Notes on the Conformal Invariance of Fluctuations

Conformal properties of $G_{\mu\nu}$ and $W_{\mu\nu}$

Under conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},$$

the Ricci tensor transforms as

$$R_{\mu\nu}(g_{\mu\nu}) \to \bar{R}_{\mu\nu}(\bar{g}_{\mu\nu}) = R_{\mu\nu}(g_{\mu\nu}) + \tilde{S}_{\mu\nu}(g_{\mu\nu})$$

where $\tilde{S}_{\mu\nu}$ involves terms with covariant derivatives of Ω. It follows that the Ricci scalar transforms as

$$g^{\alpha\beta} R_{\alpha\beta}(g_{\mu\nu}) \to \bar{R}(\bar{g}_{\mu\nu}) = \Omega^{-2}[R(g_{\mu\nu}) + g^{\alpha\beta}\tilde{S}_{\alpha\beta}(g_{\mu\nu})]$$

and thus

$$g_{\mu\nu} R \to \bar{g}_{\mu\nu}\bar{R} = g_{\mu\nu} R + S'_{\mu\nu}.$$

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ must then transform as

$$G_{\mu\nu}(g_{\mu\nu}) \to \bar{G}_{\mu\nu}(\bar{g}_{\mu\nu}) = G_{\mu\nu}(g_{\mu\nu}) + S_{\mu\nu}(g_{\mu\nu})$$

where again $S_{\mu\nu}$ is some arbitrary tensor of Ω and $g_{\mu\nu}$. Now expanding to first order in the gravitational perturbation

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

we have

$$\begin{aligned}\bar{G}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu}) &= \bar{G}^{(0)}_{\mu\nu}(\bar{g}^{(0)}_{\mu\nu}) + \delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) \\ &= G^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) + S^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta S_{\mu\nu}(h_{\mu\nu}).\end{aligned}$$

Now looking at the first order contribution,

$$\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) = \delta G_{\mu\nu}(h_{\mu\nu}) + \delta S_{\mu\nu}(h_{\mu\nu}),$$

we note that diagonality in $\bar{h}_{\mu\nu}$ of $\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu})$ requires the sum of $\delta G_{\mu\nu}(h_{\mu\nu})$ and $\delta S_{\mu\nu}(h_{\mu\nu})$ to be diagonal in $h_{\mu\nu}$.

Specifically, we may calculate $S_{\mu\nu}$ to be

$$S_{\mu\nu} = \Omega^{-1}(g_{\mu\nu}\nabla_\alpha\nabla^\alpha\Omega + 2\nabla_\nu\nabla_\mu\Omega) + \Omega^{-2}(g_{\mu\nu}\nabla_\alpha\Omega\nabla^\alpha\Omega - 4\nabla_\mu\Omega\nabla_\nu\Omega)$$

and expanding to first order (here $g_{\mu\nu} = g^{(0)}_{\mu\nu}$)

$$\begin{aligned}\delta S_{\mu\nu} = \Omega^{-1}[&-g_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta h^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta h^\gamma{}_\gamma + g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta h_{\mu\nu} - g_{\mu\nu} h^{\alpha\beta}\nabla_\beta\nabla_\alpha\Omega \\ &+ g^{\alpha\beta} h_{\mu\nu}\nabla_\beta\nabla_\alpha\Omega - \nabla_\alpha\Omega\nabla_\mu h^\alpha{}_\nu - \nabla_\alpha\Omega\nabla_\nu h^\alpha{}_\mu] \\ &+ \Omega^{-2}[g^{\alpha\beta} h_{\mu\nu}\nabla_\alpha\Omega\nabla_\beta\Omega - g_{\mu\nu} h^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega].\end{aligned}$$

