RW Coordinate Transformations

1 RW $k = -L^{-2}$

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \tag{1.1}$$

$$k = -L^{-2}, \qquad p = \frac{\tau}{L} = \frac{1}{L}\int \frac{dt}{a(t)}, \qquad \sinh\chi = \frac{r}{L} \tag{1.2}$$

$$\implies ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.3}$$

1.1 $\Omega(X^2) = \Omega(T^2 - R^2)$

$$ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.4}$$

$$T = e^p \cosh\chi, \qquad R = e^p \sinh\chi, \qquad X^2 \equiv T^2 - R^2 \tag{1.5}$$

$$\implies p = \frac{1}{2}\ln(X^2), \qquad \frac{r^2}{L^2} = \frac{R^2}{X^2}$$

$$\implies ds^2 = \frac{a\left(\frac{1}{2}\ln X^2\right)^2}{X^2}\left[-dT^2 + dR^2 + R^2 d\theta^2 + R^2\sin^2\theta d\phi^2\right] \tag{1.6}$$

$$\Omega(X^2)^2 = \frac{a\left(\frac{1}{2}\ln X^2\right)^2}{X^2} \tag{1.7}$$

1.1.1 Transformation Functions

For convenience, we denote $r_l \equiv \frac{r}{L}$.

$$\frac{\partial T}{\partial p} = T, \qquad \frac{\partial R}{\partial p} = R, \qquad \frac{\partial T}{\partial r_l} = \frac{RX}{T}, \qquad \frac{\partial R}{\partial r_l} = X \tag{1.8}$$

$$\frac{\partial}{\partial p} = \frac{\partial T}{\partial p}\frac{\partial}{\partial T} + \frac{\partial R}{\partial p}\frac{\partial}{\partial R} = T\frac{\partial}{\partial T} + R\frac{\partial}{\partial R} \tag{1.9}$$

$$\frac{\partial}{\partial r_l} = \frac{\partial T}{\partial r_l}\frac{\partial}{\partial T} + \frac{\partial R}{\partial r_l}\frac{\partial}{\partial R} = \left(\frac{RX}{T}\right)\frac{\partial}{\partial T} + X\frac{\partial}{\partial R} \tag{1.10}$$

1.1.2 Tensor Component Transformation

We take $L = 1$ such that $r/L = r$ and transform from the coordinates from (1.4) to (1.6). (See (A.2) for transformation behavior).

$$x^\mu(p, r, \theta, \phi) \quad \rightarrow \quad x'^\mu(T, R, \theta, \phi) \tag{1.11}$$

$$
h_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} h'_{\alpha\beta} \tag{1.12}
$$

$$
\begin{aligned}
h_{00} &= T^2 h'_{00} + 2TRh'_{0r} + R^2 h'_{rr} \\
h_{0r} &= RXh'_{00} + X(T+R)h'_{0r} \\
h_{0\theta} &= Th'_{0\theta} + Rh'_{r\theta} \\
h_{0\phi} &= Th'_{0\phi} + Rh'_{r\phi} \\
h_{rr} &= \left(\frac{R^2X^2}{T^2}\right) h'_{00} + 2X^2 \left(\frac{R}{T}\right) h'_{0r} + X^2 h'_{rr} \\
h_{r\theta} &= \left(\frac{RX}{T}\right) h'_{00} + Xh'_{r\theta} \\
h_{r\phi} &= \left(\frac{RX}{T}\right) h'_{00} + Xh'_{r\phi} \\
h_{\theta\phi} &= h'_{\theta\phi} \\
h_{\theta\theta} &= h'_{\theta\theta} \\
h_{\phi\phi} &= h'_{\phi\phi}
\end{aligned} \tag{1.13}
$$

