General Relativity
HW 2

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Due: September 29

1.

$$\partial_\alpha F^{\alpha\beta} = J^\beta \tag{1}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\beta\alpha} = 0 \tag{2}$$

$$J^\alpha = (\rho, J^i), \quad F^{i0} = E^{i0}, F^{ij} = -\epsilon^{ijk} B_k$$

$$\partial_i F^{i0} = J^0 \to \boxed{\nabla \cdot \mathbf{E} = \rho}$$

$$\partial_\alpha F^{\alpha i} = \frac{\partial t}{\partial F}^{0i} + \partial_j F^{ji} = J^i$$

$$\frac{\partial F^{0i}}{\partial t} + \epsilon^{jik} \partial_j B_k = J^i \to \boxed{-\frac{\partial E^i}{\partial t} + (\nabla \times \mathbf{B})^i = J^i}$$

For $\alpha = 1, \beta = 2, \gamma = 3$, (2) takes the form

$$\frac{\partial}{\partial x} F_{23} + \frac{\partial}{\partial y} F_{31} + \frac{\partial}{\partial z} F_{21} = 0$$

$$\frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z = 0$$

$$\to \boxed{\nabla \cdot \mathbf{B} = 0}.$$

For $\alpha = 0, \beta = i, \gamma = j$, (2) takes the form

$$\partial_0 F_{ij} + \partial_i F_{j0} j + \partial_j F_{0i} = 0$$

$$\frac{\partial}{\partial t} F_{ij} + \partial_i E_j + \partial_j E_i = 0$$

Now if we sum over all values of i and j (noting the antisymmetry) we have

$$\boxed{\frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} = 0}$$

2.

$$x = r \cos\phi, \qquad y = r \sin\phi$$

Line element:

$$dl^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2.$$

Metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Connection:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\partial_\lambda g_{\nu\mu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu} \right].$$

Since $g_{\mu\nu}$ is diagonal, from the above we must have $g^{\nu\sigma} = g^{\sigma\sigma}$. Also, we see the only non-zero derivative of $g_{\mu\nu}$ is $\partial_0 g_{11} = 2r$. It follows that $\lambda = 1$ and/or $\mu = 1$. If either is 1 then $\sigma = 1$, but if both $\mu = \lambda = 1$, then $\sigma = 0$. So we are left with two (due to symmetry) we need to compute:

$$\Gamma^0_{11} = -\frac{1}{2} g^{00} (\partial_o g_{11}) = -r$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11} \partial_0 g_{11} = \frac{1}{r}$$

Equation of motion:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

leads to

$$\frac{d^2 r}{d\tau^2} - r \left(\frac{d\phi}{d\tau} \right)^2 = 0$$

$$\frac{d^2 \phi}{d\tau^2} + \frac{2}{r} \frac{d\phi}{d\tau} \frac{dr}{d\tau} = 0.$$

Adding these together

$$\frac{d^2 r}{d\tau^2} + \frac{d^2 \phi}{d\tau^2} (1 - r) + \frac{2}{r} \frac{d\phi}{d\tau} \frac{dr}{d\tau} = 0.$$

3.

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \cos\phi$$

$$\frac{\partial r}{\partial y} = \sin\phi$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{x^2} \left(\frac{y}{1 + (y/x)^2} \right) = -\frac{y}{r^2} = -\frac{\sin\phi}{r}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} \left(\frac{1}{1 + (y/x)^2} \right) = \frac{x}{r} = \frac{\cos\phi}{r}$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix}.$$

4. (a) In cartesian coordinates, the connection vanishes and so

$$V^\mu{}_{;\nu} = \partial_\nu V^\mu.$$

$$\partial_0 V^0 = 2x$$

$$\partial_0 V^1 = 3$$

$$\partial_1 V^0 = 3$$

$$\partial_1 V^1 = 2y.$$

So in matrix form

$$V^\mu{}_{;\nu} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix}$$

(b) Under a change from cartesian to polar coordinates, $x \to x'$, the mixed rank tensor transforms as

$$V'^{\mu}{}_{;\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} V^{\rho}{}_{;\sigma}.$$

In matrix form

$$\begin{aligned} V'^{\mu}{}_{;\nu} &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix} \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix} \begin{pmatrix} (2x\cos\phi + 3\sin\phi) & (-2xr\sin\phi + 3r\cos\phi) \\ (3\cos\phi + 2y\sin\phi) & (-3r\sin\phi + 2yr\cos\phi) \end{pmatrix} \\ &= \begin{pmatrix} (2x\cos^2\phi + 3\sin\phi\cos\phi + 3\sin\phi\cos\phi + 2y\sin^2\phi) & (-2xr\sin\phi\cos\phi + 3r\cos^2\phi - 3r\sin^2\phi + 2yr\sin\phi\,\mathrm{co} \\ \frac{1}{r}(-2x\sin\phi\cos\phi - 3\sin^2\phi + 3\cos^2\phi + 2y\sin\phi\cos\phi) & \frac{1}{r}(2xr\sin^2\phi - 3r\sin\phi\cos\phi - 3r\sin\phi\cos\phi + 2yr\,\mathrm{cos} \end{pmatrix} \end{aligned}$$

