# RW SVT4 $k \neq 0$ v6

## 1 Background

$$ds^2 = \Omega^2(\tau)\left(-d\tau^2 + \tilde{g}_{ij}dx^i dx^j\right), \qquad R_{ij} = -2k\tilde{g}_{ij} \tag{1.1}$$

$$R_{\lambda\mu\nu\kappa} = -\tfrac{1}{6}g_{\lambda\nu}g_{\mu\kappa}R + \tfrac{1}{6}g_{\lambda\kappa}g_{\mu\nu}R - \tfrac{1}{2}g_{\mu\nu}R_{\lambda\kappa} + \tfrac{1}{2}g_{\mu\kappa}R_{\lambda\nu} + \tfrac{1}{2}g_{\lambda\nu}R_{\mu\kappa} - \tfrac{1}{2}g_{\lambda\kappa}R_{\mu\nu} \tag{1.2}$$

$$R_{\mu\nu} = (A+B)U_\mu U_\nu + g_{\mu\nu}B, \qquad R = 3B - A \tag{1.3}$$

$$G_{\mu\nu} = \tfrac{1}{2}Ag_{\mu\nu} - \tfrac{1}{2}Bg_{\mu\nu} + AU_\mu U_\nu + BU_\mu U_\nu \tag{1.4}$$

$$g^{\mu\nu}G_{\mu\nu} = A - 3B \tag{1.5}$$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \tag{1.6}$$

$$g^{\mu\nu}T_{\mu\nu} = 3p - \rho \tag{1.7}$$

$$\Delta^{(0)}_{\mu\nu} = \tfrac{1}{2}Ag_{\mu\nu} - \tfrac{1}{2}Bg_{\mu\nu} + g_{\mu\nu}p + AU_\mu U_\nu + BU_\mu U_\nu + pU_\mu U_\nu + U_\mu U_\nu\rho \tag{1.8}$$

$$g^{\mu\nu}\Delta^{(0)}_{\mu\nu} = A - 3B + 3p - \rho \tag{1.9}$$

$$\begin{aligned} A &= -\tfrac{1}{2}(3p+\rho) \\ &= -3\dot{\Omega}^2\Omega^{-4} + 3\ddot{\Omega}\Omega^{-3} \end{aligned} \tag{1.10}$$

$$\begin{aligned} B &= \tfrac{1}{2}(p-\rho) \\ &= -\dot{\Omega}^2\Omega^{-4} - \ddot{\Omega}\Omega^{-3} - 2k\Omega^{-2} \end{aligned} \tag{1.11}$$

$$\begin{aligned} \rho &= \tfrac{1}{2}(-A-3B) \\ &= 3\dot{\Omega}^2\Omega^{-4} + 3k\Omega^{-2} \end{aligned} \tag{1.12}$$

$$\begin{aligned} p &= \tfrac{1}{2}(-A+B) \\ &= \dot{\Omega}^2\Omega^{-4} - 2\ddot{\Omega}\Omega^{-3} - k\Omega^{-2} \end{aligned} \tag{1.13}$$

### 1.1 Identities

$A$ and $B$ are functions only of coordinate $x^0$.

$$U^\alpha U^\beta \nabla_\alpha F \nabla_\beta A = -\nabla^\alpha F \nabla_\alpha A \tag{1.14}$$

$$F^\alpha U_\alpha U^\beta \nabla_\beta A = -F^\alpha \nabla_\alpha A \tag{1.15}$$

$$U^\alpha \nabla_\alpha U^\mu = 0 \tag{1.16}$$

$$\nabla_\mu U_\nu = \dot{\Omega}\Omega^{-2}(g_{\mu\nu} + U_\mu U_\nu) \tag{1.17}$$

## 2 Fluctuations

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{2.1}$$

$$h_{\mu\nu} = -2g_{\mu\nu}\chi + 2\nabla_\mu\nabla_\nu F + \nabla_\mu F_\nu + \nabla_\nu F_\mu + 2F_{\mu\nu} \tag{2.2}$$

$$g^{\mu\nu}F_{\mu\nu} = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad \nabla^\mu F_\mu = 0 \tag{2.3}$$

$$U^\mu \delta U_\mu = \tfrac{1}{2}U^\mu U^\nu h_{\mu\nu}, \qquad U^\mu U_\mu = -1 \tag{2.4}$$

$$\delta U_\mu = (V_\mu + \nabla_\mu V) + U_\mu U^\alpha (V_\alpha + \nabla_\alpha V) - U_\mu \left(\tfrac{1}{2}U^\alpha U^\beta h_{\alpha\beta}\right) \tag{2.5}$$