In the conformal to flat case, $\delta S_{\mu\nu}$ simplifies to

$$\delta S_{\mu\nu} = \Omega^{-1}[\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\beta h_{\gamma\eta} + \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} + \eta^{\alpha\beta} h_{\mu\nu}\partial_\beta\partial_\alpha\Omega$$

$$- \eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}\partial_\alpha\Omega\partial_\eta h_{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\eta}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\eta\partial_\beta\Omega - \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\partial_\alpha\Omega\partial_\nu h_{\mu\beta}]$$
$$+ \Omega^{-2}[\eta^{\alpha\beta}h_{\mu\nu}\partial_\alpha\Omega\partial_\beta\Omega - \eta^{\alpha\gamma}\eta^{\beta\eta}\eta_{\mu\nu}h^{\gamma\eta}\partial_\alpha\Omega\partial_\beta\Omega].$$

In the harmonic gauge, the extra term $\delta S_{\mu\nu}(g_{\mu\nu})$ does not vanish, and thus does not yield

$$\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu}) = \delta G_{\mu\nu}(h_{\mu\nu}).$$

If the harmonic gauge did in fact cause $\delta S_{\mu\nu}$ to vanish, then we would be able to use the conformally transformed harmonic condition directly within $\delta\bar{G}_{\mu\nu}(\bar{h}_{\mu\nu})$ to obain (nearly) diagonal equations of motion (or just as diagonal as can be found using harmonic in the flat fluctuations).

In C^2 theory, however, we have

$$W_{\mu\nu} \to \bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2}W_{\mu\nu}(g_{\mu\nu})$$

and thus

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2}W_{\mu\nu}(\Omega^{-2}\bar{g}_{\mu\nu}).$$

Taking the first order fluctuations in the same manner as above, we arrive at

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(h_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(\Omega^{-2}\bar{h}_{\mu\nu}).$$

Hence if the fluctuations $\delta W_{\mu\nu}(h_{\mu\nu})$ are diagonal in $h_{\mu\nu}$, it immediately follows they will remain so under conformal transformations.

Trace Considerations

We can continue to use conformal invariance to determine the trace depedendent properties of $W_{\mu\nu}$. Taking h as a first order perturbation in the metric and using the conformal invariance, we find up to first order

$$W_{\mu\nu}\left(g^{(0)}_{\mu\nu} + \frac{h}{4}g^{(0)}_{\mu\nu}\right) = W_{\mu\nu}\left[\left(1+\frac{h}{4}\right)g^{(0)}_{\mu\nu}\right] = W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right)$$
$$= \left(1-\frac{h}{4}\right)W_{\mu\nu}(g^{(0)}_{\mu\nu}),$$

and hence

$$-\frac{h}{4}W_{\mu\nu}(g^{(0)}_{\mu\nu}) = \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right). \tag{1}$$

Now, decomposing $h_{\mu\nu}$ into a trace and trace free components

$$h_{\mu\nu} = K_{\mu\nu} + g_{\mu\nu}\frac{h}{4}$$

(where $g^{(0)\mu\nu}K_{\mu\nu} = 0$, $h = g^{(0)\mu\nu}h_{\mu\nu}$), substitute the above in, again keeping only first order terms

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}\left(K_{\mu\nu} + \frac{h}{4}g^{(0)}_{\mu\nu}\right) = \delta W_{\mu\nu}(K_{\mu\nu}) + \delta W_{\mu\nu}\left(\frac{h}{4}g^{(0)}_{\mu\nu}\right). \tag{2}$$

If we work in a background that is conformal to flat, then (1) will vanish which implies from (2) that

$$\delta W_{\mu\nu}(h_{\mu\nu}) = \delta W_{\mu\nu}(K_{\mu\nu}).$$

We may also find a relationship in the trace of entire fluctuation $\delta W_{\mu\nu}$. The tracelessness of $W_{\mu\nu}$ implies

$$g^{\mu\nu}W_{\mu\nu}(g_{\mu\nu}) = \left(g^{(0)\mu\nu} - h^{\mu\nu}\right)\left(W^{(0)}_{\mu\nu} + \delta W_{\mu\nu}\right) = 0.$$

To first order,

$$-h^{\mu\nu}W^{(0)}_{\mu\nu} + g^{(0)\mu\nu}\delta W_{\mu\nu} = 0$$

and thus

$$g^{(0)\mu\nu}\delta W_{\mu\nu}(h_{\mu\nu}) = h^{\mu\nu}W_{\mu\nu}(g^{(0)}_{\mu\nu}). \tag{3}$$

Once again, in a conformal to flat background, the trace of the fluctuations will vanish.

