

# Special Gauge v5 Matthew

## Transverse Gauge

Under the perturbation  $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ , the Einstein tensor is

$$\begin{aligned}\delta G_{\mu\nu} = & -\frac{1}{2}h_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} + \frac{1}{2}h_{\nu}{}^{\alpha}R_{\mu\alpha} + \frac{1}{2}h_{\mu}{}^{\alpha}R_{\nu\alpha} - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \frac{1}{2}\nabla_{\alpha}\nabla^{\alpha}h_{\mu\nu} \\ & - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}h + \frac{1}{2}g_{\mu\nu}\nabla_{\beta}\nabla_{\alpha}h^{\alpha\beta} - \frac{1}{2}\nabla_{\mu}\nabla_{\alpha}h_{\nu}{}^{\alpha} - \frac{1}{2}\nabla_{\nu}\nabla_{\alpha}h_{\mu}{}^{\alpha} + \frac{1}{2}\nabla_{\nu}\nabla_{\mu}h.\end{aligned}\quad (1)$$

If we now substitute the covariant transverse gauge, i.e

$$\nabla^{\mu}h_{\mu\nu} = \frac{1}{2}\nabla_{\nu}h \quad (2)$$

then the perturbation equation becomes

$$\delta G_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}h^{\alpha\beta}R_{\alpha\beta} + \frac{1}{2}h_{\nu}{}^{\alpha}R_{\mu\alpha} + \frac{1}{2}h_{\mu}{}^{\alpha}R_{\nu\alpha} - h^{\alpha\beta}R_{\mu\alpha\nu\beta} + \frac{1}{2}\nabla_{\alpha}\nabla^{\alpha}h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}h. \quad (3)$$

Now set the background metric as  $g_{\mu\nu}^{(0)} = \Omega^2\eta_{\mu\nu}$ , such that (3) becomes

$$\begin{aligned}\delta G_{\mu\nu} = & -\eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\beta}h_{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_{\alpha}h\partial_{\beta}\Omega + \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\beta}\Omega \\ & + 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\beta}\Omega - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_{\alpha}\Omega\partial_{\beta}\Omega + \frac{1}{2}\eta^{\alpha\beta}\Omega^{-2}\partial_{\beta}\partial_{\alpha}h_{\mu\nu} \\ & - \frac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_{\beta}\partial_{\alpha}h - 3\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-3}\partial_{\beta}\partial_{\alpha}\Omega + \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_{\beta}\partial_{\alpha}\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\Omega^{-3}\partial_{\beta}\partial_{\mu}\Omega \\ & + 2\eta^{\alpha\beta}h_{\mu\alpha}\Omega^{-3}\partial_{\beta}\partial_{\nu}\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\mu}h_{\nu\beta} - 6\eta^{\alpha\beta}h_{\nu\beta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\mu}\Omega \\ & - \eta^{\alpha\beta}\Omega^{-3}\partial_{\beta}h_{\nu\alpha}\partial_{\mu}\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\nu}h_{\mu\beta} - 6\eta^{\alpha\beta}h_{\mu\beta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\nu}\Omega \\ & - \eta^{\alpha\beta}\Omega^{-3}\partial_{\beta}h_{\mu\alpha}\partial_{\nu}\Omega + 3h\Omega^{-2}\partial_{\mu}\Omega\partial_{\nu}\Omega - h\Omega^{-1}\partial_{\nu}\partial_{\mu}\Omega.\end{aligned}\quad (4)$$

where  $h = g_{(0)}^{\mu\nu}h_{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}h_{\mu\nu}$ . Note that the covariant box term  $\frac{1}{2}\nabla_{\alpha}\nabla^{\alpha}$  yields another gauge condition to be substituted from the  $-\eta^{\alpha\beta}\Omega^{-3}\partial_{\beta}h_{\nu\alpha}\partial_{\mu}\Omega$  and  $-\eta^{\alpha\beta}\Omega^{-3}\partial_{\beta}h_{\mu\alpha}\partial_{\nu}\Omega$  terms. As such, we will need an expression for the transverse gauge with respect to  $g_{\mu\nu}^{(0)} = \Omega^2\eta_{\mu\nu}$ . This is

$$\eta^{\alpha\beta}\partial_{\alpha}h_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}h_{\nu\alpha}\partial_{\beta}\Omega + \frac{1}{2}\Omega^2\partial_{\nu}h + \Omega h\partial_{\nu}\Omega. \quad (5)$$