$$\begin{aligned}\delta T_{\mu\nu} = {} & \delta p g_{\mu\nu} + \delta p U_\mu U_\nu + \delta\rho U_\mu U_\nu - 2g_{\mu\nu}p\chi + 2p\nabla_\nu\nabla_\mu F + \delta U_\nu p U_\mu + \delta U_\mu p U_\nu + \delta U_\nu U_\mu \rho \\ & + \delta U_\mu U_\nu \rho + p\nabla_\mu F_\nu + p\nabla_\nu F_\mu + 2F_{\mu\nu}p\end{aligned} \tag{2.6}$$

$$\begin{aligned}g^{\mu\nu}\delta T_{\mu\nu} = {} & 3\delta p - \delta\rho - 6p\chi + 2\rho\chi + 2p\nabla_\alpha\nabla^\alpha F + 2pU^\alpha U^\beta \nabla_\beta\nabla_\alpha F + 2U^\alpha U^\beta \rho \nabla_\beta\nabla_\alpha F + 2pU^\alpha U^\beta \nabla_\beta F_\alpha \\ & + 2U^\alpha U^\beta \rho \nabla_\beta F_\alpha + 2F_{\alpha\beta}pU^\alpha U^\beta + 2F_{\alpha\beta}U^\alpha U^\beta \rho\end{aligned} \tag{2.7}$$

