SVT3 RW Radiation $\Omega(x)$ $k<0$

1 Background

1.1 Comoving $a(t)$

First, determine the form of $a(t)$ for $\rho = 3p$ radiation in comoving coordinates

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) = -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \tag{1.1}$$

$$T_{\mu\nu} = p(4U_\mu U_\nu + g_{\mu\nu}), \qquad U_\mu = \delta^0_\mu \tag{1.2}$$

$$T_{00} = 3p, \qquad T_{ij} = a^2(t) p \tilde{g}_{ij} \tag{1.3}$$

$$G_{00} = -3ka^{-2} - 3\dot{a}^2 a^{-2}, \qquad G_{ij} = \tilde{g}_{ij}(k + \dot{a}^2 + 2a\ddot{a}) \tag{1.4}$$

$$\Delta_{\mu\nu} = G_{\mu\nu} + T_{\mu\nu} = 0 \tag{1.5}$$

$$\Delta_{00} = 3(p - ka^{-2} - \dot{a}^2 a^{-2}), \qquad \Delta_{ij} = \tilde{g}_{ij}(a^2 p + k + \dot{a}^2 + 2a\ddot{a}) \tag{1.6}$$

$$\rightarrow \qquad \boxed{p = ka^{-2} + \dot{a}^2 a^{-2}} \qquad \boxed{0 = k + \dot{a}^2 + a\ddot{a}} \tag{1.7}$$

With $k = -1/L^2$, we will follow APM (B1) and take

$$a^2(t) = \frac{d^2}{L^2}\left(1 + \frac{t^2}{d^2}\right) \tag{1.8}$$

$$p = -\frac{1}{d^2(1+t^2/d^2)^2} = -\frac{d^2}{L^4 a^4} \tag{1.9}$$

1.2 Conformal T, R Coordinates

Given $a(t)$ in the form (1.8), we may transform the metric from

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{1.10}$$

to the conformal flat form

$$ds^2 = \Omega^2(X)(-dT^2 + dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2), \tag{1.11}$$

via the coordinate transformation below with the corresponding scale factors:

$$T = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\left(1 + \left(\frac{r}{L}\right)^2\right)^{1/2} \qquad R = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1}\right)\frac{r}{L} \tag{1.12}$$

$$X^2 = T^2 - R^2 = \left(\frac{t}{d} + \sqrt{\left(\frac{t}{d}\right)^2 + 1} \right)^2 \tag{1.13}$$

$$a^2(X) = \frac{d^2}{L^2} \frac{(X^2+1)^2}{4X^2} \tag{1.14}$$

$$\Omega^2(X) = L^2 \frac{a^2(X)}{X^2} = \frac{d^2}{4}(1+X^{-2})^2 = d^2 \frac{(1+t^2/d^2)}{(t/d+(1+t^2/d^2)^{1/2})^2} \tag{1.15}$$

$$\frac{t}{d} = \pm\frac{(X^2-1)}{2X}, \qquad \frac{r}{L} = \begin{cases} RX, & \text{for} \quad u = -\frac{(X^2-1)}{2X} \\ \\ \frac{R}{X} & \text{for} \quad u = +\frac{(X^2-1)}{2X} \end{cases} \tag{1.16}$$

Transformation Matrix (taking + in (1.16)):

$$x'^{\mu} = (T, R, \theta, \phi), \qquad x^{\mu} = (t, r, \theta, \phi) \tag{1.17}$$

$$A'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} A_{\alpha\beta} \frac{\partial x^{\beta}}{\partial x'^{\nu}} = J^T A J \tag{1.18}$$

$$J = \frac{\partial x^{\sigma}}{\partial x'^{\rho}} = \begin{pmatrix} \frac{dT\left(-R^2+T^2+1\right)}{2(T^2-R^2)^{3/2}} & \frac{dR\left(R^2-T^2-1\right)}{2(T^2-R^2)^{3/2}} & 0 & 0 \\ -\frac{LRT}{(T^2-R^2)^{3/2}} & \frac{LT^2}{(T^2-R^2)^{3/2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.19}$$