Within the above gauge, the Einstein tensor becomes

$$\begin{aligned}\delta G_{\mu\nu} = & -\eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\beta}h_{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_{\alpha}h\partial_{\beta}\Omega + \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\beta}\Omega \\ & + 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\beta}\Omega - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_{\alpha}\Omega\partial_{\beta}\Omega + \frac{1}{2}\eta^{\alpha\beta}\Omega^{-2}\partial_{\beta}\partial_{\alpha}h_{\mu\nu} \\ & - \frac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_{\beta}\partial_{\alpha}h - 3\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-3}\partial_{\beta}\partial_{\alpha}\Omega + \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_{\beta}\partial_{\alpha}\Omega + 2\eta^{\alpha\beta}h_{\nu\alpha}\Omega^{-3}\partial_{\beta}\partial_{\mu}\Omega \\ & + 2\eta^{\alpha\beta}h_{\mu\alpha}\Omega^{-3}\partial_{\beta}\partial_{\nu}\Omega + \eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\mu}h_{\nu\beta} - 4\eta^{\alpha\beta}h_{\nu\beta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\mu}\Omega \\ & + \eta^{\alpha\beta}\Omega^{-3}\partial_{\alpha}\Omega\partial_{\nu}h_{\mu\beta} - \frac{1}{2}\Omega^{-1}\partial_{\mu}\Omega\partial_{\nu}h - 4\eta^{\alpha\beta}h_{\mu\beta}\Omega^{-4}\partial_{\alpha}\Omega\partial_{\nu}\Omega - \frac{1}{2}\Omega^{-1}\partial_{\mu}h\partial_{\nu}\Omega \\ & + h\Omega^{-2}\partial_{\mu}\Omega\partial_{\nu}\Omega - h\Omega^{-1}\partial_{\nu}\partial_{\mu}\Omega.\end{aligned}\quad (6)$$

Evaluating for  $\Omega(\tau)$  yields

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-8\Omega^{-6}\dot{\Omega}^2 + 4\Omega^{-5}\ddot{\Omega})h_{00} + (-2\Omega^{-4}\dot{\Omega}^2 + \Omega^{-3}\ddot{\Omega} - \frac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_{\mu}\partial_{\nu} + \Omega^{-3}\dot{\Omega}\partial_0)h \quad (7)$$

$$\delta G_{00} = (5\Omega^{-4}\dot{\Omega}^2 - \Omega^{-3}\ddot{\Omega} + \frac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-3}\dot{\Omega}\partial_0)h_{00} + (\frac{1}{2}\Omega^{-2}\dot{\Omega}^2 - \frac{1}{2}\Omega^{-1}\ddot{\Omega} + \frac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{3}{2}\Omega^{-1}\dot{\Omega}\partial_0)h. \quad (8)$$

$$\delta G_{0i} = -\Omega^{-3}\dot{\Omega}\partial_i h_{00} + (2\Omega^{-4}\dot{\Omega}^2 + \Omega^{-3}\ddot{\Omega} + \frac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu)h_{0i} - \frac{1}{2}\Omega^{-1}\dot{\Omega}\partial_i h. \quad (9)$$

$$\delta G_{11} = \Omega^{-4}\dot{\Omega}^2 h_{00} - 2\Omega^{-3}\dot{\Omega}\partial_1 h_{01} + (-2\Omega^{-4}\dot{\Omega}^2 + 3\Omega^{-3}\ddot{\Omega} + \frac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)h_{11} + (\frac{1}{2}\Omega^{-2}\dot{\Omega}^2 - \frac{1}{2}\Omega^{-1}\ddot{\Omega} - \frac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{2}\Omega^{-1}\dot{\Omega}\partial_0)h. \quad (10)$$

$$\delta G_{12} = -\Omega^{-3}\dot{\Omega}\partial_2 h_{01} - \Omega^{-3}\dot{\Omega}\partial_1 h_{02} + (-2\Omega^{-4}\dot{\Omega}^2 + 3\Omega^{-3}\ddot{\Omega} + \frac{1}{2}\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \Omega^{-3}\dot{\Omega}\partial_0)h_{12}. \quad (11)$$

