

# Vector, Tensor RW PDE's

## 1 Vector $\sinh \chi$ RW

$$\phi = -\frac{1}{2}h_{00} \quad (1.1)$$

$$B = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \tilde{\nabla}_y^k \tilde{\nabla}_k^y h_{0k} \quad (1.2)$$

$$B_i = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \left[ \tilde{\nabla}_y^k \tilde{\nabla}_k^y h_{0i} - \tilde{\nabla}_i^y \tilde{\nabla}_y^k h_{0k} \right] \quad (1.3)$$

$$E_i = \int d^3y D^{(3)}(\mathbf{x} - \mathbf{y}) \int d^3z D^{(3)}(\mathbf{y} - \mathbf{z}) \left[ \tilde{\nabla}_z^k \tilde{\nabla}_k^z \tilde{\nabla}_z^j h_{ij} - \tilde{\nabla}_i^z \tilde{\nabla}_z^k \tilde{\nabla}_z^l h_{kl} \right] \quad (1.4)$$

$$\tilde{g}_{ij} = \text{Diag} (1, \sinh^2 \chi, \sinh^2 \chi \sin^2 \theta) \quad (1.5)$$

$$A^i = \tilde{\nabla}_a \tilde{\nabla}^a B^i \quad (1.6)$$

$$\tilde{\nabla}_a B^a = \frac{B_2 \cos \theta}{\sin \theta \sinh^2 \chi} + \frac{2B_1 \cosh \chi}{\sinh \chi} + \partial_1 B_1 + \frac{\partial_2 B_2}{\sinh^2 \chi} + \frac{\partial_3 B_3}{\sin^2 \theta \sinh^2 \chi} = 0 \quad (1.7)$$

$$\begin{aligned} A^1 = & B_1 \left( -2 - \frac{2}{\sinh^2 \chi} \right) - \frac{2B_2 \cos \theta \cosh \chi}{\sin \theta \sinh^3 \chi} + \frac{2 \cosh \chi \partial_1 B_1}{\sinh \chi} + \partial_1 \partial_1 B_1 + \frac{\cos \theta \partial_2 B_1}{\sin \theta \sinh^2 \chi} \\ & - \frac{2 \cosh \chi \partial_2 B_2}{\sinh^3 \chi} + \frac{\partial_2 \partial_2 B_1}{\sinh^2 \chi} - \frac{2 \cosh \chi \partial_3 B_3}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_3 \partial_3 B_1}{\sin^2 \theta \sinh^2 \chi} \end{aligned} \quad (1.8)$$

$$\begin{aligned} A^2 = & B_2 \left( -\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{\partial_1 \partial_1 B_2}{\sinh^2 \chi} + \frac{2 \cosh \chi \partial_2 B_1}{\sinh^3 \chi} + \frac{\cos \theta \partial_2 B_2}{\sin \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 B_2}{\sinh^4 \chi} \\ & - \frac{2 \cos \theta \partial_3 B_3}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 B_2}{\sin^2 \theta \sinh^4 \chi} \end{aligned} \quad (1.9)$$

$$\begin{aligned} A^3 = & -\frac{2B_3}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 B_3}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 B_3}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 B_3}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_3 B_1}{\sin^2 \theta \sinh^3 \chi} \\ & + \frac{2 \cos \theta \partial_3 B_2}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 B_3}{\sin^4 \theta \sinh^4 \chi} \end{aligned} \quad (1.10)$$

With imposition of trace and transverse conditions:

$$A^1 = B_1 \left( 2 + \frac{2}{\sinh^2 \chi} \right) + \frac{4 \cosh \chi \partial_1 B_1}{\sinh \chi} + \partial_1 \partial_1 B_1 + \frac{\cos \theta \partial_2 B_1}{\sin \theta \sinh^2 \chi} + \frac{\partial_2 \partial_2 B_1}{\sinh^2 \chi} + \frac{\partial_3 \partial_3 B_1}{\sin^2 \theta \sinh^2 \chi} \quad (1.11)$$

$$A^2 = B_2 \left( -\frac{2}{\sinh^4 \chi} + \frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{4B_1 \cos \theta \cosh \chi}{\sin \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_1 B_1}{\sin \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 B_2}{\sinh^2 \chi}$$

$$+\frac{2 \cosh \chi \partial_2 B_1}{\sinh^3 \chi} + \frac{3 \cos \theta \partial_2 B_2}{\sin \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 B_2}{\sinh^4 \chi} + \frac{\partial_3 \partial_3 B_2}{\sin^2 \theta \sinh^4 \chi} \quad (1.12)$$

$$\begin{aligned} A^3 = & -\frac{2B_3}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_1 \partial_1 B_3}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 B_3}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 B_3}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_3 B_1}{\sin^2 \theta \sinh^3 \chi} \\ & + \frac{2 \cos \theta \partial_3 B_2}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 B_3}{\sin^4 \theta \sinh^4 \chi} \end{aligned} \quad (1.13)$$

**1.1**  $B_i = g_i(\chi)$

$$\tilde{\nabla}_a B^a = g'_1 + \frac{g_2 \cos \theta}{\sin \theta \sinh^2 \chi} + \frac{2g_1 \cosh \chi}{\sinh \chi} = 0 \quad (1.14)$$

$$A^1 = g''_1 + g_1 \left( -2 - \frac{2}{\sinh^2 \chi} \right) - \frac{2g_2 \cos \theta \cosh \chi}{\sin \theta \sinh^3 \chi} + \frac{2 \cosh \chi g'_1}{\sinh \chi} \quad (1.15)$$

$$\text{Substituting } g_2 \quad (1.16)$$

$$\begin{aligned} A^1 = & \cos \theta g''_1 + g_1 \left[ \frac{\sin^2 \theta (-8 - \frac{8}{\sinh^2 \chi})}{\cos \theta} + \cos \theta \left( -2 - \frac{4}{\sinh^2 \chi} \right) + \frac{4 + \frac{4}{\sinh^2 \chi}}{\cos \theta} \right] \\ & + g'_1 \left( \frac{2 \cosh \chi}{\cos \theta \sinh \chi} + \frac{2 \cos \theta \cosh \chi}{\sinh \chi} - \frac{4 \cosh \chi \sin^2 \theta}{\cos \theta \sinh \chi} \right) \end{aligned} \quad (1.17)$$

$$A^2 = g_2 \left( -\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{g''_2}{\sinh^2 \chi} \quad (1.18)$$

$$A^3 = -\frac{2g_3}{\sin^2 \theta \sinh^2 \chi} + \frac{g''_3}{\sin^2 \theta \sinh^2 \chi} \quad (1.19)$$

