Harmonic Flat Space Conformal Transformation

Harmonic condition:

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\mu{}_\mu = 0$$

$$\nabla_\mu h^\mu{}_\nu - \frac{1}{2}\nabla_\nu h^\mu{}_\mu = 0.$$

Conformal transformation:

$$\Omega^2 g_{\mu\nu} = \bar{g}_{\mu\nu}$$

$$\Omega^{-2} g^{\mu\nu} = \bar{g}^{\mu\nu}$$

$$h^\mu{}_\nu = g^{(0)}_{\rho\nu} h^{\mu\rho} = (\Omega^{-2}\bar{g}^{(0)}_{\rho\nu})(\Omega^2 \bar{h}^{\mu\rho}) = \bar{h}^\mu{}_\nu$$

The following will be useful within our gauge transformation:

$$\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} - \Omega^{-1}\left(\delta^\lambda_\nu \partial_\mu\Omega + \delta^\lambda_\mu \partial_\nu\Omega - n_{\mu\nu} n^{\lambda\rho}\partial_\rho\Omega\right)$$

$$\begin{aligned}
\nabla_\mu h^\mu{}_\nu - \frac{1}{2}\nabla_\nu h^\mu{}_\mu &= \partial_\mu h^\mu{}_\nu + \Gamma^\mu_{\mu\rho} h^\rho{}_\nu - \Gamma^\rho_{\mu\nu} h^\mu{}_\rho - \frac{1}{2}\partial_\nu h^\mu{}_\mu \\
&= \partial_\mu \bar{h}^\mu{}_\nu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu} h^\mu{}_\rho - \frac{1}{2}\partial_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega + \Omega^{-1}\bar{h}^\mu{}_\rho\left(\delta^\rho_\nu\partial_\mu\Omega + \delta^\rho_\mu\partial_\nu\Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha\Omega\right) \\
&= \bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega + \Omega^{-1}\bar{h}^\mu{}_\rho\left(\delta^\rho_\nu\partial_\mu\Omega + \delta^\rho_\mu\partial_\nu\Omega - \eta^{\rho\alpha}\eta_{\mu\nu}\partial_\alpha\Omega\right) \\
&= \bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu - 4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega + \Omega^{-1}\bar{h}^\mu{}_\mu\partial_\nu\Omega
\end{aligned}$$

In a conformal to flat space, we need to work in the gauge

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu = 4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\mu{}_\mu\partial_\nu\Omega$$

Perturbation of Ricci Tensor:

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda \equiv S_{\mu\nu}$$

$$\delta R_{\mu\nu} = \delta S_{\mu\nu}$$

Weinberg (10.9.3)

$$\begin{aligned}
\delta R_{\mu\nu} &= (\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \\
&= \frac{1}{2}g^{\lambda\rho}\left[(h_{\lambda\rho})_{;\mu;\nu} - (h_{\rho\mu})_{;\nu;\lambda} - (h_{\rho\nu})_{;\mu;\lambda} + (h_{\mu\nu})_{;\rho;\lambda}\right] \\
&= \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda\nabla_\nu h^\lambda{}_\mu - \nabla_\lambda\nabla_\mu h^\lambda{}_\nu + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right)
\end{aligned}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \nabla_\lambda\nabla_\nu h^\lambda{}_\mu - \nabla_\lambda\nabla_\mu h^\lambda{}_\nu + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right)$$

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu = 4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\mu{}_\mu\partial_\nu\Omega$$

Referring to Mannheim (35), we may use the covariant interchange identity to express the Ricci variation as

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \nabla_\nu\nabla_\lambda h^\lambda{}_\mu - \nabla_\mu\nabla_\lambda h^\lambda{}_\nu + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right).$$

Substituting our gauge choice for the middle two covariant derivative terms

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \nabla_\nu\nabla_\lambda h^\lambda{}_\mu - \nabla_\mu\nabla_\lambda h^\lambda{}_\nu + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \frac{1}{2}\nabla_\nu\nabla_\mu h^\lambda{}_\lambda - \frac{1}{2}\nabla_\mu\nabla_\nu h^\lambda{}_\lambda + \nabla_\lambda\nabla^\lambda h_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&\quad - \nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)\\
&= \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&\quad - \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)
\end{aligned}
$$