SVT Decomposition of $\delta W_{\mu\nu}$

Under conformal transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $W_{\mu\nu}$ transforms as

$$\bar{W}_{\mu\nu}(\bar{g}_{\mu\nu}) = \Omega^{-2}W_{\mu\nu}(g_{\mu\nu}).$$

Perturbing the metric,

$$\bar{g}_{\mu\nu} = \bar{g}^{(0)}_{\mu\nu} + \bar{h}_{\mu\nu} = \Omega^2 g^{(0)}_{\mu\nu} + \Omega^2 h_{\mu\nu}$$

it follows that to first order

$$\delta\bar{W}_{\mu\nu}(\bar{h}_{\mu\nu}) = \Omega^{-2}\delta W_{\mu\nu}(h_{\mu\nu}). \tag{4}$$

Under an infinitesimal oordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, the perturbed tensor $\delta W_{\mu\nu}$ transforms as

$$\delta W_{\mu\nu}(h_{\mu\nu}) \to \delta W'_{\mu\nu}(h'_{\mu\nu}) = \delta W_{\mu\nu}(h_{\mu\nu}) - \delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu})$$

At the same time, we also consider the transformation of the entire $W_{\mu\nu}$ under the infinitesimal coordinate transformation

$$W_{\mu\nu} \to W'_{\mu\nu} = W_{\mu\nu} - \mathcal{L}_e(W_{\mu\nu}) \tag{5}$$

where the Lie derivative $\mathcal{L}_e$ for the rank 2 tensor is

$$\mathcal{L}_e(W_{\mu\nu}) = W^\lambda{}_\mu\epsilon_{\lambda;\nu} + W^\lambda{}_\nu\epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda.$$

Defining $\delta W_{\mu\nu}(\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}) \equiv \delta W_{\mu\nu}(\epsilon)$, if we expand eq (5) to first order (that is $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$), we get

$$W_{\mu\nu} \to W'_{\mu\nu} = W^{(0)}_{\mu\nu}(g^{(0)}_{\mu\nu}) + \delta W_{\mu\nu}(h_{\mu\nu}) - \mathcal{L}_e(W_{\mu\nu})$$

and conclude that

$$\delta W_{\mu\nu}(\epsilon) = \mathcal{L}_e(W_{\mu\nu}) = W^\lambda{}_\mu\epsilon_{\lambda;\nu} + W^\lambda_\nu\epsilon_{\lambda;\mu} + W_{\mu\nu;\lambda}\epsilon^\lambda.$$

Hence, in any background that is conformal to flat, the Lie derivative vanishes and thus $\delta W_{\mu\nu}$ must be gauge invariant. As such, it must always be possible to express $\delta W_{\mu\nu}$ in terms of 5 gauge invariant quantities (10 symmetric components - 4 coordinate transformation - 1 traceless condition = 5). This is shown below. Alternatively, we may also fix the gauge, as we have done to make $\delta W_{\mu\nu}$ diagonal in its indicies.

Now decomposing $h_{\mu\nu}$ according to

$$ds^2 = \Omega^2\left\{-(1+2\phi)d\tau^2 + (\nabla_i + B_i)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}]dx^i dx^j\right\}$$

we have in flat space $\delta W_{\mu\nu}(h_{\mu\nu})$ in arbitrary coordinate system

Scalars:

$$\begin{aligned}
\delta W_{00} &= -\frac{2}{3\Omega^2}\nabla^4(\phi+\psi-(E'-B)') \\
\delta W_{0i} &= -\frac{2}{3\Omega^2}\nabla^4\dot{(}\phi+\psi-(E'-B)') \\
\delta W_{ij} &= \frac{1}{3\Omega^2}\big[g_{ij}\nabla^2\ddot{(}\phi+\psi-(E'-B)')+\nabla^2\nabla_i\nabla_j(\phi+\psi-(E'-B)') \\
&\qquad -g_{ij}\nabla^4(\phi+\psi-(E'-B)')-3\nabla_i\nabla_j\ddot{(}\phi+\psi-(E'-B)')\big]
\end{aligned}$$