**1.2**  $B_i = h_i(\chi) \cos \theta$

$$\tilde{\nabla}_a B^a = \cos \theta h'_1 + h_2 \left( \frac{1}{\sin \theta \sinh^2 \chi} - \frac{2 \sin \theta}{\sinh^2 \chi} \right) + \frac{2h_1 \cos \theta \cosh \chi}{\sinh \chi} = 0 \quad (1.20)$$

$$A^1 = \cos \theta h''_1 + h_2 \left( -\frac{2 \cosh \chi}{\sin \theta \sinh^3 \chi} + \frac{4 \cosh \chi \sin \theta}{\sinh^3 \chi} \right) + h_1 \cos \theta \left( -2 - \frac{4}{\sinh^2 \chi} \right) + \frac{2 \cos \theta \cosh \chi h'_1}{\sinh \chi} \quad (1.21)$$

$$\text{Substituting } h_2 \quad (1.22)$$

$$\begin{aligned} A^1 = & \cos \theta h''_1 \\ & + h_1 \left( -2 \cos \theta - \frac{4 \cos \theta}{-1 + 2 \sin^2 \theta} + \frac{8 \cos \theta \sin^2 \theta}{-1 + 2 \sin^2 \theta} - \frac{4 \cos \theta}{\sinh^2 \chi} - \frac{4 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} + \frac{8 \cos \theta \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} \right) \\ & + h'_1 \left( \frac{2 \cos \theta \cosh \chi}{\sinh \chi} - \frac{2 \cos \theta \cosh \chi}{(-1 + 2 \sin^2 \theta) \sinh \chi} + \frac{4 \cos \theta \cosh \chi \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh \chi} \right) \end{aligned} \quad (1.23)$$

$$A^2 = h_2 \left[ \cos \theta \left( -\frac{2}{\sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) - \frac{\cos \theta}{\sin^2 \theta \sinh^4 \chi} \right] - \frac{2h_1 \cosh \chi \sin \theta}{\sinh^3 \chi} + \frac{\cos \theta h''_2}{\sinh^2 \chi} \quad (1.24)$$

$$A^3 = -\frac{2h_3 \cos \theta}{\sin^2 \theta \sinh^2 \chi} + \frac{\cos \theta h''_3}{\sin^2 \theta \sinh^2 \chi} \quad (1.25)$$

## 2 Tensor $\sinh \chi$ RW

$$\tilde{g}_{ij} = \text{Diag} (1, \sinh^2 \chi, \sinh^2 \chi \sin^2 \theta) \quad (2.1)$$

$$A^{ij} = \tilde{\nabla}_a \tilde{\nabla}^a E^{ij} \quad (2.2)$$

$$\tilde{g}^{ab} E_{ab} = E_{11} + \frac{E_{22}}{\sinh^2 \chi} + \frac{E_{33}}{\sin^2 \theta \sinh^2 \chi} = 0 \quad (2.3)$$

$$\begin{aligned} \tilde{\nabla}_a E^{a1} = & -\frac{\cosh \chi E_{22}}{\sinh^3 \chi} - \frac{\cosh \chi E_{33}}{\sin^2 \theta \sinh^3 \chi} + \frac{\cos \theta E_{12}}{\sin \theta \sinh^2 \chi} + \frac{2 \cosh \chi E_{11}}{\sinh \chi} + \partial_1 E_{11} + \frac{\partial_2 E_{12}}{\sinh^2 \chi} \\ & + \frac{\partial_3 E_{13}}{\sin^2 \theta \sinh^2 \chi} = 0 \end{aligned} \quad (2.4)$$

$$\tilde{\nabla}_a E^{a2} = -\frac{\cos \theta E_{33}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta E_{22}}{\sin \theta \sinh^4 \chi} + \frac{2 \cosh \chi E_{12}}{\sinh^3 \chi} + \frac{\partial_1 E_{12}}{\sinh^2 \chi} + \frac{\partial_2 E_{22}}{\sinh^4 \chi} + \frac{\partial_3 E_{23}}{\sin^2 \theta \sinh^4 \chi} \quad (2.5)$$

$$\tilde{\nabla}_a E^{a3} = \frac{\cos \theta E_{23}}{\sin^3 \theta \sinh^4 \chi} + \frac{2 \cosh \chi E_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 E_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{\partial_2 E_{23}}{\sin^2 \theta \sinh^4 \chi} + \frac{\partial_3 E_{33}}{\sin^4 \theta \sinh^4 \chi} \quad (2.6)$$

$$\begin{aligned} A^{11} = & E_{11} \left( -4 - \frac{4}{\sinh^2 \chi} \right) + E_{22} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) + \frac{E_{33} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} \\ & - \frac{4 \cos \theta \cosh \chi E_{12}}{\sin \theta \sinh^3 \chi} + \frac{2 \cosh \chi \partial_1 E_{11}}{\sinh \chi} + \partial_1 \partial_1 E_{11} + \frac{\cos \theta \partial_2 E_{11}}{\sin \theta \sinh^2 \chi} - \frac{4 \cosh \chi \partial_2 E_{12}}{\sinh^3 \chi} + \frac{\partial_2 \partial_2 E_{11}}{\sinh^2 \chi} \\ & - \frac{4 \cosh \chi \partial_3 E_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_3 \partial_3 E_{11}}{\sin^2 \theta \sinh^2 \chi} \end{aligned} \quad (2.7)$$

$$\begin{aligned} A^{22} = & E_{33} \left( \frac{2}{\sin^4 \theta \sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} \right) + E_{22} \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right) \\ & + E_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) - \frac{2 \cosh \chi \partial_1 E_{22}}{\sinh^5 \chi} + \frac{\partial_1 \partial_1 E_{22}}{\sinh^4 \chi} + \frac{4 \cosh \chi \partial_2 E_{12}}{\sinh^5 \chi} + \frac{\cos \theta \partial_2 E_{22}}{\sin \theta \sinh^6 \chi} \\ & + \frac{\partial_2 \partial_2 E_{22}}{\sinh^6 \chi} - \frac{4 \cos \theta \partial_3 E_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 E_{22}}{\sin^2 \theta \sinh^6 \chi} \end{aligned} \quad (2.8)$$

