3-Space Einstein Tensor Gauge Dependence v5

March 7, 2019

1 Covariant Decomposition

1.1 Geometry

Within the geometry of

$$ds^2 = (g_{ij}^{(0)} + h_{ij})dx^i dx^j \tag{1.1}$$

with maximally symmetric background

$$g_{ij}^{(0)} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.2}$$

assume the metric perturbation can be (covariant) SVT decomposed as

$$h_{ij} = -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}, \tag{1.3}$$

with 3-trace

$$h = -6\psi + 2\nabla^a\nabla_a E. \tag{1.4}$$

1.2 Background $G_{ij}^{(0)} = -\kappa_3^2 T_{ij}^{(0)}$

$$\begin{aligned} G_{ij}^{(0)} &= -\kappa_3^2 T_{ij}^{(0)} \\ kg_{ij} &= -\kappa_3^2 \Lambda g_{ij} \\ \rightarrow \quad \Lambda &= -\frac{k}{\kappa_3^2} \end{aligned} \tag{1.5}$$

1.3 Perturbed $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$

We choose to perturb only the background $T_{\mu\nu}^{(0)}$ to yield

$$-\kappa_3^2 \delta T_{ij} = kh_{ij}. \tag{1.6}$$

The perturbed Einstein equations $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ then take the form

$$\begin{aligned} &-\tfrac{1}{2}h_{ij}R + \tfrac{1}{2}g_{ij}h^{ab}R_{ab} + \tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h - \tfrac{1}{2}\nabla_a\nabla_i h_j{}^a - \tfrac{1}{2}\nabla_a\nabla_j h_i{}^a \\ &+\tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} + \tfrac{1}{2}\nabla_i\nabla_j h = kh_{ij}. \end{aligned} \tag{1.7}$$

In SVT terms this evaluates to:

$$\begin{aligned} &\nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi = \\ &k(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}), \end{aligned} \tag{1.8}$$

which may be simplified as

$$\boxed{(\nabla_a\nabla^a - 2k)E_{ij} + g_{ij}\nabla_a\nabla^a\psi - \nabla_j\nabla_i\psi + 2kg_{ij}\psi = 0.} \tag{1.9}$$

Taking the trace gives the solution for ψ

$$\boxed{(\nabla_a\nabla^a + 3k)\psi = 0} \tag{1.10}$$

In an attempt to isolate the transverse traceless contribution to (1.9), we take the combination of projected covariant derivatives

$$(\nabla_\alpha\nabla^\alpha - 3k)\Delta_{ij} + \frac{1}{2}(\nabla_i\nabla_j + 2kg_{ij} - g_{ij}\nabla_\alpha\nabla^\alpha)\Delta = 0 \tag{1.11}$$

which yields the result

$$\boxed{6k^2E_{ij} - 5k\nabla_a\nabla^a E_{ij} + \nabla_b\nabla^b\nabla_a\nabla^a E_{ij} = 0} \tag{1.12}$$

where $\Delta = g^{ab}\Delta_{ab}$. Though the equation is now fourth order, the transverse traceless E_{ij} can be decoupled from ψ.

1.4 Gauge Structure

Under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ in the RW geometry we decompose $\epsilon_i(x)$ into longitudinal and transverse components viz

$$\epsilon_i = \underbrace{\epsilon_i - \nabla_i\int D\nabla^j\epsilon_j}_{L_i} + \nabla_i\underbrace{\int D\nabla^j\epsilon_j}_{L} \tag{1.13}$$

$$\nabla_i\epsilon_j = \nabla_i L_j + \nabla_i\nabla_j L \tag{1.14}$$

For the metric

$$\begin{aligned}\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\ &= \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.15}$$

Now form the gauge transformation equation

$$\begin{aligned}-2\bar{\psi}g_{ij} + 2\nabla_i\nabla_j\bar{E} + \nabla_i\bar{E}_j + \nabla_j\bar{E}_i + 2\bar{E}_{ij} &= -2\psi g_{ij} + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij} \\ &\quad + \nabla_i L_j + \nabla_j L_i + 2\nabla_i\nabla_j L\end{aligned} \tag{1.16}$$

The trace of (1.16) yields

$$-6\bar{\psi} + 2\nabla_a\nabla^a\bar{E} = -6\psi + 2\nabla_a\nabla^a E + 2\nabla_a\nabla^a L. \tag{1.17}$$

Using (**??**), the double transverse component of (1.16) yields

$$-2\nabla_a\nabla^a\bar{\psi} + 2\nabla_a\nabla^a\nabla_b\nabla^b\bar{E} + 4k\nabla_a\nabla^a\bar{E} = -2\nabla_a\nabla^a\psi + 2\nabla_a\nabla^a\nabla_b\nabla^b(E+L) + 4k\nabla_a\nabla^a(E+L). \tag{1.18}$$

Using (1.17) to eliminate ψ in the above, we arrive at an equation in terms of E and L

$$\tfrac{2}{3}\nabla^4\bar{E} + k\nabla^2\bar{E} = \tfrac{2}{3}\nabla^4(E+L) + k\nabla^2(E+L). \tag{1.19}$$

For quantities $\bar{E}$, E, and L that vanish on the boundary, we may integrate the associated Green's function by parts to show $\bar{E} = E + L$. Substitution into (1.17) then yields $\bar{\psi} = \psi$. The remaining transverse component of (1.16) is then

$$\nabla_a\nabla^a\bar{E}_i = \nabla_a\nabla^a(E_i + L_i). \tag{1.20}$$

With $\bar{E}_i$, E_i, and L_i vanishing on the boundary we have $\bar{E}_i = E_i + L_i$. In summary,

$$\begin{aligned}
\bar{\psi} &= \psi \\
\bar{E} &= E + L \\
\bar{E}_i &= E_i + L_i \\
\bar{E}_{ij} &= E_{ij}.
\end{aligned} \tag{1.21}$$

As E_i and E are not gauge invariant, the field equations $\delta G_{\mu\nu} = -\kappa_3^2 \delta T_{\mu\nu}$ can only depend on ψ and E_{ij}, which agrees with (1.9). With the six components of h_{ij} we are free to make three coordinate transformation to reduce h_{ij} to three gauge invariant components, i.e. ψ and E_{ij}.

