

Cosmological Fluctuations in Standard and Conformal Gravity

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Doctoral Degree Final Examination



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- Introduction and Formalism
- Three Dimensional Scalar, Vector, Tensor Decomposition (SVT3)
- Four Dimensional Scalar, Vector, Tensor Decomposition (SVT4)
- Conformal Gravity (SVT and Conformal to Flat Backgrounds)
- Conformal Gravity Robertson-Walker Radiation Era Solution
- Computational Methods
- Conclusions

- Introduction and Formalism
 - Cosmological Geometries
 - Einstein Gravity
 - Perturbation Theory
 - Gauge Transformations
 - Solution Methods

- Cosmological Principle: Structure of spacetime is homogeneous and isotropic at large scales
- Geometries: Robertson Walker (flat, spherical, hyperbolic), de Sitter ($dS_4 \subset RW$)
- All background geometries relevant to cosmology can be expressed as conformal to flat

$$ds^2 = \Omega(x)^2 (-dt^2 + dx^2 + dy^2 + dz^2)$$



Hubble Ultra-Deep Field. NASA and the European Space Agency.

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \quad R_{ij} = -3k\tilde{g}_{ij}, \quad R = -6k$$

with $k \in \{-1, 0, 1\}$. Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

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$$\tau = \int \frac{dt}{a(t)},$$

set $k = 0$ (flat), simple conformal to flat form

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

$k = 1$ (spherical)

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

Set $\sin \chi = r$, $p = \tau$,

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p + \chi)/2], & p' - r' &= \tan[(p - \chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos \chi}, & r' &= \frac{\sin \chi}{\cos p + \cos \chi} \end{aligned}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2]$$

$k = -1$ (hyperbolic)

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

Set $\sin \chi = r$, $p = \tau$,

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right]$$

Introduce coordinates

$$p' + r' = \tanh[(p + \chi)/2], \quad p' - r' = \tanh[(p - \chi)/2]$$

$$p' = \frac{\sinh p}{\cosh p + \cosh \chi}, \quad r' = \frac{\sinh \chi}{\cosh p + \cosh \chi}$$

$$\implies ds^2 = \frac{4a^2(p)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2]$$

Einstein Hilbert action

$$I_{\text{EH}} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} g^{\mu\nu} R_{\mu\nu}.$$

Functional variation w.r.t $g_{\mu\nu}$ yields Einstein tensor,

$$\frac{16\pi G}{(-g)^{1/2}} \frac{\delta I_{\text{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha,$$

likewise, variation of matter action I_M w.r.t $g_{\mu\nu}$ yields Energy Momentum tensor

$$\frac{2}{(-g)^{1/2}} \frac{\delta I_M}{\delta g_{\mu\nu}} = T_{\mu\nu}.$$

Requiring sum of actions to be stationary gives us Einstein field equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha = -8\pi G T^{\mu\nu},$$

subject to Bianchi identity

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R^\mu{}_\mu \implies \nabla_\mu G^{\mu\nu} = 0.$$

Cosmological Perturbation Theory

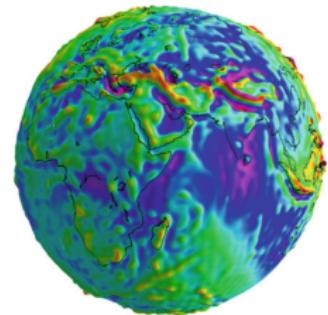
Decompose metric into background and fluctuation, truncating at linear order

$$g_{\mu\nu}(x) = g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x), \quad g_{(0)}^{\mu\nu} h_{\mu\nu} \equiv h$$

$$G_{\mu\nu} = G_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta G_{\mu\nu}(h_{\mu\nu})$$

$$G_{\mu\nu}^{(0)} = R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R_{\alpha}^{(0)\alpha}$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R_{\alpha}^{(0)\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}_{\alpha}.$$



Likewise perturb $T_{\mu\nu}$ around background

$$T_{\mu\nu} = T_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta T_{\mu\nu}(h_{\mu\nu})$$

Form background and first order equations of motion (upon setting $8\pi G = 1$)

$$\Delta_{\mu\nu}^0 = G_{\mu\nu}^{(0)} + T_{\mu\nu}^{(0)} = 0$$

$$\Delta_{\mu\nu} = \delta G_{\mu\nu}^{(0)} + \delta T_{\mu\nu}^{(0)} = 0$$

- Under coordinate transformation $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$, with $\epsilon^\mu \sim \mathcal{O}(h)$, the perturbed metric transforms as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$$