$$\begin{aligned}\delta G_{\mu\nu} = {} & 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + \tfrac{2}{3}g_{\mu\nu}\nabla_\alpha F\nabla^\alpha A + \tfrac{1}{2}U_\mu U_\nu \nabla_\alpha F\nabla^\alpha A + \tfrac{1}{2}U_\mu U_\nu \nabla_\alpha F\nabla^\alpha B + \tfrac{1}{2}AU_\nu\nabla_\alpha U_\mu \nabla^\alpha F \\ & + \tfrac{1}{2}BU_\nu\nabla_\alpha U_\mu\nabla^\alpha F + \tfrac{1}{2}AU_\mu\nabla_\alpha U_\nu\nabla^\alpha F + \tfrac{1}{2}BU_\mu\nabla_\alpha U_\nu\nabla^\alpha F - \tfrac{1}{2}Ag_{\mu\nu}U^\alpha\nabla_\alpha F\nabla_\beta U^\beta \\ & - \tfrac{1}{2}Bg_{\mu\nu}U^\alpha\nabla_\alpha F\nabla_\beta U^\beta + \tfrac{1}{2}U^\alpha U_\nu\nabla_\alpha F\nabla_\mu A + \tfrac{1}{2}U^\alpha U_\nu\nabla_\alpha F\nabla_\mu B + \tfrac{1}{2}AU_\nu\nabla^\alpha F\nabla_\mu U_\alpha \\ & + \tfrac{1}{2}BU_\nu\nabla^\alpha F\nabla_\mu U_\alpha + \tfrac{1}{2}AU^\alpha\nabla_\alpha F\nabla_\mu U_\nu + \tfrac{1}{2}BU^\alpha\nabla_\alpha F\nabla_\mu U_\nu + AU^\alpha U_\nu\nabla_\mu\nabla_\alpha F \\ & + BU^\alpha U_\nu\nabla_\mu\nabla_\alpha F - \tfrac{1}{2}U^\alpha U_\mu\nabla_\alpha A\nabla_\nu F - \tfrac{1}{2}U^\alpha U_\mu\nabla_\alpha B\nabla_\nu F - \tfrac{1}{2}AU_\mu\nabla_\alpha U^\alpha\nabla_\nu F \\ & - \tfrac{1}{2}BU_\mu\nabla_\alpha U^\alpha\nabla_\nu F - \tfrac{1}{6}\nabla_\mu A\nabla_\nu F + \tfrac{1}{2}\nabla_\mu B\nabla_\nu F + AU^\alpha U_\mu\nabla_\nu\nabla_\alpha F + BU^\alpha U_\mu\nabla_\nu\nabla_\alpha F \\ & + A\nabla_\nu\nabla_\mu F - B\nabla_\nu\nabla_\mu F - 2\nabla_\nu\nabla_\mu\chi + \tfrac{2}{3}F^\alpha g_{\mu\nu}\nabla_\alpha A - \tfrac{1}{4}F_\nu U^\alpha U_\mu\nabla_\alpha A - \tfrac{1}{4}F_\mu U^\alpha U_\nu\nabla_\alpha A \\ & + \tfrac{1}{2}F^\alpha U_\mu U_\nu\nabla_\alpha A - \tfrac{1}{4}F_\nu U^\alpha U_\mu\nabla_\alpha B - \tfrac{1}{4}F_\mu U^\alpha U_\nu\nabla_\alpha B + \tfrac{1}{2}F^\alpha U_\mu U_\nu\nabla_\alpha B - \tfrac{1}{4}AF_\nu U_\mu\nabla_\alpha U^\alpha \\ & - \tfrac{1}{4}BF_\nu U_\mu\nabla_\alpha U^\alpha - \tfrac{1}{4}AF_\mu U_\nu\nabla_\alpha U^\alpha - \tfrac{1}{4}BF_\mu U_\nu\nabla_\alpha U^\alpha + \tfrac{1}{2}AF^\alpha U_\nu\nabla_\alpha U_\mu + \tfrac{1}{2}BF^\alpha U_\nu\nabla_\alpha U_\mu \\ & + \tfrac{1}{2}AF^\alpha U_\mu\nabla_\alpha U_\nu + \tfrac{1}{2}BF^\alpha U_\mu\nabla_\alpha U_\nu - \tfrac{1}{2}AF^\alpha g_{\mu\nu}U_\alpha\nabla_\beta U^\beta - \tfrac{1}{2}BF^\alpha g_{\mu\nu}U_\alpha\nabla_\beta U^\beta - \tfrac{1}{12}F_\nu\nabla_\mu A \\ & + \tfrac{1}{4}F^\alpha U_\alpha U_\nu\nabla_\mu A + \tfrac{1}{4}F_\nu\nabla_\mu B + \tfrac{1}{4}F^\alpha U_\alpha U_\nu\nabla_\mu B + AU^\alpha U_\nu\nabla_\mu F_\alpha + BU^\alpha U_\nu\nabla_\mu F_\alpha \\ & + \tfrac{1}{2}A\nabla_\mu F_\nu - \tfrac{1}{2}B\nabla_\mu F_\nu + \tfrac{1}{4}AF^\alpha U_\nu\nabla_\mu U_\alpha + \tfrac{1}{4}BF^\alpha U_\nu\nabla_\mu U_\alpha + \tfrac{1}{4}AF^\alpha U_\alpha\nabla_\mu U_\nu \\ & + \tfrac{1}{4}BF^\alpha U_\alpha\nabla_\mu U_\nu - \tfrac{1}{12}F_\mu\nabla_\nu A + \tfrac{1}{4}F^\alpha U_\alpha U_\mu\nabla_\nu A + \tfrac{1}{4}F_\mu\nabla_\nu B + \tfrac{1}{4}F^\alpha U_\alpha U_\mu\nabla_\nu B \\ & + AU^\alpha U_\mu\nabla_\nu F_\alpha + BU^\alpha U_\mu\nabla_\nu F_\alpha + \tfrac{1}{2}A\nabla_\nu F_\mu - \tfrac{1}{2}B\nabla_\nu F_\mu + \tfrac{1}{4}AF^\alpha U_\mu\nabla_\nu U_\alpha\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&+\tfrac{1}{4}BF^\alpha U_\mu \nabla_\nu U_\alpha + \tfrac{1}{4}AF^\alpha U_\alpha \nabla_\nu U_\mu + \tfrac{1}{4}BF^\alpha U_\alpha \nabla_\nu U_\mu + \tfrac{4}{3}AF_{\mu\nu} + 2AF_{\nu\alpha}U^\alpha U_\mu \\
&+2BF_{\nu\alpha}U^\alpha U_\mu + 2AF_{\mu\alpha}U^\alpha U_\nu + 2BF_{\mu\alpha}U^\alpha U_\nu + \nabla_\alpha \nabla^\alpha F_{\mu\nu} \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
g^{\mu\nu}\delta G_{\mu\nu} \;=\;& A\nabla_\alpha\nabla^\alpha F - B\nabla_\alpha\nabla^\alpha F + 6\nabla_\alpha\nabla^\alpha\chi + 2\nabla_\alpha F\nabla^\alpha A - 2AU^\alpha\nabla_\alpha F\nabla_\beta U^\beta - 2BU^\alpha\nabla_\alpha F\nabla_\beta U^\beta \\
&+2AU^\alpha U^\beta\nabla_\beta\nabla_\alpha F + 2BU^\alpha U^\beta\nabla_\beta\nabla_\alpha F + AU^\alpha\nabla_\beta U_\alpha\nabla^\beta F + BU^\alpha\nabla_\beta U_\alpha\nabla^\beta F + 2F^\alpha\nabla_\alpha A \\
&+AF^\alpha U^\beta\nabla_\alpha U_\beta + BF^\alpha U^\beta\nabla_\alpha U_\beta + 2AU^\alpha U^\beta\nabla_\beta F_\alpha + 2BU^\alpha U^\beta\nabla_\beta F_\alpha - 2AF^\alpha U_\alpha\nabla_\beta U^\beta \\
&-2BF^\alpha U_\alpha\nabla_\beta U^\beta + 4AF_{\alpha\beta}U^\alpha U^\beta + 4BF_{\alpha\beta}U^\alpha U^\beta \qquad (2.9)
\end{aligned}
$$