2 3-Space Scalar Eigenfunctions

In order for E_{ij} and ψ to decouple from (1.9), we assess whether every solution to

$$(\nabla_a \nabla^a + 3k)\psi = 0 \tag{2.1}$$

obeys

$$\nabla_i \nabla_j \psi \stackrel{!}{=} -k g_{ij} \psi. \tag{2.2}$$

Thus we first seek to find the general solution to curved space harmonic eigenfunction

$$(g^{ij} \nabla_i \nabla_j + \lambda^2)\psi = 0 \tag{2.3}$$

where $\lambda^2 = 3k$. Evaluating the above in the 3-space geometry, we find

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) - 3kr\frac{\partial \psi}{\partial r} - kr^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right] + \lambda^2 \psi = 0. \tag{2.4}$$

Noting the angular portion of the Laplacian, we take as solution $\psi = f_l(r) Y^l_m(\theta, \phi)$ to find a radial equation of

$$r^2 \frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - 3kr^3 \frac{df}{dr} - kr^4 \frac{d^2 f}{dr^2} + [\lambda^2 r^2 - l(l+1)] f_l(r) = 0. \tag{2.5}$$

We may rewrite the above as

$$\left[(1 - kr^2)\frac{d^2}{dr^2} + \left(\frac{2}{r} - 3kr\right)\frac{d}{dr} - \frac{l(l+1)}{r^2} + \lambda^2\right] f_l(r) = 0. \tag{2.6}$$

This coincides with [1] eq. (19), with the exception that λ^2 is not a separation constant in our case, but rather assumes the value of $\lambda^2 = 3k$.

2.1 Hyperbolic Geometry $k = -1$

For $k = -1$ we set $r = \sinh\chi$ to bring the RW geometry to the form

$$\begin{aligned}
ds^2 &= \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \\
&= d\chi^2 + \chi^2 d\Omega^2.
\end{aligned} \tag{2.7}$$

In the hyperbolic coordinates, taking $\psi = \psi(\chi)$, the scalar equation

$$\nabla_a \nabla^a \psi = -3k\psi \tag{2.8}$$

takes the form

$$2\frac{\cosh\chi}{\sinh\chi}\frac{d\psi}{d\chi}+\frac{d^2\psi}{d\chi^2}=-3k\psi. \tag{2.9}$$

Now taking ψ as

$$\psi(\chi)=\frac{e^{\pm\nu\chi}}{\sinh\chi} \tag{2.10}$$

it follows

$$2\frac{\cosh\chi}{\sinh\chi}\frac{d\psi}{d\chi}+\frac{d^2\psi}{d\chi^2}=(\nu^2-1)\psi. \tag{2.11}$$

Hence $\nu=2$ and the two radial solutions are

$$\psi_1=\frac{e^{-2\chi}}{\sinh\chi}, \qquad \psi_2=\frac{e^{2\chi}}{\sinh\chi}$$
$$\psi_1=\frac{1}{r[r+(1+r^2)^{1/2}]^2}, \qquad \psi_2=\frac{[r+(1+r^2)^{1/2}]^2}{r}. \tag{2.12}$$

As $r\to\infty$,

$$\psi_1\to 0 \qquad \psi_2\to\infty. \tag{2.13}$$

Taking the asymptotically convergent ψ_1 and evaluating

$$\nabla_i\nabla_j\psi_1+kg_{ij}\psi_1\stackrel{!}{=}0 \tag{2.14}$$

we find that for $i=1$, $j=1$

$$\nabla_1\nabla_1\psi_1-g_{11}\psi_1=\frac{2}{\sinh^3\chi}. \tag{2.15}$$

Therefore it would appear we have found a solution with well behaved asymptotics obeying

$$(\nabla_a\nabla^a+3k)\psi=0 \tag{2.16}$$

that does not obey

$$\nabla_i\nabla_j\psi_1+kg_{ij}\psi_1=0, \tag{2.17}$$

implying ψ and E_{ij} may not necessarily decouple from (1.9) for $k<0$.

2.2 Spherical Geometry $k=1$

For $k=1$ we set $r=\sin\chi$ to bring the RW geometry to the form

$$\begin{aligned} ds^2 &= \frac{dr^2}{1-r^2}+r^2d\Omega^2 \\ &= d\chi^2+\chi^2d\Omega^2. \end{aligned} \tag{2.18}$$

In the spherical coordinates, taking $\psi=\psi(\chi)$, the scalar equation

$$\nabla_a\nabla^a\psi=-3k\psi \tag{2.19}$$

takes the form

$$2\frac{\cos\chi}{\sin\chi}\frac{d\psi}{d\chi}+\frac{d^2\psi}{d\chi^2}=-3k\psi. \tag{2.20}$$

Now taking ψ as

$$\psi(\chi) = \frac{e^{\pm i\nu\chi}}{\sin\chi} \tag{2.21}$$

it follows

$$2\frac{\cos\chi}{\sin\chi}\frac{d\psi}{d\chi} + \frac{d^2\psi}{d\chi^2} = (1-\nu^2)\psi. \tag{2.22}$$

Hence $\nu = \sqrt{2}$ and the two radial solutions are

$$\begin{aligned} \psi_1 &= \frac{e^{-i\sqrt{2}\chi}}{\sin\chi}, \qquad \psi_2 = \frac{e^{i\sqrt{2}\chi}}{\sin\chi} \\ \psi_1 &= \frac{e^{-\sqrt{2}\sin^{-1}r}}{r}, \qquad \psi_2 = \frac{e^{\sqrt{2}\sin^{-1}r}}{r} \end{aligned} \tag{2.23}$$

Taking the real part of ψ_2 and evaluating

$$\nabla_i\nabla_j\psi_2 + kg_{ij}\psi_2 \overset{!}{=} 0 \tag{2.24}$$

we find that for $i = 1$, $j = 1$

$$\nabla_1\nabla_1\psi_2 + g_{11}\psi_2 = 2\cos\chi\left[\cos(\sqrt{2}\chi)\frac{\cos\chi}{\sin\chi} + \sqrt{2}\sin(\sqrt{2}\chi)\right]\sin^{-2}\chi. \tag{2.25}$$

As with the hyperbolic case, it would appear we have found a solution to

$$(\nabla_a\nabla^a + 3k)\psi = 0 \tag{2.26}$$

that does not obey

$$\nabla_i\nabla_j\psi_1 + kg_{ij}\psi_1 = 0, \tag{2.27}$$

implying ψ and E_{ij} may not necessarily decouple from (1.9) for $k > 0$.

