RW SVT4 $k \neq 0$ v2

1 Background

1.1 Conformal $\Omega(\tau)$

$$ds^2 = \Omega^2(\tau)\tilde{g}_{\mu\nu}dx^\mu dx^\nu, \qquad \tilde{g}_{\mu\nu} = \text{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) \tag{1.1}$$

$$G_{00} = -3k - 3\dot{\Omega}^2\Omega^{-2} \qquad G_{ij} = k\tilde{g}_{ij} - \dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 2\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij} \tag{1.2}$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\Omega^2\tilde{g}_{\mu\nu}, \qquad U_\mu = -\Omega\delta_\mu^0 \qquad \text{[Evaluated in (1.1)]} \tag{1.3}$$

$$\Delta_{00}^{(0)} = -3k - 3\dot{\Omega}^2\Omega^{-2} + \Omega^2\rho \tag{1.4}$$

$$\rightarrow \quad \boxed{\rho = 3k\Omega^{-2} + 3\dot{\Omega}^2\Omega^{-4}} \tag{1.5}$$

$$\Delta_{ij}^{(0)} = k\tilde{g}_{ij} - \dot{\Omega}^2\Omega^{-2}\tilde{g}_{ij} + 2\ddot{\Omega}\Omega^{-1}\tilde{g}_{ij} + \Omega^2 p\tilde{g}_{ij} \tag{1.6}$$

$$\rightarrow \quad \boxed{p = -k\Omega^{-2} + \dot{\Omega}^2\Omega^{-4} - 2\ddot{\Omega}\Omega^{-3}} \tag{1.7}$$

$$\begin{aligned}\nabla_\mu T^{\mu 0} &= \Omega^{-5}\left(\tilde{g}^{ab}T_{ab}\dot{\Omega} + T_{00}\dot{\Omega} + \dot{T}_{00}\Omega - \Omega\tilde{\nabla}_a T_0{}^a\right) \\ &= 3\dot{\Omega}\Omega^{-3}p + 3\dot{\Omega}\Omega^{-3}\rho + \Omega^{-2}\dot{\rho}\end{aligned} \tag{1.8}$$

$$\begin{aligned}\nabla_\mu T^{\mu i} &= \Omega^{-5}\left(-2T_0{}^i\dot{\Omega} - \dot{T}_0{}^i\Omega + \Omega\tilde{\nabla}_a T^{ia}\right) \\ &= 0\end{aligned} \tag{1.9}$$

2 Fluctuations

$$ds^2 = \Omega^2(\tau)[\tilde{g}_{\mu\nu} + f_{\mu\nu}]dx^\mu dx^\nu \tag{2.1}$$

$$\tilde{g}_{\mu\nu} = \text{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) \tag{2.2}$$

$$f_{\mu\nu} = -2\tilde{g}_{\mu\nu}\chi + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu F + \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu + 2F_{\mu\nu} \tag{2.3}$$

$$f_{00} = 2\chi + 2\ddot{F} + 2\dot{F}_0 + 2F_{00} \tag{2.4}$$

$$f_{0i} = 2\tilde{\nabla}_i\dot{F} + \dot{F}_i + \tilde{\nabla}_i F_0 + 2F_{0i} \tag{2.5}$$

$$f_{ij} = -2\tilde{g}_{ij}\chi + 2\tilde{\nabla}_i\tilde{\nabla}_j F + \tilde{\nabla}_i F_j + \tilde{\nabla}_j F_i + 2F_{ij} \tag{2.6}$$

$$\tilde{g}^{\mu\nu}F_{\mu\nu} = 0, \qquad \tilde{\nabla}^\mu F_{\mu\nu} = 0, \qquad \tilde{\nabla}^\mu F_\mu = 0 \tag{2.7}$$

$$\delta U_0 = \tfrac{1}{2}\Omega f_{00} \tag{2.8}$$

$$\delta T_{00} = (-6k - 6\dot{\Omega}^2\Omega^{-2})\ddot{F} + \Omega^2\delta\rho$$
$$+ (-6k - 6\dot{\Omega}^2\Omega^{-2})\chi + (-6k - 6\dot{\Omega}^2\Omega^{-2})\dot{F}_0 + (-6k - 6\dot{\Omega}^2\Omega^{-2})F_{00} \tag{2.9}$$

$$\delta T_{0i} = (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\dot{F}$$
$$+ (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\dot{F}_i$$
$$+ (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i$$
$$+ (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_0 + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})F_{0i} \tag{2.10}$$

$$\delta T_{ij} = \Omega^2\delta p\,\tilde{g}_{ij} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\chi$$
$$+ (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\tilde{\nabla}_j F + (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_j$$
$$+ (-k + \dot{\Omega}^2\Omega^{-2} - 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j F_i + (-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})F_{ij} \tag{2.11}$$

