Coordinate Transformations RW $k < 0$ v1

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We may form a 3-space of constant curvature by embedding within a flat 4-space, just as we may embed a 2-sphere or 2 dimensional hyperbola (or also a flat plane) within 3 dimensional space. Constraining to a space of constant curvature, we have

$$\mathbf{x}^2 + z^2 = C^2. \tag{1}$$

Here C^2 represents the degree and sign of curvature, with dimension of length $C \sim [L]$. For C^2 positive, we have a bound 3-sphere, while for $C^2 = 0$, we have unbound Euclidean geometry, and for $C^2 < 0$ we have an unbound hyperbolic geometry. Constructing the flat 4-space line element,

$$ds^2 = d\mathbf{x}^2 + dz^2. \tag{2}$$

Taking the differential of (1) allows us to relate dz to the three space variables $\mathbf{x}$ via

$$dz^2 = \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{C^2 - \mathbf{x}^2} \tag{3}$$

Substituting into the line element we have

$$ds^2 = d\mathbf{x}^2 + \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{C^2 - \mathbf{x}^2} \tag{4}$$

Adopting polar coordinates, this becomes

$$ds^2 = \frac{dr^2}{1 - r^2/C^2} + r^2 d\Omega^2 \tag{5}$$

With the above general form for a maximally symmetric 3-space with constant curvature, we may form the invariant spacetime interval as

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - r^2/C^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{6}$$

where $a(t)$ is an arbitrary function of time to be set by dynamics. Worth noting is that if we rescale $r' = r/|C|$, radial distances will be dimensionless and $a_{rescaled}(t) = a(t)/|C|$ will have dimension of $[L]$. Such a rescaling is necessary for the metric convention in which $\frac{dr^2}{1-Kr^2}$ for $K \in [-1, 0, 1]$. However, cosmological convention utilizes a dimensionless $a(t)$, thus we leave in the form of r^2/C^2.

By a coordinate transformation upon t via

$$\tau = \int \frac{dt}{a(t)}, \tag{7}$$

we may express (6) in terms of conformal time τ as

$$ds^2 = a^2(\tau) \left(d\tau^2 - \frac{dr^2}{1 - r^2/C^2} + r^2 d\Omega^2 \right) \tag{8}$$

RW to Conformal to Flat Form

First Transformation

As the first step towards bringing the metric to conformal-flat form for $C^2 < 0$, we introduce curvature magnitude $L^2 = -C^2$ (an inherently positive quantity) and we make coordinate transformations

$$p = \frac{\tau}{L}, \qquad \sinh\chi = \frac{r}{L}, \tag{9}$$

which take the line element of (8) into

$$ds^2 = L^2 a^2(p)\left(dp^2 - d\chi^2 - \sinh^2\chi d\Omega^2\right). \tag{10}$$

In this form, all length dimension lies within L^2.

Second Transformation (Alternative)

To finally bring (10) to the flat form, we make coordinate substitutions

$$T = e^p\cosh\chi, \qquad R = e^p\sinh\chi. \tag{11}$$

It is convenient to introduce a somewhat 'light-like' coordinate defined by

$$X^2 \equiv T^2 - R^2. \tag{12}$$

The coordinate relation for the time coordinate $p(T,R)$ is in fact only a function of X^2, viz.

$$e^{2p} = X^2, \qquad p = \frac{1}{2}\ln(X^2). \tag{13}$$

For the radial coordinate $\chi(T,R)$ we have the relations

$$\sinh\chi = \frac{R}{X}, \qquad \cosh\chi = \frac{T}{X}. \tag{14}$$

Though not as useful, we may invert (14) to find $\chi(T,R)$ as

$$\chi = \ln\left(\frac{T+R}{X}\right) \tag{15}$$

To aid in determining the differentials, we note

$$dX = \frac{\partial X}{\partial T}dT + \frac{\partial X}{\partial R}dR = \frac{TdT - RdR}{X}. \tag{16}$$