3 Conformal to Flat

The 3-space of constant curvature can be expressed in the conformal flat form as in (A.1)

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{4}{\left(1+k\rho^2\right)^2}\left(d\rho^2+\rho^2 d\Omega^2\right) \end{aligned} \tag{3.1}$$

3.1 Background $G_{ij}^{(0)} = -\kappa_3^2 T_{ij}^{(0)}$

From (B.3) we see since $G_{\mu\nu}$ vanishes in a flat geometry, the background equation is given as

$$g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = -\kappa_3^2\Lambda\Omega^2 g_{ij}. \tag{3.2}$$

Taking the trace

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = -3\kappa_3^2\Lambda\Omega^2. \tag{3.3}$$

In the covariant formulation, we saw from (1.5) that $-\kappa_3^2\Lambda = k$, a constant relation independent of choice of coordinate system. As such we expect the above to obey

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = 3\Omega^2 k \tag{3.4}$$

Calculation of the above indeed yields

$$-2\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \frac{12k}{(1+k\rho^2)^2} = 3\Omega^2 k \tag{3.5}$$

The two background equations that will prove useful are:

$$-\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega = \Omega^2 k \tag{3.6}$$

$$g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega) + \Omega^{-1}\nabla_i\nabla_j\Omega - 2\Omega^{-2}\nabla_i\Omega\nabla_j\Omega = k\Omega^2 g_{ij}. \tag{3.7}$$

3.2 $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$

Within geometry

$$ds^2 = \Omega^2(\rho)(g_{ij}+f_{ij})dx^i dx^j, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E, +\tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{3.8}$$

the perturbed Einstein tensor takes the form (with ∇ denoting flat space derivative)

$$
\begin{aligned}
\delta G_{ij} &= g_{ij}\nabla_a\nabla^a\psi + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E - 2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega \\
&+2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega - 2\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\
&+2\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\
&\\
&+g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\
&-\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j - \Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i + \Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i \\
&-\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a \\
&\\
&+\nabla_a\nabla^a E_{ij} - 2E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + 2E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
&+2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}. \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
-\kappa_3^2\delta T_{ij} &= -\kappa_3^2\Lambda\Omega^2 h_{ij} \\
&= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\
&\\
-\kappa_3^2 g^{ij}\delta T_{ij} &= k\Omega^2(-6\psi + 2\nabla_a\nabla^a E) \qquad (3.10)
\end{aligned}
$$

3.3 Gauge Structure

Within the conformal flat geometry of (3.1) under coordinate transformation $x^i \to \bar{x}^i = x^i - \epsilon^i(x)$ we take the general $\epsilon_i(x)$ as $\epsilon_i = \Omega^2 f_i$ with

$$
f_i = \underbrace{f_i - \tilde{\nabla}_i \int D\tilde{\nabla}^j f_j}_{L_i} + \tilde{\nabla}_i \underbrace{\int D\tilde{\nabla}^j f_j}_{L} \qquad (3.11)
$$

It will be helpful to calculate $\nabla_i\epsilon_j$ in terms of f_i,

$$
\begin{aligned}
\nabla_i\epsilon_j &= \partial_i\epsilon_j - \Gamma^k_{ij}\epsilon_k \\
&= \partial_i\epsilon_j - \epsilon_k\left[\tilde{\Gamma}^k_{ij} + \Omega^{-1}(\delta^k_i\partial_j + \delta^k_j\partial_i - g_{ij}g^{kl}\partial_l)\Omega\right]) \\
&= \Omega^2\nabla_i f_j - \Omega(f_i\tilde{\nabla}_j\Omega - f_j\tilde{\nabla}_i\Omega - \tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega) \qquad (3.12)
\end{aligned}
$$

It then follows

$$
\begin{aligned}
\Delta_\epsilon h_{ij} &= \nabla_i\epsilon_j + \nabla_j\epsilon_i \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + 2\Omega^{-1}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega) \\
&= \Omega^2(\tilde{\nabla}_i f_j + \tilde{\nabla}_j f_i + \Omega^{-2}\tilde{g}_{ij}f_k\tilde{\nabla}^k\Omega^2). \qquad (3.13)
\end{aligned}
$$

The transformation of f_{ij} is then

$$
\bar{f}_{ij} = f_{ij} + \tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\tilde{\nabla}_i\tilde{\nabla}_j L + \Omega^{-2}\tilde{g}_{ij}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega^2 \qquad (3.14)
$$

Instead of taking the trace and transverse components of (3.14) as we did for the covariant case, since we know the projectors in flat space, we can instead make use of the defining conditions for SVT quantities and find their gauge structure. Enforcing the SVT quantities vanish on the spatial boundary at infinity, we use the decomposition defined in APM-CPII (66) and solve the gauge transformation of ψ

$$
\bar{\psi} = \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega. \qquad (3.15)
$$

Substituting the above into $\tilde{\nabla}^i\tilde{\nabla}^j\bar{f}_{ij}$ yields the relation for $\bar{E}$

$$\bar{E} \quad = E + L. \tag{3.16}$$

Then substitution of $\bar{E}$ into the $\tilde{\nabla}^j\bar{f}_{ij}$ yields the expression for $\bar{E}_i$

$$\bar{E}_i \quad = \quad E_i + L_i. \tag{3.17}$$

In summary

$$\begin{aligned}
\bar{\psi} &= \psi - \Omega^{-1}(\tilde{\nabla}_k L + L_k)\tilde{\nabla}^k\Omega \\
\bar{E} &= E + L \\
\bar{E}_i &= E_i + L_i \\
\bar{E}_{ij} &= E_{ij}
\end{aligned} \tag{3.18}$$

We find two gauge invariant quantities

$$\begin{aligned}
\bar{\psi} + \Omega^{-1}(\tilde{\nabla}_k\bar{E} + \bar{E}_k)\tilde{\nabla}^k\Omega &= \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k\Omega \\
\bar{E}_{ij} &= E_{ij}
\end{aligned} \tag{3.19}$$

We will denote

$$\Psi \equiv \psi + \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k\Omega. \tag{3.20}$$

3.4 $\delta G_{\mu\nu} = -\kappa_3^2\delta T_{\mu\nu}$ Simplification to Gauge Invariant Form

First we transform $\delta T_{\mu\nu}$ using the background equations

$$\begin{aligned}
-\kappa_3^2\Lambda\delta T_{ij} &= k\Omega^2(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}) \\
&= -2g_{ij}(\Omega^{-2}\nabla_a\Omega\nabla^a\Omega - \Omega^{-1}\nabla_a\nabla^a\Omega)\psi - 2\Omega^{-1}\nabla_i\nabla_j\Omega\psi + 4\Omega^{-2}\nabla_i\Omega\nabla_j\Omega\psi \\
&\quad -\tfrac{4}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i\nabla_j E + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i\nabla_j E \\
&\quad -\tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) + \tfrac{1}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\
&\quad -\tfrac{4}{3}\Omega^{-1}\nabla_a\nabla^a\Omega E_{ij} + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega E_{ij}
\end{aligned} \tag{3.21}$$