# 3 Field Equations

$$
\begin{aligned}
\Delta_{\mu\nu} \;=\;& \delta p g_{\mu\nu} + \delta p U_\mu U_\nu + \delta\rho U_\mu U_\nu + A g_{\mu\nu}\chi - B g_{\mu\nu}\chi + 2AU_\mu U_\nu\chi + 2BU_\mu U_\nu\chi - 2AU^\alpha U_\mu U_\nu\nabla_\alpha V \\
&-2BU^\alpha U_\mu U_\nu\nabla_\alpha V + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + \tfrac{2}{3}g_{\mu\nu}\nabla_\alpha F\nabla^\alpha A + \tfrac{1}{2}U_\mu U_\nu\nabla_\alpha F\nabla^\alpha A \\
&+\tfrac{1}{2}U_\mu U_\nu\nabla_\alpha F\nabla^\alpha B + \tfrac{1}{2}AU_\nu\nabla_\alpha U_\mu\nabla^\alpha F + \tfrac{1}{2}BU_\nu\nabla_\alpha U_\mu\nabla^\alpha F + \tfrac{1}{2}AU_\mu\nabla_\alpha U_\nu\nabla^\alpha F \\
&+\tfrac{1}{2}BU_\mu\nabla_\alpha U_\nu\nabla^\alpha F - \tfrac{1}{2}Ag_{\mu\nu}U^\alpha\nabla_\alpha F\nabla_\beta U^\beta - \tfrac{1}{2}Bg_{\mu\nu}U^\alpha\nabla_\alpha F\nabla_\beta U^\beta \\
&+2AU^\alpha U^\beta U_\mu U_\nu\nabla_\beta\nabla_\alpha F + 2BU^\alpha U^\beta U_\mu U_\nu\nabla_\beta\nabla_\alpha F + \tfrac{1}{2}U^\alpha U_\nu\nabla_\alpha F\nabla_\mu A + \tfrac{1}{2}U^\alpha U_\nu\nabla_\alpha F\nabla_\mu B \\
&+\tfrac{1}{2}AU_\nu\nabla^\alpha F\nabla_\mu U_\alpha + \tfrac{1}{2}BU_\nu\nabla^\alpha F\nabla_\mu U_\alpha + \tfrac{1}{2}AU^\alpha\nabla_\alpha F\nabla_\mu U_\nu + \tfrac{1}{2}BU^\alpha\nabla_\alpha F\nabla_\mu U_\nu \\
&-AU_\nu\nabla_\mu V - BU_\nu\nabla_\mu V + AU^\alpha U_\nu\nabla_\mu\nabla_\alpha F + BU^\alpha U_\nu\nabla_\mu\nabla_\alpha F - \tfrac{1}{2}U^\alpha U_\mu\nabla_\alpha A\nabla_\nu F \\
&-\tfrac{1}{2}U^\alpha U_\mu\nabla_\alpha B\nabla_\nu F - \tfrac{1}{2}AU_\mu\nabla_\alpha U^\alpha\nabla_\nu F - \tfrac{1}{2}BU_\mu\nabla_\alpha U^\alpha\nabla_\nu F - \tfrac{1}{6}\nabla_\mu A\nabla_\nu F + \tfrac{1}{2}\nabla_\mu B\nabla_\nu F \\
&-AU_\mu\nabla_\nu V - BU_\mu\nabla_\nu V + AU^\alpha U_\mu\nabla_\nu\nabla_\alpha F + BU^\alpha U_\mu\nabla_\nu\nabla_\alpha F - 2\nabla_\nu\nabla_\mu\chi - 2AU^\alpha U_\mu U_\nu V_\alpha \\
&-2BU^\alpha U_\mu U_\nu V_\alpha - AU_\nu V_\mu - BU_\nu V_\mu - AU_\mu V_\nu - BU_\mu V_\nu + \tfrac{2}{3}F^\alpha g_{\mu\nu}\nabla_\alpha A - \tfrac{1}{4}F_\nu U^\alpha U_\mu\nabla_\alpha A \\
&-\tfrac{1}{4}F_\mu U^\alpha U_\nu\nabla_\alpha A + \tfrac{1}{2}F^\alpha U_\mu U_\nu\nabla_\alpha A - \tfrac{1}{4}F_\nu U^\alpha U_\mu\nabla_\alpha B - \tfrac{1}{4}F_\mu U^\alpha U_\nu\nabla_\alpha B + \tfrac{1}{2}F^\alpha U_\mu U_\nu\nabla_\alpha B \\
&-\tfrac{1}{4}AF_\nu U_\mu\nabla_\alpha U^\alpha - \tfrac{1}{4}BF_\nu U_\mu\nabla_\alpha U^\alpha - \tfrac{1}{4}AF_\mu U_\nu\nabla_\alpha U^\alpha - \tfrac{1}{4}BF_\mu U_\nu\nabla_\alpha U^\alpha + \tfrac{1}{2}AF^\alpha U_\nu\nabla_\alpha U_\mu \\
&+\tfrac{1}{2}BF^\alpha U_\nu\nabla_\alpha U_\mu + \tfrac{1}{2}AF^\alpha U_\mu\nabla_\alpha U_\nu + \tfrac{1}{2}BF^\alpha U_\mu\nabla_\alpha U_\nu + 2AU^\alpha U^\beta U_\mu U_\nu\nabla_\beta F_\alpha \\
&+2BU^\alpha U^\beta U_\mu U_\nu\nabla_\beta F_\alpha - \tfrac{1}{2}AF^\alpha g_{\mu\nu}U_\alpha\nabla_\beta U^\beta - \tfrac{1}{2}BF^\alpha g_{\mu\nu}U_\alpha\nabla_\beta U^\beta - \tfrac{1}{12}F_\nu\nabla_\mu A \\