Vectors:

$$\begin{aligned}
\delta W_{0i} &= \frac{1}{2\Omega^2}\left[\nabla^4(B_i-E_i')-\nabla^2(B_i-E_i')''\right] \\
\delta W_{ij} &= \frac{1}{2\Omega^2}\left[\nabla^2\nabla_i(B_j-E_j')'+\nabla^2\nabla_j(B_i-E_i')'-\nabla_i(B_j-E_j')''-\nabla_j(B_i-E_i')''\right]
\end{aligned}$$

Tensors:

$$\delta W_{ij} = \frac{1}{\Omega^2}\left(E_{ij}-2\nabla^2\ddot{E}_{ij}+\nabla^4 E_{ij}\right)$$

According to eq. (4), we may find $\delta W_{\mu\nu}$ based on a conformal to flat background by simply multiplying the above by a factor of Ω^{-2}.

Under coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu$ where $\epsilon^\mu = (T, \partial^i L + L^i)$ the SVT quantities in the RW $K=0$ background transform as ($\mathcal{H} = \frac{\dot{\Omega}}{\Omega}$)

$$\tilde{\phi} = \phi - T' - \mathcal{H}T \tag{6}$$

$$\tilde{\psi} = \psi + \mathcal{H}T \tag{7}$$

$$\tilde{E} = E - L \tag{8}$$

$$\tilde{B} = B + T - L' \tag{9}$$

$$\tilde{B}_i = B_i - L_i' \tag{10}$$

$$\tilde{E}_i = E_i - L_i \tag{11}$$

$$\tilde{E}_{ij} = E_{ij} \tag{12}$$

in which the gauge invariant combinations are

$$\Phi = \phi - \mathcal{H}(E'-B) - (E'-B)' \tag{13}$$

$$\Psi = \psi + \mathcal{H}(E'-B) \tag{14}$$

$$\mathcal{Q}_i = B_i - E_i' \tag{15}$$

$$E_{ij} = E_{ij}. \tag{16}$$

and, importantly for the Weyl tensor

$$\Sigma = \Phi + \Psi = \phi + \psi - (E'-B)'. \tag{17}$$

Now, if we generalize the conformal factor $\Omega(\tau) \to \Omega(x)$ we can calculate the gauge transformations by effectively sending

$$T\mathcal{H} \to \tilde{H} = \frac{\epsilon^\mu \partial_\mu \Omega}{\Omega} = T\mathcal{H} + (\partial^i L + L^i)\frac{\partial_i \Omega}{\Omega}.$$

That this is true can be seen from the first order contrubtion of $\Omega(x^\mu + \epsilon^\mu)$. As such, the analogous SVT quantities under the coordinate transformation are

$$\tilde{\phi} = \phi - T' - \tilde{H} \tag{18}$$

$$\tilde{\psi} = \psi + \tilde{H} \tag{19}$$

$$\tilde{E} = E - L \tag{20}$$

$$\tilde{B} = B + T - L' \tag{21}$$

$$\tilde{B}_i = B_i - L_i' \tag{22}$$

$$\tilde{E}_i = E_i - L_i \tag{23}$$

$$\tilde{E}_{ij} = E_{ij} \tag{24}$$

The gauge invariant combinations can then only possibly differ from that of RW for those involving ψ and ϕ and in the Weyl case we only care about

$$\Sigma = \phi + \psi - (E' - B)'. \tag{25}$$

But we note that the $\tilde{H}$ terms drop out identically, and thus in the general conformal case and thus the same form for Σ remains invariant. Thus the gauge invariant quantities for any conformal factor are:

$$\Sigma = \phi + \psi - (E' - B)' \tag{26}$$

$$\mathcal{Q}_i = B_i - E_i' \tag{27}$$

$$E_{ij} = E_{ij}. \tag{28}$$

This brings us to 5 independent components in total, as mentioned above, and the gauge invariant combinations within $\delta W_{\mu\nu}$ drop out very clearly.