We first determine dp:

$$dp = \frac{T}{X^2}dT - \frac{R}{X^2}dR. \tag{17}$$

To find $d\chi$, we differentiate $\sinh\chi$:

$$d(\sinh\chi) = \cosh\chi d\chi = \frac{dR}{X} - \frac{R}{X^3}(TdT - RdR) \tag{18}$$

$$\frac{T}{X}d\chi = \frac{dR}{X} - \frac{TR}{X^3}dT + \frac{R^2}{X^3}dR, \tag{19}$$

hence

$$d\chi = \frac{dR}{T} - \frac{R}{X^2}dT + \frac{R^2}{TX^2}dR. \tag{20}$$

After repeated usage of $X^2 = T^2 - R^2$, we find the coordinate relation between infinitesimals

$$dp^2 - d\chi^2 = \frac{1}{X^2}\left(dT^2 - dR^2\right). \tag{21}$$

Finally, with $\sinh^2\chi = \frac{R^2}{X^2}$, we may write the line element in these new coordinates:

$$ds^2 = L^2\frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right) \tag{22}$$

Conformal Flat to RW Coordinates

Conformal Factor

We note that the conformal factor in the flat T, R coordinates is only a function of $X^2 = T^2 - R^2$. The factor is simply

$$\Omega(X)^2 = L^2 \frac{a^2(X)}{X^2} \tag{23}$$

where

$$a(X) = a\left(\frac{1}{2}\ln(X^2)\right). \tag{24}$$

The relation of the conformal factor to the p, χ geometry is simple,

$$\Omega^2(X) \equiv \Omega^2(p,\chi) = L^2 a^2(p) e^{-2p}. \tag{25}$$

Interestingly, it is a function entirely of time coordinate p. We may bring this to the comoving RW form by successive transformations

$$p = \frac{\tau}{L}, \qquad \tau = \int \frac{a(t)}{dt}, \tag{26}$$

in which the conformal factor becomes

$$\Omega^2(X) \equiv \Omega^2(t) = L^2 a^2(t) \exp\left[-\frac{2}{L}\int \frac{dt}{a(t)}\right] \tag{27}$$

Two Step Transformation

From the relations

$$T = e^p \cosh\chi, \qquad R = e^p \sinh\chi \tag{28}$$

and

$$p = \frac{\tau}{L}, \qquad \sinh\chi = \frac{r}{L} \tag{29}$$

we see that we could enact a coordinate transformation from conformal time (τ) RW geometry

$$ds^2 = a^2(\tau)\left(d\tau^2 - \frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2\right) \tag{30}$$

to conformal to flat (polar) geometry

$$ds^2 = L^2 \frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right) \tag{31}$$

via the effective transformation

$$T = \exp\left(\frac{\tau}{L}\right)\left(1 + \left(\frac{r}{L}\right)^2\right)^{1/2}, \qquad R = \exp\left(\frac{\tau}{L}\right)\frac{r}{L}, \qquad X^2 \equiv T^2 - R^2 = \exp\left(\frac{2\tau}{L}\right) \tag{32}$$

One Step Transformation

Lastly, we may substitute the transformation of τ viz

$$\tau = \int \frac{dt}{a(t)}, \tag{33}$$

to finally bring us to comoving coordinates. That is, via coordinate transformation

$$T = \exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right)\left(1+\left(\frac{r}{L}\right)^2\right), \qquad R = \exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right)\frac{r}{L}, \qquad X^2 \equiv T^2 - R^2 = \exp\left(\frac{2}{L}\int \frac{dt}{a(t)}\right) \tag{34}$$

we may transform from comoving coordinates

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1+r^2/L^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{35}$$

to conformal flat (polar) coordinates

$$ds^2 = L^2\frac{a^2(X)}{X^2}\left(dT^2 - dR^2 - R^2 d\Omega^2\right). \tag{36}$$

When $a(t)$ is specified apriori via a dynamics, exponential factors will simplify, especially for a τ which behaves logarithmically. For example, in the early universe radiation era, we have determined τ as

$$\tau = L\int_0^t \frac{dt}{(d^2+t^2)^{1/2}} = L\ \mathrm{arcsinh}\left(\frac{t}{d}\right). \tag{37}$$

This is equivalent to

$$\tau = L\ln\left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right) \tag{38}$$

in which our exponential calculates to

$$\exp\left(\frac{1}{L}\int \frac{dt}{a(t)}\right) = \frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}. \tag{39}$$