From here we would like to evaluate the Riemann tensor for a conformal to flat metric. From Weinberg (6.1.5) we have

$$
\begin{aligned}
\Omega^2 g_{\mu\nu} &= \bar{g}_{\mu\nu}\\
\Omega^{-2} g^{\mu\nu} &= \bar{g}^{\mu\nu}
\end{aligned}
$$

$$
R^\lambda{}_{\mu\nu\kappa} = \partial_\kappa\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}.
$$

We will need an expression for the Christoffel symbol:

$$
\Gamma^\lambda_{\mu\nu} = \Omega^{-1}\left(\delta^\lambda_\nu\partial_\mu\Omega + \delta^\lambda_\mu\partial_\nu\Omega - n_{\mu\nu}n^{\lambda\rho}\partial_\rho\Omega\right).
$$

Now form the Riemann tensor

$$
\begin{aligned}
R_{\lambda\mu\nu\kappa} &= g_{\lambda\rho}(\partial_\kappa\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\rho_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\rho_{\nu\eta})\\
&= \Omega\left(\eta_{\lambda\nu}\partial_\mu\partial_\kappa\Omega + \eta_{\kappa\mu}\partial_\nu\partial_\lambda\Omega - \eta_{\mu\nu}\partial_\lambda\partial_\kappa\Omega - \eta_{\kappa\lambda}\partial_\mu\partial_\nu\Omega\right) + \eta_{\mu\kappa}\eta_{\lambda\nu}\partial_\alpha\Omega\partial^\alpha\Omega - \eta_{\kappa\lambda}\eta_{\mu\nu}\partial_\alpha\Omega\partial^\alpha\Omega\\
&\quad + 2\eta_{\mu\nu}\partial_\kappa\Omega\partial_\lambda\Omega - 2\eta_{\lambda\nu}\partial_\kappa\Omega\partial_\mu\Omega - 2\eta_{\kappa\mu}\partial_\lambda\Omega\partial_\nu\Omega + 2\eta_{\kappa\lambda}\partial_\mu\Omega\partial_\nu\Omega
\end{aligned}
$$

$$
\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\
&\quad - \nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) &= 4\Omega^{-1}\left(\nabla_\nu\bar{h}^\rho{}_\mu\partial_\rho\Omega + \bar{h}^\rho{}_\mu\nabla_\nu\nabla_\rho\Omega - \Omega^{-1}\bar{h}^\rho{}_\mu\partial_\nu\Omega\partial_\rho\Omega\right)\\
&\quad - \Omega^{-1}\left(\partial_\nu\bar{h}^\lambda{}_\lambda\partial_\mu\Omega + \bar{h}^\lambda{}_\lambda\nabla_\nu\nabla_\mu\Omega - \Omega^{-1}\bar{h}^\lambda{}_\lambda\partial_\nu\Omega\partial_\mu\Omega\right)\\
&= 4\Omega^{-3}\eta^{\rho\kappa}\left(\nabla_\nu\bar{h}_{\kappa\mu}\partial_\rho\Omega + \bar{h}_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega - \Omega^{-1}\bar{h}_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega\right)\\
&\quad - \Omega^{-3}\eta^{\lambda\kappa}\left(\partial_\nu\bar{h}_{\kappa\lambda}\partial_\mu\Omega + \bar{h}_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega - \Omega^{-1}\bar{h}_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right)\\
&= \Omega^{-3}\left(4\eta^{\rho\kappa}\nabla_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\nabla_\nu\nabla_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\nabla_\nu\nabla_\mu\Omega\right)\\
&\quad \Omega^{-4}\left(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu h_{\kappa\mu} &= \partial_\nu h_{\kappa\mu} + \Omega^{-1}\left(\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega + \eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega - h_{\mu\nu}\partial_\kappa\Omega - h_{\kappa\nu}\partial_\mu\Omega - 2h_{\kappa\mu}\partial_\nu\Omega\right)\\
\nabla_\nu\nabla_\rho\Omega &= \partial_\rho\partial_\nu\Omega + \Omega^{-1}(\eta^{\alpha\beta}\eta_{\nu\rho}\partial_\alpha\Omega\partial_\beta\Omega - 2\partial_\nu\Omega\partial_\rho\Omega)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}\bar{h}^\lambda{}_\lambda\partial_\mu\Omega\right) &=\\
&\Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right)\\
&+ \Omega^{-4}(-4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\kappa\nu}h_{\mu\alpha}\partial_\beta\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\mu\nu}\partial_\kappa\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\
&- 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\nu\rho}h_{\kappa\mu}\partial_\alpha\Omega\partial_\beta\Omega - 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega \\
&- \eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega + 2\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Omega^{-3}\left(4\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 4\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right) \\
&+ \Omega^{-4}(3\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 4\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 4\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega \\
&- 16\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega)
\end{aligned}
$$