Since that got cutoff,

$$\begin{aligned} V^0{}_{;0} &= 2x\cos^2\phi + 3\sin\phi\cos\phi + 3\sin\phi\cos\phi + 2y\sin^2\phi \\ &= 2r\cos^2\phi + 6\sin\phi\cos\phi + 2r\sin^2\phi \end{aligned}$$

$$\begin{aligned} V^0{}_{;1} &= -2xr\sin\phi\cos\phi + 3r\cos^2\phi - 3r\sin^2\phi + 2yr\sin\phi\cos\phi \\ &= -2r^2\sin\phi\cos^2\phi + 3r\cos^2\phi - 3r\sin^2\phi + 2r^2\sin^2\phi\cos\phi \end{aligned}$$

$$\begin{aligned} V^1{}_{;0} &= \frac{1}{r}(-2x\sin\phi\cos\phi - 3\sin^2\phi + 3\cos^2\phi + 2y\sin\phi\cos\phi) \\ &= -2\sin\phi\cos^2\phi + \frac{3}{r}\cos^2\phi - \frac{3}{r}\sin^2\phi + 2\sin\phi\cos^2\phi \end{aligned}$$

$$\begin{aligned} V^1{}_{;1} &= \frac{1}{r}(2xr\sin^2\phi - 3r\sin\phi\cos\phi - 3r\sin\phi\cos\phi + 2yr\cos^2\phi) \\ &= 2r\cos\phi\sin^2\phi - 6\sin\phi\cos\phi + 2r\sin\phi\cos^2\phi \end{aligned}$$

(c)

$$V^{\mu}{}_{;\nu} = \partial_\nu V^\mu + \Gamma^{\mu}_{\lambda\nu} V^\lambda$$

From question 2, we have

$$\Gamma^0_{11} = -r, \qquad \Gamma^1_{01} = \frac{1}{r}.$$

We must also convert V^μ to polar

$$V^\mu = (r^2\cos^2\phi + 3r\sin\phi, r^2\sin^2\phi + 3r\cos\phi).$$

Taking each appropriate derivative with respective Christoffel symbol (summing over λ), we find

$$\begin{aligned} V^0{}_{;0} &= 2r\cos^2\phi + 6\sin\phi\cos\phi + 2r\sin^2\phi \\ V^0{}_{;1} &== -2r^2\sin\phi\cos^2\phi + 3r\cos^2\phi - 3r\sin^2\phi + 2r^2\sin^2\phi\cos\phi \\ V^1{}_{;0} &= -2\sin\phi\cos^2\phi + \frac{3}{r}\cos^2\phi - \frac{3}{r}\sin^2\phi + 2\sin\phi\cos^2\phi \\ V^1{}_{;1} &= 2r\cos\phi\sin^2\phi - 6\sin\phi\cos\phi + 2r\sin\phi\cos^2\phi, \end{aligned}$$

which coincides with what we calculated in part b.

5. (a) In the inertial system, the proper time interval is given as (constrained to 1D motion)

$$d\tau^2 = dt^2 - dx^2.$$

With the coordinate transformation parametrized by λ

$$t(\lambda) = a\sinh(\lambda), \qquad x(\lambda) = a\cosh(\lambda)$$

it follows that

$$dt = a\cosh(\lambda)d\lambda, \qquad dx = a\sinh(\lambda)d\lambda$$

and thus

$$d\tau^2 = a^2 d\lambda^2[\cosh^2(\lambda) - \sinh^2(\lambda)] = a^2 d\lambda^2.$$

For a finite proper time, taking $\tau_0 = 0$, we integrate

$$\tau = a\lambda.$$

(b) For an interial observer, a spacelike line can be expressed as

$$t = xb, \qquad -1 < b < 1.$$

For the accelerated observer, his coordinates satisfy the relation

$$x^2 - t^2 = a^2$$

for any fixed λ. Different values of a will generate different hyperbolic curves in the $x - t$ plane. Solving for the point of intersection between these two worldlines,

$$x_0 = \frac{a}{\sqrt{1-b^2}}, \qquad t_0 = \frac{ab}{\sqrt{1-b^2}}.$$

The two lines will be perpendicular if their derivatives are negative inverses. For the inertial observer,

$$\frac{dt}{dx} = b$$

while for the accelerated observer

$$2x - 2t\frac{dt}{dx} = 0 \to \frac{dt}{dx} = \left.\frac{x}{t}\right|_{t_0,x_0} = \frac{1}{b}.$$

This would show they were orthogonal if the slopes were of opposite sign...