In a deSitter background  $\Omega(\tau) = \frac{1}{H\tau}$ ,  $\delta G_{\mu\nu}$  evaluates to

$$\Omega^{-2}\eta^{\mu\nu}\delta G_{\mu\nu} = (-\frac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h \quad (12)$$

$$\delta G_{00} = (3H^2 + \frac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu + H^2\tau\partial_0)h_{00} + (-\frac{1}{2}\tau^{-2} + \frac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{3}{2}\tau^{-1}\partial_0)h \quad (13)$$

$$\delta G_{0i} = H^2\tau\partial_i h_{00} + (4H^2 + \frac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu)h_{0i} + \frac{1}{2}\tau^{-1}\partial_i h \quad (14)$$

$$\delta G_{11} = H^2 h_{00} + 2H^2\tau\partial_1 h_{01} + (4H^2 + \frac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h_{11} + (-\frac{1}{2}\tau^{-2} - \frac{1}{4}\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{1}{2}\tau^{-1}\partial_0)h \quad (15)$$

$$\delta G_{12} = H^2\tau\partial_2 h_{01} + H^2\tau\partial_1 h_{02} + (4H^2 + \frac{1}{2}H^2\tau^2\eta^{\mu\nu}\partial_\mu\partial_\nu - H^2\tau\partial_0)h_{12}. \quad (16)$$

## Special $K_{\mu\nu}$ Gauge

The perturbed Einstein tensor  $\delta G_{\mu\nu}(h_{\mu\nu})$  evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \quad (17)$$

is calculated as

$$\begin{aligned} \delta G_{\mu\nu} = & \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \frac{1}{2}\partial_\nu\partial_\mu h. \end{aligned} \quad (18)$$

Now we split  $h_{\mu\nu}$  into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \quad (19)$$

where  $h = \eta^{\mu\nu} h_{\mu\nu}$ . With this substitution, (2) takes the form

$$\begin{aligned}
\delta G_{\mu\nu} = & -2\eta^{\alpha\beta} K_{\mu\beta} \Omega^{-1} \partial_\alpha \partial_\nu \Omega + \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\beta K_{\mu\nu} - \eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega \\
& + \eta^{\alpha\beta} K_{\mu\nu} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega + \frac{1}{2} \eta^{\alpha\beta} \partial_\beta \partial_\alpha K_{\mu\nu} + 2\eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-1} \partial_\beta \partial_\alpha \Omega \\
& - 2\eta^{\alpha\beta} K_{\mu\nu} \Omega^{-1} \partial_\beta \partial_\alpha \Omega + 2\eta^{\alpha\beta} \eta^{\gamma\zeta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha \Omega \partial_\zeta K_{\beta\gamma} + \frac{1}{2} \eta^{\alpha\beta} \eta^{\gamma\zeta} \eta_{\mu\nu} \partial_\zeta \partial_\beta K_{\alpha\gamma} \\
& - \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\mu K_{\nu\beta} - \frac{1}{2} \eta^{\alpha\beta} \partial_\mu \partial_\beta K_{\nu\alpha} - \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\nu K_{\mu\beta} + 2\eta^{\alpha\beta} K_{\mu\beta} \Omega^{-1} \partial_\nu \partial_\alpha \Omega \\
& - \frac{1}{2} \eta^{\alpha\beta} \partial_\nu \partial_\beta K_{\mu\alpha} + \frac{3}{4} \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha \Omega \partial_\beta h - \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha h \partial_\beta \Omega - \frac{1}{4} \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\beta \partial_\alpha h \\
& - \frac{1}{8} \partial_\mu \partial_\nu h - \frac{1}{4} \Omega^{-1} \partial_\mu \Omega \partial_\nu h - \frac{1}{4} \Omega^{-1} \partial_\mu h \partial_\nu \Omega + \frac{3}{8} \partial_\nu \partial_\mu h.
\end{aligned} \tag{20}$$