$$\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)$$
$$- \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)$$

$$\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} = \Omega^{-2}\left(\frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu}\right)$$
$$+ \Omega^{-3}\Big(- \eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega - \eta^{\alpha\gamma}\partial_\alpha h_{\mu\nu}\partial_\gamma\Omega + \eta^{\alpha\eta}\partial_\mu h_{\nu\alpha}\partial_\eta\Omega - \eta^{\alpha\beta}\partial_\beta h_{\nu\alpha}\partial_\mu\Omega$$
$$+ \eta^{\alpha\lambda}\partial_\nu h_{\mu\alpha}\partial_\lambda\Omega - \eta^{\alpha\beta}\partial_\beta h_{\mu\alpha}\partial_\nu\Omega\Big)$$
$$+ \Omega^{-4}\Big(\eta^{\alpha\eta}\eta^{\beta\rho}\eta_{\mu\nu}h_{\alpha\beta}\partial_\eta\Omega\partial_\rho\Omega - 2\eta^{\alpha\kappa}h_{\nu\alpha}\partial_\kappa\Omega\partial_\mu\Omega + \eta^{\alpha\beta}h_{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega - 2\eta^{\alpha\rho}h_{\mu\alpha}\partial_\nu\Omega\partial_\rho\Omega\Big)$$

$$\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) =$$
$$\Omega^{-3}\Big(\eta^{\alpha\beta}h_{\mu\nu}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\partial_\beta\partial_\mu\Omega + 2\eta^{\alpha\beta}h_{\mu\alpha}\partial_\beta\partial_\nu\Omega$$
$$- \eta^{\alpha\beta}\eta^{\eta\gamma}\eta_{\mu\nu}h_{\alpha\gamma}\partial_\eta\partial_\beta\Omega - \eta^{\alpha\beta}h_{\alpha\beta}\partial_\nu\partial_\mu\Omega\Big)$$
$$+ \Omega^{-4}\Big(- \eta^{\alpha\eta}\eta^{\gamma\beta}\eta_{\mu\nu}h_{\beta\gamma}\partial_\alpha\Omega\partial_\eta\Omega + 2\eta^{\alpha\kappa}h_{\mu\nu}\partial_\alpha\Omega\partial_\kappa\Omega$$
$$- 4\eta^{\alpha\rho}h_{\nu\alpha}\partial_\rho\Omega\partial_\mu\Omega - 4\eta^{\alpha\eta}h_{\mu\alpha}\partial_\eta\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}h_{\alpha\beta}\partial_\mu\Omega\partial_\nu\Omega$$
$$+ 2\eta^{\alpha\lambda}\eta^{\beta\rho}\eta_{\mu\nu}h_{\alpha\beta}\partial_\lambda\Omega\partial_\rho\Omega\Big)$$

$$-\frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right) =$$
$$\Omega^{-3}\left(2\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 2\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \frac{1}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right)$$
$$+ \Omega^{-4}\Big(\frac{3}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 2\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 2\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega$$
$$- 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\Big)$$
$$+ (\mu \leftrightarrow \nu)$$

$$\frac{1}{2}\Box\left(\Omega^{-2}\bar{h}_{\mu\nu}\right) = \frac{1}{2}\Omega^{-2}\Box\bar{h}_{\mu\nu} - \Omega^{-3}\bar{h}_{\mu\nu}\Box\Omega$$

$$\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)$$
$$- \frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right)$$

