Coordinate Transformations RW $k < 0$ v3

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We may form a 3-space of constant curvature by embedding within a flat 4-space, just as we may embed a 2-sphere or 2 dimensional hyperbola (or also a flat plane) within 3 dimensional space. Constraining to a space of constant curvature, we have

$$\mathbf{x}^2 + z^2 = C^2. \tag{1}$$

Here C^2 represents the degree and sign of curvature, with dimension of length $C \sim [L]$. For C^2 positive, we have a bound 3-sphere, while for $C^2 = 0$, we have unbound Euclidean geometry, and for $C^2 < 0$ we have an unbound hyperbolic geometry. Constructing the flat 4-space line element,

$$ds^2 = d\mathbf{x}^2 + dz^2. \tag{2}$$

Taking the differential of (1) allows us to relate dz to the three space variables $\mathbf{x}$ via

$$dz^2 = \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{C^2 - \mathbf{x}^2} \tag{3}$$

Substituting into the line element we have

$$ds^2 = d\mathbf{x}^2 + \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{C^2 - \mathbf{x}^2} \tag{4}$$

Adopting polar coordinates, this becomes

$$ds^2 = \frac{dr^2}{1 - r^2/C^2} + r^2 d\Omega^2 \tag{5}$$

With the above general form for a maximally symmetric 3-space with constant curvature, we may form the invariant spacetime interval as

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - r^2/C^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{6}$$

where $a(t)$ is an arbitrary function of time to be set by dynamics. Worth noting is that if we rescale $r' = r/|C|$, radial distances will be dimensionless and $a_{rescaled}(t) = a(t)/|C|$ will have dimension of $[L]$. Such a rescaling is necessary for the metric convention in which $\frac{dr^2}{1-Kr^2}$ for $K \in [-1, 0, 1]$. However, cosmological convention utilizes a dimensionless $a(t)$, thus we leave in the form of r^2/C^2.

By a coordinate transformation upon t via

$$\tau = \int \frac{dt}{a(t)}, \tag{7}$$

we may express (6) in terms of conformal time τ as

$$ds^2 = a^2(\tau) \left(d\tau^2 - \frac{dr^2}{1 - r^2/C^2} + r^2 d\Omega^2 \right) \tag{8}$$

RW to Conformal to Flat Form

First Transformation

As the first step towards bringing the metric to conformal-flat form for $C^2 < 0$, we introduce curvature magnitude $L^2 = -C^2$ (an inherently positive quantity) and we make coordinate transformations

$$p = \frac{\tau}{L}, \qquad \sinh\chi = \frac{r}{L}, \tag{9}$$

which take the line element of (8) into

$$ds^2 = L^2 a^2(p)\left(dp^2 - d\chi^2 - \sinh^2\chi d\Omega^2\right). \tag{10}$$

In this form, all length dimension lies within L^2.

Second Transformation (Alternative)

To finally bring (10) to the flat form, we make coordinate substitutions

$$T = e^p \cosh\chi, \qquad R = e^p \sinh\chi. \tag{11}$$

It is convenient to introduce a somewhat 'light-like' coordinate defined by

$$X^2 \equiv T^2 - R^2. \tag{12}$$

The coordinate relation for the time coordinate $p(T,R)$ is in fact only a function of X^2, viz.

$$e^{2p} = X^2, \qquad p = \frac{1}{2}\ln(X^2). \tag{13}$$

For the radial coordinate $\chi(T,R)$ we have the relations

$$\sinh\chi = \frac{R}{X}, \qquad \cosh\chi = \frac{T}{X}. \tag{14}$$

Though not as useful, we may invert (14) to find $\chi(T,R)$ as

$$\chi = \ln\left(\frac{T+R}{X}\right) \tag{15}$$

To aid in determining the differentials, we note

$$dX = \frac{\partial X}{\partial T}dT + \frac{\partial X}{\partial R}dR = \frac{TdT - RdR}{X}. \tag{16}$$

We first determine dp:

$$dp = \frac{T}{X^2}dT - \frac{R}{X^2}dR. \tag{17}$$

To find $d\chi$, we differentiate $\sinh\chi$:

$$d(\sinh\chi) = \cosh\chi d\chi = \frac{dR}{X} - \frac{R}{X^3}(TdT - RdR) \tag{18}$$

$$\frac{T}{X}d\chi = \frac{dR}{X} - \frac{TR}{X^3}dT + \frac{R^2}{X^3}dR, \tag{19}$$

hence

$$d\chi = \frac{dR}{T} - \frac{R}{X^2}dT + \frac{R^2}{TX^2}dR. \tag{20}$$

After repeated usage of $X^2 = T^2 - R^2$, we find the coordinate relation between infinitesimals

$$dp^2 - d\chi^2 = \frac{1}{X^2}\left(dT^2 - dR^2\right). \tag{21}$$

Finally, with $\sinh^2\chi = \frac{R^2}{X^2}$, we may write the line element in these new coordinates:

$$ds^2 = L^2\frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right) \tag{22}$$

Conformal Flat to RW Coordinates

Conformal Factor

We note that the conformal factor in the flat T, R coordinates is only a function of $X^2 = T^2 - R^2$. The factor is simply

$$\Omega(X)^2 = L^2 \frac{a^2(X)}{X^2} \tag{23}$$

where

$$a(X) = a\left(\frac{1}{2}\ln(X^2)\right). \tag{24}$$

The relation of the conformal factor to the p, χ geometry is simple,

$$\Omega^2(X) \equiv \Omega^2(p,\chi) = L^2 a^2(p) e^{-2p}. \tag{25}$$

Interestingly, it is a function entirely of time coordinate p. We may bring this to the comoving RW form by successive transformations

$$p = \frac{\tau}{L}, \qquad \tau = \int \frac{a(t)}{dt}, \tag{26}$$

in which the conformal factor becomes

$$\Omega^2(X) \equiv \Omega^2(t) = L^2 a^2(t) \exp\left[-\frac{2}{L}\int \frac{dt}{a(t)}\right] \tag{27}$$