Now we impose a generalized gauge of the form

$$\eta^{\alpha\beta} \partial_\alpha K_{\beta\nu} = \Omega^{-1} J \eta^{\alpha\beta} K_{\nu\alpha} \partial_\beta \Omega + P \partial_\nu h + R \Omega^{-1} h \partial_\nu \Omega. \tag{21}$$

Within this gauge, (4) is evaluated as

$$\begin{aligned}
\delta G_{\mu\nu} = & -2\eta^{\alpha\beta} K_{\mu\beta} \Omega^{-1} \partial_\alpha \partial_\nu \Omega + \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\beta K_{\mu\nu} - \frac{1}{4} \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha h \partial_\beta \Omega \\
& + 2P \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha h \partial_\beta \Omega + \frac{1}{2} J P \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha h \partial_\beta \Omega + \frac{1}{2} R \eta^{\alpha\beta} \eta_{\mu\nu} \Omega^{-1} \partial_\alpha h \partial_\beta \Omega \\
& - \eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega + \frac{3}{2} J \eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega \\
& + \frac{1}{2} J^2 \eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega + \eta^{\alpha\beta} K_{\mu\nu} \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega + \frac{3}{2} R \eta^{\alpha\beta} \eta_{\mu\nu} h \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega \\
& + \frac{1}{2} J R \eta^{\alpha\beta} \eta_{\mu\nu} h \Omega^{-2} \partial_\alpha \Omega \partial_\beta \Omega + \frac{1}{2} \eta^{\alpha\beta} \partial_\beta \partial_\alpha K_{\mu\nu} - \frac{1}{4} \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\beta \partial_\alpha h + \frac{1}{2} P \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\beta \partial_\alpha h \\
& + 2\eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-1} \partial_\beta \partial_\alpha \Omega + \frac{1}{2} J \eta^{\alpha\gamma} \eta^{\beta\zeta} \eta_{\mu\nu} K_{\gamma\zeta} \Omega^{-1} \partial_\beta \partial_\alpha \Omega - 2\eta^{\alpha\beta} K_{\mu\nu} \Omega^{-1} \partial_\beta \partial_\alpha \Omega \\
& + \frac{1}{2} R \eta^{\alpha\beta} \eta_{\mu\nu} h \Omega^{-1} \partial_\beta \partial_\alpha \Omega - \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\mu K_{\nu\beta} - \frac{1}{2} J \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\mu K_{\nu\beta} \\
& + \frac{1}{2} J \eta^{\alpha\beta} K_{\nu\beta} \Omega^{-2} \partial_\alpha \Omega \partial_\mu \Omega - \frac{1}{2} J \eta^{\alpha\beta} K_{\nu\beta} \Omega^{-1} \partial_\mu \partial_\alpha \Omega - \frac{1}{8} \partial_\mu \partial_\nu h - \frac{1}{2} P \partial_\mu \partial_\nu h \\
& - \frac{1}{2} R h \Omega^{-1} \partial_\mu \partial_\nu \Omega - \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\nu K_{\mu\beta} - \frac{1}{2} J \eta^{\alpha\beta} \Omega^{-1} \partial_\alpha \Omega \partial_\nu K_{\mu\beta} - \frac{1}{4} \Omega^{-1} \partial_\mu \Omega \partial_\nu h \\
& - \frac{1}{2} R \Omega^{-1} \partial_\mu \Omega \partial_\nu h + \frac{1}{2} J \eta^{\alpha\beta} K_{\mu\beta} \Omega^{-2} \partial_\alpha \Omega \partial_\nu \Omega - \frac{1}{4} \Omega^{-1} \partial_\mu h \partial_\nu \Omega - \frac{1}{2} R \Omega^{-1} \partial_\mu h \partial_\nu \Omega \\
& + R h \Omega^{-2} \partial_\mu \Omega \partial_\nu \Omega + 2\eta^{\alpha\beta} K_{\mu\beta} \Omega^{-1} \partial_\nu \partial_\alpha \Omega - \frac{1}{2} J \eta^{\alpha\beta} K_{\mu\beta} \Omega^{-1} \partial_\nu \partial_\alpha \Omega + \frac{3}{8} \partial_\nu \partial_\mu h \\
& - \frac{1}{2} P \partial_\nu \partial_\mu h - \frac{1}{2} R h \Omega^{-1} \partial_\nu \partial_\mu \Omega.
\end{aligned} \tag{22}$$