&+\tfrac{1}{4}F^\alpha U_\alpha U_\nu\nabla_\mu A + \tfrac{1}{4}F_\nu\nabla_\mu B + \tfrac{1}{4}F^\alpha U_\alpha U_\nu\nabla_\mu B + AU^\alpha U_\nu\nabla_\mu F_\alpha + BU^\alpha U_\nu\nabla_\mu F_\alpha \\
&+\tfrac{1}{4}AF^\alpha U_\nu\nabla_\mu U_\alpha + \tfrac{1}{4}BF^\alpha U_\nu\nabla_\mu U_\alpha + \tfrac{1}{4}AF^\alpha U_\alpha\nabla_\mu U_\nu + \tfrac{1}{4}BF^\alpha U_\alpha\nabla_\mu U_\nu - \tfrac{1}{12}F_\mu\nabla_\nu A \\
&+\tfrac{1}{4}F^\alpha U_\alpha U_\mu\nabla_\nu A + \tfrac{1}{4}F_\mu\nabla_\nu B + \tfrac{1}{4}F^\alpha U_\alpha U_\mu\nabla_\nu B + AU^\alpha U_\mu\nabla_\nu F_\alpha + BU^\alpha U_\mu\nabla_\nu F_\alpha \\
&+\tfrac{1}{4}AF^\alpha U_\mu\nabla_\nu U_\alpha + \tfrac{1}{4}BF^\alpha U_\mu\nabla_\nu U_\alpha + \tfrac{1}{4}AF^\alpha U_\alpha\nabla_\nu U_\mu + \tfrac{1}{4}BF^\alpha U_\alpha\nabla_\nu U_\mu + \tfrac{1}{3}AF_{\mu\nu} + BF_{\mu\nu} \\
&+2AF_{\nu\alpha}U^\alpha U_\mu + 2BF_{\nu\alpha}U^\alpha U_\mu + 2AF_{\mu\alpha}U^\alpha U_\nu + 2BF_{\mu\alpha}U^\alpha U_\nu + 2AF_{\alpha\beta}U^\alpha U^\beta U_\mu U_\nu \\
&+2BF_{\alpha\beta}U^\alpha U^\beta U_\mu U_\nu + \nabla_\alpha\nabla^\alpha F_{\mu\nu} \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
g^{\mu\nu}\Delta_{\mu\nu} \;=\;& 3\delta p - \delta\rho + 2A\chi - 6B\chi + 6\nabla_\alpha\nabla^\alpha\chi + 2\nabla_\alpha F\nabla^\alpha A - 2AU^\alpha\nabla_\alpha F\nabla_\beta U^\beta - 2BU^\alpha\nabla_\alpha F\nabla_\beta U^\beta \\
&+AU^\alpha\nabla_\beta U_\alpha\nabla^\beta F + BU^\alpha\nabla_\beta U_\alpha\nabla^\beta F + 2F^\alpha\nabla_\alpha A + AF^\alpha U^\beta\nabla_\alpha U_\beta + BF^\alpha U^\beta\nabla_\alpha U_\beta \\
&-2AF^\alpha U_\alpha\nabla_\beta U^\beta - 2BF^\alpha U_\alpha\nabla_\beta U^\beta + 2AF_{\alpha\beta}U^\alpha U^\beta + 2BF_{\alpha\beta}U^\alpha U^\beta \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
U^\mu U^\nu\Delta_{\mu\nu} \;=\;& \delta\rho + A\chi + 3B\chi - 2\nabla_\alpha\nabla^\alpha\chi + AU^\alpha\nabla_\alpha F\nabla_\beta U^\beta + BU^\alpha\nabla_\alpha F\nabla_\beta U^\beta - 2U^\alpha U^\beta\nabla_\beta\nabla_\alpha\chi \\
&-AU^\alpha\nabla_\beta U_\alpha\nabla^\beta F - BU^\alpha\nabla_\beta U_\alpha\nabla^\beta F - AF^\alpha U^\beta\nabla_\alpha U_\beta - BF^\alpha U^\beta\nabla_\alpha U_\beta + AF^\alpha U_\alpha\nabla_\beta U^\beta \\
&+BF^\alpha U_\alpha\nabla_\beta U^\beta - \tfrac{5}{3}AF_{\alpha\beta}U^\alpha U^\beta - BF_{\alpha\beta}U^\alpha U^\beta + U^\alpha U^\beta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
(U^\mu U^\nu + g^{\mu\nu})\Delta_{\mu\nu} \;=\;& 3\delta p + 3A\chi - 3B\chi + 4\nabla_\alpha\nabla^\alpha\chi + 2\nabla_\alpha F\nabla^\alpha A - AU^\alpha\nabla_\alpha F\nabla_\beta U^\beta - BU^\alpha\nabla_\alpha F\nabla_\beta U^\beta \\
&-2U^\alpha U^\beta\nabla_\beta\nabla_\alpha\chi + 2F^\alpha\nabla_\alpha A - AF^\alpha U_\alpha\nabla_\beta U^\beta - BF^\alpha U_\alpha\nabla_\beta U^\beta + \tfrac{1}{3}AF_{\alpha\beta}U^\alpha U^\beta \\
&+BF_{\alpha\beta}U^\alpha U^\beta + U^\alpha U^\beta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \qquad (3.4)
\end{aligned}
$$