Two Step Transformation

From the relations

$$T = e^p \cosh\chi, \qquad R = e^p \sinh\chi \tag{28}$$

and

$$p = \frac{\tau}{L}, \qquad \sinh\chi = \frac{r}{L} \tag{29}$$

we see that we could enact a coordinate transformation from conformal time (τ) RW geometry

$$ds^2 = a^2(\tau)\left(d\tau^2 - \frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2\right) \tag{30}$$

to conformal to flat (polar) geometry

$$ds^2 = L^2 \frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right) \tag{31}$$

via the effective transformation

$$T = \exp\left(\frac{\tau}{L}\right)\left(1 + \left(\frac{r}{L}\right)^2\right)^{1/2}, \qquad R = \exp\left(\frac{\tau}{L}\right)\frac{r}{L}, \qquad X^2 \equiv T^2 - R^2 = \exp\left(\frac{2\tau}{L}\right) \tag{32}$$

One Step Transformation

Lastly, we may substitute the transformation of τ viz

$$\tau = \int \frac{dt}{a(t)}, \tag{33}$$

to finally bring us to comoving coordinates. That is, via coordinate transformation

$$T = \exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right)\left(1+\left(\frac{r}{L}\right)^2\right), \qquad R = \exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right)\frac{r}{L}, \qquad X^2 \equiv T^2 - R^2 = \exp\left(\frac{2}{L}\int \frac{dt}{a(t)}\right) \tag{34}$$

we may transform from comoving coordinates

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1+r^2/L^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{35}$$

to conformal flat (polar) coordinates

$$ds^2 = L^2\frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right). \tag{36}$$

When $a(t)$ is specified apriori via a dynamics, exponential factors will simplify, especially for a τ which behaves logarithmically. For example, in the early universe radiation era, we have determined τ as

$$\tau = L\int_0^t \frac{dt}{(d^2+t^2)^{1/2}} = L\ \mathrm{arcsinh}\left(\frac{t}{d}\right). \tag{37}$$

This is equivalent to

$$\tau = L\ln\left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right) \tag{38}$$

in which our exponential calculates to

$$\exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right) = \frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}. \tag{39}$$

In the (conformal) early universe then, the conformal factor $\Omega(X)$ goes as

$$\Omega^2(X) = L^2 a^2(t)\exp\left[-\frac{2}{L}\int \frac{dt}{a(t)}\right] \tag{40}$$

$$= (d^2+t^2)\left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)^{-2} \tag{41}$$

The flat space coordinate transformations T and R then are specified as

$$T = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\left(1+\left(\frac{r}{L}\right)^2\right)^{1/2}, \qquad R = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\frac{r}{L} \tag{42}$$

$$X^2 \equiv T^2 - R^2 = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)^2 \tag{43}$$

$$a^2(X) = \frac{d^2}{L^2}\frac{(X^2+1)^2}{4X^2} \tag{44}$$

$$\Omega^2(X) = L^2\frac{a^2(X)}{X^2} = \left[\frac{d}{2}\left(1+\frac{1}{X^2}\right)\right]^2 \tag{45}$$

$$\Omega(X) = \frac{d}{2}(1+X^{-2}) \tag{46}$$

Cartesian to Polar

Transformation Matrices

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{48}$$

Time-Time

$$K'_{00} = K_{00} \tag{49}$$

Time-Space

$$K'_{0i} = \frac{\partial x^j}{\partial x'^i} K_{0j} \tag{50}$$

$$\begin{pmatrix} K'_{01} \\ K'_{02} \\ K'_{03} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{01} \\ K_{02} \\ K_{03} \end{pmatrix} \tag{51}$$

$$K'_{01} = K_{01}\sin(\theta)\cos(\phi) + K_{02}\sin(\theta)\sin(\phi) + K_{03}\cos(\theta) \tag{52}$$

$$K'_{02} = K_{01}r\cos(\theta)\cos(\phi) + K_{02}r\cos(\theta)\sin(\phi) - K_{03}r\sin(\theta) \tag{53}$$

$$K'_{03} = -K_{01}r\sin(\theta)\sin(\phi) + K_{02}r\sin(\theta)\cos(\phi) \tag{54}$$

Space-Space

$$K'_{ij} = \frac{\partial x^k}{\partial x'^i} K_{kl} \frac{\partial x^l}{\partial x'^j} \tag{55}$$

$$\begin{pmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{21} & K'_{22} & K'_{23} \\ K'_{31} & K'_{32} & K'_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}^T \tag{56}$$

$$\begin{aligned} K'_{11} = {} & K_{11}\sin^2(\theta)\cos^2(\phi) + K_{12}\sin^2(\theta)\sin(2\phi) + K_{13}\sin(2\theta)\cos(\phi) + K_{22}\sin^2(\theta)\sin^2(\phi) \\ & + K_{23}\sin(2\theta)\sin(\phi) + K_{33}\cos^2(\theta) \end{aligned} \tag{57}$$

$$\begin{aligned} K'_{22} = {} & K_{11}r^2\cos^2(\theta)\cos^2(\phi) + K_{12}r^2\cos^2(\theta)\sin(2\phi) - K_{13}r^2\sin(2\theta)\cos(\phi) + K_{22}r^2\cos^2(\theta)\sin^2(\phi) \\ & - K_{23}r^2\sin(2\theta)\sin(\phi) + K_{33}r^2\sin^2(\theta) \end{aligned} \tag{58}$$

$$K'_{33} = K_{11}r^2\sin^2(\theta)\sin^2(\phi) - 2K_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{22}r^2\sin^2(\theta)\cos^2(\phi) \tag{59}$$

$$K'_{12} = K_{11} r \sin(\theta)\cos(\theta)\cos^2(\phi) + K_{12} r \sin(\theta)\cos(\theta)\sin(2\phi) + K_{13} r \cos(2\theta)\cos(\phi) + K_{22} r \sin(\theta)\cos(\theta)\sin^2(\phi) + K_{23} r \cos(2\theta)\sin(\phi) - K_{33} r \sin(\theta)\cos(\theta) \tag{60}$$