Upon taking  $J = -2$ ,  $P = \frac{1}{2}$ , and  $R = 0$ , viz.

$$\eta^{\alpha\beta} \partial_\alpha K_{\beta\nu} = -2\Omega^{-1} \eta^{\alpha\beta} K_{\nu\alpha} \partial_\beta \Omega + \frac{1}{2} \partial_\nu h, \tag{23}$$

for a strictly time dependent conformal factor  $\Omega(\tau)$ , we find the fluctuations take the form

$$\Omega^{-2} \eta^{\mu\nu} \delta G_{\mu\nu} = (-10\Omega^{-4} \dot{\Omega}^2 + 6\Omega^{-3} \ddot{\Omega}) K_{00} + (-\frac{1}{4} \Omega^{-2} \eta^{\mu\nu} \partial_\mu \partial_\nu - \frac{1}{2} \Omega^{-3} \dot{\Omega} \partial_0) h \tag{24}$$

$$\delta G_{00} = (3\Omega^{-2} \dot{\Omega}^2 - \Omega^{-1} \ddot{\Omega} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu - \Omega^{-1} \dot{\Omega} \partial_0) K_{00} + (-\frac{1}{4} \Omega^{-1} \dot{\Omega} \partial_0 - \frac{1}{4} \partial_0 \partial_0) h. \tag{25}$$

$$\delta G_{0i} = (\Omega^{-1} \ddot{\Omega} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu - \Omega^{-1} \dot{\Omega} \partial_0) K_{0i} + (-\frac{1}{4} \Omega^{-1} \dot{\Omega} \partial_i - \frac{1}{4} \partial_i \partial_0) h. \tag{26}$$

$$\begin{aligned}
\delta G_{11} = & (-2\Omega^{-2} \dot{\Omega}^2 + \Omega^{-1} \ddot{\Omega}) K_{00} + (-\Omega^{-2} \dot{\Omega}^2 + 2\Omega^{-1} \ddot{\Omega} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu - \Omega^{-1} \dot{\Omega} \partial_0) K_{11} \\
& + (-\frac{1}{4} \Omega^{-1} \dot{\Omega} \partial_0 - \frac{1}{4} \partial_1 \partial_1) h.
\end{aligned} \tag{27}$$

$$\delta G_{12} = (-\Omega^{-2} \dot{\Omega}^2 + 2\Omega^{-1} \ddot{\Omega} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu - \Omega^{-1} \dot{\Omega} \partial_0) K_{12} - \frac{1}{4} \partial_2 \partial_1 h. \tag{28}$$

In the deSitter background, we take  $\Omega(\tau) = \frac{1}{H\tau}$ , in which  $\delta G_{\mu\nu}$  reduces to

$$\Omega^{-2} \eta^{\mu\nu} \delta G_{\mu\nu} = 2H^2 K_{00} + (-\frac{1}{4} H^2 \tau^2 \eta^{\mu\nu} \partial_\mu \partial_\nu + \frac{1}{2} H^2 \tau \partial_0) h \tag{29}$$

$$\delta G_{00} = (\tau^{-2} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{00} + (\frac{1}{4}\tau^{-1}\partial_0 - \frac{1}{4}\partial_0\partial_0)h \quad (30)$$

$$\delta G_{0i} = (2\tau^{-2} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{0i} + (\frac{1}{4}\tau^{-1}\partial_i - \frac{1}{4}\partial_i\partial_0)h \quad (31)$$

$$\delta G_{11} = (3\tau^{-2} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{11} + (\frac{1}{4}\tau^{-1}\partial_0 - \frac{1}{4}\partial_1\partial_1)h. \quad (32)$$

$$\delta G_{12} = (3\tau^{-2} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{12} - \frac{1}{4}\partial_2\partial_1h. \quad (33)$$