(c) The coordinate and inverse coordinate transformation is

$$t = a\sinh\lambda, \qquad x = a\cosh\lambda$$

$$\lambda = \tanh^{-1}\left(\frac{t}{x}\right), \qquad a = \sqrt{x^2 - t^2}.$$

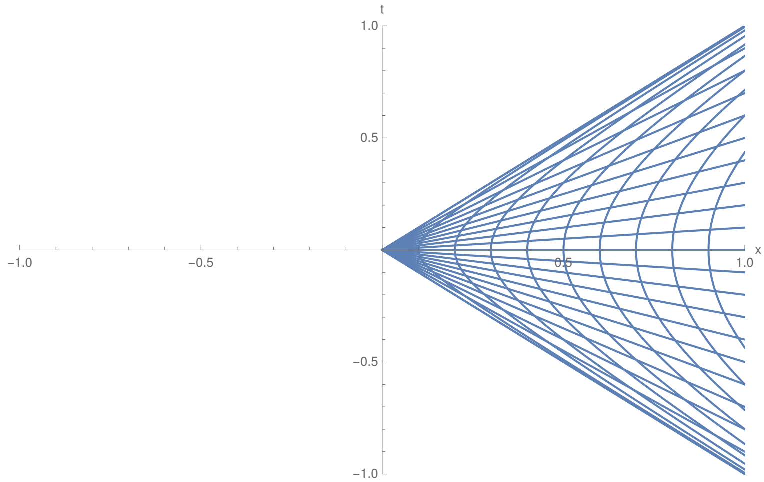
From our earlier equation

$$x^2 - t^2 = a^2,$$

we see that curves of constant λ will yield a family of hyperbolas. Since $a^2 \geq 0$, these will be contrained to the $x \geq 0$ region, thus only covering half of the $x - t$ plane. Meanwhile, lines of constant a take the form

$$t = \tanh(\lambda)x.$$

For $-\infty < \lambda < \infty$, it follows that the above equation forms straight lines with slopes from $(-1, 1)$. Note that these slopes specifically exclude $\{-1, 1\}$ from the interval, and since they can only be reached in the limit of infinity, we consider these "bad" coordinates. In total we will have something like this:



(d) Recalling that

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta},$$

for our 2d case $(t, x) \to (\lambda, a)$ this becomes

$$g_{\mu\nu} = -\frac{\partial t}{\partial x^\mu}\frac{\partial t}{\partial x^\nu} + \frac{\partial x}{\partial x^\mu}\frac{\partial x}{\partial x^\nu}.$$

Taking each component

$$\begin{aligned} g_{00} &= -\left(\frac{\partial t}{\partial \lambda}\right)^2 + \left(\frac{\partial x}{\partial \lambda}\right)^2 \\ &= a^2(\sinh^2\lambda - \cosh^2\lambda) \\ &= -a^2 \end{aligned}$$

$$\begin{aligned} g_{01} = g_{10} &= -\frac{\partial t}{\partial a}\frac{\partial t}{\partial \lambda} + \frac{\partial x}{\partial a}\frac{\partial x}{\partial \lambda} \\ &= -a\sinh\lambda\cosh\lambda + a\sinh\lambda\cosh\lambda \\ &= 0 \end{aligned}$$

$$\begin{aligned} g_{11} &= -\left(\frac{\partial t}{\partial a}\right)^2 + \left(\frac{\partial x}{\partial a}\right)^2 \\ &= -\sinh^2\lambda + \cosh^2\lambda \\ &= 1. \end{aligned}$$

Thus

$$g_{\mu\nu} = \begin{pmatrix} -a^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Christoffel symbols are

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial \lambda} = 0 \\ \Gamma^1_{11} &= \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial a} = 0 \\ \Gamma^0_{01} &= \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial a} = \frac{1}{a} \end{aligned}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11} \frac{\partial g_{10}}{\partial \lambda} = 0$$
$$\Gamma^0_{11} = \frac{1}{2} g^{00} \left[2\frac{\partial g_{01}}{\partial a} - \frac{\partial g_{11}}{\partial \lambda} \right] = 0$$
$$\Gamma^1_{00} = \frac{1}{2} g^{11} \left[2\frac{\partial g_{01}}{\partial \lambda} - \frac{\partial g_{00}}{\partial a} \right] = a.$$

6.

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dx = (\sin\theta\cos\phi)dr + (r\cos\theta\cos\phi)d\theta + (-r\sin\theta\sin\phi)d\phi$$
$$dy = (\sin\theta\sin\phi)dr + (r\cos\theta\sin\phi)d\theta + (r\sin\theta\cos\phi)d\phi$$
$$dz = \cos\theta dr + (-r\sin\theta)d\theta$$

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dr^2(\sin^2\theta + \cos^2\theta) + d\theta^2(r^2) + d\phi^2(r^2) \\ &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned}$$

Setting $dr = 0$, we are left with only the angular part (at a fixed radius) $d\Omega^2$

$$g_{\mu\nu} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}.$$