$$K'_{13} = -K_{11} r \sin^2(\theta)\sin(\phi)\cos(\phi) + K_{12} r \sin^2(\theta)\cos(2\phi) - K_{13} r \sin(\theta)\cos(\theta)\sin(\phi) + K_{22} r \sin^2(\theta)\sin(\phi)\cos(\phi) + K_{23} r \sin(\theta)\cos(\theta)\cos(\phi) \tag{61}$$

$$K'_{23} = -K_{11} r^2 \sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + K_{12} r^2 \sin(\theta)\cos(\theta)\cos(2\phi) + K_{13} r^2 \sin^2(\theta)\sin(\phi) + K_{22} r^2 \sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) - K_{23} r^2 \sin^2(\theta)\cos(\phi) \tag{62}$$

Early Universe Setup

Given the geometry

$$ds^2 = (g_{\mu\nu} + K_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu, \tag{63}$$

upon imposing the conformal gauge condition $\nabla_\nu K^{\mu\nu} - \frac{1}{2}K^{\mu\nu} g^{\alpha\beta}_{(0)} \partial_\nu g^{(0)}_{\alpha\beta} = 0$, solutions to the first order source free Bach tensor $\delta W_{\mu\nu} = 0$ are found to obey

$$\frac{1}{2}\Omega^{-2}\Box^2 k_{\mu\nu} = 0 \tag{64}$$

After performing residual gauge transformations to eliminate gauge degrees of freedom, the general momentum eigenstate solution to (64) for a given k-mode is

$$k_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \begin{pmatrix} 0 & B_{01} & B_{02} & 0 \\ B_{01} & B_{11} & B_{12} & 0 \\ B_{02} & B_{12} & -B_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} n_\alpha x^\alpha e^{ikx} \tag{65}$$

with timelike $n_\alpha = (1, 0, 0, 0)$. The full solution for $K_{\mu\nu}$ is then given as

$$K_{\mu\nu} = \Omega^2 k_{\mu\nu}. \tag{66}$$

The $k < 0$ R.W. line element is given in comoving coordinates as

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{67}$$

where $k = -1/L^2$ (with $k < 0$). By coordinate transformation, the hyperbolic R.W. background geometry may be expressed in the form of $g^{(0)}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$, with the general conformal factor Ω having time and spatial dependence in the Minkowski coordinates.

Within the early universe radiation era, the perfect fluid energy momentum tensor obeys $\rho = 3p$, $\rho = A/a^4(t)$, $A > 0$, with $a(t)$ following the evolution equation

$$\begin{aligned} \dot{a}^2 - \frac{1}{L^2} &= \alpha a^2 - \frac{2A}{S_0^2 a^2} \\ &= -2\frac{a^2}{S_0^2}\left(\lambda_S S_0^4 + \frac{A}{a^4}\right) \end{aligned} \tag{68}$$

With the radiation dominating over the cosmological constant in the early universe (since $a(t)$ is small), i.e.

$$\frac{A}{a^4} \gg \lambda_S S_0^4, \tag{69}$$

the evolution equation can then be brought to the form

$$L^2\dot{a}^2 = 1 - \frac{d^2}{L^2}\left(\frac{1}{a^2}\right), \tag{70}$$

in which the solution $a(t)$ is

$$a^2(t) = \frac{1}{L^2}(d^2 + t^2) \tag{71}$$

where we have defined

$$d^2 \equiv \frac{2AL^4}{S_0^2}. \tag{72}$$

(With $A \sim [L]^{-4}$ and $S_0 \sim [L]^{-1}$ fixed early on, the relevant quantities to compare in the radiation dominated era should be the dimensionless $a(t)$ and λ_S).

Notation

From the original form of the scale factor

$$a^2(t) = \frac{2AL^2}{S_0^2} + \frac{t^2}{L^2} \tag{73}$$

we see that for setting up a definition for large t, we should take

$$\frac{t^2}{L^2} \gg \frac{2AL^2}{S_0^2}. \tag{74}$$

This is equivalent to requiring $t \gg d$. If the scale behaves such that $2AL^2/S_0^2 \ll 1$, then $t \gg d$ does not necessarily imply $t \gg L$. Noting in addition the R.W. comoving geometry distance r/L, we introduce two scales of comparison

$$u \equiv \frac{t}{d}, \qquad v \equiv \frac{r}{L}. \tag{75}$$

Thus we define large t behavior as taking $u \gg 1$, holding v finite.

In terms of u and v, the scale factor takes the form

$$a^2(u) = \frac{d^2}{L^2}(1 + u^2) \tag{76}$$

comoving R.W. metric takes the form

$$\begin{aligned} ds^2 &= dt^2 - a(t)^2\left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &= d^2\left[du^2 - (1 + u^2)\left(\frac{dv^2}{1 + v^2} + v^2 d\Omega^2\right)\right] \end{aligned} \tag{77}$$

Coordinate Transformations

Cartesian to Polar

In going from the geometry of

$$ds^2 = \Omega^2(\eta_{\mu\nu} + k_{\mu\nu})dx^\mu dx^\nu \tag{78}$$

to

$$ds^2 = \Omega^2(dt^2 - dr^2 - r^2 d\Omega^2 + k^{(P)}_{\mu\nu} dx^\mu dx^\nu), \tag{79}$$

we must perform the appropriate coordinate transformation (given in the Appendix). Denoting the polar coordinate system as $x^{(P)}$, we find, after imposing the transverse and residual relations, the following:

$$\begin{aligned}
k^{(P)}_{00} &= 0 \\
k^{(P)}_{01} &= k_{01}\sin(\theta)\cos(\phi) + k_{02}\sin(\theta)\sin(\phi) \\
k^{(P)}_{02} &= k_{01}r\cos(\theta)\cos(\phi) + k_{02}r\cos(\theta)\sin(\phi) \\
k^{(P)}_{03} &= -k_{01}r\sin(\theta)\sin(\phi) + k_{02}r\sin(\theta)\cos(\phi) \\
k^{(P)}_{11} &= k_{11}\sin^2(\theta)\cos(2\phi) + k_{12}\sin^2(\theta)\sin(2\phi) \\
k^{(P)}_{22} &= k_{11}r^2\cos^2(\theta)\cos(2\phi) + k_{12}r^2\cos^2(\theta)\sin(2\phi) \\
k^{(P)}_{33} &= -k_{11}r^2\sin^2(\theta)\cos(2\phi) - 2k_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) \\
k^{(P)}_{12} &= \frac{1}{2}k_{11}r\sin(2\theta)\cos(2\phi) + k_{12}r\sin(\theta)\cos(\theta)\sin(2\phi) \\
k^{(P)}_{13} &= -2k_{11}r\sin^2(\theta)\sin(\phi)\cos(\phi) + k_{12}r\sin^2(\theta)\cos(2\phi) \\
k^{(P)}_{23} &= -2k_{11}r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + k_{12}r^2\sin(\theta)\cos(\theta)\cos(2\phi)
\end{aligned} \tag{80}$$

Since the $\Box^2 k_{\mu\nu} = 0$ is only valid in a conformal to Minkowski background, upon transforming the solution for $k_{\mu\nu}$ to polar coordinates, we must account for the factors of $R(t,r)$ and $r'(t,r)$ in regards to the asymptotic time behavior. As a rule, every angular index gets a power of r.

Original Coordinates

Performing coordinate transformations

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \qquad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{81}$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(p', r')(dp'^2 - dr'^2 - r'^2 d\Omega^2) \tag{82}$$

with conformal factor

$$\Omega^2(p', r') = \frac{4L^2 a^2}{(1-(p'+r')^2)(1-(p'+r')^2)} = d^2(1+u^2)\left[(1+u^2)^{1/2} + (1+v^2)^{1/2}\right]^2. \tag{83}$$

We will soon make use of the coordinate relations

$$\begin{aligned}
\frac{\partial p'}{\partial t} &= \frac{1}{d}\frac{\partial p'}{\partial u} = \left(\frac{1}{d}\right)\frac{1 + (1+u^2)^{1/2}(1+v^2)^{1/2}}{(1+u^2)^{1/2}((1+u^2)^{1/2} + (1+v^2)^{1/2})^2} \\
\frac{\partial p'}{\partial r} &= \frac{1}{L}\frac{\partial p'}{\partial v} = -\left(\frac{1}{L}\right)\frac{uv}{(1+v^2)^{1/2}((1+u^2)^{1/2} + (1+v^2)^{1/2})^2} \\
\frac{\partial r'}{\partial t} &= \frac{1}{d}\frac{\partial r'}{\partial u} = -\left(\frac{1}{d}\right)\frac{uv}{(1+u^2)^{1/2}((1+u^2)^{1/2} + (1+v^2)^{1/2})^2} \\
\frac{\partial r'}{\partial r} &= \frac{1}{L}\frac{\partial r'}{\partial v} = \left(\frac{1}{L}\right)\frac{1 + (1+u^2)^{1/2}(1+v^2)^{1/2}}{(1+v^2)^{1/2}((1+u^2)^{1/2} + (1+v^2)^{1/2})^2}
\end{aligned} \tag{84}$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial p'}{\partial t}\frac{\partial r'}{\partial t}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial p'}{\partial t}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial p'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial r'}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial p'}{\partial t}k_{02}^{(P)} + \frac{\partial r'}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial p'}{\partial t}k_{03}^{(P)} + \frac{\partial r'}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial p'}{\partial r}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial p'}{\partial r}k_{02}^{(P)} + \frac{\partial r'}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial p'}{\partial r}k_{03}^{(P)} + \frac{\partial r'}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{85}$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the z' axis is

$$K_{\mu\nu} = \Omega^2(p', r')\left[C_{\mu\nu}p'\cos(k(r'\cos\theta - p')) + D_{\mu\nu}\sin(k(r'\cos\theta - p'))\right] \tag{86}$$

where $k_\mu = (-k, 0, 0, k)$, $z' = r'\cos\theta$, $C_{\mu\nu} = B_{\mu\nu} + B^*_{\mu\nu}$, and $D_{\mu\nu} = i(B_{\mu\nu} - B^*_{\mu\nu})$.

Up to leading order in u, we have:

$$p' \sim 1, \qquad r' \sim \frac{1}{u}, \qquad \Omega^2(p', r') \sim d^2u^4. \tag{87}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{d}\left(\frac{1}{u^2}\right), \qquad \frac{\partial p'}{\partial r} \sim -\frac{1}{L}\left(\frac{1}{u}\right), \qquad \frac{\partial r'}{\partial t} \sim -\frac{1}{d}\left(\frac{1}{u^2}\right), \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{L}\left(\frac{1}{u}\right). \tag{88}$$

For the plane wave $\sin(k(z' - p'))$, the phase equates to

$$z' - p' = \frac{v\cos\theta - u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}. \tag{89}$$

For $u \to \infty$, the above converges and has asymptotic expansion

$$z' - p' \approx -1 + \frac{1}{u}(v + (1+v^2)^{1/2})\cos\theta - \frac{1}{u^2}\left(\frac{1}{2} + v^2 + v(1+v^2)^{1/2}\cos\theta\right) + O\left(\frac{1}{u^3}\right). \tag{90}$$