- For every solution $h_{\mu\nu}$ to $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$, a transformed $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ will also serve as a solution
- Set of four $\epsilon^\mu(x)$ define gauge freedom under coordinate transformation
- 10 components in $h_{\mu\nu}$, 4 coordinate transformations, leads to 6 independent degrees of freedom
- Under $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$, the perturbed tensors transform as

$$\begin{aligned}\delta G_{\mu\nu} &\rightarrow \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda G_{\mu\nu}^{(0)} \epsilon^\lambda \\ \delta T_{\mu\nu} &\rightarrow \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda T_{\mu\nu}^{(0)} \epsilon^\lambda.\end{aligned}$$

- If background $G_{\mu\nu}^{(0)} = 0$, then $\delta G_{\mu\nu}$ separately gauge invariant; likewise for vanishing background energy momentum tensor
- If $G_{\mu\nu}^{(0)} \neq 0$, then only the entire $\Delta_{\mu\nu} = \delta G_{\mu\nu} + T_{\mu\nu}$ is gauge invariant

- Perturbed field equations $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ form a rather complex and extensive set of coupled tensor PDE's
- Much effort involved in simplifying, decoupling, and solving them

$$\begin{aligned}
 \delta G_{ij} = & -\frac{1}{2}\ddot{h}_{ij} + \frac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \frac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2 \tilde{g}_{ij}h_{00}\Omega^{-2} \\
 & - \dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
 & + 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
 & - 4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
 & - \tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
 & + \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \frac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
 & - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
 & + \frac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \frac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
 & - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \frac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
 & + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \frac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h,
 \end{aligned}$$

- Three-dimensional Scalar, Vector, Tensor Basis (SVT3)
 - SVT3 Decomposition
 - Einstein Fluctuations in a de Sitter Background
 - Integral Formalism

SVT3 Decomposition

Decompose the metric perturbation $h_{\mu\nu}$ into a set of scalars, vectors, and tensors according to their transformation behavior under 3D rotations

- Define $h_{\mu\nu} = \Omega^2(x) f_{\mu\nu}$, perform 3 + 1 decomposition

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = (g_{\mu\nu}^{(0)} + h_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) (\tilde{g}_{\mu\nu}^{(0)} + f_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) [(-1 + f_{00}) dt^2 + 2f_{0i} dt dx^i + (\tilde{g}_{ij} + f_{ij})] dx^i dx^j \end{aligned}$$

- Decompose f_{00} , f_{0i} , and f_{ij} in terms of 3-dimensional scalars, vectors, and tensors

$$\begin{aligned} f_{00} &= -2\phi, & f_{0i} &= B_i + \tilde{\nabla}_i B \\ f_{ij} &= -2\psi\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned}$$

with vectors and tensors obeying

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0.$$

$$\begin{aligned} ds^2 &= \Omega^2(x) [- (1 + 2\phi) dt^2 + 2(B_i + \tilde{\nabla}_i B) dt dx^i \\ &\quad + ((1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}) dx^i dx^j], \end{aligned}$$

- 10 components in total

SVT3 $\delta G_{\mu\nu}$ in a de Sitter Background

- Insert the SVT3 decomposed $h_{\mu\nu}$ into $\delta G_{\mu\nu}$

$$\delta G_{00} = -2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi,$$

$$\delta G_{0i} = -2\tilde{\nabla}_i\dot{\psi} + \frac{1}{2}\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(B_i - \dot{E}_i),$$

$$\begin{aligned}\delta G_{ij} &= -2\delta_{ij}\ddot{\psi} - \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}) + \delta^{ab}\delta_{ij}\tilde{\nabla}_b\tilde{\nabla}_a\psi + \tilde{\nabla}_j\tilde{\nabla}_i(\phi + \dot{B} - \ddot{E}) \\ &\quad - \tilde{\nabla}_j\tilde{\nabla}_i\psi + \frac{1}{2}\tilde{\nabla}_i(\dot{B}_j - \ddot{E}_j) + \frac{1}{2}\tilde{\nabla}_j(\dot{B}_i - \ddot{E}_i) - \ddot{E}_{ij} + \delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a E_{ij},\end{aligned}$$

$$g^{\mu\nu}\delta G_{\mu\nu} = -\delta G_{00} + \delta^{ij}\delta G_{ij} = 4\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a\psi - 6\ddot{\psi} - 2\delta^{ab}\tilde{\nabla}_b\tilde{\nabla}_a(\phi + \dot{B} - \ddot{E}),$$

$$\begin{aligned} I_W &= -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\ &\equiv -2\alpha_g \int d^4x (-g)^{1/2} [R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2], \end{aligned}$$

$$\begin{aligned} C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} - \frac{1}{2} (g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}) \\ &\quad + \frac{1}{6} R^\alpha{}_\alpha (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) \end{aligned}$$

$$-\frac{2}{(-g)^{1/2}} \frac{\delta I_W}{\delta g_{\mu\nu}} = 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2\nabla_\kappa \nabla_\lambda C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} C^{\mu\lambda\nu\kappa} \right].$$