In the (conformal) early universe then, the conformal factor $\Omega(X)$ goes as

$$\Omega^2(X) = L^2 a^2(t)\exp\left[-\frac{2}{L}\int \frac{dt}{a(t)}\right] \tag{40}$$

$$= (d^2+t^2)\left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)^{-2} \tag{41}$$

The flat space coordinate transformations T and R then are specified as

$$T = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\left(1+\left(\frac{r}{L}\right)^2\right)^{1/2}, \qquad R = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\frac{r}{L} \tag{42}$$

$$X^2 \equiv T^2 - R^2 = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)^2 \tag{43}$$

$$a^2(X) = \frac{d^2}{L^2}\frac{(X^2+1)^2}{4X^2} \tag{44}$$

$$\Omega^2(X) = L^2\frac{a^2(X)}{X^2} = \left[\frac{d}{2}\left(1+\frac{1}{X^2}\right)\right]^2 \tag{45}$$

$$\Omega(X) = \frac{d}{2}(1+X^{-2}) \tag{46}$$

Cartesian to Polar

Transformation Matrices

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{48}$$

Time-Time

$$K'_{00} = K_{00} \tag{49}$$

Time-Space

$$K'_{0i} = \frac{\partial x^j}{\partial x'^i} K_{0j} \tag{50}$$

$$\begin{pmatrix} K'_{01} \\ K'_{02} \\ K'_{03} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{01} \\ K_{02} \\ K_{03} \end{pmatrix} \tag{51}$$

$$K'_{01} = K_{01}\sin(\theta)\cos(\phi) + K_{02}\sin(\theta)\sin(\phi) + K_{03}\cos(\theta) \tag{52}$$

$$K'_{02} = K_{01}r\cos(\theta)\cos(\phi) + K_{02}r\cos(\theta)\sin(\phi) - K_{03}r\sin(\theta) \tag{53}$$

$$K'_{03} = -K_{01}r\sin(\theta)\sin(\phi) + K_{02}r\sin(\theta)\cos(\phi) \tag{54}$$

Space-Space

$$K'_{ij} = \frac{\partial x^k}{\partial x'^i} K_{kl} \frac{\partial x^l}{\partial x'^j} \tag{55}$$

$$\begin{pmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{21} & K'_{22} & K'_{23} \\ K'_{31} & K'_{32} & K'_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^3}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^3}{\partial x'^2} \\ \frac{\partial x^1}{\partial x'^3} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}^T \tag{56}$$

Example:

$$\begin{aligned} K'_{11} = {} & K_{11}\sin^2(\theta)\cos^2(\phi) + K_{12}\sin^2(\theta)\sin(2\phi) + K_{13}\sin(2\theta)\cos(\phi) + K_{22}\sin^2(\theta)\sin^2(\phi) \\ & + K_{23}\sin(2\theta)\sin(\phi) + K_{33}\cos^2(\theta) \end{aligned} \tag{57}$$

$$\begin{aligned} K'_{22} = {} & K_{11}r^2\cos^2(\theta)\cos^2(\phi) + K_{12}r^2\cos^2(\theta)\sin(2\phi) - K_{13}r^2\sin(2\theta)\cos(\phi) + K_{22}r^2\cos^2(\theta)\sin^2(\phi) \\ & - K_{23}r^2\sin(2\theta)\sin(\phi) + K_{33}r^2\sin^2(\theta) \end{aligned} \tag{58}$$

$$K'_{33} = K_{11}r^2\sin^2(\theta)\sin^2(\phi) - 2K_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{22}r^2\sin^2(\theta)\cos^2(\phi) \tag{59}$$

$$K'_{12} = K_{11} r\sin(\theta)\cos(\theta)\cos^2(\phi) + K_{12} r\sin(\theta)\cos(\theta)\sin(2\phi) + K_{13} r\cos(2\theta)\cos(\phi) + K_{22} r\sin(\theta)\cos(\theta)\sin^2(\phi) + K_{23} r\cos(2\theta)\sin(\phi) - K_{33} r\sin(\theta)\cos(\theta) \tag{60}$$

$$K'_{13} = -K_{11} r\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{12} r\sin^2(\theta)\cos(2\phi) - K_{13} r\sin(\theta)\cos(\theta)\sin(\phi) + K_{22} r\sin^2(\theta)\sin(\phi)\cos(\phi) + K_{23} r\sin(\theta)\cos(\theta)\cos(\phi) \tag{61}$$

$$K'_{23} = -K_{11} r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + K_{12} r^2\sin(\theta)\cos(\theta)\cos(2\phi) + K_{13} r^2\sin^2(\theta)\sin(\phi) + K_{22} r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) - K_{23} r^2\sin^2(\theta)\cos(\phi) \tag{62}$$

