Coordinate Transformations RW $k < 0$ v7

For $K < 1$ FRW cosmology with $L^2 a^2 = t^2 + d^2$, the line element takes the form

$$\begin{aligned} ds^2 &= dt^2 - a(t)^2 \left(\frac{dr^2}{1 + r^2/L^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \\ &= d^2 \left[du^2 - (1 + u^2) \left(\frac{dv^2}{1 + v^2} + v^2 d\Omega^2 \right) \right], \end{aligned} \tag{1}$$

where we have introduced

$$u = \frac{t}{d}, \qquad v = \frac{r}{L}. \tag{2}$$

Original Coordinates

Transformations and Asymptotics:

$$p' = \frac{u}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}}, \qquad r' = \frac{v}{(1 + u^2)^{1/2} + (1 + v^2)^{1/2}} \tag{3}$$

$$u^2 = \frac{4p'^2}{(1 - (p' + r')^2)(1 - (p' - r')^2)}, \qquad v = \left(\frac{r'}{p'} \right) u \tag{4}$$

$$\Omega^2(p', r') = \frac{4L^2 a^2}{(1 - (p' + r')^2)(1 - (p' - r')^2)} = d^2 (1 + u^2) \left[(1 + u^2)^{1/2} + (1 + v^2)^{1/2} \right]^2 \tag{5}$$

Null Trajectory

In the u, v geometry, the condition for null separation (at fixed angle) is $u = v$. Inspection of coordinate transformation (3-5) shows the leading order ($u \gg 1$) contributions for null separation:

$$p' \sim 1, \qquad r' \sim 1, \qquad \Omega^2 \sim u^4. \tag{6}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{u} \qquad \frac{\partial p'}{\partial r} \sim \frac{1}{u}, \qquad \frac{\partial r'}{\partial t} \sim \frac{1}{u} \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{u}. \tag{7}$$

The leading behavior for the full $K^{(cm)}_{\mu\nu}$ behaves as

$$
\begin{aligned}
&K^{(cm)}_{00} \sim u^2 \\
&K^{(cm)}_{01} \sim u^2 \\
&K^{(cm)}_{02} \sim u^3 \\
&K^{(cm)}_{03} \sim u^3 \\
&K^{(cm)}_{11} \sim u^2 \\
&K^{(cm)}_{22} \sim u^4 \\
&K^{(cm)}_{33} \sim u^4 \\
&K^{(cm)}_{12} \sim u^3 \\
&K^{(cm)}_{13} \sim u^3 \\
&K^{(cm)}_{23} \sim u^4
\end{aligned} \tag{8}
$$

The purely angular sector of this result coincides with the null configuration given in PRD 2012.

Timelike Trajectory

For coordinate separations which are timelike we take $u \gg v$. In order to find the leading contribution in u, we will effectively take v to be finite on the order $\mathcal{O}(1)$, and take $u \gg 1$. These results yield a leading behavior of:

$$p' \sim 1, \qquad r' \sim \frac{1}{u}, \qquad \Omega^2 \sim u^4. \tag{9}$$

$$\frac{\partial p'}{\partial t} \sim \frac{1}{u^2} \qquad \frac{\partial p'}{\partial r} \sim \frac{1}{u}, \qquad \frac{\partial r'}{\partial t} \sim \frac{1}{u^2} \qquad \frac{\partial r'}{\partial r} \sim \frac{1}{u}. \tag{10}$$

The leading behavior for the full $K^{(cm)}_{\mu\nu}$ behaves as

$$
\begin{aligned}
&K^{(cm)}_{00} \sim 1 \\
&K^{(cm)}_{01} \sim u \\
&K^{(cm)}_{02} \sim u \\
&K^{(cm)}_{03} \sim u \\
&K^{(cm)}_{11} \sim u^2 \\
&K^{(cm)}_{22} \sim u^2 \\
&K^{(cm)}_{33} \sim u^2 \\
&K^{(cm)}_{12} \sim u^2 \\
&K^{(cm)}_{13} \sim u^2 \\
&K^{(cm)}_{23} \sim u^2
\end{aligned} \tag{11}
$$

Email Comment

The difference between the revised Appendix B and the results above resides only in the timelike $K_{t\theta}$ and $K_{t\phi}$ components. In the notation of Appendix B, when transforming from Cartesian to polar, a prefactor analgous to (B7) should be included for these modes, i.e.

$$K_{t'\theta} = \frac{\partial x^\alpha}{\partial \theta} K_{t'\alpha} = r' \cos\theta \cos\phi K_{t'x'} + r' \cos\theta \sin\phi K_{t'y'} - r \sin\theta K_{t'z'} \sim \frac{1}{t}. \tag{12}$$

If such a prefactor is included, then $K_{t\phi}$ and $K_{t\theta}$ have an overall supression of $1/t^3$, and then when multiplied by $p'\Omega^2$ behave in total as $\sim t$.

Concerning the difference between PRD 2012 and the results above, regarding $k_{\theta\theta}$ in particular, we note that eq. (114) in PRD has solution

$$k_{\theta\theta} \propto r'p'e^{iq(r'-p')}. \tag{13}$$

However, in solving for $k_{\theta\theta}$ in APM (via coordinate transformation from the flat $\Box^2 k_{\mu\nu} = 0$) we found solutions to obey

$$k_{\theta\theta} \propto r'^2 p'e^{iq(r'-p')}. \tag{14}$$

This additional factor of r' differentiates PRD and APM. In null configuations $r' \sim 1$, and thus the two results agree asymptotically. However, for lightlike configurations, $r' \sim 1/t$ and thus the leading angular sector behavior in PRD will behave as t^3 while in APM as t^2.

It still remains for me to figure out a). why the synchronous condition would yield a different r' dependence and b). why it appears the asymptotic behavior differs when working in the new coordinates system $(\Omega(T, R))$. Insight into the latter is expected to be found in the gauging procedure for each coordinate system