$$A'_{\mu\nu} = \tag{1.20}$$

$$\begin{pmatrix} -\frac{T^2\left(d^2A_{00}\left(-R^2+T^2+1\right)^2+4LR\left(d\left(R^2-T^2-1\right)A_{01}+LRA_{11}\right)\right)}{4(R^2-T^2)^3} & \frac{T\left(d^2RA_{00}\left(-R^2+T^2+1\right)^2+2L\left(2LRA_{11}T^2+d\left(R^4-R^2-T^2\left(T^2+1\right)\right)A_{01}\right)\right)}{4(R^2-T^2)^3} & \frac{T\left(d\left(-R^2+T^2+1\right)A_{02}-2LRA_{12}\right)}{2(T^2-R^2)^{3/2}} & \frac{T\left(d\left(-R^2+T^2+1\right)A_{03}-2LRA_{13}\right)}{2(T^2-R^2)^{3/2}} \\ \frac{T\left(d^2RA_{00}\left(-R^2+T^2+1\right)^2+2L\left(2LRA_{11}T^2+d\left(R^4-R^2-T^2\left(T^2+1\right)\right)A_{01}\right)\right)}{4(R^2-T^2)^3} & \frac{4LT^2\left(dR\left(-R^2+T^2+1\right)A_{01}-LT^2A_{11}\right)-d^2R^2\left(-R^2+T^2+1\right)^2A_{00}}{4(R^2-T^2)^3} & -\frac{dR\left(-R^2+T^2+1\right)A_{02}-2LT^2A_{12}}{2(T^2-R^2)^{3/2}} & -\frac{dR\left(-R^2+T^2+1\right)A_{03}-2LT^2A_{13}}{2(T^2-R^2)^{3/2}} \\ \frac{T\left(d\left(-R^2+T^2+1\right)A_{02}-2LRA_{12}\right)}{2(T^2-R^2)^{3/2}} & -\frac{dR\left(-R^2+T^2+1\right)A_{02}-2LT^2A_{12}}{2(T^2-R^2)^{3/2}} & A_{22} & A_{23} \\ \frac{T\left(d\left(-R^2+T^2+1\right)A_{03}-2LRA_{13}\right)}{2(T^2-R^2)^{3/2}} & -\frac{dR\left(-R^2+T^2+1\right)A_{03}-2LT^2A_{13}}{2(T^2-R^2)^{3/2}} & A_{23} & A_{33} \end{pmatrix}$$

1.3 $T'_{\mu\nu}(T, R)$

$$T'_{\mu\nu} = p(4U'_{\mu}U'_{\nu} + g'_{\mu\nu}) \tag{1.21}$$

$$p = -d^2\Omega^{-4}X^{-4} \tag{1.22}$$

$$\begin{aligned} U'_{\mu} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} U_{\alpha} = UJ \\ &= \Omega\left(-\frac{T}{X}, \frac{R}{X}, 0, 0\right) \end{aligned} \tag{1.23}$$

$$g'_{\mu\nu} = \Omega^2 \mathrm{diag}\left(-1, 1, R^2, R^2\sin^2\theta\right) \tag{1.24}$$

2 Fluctuations

$$ds^2 = \Omega^2(x)(-d\tau^2 + \tilde{g}_{ij}dx^i dx^j + f_{\mu\nu}dx^\mu dx^\nu) \tag{2.1}$$

$$\tilde{g}_{ij} = \mathrm{diag}(-1, R^2, R^2\sin^2\theta) \tag{2.2}$$

$$f_{00} = -2\phi, \qquad f_{0i} = \tilde{\nabla}_i B + B_i, \qquad f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{2.3}$$

2.1 $\delta G_{\mu\nu}$

$$\begin{aligned}\delta G_{00} = {} & 6\dot{\psi}\dot{\Omega}\Omega^{-1} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} - 2\tilde{\nabla}_a\tilde{\nabla}^a\psi + 4\phi\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega \\ & +4\psi\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega + 4\Omega^{-1}\tilde{\nabla}_a\dot{\Omega}\tilde{\nabla}^a B - 2\dot{\Omega}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a B - 2\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\psi \\ & -2\phi\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega - 2\psi\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E \\ & +2\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}_a E\tilde{\nabla}^b\Omega - 4\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E \\ & +4B^a\Omega^{-1}\tilde{\nabla}_a\dot{\Omega} - 2B^a\dot{\Omega}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a + 2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^b E^a \\ & -4\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a - 4E^{ab}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega + 2E_{ab}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega\end{aligned} \tag{2.4}$$