Now forming $\Delta_{ij} \equiv \delta G_{ij} + \kappa_3^2\delta T_{ij} = 0$

$$\begin{aligned}
\Delta_{ij} &= g_{ij}\nabla_a\nabla^a\psi - 2g_{ij}\psi\Omega^{-1}\nabla_a\nabla^a\Omega + 2g_{ij}\psi\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E \\
&\quad -2g_{ij}\Omega^{-2}\nabla^a\Omega\nabla_b\nabla_a E\nabla^b\Omega + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b\nabla^a E + \Omega^{-1}\nabla_i\Omega\nabla_j\psi + \Omega^{-1}\nabla_i\psi\nabla_j\Omega \\
&\quad -4\psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E + \tfrac{4}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - \nabla_j\nabla_i\psi \\
&\quad +2\psi\Omega^{-1}\nabla_j\nabla_i\Omega - \Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i\nabla_a E \\
&\quad +g_{ij}\Omega^{-1}\nabla^a\Omega\nabla_b\nabla^b E_a - 2g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\Omega\nabla^b E^a + 2g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega\nabla^b E^a \\
&\quad -\Omega^{-1}\nabla^a\Omega\nabla_j\nabla_i E_a - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\
&\quad +\tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega(\nabla_i E_j + \nabla_j E_i) \\
&\quad +\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
&\quad +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{3.22}$$

To facilitate simplification into gauge invariant components, we make substitution

$$\psi \quad = \quad \Psi - \Omega^{-1}(\tilde{\nabla}_k E + E_k)\tilde{\nabla}^k\Omega \tag{3.23}$$

in which Δ_{ij} then becomes

$$\begin{aligned}
\Delta_{ij} \quad = \quad & g_{ij}\nabla_a\nabla^a\Psi - 2\Psi g_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Psi g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + \Omega^{-1}\nabla_i\Omega\nabla_j\Psi + \Omega^{-1}\nabla_i\Psi\nabla_j\Omega \\
& -4\Psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \nabla_j\nabla_i\Psi + 2\Psi\Omega^{-1}\nabla_j\nabla_i\Omega
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
& +3g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a E\nabla_b\nabla^b\Omega - g_{ij}\Omega^{-1}\nabla^a E\nabla_b\nabla^b\nabla_a\Omega - 4g_{ij}\Omega^{-3}\nabla_a\Omega\nabla^a E\nabla_b\Omega\nabla^b\Omega \\
& +2g_{ij}\Omega^{-2}\nabla^a E\nabla_b\nabla_a\Omega\nabla^b\Omega + \Omega^{-1}\nabla^a\nabla_j E\nabla_i\nabla_a\Omega + 8\Omega^{-3}\nabla_a\Omega\nabla^a E\nabla_i\Omega\nabla_j\Omega \\
& -2\Omega^{-2}\nabla^a\Omega\nabla_i\nabla_a E\nabla_j\Omega - 2\Omega^{-2}\nabla^a E\nabla_i\nabla_a\Omega\nabla_j\Omega - 2\Omega^{-2}\nabla^a\Omega\nabla_i\Omega\nabla_j\nabla_a E \\
& +\Omega^{-1}\nabla^a\nabla_i E\nabla_j\nabla_a\Omega - 2\Omega^{-2}\nabla^a E\nabla_i\Omega\nabla_j\nabla_a\Omega - \tfrac{2}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j\nabla_i E \\
& +\tfrac{4}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j\nabla_i E - 3\Omega^{-2}\nabla_a\Omega\nabla^a E\nabla_j\nabla_i\Omega + \Omega^{-1}\nabla^a E\nabla_j\nabla_i\nabla_a\Omega
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
& +3E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega - E^a g_{ij}\Omega^{-1}\nabla_b\nabla^b\nabla_a\Omega - 4E^a g_{ij}\Omega^{-3}\nabla_a\Omega\nabla_b\Omega\nabla^b\Omega \\
& +2E^a g_{ij}\Omega^{-2}\nabla_b\nabla_a\Omega\nabla^b\Omega - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_i E_j + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_i E_j \\
& -2\Omega^{-2}\nabla_a\Omega\nabla_i\Omega\nabla_j E^a + \Omega^{-1}\nabla_i\nabla_a\Omega\nabla_j E^a - \tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega\nabla_j E_i \\
& +\tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i - 2\Omega^{-2}\nabla_a\Omega\nabla_i E^a\nabla_j\Omega + 8E^a\Omega^{-3}\nabla_a\Omega\nabla_i\Omega\nabla_j\Omega \\
& -2E^a\Omega^{-2}\nabla_i\nabla_a\Omega\nabla_j\Omega + \Omega^{-1}\nabla_i E^a\nabla_j\nabla_a\Omega - 2E^a\Omega^{-2}\nabla_i\Omega\nabla_j\nabla_a\Omega \\
& -3E^a\Omega^{-2}\nabla_a\Omega\nabla_j\nabla_i\Omega + E^a\Omega^{-1}\nabla_j\nabla_i\nabla_a\Omega.
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
& +\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
& +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{3.27}$$

Inputting the explicit form of $\Omega(\rho)$, expanding covariant derivatives, and evaluating component by component we find that (3.25) and (3.26) vanish identically. To show an example of the interplay between contracted vector quantities like $E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega$ and free vectors such as $\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\nabla_j E_i$, we first isolate the free vector contribution to Δ_{ij}:

$$\begin{aligned}
\Delta_{ij}^{(V_1)} \quad &= \quad \left(-\tfrac{1}{3}\Omega^{-1}\nabla_a\nabla^a\Omega + \tfrac{2}{3}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega\right)(\nabla_i E_j + \nabla_j E_i) \\
&= \quad \left(\frac{2k}{1-k\rho^2}\right)(\nabla_i E_j + \nabla_j E_i)
\end{aligned} \tag{3.28}$$

The remaining contracted vector contribution to $\Delta_{ij}^{(V)} = \Delta_{ij}^{(V_1)} + \Delta_{ij}^{(V_2)}$ is

$$\begin{aligned}
\Delta_{ij}^{(V_2)} \quad = \quad & 3E^a g_{ij}\Omega^{-2}\nabla_a\Omega\nabla_b\nabla^b\Omega - E^a g_{ij}\Omega^{-1}\nabla_b\nabla^b\nabla_a\Omega - 4E^a g_{ij}\Omega^{-3}\nabla_a\Omega\nabla_b\Omega\nabla^b\Omega + 2E^a g_{ij}\Omega^{-2}\nabla_b\nabla_a\Omega\nabla^b\Omega \\
& -2\Omega^{-2}\nabla_a\Omega\nabla_i\Omega\nabla_j E^a + \Omega^{-1}\nabla_i\nabla_a\Omega\nabla_j E^a - 2\Omega^{-2}\nabla_a\Omega\nabla_i E^a\nabla_j\Omega + 8E^a\Omega^{-3}\nabla_a\Omega\nabla_i\Omega\nabla_j\Omega \\
& -2E^a\Omega^{-2}\nabla_i\nabla_a\Omega\nabla_j\Omega + \Omega^{-1}\nabla_i E^a\nabla_j\nabla_a\Omega - 2E^a\Omega^{-2}\nabla_i\Omega\nabla_j\nabla_a\Omega - 3E^a\Omega^{-2}\nabla_a\Omega\nabla_j\nabla_i\Omega \\
& +E^a\Omega^{-1}\nabla_j\nabla_i\nabla_a\Omega.
\end{aligned} \tag{3.29}$$