$$g^{\mu\nu}\delta T_{\mu\nu} = (6\dot{\Omega}^2\Omega^{-4} + 6k\Omega^{-2})\ddot{F} + 3\delta p - \delta\rho + (12\ddot{\Omega}\Omega^{-3} + 12k\Omega^{-2})\chi$$
$$+ (2\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} - 2k\Omega^{-2})\tilde{\nabla}_a\tilde{\nabla}^a F + (8\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} + 4k\Omega^{-2})\dot{F}_0 + (8\dot{\Omega}^2\Omega^{-4} - 4\ddot{\Omega}\Omega^{-3} + 4k\Omega^{-2})F_{00}$$
$$\tag{2.12}$$

$$\delta G_{00} = 6k\ddot{F} + 6\dot{\Omega}\Omega^{-1}\dot{\chi} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} - 2\tilde{\nabla}_a\tilde{\nabla}^a\chi + 6k\dot{F}_0 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a F_0 - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00}$$
$$+ 8kF_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{00} \tag{2.13}$$

$$\delta G_{0i} = 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\ddot{F} + (4k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\dot{F} - 2\tilde{\nabla}_i\dot{\chi}$$
$$+ 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\chi + (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\dot{F}_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{F}_0$$
$$+ (3k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_0 - \ddot{F}_{0i} + (4k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})F_{0i} + \tilde{\nabla}_a\tilde{\nabla}^a F_{0i}$$
$$+ 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00} \tag{2.14}$$

$$\delta G_{ij} = 2\dot{\Omega}\Omega^{-1}\ddot{F}\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\ddot{F}\tilde{g}_{ij} - 2\ddot{\chi}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\chi}\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F}$$
$$+ 2\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\chi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{F} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\tilde{\nabla}_i F$$
$$- 2\tilde{\nabla}_j\tilde{\nabla}_i\chi + 2\dot{\Omega}\Omega^{-1}\ddot{F}_0\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\dot{F}_0\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a F_0$$
$$+ (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i F_j + (k - \dot{\Omega}^2\Omega^{-2} + 2\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j F_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i F_0 - \ddot{F}_{ij}$$
$$- 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})F_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{ij}$$
$$+ 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i} \tag{2.15}$$

$$g^{\mu\nu}\delta G_{\mu\nu} = 6\dot{\Omega}\Omega^{-3}\ddot{F} + (-6\dot{\Omega}^2\Omega^{-4} + 12\ddot{\Omega}\Omega^{-3} - 6k\Omega^{-2})\ddot{F} - 6\Omega^{-2}\ddot{\chi} - 12\dot{\Omega}\Omega^{-3}\dot{\chi} - 6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F}$$
$$+ (-2\dot{\Omega}^2\Omega^{-4} + 4\ddot{\Omega}\Omega^{-3} + 2k\Omega^{-2})\tilde{\nabla}_a\tilde{\nabla}^a F + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\chi + 6\dot{\Omega}\Omega^{-3}\ddot{F}_0$$

$$+(-8\dot{\Omega}^2\Omega^{-4} + 16\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})\dot{F}_0$$
$$-6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a F_0 + (-8\dot{\Omega}^2\Omega^{-4} + 16\ddot{\Omega}\Omega^{-3} - 8k\Omega^{-2})F_{00} \tag{2.16}$$

3 Field Equations

$$\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} \tag{3.1}$$

$$\begin{aligned}
\Delta_{00} =\ & -6\dot{\Omega}^2\Omega^{-2}\ddot{F} + \Omega^2\delta\rho + 6\dot{\Omega}\Omega^{-1}\dot{\chi} + (-6k - 6\dot{\Omega}^2\Omega^{-2})\chi + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} \\
& -2\tilde{\nabla}_a\tilde{\nabla}^a\chi - 6\dot{\Omega}^2\Omega^{-2}\dot{F}_0 + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a F_0 - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00} + (2k - 6\dot{\Omega}^2\Omega^{-2})F_{00} \\
& +\tilde{\nabla}_a\tilde{\nabla}^a F_{00}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\Delta_{0i} =\ & 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\ddot{F} + 2k\tilde{\nabla}_i\dot{F} - 2\tilde{\nabla}_i\dot{\chi} + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V \\
& +2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\chi + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{F}_0 + 2k\tilde{\nabla}_i F_0 - \ddot{F}_{0i} + 2kF_{0i} \\
& +\tilde{\nabla}_a\tilde{\nabla}^a F_{0i} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\Delta_{ij} =\ & 2\dot{\Omega}\Omega^{-1}\dddot{F}\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\ddot{F}\tilde{g}_{ij} - 2\ddot{\chi}\tilde{g}_{ij} + \Omega^2\delta p\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{\chi}\tilde{g}_{ij} \\
& +(2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\chi - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} + 2\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\chi \\
& +(-2k + 2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_i\tilde{\nabla}_j F + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{F} + (2k - 2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{\nabla}_j\tilde{\nabla}_i F \\
& -2\tilde{\nabla}_j\tilde{\nabla}_i\chi + 2\dot{\Omega}\Omega^{-1}\ddot{F}_0\tilde{g}_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\dot{F}_0\tilde{g}_{ij} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a F_0 \\
& +2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i F_0 - \ddot{F}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} - 2kF_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{ij} \\
& +2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i}
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} =\ & 6\dot{\Omega}\Omega^{-3}\dddot{F} + 12\ddot{\Omega}\Omega^{-3}\ddot{F} - 6\Omega^{-2}\ddot{\chi} + 3\delta p - \delta\rho - 12\dot{\Omega}\Omega^{-3}\dot{\chi} + (12\ddot{\Omega}\Omega^{-3} + 12k\Omega^{-2})\chi \\
& -6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{F} + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\chi + 6\dot{\Omega}\Omega^{-3}\ddot{F}_0 + 12\ddot{\Omega}\Omega^{-3}\dot{F}_0 \\
& -6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a F_0 + (12\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})F_{00}
\end{aligned} \tag{3.5}$$