$$\begin{aligned} A^{33} = & E_{22} \left( \frac{2}{\sin^4 \theta \sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} \right) + E_{33} \left( \frac{2}{\sin^6 \theta \sinh^6 \chi} - \frac{2}{\sin^4 \theta \sinh^4 \chi} \right) \\ & + \frac{E_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} + \frac{4 \cos \theta \cosh \chi E_{12}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cosh \chi \partial_1 E_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 E_{33}}{\sin^4 \theta \sinh^4 \chi} \\ & - \frac{3 \cos \theta \partial_2 E_{33}}{\sin^5 \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 E_{33}}{\sin^4 \theta \sinh^6 \chi} + \frac{4 \cosh \chi \partial_3 E_{13}}{\sin^4 \theta \sinh^5 \chi} + \frac{4 \cos \theta \partial_3 E_{23}}{\sin^5 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 E_{33}}{\sin^6 \theta \sinh^6 \chi} \end{aligned} \quad (2.9)$$

$$\begin{aligned} A^{12} = & E_{12} \left( -\frac{4}{\sinh^4 \chi} - \frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right) + \frac{2 \cos \theta \cosh \chi E_{33}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cos \theta \cosh \chi E_{22}}{\sin \theta \sinh^5 \chi} \\ & + \frac{\partial_1 \partial_1 E_{12}}{\sinh^2 \chi} + \frac{2 \cosh \chi \partial_2 E_{11}}{\sinh^3 \chi} + \frac{\cos \theta \partial_2 E_{12}}{\sin \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_2 E_{22}}{\sinh^5 \chi} + \frac{\partial_2 \partial_2 E_{12}}{\sinh^4 \chi} - \frac{2 \cos \theta \partial_3 E_{13}}{\sin^3 \theta \sinh^4 \chi} \\ & - \frac{2 \cosh \chi \partial_3 E_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\partial_3 \partial_3 E_{12}}{\sin^2 \theta \sinh^4 \chi} \end{aligned} \quad (2.10)$$

$$A^{13} = \frac{E_{13} \left( -\frac{4}{\sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right)}{\sin^2 \theta} - \frac{2 \cos \theta \cosh \chi E_{23}}{\sin^3 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 E_{13}}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 E_{13}}{\sin^3 \theta \sinh^4 \chi}$$

$$\begin{aligned}
& -\frac{2 \cosh \chi \partial_2 E_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\partial_2 \partial_2 E_{13}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_3 E_{11}}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 E_{12}}{\sin^3 \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_3 E_{33}}{\sin^4 \theta \sinh^5 \chi} \\
& + \frac{\partial_3 \partial_3 E_{13}}{\sin^4 \theta \sinh^4 \chi}
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
A^{23} = & E_{23} \left( \frac{\frac{4}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi}}{\sin^2 \theta} - \frac{3}{\sin^4 \theta \sinh^6 \chi} \right) - \frac{4 \cos \theta \cosh \chi E_{13}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cosh \chi \partial_1 E_{23}}{\sin^2 \theta \sinh^5 \chi} \\
& + \frac{\partial_1 \partial_1 E_{23}}{\sin^2 \theta \sinh^4 \chi} + \frac{2 \cosh \chi \partial_2 E_{13}}{\sin^2 \theta \sinh^5 \chi} - \frac{\cos \theta \partial_2 E_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 E_{23}}{\sin^2 \theta \sinh^6 \chi} + \frac{2 \cosh \chi \partial_3 E_{12}}{\sin^2 \theta \sinh^5 \chi} \\
& + \frac{2 \cos \theta \partial_3 E_{22}}{\sin^3 \theta \sinh^6 \chi} - \frac{2 \cos \theta \partial_3 E_{33}}{\sin^5 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 E_{23}}{\sin^4 \theta \sinh^6 \chi}
\end{aligned} \tag{2.12}$$

With imposition of trace and transverse conditions:

$$A^{11} = E_{11} \left( 6 + \frac{6}{\sinh^2 \chi} \right) + \frac{6 \cosh \chi \partial_1 E_{11}}{\sinh \chi} + \partial_1 \partial_1 E_{11} + \frac{\cos \theta \partial_2 E_{11}}{\sin \theta \sinh^2 \chi} + \frac{\partial_2 \partial_2 E_{11}}{\sinh^2 \chi} + \frac{\partial_3 \partial_3 E_{11}}{\sin^2 \theta \sinh^2 \chi} \tag{2.13}$$