Evaluating (3.29) component by component we find

$$\begin{aligned}
\Delta^{(V_2)}_{rr} &= -\left(\frac{4k}{1+k\rho^2}\right)\partial_r E_r \\
\Delta^{(V_2)}_{\theta\theta} &= -\left(\frac{4k}{1+k\rho^2}\right)(rE_r + \partial_\theta E_\theta) \\
\Delta^{(V_2)}_{\phi\phi} &= -\left(\frac{4k}{1+k\rho^2}\right)(r\sin^2\theta E_r + \sin\theta\cos\theta E_\theta + \partial_\phi E_\phi) \\
\Delta^{(V_2)}_{r\theta} &= \left(\frac{2k}{1+k\rho^2}\right)\left(\frac{2E_\theta}{r} - \partial_r E_\theta - \partial_\theta E_r\right) \\
\Delta^{(V_2)}_{r\phi} &= \left(\frac{2k}{1+k\rho^2}\right)\left(\frac{2E_\phi}{r} - \partial_r E_\phi - \partial_\phi E_r\right) \\
\Delta^{(V_2)}_{\theta\phi} &= \left(\frac{2k}{1+k\rho^2}\right)\left(2\sin^{-1}\theta\cos\theta E_\phi - \partial_\theta E_\phi - \partial_\phi E_\theta\right)
\end{aligned} \tag{3.30}$$

Comparison of (3.30) to (3.28) illustrates the vanishing of the entire vector portion $\Delta^{(V)}_{ij}$. When evaluated component by component, the remaining scalar piece (3.25) similarly vanishes, and thus we are left with the gauge invariant form

$$\begin{aligned}
\Delta_{ij} &= g_{ij}\nabla_a\nabla^a\Psi - 2\Psi g_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Psi g_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega + \Omega^{-1}\nabla_i\Omega\nabla_j\Psi + \Omega^{-1}\nabla_i\Psi\nabla_j\Omega \\
&\quad -4\Psi\Omega^{-2}\nabla_i\Omega\nabla_j\Omega - \nabla_j\nabla_i\Psi + 2\Psi\Omega^{-1}\nabla_j\nabla_i\Omega \\
&\quad +\nabla_a\nabla^a E_{ij} - \tfrac{2}{3}E_{ij}\Omega^{-1}\nabla_a\nabla^a\Omega + \Omega^{-1}\nabla_a E_{ij}\nabla^a\Omega + \tfrac{4}{3}E_{ij}\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
&\quad +2E^{ab}g_{ij}\Omega^{-1}\nabla_b\nabla_a\Omega - 2E_{ab}g_{ij}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega - \Omega^{-1}\nabla^a\Omega\nabla_i E_{ja} - \Omega^{-1}\nabla^a\Omega\nabla_j E_{ia}.
\end{aligned} \tag{3.31}$$

Taking the trace of the above, we find

$$\begin{aligned}
g^{ij}\Delta_{ij} &= 2\nabla_a\nabla^a\Psi - 4\Psi\Omega^{-1}\nabla_a\nabla^a\Omega + 2\Omega^{-1}\nabla_a\Omega\nabla^a\Psi + 2\Psi\Omega^{-2}\nabla_a\Omega\nabla^a\Omega \\
&\quad +6E^{ab}\Omega^{-1}\nabla_b\nabla_a\Omega - 6E_{ab}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega
\end{aligned} \tag{3.32}$$

Since the trace of the covariant Δ_{ij} depended only upon ψ, one might expect a similar result for in the conformal to flat svt decomposition. However, we find the tensor quantity residing in the trace does not vanish and calculates to the single term

$$6E^{ab}\Omega^{-1}\nabla_b\nabla_a\Omega - 6E_{ab}\Omega^{-2}\nabla^a\Omega\nabla^b\Omega = 24\frac{k^2\rho^2}{(1+k\rho^2)^2}E_{\rho\rho}. \tag{3.33}$$

In the same manner that the transverse traceless E_{ij} was isolated in (1.12), we attempt to perform a similar projection upon (3.31). In applying the derivative projection

$$(\nabla_\alpha\nabla^\alpha - 3k)\Delta_{ij} + \frac{1}{2}(\nabla_i\nabla_j + 2kg_{ij} - g_{ij}\nabla_\alpha\nabla^\alpha)\Delta = 0 \tag{3.34}$$

to the conformal flat (3.31), we expand all covariant derivatives into their conformal-flat components. Such a projection now takes the equivalent form

$$\begin{aligned}
0 &= \tfrac{1}{2}g_{ij}\Omega^{-2}\nabla_a\nabla^a\Delta + \Omega^{-2}\nabla_a\nabla^a\Delta_{ij} + \tfrac{1}{3}\Delta g_{ij}\Omega^{-3}\nabla_a\nabla^a\Omega + 2g_{ij}\Omega^{-3}\nabla_a\Delta\nabla^a\Omega \\
&\quad -3\Omega^{-3}\nabla_a\Delta_{ij}\nabla^a\Omega + \Delta_{ij}\Omega^{-4}\nabla_a\Omega\nabla^a\Omega - \tfrac{8}{3}\Delta g_{ij}\Omega^{-4}\nabla_a\Omega\nabla^a\Omega + 2\Delta_{ab}g_{ij}\Omega^{-4}\nabla^a\Omega\nabla^b\Omega \\
&\quad +2\Omega^{-3}\nabla^a\Omega\nabla_i\Delta_{ja} - 2\Omega^{-3}\nabla_a\Delta_j{}^a\nabla_i\Omega - 3\Delta_{ja}\Omega^{-4}\nabla^a\Omega\nabla_i\Omega - \tfrac{3}{2}\Omega^{-3}\nabla_i\Omega\nabla_j\Delta \\
&\quad +2\Omega^{-3}\nabla^a\Omega\nabla_j\Delta_{ia} - 2\Omega^{-3}\nabla_a\Delta_i{}^a\nabla_j\Omega - 3\Delta_{ia}\Omega^{-4}\nabla^a\Omega\nabla_j\Omega - \tfrac{3}{2}\Omega^{-3}\nabla_i\Delta\nabla_j\Omega \\
&\quad +7\Delta\Omega^{-4}\nabla_i\Omega\nabla_j\Omega + \tfrac{1}{2}\Omega^{-2}\nabla_j\nabla_i\Delta - \Delta\Omega^{-3}\nabla_j\nabla_i\Omega
\end{aligned} \tag{3.35}$$