$$\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\{$$
$$[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} + [\Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\nu} - \partial_\lambda\Gamma^\kappa_{\rho\nu} - \Gamma^\kappa_{\rho\nu}\partial_\lambda - \Gamma^\kappa_{\lambda\nu}\partial_\rho]h_{\kappa\mu}$$
$$+ [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\mu} - \partial_\lambda\Gamma^\kappa_{\rho\mu} - \Gamma^\kappa_{\rho\mu}\partial_\lambda - \Gamma^\kappa_{\lambda\mu}\partial_\rho]h_{\kappa\nu}\}$$

$$\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}(h_{\sigma\rho}R^\sigma{}_{\nu\mu\lambda})$$
$$= \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\rho}$$

$$\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda}) = \frac{1}{2}g^{\lambda\rho}[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}$$

$$\frac{1}{2}g^{\lambda\rho}(-h^\sigma{}_\mu R_{\rho\sigma\nu\lambda}) = \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\mu R_{\sigma\rho\nu\lambda}) = \frac{1}{2}g^{\lambda\rho}(h_{\sigma\mu}R^\sigma{}_{\rho\nu\lambda})$$
$$= \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu}$$

$$\frac{1}{2}g^{\lambda\rho}(-h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}[\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}$$

$$\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = \frac{1}{2}g^{\lambda\rho}\{$$
$$[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}$$
$$+ [\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu}$$
$$+ [\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}\}$$

$$-\frac{1}{2}\nabla_\nu\left(4\Omega^{-1}\bar{h}^\rho{}_\mu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\mu\Omega\right) - \frac{1}{2}\nabla_\mu\left(4\Omega^{-1}\bar{h}^\rho{}_\nu\partial_\rho\Omega - \Omega^{-1}h^\lambda{}_\lambda\partial_\nu\Omega\right) =$$
$$\Omega^{-3}\left(2\eta^{\rho\kappa}\partial_\nu h_{\kappa\mu}\partial_\rho\Omega + 2\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\partial_\nu h_{\kappa\lambda}\partial_\mu\Omega - \frac{1}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\partial_\mu\Omega\right)$$
$$+ \Omega^{-4}\Bigg(\frac{3}{2}\eta^{\lambda\kappa}h_{\kappa\lambda}\partial_\nu\Omega\partial_\mu\Omega + 2\eta^{\rho\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\alpha}\partial_\beta\Omega\partial_\rho\Omega - 2\eta^{\rho\kappa}h_{\kappa\nu}\partial_\mu\Omega\partial_\rho\Omega$$
$$- 8\eta^{\rho\kappa}h_{\kappa\mu}\partial_\nu\Omega\partial_\rho\Omega - \frac{1}{2}\eta^{\lambda\kappa}\eta^{\alpha\beta}\eta_{\mu\nu}h_{\kappa\lambda}\partial_\alpha\Omega\partial_\beta\Omega\Bigg)$$
$$+ (\mu \leftrightarrow \nu)$$

$$\frac{1}{2}\Box\left(\Omega^{-2}\bar{h}_{\mu\nu}\right) = \frac{1}{2}\Omega^{-2}\Box\bar{h}_{\mu\nu} - \Omega^{-3}\bar{h}_{\mu\nu}\Box\Omega$$

Gauge:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu}\bar{h}^\mu{}_\rho$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\ &+ \frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \frac{1}{2}\nabla_\nu(\Gamma^\sigma_{\rho\mu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\mu)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} = &\frac{1}{2}g^{\lambda\rho}\{\\ &[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma} + [\Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\nu} - \partial_\lambda\Gamma^\kappa_{\rho\nu} - \Gamma^\kappa_{\rho\nu}\partial_\lambda - \Gamma^\kappa_{\lambda\nu}\partial_\rho]h_{\kappa\mu}\\ &+ [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\sigma} + \Gamma^\sigma_{\lambda\rho}\Gamma^\kappa_{\sigma\mu} - \partial_\lambda\Gamma^\kappa_{\rho\mu} - \Gamma^\kappa_{\rho\mu}\partial_\lambda - \Gamma^\kappa_{\lambda\mu}\partial_\rho]h_{\kappa\nu}\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}) = &\frac{1}{2}g^{\lambda\rho}\big\{\\ &[2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}\\ &+ [\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\mu}\\ &+ [\partial_\lambda\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\lambda\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\rho}\Gamma^\sigma_{\mu\alpha}]h_{\sigma\nu}\big\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda})\\ = \frac{1}{2}g^{\lambda\rho}\big\{&[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma}\\ &+ [2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}\\ &+ [-\partial_\nu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\nu}\partial_\lambda - \Gamma^\sigma_{\lambda\nu}\partial_\rho]h_{\sigma\mu}\\ &+ [-\partial_\mu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\mu}\partial_\lambda - \Gamma^\sigma_{\lambda\mu}\partial_\rho]h_{\sigma\nu}\big\}\end{aligned}$$