# 4 Field Equations (G.I. Form)

$$(Q_\mu \equiv F_\mu + \nabla_\mu F) \tag{4.1}$$

$$\delta\rho^{GI} = \delta\rho + (A+B)(Q^\alpha U_\alpha \nabla_\beta U^\beta - Q^\alpha U^\beta \nabla_\alpha U_\beta) \tag{4.2}$$

$$\delta p^{GI} = \delta p + \tfrac{2}{3} Q^\alpha \nabla_\alpha A - \tfrac{1}{3}(A+B) Q^\alpha U_\alpha \nabla_\beta U^\beta \tag{4.3}$$

$$V^{GI} = V - U^\alpha Q_\alpha, \qquad \chi, \quad F_{\mu\nu} \quad V_\mu \tag{4.4}$$

## 4.1 $\Delta(A,B)$

$$\begin{aligned}\Delta_{\mu\nu} = {} & (g_{\mu\nu}+U_\mu U_\nu)\delta p^{GI} + U_\mu U_\nu \delta\rho^{GI} + \big((A-B)g_{\mu\nu} + 2(A+B)U_\mu U_\nu\big)\chi \\ & -2(A+B)U_\mu U_\nu U^\alpha \nabla_\alpha V^{GI} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi - (A+B)U_\nu\nabla_\mu V^{GI} - (A+B)U_\mu\nabla_\nu V^{GI} \\ & -2\nabla_\nu\nabla_\mu\chi - 2(A+B)U_\mu U_\nu U^\alpha V_\alpha - (A+B)U_\nu V_\mu \\ & -(A+B)U_\mu V_\nu + 2(A+B)U_\mu U_\nu U^\alpha U^\beta F_{\alpha\beta} + 2(A+B)U_\nu U^\alpha F_{\mu\alpha} + (\tfrac{1}{3}A+B)F_{\mu\nu} \\ & +2(A+B)U_\mu U^\alpha F_{\nu\alpha} + \nabla_\alpha\nabla^\alpha F_{\mu\nu}\end{aligned} \tag{4.5}$$

$$g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p^{GI} - \delta\rho^{GI} + 2(A-3B)\chi + 6\nabla_\alpha\nabla^\alpha\chi + 2(A+B)U^\alpha U^\beta F_{\alpha\beta} \tag{4.6}$$