$$\begin{aligned}
A^{22} = & \frac{4E_{22}}{\sinh^6 \chi} - \frac{4E_{22}}{\sin^2 \theta \sinh^6 \chi} + \frac{4E_{11}}{\sinh^4 \chi} - \frac{2E_{22}}{\sinh^4 \chi} - \frac{2E_{11}}{\sin^2 \theta \sinh^4 \chi} + \frac{2E_{11}}{\sinh^2 \chi} - \frac{2 \cosh \chi \partial_1 E_{22}}{\sinh^5 \chi} \\
& + \frac{\partial_1 \partial_1 E_{22}}{\sinh^4 \chi} + \frac{4 \cosh \chi \partial_2 E_{12}}{\sinh^5 \chi} + \frac{\cos \theta \partial_2 E_{22}}{\sin \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 E_{22}}{\sinh^6 \chi} - \frac{4 \cos \theta \partial_3 E_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_3 \partial_3 E_{22}}{\sin^2 \theta \sinh^6 \chi}
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
A^{33} = & \frac{E_{33} \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^4 \theta} + E_{11} \left( \frac{2}{\sin^4 \theta \sinh^4 \chi} + \frac{2}{\sin^2 \theta \sinh^2 \chi} \right) - \frac{4 \cos \theta \cosh \chi E_{12}}{\sin^3 \theta \sinh^5 \chi} \\
& - \frac{4 \cos \theta \partial_1 E_{12}}{\sin^3 \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_1 E_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 E_{33}}{\sin^4 \theta \sinh^4 \chi} + \frac{4 \cos \theta \partial_2 E_{11}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta \partial_2 E_{33}}{\sin^5 \theta \sinh^6 \chi} \\
& + \frac{\partial_2 \partial_2 E_{33}}{\sin^4 \theta \sinh^6 \chi} + \frac{4 \cosh \chi \partial_3 E_{13}}{\sin^4 \theta \sinh^5 \chi} + \frac{\partial_3 \partial_3 E_{33}}{\sin^6 \theta \sinh^6 \chi}
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
A^{12} = & E_{12} \left( -\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{2 \cosh \chi \partial_1 E_{12}}{\sinh^3 \chi} + \frac{\partial_1 \partial_1 E_{12}}{\sinh^2 \chi} + \frac{2 \cosh \chi \partial_2 E_{11}}{\sinh^3 \chi} \\
& + \frac{\cos \theta \partial_2 E_{12}}{\sin \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 E_{12}}{\sinh^4 \chi} - \frac{2 \cos \theta \partial_3 E_{13}}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 E_{12}}{\sin^2 \theta \sinh^4 \chi}
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
A^{13} = & -\frac{2E_{13}}{\sin^2 \theta \sinh^2 \chi} + \frac{2 \cosh \chi \partial_1 E_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\partial_1 \partial_1 E_{13}}{\sin^2 \theta \sinh^2 \chi} - \frac{\cos \theta \partial_2 E_{13}}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_2 \partial_2 E_{13}}{\sin^2 \theta \sinh^4 \chi} \\
& + \frac{2 \cosh \chi \partial_3 E_{11}}{\sin^2 \theta \sinh^3 \chi} + \frac{2 \cos \theta \partial_3 E_{12}}{\sin^3 \theta \sinh^4 \chi} + \frac{\partial_3 \partial_3 E_{13}}{\sin^4 \theta \sinh^4 \chi}
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
A^{23} = & E_{23} \left( \frac{\frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi}}{\sin^2 \theta} - \frac{1}{\sin^4 \theta \sinh^6 \chi} \right) + \frac{2 \cos \theta \partial_1 E_{13}}{\sin^3 \theta \sinh^4 \chi} - \frac{2 \cosh \chi \partial_1 E_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\partial_1 \partial_1 E_{23}}{\sin^2 \theta \sinh^4 \chi} \\
& + \frac{2 \cosh \chi \partial_2 E_{13}}{\sin^2 \theta \sinh^5 \chi} + \frac{\cos \theta \partial_2 E_{23}}{\sin^3 \theta \sinh^6 \chi} + \frac{\partial_2 \partial_2 E_{23}}{\sin^2 \theta \sinh^6 \chi} + \frac{2 \cosh \chi \partial_3 E_{12}}{\sin^2 \theta \sinh^5 \chi} + \frac{2 \cos \theta \partial_3 E_{22}}{\sin^3 \theta \sinh^6 \chi} \\
& + \frac{\partial_3 \partial_3 E_{23}}{\sin^4 \theta \sinh^6 \chi}
\end{aligned} \tag{2.18}$$

## 2.1 $E_{ij} = g_{ij}(\chi)$

$$\tilde{g}^{ab} E_{ab} = g_{11} + \frac{g_{22}}{\sinh^2 \chi} + \frac{g_{33}}{\sin^2 \theta \sinh^2 \chi} = 0 \tag{2.19}$$

$$\tilde{\nabla}_a E^{a1} = g'_{11} - \frac{\cosh \chi g_{22}}{\sinh^3 \chi} - \frac{\cosh \chi g_{33}}{\sin^2 \theta \sinh^3 \chi} + \frac{\cos \theta g_{12}}{\sin \theta \sinh^2 \chi} + \frac{2 \cosh \chi g_{11}}{\sinh \chi} \quad (2.20)$$

$$\tilde{\nabla}_a E^{a2} = -\frac{\cos \theta g_{33}}{\sin^3 \theta \sinh^4 \chi} + \frac{\cos \theta g_{22}}{\sin \theta \sinh^4 \chi} + \frac{2 \cosh \chi g_{12}}{\sinh^3 \chi} + \frac{g'_{12}}{\sinh^2 \chi} \quad (2.21)$$

$$\tilde{\nabla}_a E^{a3} = \frac{\cos \theta g_{23}}{\sin^3 \theta \sinh^4 \chi} + \frac{2 \cosh \chi g_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{g'_{13}}{\sin^2 \theta \sinh^2 \chi} \quad (2.22)$$

Without imposing trace or transverse conditions:

$$\begin{aligned} A^{11} = & g''_{11} + g_{11} \left( -4 - \frac{4}{\sinh^2 \chi} \right) + g_{22} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) + \frac{g_{33} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} \\ & - \frac{4 \cos \theta \cosh \chi g_{12}}{\sin \theta \sinh^3 \chi} + \frac{2 \cosh \chi g'_{11}}{\sinh \chi} \end{aligned} \quad (2.23)$$

$$\begin{aligned} A^{22} = & g_{33} \left( \frac{2}{\sin^4 \theta \sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} \right) + g_{22} \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right) \\ & + g_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) - \frac{2 \cosh \chi g'_{22}}{\sinh^5 \chi} + \frac{g''_{22}}{\sinh^4 \chi} \end{aligned} \quad (2.24)$$

$$\begin{aligned} A^{33} = & g_{22} \left( \frac{2}{\sin^4 \theta \sinh^6 \chi} - \frac{2}{\sin^2 \theta \sinh^6 \chi} \right) + g_{33} \left( \frac{2}{\sin^6 \theta \sinh^6 \chi} - \frac{2}{\sin^4 \theta \sinh^4 \chi} \right) \\ & + \frac{g_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} - \frac{2 \cosh \chi g'_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{4 \cos \theta \cosh \chi g_{12}}{\sin^3 \theta \sinh^5 \chi} + \frac{g''_{33}}{\sin^4 \theta \sinh^4 \chi} \end{aligned} \quad (2.25)$$

$$A^{12} = g_{12} \left( -\frac{4}{\sinh^4 \chi} - \frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right) + \frac{2 \cos \theta \cosh \chi g_{33}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cos \theta \cosh \chi g_{22}}{\sin \theta \sinh^5 \chi} + \frac{g'_{12}}{\sinh^2 \chi} \quad (2.26)$$

$$A^{13} = \frac{g_{13} \left( -\frac{4}{\sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right)}{\sin^2 \theta} - \frac{2 \cos \theta \cosh \chi g_{23}}{\sin^3 \theta \sinh^5 \chi} + \frac{g'_{13}}{\sin^2 \theta \sinh^2 \chi} \quad (2.27)$$

$$A^{23} = g_{23} \left( \frac{\frac{4}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi}}{\sin^2 \theta} - \frac{3}{\sin^4 \theta \sinh^6 \chi} \right) - \frac{4 \cos \theta \cosh \chi g_{13}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cosh \chi g'_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{g''_{23}}{\sin^2 \theta \sinh^4 \chi} \quad (2.28)$$