1.2 $\Omega(p', r')$

$$
ds^2 = a(p)^2 \left[-dp^2 + d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.14}
$$

$$
p' + r' = \tanh\left[\frac{p+\chi}{2}\right], \quad p' - r' = \tanh\left[\frac{p-\chi}{2}\right], \quad p' = \frac{\sinh p}{\cosh p + \cosh\chi}, \quad r' = \frac{\sinh\chi}{\cosh p + \cosh\chi} \tag{1.15}
$$

$$
\Longrightarrow p = \tanh^{-1}(p'+r') + \tanh^{-1}(p'-r'), \qquad \chi = \tanh^{-1}(p'+r') - \tanh^{-1}(p'-r') \tag{1.16}
$$

$$
\frac{r}{L} = \frac{2r'}{\left[(1-(p'+r')^2)(1-(p'-r')^2)\right]^{1/2}} \tag{1.17}
$$

$$
\Longrightarrow ds^2 = \frac{4L^2 a(p',r')^2}{[1-(p'+r')^2][1-(p'-r')^2]} \left[-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2\right] \tag{1.18}
$$

$$
\Omega(p',r')^2 = \frac{4L^2 a(p',r')^2}{[1-(p'+r')^2][1-(p'-r')^2]} \tag{1.19}
$$

1.2.1 Transformation Functions

For convenience, we denote $r_l \equiv \frac{r}{L}$.

$$
\frac{\partial p'}{\partial p} = \frac{1 + (1+r_l^2)^{1/2}\cosh(p)}{\left[(1+r_l^2)^{1/2} + \cosh(p)\right]^2} = \frac{1}{2}\left[1 - (p'^2 + r'^2)\right] = \frac{1}{2} n(x') \tag{1.20}
$$

$$
\frac{\partial r'}{\partial p} = -\frac{r_l \sinh(p)}{\left[(1+r_l^2)^{1/2} + \cosh(p)\right]^2} = -p'r' \tag{1.21}
$$

$$
\begin{aligned}
\frac{\partial p'}{\partial r_l} &= \frac{\partial p'}{\partial \chi}\frac{\partial \chi}{\partial r_l} = -\frac{r_l \sinh p}{(1+r_l^2)^{1/2}\left[(1+r_l^2)^{1/2} + \cosh p\right]^2} = -\frac{p'r'\left[1-(p'+r')^2\right]^{1/2}\left[1-(p'-r')^2\right]^{1/2}}{1-(p'^2-r'^2)} \\
&= -p'r'm(x')
\end{aligned} \tag{1.22}
$$

$$\begin{aligned}
\frac{\partial r'}{\partial r_l} &= \frac{\partial r'}{\partial \chi}\frac{\partial \chi}{\partial r_l} = \frac{1+(1+r_l^2)^{1/2}\cosh p}{(1+r_l^2)^{1/2}\left[(1+r_l^2)^{1/2}+\cosh p\right]^2} = \frac{1}{2}\frac{\left[1-(p'^2+r'^2)\right]\left[1-(p'+r')^2\right]^{1/2}\left[1-(p'-r')^2\right]^{1/2}}{1-(p'^2-r'^2)} \\
&= \frac{1}{2}m(x')n(x') \qquad (1.23)
\end{aligned}$$

$$\frac{\partial}{\partial p} = \frac{\partial p'}{\partial p}\frac{\partial}{\partial p'} + \frac{\partial r'}{\partial p}\frac{\partial}{\partial r'} = \frac{1}{2}n(x')\frac{\partial}{\partial p'} - p'r'\frac{\partial}{\partial r'} \qquad (1.24)$$

$$\frac{\partial}{\partial r_l} = \frac{\partial p'}{\partial r_l}\frac{\partial}{\partial p'} + \frac{\partial r'}{\partial r_l}\frac{\partial}{\partial r'} = -p'r'm(x')\frac{\partial}{\partial p'} + \frac{1}{2}m(x')n(x')\frac{\partial}{\partial r'} \qquad (1.25)$$

$$m(x') \equiv \frac{\left[1-(p'+r')^2\right]^{1/2}\left[1-(p'-r')^2\right]^{1/2}}{1-(p'^2-r'^2)} \qquad (1.26)$$

$$n(x') \equiv 1-(p'^2+r'^2) \qquad (1.27)$$

1.2.2 Tensor Component Transformation

We take $L = 1$ such that $r/L = r$ and transform from the coordinates from (1.14) to (1.18). (See (A.2) for transformation behavior).