$$\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) = \frac{1}{2}g^{\lambda\rho}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma} - \Gamma^\rho_{\rho\sigma}h_{\lambda\nu})$$

$$\frac{1}{2}g^{\lambda\rho}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma}) = \frac{1}{2}g^{\lambda\rho}(h_{\lambda\sigma}\nabla_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\nabla_\mu h_{\lambda\sigma})$$

$$\begin{aligned}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) \equiv \nabla_\mu T_\nu &= \partial_\mu T_\nu - \Gamma^\lambda_{\mu\nu}T_\lambda\\ &= \partial_\mu\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\rho\lambda}h^\rho{}_\sigma\end{aligned}$$

Gauge:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + \bar{\Gamma}^\mu_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{\Gamma}^\rho_{\mu\nu}\bar{h}^\mu{}_\rho$$

We can also write this as:

$$\bar{\nabla}_\mu \bar{h}^\mu{}_\nu = \frac{1}{2}\bar{\nabla}_\nu \bar{h}^\mu{}_\mu + (\bar{\nabla}_\mu - \partial_\mu)\bar{h}^\mu{}_\nu$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right)\\ &+ \frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \frac{1}{2}\nabla_\nu(\Gamma^\sigma_{\rho\mu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\mu)\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda})\\ &\quad= \frac{1}{2}g^{\lambda\rho}\big\{[\partial_\lambda\partial_\rho - \Gamma^\sigma_{\lambda\rho}\partial_\sigma]h_{\mu\nu} + [\Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\rho\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\kappa_{\rho\mu}]h_{\kappa\sigma}\\ &\qquad+ [2\partial_\lambda\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\lambda\nu} - \partial_\nu\Gamma^\sigma_{\lambda\mu} + 2\Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\alpha_{\lambda\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\sigma_{\nu\alpha}]h_{\sigma\rho}\\ &\qquad+ [-\partial_\nu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\nu}\partial_\lambda - \Gamma^\sigma_{\lambda\nu}\partial_\rho]h_{\sigma\mu}\\ &\qquad+ [-\partial_\mu\Gamma^\sigma_{\lambda\rho} + 2\Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\lambda\alpha} - \Gamma^\sigma_{\rho\mu}\partial_\lambda - \Gamma^\sigma_{\lambda\mu}\partial_\rho]h_{\sigma\nu}\big\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) = &\frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\lambda\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\lambda\kappa} + \Gamma^\sigma_{\mu\nu}\Gamma^\kappa_{\kappa\lambda})h_{\sigma\rho} - (\partial_\mu\Gamma^\sigma_{\sigma\lambda} + \Gamma^\sigma_{\sigma\lambda}\partial_\mu)h_{\rho\nu}\right]\\ &+ \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\lambda\nu}h_{\rho\sigma} - \Gamma^\sigma_{\sigma\lambda}h_{\rho\nu}\right)\end{aligned}$$

$$\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma = \Omega^{-1}h^\sigma{}_\sigma\nabla_\nu\Omega$$
$$\Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu = 4\Omega^{-1}h^\rho{}_\nu\nabla_\rho\Omega$$

$$\begin{aligned}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) \equiv \nabla_\mu T_\nu &= \partial_\mu T_\nu - \Gamma^\lambda_{\mu\nu}T_\lambda\\ &= \partial_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\rho\lambda}h^\rho{}_\sigma\end{aligned}$$