With imposition of trace and transverse conditions:

$$A^{11} = g''_{11} + g_{11} \left( 6 + \frac{6}{\sinh^2 \chi} \right) + \frac{6 \cosh \chi g'_{11}}{\sinh \chi} \quad (2.29)$$

$$\begin{aligned} A^{22} = & g_{22} \left( \frac{4}{\sinh^6 \chi} - \frac{4}{\sin^2 \theta \sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right) + g_{11} \left( \frac{4}{\sinh^4 \chi} - \frac{2}{\sin^2 \theta \sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) \\ & - \frac{2 \cosh \chi g'_{22}}{\sinh^5 \chi} + \frac{g''_{22}}{\sinh^4 \chi} \end{aligned} \quad (2.30)$$

$$\begin{aligned} A^{33} = & \frac{g_{33} \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^4 \theta} + g_{11} \left( -\frac{\frac{8}{\sinh^4 \chi} + \frac{10}{\sinh^2 \chi}}{\sin^2 \theta} - \frac{2}{\sin^4 \theta \sinh^4 \chi} \right) - \frac{2 \cosh \chi g'_{33}}{\sin^4 \theta \sinh^5 \chi} \\ & + \frac{g''_{33}}{\sin^4 \theta \sinh^4 \chi} - \frac{4 \cosh \chi g'_{11}}{\sin^2 \theta \sinh^3 \chi} \end{aligned} \quad (2.31)$$

$$A^{12} = g_{12} \left( -\frac{1}{\sin^2 \theta \sinh^4 \chi} - \frac{2}{\sinh^2 \chi} \right) + \frac{2 \cosh \chi g'_{12}}{\sinh^3 \chi} + \frac{g'_{12}}{\sinh^2 \chi} \quad (2.32)$$

$$A^{13} = \frac{2 \cosh \chi g'_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{g'_{13}}{\sin^2 \theta \sinh^2 \chi} - \frac{2 g_{13}}{\sin^2 \theta \sinh^2 \chi} \quad (2.33)$$

$$A^{23} = g_{23} \left( \frac{\frac{4}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi}}{\sin^2 \theta} - \frac{3}{\sin^4 \theta \sinh^6 \chi} \right) - \frac{4 \cos \theta \cosh \chi g_{13}}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cosh \chi g'_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{g''_{23}}{\sin^2 \theta \sinh^4 \chi} \quad (2.34)$$

$$2.2 \quad E_{ij} = h_{ij}(\chi) \cos \theta$$

$$\tilde{g}^{ab} E_{ab} = h_{11} + \frac{h_{22}}{\sinh^2 \chi} + \frac{h_{33}}{\sin^2 \theta \sinh^2 \chi} = 0 \quad (2.35)$$

$$\begin{aligned} \tilde{\nabla}_a E^{a1} = & \cos \theta h'_{11} + h_{12} \left( \frac{1}{\sin \theta \sinh^2 \chi} - \frac{2 \sin \theta}{\sinh^2 \chi} \right) - \frac{\cos \theta \cosh \chi h_{22}}{\sinh^3 \chi} - \frac{\cos \theta \cosh \chi h_{33}}{\sin^2 \theta \sinh^3 \chi} \\ & + \frac{2 \cos \theta \cosh \chi h_{11}}{\sinh \chi} \end{aligned} \quad (2.36)$$

$$\begin{aligned} \tilde{\nabla}_a E^{a2} = & h_{33} \left( -\frac{1}{\sin^3 \theta \sinh^4 \chi} + \frac{1}{\sin \theta \sinh^4 \chi} \right) + h_{22} \left( \frac{1}{\sin \theta \sinh^4 \chi} - \frac{2 \sin \theta}{\sinh^4 \chi} \right) + \frac{2 \cos \theta \cosh \chi h_{12}}{\sinh^3 \chi} \\ & + \frac{\cos \theta h'_{12}}{\sinh^2 \chi} \end{aligned} \quad (2.37)$$

$$\tilde{\nabla}_a E^{a3} = h_{23} \left( \frac{1}{\sin^3 \theta \sinh^4 \chi} - \frac{2}{\sin \theta \sinh^4 \chi} \right) + \frac{2 \cos \theta \cosh \chi h_{13}}{\sin^2 \theta \sinh^3 \chi} + \frac{\cos \theta h'_{13}}{\sin^2 \theta \sinh^2 \chi} \quad (2.38)$$

Without imposing trace or transverse conditions:

$$\begin{aligned} A^{11} = & \cos \theta h''_{11} + h_{12} \left( -\frac{4 \cosh \chi}{\sin \theta \sinh^3 \chi} + \frac{8 \cosh \chi \sin \theta}{\sinh^3 \chi} \right) + \cos \theta h_{11} \left( -4 - \frac{6}{\sinh^2 \chi} \right) \\ & + \cos \theta h_{22} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) + \frac{\cos \theta h_{33} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} + \frac{2 \cos \theta \cosh \chi h'_{11}}{\sinh \chi} \end{aligned} \quad (2.39)$$

$$\begin{aligned} A^{22} = & h_{33} \left( \frac{2 \cos \theta}{\sin^4 \theta \sinh^6 \chi} - \frac{2 \cos \theta}{\sin^2 \theta \sinh^6 \chi} \right) + h_{22} \left( -\frac{2 \cos \theta}{\sin^2 \theta \sinh^6 \chi} - \frac{2 \cos \theta}{\sinh^4 \chi} \right) \\ & + \cos \theta h_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) - \frac{2 \cos \theta \cosh \chi h'_{22}}{\sinh^5 \chi} - \frac{4 \cosh \chi h_{12} \sin \theta}{\sinh^5 \chi} + \frac{\cos \theta h''_{22}}{\sinh^4 \chi} \end{aligned} \quad (2.40)$$

