Group Theory Notes

A group is defined as a set of elements that
1. Are closed under composition (multiplication)
2. Contain all inverses
3. Associative
4. Existence of identity

If a group is parameterized by continuous parameters (such as θ in 2D rotations), then this is a Lie group. If two elements of a Lie group commute, the group is said to be abelian. Otherwise, non-abelian.

Infinitesimal transformations can be found by taking derivatives with respect to the continuous parameters. These infinitesimal generators form a group as well, and if elements under commutation lie within the vector space spanned by the group, then they are said for form a Lie algebra.

Most often we take groups to be transformations acting on a vector space V. On such group is the general linear group which consists of all *invertible* linear operators acting on a space V

$$GL(V) \subset \mathcal{L}(V).$$

For a vector space with scalar field C and dimension n, then $GL(V)$ may be represented by a subset of invertable $n \times n$ matrices denoted

$$GL(n, C) \subset M_n(C).$$

For $C = \mathbb{C}, \mathbb{R}$, we have the complex general linear group and real general linear group in n dimensions respectively.

If our vector space is equipped with a non-degenerate hermitian form (defined via an "inner product"), there exist an important subset of $GL(V)$ called isometries, $\mathrm{Isom}(V)$ which by definition preserve the inner product. For $T \in \mathrm{Isom}(V)$,

$$(Tu, Tw) = (u, w) \ \forall u, w \in V.$$

These form a group.

We can show that for a vector space in n dimensions with an inner product space ($(v, v) > 0 \ \forall v \neq 0$), T consists of all $n \times n$ matrices such that

$$T^{-1} = T^{\dagger}$$

where for real vector spaces this simplifies to $T^{-1} = T^T$. These groups correspond to $U(n)$ and $O(n)$ respectively. If we further restrict to matrices with determinant equal to unity, then we have $SU(n)$ and $SO(n)$ which are important subgroups.
Real vector space nd hermitian forms are symmetric, called a metric

For the Minkowski metric, the group $\mathrm{Isom}(V)$ can be shown to be transformations that satisfy

$$T_{\mu}{}^{\rho} T_{\nu}{}^{\sigma} \eta_{\rho\sigma} = \eta_{\mu\nu}$$

i.e. Lorentz transformations denoted as $O(n-1, 1)$. The restricted Lorentz group in four dimensions $SO(3, 1)_{\sigma}$ is the restriction that for $A \in O(3, 1)$

$$|A| = 1, \quad A_{11} > 0.$$

This preserves the orientation of space and time. Also, any element $A \in SO(3,1)_\sigma$ can be expressed as a rotation and a Lorentz boost $R'L$ where

$$R' = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$$

for $R \in SO(3)$. If we add the parity and time reversal transformations to the $SO(3,1)_\sigma$ group, we recover the full $O(3,1)$ group.