$$x^\mu(p, r, \theta, \phi) \rightarrow x'^\mu(p', r', \theta, \phi) \qquad (1.28)$$

$$h_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}h'_{\alpha\beta} \qquad (1.29)$$

$$\begin{aligned}
h_{00} &= \tfrac{1}{4}n(x')^2 h'_{00} - p'r'h'_{0r} + p'^2r'^2h'_{rr} \\
h_{0r} &= -\tfrac{1}{2}p'r'm(x')n(x')h'_{00} + \tfrac{1}{2}n(x')\left(\tfrac{1}{2}m(x')n(x') - p'r'm(x')\right)h'_{0r} \\
h_{0\theta} &= \tfrac{1}{2}n(x')h'_{0\theta} - p'r'h'_{r\theta} \\
h_{0\phi} &= \tfrac{1}{2}n(x')h'_{0\phi} - p'r'h'_{r\phi} \\
h_{rr} &= p'^2r'^2m(x')^2h'_{00} - p'r'm(x')^2n(x'))h'_{0r} + \tfrac{1}{4}m(x')^2n(x')^2h'_{rr} \\
h_{r\theta} &= -p'r'm(x')h'_{00} + \tfrac{1}{2}m(x')n(x')h'_{r\theta} \\
h_{r\phi} &= -p'r'm(x')h'_{00} + \tfrac{1}{2}m(x')n(x')h'_{r\phi} \\
h_{\theta\phi} &= h'_{\theta\phi} \\
h_{\theta\theta} &= h'_{\theta\theta} \\
h_{\phi\phi} &= h'_{\phi\phi} \qquad (1.30)
\end{aligned}$$

Appendix A $h_{\mu\nu}$ Coordinate Transformation

We transform from $x^\mu(p, r, \theta, \phi) \rightarrow x'^\mu(T, R, \theta, \phi)$. This also serves as a template for transforming from $x^\mu(p, r, \theta, \phi) \rightarrow x'^\mu(p', r', \theta, \phi)$.

$$h_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}h'_{\alpha\beta} \qquad (A.1)$$

$$\begin{aligned}
h_{00} &= \left(\frac{\partial T}{\partial p}\right)^2 h'_{00} + 2\left(\frac{\partial T}{\partial p}\right)\left(\frac{\partial R}{\partial p}\right)h'_{0r} + \left(\frac{\partial R}{\partial p}\right)^2 h'_{rr} \\
h_{0r} &= \left(\frac{\partial T}{\partial p}\right)\left(\frac{\partial T}{\partial r}\right)h'_{00} + \left[\left(\frac{\partial T}{\partial p}\right)\left(\frac{\partial T}{\partial r}\right) + \left(\frac{\partial T}{\partial p}\right)\left(\frac{\partial R}{\partial r}\right)\right]h'_{0r} \\
h_{0\theta} &= \left(\frac{\partial T}{\partial p}\right)h'_{0\theta} + \left(\frac{\partial R}{\partial p}\right)h'_{r\theta}
\end{aligned}$$

$$
\begin{aligned}
h_{0\phi} &= \left(\frac{\partial T}{\partial p}\right) h'_{0\phi} + \left(\frac{\partial R}{\partial p}\right) h'_{r\phi} \\
h_{rr} &= \left(\frac{\partial T}{\partial r}\right)^2 h'_{00} + 2\left(\frac{\partial T}{\partial r}\right)\left(\frac{\partial R}{\partial r}\right) h'_{0r} + \left(\frac{\partial R}{\partial r}\right)^2 h'_{rr} \\
h_{r\theta} &= \left(\frac{\partial T}{\partial r}\right) h'_{00} + \left(\frac{\partial R}{\partial r}\right) h'_{r\theta} \\
h_{r\phi} &= \left(\frac{\partial T}{\partial r}\right) h'_{00} + \left(\frac{\partial R}{\partial r}\right) h'_{r\phi} \\
h_{\theta\phi} &= h'_{\theta\phi} \\
h_{\theta\theta} &= h'_{\theta\theta} \\
h_{\phi\phi} &= h'_{\phi\phi}
\end{aligned}
\tag{A.2}
$$