$$\begin{aligned}\delta G_{0i} = {} & -\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i B + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i B - 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_i B + \Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_i B - 2\tilde{\nabla}_i\dot{\psi} \\ & -2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi + 2\dot{\psi}\Omega^{-1}\tilde{\nabla}_i\Omega - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_i\tilde{\nabla}_a\dot{E} - B_i\dot{\Omega}^2\Omega^{-2} + 2B_i\ddot{\Omega}\Omega^{-1} \\ & +\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a B_i - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i - 2B_i\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega + \Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a B_i - \Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\dot{E}_i \\ & +B_i\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega - \Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i B^a - \Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i\dot{E}^a - 2\dot{E}_{ia}\Omega^{-1}\tilde{\nabla}^a\Omega\end{aligned} \tag{2.5}$$

$$\begin{aligned}\delta G_{ij} = {} & -2\ddot{\psi}\tilde{g}_{ij} + 2\dot{\Omega}^2\tilde{g}_{ij}\phi\Omega^{-2} + 2\dot{\Omega}^2\tilde{g}_{ij}\psi\Omega^{-2} - 2\dot{\phi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\dot{\psi}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} - 4\ddot{\Omega}\tilde{g}_{ij}\phi\Omega^{-1} \\ & -4\ddot{\Omega}\tilde{g}_{ij}\psi\Omega^{-1} - 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a B - \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E} \\ & -\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi + \tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi - 4\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\dot{\Omega}\tilde{\nabla}^a B + 2\dot{\Omega}\tilde{g}_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a B \\ & -2\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\dot{B} - 2\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\phi + 2\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E \\ & -2\tilde{g}_{ij}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}_a E\tilde{\nabla}^b\Omega + 4\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E + 2\Omega^{-1}\tilde{\nabla}_i\Omega\tilde{\nabla}_j\psi \\ & +2\Omega^{-1}\tilde{\nabla}_i\psi\tilde{\nabla}_j\Omega + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i B + \tilde{\nabla}_j\tilde{\nabla}_i\dot{B} - \tilde{\nabla}_j\tilde{\nabla}_i\ddot{E} - 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E} \\ & -2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_i E + 4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i E - 4\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E + 2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E \\ & +\tilde{\nabla}_j\tilde{\nabla}_i\phi - \tilde{\nabla}_j\tilde{\nabla}_i\psi - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i\tilde{\nabla}_a E \\ & -4B^a\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\dot{\Omega} + 2B^a\dot{\Omega}\tilde{g}_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{B}^a\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega + 2\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a \\ & -2\tilde{g}_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^b E^a + 4\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i B_j + \tfrac{1}{2}\tilde{\nabla}_i\dot{B}_j - \tfrac{1}{2}\tilde{\nabla}_i\ddot{E}_j \\ & -\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_i E_j + 2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_i E_j - 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_j \\ & +\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_j + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j B_i + \tfrac{1}{2}\tilde{\nabla}_j\dot{B}_i - \tfrac{1}{2}\tilde{\nabla}_j\ddot{E}_i - \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i - \dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j E_i \\ & +2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j E_i - 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_i + \Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_i - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E_a \\ & -\ddot{E}_{ij} - 2\dot{\Omega}^2 E_{ij}\Omega^{-2} - 2\dot{E}_{ij}\dot{\Omega}\Omega^{-1} + 4\ddot{\Omega}E_{ij}\Omega^{-1} + \tilde{\nabla}_a\tilde{\nabla}^a E_{ij} - 4E_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega \\ & +2\Omega^{-1}\tilde{\nabla}_a E_{ij}\tilde{\nabla}^a\Omega + 2E_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega + 4E^{ab}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\ & -2E_{ab}\tilde{g}_{ij}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_{ja} - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_{ia}\end{aligned} \tag{2.6}$$

$$\begin{aligned}g^{\mu\nu}\delta G_{\mu\nu} &= \Omega^{-2}(-\delta G_{00} + \tilde{g}^{ab}\delta G_{ab}) \\ &= 6\dot{\Omega}^2\phi\Omega^{-4} + 6\dot{\Omega}^2\psi\Omega^{-4} - 6\dot{\phi}\dot{\Omega}\Omega^{-3} - 18\dot{\psi}\dot{\Omega}\Omega^{-3} - 12\ddot{\Omega}\phi\Omega^{-3} - 12\ddot{\Omega}\psi\Omega^{-3} - 6\ddot{\psi}\Omega^{-2} \\ &\quad -6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a B - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B} + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E} + 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}\end{aligned}$$