Hence, to second leading order, the (p', z') plane wave behave asymptotically as

$$\begin{aligned}
\sin(k(z' - p')) &\approx -\sin(k) + \frac{k\cos(k)}{u}(v\cos\theta + (1+v^2)^{1/2}) \\
\cos(k(z' - p')) &\approx \cos(k) + \frac{k\sin(k)}{u}(v\cos\theta + (1+v^2)^{1/2})
\end{aligned} \tag{91}$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim r'$, the leading large u behavior of $B_{\mu\nu}^{(cm)}$ is calculated as:

$$B_{00}^{(cm)} \sim \frac{1}{d^2}\left(\frac{1}{u^4}\right), \qquad B_{01}^{(cm)} \sim \frac{1}{dL}\left(\frac{1}{u^3}\right), \qquad B_{02}^{(cm)} \sim \frac{1}{d}\left(\frac{1}{u^3}\right), \qquad B_{03}^{(cm)} \sim \frac{1}{d}\left(\frac{1}{u^3}\right)$$
$$B_{11}^{(cm)} \sim \frac{1}{L^2}\left(\frac{1}{u^2}\right), \quad B_{22}^{(cm)} \sim \frac{1}{u^2}, \quad B_{33}^{(cm)} \sim \frac{1}{u^2}, \quad B_{12}^{(cm)} \sim \frac{1}{L}\left(\frac{1}{u^2}\right), \quad B_{13}^{(cm)} \sim \frac{1}{L}\left(\frac{1}{u^2}\right), \quad B_{23}^{(cm)} \sim \frac{1}{u^2} \tag{92}$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K_{\mu\nu}^{(cm)}$, which follows

$$K_{\mu\nu}^{(cm)} = \Omega^2(p', r') B_{\mu\nu}^{(cm)} r' \sin(k(z' - p')) \sim d^2 u^4 B_{\mu\nu}^{(cm)}. \tag{93}$$

$$\begin{aligned}
K_{00}^{(cm)} &\sim 1 \\
K_{01}^{(cm)} &\sim \frac{d}{L} u \\
K_{02}^{(cm)} &\sim \frac{1}{d} u \\
K_{03}^{(cm)} &\sim \frac{1}{d} u \\
K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^2) \\
K_{22}^{(cm)} &\sim d^2(u^2) \\
K_{33}^{(cm)} &\sim d^2(u^2) \\
K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^2) \\
K_{23}^{(cm)} &\sim d^2(u^2)
\end{aligned} \tag{94}$$

New Coordinates

Performing coordinate transformations

$$T = \left[u + (1+u^2)^{1/2}\right](1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2}\right] v, \qquad X^2 = T^2 - R^2, \tag{95}$$

transforms the comoving R.W. line element to the conformal to flat (polar)

$$ds^2 = \Omega^2(T, R)(dT^2 - dR^2 - R^2 d\Omega^2) \tag{96}$$

with conformal factor

$$\Omega^2(T, R) = \frac{L^2 a^2}{T^2 - R^2} = d^2(1+u^2)((1+u^2)^{1/2} - u)^2. \tag{97}$$

We will soon make use of the coordinate relations

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{1}{d}\frac{\partial T}{\partial u} = \left(\frac{1}{d}\right)\frac{(u + (1+u^2)^{1/2})(1+v^2)^{1/2}}{(1+u^2)^{1/2}} \\
\frac{\partial T}{\partial r} &= \frac{1}{L}\frac{\partial T}{\partial v} = \left(\frac{1}{L}\right)\frac{(u + (1+u^2)^{1/2})v}{(1+v^2)^{1/2}} \\
\frac{\partial R}{\partial t} &= \frac{1}{d}\frac{\partial R}{\partial u} = \left(\frac{1}{d}\right)\frac{(u + (1+u^2)^{1/2})v}{(1+u^2)^{1/2}} \\
\frac{\partial R}{\partial r} &= \frac{1}{L}\frac{\partial R}{\partial v} = \left(\frac{1}{L}\right)(u + (1+u^2)^{1/2})
\end{aligned} \tag{98}$$

After transforming from Minkowski to polar, it remains to transform the $k_{\mu\nu}$ from polar to comoving coordinates. We note that angular coordinates are unaffected. In calculating the transformation (given in the appendix), we have

$$\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial T}{\partial t}\frac{\partial R}{\partial t}k_{01}^{(P)} + \left(\frac{\partial R}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial T}{\partial t}\frac{\partial R}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial T}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial T}{\partial t}k_{02}^{(P)} + \frac{\partial R}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial T}{\partial t}k_{03}^{(P)} + \frac{\partial R}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial T}{\partial r}\frac{\partial R}{\partial r}k_{01}^{(P)} + \left(\frac{\partial R}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial T}{\partial r}k_{02}^{(P)} + \frac{\partial R}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial T}{\partial r}k_{03}^{(P)} + \frac{\partial R}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{99}$$

Asymptotics

The leading order solution for $K_{\mu\nu}$ for a wave propagating along the Z axis is

$$K_{\mu\nu} = \Omega^2(T,R)\left[C_{\mu\nu}T\cos(k(R\cos\theta - T)) + D_{\mu\nu}\sin(k(R\cos\theta - T))\right] \tag{100}$$

where $k_\mu = (-k,0,0,k)$, $Z = R\cos\theta$, $C_{\mu\nu} = B_{\mu\nu} + B_{\mu\nu}^*$, and $D_{\mu\nu} = i(B_{\mu\nu} - B_{\mu\nu}^*)$.

