Quantum Mechanics III
HW 6

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4.10 (a) Take a joint system $S+E$, an arbitrary operator of the joint system A, and a trace preserving completely positive map $\mathcal{L}$ for the system S. Show that $\mathrm{Tr}_S(\mathcal{L}A) = \mathrm{Tr}_S A$.

(b) Use the result of part (a) to solve the problem 4.3.

(a) From the theorem given in eq. 4.19, the CP map $\mathcal{L}$ acting on joint operator A may be written in terms of Krauss operators K, $K^\dagger$ as

$$\mathcal{L}(A) = \sum_k K_k A K_k^\dagger.$$

In order for $\mathcal{L}$ to preserve the trace, we have the condition

$$\sum_k K_k^\dagger K = 1.$$

Now we form the trace

$$\mathrm{Tr}_S(\mathcal{L}A) = \mathrm{Tr}_S\left(\sum_k K_k A K_k^\dagger\right) = \mathrm{Tr}_S\left(\sum_k K_k^\dagger K_k A\right) = \mathrm{Tr}_S A.$$

Thus

$$\mathrm{Tr}_S(\mathcal{L}A) = \mathrm{Tr}_S A.$$

(b) State 2 before measurement is given by

$$\rho_2 = \mathrm{Tr}_1(\rho)$$

where $\rho = |\psi\rangle\langle\psi|$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)$. After measurement, the density operator is mapped into another positive operator via the CP linear mapping $\mathcal{L}$, i.e.

$$\rho \to \rho' = \sum_k P(k)\rho_k = \mathcal{L}(\rho).$$

Now state 2 is

$$\rho_2' = \mathrm{Tr}_1(\mathcal{L}\rho).$$

By the result from part (a), we then have

$$\mathrm{Tr}_1(\rho) = \mathrm{Tr}_1(\mathcal{L}\rho) \Rightarrow \rho_2 = \rho_2'.$$

Thus state 2 does not change after measurement of state 1.

5.2 Consider a simple harmonic oscillator with the Hamiltonian $H_0 = \hbar\omega a^\dagger a$ and a damping constant γ, so that the density operator has the master equation

$$\dot{\rho} = \frac{1}{i\hbar}[H_0, \rho] + \gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a).$$

(a) Argue that the relaxation term in fact is of the proper Lindblad form.

(b) Show (using the cyclic invariance of trace) that the expectation value of the (nonhermitian) operator a satisfies the equation of motion $\frac{d}{dt}\langle a\rangle = -i\omega\langle a\rangle - \gamma\langle a\rangle$.

(c) Find the equations of motion for the expectation values $\langle x\rangle$ and $\langle p\rangle$.

(a) The Lindblad form of the relaxation term $\mathcal{L}\rho$ is

$$\mathcal{L}\rho = \sum_k [2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k].$$

Now if we denote

$$L = \gamma^{1/2}a; \qquad L^\dagger = \gamma^{1/2}a^\dagger$$

we see that the relaxation term $\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$ is indeed

$$\mathcal{L}\rho = 2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L$$

which follows the Linblad form (as a single term in the summation).

(b) Multiply the master equation by a, take the trace, and commute things around from the cyclic invariance of the trace

$$\begin{aligned}
\mathrm{Tr}\left(a\frac{d}{dt}\rho\right) &= \mathrm{Tr}\left(\frac{1}{i\hbar}[H_0,\rho] + \gamma(2aa\rho a^\dagger - aa^\dagger a\rho - a\rho a^\dagger a)\right)\\
\mathrm{Tr}\left(\frac{d}{dt}\rho a\right) &= -i\omega\mathrm{Tr}(\rho[a,a^\dagger]a) + 2\gamma\mathrm{Tr}\left(\rho a^\dagger aa\right) - \gamma\mathrm{Tr}\left(\rho aa^\dagger a\right) - \gamma\mathrm{Tr}\left(\rho a^\dagger aa\right)\\
\frac{d}{dt}\mathrm{Tr}(\rho a) &= -i\omega\mathrm{Tr}(\rho a) + \gamma\mathrm{Tr}(\rho[a^\dagger, a]a)\\
\frac{d}{dt}\langle a\rangle &= -i\omega\langle a\rangle - \gamma\langle a\rangle
\end{aligned}$$

where we have used the commutation relation of the ladder operators $[a, a^\dagger] = 1$.