## 4.2 $\Delta(\rho,p)$

$$\begin{aligned}\Delta_{\mu\nu} = {} & (g_{\mu\nu}+U_\mu U_\nu)\delta p^{GI} + U_\mu U_\nu \delta\rho^{GI} + \big(-2pg_{\mu\nu} - 2(p+\rho)U_\mu U_\nu\big)\chi + 2(p+\rho)U_\mu U_\nu U^\alpha\nabla_\alpha V^{GI} \\ & +2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi + (p+\rho)U_\nu\nabla_\mu V^{GI} + (p+\rho)U_\mu\nabla_\nu V^{GI} - 2\nabla_\nu\nabla_\mu\chi + 2(p+\rho)U_\mu U_\nu U^\alpha V_\alpha \\ & +(p+\rho)U_\nu V_\mu + (p+\rho)U_\mu V_\nu - 2(p+\rho)U_\mu U_\nu U^\alpha U^\beta F_{\alpha\beta} - 2(p+\rho)U_\nu U^\alpha F_{\mu\alpha} - \tfrac{2}{3}\rho F_{\mu\nu} \\ & -2(p+\rho)U_\mu U^\alpha F_{\nu\alpha} + \nabla_\alpha\nabla^\alpha F_{\mu\nu}\end{aligned} \tag{4.7}$$

$$g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p^{GI} - \delta\rho^{GI} + (-6p+2\rho)\chi + 6\nabla_\alpha\nabla^\alpha\chi - 2(p+\rho)U^\alpha U^\beta F_{\alpha\beta} \tag{4.8}$$

## 4.3 $\Delta(\Omega,k)$

$$\begin{aligned}\Delta_{\mu\nu} = {} & (g_{\mu\nu}+U_\mu U_\nu)\delta p^{GI} + U_\mu U_\nu\delta\rho^{GI} \\ & +\Big(\Omega^{-4}\big(-2\dot\Omega^2 + 2\Omega(2\ddot\Omega + k\Omega)\big)g_{\mu\nu} - 4\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu U_\nu\Big)\chi \\ & +4\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu U_\nu U^\alpha\nabla_\alpha V^{GI} + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi \\ & +2\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\nu\nabla_\mu V^{GI} + 2\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu\nabla_\nu V^{GI} \\ & -2\nabla_\nu\nabla_\mu\chi + 4\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu U_\nu U^\alpha V_\alpha + 2\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\nu V_\mu \\ & +2\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu V_\nu - 4\Omega^{-4}\big(2\dot\Omega^2 + \Omega(-\ddot\Omega + k\Omega)\big)U_\mu U_\nu U^\alpha U^\beta F_{\alpha\beta}\end{aligned}$$

$$
\begin{aligned}
&-4\Omega^{-4}\big(2\dot\Omega^2+\Omega(-\ddot\Omega+k\Omega)\big)U_\nu U^\alpha F_{\mu\alpha}-2\Omega^{-4}(\dot\Omega^2+k\Omega^2)F_{\mu\nu}\\
&-4\Omega^{-4}\big(2\dot\Omega^2+\Omega(-\ddot\Omega+k\Omega)\big)U_\mu U^\alpha F_{\nu\alpha}+\nabla_\alpha\nabla^\alpha F_{\mu\nu}
\end{aligned}
\tag{4.9}
$$

$$
g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p^{GI}-\delta\rho^{GI}+12\Omega^{-3}(\ddot\Omega+k\Omega)\chi+6\nabla_\alpha\nabla^\alpha\chi-4\Omega^{-4}\big(2\dot\Omega^2+\Omega(-\ddot\Omega+k\Omega)\big)U^\alpha U^\beta F_{\alpha\beta} \tag{4.10}
$$

# 5 Separation

$$
\begin{aligned}
\Delta_{\mu\nu} = {}& (g_{\mu\nu}+U_\mu U_\nu)\delta p^{GI}+U_\mu U_\nu\delta\rho^{GI}+\big(-2pg_{\mu\nu}-2(p+\rho)U_\mu U_\nu\big)\chi+2(p+\rho)U_\mu U_\nu U^\alpha\nabla_\alpha V^{GI}\\
&+2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi+(p+\rho)U_\nu\nabla_\mu V^{GI}+(p+\rho)U_\mu\nabla_\nu V^{GI}-2\nabla_\nu\nabla_\mu\chi+2(p+\rho)U_\mu U_\nu U^\alpha V_\alpha\\
&+(p+\rho)U_\nu V_\mu+(p+\rho)U_\mu V_\nu-2(p+\rho)U_\mu U_\nu U^\alpha U^\beta F_{\alpha\beta}-2(p+\rho)U_\nu U^\alpha F_{\mu\alpha}-\tfrac{2}{3}\rho F_{\mu\nu}\\
&-2(p+\rho)U_\mu U^\alpha F_{\nu\alpha}+\nabla_\alpha\nabla^\alpha F_{\mu\nu}
\end{aligned}
\tag{5.1}
$$