Early Universe (Radiation Era)

In the conformal to Minkowski coordinate system of

$$ds^2 = \Omega^2(X^2)(dT^2 - dx^2 - dy^2 - dz^2), \qquad X^2 = T^2 - (x^2 + y^2 + z^2) \tag{63}$$

when we impose the transverse gauge, solutions to conformal gravity $\delta W_{\mu\nu} = 0$ are found to obey

$$\frac{1}{2}\Omega^{-2}\Box^2 k_{\mu\nu} = 0 \tag{64}$$

where $\Omega^2 k_{\mu\nu} = K_{\mu\nu}$. Upon performing residual gauge transformations to eliminate gauge degrees of freedom, the general solution to (64) for a given k-mode is then

$$k_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \begin{pmatrix} 0 & B_{01} & B_{02} & 0 \\ B_{01} & B_{11} & B_{12} & 0 \\ B_{02} & B_{12} & -B_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} T e^{ikx} \tag{65}$$

Since we will soon find that $T \sim t$, with t the comoving time, we see the $B_{\mu\nu}$ are leading order in t. Hence for early universe fluctuations, the leading t order solution to $K_{\mu\nu}$ will be

$$K_{\mu\nu} \sim \Omega^2 T B_{\mu\nu} e^{ikx}, \tag{66}$$

where

$$B_{22} = -B_{11}, \qquad B_{0\mu} = B_{33} = 0 \tag{67}$$

Conformal Minkoski to Polar RW Comoving

In going from the geometry of

$$ds^2 = \Omega^2(dT^2 - dx^2 - dy^2 - dz^2) \tag{68}$$

to

$$ds^2 = \Omega^2(dT^2 - dR^2 - R^2 d\Omega^2), \tag{69}$$

we utilize the Cartesian to polar conversions given in the Appendix. Denoting the polar coordinate system as $x^{(P)}$, we find, after imposing the transverse and residual relations, the following:

$$
\begin{aligned}
k_{00}^{(P)} &= 0 \\
k_{01}^{(P)} &= k_{01}\sin(\theta)\cos(\phi) + k_{02}\sin(\theta)\sin(\phi) \\
k_{02}^{(P)} &= k_{01}r\cos(\theta)\cos(\phi) + k_{02}r\cos(\theta)\sin(\phi) \\
k_{03}^{(P)} &= -k_{01}r\sin(\theta)\sin(\phi) + k_{02}r\sin(\theta)\cos(\phi) \\
k_{11}^{(P)} &= k_{11}\sin^2(\theta)\cos(2\phi) + k_{12}\sin^2(\theta)\sin(2\phi) \\
k_{22}^{(P)} &= k_{11}r^2\cos^2(\theta)\cos(2\phi) + k_{12}r^2\cos^2(\theta)\sin(2\phi) \\
k_{33}^{(P)} &= -k_{11}r^2\sin^2(\theta)\cos(2\phi) - 2k_{12}r^2\sin^2(\theta)\sin(\phi)\cos(\phi) \\
k_{12}^{(P)} &= \frac{1}{2}k_{11}r\sin(2\theta)\cos(2\phi) + k_{12}r\sin(\theta)\cos(\theta)\sin(2\phi) \\
k_{13}^{(P)} &= -2k_{11}r\sin^2(\theta)\sin(\phi)\cos(\phi) + k_{12}r\sin^2(\theta)\cos(2\phi) \\
k_{23}^{(P)} &= -2k_{11}r^2\sin(\theta)\cos(\theta)\sin(\phi)\cos(\phi) + k_{12}r^2\sin(\theta)\cos(\theta)\cos(2\phi)
\end{aligned} \tag{70}
$$

$$K'_{\mu\nu}(t,r,\theta,\phi) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}k_{\alpha\beta}(T,R,\theta,\phi) \tag{71}$$

