3-Space Einstein Tensor Gauge Dependence v1

1 Covariant Form $\delta G_{ij} = \delta T_{ij}$

Within the geometry of

$$ds^2 = (g_{ij}^{(0)} + h_{ij})dx^i dx^j \tag{1.1}$$

with maximally symmetric background

$$g_{ij}^{(0)} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.2}$$

assume the metric perturbation can be (covariant) SVT decomposed as

$$h_{ij} = -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}, \tag{1.3}$$

with 3-trace

$$h = -6\psi + 2\nabla^a\nabla_a E. \tag{1.4}$$

The three dimensional Einstein background field equations take the form $G_{ij}^{(0)} = -T_{ij}^{(0)}$. Since the background is maximally symmetric, the solution to the zeroth order Einstein equations yields energy momentum tensor $T_{ij}^{(0)} = \Lambda g_{ij}^{(0)} = k g_{ij}^{(0)}$.

The perturbed Einstein equations then take the form,

$$\begin{aligned} \delta G_{ij} &= -\delta T_{ij} && (1.5) \\ &= k h_{ij} && (1.6) \end{aligned}$$

Evaluating the Einstein tensor in terms of (3) yields

$$\delta G_{ij} = \tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h + \tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} - \tfrac{1}{2}\nabla_i\nabla_a h_j{}^a - \tfrac{1}{2}\nabla_j\nabla_a h_i{}^a + \tfrac{1}{2}\nabla_j\nabla_i h, \tag{1.7}$$

which takes the SVT form

$$\delta G_{ij} = \nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi. \tag{1.8}$$

Composing the field equation $\delta G_{\mu\nu} = -\delta T_{\mu\nu}$ yields

$$\begin{aligned} &\nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi = \\ &k(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}), \end{aligned} \tag{1.9}$$

which may be simplified as

$$(\nabla_a\nabla^a - 2k)E_{ij} + g_{ij}\nabla_a\nabla^a\psi - \nabla_j\nabla_i\psi + 2kg_{ij}\psi = 0. \tag{1.10}$$

Taking the trace gives the solution for ψ

$$(\nabla_a \nabla^a + 3k)\psi = 0 \tag{1.11}$$

As the above equation is traceless, it is not clear how to decouple ψ from the tensor mode E_{ij}.

$$\nabla^i \nabla^j h_{ij} = -2\nabla^i \nabla_i \psi + 2\nabla^i \nabla_i \nabla^j \nabla_j E + 2k\nabla_i \nabla^i E \tag{1.12}$$

$$\nabla^j \delta G_{ij} = -2k\nabla_i \psi + k(\nabla^a \nabla_a + 2k)E_i + 2k\nabla^a \nabla_a \nabla_i E \tag{1.13}$$

$$\nabla^i \nabla^j \delta G_{ij} = -2k\nabla^a \nabla_a \psi + 2k\nabla^a \nabla_a (\nabla^b \nabla_b + 2k)E \tag{1.14}$$

2 Conformal to Flat $\delta G_{ij} = \delta T_{ij}$

The 3-space of constant curvature can be expressed in the conformal flat form as in (**??**)

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right) \\ &= \frac{4}{(1+k\rho^2)^2}\left(d\rho^2 + \rho^2 d\Omega^2\right) \end{aligned} \tag{2.1}$$

Within the above geometry, the perturbed Einstein tensor takes the form (with $\tilde{\nabla}$ denoting flat space derivative)

