Mathematical Methods Physics 5101 Midterm Exam 10/17/17 6:30pm–9:30pm PB 121

Answer all **THREE** problems, each one in a separate blue answer book, with each book marked with your name and the number of the problem on the front. Show all your work in the answer books so partial credit may be given as appropriate.

(1) **(a)** Consider the totally antisymmetric Levi-Civita symbol ϵ_{ij} in a 2-dimensional space. Construct a 2×2 matrix that represents a rotation in the 2-dimensional space through an angle θ. Show that ϵ_{ij} is left invariant by this rotation.

(b) In terms of unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ in the x, y and z directions in a 3-dimensional space, one can construct polar coordinate unit vectors:

$$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\theta\mathbf{e}_z,$$

$$\mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{e}_x + \cos\theta\sin\phi\mathbf{e}_y - \sin\theta\mathbf{e}_z, \quad \mathbf{e}_\phi = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y.$$

Show that in terms of them the angular momentum operator $\mathbf{L} = -i\mathbf{r} \times \nabla$ can be written as

$$\mathbf{L} = i\mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - i\mathbf{e}_\phi \frac{\partial}{\partial\theta}.$$

Then show that the total angular momentum operator can be written as:

$$\mathbf{L}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

(c) With the form for $\mathbf{L}$ given in part (**b**) determine the commutator $[L_x, L_y]$.

(2) **(a)** In the interval $-\pi < x < \pi$ one can expand a general function $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

with coefficients a_n, b_n, where $n = 1,\ 2, ...,\ \infty$. For the step function

$$f(-\pi < x < 0) = -\frac{h}{2}, \quad f(0 < x < \pi) = +\frac{h}{2}$$

determine all the a_n and b_n coefficients.

(b) For this step function show that

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{h^2\pi}{2} = \frac{4h^2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

(c) The Riemann zeta function $\zeta(2)$ is given by

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Use an appropriate convergence test to show that $\zeta(2)$ is convergent.

(d) Use the result of part **(b)** to show that $\zeta(2) = \pi^2/6$.

(3) Consider the matrix M given by

$$M = \begin{pmatrix} 0 & \alpha+\beta \\ \alpha-\beta & 0 \end{pmatrix} = \alpha\sigma_1 + i\beta\sigma_2$$

where α and β are real and positive and $\alpha > \beta$.

(a) Determine the eigenvalues and eigenvectors of M.

(b) Use these eigenvectors to construct the matrix S for which $H = SMS^{-1}$ obeys $H = H^\dagger$.

(c) Construct the operator V that effects $VMV^{-1} = M^\dagger$.

(d) Show that the u_i $(i = 1,\ 2)$ eigenvectors of M are orthogonal with respect to V, i.e that $u_i^\dagger V u_j = \delta_{ij}$.

(e) In what way is V related to S?