$$J_{\mu\nu} = \frac{\partial x^\nu}{\partial x'^\mu}, \qquad \text{where} \quad x(T,R,\theta,\phi) \quad x'(t,r,\theta,\phi) \tag{72}$$

$$J_{\mu\nu} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{73}$$

$$k_{\mu\nu}^{(cm)} = \frac{\partial x^k_{(P)}}{\partial x^i_{(cm)}}k_{kl}^{(P)}\frac{\partial x^l_{(P)}}{\partial x^j_{(cm)}} \tag{74}$$

$$\begin{pmatrix} k_{00}^{(cm)} & k_{01}^{(cm)} & k_{02}^{(cm)} & k_{03}^{(cm)} \\ k_{10}^{(cm)} & k_{11}^{(cm)} & k_{12}^{(cm)} & k_{13}^{(cm)} \\ k_{20}^{(cm)} & k_{21}^{(cm)} & k_{22}^{(cm)} & k_{23}^{(cm)} \\ k_{30}^{(cm)} & k_{31}^{(cm)} & k_{32}^{(cm)} & k_{33}^{(cm)} \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{00}^{(P)} & k_{01}^{(P)} & k_{02}^{(P)} & k_{03}^{(P)} \\ k_{10}^{(P)} & k_{11}^{(P)} & k_{12}^{(P)} & k_{13}^{(P)} \\ k_{20}^{(P)} & k_{21}^{(P)} & k_{22}^{(P)} & k_{23}^{(P)} \\ k_{30}^{(P)} & k_{31}^{(P)} & k_{32}^{(P)} & k_{33}^{(P)} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial R}{\partial t} & 0 & 0 \\ \frac{\partial T}{\partial r} & \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \tag{75}$$

$$
\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial T}{\partial t}\frac{\partial R}{\partial t}k_{01}^{(P)} + \left(\frac{\partial R}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial T}{\partial t}\frac{\partial R}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial T}{\partial r}k_{01}^{(P)} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial T}{\partial t}k_{02}^{(P)} + \frac{\partial R}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial T}{\partial t}k_{03}^{(P)} + \frac{\partial R}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial T}{\partial r}\frac{\partial R}{\partial r}k_{01}^{(P)} + \left(\frac{\partial R}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial T}{\partial r}k_{02}^{(P)} + \frac{\partial R}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial T}{\partial r}k_{03}^{(P)} + \frac{\partial R}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned} \tag{76}
$$

$$
T = \left(\frac{t}{d} + \sqrt{1+\left(\frac{t}{d}\right)^2}\right)\sqrt{1+\left(\frac{r}{L}\right)^2}, \qquad R = \left(\frac{t}{d} + \sqrt{1+\left(\frac{t}{d}\right)^2}\right)\frac{r}{L} \tag{77}
$$

$$
\frac{\partial T}{\partial t} = \frac{1}{d}\left(1 + \frac{\frac{t}{d}}{\sqrt{1+\left(\frac{t}{d}\right)^2}}\right)\sqrt{1+\left(\frac{r}{L}\right)^2} \tag{78}
$$

$$
\frac{\partial R}{\partial t} = \frac{1}{d}\left(1 + \frac{\frac{t}{d}}{\sqrt{1+\left(\frac{t}{d}\right)^2}}\right)\frac{r}{L} \tag{79}
$$

$$
\frac{\partial T}{\partial r} = \frac{1}{L}\left(\frac{t}{d} + \sqrt{1+\frac{t^2}{d^2}}\right)\left(\frac{\frac{r}{L}}{\sqrt{1+\left(\frac{r}{L}\right)^2}}\right) \tag{80}
$$

$$
\frac{\partial R}{\partial r} = \frac{1}{L}\left(\frac{t}{d} + \sqrt{1+\frac{t^2}{d^2}}\right) \tag{81}
$$

For late times such that $t \gg d$, the large time behavior goes as:

$$
T \sim \frac{t}{d}, \quad R \sim \frac{t}{d}, \quad \frac{\partial T}{\partial t} \sim \frac{1}{d}\left(\frac{t}{d}\right)^0, \quad \frac{\partial R}{\partial t} \sim \frac{1}{d}\left(\frac{t}{d}\right)^0, \quad \frac{\partial T}{\partial r} \sim \frac{1}{L}\left(\frac{t}{d}\right), \quad \frac{\partial R}{\partial r} \sim \frac{1}{L}\left(\frac{t}{d}\right) \tag{82}
$$