$$
g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p^{GI}-\delta\rho^{GI}+(-6p+2\rho)\chi+6\nabla_\alpha\nabla^\alpha\chi-2(p+\rho)U^\alpha U^\beta F_{\alpha\beta} \tag{5.2}
$$

$$
U^\mu U^\nu\Delta_{\mu\nu} = \delta\rho^{GI}-2\rho\chi-2\nabla_\alpha\nabla^\alpha\chi-2U^\alpha U^\beta\nabla_\beta\nabla_\alpha\chi+\tfrac{2}{3}(3p+2\rho)U^\alpha U^\beta F_{\alpha\beta}+U^\alpha U^\beta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \tag{5.3}
$$

$$
(g^{\mu\nu}+U^\mu U^\nu)\Delta_{\mu\nu} = 3\delta p^{GI}-6p\chi+4\nabla_\alpha\nabla^\alpha\chi-2U^\alpha U^\beta\nabla_\beta\nabla_\alpha\chi-\tfrac{2}{3}\rho U^\alpha U^\beta F_{\alpha\beta}+U^\alpha U^\beta\nabla_\gamma\nabla^\gamma F_{\alpha\beta} \tag{5.4}
$$

# 6 Validation

## 6.1 deSitter

$$
ds^2 = \frac{1}{H^2\tau^2}\left(-d\tau^2+\delta^{ij}dx^idx^j\right) \tag{6.1}
$$

$$
\Omega = \frac{1}{H\tau},\qquad \dot\Omega=-\frac{1}{H\tau^2},\qquad \ddot\Omega=\frac{2}{H\tau^3} \tag{6.2}
$$

$$
\delta\rho^{GI} = \delta\rho,\qquad \delta p^{GI}=\delta p \tag{6.3}
$$

$$
\begin{aligned}
\Delta_{\mu\nu} = {}& \delta\rho^{GI}U_\mu U_\nu+\delta p^{GI}(g_{\mu\nu}+U_\mu U_\nu)+6H^2g_{\mu\nu}\chi+2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi-2\nabla_\nu\nabla_\mu\chi-2H^2F_{\mu\nu}\\
&+\nabla_\alpha\nabla^\alpha F_{\mu\nu}
\end{aligned}
\tag{6.4}
$$

$$
g^{\mu\nu}\Delta_{\mu\nu} = 3\delta p^{GI}-\delta\rho^{GI}+24H^2\chi+6\nabla_\alpha\nabla^\alpha\chi \tag{6.5}
$$

In $dS_4$, we have $\rho=-p=const.$, such that $T_{\mu\nu}=pg_{\mu\nu}$ and $\delta T_{\mu\nu}=ph_{\mu\nu}$. For $\Omega(\tau)=(H\tau)^{-1}$, it follows that $p=\dot\Omega^2\Omega^{-4}-2\ddot\Omega\Omega^{-3}=-3H^2$. Setting $\delta\rho=\delta p=0$, (6.4) and (6.5) become

$$
\Delta_{\mu\nu} = 6H^2g_{\mu\nu}\chi+2g_{\mu\nu}\nabla_\alpha\nabla^\alpha\chi-2\nabla_\nu\nabla_\mu\chi-2H^2F_{\mu\nu}+\nabla_\alpha\nabla^\alpha F_{\mu\nu} \tag{6.6}
$$

$$
g^{\mu\nu}\Delta_{\mu\nu} = 24H^2\chi+6\nabla_\alpha\nabla^\alpha\chi \tag{6.7}
$$

## 6.2 Conservation

$$\begin{aligned}\delta(\nabla_\mu G^{\mu\nu}) &= \tfrac{1}{2}G^\nu{}_\alpha\nabla^\alpha h - h^{\nu\alpha}\nabla_\beta G_\alpha{}^\beta - h^{\alpha\beta}\nabla_\beta G^\nu{}_\alpha - G^{\nu\alpha}\nabla_\beta h_\alpha{}^\beta + g^{\beta\gamma}g^{\nu\alpha}\nabla_\gamma\delta G_{\alpha\beta} - \tfrac{1}{2}G^{\alpha\beta}\nabla^\nu h_{\alpha\beta} \\ &= 0 \end{aligned} \tag{6.8}$$

$$\delta(\nabla_\mu \Delta^{\mu\nu}) = \delta(\nabla_\mu T^{\mu\nu}) \tag{6.9}$$