$$\begin{aligned}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) \equiv \nabla_\mu T_\nu &= \partial_\mu T_\nu - \Gamma^\lambda_{\mu\nu}T_\lambda \\ &= \partial_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\rho\lambda}h^\rho{}_\sigma\end{aligned}$$

$$\frac{1}{2}\nabla_\mu(\nabla_\rho - \partial_\rho)h^\rho{}_\nu = \frac{1}{2}g^{\lambda\rho}[\nabla_\mu\nabla_\rho h_{\lambda\nu} - \nabla_\mu(\partial_\rho g^{\lambda\rho}h_{\lambda\nu})]$$

$$\begin{aligned}\delta R_{\mu\nu} = &\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left(h^\sigma{}_\rho R_{\sigma\nu\mu\lambda} + h^\sigma{}_\rho R_{\sigma\mu\nu\lambda} - h^\sigma{}_\mu R_{\rho\sigma\nu\lambda} - h^\sigma{}_\nu R_{\rho\sigma\mu\lambda}\right) \\ &- \frac{1}{2}\nabla_\mu(\nabla_\lambda - \partial_\lambda)h^\lambda{}_\nu - \frac{1}{2}\nabla_\nu(\nabla_\lambda - \partial_\lambda)h^\lambda{}_\mu\end{aligned}$$

$$\delta R_{\mu\nu} = \frac{1}{2}(\nabla_\lambda\nabla^\lambda h_{\mu\nu} - \nabla_\lambda\nabla_\mu h^\lambda{}_\nu - \nabla_\lambda\nabla_\nu h^\lambda{}_\mu) - \nabla_\mu\partial_\lambda h^\lambda{}_\nu - \nabla_\nu\partial_\lambda h^\lambda{}_\mu$$

$$\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma = \Omega^{-1}h^\sigma{}_\sigma\nabla_\nu\Omega$$
$$\Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu = 4\Omega^{-1}h^\rho{}_\nu\nabla_\rho\Omega$$

$$\begin{aligned}\frac{1}{2}\nabla_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma - \Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) &= \frac{1}{2}\left[\partial_\mu(\Gamma^\sigma_{\rho\nu}h^\rho{}_\sigma) - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa}h^\rho{}_\sigma - \partial_\mu(\Gamma^\sigma_{\sigma\rho}h^\rho{}_\nu) + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h^\rho{}_\kappa\right] \\ &= \frac{1}{2}\left[(\partial_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa})h^\rho{}_\sigma - (\partial_\mu\Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\sigma\rho}\partial_\mu)h^\rho{}_\nu + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h^\rho{}_\kappa\right] \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\rho\nu} + \Gamma^\sigma_{\rho\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\rho\kappa})h_{\lambda\sigma} - (\partial_\mu\Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\sigma\rho}\partial_\mu)h_{\lambda\nu} + \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\sigma\rho}h_{\lambda\kappa}\right] \\ &\quad + \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\rho\nu}h_{\lambda\sigma} - \Gamma^\sigma_{\sigma\rho}h_{\lambda\nu}\right) \\ &= \frac{1}{2}g^{\lambda\rho}\left[(\partial_\mu\Gamma^\sigma_{\lambda\nu} + \Gamma^\sigma_{\lambda\nu}\partial_\mu - \Gamma^\kappa_{\mu\nu}\Gamma^\sigma_{\lambda\kappa} + \Gamma^\sigma_{\mu\nu}\Gamma^\kappa_{\kappa\lambda})h_{\sigma\rho} - (\partial_\mu\Gamma^\sigma_{\sigma\lambda} + \Gamma^\sigma_{\sigma\lambda}\partial_\mu)h_{\rho\nu}\right] \\ &\quad + \frac{1}{2}\partial_\mu g^{\lambda\rho}\left(\Gamma^\sigma_{\lambda\nu}h_{\rho\sigma} - \Gamma^\sigma_{\sigma\lambda}h_{\rho\nu}\right)\end{aligned}$$

$$ds^2 = -(1+2\phi)d\tau^2 + (B_i + \nabla_i B)dx^i d\tau + [(1-2\psi)\delta_{ij} + 2[(\nabla_i\nabla_j - 1/3\nabla^2)E + 1/2(\nabla_i E_j + \nabla_j E_i) + E_{ij}]]dx^i dx^j$$