_i = B_i - \dot{E}_i, \qquad E_{ij}.$$
(1.4)

$$\tilde{\nabla}_{a}\tilde{\nabla}^{a}\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega = -\tilde{\Omega} - 6\tilde{\Omega}\dot{\Omega}^{2}\Omega^{-2} + 3\tilde{\Omega}^{2}\Omega^{-1} + 4\tilde{\Omega}\dot{\Omega}\Omega^{-1} + 2\tilde{\nabla}_{a}\tilde{\nabla}^{a}\tilde{\Omega} - 4\dot{\Omega}\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\dot{\Omega} - 2k\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega + 6\dot{\Omega}^{2}\Omega^{-2}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega - 6\tilde{\Omega}\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega - 4\Omega^{-1}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\tilde{\Omega} + 6\tilde{\Omega}\Omega^{-2}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega + 3\Omega^{-1}\tilde{\nabla}_{a}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega - 6\Omega^{-2}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega + 4\Omega^{-1}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\tilde{\nabla}_{a}\Omega \qquad (1.5)$$

$$\begin{split} \tilde{\nabla}_{a}\tilde{\nabla}^{a}\tilde{\nabla}_{b}\tilde{\nabla}^{b}\dot{\Omega} &= -\overset{\cdots}{\Omega} + 12\overset{\circ}{\Omega}\dot{\Omega}^{3}\Omega^{-3} - 15\overset{\circ}{\Omega}^{2}\dot{\Omega}\Omega^{-2} - 10\overset{\circ}{\Omega}\dot{\Omega}^{2}\Omega^{-2} + 10\overset{\circ}{\Omega}\overset{\circ}{\Omega}\Omega^{-1} + 4\overset{\circ}{\Omega}\dot{\Omega}\Omega^{-1} + 2\overset{\circ}{\nabla}_{a}\tilde{\nabla}^{a}\overset{\circ}{\Omega} \\ &- 4\dot{\Omega}\Omega^{-1}\overset{\circ}{\nabla}_{a}\tilde{\nabla}^{a}\overset{\circ}{\Omega} + 10\dot{\Omega}^{2}\Omega^{-2}\overset{\circ}{\nabla}_{a}\tilde{\nabla}^{a}\overset{\circ}{\Omega} - 10\overset{\circ}{\Omega}\Omega^{-1}\overset{\circ}{\nabla}_{a}\tilde{\nabla}^{a}\overset{\circ}{\Omega} - 12\overset{\circ}{\Omega}^{3}\Omega^{-3}\overset{\circ}{\nabla}_{a}\overset{\circ}{\Omega}^{a}\Omega \\ &+ 18\overset{\circ}{\Omega}\dot{\Omega}\Omega^{-2}\overset{\circ}{\nabla}_{a}\overset{\circ}{\Omega}^{a}\Omega - 6\overset{\circ}{\Omega}\Omega^{-1}\overset{\circ}{\nabla}_{a}\tilde{\nabla}^{a}\Omega - 4\Omega^{-1}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega} - 4\Omega^{-1}\overset{\circ}{\nabla}_{a}\dot{\Omega}\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega} + 4\overset{\circ}{\Omega}\Omega^{-2}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega} \\ &+ 12\overset{\circ}{\Omega}\Omega^{-2}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega} - 12\overset{\circ}{\Omega}\dot{\Omega}\Omega^{-3}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\Omega + 6\overset{\circ}{\Omega}\Omega^{-2}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\Omega - 6\Omega^{-2}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega}\overset{\circ}{\nabla}^{b}\overset{\circ}{\Omega} \\ &+ 6\Omega^{-1}\overset{\circ}{\nabla}_{a}\overset{\circ}{\Omega}\overset{\circ}{\nabla}_{b}\overset{\circ}{\nabla}^{b}\Omega - 3\overset{\circ}{\Omega}\Omega^{-2}\overset{\circ}{\nabla}_{a}\overset{\circ}{\nabla}^{a}\Omega\overset{\circ}{\nabla}_{b}\overset{\circ}{\nabla}^{b}\Omega - 12\Omega^{-2}\overset{\circ}{\nabla}_{a}\Omega\overset{\circ}{\nabla}^{a}\overset{\circ}{\Omega}\overset{\circ}{\nabla}_{b}\overset{\circ}{\nabla}^{b}\Omega \end{split}$$

$$+12\dot{\Omega}\Omega^{-3}\tilde{\nabla}_{a}\Omega\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\Omega + 4\Omega^{-1}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\tilde{\nabla}_{a}\dot{\Omega} + 4\Omega^{-1}\tilde{\nabla}^{a}\dot{\Omega}\tilde{\nabla}_{b}\tilde{\nabla}^{b}\tilde{\nabla}_{a}\Omega - 4\dot{\Omega}\Omega^{-2}\tilde{\nabla}^{a}\Omega\tilde{\nabla}_{b}\tilde{\nabla}^{b}\tilde{\nabla}_{a}\Omega.$$

$$(1.6)$$

The perturbed trace $q^{\mu\nu}\delta W_{\mu\nu}^{(1)}$ can then be expressed entirely in terms of the gauge invariants as

$$\begin{split} g^{\mu\nu}\delta W_{\mu\nu}^{(1)} &= \bar{\alpha}(144\Omega^2\Omega^{-6} - 144\Omega\Omega^{-5}) - 36\bar{\alpha}\Omega\Omega^{-5} - 36\bar{\gamma}\Omega^{-4} - 12\Omega^{-4}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\alpha} + 72\Omega^{-4}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} \\ &+ 60\Omega\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} + 32\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} + (288\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}) - 576\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi})\bar{\chi}^{\alpha}\dot{\gamma} \\ &- 288\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} + \bar{\chi}_{\alpha}\alpha(72\Omega^{-5}\bar{\chi}^{\alpha}\dot{\chi}) + 144\Omega\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 288\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 432\Omega^{2}\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\Omega - 432\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} - 288\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} + 432\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 432\Omega^{2}\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\Omega - 432\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\gamma} - 288\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 144\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 432\Omega^{2}\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 432\Omega\Omega^{-6} - 144\Omega\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 288\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 144\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 432\Omega^{2}\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 432\Omega\Omega^{-6} - 144\Omega\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\Omega - 288\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} - 432\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 216\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi}^{\alpha} - 144\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\Omega - 288\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi}^{\alpha}\dot{\chi} + 322\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 216\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi}^{\alpha}\dot{\chi}^{\alpha} - 72\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi}^{\alpha}\dot{\chi} - 24\Omega^{-5}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi}^{\alpha}\dot{\chi}^{\alpha}\dot{\chi} + 32\Omega\Omega^{-7}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi} \\ &+ 72\Omega^{-6}\bar{\chi}_{\alpha}\bar{\chi}^{\alpha}\dot{\chi$$

(1.7)