Up to leading order in u, we have:

$$T \sim u, \qquad R \sim u, \qquad \Omega^2(T,R) \sim d^2 \tag{101}$$

$$\frac{\partial T}{\partial t} \sim \frac{1}{d}, \qquad \frac{\partial T}{\partial r} \sim \frac{u}{L}, \qquad \frac{\partial R}{\partial t} \sim \frac{1}{d}, \qquad \frac{\partial R}{\partial r} \sim \frac{u}{L} \tag{102}$$

For the plane wave $\sin(k(Z-T))$, the phase equates to

$$Z - T = \left[u + (1+u^2)^{1/2}\right]\left[v\cos\theta - (1+v^2)^{1/2}\right] \tag{103}$$

For $u \to \infty$, the above diverges and has asymptotic expansion

$$Z - T \approx 2u\left(v\cos\theta - (1+v^2)^{1/2}\right) + \frac{1}{2u}\left(v\cos\theta - (1+v^2)^{1/2}\right) + O\left(\frac{1}{u^3}\right) \tag{104}$$

Hence, in the (T,Z) coordinate system, plane waves remain at least periodic with asymptotic form

$$\begin{aligned}
\sin(k(Z-T)) &\approx \sin\left[2ku\left(v\cos\theta - (1+v^2)^{1/2}\right)\right] \\
\cos(k(Z-T)) &\approx \cos\left[2ku\left(v\cos\theta - (1+v^2)^{1/2}\right)\right]
\end{aligned} \tag{105}$$

For the tensor transformation behavior, recalling that each angular index goes as $\sim R$, the leading large u behavior of $B_{\mu\nu}^{(cm)}$ is calculated as:

$$\begin{aligned}
&B_{00}^{(cm)} \sim \frac{1}{d^2}, \qquad B_{01}^{(cm)} \sim \frac{1}{d^2}, \qquad B_{02}^{(cm)} \sim \frac{u}{d}, \qquad B_{03}^{(cm)} \sim \frac{u}{d}, \qquad B_{11}^{(cm)} \sim \frac{u^2}{L^2} \\
&B_{22}^{(cm)} \sim u^2, \qquad B_{33}^{(cm)} \sim u^2, \qquad B_{12}^{(cm)} \sim \frac{u^2}{L}, \qquad B_{13}^{(cm)} \sim \frac{u^2}{L}, \qquad B_{23}^{(cm)} \sim u^2
\end{aligned} \tag{106}$$

Finally, we calculate the leading $u = t/d$ behavior for the comoving $K_{\mu\nu}^{(cm)}$, which follows

$$K_{\mu\nu}^{(cm)} = \Omega^2(T,R)B_{\mu\nu}^{(cm)}T\sin(k(Z-T)) \sim d^2uB_{\mu\nu}^{(cm)}. \tag{107}$$

$$\begin{aligned}
K_{00}^{(cm)} &\sim u \\
K_{01}^{(cm)} &\sim u \\
K_{02}^{(cm)} &\sim d(u^2) \\
K_{03}^{(cm)} &\sim d(u^2) \\
K_{11}^{(cm)} &\sim \frac{d^2}{L^2}(u^3) \\
K_{22}^{(cm)} &\sim d^2(u^3) \\
K_{33}^{(cm)} &\sim d^2(u^3) \\
K_{12}^{(cm)} &\sim \frac{d^2}{L}(u^3) \\
K_{13}^{(cm)} &\sim \frac{d^2}{L}(u^3) \\
K_{23}^{(cm)} &\sim d^2(u^3)
\end{aligned} \tag{108}$$

Conformal Minkoski to Polar RW Comoving

In going from the geometry of

$$ds^2 = \Omega^2(dT^2 - dx^2 - dy^2 - dz^2) \tag{109}$$

to

$$ds^2 = \Omega^2(dT^2 - dR^2 - R^2d\Omega^2), \tag{110}$$

we utilize the Cartesian to polar conversions given in the Appendix. Denoting the polar coordinate system as $x^{(P)}$, we find, after imposing the transverse and residual relations, the following:

$$\begin{aligned}
k_{00}^{(P)} &= 0 \\
k_{01}^{(P)} &= k_{01}\sin(\theta)\cos(\phi) + k_{02}\sin(\theta)\sin(\phi) \\
k_{02}^{(P)} &= k_{01}r\cos(\theta)\cos(\phi) + k_{02}r\cos(\theta)\sin(\phi) \\
k_{03}^{(P)} &= -k_{01}r\sin(\theta)\sin(\phi) + k_{02}r\sin(\theta)\cos(\phi) \\
k_{11}^{(P)} &= k_{11}\sin^2(\theta)\cos(2\phi) + k_{12}\sin^2(\theta)\sin(2\phi) \\
k_{22}^{(P)} &= k_{11}r^2\cos^2(\theta)\cos(2\phi) + k_{12}r^2\cos^2(\theta)\sin(2\phi) \\
k_{33}^{(P)} &= -k_{11}r^2\sin^2(\theta)\cos(2\phi) - 2k_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) \\
k_{12}^{(P)} &= \frac{1}{2}k_{11}r\sin(2\theta)\cos(2\phi) + k_{12}r\sin(\theta)\cos(\theta)\sin(2\phi) \\
k_{13}^{(P)} &= -2k_{11}r\sin^2(\theta)\sin(\phi)\cos(\phi) + k_{12}r\sin^2(\theta)\cos(2\phi) \\
k_{23}^{(P)} &= -2k_{11}r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + k_{12}r^2\sin(\theta)\cos(\theta)\cos(2\phi)
\end{aligned} \tag{111}$$

$$K'_{\mu\nu}(t,r,\theta,\phi) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}k_{\alpha\beta}(T,R,\theta,\phi) \tag{112}$$

$$J_{\mu\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}, \qquad \text{where} \quad x(T, R, \theta, \phi) \quad x'(t, r, \theta, \phi) \tag{113}$$

$$J_{\mu\nu} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{114}$$

$$k^{(cm)}_{\mu\nu} = \frac{\partial x^{k}_{(P)}}{\partial x^{i}_{(cm)}} k^{(P)}_{kl} \frac{\partial x^{l}_{(P)}}{\partial x^{j}_{(cm)}} \tag{115}$$

$$\begin{pmatrix} k^{(cm)}_{00} & k^{(cm)}_{01} & k^{(cm)}_{02} & k^{(cm)}_{03} \\ k^{(cm)}_{10} & k^{(cm)}_{11} & k^{(cm)}_{12} & k^{(cm)}_{13} \\ k^{(cm)}_{20} & k^{(cm)}_{21} & k^{(cm)}_{22} & k^{(cm)}_{23} \\ k^{(cm)}_{30} & k^{(cm)}_{31} & k^{(cm)}_{32} & k^{(cm)}_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^{(P)}_{00} & k^{(P)}_{01} & k^{(P)}_{02} & k^{(P)}_{03} \\ k^{(P)}_{10} & k^{(P)}_{11} & k^{(P)}_{12} & k^{(P)}_{13} \\ k^{(P)}_{20} & k^{(P)}_{21} & k^{(P)}_{22} & k^{(P)}_{23} \\ k^{(P)}_{30} & k^{(P)}_{31} & k^{(P)}_{32} & k^{(P)}_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{T} \tag{116}$$