(c) From $\langle a\rangle^\dagger = \langle a^\dagger\rangle$ we may form the adjoint equation of motion for $a^\dagger$

$$\frac{d}{dt}\langle a^\dagger\rangle = (-i\omega\langle a\rangle - \gamma\langle a\rangle)^\dagger = i\omega\langle a^\dagger\rangle - \gamma\langle a^\dagger\rangle.$$

The operators x and p are related to the ladder operators by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a); \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

To find the equations of motion, lets add the derivatives

$$\begin{aligned}
\frac{d}{dt}\langle a\rangle + \frac{d}{dt}\langle a^\dagger\rangle &= i\omega(\langle a^\dagger\rangle - \langle a\rangle) - \gamma(\langle a^\dagger\rangle + \langle a\rangle)\\
\sqrt{\frac{2m\omega}{\hbar}}\frac{d}{dt}\langle x\rangle &= \frac{1}{m}\sqrt{\frac{2m\omega}{\hbar}}\langle p\rangle - \gamma\sqrt{\frac{2m\omega}{\hbar}}\langle x\rangle
\end{aligned}$$

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m} - \gamma\langle x\rangle .$$

Now subtract the derivatives

$$\begin{aligned}
\frac{d}{dt}\langle a^\dagger\rangle - \frac{d}{dt}\langle a\rangle &= i\omega(\langle a^\dagger\rangle + \langle a\rangle) - \gamma(\langle a^\dagger\rangle - \langle a\rangle) \\
-i\sqrt{\frac{2}{\hbar m\omega}}\frac{d}{dt}\langle p\rangle &= i\omega\sqrt{\frac{2}{\hbar m\omega}}\langle x\rangle + i\gamma\sqrt{\frac{2}{\hbar m\omega}}\langle p\rangle \\
\frac{d}{dt}\langle p\rangle &= -m\omega^2\langle x\rangle - \gamma\langle p\rangle .
\end{aligned}$$

Altogether then, we have

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m} - \gamma\langle x\rangle ; \qquad \frac{d}{dt}\langle p\rangle = -m\omega^2\langle x\rangle - \gamma\langle p\rangle .$$

6.1 Let ψ_a and ψ_b be two orthonormal one-particle states. Show that the two-particles wave functions

$$\psi^{\pm}_{ab}(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)\right]$$

are normalized to unity and have the proper boson (+) and fermion (−) exchange symmetries. They could be, and actually are, the many-body wave functions that express the state of affairs that one particle is in state a and the other in state a and the other in state b.

Bosons are symmetric under exchange of a and b

$$\frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) + \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)\right] \overset{ab\to ba}{\to} +\frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) + \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)\right]$$

while fermions are antisymmetric

$$\frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)\right] \overset{ab\to ba}{\to} -\frac{1}{\sqrt{2}}\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)\right].$$

Switching to dirac notation (no longer in position basis (wavefunction)), where $|n\rangle_1$ and $|n\rangle_2$ represent two (orthonormal) one-particle states

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle_1|n\rangle_2 \pm |n\rangle_2|n\rangle_1\right).$$

Now form the inner product

$$\begin{aligned}
\langle\psi|\psi\rangle &= \frac{1}{2}\left(\langle n|_2\langle n|_1 \pm \langle n|_1\langle n|_2\right)\left(|n\rangle_1|n\rangle_2 \pm |n\rangle_2|n\rangle_1\right) \\
&= \frac{1}{2}(1 \pm 0 \pm 0 + 1) \\
&= 1.
\end{aligned}$$

6.2 Consider a system of two identical particles (or two particles with the same fixed value of the z component of the spin) that interact with a potential that is a function of the absolute value of the distance between the particles $|\mathbf{r}_1 - \mathbf{r}_2|$. As is well known, the center-of-mass degree of freedom and the relative motion of the two particles may then be separated. Show that in such a product form the wave function of the relative motion must be an even function of the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ for bosons, and an odd function for fermions.

For the interacting two particle system, the Hamiltonian is

$$H = \frac{p_1^2 + p_2^2}{2m} + V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

which may be converted into the center of mass degree of freedom and relative motion Hamiltonian. The center of mass is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}$$

and its derivative

$$\dot{\mathbf{R}} = \frac{1}{2}\dot{\mathbf{r}_1} + \dot{\mathbf{r}_2}.$$

Now with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mu = m_1 m_2/(m_1 + m_2) = m/2$, and $M = 2m$, the Hamiltonian can then be brought to the form

$$H = \frac{\mathbf{P}_{cm}}{2M} + \frac{\mathbf{p}}{2\mu} + V(r) = H(\mathbf{R}) + H(\mathbf{r}).$$

To clarify, $\mathbf{P}$ is the momentum conjugate to $\mathbf{R}$ and $\mathbf{p}$ is the momentum conjugate to $\mathbf{r}$. As the Hamiltonian is now separated into two terms, we may write the wavefunction as the product of the two coordinates

$$\psi(\mathbf{R}, \mathbf{r}) = \psi_1(\mathbf{R})\psi_2(\mathbf{r})$$

(one can also think of this as separation of variables in the Schrodinger eq.). Given this wavefunction, under particle exchange we see that $\mathbf{R}$ is symmetric since $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$. The exchange effects must then lie all within $\psi_2(\mathbf{r})$. Since bosons (fermions) are even (odd) under particle exchange, we deduce that

$$\psi_2(\mathbf{r}) \stackrel{r_1 r_2 \to r_2 r_1}{\to} +\psi_2(\mathbf{r}) \quad \text{for bosons}$$

$$\psi_2(\mathbf{r}) \stackrel{r_1 r_2 \to r_2 r_1}{\to} -\psi_2(\mathbf{r}) \quad \text{for fermions.}$$

Thus product form the wave function of the relative motion must be an even function of the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ for bosons, and an odd function for fermions.