$$\begin{aligned} A^{33} = & h_{33} \left( \frac{\cos \theta \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^4 \theta} + \frac{2 \cos \theta}{\sin^6 \theta \sinh^6 \chi} \right) + h_{12} \left( \frac{4 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} - \frac{4 \cosh \chi}{\sin \theta \sinh^5 \chi} \right) \\ & + \frac{\cos \theta h_{11} \left( \frac{2}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right)}{\sin^2 \theta} + \frac{2 \cos^3 \theta h_{22}}{\sin^4 \theta \sinh^6 \chi} - \frac{2 \cos \theta \cosh \chi h'_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\cos \theta h''_{33}}{\sin^4 \theta \sinh^4 \chi} \end{aligned} \quad (2.41)$$

$$\begin{aligned} A^{12} = & h_{33} \left( \frac{2 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} - \frac{2 \cosh \chi}{\sin \theta \sinh^5 \chi} \right) + h_{22} \left( -\frac{2 \cosh \chi}{\sin \theta \sinh^5 \chi} + \frac{4 \cosh \chi \sin \theta}{\sinh^5 \chi} \right) \\ & + h_{12} \left( \cos \theta \left( -\frac{6}{\sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right) - \frac{\cos \theta}{\sin^2 \theta \sinh^4 \chi} \right) - \frac{2 \cosh \chi h_{11} \sin \theta}{\sinh^3 \chi} + \frac{\cos \theta h''_{12}}{\sinh^2 \chi} \end{aligned} \quad (2.42)$$

$$A^{13} = h_{23} \left( -\frac{2 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} + \frac{4 \cosh \chi}{\sin \theta \sinh^5 \chi} \right) + \frac{\cos \theta h_{13} \left( -\frac{4}{\sinh^4 \chi} - \frac{6}{\sinh^2 \chi} \right)}{\sin^2 \theta} + \frac{\cos \theta h''_{13}}{\sin^2 \theta \sinh^2 \chi} \quad (2.43)$$

$$\begin{aligned} A^{23} = & h_{23} \left( \frac{\cos \theta \left( \frac{4}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^2 \theta} - \frac{3 \cos \theta}{\sin^4 \theta \sinh^6 \chi} \right) + h_{13} \left( -\frac{4 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} + \frac{2 \cosh \chi}{\sin \theta \sinh^5 \chi} \right) \\ & - \frac{2 \cos \theta \cosh \chi h'_{23}}{\sin^2 \theta \sinh^5 \chi} + \frac{\cos \theta h''_{23}}{\sin^2 \theta \sinh^4 \chi} \end{aligned} \quad (2.44)$$

With imposition of trace and transverse conditions:

$$\begin{aligned}
A^{11} = & \cos \theta h''_{11} \\
& + h_{11} \left( -6 \cos \theta - \frac{12 \cos \theta}{-1 + 2 \sin^2 \theta} + \frac{24 \cos \theta \sin^2 \theta}{-1 + 2 \sin^2 \theta} - \frac{8 \cos \theta}{\sinh^2 \chi} - \frac{12 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} + \frac{24 \cos \theta \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} \right) \\
& + h'_{11} \left( \frac{2 \cos \theta \cosh \chi}{\sinh \chi} - \frac{4 \cos \theta \cosh \chi}{(-1 + 2 \sin^2 \theta) \sinh \chi} + \frac{8 \cos \theta \cosh \chi \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh \chi} \right)
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
A^{22} = & h_{22} \left( \cos \theta \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right) - \frac{4 \cos \theta}{\sin^2 \theta \sinh^6 \chi} \right) \\
& + h_{11} \left( \cos \theta \left( \frac{4}{\sinh^4 \chi} + \frac{2}{\sinh^2 \chi} \right) - \frac{2 \cos \theta}{\sin^2 \theta \sinh^4 \chi} \right) - \frac{2 \cos \theta \cosh \chi h'_{22}}{\sinh^5 \chi} - \frac{4 \cosh \chi h_{12} \sin \theta}{\sinh^5 \chi} \\
& + \frac{\cos \theta h''_{22}}{\sinh^4 \chi}
\end{aligned} \tag{2.46}$$

$$\begin{aligned}
A^{33} = & h_{33} \left( \frac{\frac{2 \cos \theta}{\sinh^6 \chi} - \frac{2 \cos^3 \theta}{\sinh^6 \chi}}{\sin^6 \theta} + \frac{\cos \theta \left( \frac{2}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^4 \theta} \right) \\
& + h'_{11} \left( -\frac{4 \cos \theta \cosh \chi}{(-1 + 2 \sin^2 \theta) \sinh^3 \chi} + \frac{4 \cos \theta \cosh \chi}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^3 \chi} \right) \\
& + h_{11} \left( -\frac{2 \cos^3 \theta}{\sin^4 \theta \sinh^4 \chi} + \frac{2 \cos \theta}{\sin^2 \theta \sinh^4 \chi} - \frac{12 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^4 \chi} \right. \\
& + \frac{12 \cos \theta}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^4 \chi} + \frac{2 \cos \theta}{\sin^2 \theta \sinh^2 \chi} - \frac{12 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} + \left. \frac{12 \cos \theta}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^2 \chi} \right) \\
& - \frac{2 \cos \theta \cosh \chi h'_{33}}{\sin^4 \theta \sinh^5 \chi} + \frac{\cos \theta h''_{33}}{\sin^4 \theta \sinh^4 \chi}
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
A^{12} = & h_{33} \left( \frac{2 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} + \frac{2 \cosh \chi}{\sin^3 \theta (-1 + 2 \sin^2 \theta) \sinh^5 \chi} - \frac{8 \cosh \chi}{\sin \theta (-1 + 2 \sin^2 \theta) \sinh^5 \chi} + \frac{6 \cosh \chi \sin \theta}{(-1 + 2 \sin^2 \theta) \sinh^5 \chi} \right) \\
& + h'_{12} \left( -\frac{2 \cos \theta \cosh \chi}{(-1 + 2 \sin^2 \theta) \sinh^3 \chi} + \frac{6 \cos \theta \cosh \chi \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh^3 \chi} \right) \\
& + h_{12} \left( -\frac{6 \cos \theta}{\sinh^4 \chi} - \frac{\cos \theta}{\sin^2 \theta \sinh^4 \chi} - \frac{4 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^4 \chi} \right. \\
& + \frac{12 \cos \theta \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh^4 \chi} - \frac{6 \cos \theta}{\sinh^2 \chi} - \frac{4 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} + \left. \frac{12 \cos \theta \sin^2 \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} \right) \\
& + \frac{\cos \theta h''_{12}}{\sinh^2 \chi}
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
A^{13} = & h'_{13} \left( \frac{4 \cos \theta \cosh \chi}{(-1 + 2 \sin^2 \theta) \sinh^3 \chi} - \frac{2 \cos \theta \cosh \chi}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^3 \chi} \right) \\
& + h_{13} \left( -\frac{4 \cos \theta}{\sin^2 \theta \sinh^4 \chi} + \frac{8 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^4 \chi} - \frac{4 \cos \theta}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^4 \chi} \right. \\
& - \frac{6 \cos \theta}{\sin^2 \theta \sinh^2 \chi} + \frac{8 \cos \theta}{(-1 + 2 \sin^2 \theta) \sinh^2 \chi} - \frac{4 \cos \theta}{\sin^2 \theta (-1 + 2 \sin^2 \theta) \sinh^2 \chi} \left. \right) \\
& + \frac{\cos \theta h''_{13}}{\sin^2 \theta \sinh^2 \chi}
\end{aligned} \tag{2.49}$$