4 Field Equations (G.I. Form)

$$\alpha = \chi - \dot{\Omega}\Omega^{-1}\dot{F} - \dot{\Omega}\Omega^{-1}F_0, \qquad F_{\mu\nu}, \qquad V_i \tag{4.1}$$

$$V^{GI} = V - \Omega\dot{F} - \Omega F_0 \tag{4.2}$$

$$\delta\rho^{GI} = \delta\rho - 12\dot{\Omega}^2\chi\Omega^{-4} + 6\ddot{\Omega}\chi\Omega^{-3} - 6k\chi\Omega^{-2} \tag{4.3}$$

$$\delta p^{GI} = \delta p - 4\dot{\Omega}^2\chi\Omega^{-4} - 2\dddot{\Omega}\dot{F}\Omega^{-3} - 2\dddot{\Omega}F_0\Omega^{-3} + 8\ddot{\Omega}\chi\Omega^{-3} + 2k\chi\Omega^{-2} \tag{4.4}$$

$$\begin{aligned}
\Delta_{00} =\ & \Omega^2\delta\rho^{GI} + 6\dot{\Omega}\Omega^{-1}\dot{\alpha} + (6\dot{\Omega}^2\Omega^{-2} - 6\ddot{\Omega}\Omega^{-1})\alpha - 2\tilde{\nabla}_a\tilde{\nabla}^a\alpha - \ddot{F}_{00} + 2\dot{\Omega}\Omega^{-1}\dot{F}_{00} \\
& +(2k - 6\dot{\Omega}^2\Omega^{-2})F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{00}
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\Delta_{0i} =\ & -2\tilde{\nabla}_i\dot{\alpha} + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})\tilde{\nabla}_i V^{GI} \\
& +2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\alpha + (-4\dot{\Omega}^2\Omega^{-3} + 2\ddot{\Omega}\Omega^{-2} - 2k\Omega^{-1})V_i - \ddot{F}_{0i} + 2kF_{0i} + \tilde{\nabla}_a\tilde{\nabla}^a F_{0i}
\end{aligned}$$

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$$+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{00} \tag{4.6}$$

$$
\begin{aligned}
\Delta_{ij} = {}& -2\tilde{g}_{ij}\ddot{\alpha} + \Omega^2 \tilde{g}_{ij}\delta p^{GI} - 2\dot{\Omega}\Omega^{-1}\tilde{g}_{ij}\dot{\alpha} + (2\dot{\Omega}^2\Omega^{-2} - 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}\alpha + 2\tilde{g}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\alpha - 2\tilde{\nabla}_j\tilde{\nabla}_i\alpha \\
& -\ddot{F}_{ij} - 2\dot{\Omega}\Omega^{-1}\dot{F}_{ij} - 2kF_{ij} + (-2\dot{\Omega}^2\Omega^{-2} + 4\ddot{\Omega}\Omega^{-1})\tilde{g}_{ij}F_{00} + \tilde{\nabla}_a\tilde{\nabla}^a F_{ij} + 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i F_{0j} \\
& +2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j F_{0i}
\end{aligned} \tag{4.7}
$$

$$g^{\mu\nu}\Delta_{\mu\nu} = -6\Omega^{-2}\ddot{\alpha} + 3\delta p^{GI} - \delta\rho^{GI} - 12\dot{\Omega}\Omega^{-3}\dot{\alpha} - 6\ddot{\Omega}\Omega^{-3}\alpha + 6\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\alpha + (12\ddot{\Omega}\Omega^{-3} - 4k\Omega^{-2})F_{00} \tag{4.8}$$