where $\Delta = g^{ab}\Delta_{ab}$. To be clear, ∇ in this context denotes the flat space derivative and g_{ab} the flat space metric. Now inserting (3.31) into (3.35), we find all terms involving Ψ vanish and we are left with a lengthy transverse traceless relation involving only E_{ij} as

$$
\begin{aligned}
0 \quad = \quad & -\tfrac{2}{3}\Omega^{-3}\nabla_a\nabla^a\Omega\nabla_b\nabla^b E_{ij} + \tfrac{7}{3}\Omega^{-4}\nabla_a\Omega\nabla^a\Omega\nabla_b\nabla^b E_{ij} + \tfrac{2}{3}E_{ij}\Omega^{-4}\nabla_a\nabla^a\Omega\nabla_b\nabla^b\Omega \\
& +\tfrac{7}{3}\Omega^{-4}\nabla_a E_{ij}\nabla^a\Omega\nabla_b\nabla^b\Omega - \tfrac{20}{3}E_{ij}\Omega^{-5}\nabla_a\Omega\nabla^a\Omega\nabla_b\nabla^b\Omega - 2\Omega^{-3}\nabla^a\Omega\nabla_b\nabla^b\nabla_a E_{ij} \\
& -\tfrac{1}{3}\Omega^{-3}\nabla^a E_{ij}\nabla_b\nabla^b\nabla_a\Omega + 6E_{ij}\Omega^{-4}\nabla^a\Omega\nabla_b\nabla^b\nabla_a\Omega + \Omega^{-2}\nabla_b\nabla^b\nabla_a\nabla^a E_{ij} \\
& -\tfrac{2}{3}E_{ij}\Omega^{-3}\nabla_b\nabla^b\nabla_a\nabla^a\Omega - \tfrac{10}{3}\Omega^{-5}\nabla_a\Omega\nabla^a\Omega\nabla_b E_{ij}\nabla^b\Omega + \tfrac{52}{3}E_{ij}\Omega^{-6}\nabla_a\Omega\nabla^a\Omega\nabla_b\Omega\nabla^b\Omega \\
& -5\Omega^{-4}\nabla^a\Omega\nabla_b\nabla_a E_{ij}\nabla^b\Omega - \tfrac{56}{3}E_{ij}\Omega^{-5}\nabla^a\Omega\nabla_b\nabla_a\Omega\nabla^b\Omega + \tfrac{1}{3}\Omega^{-4}\nabla^a\Omega\nabla_b E_{ij}\nabla^b\nabla_a\Omega \\
& +2\Omega^{-3}\nabla_b\nabla_a E_{ij}\nabla^b\nabla^a\Omega + \tfrac{8}{3}E_{ij}\Omega^{-4}\nabla_b\nabla_a\Omega\nabla^b\nabla^a\Omega - 2\Omega^{-4}\nabla_a E_{jb}\nabla^a\Omega\nabla^b\nabla_i\Omega \\
& +2\Omega^{-4}\nabla^a\Omega\nabla_b E_{ja}\nabla^b\nabla_i\Omega - 2\Omega^{-4}\nabla_a E_{ib}\nabla^a\Omega\nabla^b\nabla_j\Omega + 2\Omega^{-4}\nabla^a\Omega\nabla_b E_{ia}\nabla^b\nabla_j\Omega \\
& -18E^{bc}g_{ij}\Omega^{-5}\nabla_a\Omega\nabla^a\Omega\nabla_c\nabla_b\Omega - 20E_a{}^c g_{ij}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_c\nabla_b\Omega \\
& +8E^{bc}g_{ij}\Omega^{-4}\nabla^a\Omega\nabla_c\nabla_b\nabla_a\Omega + 3g_{ij}\Omega^{-4}\nabla^a\Omega\nabla^b\Omega\nabla_c\nabla^c E_{ab} - g_{ij}\Omega^{-3}\nabla^b\nabla^a\Omega\nabla_c\nabla^c E_{ab} \\
& +3E^{ab}g_{ij}\Omega^{-4}\nabla_b\nabla_a\Omega\nabla_c\nabla^c\Omega - \tfrac{16}{3}E_{ab}g_{ij}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_c\nabla^c\Omega \\
& +2E_a{}^b g_{ij}\Omega^{-4}\nabla^a\Omega\nabla_c\nabla^c\nabla_b\Omega - E^{ab}g_{ij}\Omega^{-3}\nabla_c\nabla^c\nabla_b\nabla_a\Omega - 2g_{ij}\Omega^{-3}\nabla_c\nabla_b\nabla_a\Omega\nabla^c E^{ab} \\
& +\tfrac{92}{3}E_{bc}g_{ij}\Omega^{-6}\nabla_a\Omega\nabla^a\Omega\nabla^b\Omega\nabla^c\Omega - 12g_{ij}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_c E_{ab}\nabla^c\Omega \\
& +2E^{ab}g_{ij}\Omega^{-4}\nabla_c\nabla_b\Omega\nabla^c\nabla_a\Omega + 8g_{ij}\Omega^{-4}\nabla_a E_{bc}\nabla^a\Omega\nabla^c\nabla^b\Omega \\
& +4g_{ij}\Omega^{-4}\nabla^a\Omega\nabla_c E_{ab}\nabla^c\nabla^b\Omega - 8\Omega^{-4}\nabla^a\Omega\nabla^b\nabla_j\Omega\nabla_i E_{ab} - \tfrac{1}{3}\Omega^{-4}\nabla^a\Omega\nabla_b\nabla^b\Omega\nabla_i E_{ja} \\
& -\Omega^{-3}\nabla_b\nabla^b\nabla_a\Omega\nabla_i E_j{}^a - \tfrac{10}{3}\Omega^{-5}\nabla_a\Omega\nabla^a\Omega\nabla^b\Omega\nabla_i E_{jb} + 5\Omega^{-4}\nabla^a\Omega\nabla^b\nabla_a\Omega\nabla_i E_{jb} \\