$$A^{23} = h_{23} \left( \frac{\cos \theta \left( \frac{4}{\sinh^6 \chi} - \frac{2}{\sinh^4 \chi} \right)}{\sin^2 \theta} - \frac{3 \cos \theta}{\sin^4 \theta \sinh^6 \chi} \right) + h_{13} \left( -\frac{4 \cosh \chi}{\sin^3 \theta \sinh^5 \chi} + \frac{2 \cosh \chi}{\sin \theta \sinh^5 \chi} \right)$$

$$-\frac{2\cos\theta\cosh\chi h'_{23}}{\sin^2\theta\sinh^5\chi}+\frac{\cos\theta h''_{23}}{\sin^2\theta\sinh^4\chi} \tag{2.50}$$



## Appendix A Contravariance

The program is currently setup such that covariant components are the default form; for example  $E^{ij} = \tilde{g}^{ik}\tilde{g}^{j\ell}E_{k\ell}$ . In the polar  $\chi$  coordinates, the calculations for  $A^i$  and  $A^{ij}$  go as

$$\begin{aligned}\tilde{\nabla}_a \tilde{\nabla}^a B^i &= -\frac{2B_2 \cos \theta \cosh \chi \tilde{g}^{i1}}{\sin \theta \sinh^3 \chi} + \frac{B_2 \tilde{g}^{i2}}{\sinh^2 \chi} + \frac{B_3 \tilde{g}^{i3}}{\sinh^2 \chi} + \frac{B_3 \tilde{g}^{i3}}{\sin^2 \theta \sinh^2 \chi} - \frac{B_2 \tilde{g}^{i2}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} \\ &\quad - \frac{B_3 \tilde{g}^{i3}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{2B_1 \tilde{g}^{i1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{2B_2 \tilde{g}^{i2}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{2B_3 \tilde{g}^{i3}(1 + \sinh^2 \chi)}{\sinh^2 \chi} \\ &\quad + \frac{2 \cosh \chi \tilde{g}^{i1} \partial_1 B_1}{\sinh \chi} + \tilde{g}^{i1} \partial_1 \partial_1 B_1 + \tilde{g}^{i2} \partial_1 \partial_1 B_2 + \tilde{g}^{i3} \partial_1 \partial_1 B_3 + \frac{\cos \theta \tilde{g}^{i1} \partial_2 B_1}{\sin \theta \sinh^2 \chi} \\ &\quad + \frac{2 \cosh \chi \tilde{g}^{i2} \partial_2 B_1}{\sinh \chi} - \frac{2 \cosh \chi \tilde{g}^{i1} \partial_2 B_2}{\sinh^3 \chi} + \frac{\cos \theta \tilde{g}^{i2} \partial_2 B_2}{\sin \theta \sinh^2 \chi} - \frac{\cos \theta \tilde{g}^{i3} \partial_2 B_3}{\sin \theta \sinh^2 \chi} + \frac{\tilde{g}^{i1} \partial_2 \partial_2 B_1}{\sinh^2 \chi} \\ &\quad + \frac{\tilde{g}^{i2} \partial_2 \partial_2 B_2}{\sinh^2 \chi} + \frac{\tilde{g}^{i3} \partial_2 \partial_2 B_3}{\sinh^2 \chi} + \frac{2 \cosh \chi \tilde{g}^{i3} \partial_3 B_1}{\sinh \chi} + \frac{2 \cos \theta \tilde{g}^{i3} \partial_3 B_2}{\sin \theta \sinh^2 \chi} - \frac{2 \cosh \chi \tilde{g}^{i1} \partial_3 B_3}{\sin^2 \theta \sinh^3 \chi} \\ &\quad - \frac{2 \cos \theta \tilde{g}^{i2} \partial_3 B_3}{\sin^3 \theta \sinh^2 \chi} + \frac{\tilde{g}^{i1} \partial_3 \partial_3 B_1}{\sin^2 \theta \sinh^2 \chi} + \frac{\tilde{g}^{i2} \partial_3 \partial_3 B_2}{\sin^2 \theta \sinh^2 \chi} + \frac{\tilde{g}^{i3} \partial_3 \partial_3 B_3}{\sin^2 \theta \sinh^2 \chi}\end{aligned}\tag{A.1}$$