$$\begin{aligned}
k^{(cm)}_{00} &= 2\frac{\partial T}{\partial t}\frac{\partial R}{\partial t}k^{(P)}_{01} + \left(\frac{\partial R}{\partial t}\right)^{2} k^{(P)}_{11} \\
k^{(cm)}_{01} &= \frac{\partial T}{\partial t}\frac{\partial R}{\partial r}k^{(P)}_{01} + \frac{\partial R}{\partial t}\frac{\partial T}{\partial r}k^{(P)}_{01} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r}k^{(P)}_{11} \\
k^{(cm)}_{02} &= \frac{\partial T}{\partial t}k^{(P)}_{02} + \frac{\partial R}{\partial t}k^{(P)}_{12} \\
k^{(cm)}_{03} &= \frac{\partial T}{\partial t}k^{(P)}_{03} + \frac{\partial R}{\partial t}k^{(P)}_{13} \\
k^{(cm)}_{11} &= 2\frac{\partial T}{\partial r}\frac{\partial R}{\partial r}k^{(P)}_{01} + \left(\frac{\partial R}{\partial r}\right)^{2} k^{(P)}_{11} \\
k^{(cm)}_{22} &= k^{(P)}_{22} \\
k^{(cm)}_{33} &= k^{(P)}_{33} \\
k^{(cm)}_{12} &= \frac{\partial T}{\partial r}k^{(P)}_{02} + \frac{\partial R}{\partial r}k^{(P)}_{12} \\
k^{(cm)}_{13} &= \frac{\partial T}{\partial r}k^{(P)}_{03} + \frac{\partial R}{\partial r}k^{(P)}_{13} \\
k^{(cm)}_{23} &= k^{(P)}_{23}
\end{aligned} \tag{117}$$

$$T = \left(\frac{t}{d} + \sqrt{1 + \left(\frac{t}{d}\right)^{2}}\right)\sqrt{1 + \left(\frac{r}{L}\right)^{2}}, \qquad R = \left(\frac{t}{d} + \sqrt{1 + \left(\frac{t}{d}\right)^{2}}\right)\frac{r}{L} \tag{118}$$

$$\frac{\partial T}{\partial t} = \frac{1}{d}\left(1 + \frac{\frac{t}{d}}{\sqrt{1 + \left(\frac{t}{d}\right)^{2}}}\right)\sqrt{1 + \left(\frac{r}{L}\right)^{2}} \tag{119}$$

$$\frac{\partial R}{\partial t} = \frac{1}{d}\left(1 + \frac{\frac{t}{d}}{\sqrt{1 + \left(\frac{t}{d}\right)^{2}}}\right)\frac{r}{L} \tag{120}$$

$$\frac{\partial T}{\partial r} = \frac{1}{L}\left(\frac{t}{d} + \sqrt{1 + \frac{t^{2}}{d^{2}}}\right)\left(\frac{\frac{r}{L}}{\sqrt{1 + \left(\frac{r}{L}\right)^{2}}}\right) \tag{121}$$

$$\frac{\partial R}{\partial r} = \frac{1}{L}\left(\frac{t}{d} + \sqrt{1+\frac{t^2}{d^2}}\right) \tag{122}$$

For late times such that $t \gg d$, the large time behavior goes as:

$$T \sim \frac{t}{d}, \quad R \sim \frac{t}{d}, \quad \frac{\partial T}{\partial t} \sim \frac{1}{d}\left(\frac{t}{d}\right)^0, \quad \frac{\partial R}{\partial t} \sim \frac{1}{d}\left(\frac{t}{d}\right)^0, \quad \frac{\partial T}{\partial r} \sim \frac{1}{L}\left(\frac{t}{d}\right), \quad \frac{\partial R}{\partial r} \sim \frac{1}{L}\left(\frac{t}{d}\right) \tag{123}$$

We note that in converting from Cartesian to polar, there reside factors of R in the Jacobian of transformation. Thus, for $k^{(P)}_{\mu\nu}$ we have, to leading order

$$\begin{aligned}
&k^{(P)}_{00} = 0 \qquad k^{(P)}_{01} \sim \left(\frac{t}{d}\right) \qquad k^{(P)}_{02} \sim \left(\frac{t}{d}\right)^2 \qquad k^{(P)}_{03} \sim \left(\frac{t}{d}\right)^2 \\
&\qquad\qquad k^{(P)}_{11} \sim \left(\frac{t}{d}\right) \qquad k^{(P)}_{22} \sim \left(\frac{t}{d}\right)^3 \qquad k^{(P)}_{33} \sim \left(\frac{t}{d}\right)^3 \\
&\qquad\qquad k^{(P)}_{12} \sim \left(\frac{t}{d}\right)^2 \qquad k^{(P)}_{13} \sim \left(\frac{t}{d}\right)^2 \qquad k^{(P)}_{23} \sim \left(\frac{t}{d}\right)^3
\end{aligned} \tag{124}$$