& -\tfrac{16}{3}E_j{}^b\Omega^{-5}\nabla^a\Omega\nabla_b\nabla_a\Omega\nabla_i\Omega - \Omega^{-4}\nabla^a\Omega\nabla_b\nabla^b E_{ja}\nabla_i\Omega + 2E_{ja}\Omega^{-5}\nabla^a\Omega\nabla_b\nabla^b\Omega\nabla_i\Omega \\
& +\tfrac{4}{3}E_j{}^a\Omega^{-4}\nabla_b\nabla^b\nabla_a\Omega\nabla_i\Omega - 4E_{jb}\Omega^{-6}\nabla_a\Omega\nabla^a\Omega\nabla^b\Omega\nabla_i\Omega + 5\Omega^{-4}\nabla^a\Omega\nabla^b\Omega\nabla_i\nabla_b E_{ja} \\
& -2\Omega^{-3}\nabla^b\nabla^a\Omega\nabla_i\nabla_b E_{ja} + \tfrac{16}{3}E_{ja}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_i\nabla_b\Omega + \Omega^{-3}\nabla^a\Omega\nabla_i\nabla_b\nabla^b E_{ja} \\
& -\tfrac{4}{3}E_{ja}\Omega^{-4}\nabla^a\Omega\nabla_i\nabla_b\nabla^b\Omega - 8\Omega^{-4}\nabla^a\Omega\nabla^b\nabla_i\Omega\nabla_j E_{ab} + 18\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_i\Omega\nabla_j E_{ab} \\
& -10\Omega^{-4}\nabla^b\nabla^a\Omega\nabla_i\Omega\nabla_j E_{ab} + 3\Omega^{-3}\nabla_i\nabla_b\nabla_a\Omega\nabla_j E^{ab} - \tfrac{1}{3}\Omega^{-4}\nabla^a\Omega\nabla_b\nabla^b\Omega\nabla_j E_{ia} \\
& -\Omega^{-3}\nabla_b\nabla^b\nabla_a\Omega\nabla_j E_i{}^a - \tfrac{10}{3}\Omega^{-5}\nabla_a\Omega\nabla^a\Omega\nabla^b\Omega\nabla_j E_{ib} + 5\Omega^{-4}\nabla^a\Omega\nabla^b\nabla_a\Omega\nabla_j E_{ib} \\
& -\tfrac{16}{3}E_i{}^b\Omega^{-5}\nabla^a\Omega\nabla_b\nabla_a\Omega\nabla_j\Omega - \Omega^{-4}\nabla^a\Omega\nabla_b\nabla^b E_{ia}\nabla_j\Omega + 2E_{ia}\Omega^{-5}\nabla^a\Omega\nabla_b\nabla^b\Omega\nabla_j\Omega \\
& +\tfrac{4}{3}E_i{}^a\Omega^{-4}\nabla_b\nabla^b\nabla_a\Omega\nabla_j\Omega - 4E_{ib}\Omega^{-6}\nabla_a\Omega\nabla^a\Omega\nabla^b\Omega\nabla_j\Omega \\
& +18\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_i E_{ab}\nabla_j\Omega - 10\Omega^{-4}\nabla^b\nabla^a\Omega\nabla_i E_{ab}\nabla_j\Omega \\
& +54E^{ab}\Omega^{-5}\nabla_b\nabla_a\Omega\nabla_i\Omega\nabla_j\Omega - 84E_{ab}\Omega^{-6}\nabla^a\Omega\nabla^b\Omega\nabla_i\Omega\nabla_j\Omega \\
& +30E_a{}^b\Omega^{-5}\nabla^a\Omega\nabla_i\nabla_b\Omega\nabla_j\Omega - 12E^{ab}\Omega^{-4}\nabla_i\nabla_b\nabla_a\Omega\nabla_j\Omega + 5\Omega^{-4}\nabla^a\Omega\nabla^b\Omega\nabla_j\nabla_b E_{ia} \\
& -2\Omega^{-3}\nabla^b\nabla^a\Omega\nabla_j\nabla_b E_{ia} + \tfrac{16}{3}E_{ia}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_j\nabla_b\Omega + 30E_a{}^b\Omega^{-5}\nabla^a\Omega\nabla_i\Omega\nabla_j\nabla_b\Omega \\
& -6E^{ab}\Omega^{-4}\nabla_i\nabla_a\Omega\nabla_j\nabla_b\Omega + 3\Omega^{-3}\nabla_i E^{ab}\nabla_j\nabla_b\nabla_a\Omega - 12E^{ab}\Omega^{-4}\nabla_i\Omega\nabla_j\nabla_b\nabla_a\Omega \\
& +\Omega^{-3}\nabla^a\Omega\nabla_j\nabla_b\nabla^b E_{ia} - \tfrac{4}{3}E_{ia}\Omega^{-4}\nabla^a\Omega\nabla_j\nabla_b\nabla^b\Omega - 7\Omega^{-4}\nabla^a\Omega\nabla^b\Omega\nabla_j\nabla_i E_{ab} \\
& +3\Omega^{-3}\nabla^b\nabla^a\Omega\nabla_j\nabla_i E_{ab} - 9E^{ab}\Omega^{-4}\nabla_b\nabla_a\Omega\nabla_j\nabla_i\Omega + 12E_{ab}\Omega^{-5}\nabla^a\Omega\nabla^b\Omega\nabla_j\nabla_i\Omega \\
& -6E_a{}^b\Omega^{-4}\nabla^a\Omega\nabla_j\nabla_i\nabla_b\Omega + 3E^{ab}\Omega^{-3}\nabla_j\nabla_i\nabla_b\nabla_a\Omega.
\end{aligned} \tag{3.36}
$$