$$\begin{aligned}\tilde{\nabla}_a \tilde{\nabla}^a E^{ij} &= \frac{2 \cos \theta \cosh \chi E_{33} \tilde{g}^{i2} \tilde{g}^{j1}}{\sin^3 \theta \sinh^3 \chi} + \frac{2 \cos \theta \cosh \chi E_{33} \tilde{g}^{i1} \tilde{g}^{j2}}{\sin^3 \theta \sinh^3 \chi} - \frac{2 \cos \theta \cosh \chi E_{12} \tilde{g}^{i1} \tilde{g}^{j1}}{\sin \theta \sinh^3 \chi} \\ &\quad - \frac{2 \cos \theta \cosh \chi E_{21} \tilde{g}^{i1} \tilde{g}^{j1}}{\sin \theta \sinh^3 \chi} - \frac{2 \cos \theta \cosh \chi E_{22} \tilde{g}^{i2} \tilde{g}^{j1}}{\sin \theta \sinh^3 \chi} - \frac{2 \cos \theta \cosh \chi E_{23} \tilde{g}^{i3} \tilde{g}^{j1}}{\sin \theta \sinh^3 \chi} \\ &\quad - \frac{2 \cos \theta \cosh \chi E_{22} \tilde{g}^{i1} \tilde{g}^{j2}}{\sin \theta \sinh^3 \chi} - \frac{2 \cos \theta \cosh \chi E_{32} \tilde{g}^{i1} \tilde{g}^{j3}}{\sin \theta \sinh^3 \chi} + \frac{E_{21} \tilde{g}^{i2} \tilde{g}^{j1}}{\sinh^2 \chi} + \frac{E_{31} \tilde{g}^{i3} \tilde{g}^{j1}}{\sinh^2 \chi} + \frac{E_{12} \tilde{g}^{i1} \tilde{g}^{j2}}{\sinh^2 \chi} \\ &\quad + \frac{2E_{22} \tilde{g}^{i2} \tilde{g}^{j2}}{\sinh^2 \chi} + \frac{2E_{32} \tilde{g}^{i3} \tilde{g}^{j2}}{\sinh^2 \chi} + \frac{E_{13} \tilde{g}^{i1} \tilde{g}^{j3}}{\sinh^2 \chi} + \frac{2E_{23} \tilde{g}^{i2} \tilde{g}^{j3}}{\sinh^2 \chi} + \frac{2E_{33} \tilde{g}^{i3} \tilde{g}^{j3}}{\sinh^2 \chi} + \frac{E_{31} \tilde{g}^{i3} \tilde{g}^{j1}}{\sin^2 \theta \sinh^2 \chi} \\ &\quad + \frac{E_{32} \tilde{g}^{i3} \tilde{g}^{j2}}{\sin^2 \theta \sinh^2 \chi} + \frac{E_{13} \tilde{g}^{i1} \tilde{g}^{j3}}{\sin^2 \theta \sinh^2 \chi} + \frac{E_{23} \tilde{g}^{i2} \tilde{g}^{j3}}{\sin^2 \theta \sinh^2 \chi} + \frac{2E_{33} \tilde{g}^{i3} \tilde{g}^{j3}}{\sin^2 \theta \sinh^2 \chi} + \frac{2E_{22} \tilde{g}^{i3} \tilde{g}^{j3}(1 - \sin^2 \theta)}{\sinh^2 \chi} \\ &\quad + \frac{2E_{33} \tilde{g}^{i2} \tilde{g}^{j2}(1 - \sin^2 \theta)}{\sin^4 \theta \sinh^2 \chi} - \frac{E_{21} \tilde{g}^{i2} \tilde{g}^{j1}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{E_{31} \tilde{g}^{i3} \tilde{g}^{j1}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} \\ &\quad - \frac{E_{12} \tilde{g}^{i1} \tilde{g}^{j2}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{2E_{22} \tilde{g}^{i2} \tilde{g}^{j2}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{2E_{23} \tilde{g}^{i3} \tilde{g}^{j2}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} \\ &\quad - \frac{2E_{32} \tilde{g}^{i3} \tilde{g}^{j2}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{E_{13} \tilde{g}^{i1} \tilde{g}^{j3}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{2E_{23} \tilde{g}^{i2} \tilde{g}^{j3}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} \\ &\quad - \frac{2E_{32} \tilde{g}^{i2} \tilde{g}^{j3}(1 - \sin^2 \theta)}{\sin^2 \theta \sinh^2 \chi} - \frac{2 \cos \theta \cosh \chi E_{13} \tilde{g}^{i3} \tilde{g}^{j2}}{\sin \theta \sinh \chi} - \frac{2 \cos \theta \cosh \chi E_{31} \tilde{g}^{i3} \tilde{g}^{j2}}{\sin \theta \sinh \chi} \\ &\quad - \frac{2 \cos \theta \cosh \chi E_{13} \tilde{g}^{i2} \tilde{g}^{j3}}{\sin \theta \sinh \chi} - \frac{2 \cos \theta \cosh \chi E_{31} \tilde{g}^{i2} \tilde{g}^{j3}}{\sin \theta \sinh \chi} + \frac{2 \cos \theta \cosh \chi E_{12} \tilde{g}^{i3} \tilde{g}^{j3} \sin \theta}{\sinh \chi} \\ &\quad + \frac{2 \cos \theta \cosh \chi E_{21} \tilde{g}^{i3} \tilde{g}^{j3} \sin \theta}{\sinh \chi} + 2E_{11} \tilde{g}^{i2} \tilde{g}^{j2}(1 + \sinh^2 \chi) + 2E_{11} \tilde{g}^{i3} \tilde{g}^{j3} \sin^2 \theta(1 + \sinh^2 \chi) \\ &\quad + \frac{2E_{22} \tilde{g}^{i1} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^4 \chi} + \frac{2E_{33} \tilde{g}^{i1} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sin^2 \theta \sinh^4 \chi} - \frac{4E_{11} \tilde{g}^{i1} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} \\ &\quad - \frac{2E_{12} \tilde{g}^{i2} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{4E_{21} \tilde{g}^{i2} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{2E_{13} \tilde{g}^{i3} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} \\ &\quad - \frac{4E_{31} \tilde{g}^{i3} \tilde{g}^{j1}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{4E_{12} \tilde{g}^{i1} \tilde{g}^{j2}(1 + \sinh^2 \chi)}{\sinh^2 \chi} - \frac{2E_{21} \tilde{g}^{i1} \tilde{g}^{j2}(1 + \sinh^2 \chi)}{\sinh^2 \chi}\end{aligned}$$

[illegible]