$$\begin{aligned} \delta G_{ij} = \quad & g_{ij}\tilde{\nabla}_a \tilde{\nabla}^a \psi + 2g_{ij}\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a E - 2g_{ij}\Omega^{-2}\tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}_a E \tilde{\nabla}^b \Omega \\ & +4g_{ij}\Omega^{-1}\tilde{\nabla}_b \tilde{\nabla}_a \Omega \tilde{\nabla}^b \tilde{\nabla}^a E + 2\Omega^{-1}\tilde{\nabla}_i \Omega \tilde{\nabla}_j \psi + 2\Omega^{-1}\tilde{\nabla}_i \psi \tilde{\nabla}_j \Omega - 4\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i E \\ & +2\Omega^{-2}\tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i E - \tilde{\nabla}_j \tilde{\nabla}_i \psi - 2\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i \tilde{\nabla}_a E \\ \\ & +2g_{ij}\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_b \tilde{\nabla}^b E_a - 2g_{ij}\Omega^{-2}\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega \tilde{\nabla}^b E^a + 4g_{ij}\Omega^{-1}\tilde{\nabla}_b \tilde{\nabla}_a \Omega \tilde{\nabla}^b E^a \\ & -2\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \Omega \tilde{\nabla}_i E_j + \Omega^{-2}\tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tilde{\nabla}_i E_j - 2\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \Omega \tilde{\nabla}_j E_i \\ & +\Omega^{-2}\tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \tilde{\nabla}_j E_i - 2\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_j \tilde{\nabla}_i E_a \\ \\ & +\tilde{\nabla}_a \tilde{\nabla}^a E_{ij} - 4E_{ij}\Omega^{-1}\tilde{\nabla}_a \tilde{\nabla}^a \Omega + 2\Omega^{-1}\tilde{\nabla}_a E_{ij}\tilde{\nabla}^a \Omega + 2E_{ij}\Omega^{-2}\tilde{\nabla}_a \Omega \tilde{\nabla}^a \Omega \\ & +4E^{ab}g_{ij}\Omega^{-1}\tilde{\nabla}_b \tilde{\nabla}_a \Omega - 2E_{ab}g_{ij}\Omega^{-2}\tilde{\nabla}^a \Omega \tilde{\nabla}^b \Omega - 2\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_i E_{ja} - 2\Omega^{-1}\tilde{\nabla}^a \Omega \tilde{\nabla}_j E_{ia} \end{aligned} \tag{2.2}$$

with energy momentum tensor

$$\delta T_{ij} = k\Omega^2 h_{ij} = k\Omega^2(-2g_{ij}\psi + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \nabla_j E_i + 2E_{ij}) \tag{2.3}$$

Appendix A Conformal to Flat Maximal 3-Space

A.1 $k < 0$

The 3-space of constant curvature can be expressed in the conformal flat form (using $-k = 1/L^2$) as

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right) && \text{(A.1)} \\ &= \frac{4}{\left(1-\rho^2/L^2\right)^2}\left(d\rho^2 + \rho^2 d\Omega^2\right) && \text{(A.2)} \\ &= \frac{dr^2}{1+r^2/L^2} + r^2 d\Omega^2 && \text{(A.3)} \end{aligned}$$

The relevant transformations are:

$$\begin{aligned} \rho(r) &= \frac{r}{1+\left(1+r^2/L^2\right)^{1/2}}, \qquad \Omega^2(r) = \left(1+\left[1+r^2/L^2\right)^{1/2}\right]^2 \\ r(\rho) &= \frac{2\rho}{1-\rho^2/L^2}, \qquad \Omega^2(\rho) = \frac{4}{\left(1-\rho^2/L^2\right)^2} && \text{(A.4)} \end{aligned}$$

A.2 $k > 0$

Now instead we set $k = 1/L^2$ to express the line element as

$$\begin{aligned} ds^2 &= \Omega^2(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right) && \text{(A.5)} \\ &= \frac{4}{\left(1+\rho^2/L^2\right)^2}\left(d\rho^2 + \rho^2 d\Omega^2\right) && \text{(A.6)} \\ &= \frac{dr^2}{1-r^2/L^2} + r^2 d\Omega^2 && \text{(A.7)} \end{aligned}$$

The relevant transformations are:

$$\begin{aligned} \rho(r) &= \frac{r}{1+\left(1-r^2/L^2\right)^{1/2}}, \qquad \Omega^2(r) = \left[1+\left(1-r^2/L^2\right)^{1/2}\right] \\ r(\rho) &= \frac{2\rho}{1+\rho^2/L^2}, \qquad \Omega^2(\rho) = \frac{4}{\left(1+\rho^2/L^2\right)^2} && \text{(A.8)} \end{aligned}$$

After calculation, we see that solutions to positive/negative geometries are affected by $L^2 \to -L^2$. This is not quite the case in 4D comoving RW, where we must make use of trigonometric and hyperbolic transformations depending on the sign of the curvature.