We note that in converting from Cartesian to polar, there reside factors of R in the Jacobian of transformation. Thus, for $k_{\mu\nu}^{(P)}$ we have, to leading order

$$
\begin{array}{llll}
k_{00}^{(P)} = 0 & k_{01}^{(P)} \sim \left(\frac{t}{d}\right) & k_{02}^{(P)} \sim \left(\frac{t}{d}\right)^2 & k_{03}^{(P)} \sim \left(\frac{t}{d}\right)^2 \\
 & k_{11}^{(P)} \sim \left(\frac{t}{d}\right) & k_{22}^{(P)} \sim \left(\frac{t}{d}\right)^3 & k_{33}^{(P)} \sim \left(\frac{t}{d}\right)^3 \\
 & k_{12}^{(P)} \sim \left(\frac{t}{d}\right)^2 & k_{13}^{(P)} \sim \left(\frac{t}{d}\right)^2 & k_{23}^{(P)} \sim \left(\frac{t}{d}\right)^3
\end{array} \tag{83}
$$

Next, we use (82-83) to determine the late time behavior in comoving coordinates:

$$\begin{aligned}
&k_{00}^{(cm)} \sim \frac{1}{d^2}\left(\frac{t}{d}\right) \qquad k_{01}^{(cm)} \sim \frac{1}{Ld}\left(\frac{t}{d}\right)^2 \qquad k_{02}^{(cm)} \sim \frac{1}{d}\left(\frac{t}{d}\right)^2 \qquad k_{03}^{(cm)} \sim \frac{1}{d}\left(\frac{t}{d}\right)^2 \\
&\qquad k_{11}^{(cm)} \sim \frac{1}{L^2}\left(\frac{t}{d}\right)^3 \qquad k_{22}^{(cm)} \sim \left(\frac{t}{d}\right)^3 \qquad k_{33}^{(cm)} \sim \left(\frac{t}{d}\right)^3 \\
&\qquad k_{12}^{(cm)} \sim \frac{1}{L}\left(\frac{t}{d}\right)^3 \qquad k_{13}^{(cm)} \sim \frac{1}{L}\left(\frac{t}{d}\right)^3 \qquad k_{23}^{(cm)} \sim \left(\frac{t}{d}\right)^3 .
\end{aligned} \tag{84}$$

Finally, with the conformal factor late time dependence behaving as

$$\Omega^2(X^2) = \frac{L^2 a^2(X^2)}{X^2} = \frac{d^2\left(1+\frac{t^2}{d^2}\right)}{\left(\frac{t}{d}+\sqrt{1+\left(\frac{t}{d}\right)^2}\right)^2} \sim d^2 \tag{85}$$

we construct the comoving $K_{\mu\nu}^{(cm)}$ as

$$K_{\mu\nu}^{(cm)} = \Omega^2 k_{\mu\nu}^{(cm)} \tag{86}$$

to thus have

$$\begin{aligned}
&K_{00}^{(cm)} \sim \left(\frac{t}{d}\right) \qquad K_{01}^{(cm)} \sim \frac{d}{L}\left(\frac{t}{d}\right)^2 \qquad K_{02}^{(cm)} \sim d\left(\frac{t}{d}\right)^2 \qquad K_{03}^{(cm)} \sim d\left(\frac{t}{d}\right)^2 \\
&\qquad K_{11}^{(cm)} \sim \frac{d^2}{L^2}\left(\frac{t}{d}\right)^3 \qquad K_{22}^{(cm)} \sim d^2\left(\frac{t}{d}\right)^3 \qquad K_{33}^{(cm)} \sim d^2\left(\frac{t}{d}\right)^3 \\
&\qquad K_{12}^{(cm)} \sim \frac{d^2}{L}\left(\frac{t}{d}\right)^3 \qquad K_{13}^{(cm)} \sim \frac{d^2}{L}\left(\frac{t}{d}\right)^3 \qquad K_{23}^{(cm)} \sim d^2\left(\frac{t}{d}\right)^3 .
\end{aligned} \tag{87}$$