$$
\begin{aligned}
&-2\dot{\Omega}^2\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^a E + 4\ddot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a E - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi + 4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi - 4\phi\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega \\
&-4\psi\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\Omega - 16\Omega^{-3}\tilde{\nabla}_a\dot{\Omega}\tilde{\nabla}^a B + 8\dot{\Omega}\Omega^{-4}\tilde{\nabla}_a\Omega\tilde{\nabla}^a B - 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\dot{B} \\
&-6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\phi + 6\Omega^{-3}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\psi + 2\phi\Omega^{-4}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega + 2\psi\Omega^{-4}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega \\
&+2\Omega^{-4}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E - 4\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a E\tilde{\nabla}_b\tilde{\nabla}^b\Omega + 6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E \\
&-8\Omega^{-4}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}_a E\tilde{\nabla}^b\Omega + 16\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E. - 16B^a\Omega^{-3}\tilde{\nabla}_a\dot{\Omega} \\
&+8B^a\dot{\Omega}\Omega^{-4}\tilde{\nabla}_a\Omega - 6\dot{B}^a\Omega^{-3}\tilde{\nabla}_a\Omega + 6\Omega^{-3}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a - 8\Omega^{-4}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^b E^a \\
&+16\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a + 16E^{ab}\Omega^{-3}\tilde{\nabla}_b\tilde{\nabla}_a\Omega - 8E_{ab}\Omega^{-4}\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega \qquad (2.7)
\end{aligned}
$$

2.2 $\delta T_{\mu\nu}$

From $\delta(g^{\mu\nu}U_\mu U_\nu) = 0$ we find

$$
\begin{aligned}
U^\mu \delta U_\mu &= \frac{1}{2}\left(h_{\mu\nu}U^\mu U^\nu\right) \\
\rightarrow T\delta U_0 + R\delta U_r &= \Omega X^{-1}\left[-\phi T^2 + (-\psi + \tilde{\nabla}_r\tilde{\nabla}_r E + \tilde{\nabla}_r E_r + E_{rr})R^2 + (\tilde{\nabla}_r B + B_r)TR\right] \qquad (2.8)
\end{aligned}
$$

$$
p = -d^2\Omega^{-4}X^{-4}, \qquad U_\mu = \Omega\left(-\frac{T}{X}, \frac{R}{X}, 0, 0\right), \quad U^\mu = \Omega^{-1}\left(\frac{T}{X}, \frac{R}{X}, 0, 0\right) \qquad (2.9)
$$

$$
\delta T_{\mu\nu} = \delta p(4U_\mu U_\nu + \Omega^2\tilde{g}_{\mu\nu}) + p\left(4\delta U_\mu U_\nu + 4U_\mu\delta U_\nu + \Omega^2 f_{\mu\nu}\right) \qquad (2.10)
$$

$$
\delta T_{00} = \Omega^2\delta p(4T^2X^{-2} - 1) - 8\Omega T X^{-1}p\delta U_0 - 2\Omega^2 p\phi \qquad (2.11)
$$

$$
\delta T_{0r} = -4\Omega^2\delta p T R X^{-2} + 4\Omega p R X^{-1}\delta U_0 - 4\Omega p T X^{-1}\delta U_r + \Omega^2 p(\tilde{\nabla}_r B + B_r) \qquad (2.12)
$$

$$
\delta T_{0\theta} = -4\Omega T X - 1p\delta U_\theta + \Omega^2 p(\tilde{\nabla}_\theta B + B_\theta) \qquad (2.13)
$$

$$
\delta T_{0\phi} = -4\Omega T X - 1p\delta U_\phi + \Omega^2 p(\tilde{\nabla}_\phi B + B_\phi) \qquad (2.14)
$$

$$
\delta T_{rr} = \Omega^2\delta p(4R^2X^{-2} + 1) + 8\Omega R X^{-1}p\delta U_r + 2\Omega^2 p(-\phi + \tilde{\nabla}_r\tilde{\nabla}_r E + \tilde{\nabla}_r E_r + E_{rr}) \qquad (2.15)
$$

$$
g^{\mu\nu}\delta T_{\mu\nu} = \Omega^{-1}X^{-1}p(T\delta U_0 + R\delta U_r) + p(2\phi - 6\psi + 2\tilde{\nabla}_a\tilde{\nabla}^a E) \qquad (2.16)
$$