Next, we use (82-83) to determine the late time behavior in comoving coordinates:

$$\begin{aligned}
&k^{(cm)}_{00} \sim \frac{1}{d^2}\left(\frac{t}{d}\right) \qquad k^{(cm)}_{01} \sim \frac{1}{Ld}\left(\frac{t}{d}\right)^2 \qquad k^{(cm)}_{02} \sim \frac{1}{d}\left(\frac{t}{d}\right)^2 \qquad k^{(cm)}_{03} \sim \frac{1}{d}\left(\frac{t}{d}\right)^2 \\
&\qquad\qquad k^{(cm)}_{11} \sim \frac{1}{L^2}\left(\frac{t}{d}\right)^3 \qquad k^{(cm)}_{22} \sim \left(\frac{t}{d}\right)^3 \qquad k^{(cm)}_{33} \sim \left(\frac{t}{d}\right)^3 \\
&\qquad\qquad k^{(cm)}_{12} \sim \frac{1}{L}\left(\frac{t}{d}\right)^3 \qquad k^{(cm)}_{13} \sim \frac{1}{L}\left(\frac{t}{d}\right)^3 \qquad k^{(cm)}_{23} \sim \left(\frac{t}{d}\right)^3.
\end{aligned} \tag{125}$$

Finally, with the conformal factor late time dependence behaving as

$$\Omega^2(X^2) = \frac{L^2 a^2(X^2)}{X^2} = \frac{d^2\left(1+\frac{t^2}{d^2}\right)}{\left(\frac{t}{d}+\sqrt{1+\left(\frac{t}{d}\right)^2}\right)^2} \sim d^2 \tag{126}$$

we construct the comoving $K^{(cm)}_{\mu\nu}$ as

$$K^{(cm)}_{\mu\nu} = \Omega^2 k^{(cm)}_{\mu\nu} \tag{127}$$

to thus have

$$\begin{aligned}
&K^{(cm)}_{00} \sim \left(\frac{t}{d}\right) \qquad K^{(cm)}_{01} \sim \frac{d}{L}\left(\frac{t}{d}\right)^2 \qquad K^{(cm)}_{02} \sim d\left(\frac{t}{d}\right)^2 \qquad K^{(cm)}_{03} \sim d\left(\frac{t}{d}\right)^2 \\
&\qquad\qquad K^{(cm)}_{11} \sim \frac{d^2}{L^2}\left(\frac{t}{d}\right)^3 \qquad K^{(cm)}_{22} \sim d^2\left(\frac{t}{d}\right)^3 \qquad K^{(cm)}_{33} \sim d^2\left(\frac{t}{d}\right)^3 \\
&\qquad\qquad K^{(cm)}_{12} \sim \frac{d^2}{L}\left(\frac{t}{d}\right)^3 \qquad K^{(cm)}_{13} \sim \frac{d^2}{L}\left(\frac{t}{d}\right)^3 \qquad K^{(cm)}_{23} \sim d^2\left(\frac{t}{d}\right)^3.
\end{aligned} \tag{128}$$

$$
\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial p'}{\partial t}\frac{\partial r'}{\partial t}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial p'}{\partial t}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial p'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial r'}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial p'}{\partial t}k_{02}^{(P)} + \frac{\partial r'}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial p'}{\partial t}k_{03}^{(P)} + \frac{\partial r'}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial p'}{\partial r}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial p'}{\partial r}k_{02}^{(P)} + \frac{\partial r'}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial p'}{\partial r}k_{03}^{(P)} + \frac{\partial r'}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned}
\tag{129}
$$

Original Transformation

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \qquad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{130}$$

The above transformation takes us from

$$ds^2 = \Omega^2(p', r')(dp'^2 - dr'^2 - r'^2 d\Omega^2) \tag{131}$$

to

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2 \right) \tag{132}$$

$$\Omega^2(p', r') \sim d^2 \frac{u^4}{4} = \frac{t^4}{4} \tag{133}$$

$$p' \sim u^0, \qquad r' \sim \frac{v}{u} \tag{134}$$

New Transformation

$$T = \left[u + (1+u^2)^{1/2} \right] (1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2} \right] v \tag{135}$$

The above transformation takes us from

$$ds^2 = \Omega^2(X^2)(dT^2 - dR^2 - R^2 d\Omega^2) \tag{136}$$

to

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2 \right). \tag{137}$$

$$\Omega^2(X^2) \sim \frac{d^2}{4} \tag{138}$$

$$T \sim 2u(1+v^2)^{1/2}, \qquad R \sim 2uv \tag{139}$$

Plane Waves

Original Coordinates

$$\sin(k(z' - p')) \tag{140}$$

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \qquad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{141}$$

$$z' = r' \cos\theta = \frac{v \cos\theta}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{142}$$

$$z' - p' = \frac{v \cos\theta - u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{143}$$

For $u \to \infty$, the above converges and has asymptotic expansion

$$z' - p' \approx -1 + \frac{1}{u}(v + (1+v^2)^{1/2}) \cos\theta - \frac{1}{u^2} \left(\frac{1}{2} + v^2 + v(1+v^2)^{1/2} \cos\theta \right) + O\left(\frac{1}{u^3} \right). \tag{144}$$

Hence, to second leading order, the (p', z') plane waves behave asymptotically as

$$\sin(k(z' - p')) \approx -\sin(k) + \cos(k) \left[\frac{1}{u}(v + (1+v^2)^{1/2}) \cos\theta \right] \tag{145}$$

New Coordinates

$$\sin(k(Z-T)) \tag{146}$$

$$T = \left[u + (1+u^2)^{1/2}\right](1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2}\right] v \tag{147}$$

$$Z = R\cos\theta = \left[u + (1+u^2)^{1/2}\right] v\cos\theta \tag{148}$$

$$Z - T = \left[u + (1+u^2)^{1/2}\right]\left[v\cos\theta - (1+v^2)^{1/2}\right] \tag{149}$$

For $u \to \infty$, the above diverges and has asymptotic expansion

$$Z - T \approx 2u\left(v\cos\theta - (1+v^2)^{1/2}\right) + \frac{1}{2u}\left(v\cos\theta - (1+v^2)^{1/2}\right) + O\left(\frac{1}{u^3}\right) \tag{150}$$

Hence, in the (T, Z) coordinate system, plane waves remain at least periodic with asymptotic form

$$\sin(k(Z-T)) \approx \sin\left[2ku\left(v\cos\theta - (1+v^2)^{1/2}\right)\right]. \tag{151}$$