$$
\begin{aligned}
k_{00}^{(cm)} &= 2\frac{\partial p'}{\partial t}\frac{\partial r'}{\partial t}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial t}\right)^2 k_{11}^{(P)} \\
k_{01}^{(cm)} &= \frac{\partial p'}{\partial t}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial p'}{\partial r}k_{01}^{(P)} + \frac{\partial r'}{\partial t}\frac{\partial r'}{\partial r}k_{11}^{(P)} \\
k_{02}^{(cm)} &= \frac{\partial p'}{\partial t}k_{02}^{(P)} + \frac{\partial r'}{\partial t}k_{12}^{(P)} \\
k_{03}^{(cm)} &= \frac{\partial p'}{\partial t}k_{03}^{(P)} + \frac{\partial r'}{\partial t}k_{13}^{(P)} \\
k_{11}^{(cm)} &= 2\frac{\partial p'}{\partial r}\frac{\partial r'}{\partial r}k_{01}^{(P)} + \left(\frac{\partial r'}{\partial r}\right)^2 k_{11}^{(P)} \\
k_{22}^{(cm)} &= k_{22}^{(P)} \\
k_{33}^{(cm)} &= k_{33}^{(P)} \\
k_{12}^{(cm)} &= \frac{\partial p'}{\partial r}k_{02}^{(P)} + \frac{\partial r'}{\partial r}k_{12}^{(P)} \\
k_{13}^{(cm)} &= \frac{\partial p'}{\partial r}k_{03}^{(P)} + \frac{\partial r'}{\partial r}k_{13}^{(P)} \\
k_{23}^{(cm)} &= k_{23}^{(P)}
\end{aligned}
\tag{88}
$$

Original Transformation

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}}, \qquad r' = \frac{v}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{89}$$

The above transformation takes us from

$$ds^2 = \Omega^2(p', r')(dp'^2 - dr'^2 - r'^2 d\Omega^2) \tag{90}$$

to

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2 \right) \tag{91}$$

$$\Omega^2(p', r') \sim d^2 \frac{u^4}{4} = \frac{t^4}{4} \tag{92}$$

$$p' \sim u^0, \qquad r' \sim \frac{v}{u} \tag{93}$$

New Transformation

$$T = \left[u + (1+u^2)^{1/2} \right] (1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2} \right] v \tag{94}$$

The above transformation takes us from

$$ds^2 = \Omega^2(X^2)(dT^2 - dR^2 - R^2 d\Omega^2) \tag{95}$$

to

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2 \right). \tag{96}$$

$$\Omega^2(X^2) \sim \frac{d^2}{4} \tag{97}$$

$$T \sim 2u(1+v^2)^{1/2}, \qquad R \sim 2uv \tag{98}$$

Plane Waves

Original Coordinates

$$\sin(k(z' - p')) \tag{99}$$

$$p' = \frac{u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{100}$$

$$z' = r' \cos\theta = \frac{v \cos\theta}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{101}$$

$$z' - p' = \frac{v \cos\theta - u}{(1+u^2)^{1/2} + (1+v^2)^{1/2}} \tag{102}$$

For $u \to \infty$, the above converges and has asymptotic expansion

$$z' - p' \approx \cos\theta - \frac{1}{u}(v + (1+v^2)^{1/2}) \cos\theta + \frac{1}{u^2}() + O\left(\frac{1}{u^3}\right) \tag{103}$$

New Coordinates

$$\sin(k(Z-T)) \tag{104}$$

$$T = \left[u + (1+u^2)^{1/2}\right](1+v^2)^{1/2}, \qquad R = \left[u + (1+u^2)^{1/2}\right] v \tag{105}$$

$$Z = R\cos\theta = \left[u + (1+u^2)^{1/2}\right] v\cos\theta \tag{106}$$

$$Z - T = \left[u + (1+u^2)^{1/2}\right]\left[v\cos\theta - (1+v^2)^{1/2}\right] \tag{107}$$

Taking $\theta = 0$,

$$Z - T = \left[u + (1+u^2)^{1/2}\right]\left[v - (1+v^2)^{1/2}\right] \tag{108}$$