In forming (3.36), we noted that the transverse Δ_{ij} may be decomposed as

$$
\Delta_{ij} \quad = \quad \Delta_{ij}^{TT} + \Delta_{ij}^{TNT}. \tag{3.37}
$$

While (3.36) is proportional to Δ_{ij}^{TT}, since the trace of Δ_{ij} (i.e. (3.32)) includes the tensor component E_{ij}, the transverse traceless Δ_{ij}^{TNT} will necessarily have E_{ij} components and will not serve to isolate Ψ alone, unlike the covariant case.

Appendix A Conformal to Flat Maximal 3-Space

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{4}{(1+k\rho^2)^2}\left(d\rho^2+\rho^2 d\Omega^2\right) \\ &= \frac{dr^2}{1-kr^2}+r^2 d\Omega^2 \end{aligned} \tag{A.1}$$

The relevant transformations are:

$$\begin{aligned} \rho(r) &= \frac{r}{1+(1-kr^2)^{1/2}}, \qquad \Omega^2(r)=\left[1+\left(1-kr^2\right)^{1/2}\right] \\ r(\rho) &= \frac{2\rho}{1+k\rho^2}, \qquad \Omega^2(\rho)=\frac{4}{(1+k\rho^2)^2} \end{aligned} \tag{A.2}$$

Appendix B δG_{ij} Under Conformal Transformation

Although the Riemann tensor transforms the same under conformal transformation, viz.

$$\begin{aligned} R_{\lambda\mu\nu\kappa} \rightarrow\ & \Omega^2 R_{\lambda\mu\nu\kappa}+\Omega\left(-g_{\mu\nu}\nabla_\lambda\nabla_\kappa\Omega+g_{\lambda\nu}\nabla_\mu\nabla_\kappa\Omega+g_{\mu\kappa}\nabla_\nu\nabla_\lambda\Omega-g_{\lambda\kappa}\nabla_\mu\nabla_\nu\Omega\right) \\ & +2g_{\mu\nu}\nabla_\kappa\Omega\nabla_\lambda\Omega-2g_{\lambda\nu}\nabla_\kappa\Omega\nabla_\mu\Omega-2g_{\mu\kappa}\nabla_\lambda\Omega\nabla_\nu\Omega+2g_{\lambda\kappa}\nabla_\mu\Omega\nabla_\nu\Omega \\ & +(g_{\lambda\nu}g_{\mu\kappa}-g_{\lambda\kappa}g_{\mu\nu})\nabla^\rho\Omega\nabla_\rho\Omega \end{aligned} \tag{B.1}$$

its contractions do depend on the dimension under consideration. For $D=3$ $\mu,\nu=1,2,3$ the Ricci tensor and scalar transform as

$$\begin{aligned} R_{\mu\nu} &\rightarrow R_{\mu\nu}+g_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega+\Omega^{-1}\nabla_\mu\nabla_\nu\Omega-2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \\ R &\rightarrow \Omega^{-2}R+4\Omega^{-3}\nabla_\alpha\nabla^\alpha\Omega-2\Omega^{-4}\nabla_\alpha\Omega\nabla^\alpha\Omega \end{aligned} \tag{B.2}$$

and thus the Einstein tensor transforms as

$$G_{\mu\nu} \rightarrow G_{\mu\nu}+g_{\mu\nu}(\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega-\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega)+\Omega^{-1}\nabla_\mu\nabla_\nu\Omega-2\Omega^{-2}\nabla_\mu\Omega\nabla_\nu\Omega \tag{B.3}$$

$$\begin{aligned} \delta\Gamma^\lambda_{\mu\nu} &= \tfrac{1}{2}g^{\lambda\rho}\left[\nabla_\mu h_{\nu\rho}+\nabla_\nu h_{\mu\rho}-\nabla_\rho h_{\mu\nu}\right] \\ \nabla_\mu\nabla_\nu\Omega &= \partial_\mu\nabla_\nu\Omega-\Gamma^\lambda_{\mu\nu}\nabla_\lambda\Omega \\ \delta(\nabla_\mu\nabla_\nu\Omega) &= -\tfrac{1}{2}\nabla^\rho\Omega(\nabla_\mu h_{\rho\nu}+\nabla_\nu h_{\mu\rho}-\nabla_\rho h_{\mu\nu}) \end{aligned} \tag{B.4}$$

$$\delta G_{\mu\nu}\rightarrow\delta G_{\mu\nu}+\delta S_{\mu\nu} \tag{B.5}$$

$$\begin{aligned} \delta G_{\mu\nu} =\ & \tfrac{1}{2}\nabla_\alpha\nabla^\alpha h_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla^\alpha h+\tfrac{1}{2}g_{\mu\nu}\nabla_\beta\nabla_\alpha h^{\alpha\beta}-\tfrac{1}{2}\nabla_\mu\nabla_\alpha h_\nu{}^\alpha-\tfrac{1}{2}\nabla_\nu\nabla_\alpha h_\mu{}^\alpha+\tfrac{1}{2}\nabla_\nu\nabla_\mu h \\ \delta S_{\mu\nu} =\ & -h_{\mu\nu}\Omega^{-1}\nabla_\alpha\nabla^\alpha\Omega+\tfrac{1}{2}\Omega^{-1}\nabla_\alpha h_{\mu\nu}\nabla^\alpha\Omega-\tfrac{1}{2}g_{\mu\nu}\Omega^{-1}\nabla_\alpha h\nabla^\alpha\Omega+h_{\mu\nu}\Omega^{-2}\nabla_\alpha\Omega\nabla^\alpha\Omega \\ & +g_{\mu\nu}\Omega^{-1}\nabla^\alpha\Omega\nabla_\beta h_\alpha{}^\beta-g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha\Omega\nabla^\beta\Omega+g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta\nabla^\alpha\Omega \\ & -\tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\mu h_{\nu\alpha}-\tfrac{1}{2}\Omega^{-1}\nabla^\alpha\Omega\nabla_\nu h_{\mu\alpha}. \end{aligned} \tag{B.6}$$

Appendix C Maximal 3-Space Geometric Quantities

Geometry

$$ds^2 = g_{ij}dx^i dx^j = \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right): \tag{C.1}$$

$$R_{ijkl} = k(g_{jk}g_{il} - g_{ik}g_{jl}), \qquad R_{ij} = -2kg_{ij}, \qquad R = -6k \tag{C.2}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R = -2kg_{ij} - \frac{1}{2}g_{ij}(-6k) = kg_{ij} \\ g^{ij}G_{ij} &= R - \frac{3}{2}R = -\frac{1}{2}R = 3k \end{aligned} \tag{C.3}$$

$$[\nabla_i, \nabla_j]V_k = -V_l R^l{}_{jki} = -V_l(k(g_{jk}g^l{}_i - g^l{}_k g_{ij})) = k(g_{ij}V_k - g_{jk}V_i) \tag{C.4}$$

$$\begin{aligned} [\nabla_a\nabla^a, \nabla_i]E &= 2k\nabla_i E \\ [\nabla^j, \nabla_i]\nabla_j E &= 2k\nabla_i E \\ [\nabla_a\nabla^a, \nabla_i\nabla_j]E &= -2kg_{ij}\nabla_a\nabla^a E + 6k\nabla_i\nabla_j E \\ [\nabla_a\nabla^a, \nabla_i]E_j &= 2k(\nabla_i E_j + \nabla_j E_i) \\ [\nabla^i, \nabla_j]E_i &= 2kE_j \\ [\nabla^i, \nabla_a\nabla^a]E_{ij} &= 0 \end{aligned} \tag{C.5}$$

$$\begin{aligned} \Gamma^r_{rr} &= \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\ \Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \quad \text{with all others zero} \end{aligned} \tag{C.6}$$

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