Appendix B δG_{ij} Under Conformal Transformation

Under general conformal transformation $g_{ij} \to \Omega^2(x) g_{ij}$, the Einstein tensor transforms as

$$\begin{aligned} G_{ij} &\to G_{ij} + S_{ij} \\ &= G_{ij} + \Omega^{-1}\left(-2g_{ij}\tilde{\nabla}^a\tilde{\nabla}_a\Omega + 2\tilde{\nabla}_i\tilde{\nabla}_j\Omega\right) + \Omega^{-2}\left(g_{ij}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega - 4\tilde{\nabla}_i\Omega\tilde{\nabla}_j\Omega\right). && \text{(B.1)} \end{aligned}$$

Perturbing the above to first order yields the transformation of δG_{ij}:

$$\delta G_{ij} \to \delta G_{ij} + \delta S_{ij}, \tag{B.2}$$

where

$$\begin{aligned} \delta S_{ij} = {} & -2h_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega - g_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}^a h + \Omega^{-1}\tilde{\nabla}_a h_{ij}\tilde{\nabla}^a\Omega + h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega \\ & +2g_{ij}\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b h_a{}^b + 2g_{ij}h^{ab}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega - g_{ij}h_{ab}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega - \Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_i h_{ja} \\ & -\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j h_{ia}. && \text{(B.3)} \end{aligned}$$

In the conformal to flat metric (??), δG_{ij} as defined by (??) takes the same form as (??) with $k=0$, and δS_{ij} evaluates to

$$\begin{aligned}
\delta S_{ij} = \quad & 2g_{ij}\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a E - 2g_{ij}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}_a E\tilde{\nabla}^b\Omega + 4g_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b\tilde{\nabla}^a E \\
& +2\Omega^{-1}\tilde{\nabla}_i\Omega\tilde{\nabla}_j\psi + 2\Omega^{-1}\tilde{\nabla}_i\psi\tilde{\nabla}_j\Omega - 4\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E + 2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E \\
& -2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i\tilde{\nabla}_a E \\
& +2g_{ij}\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}^b E_a - 2g_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega\tilde{\nabla}^b E^a + 4g_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega\tilde{\nabla}^b E^a \\
& -2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_j + \Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_j - 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_i \\
& +\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_i - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j\tilde{\nabla}_i E_a \\
& -4E_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\Omega + 2\Omega^{-1}\tilde{\nabla}_a E_{ij}\tilde{\nabla}^a\Omega + 2E_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega + 4E^{ab}g_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
& -2E_{ab}g_{ij}\Omega^{-2}\tilde{\nabla}^a\Omega\tilde{\nabla}^b\Omega - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_i E_{ja} - 2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_j E_{ia}.
\end{aligned} \tag{B.4}$$

Appendix C Maximal 3-Space Geometric Quantities

Geometry

$$ds^2 = g_{ij}dx^i dx^j = \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right): \tag{C.1}$$

$$R_{ijkl} = k(g_{jk}g_{il} - g_{ik}g_{jl}), \qquad R_{ij} = -2kg_{ij}, \qquad R = -6k \tag{C.2}$$

$$[\nabla_i, \nabla_j]V_k = V_m R^m{}_{kij} = k(g_{ki}g^m{}_j - g^m{}_i g_{kj})V_m = k(g_{ik}V_j - g_{jk}V_i) \tag{C.3}$$

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\
\Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \quad \text{with all others zero}
\end{aligned} \tag{C.4}$$