Flat Cosmological Fluctuation Equations

We work to first order with Minkowski background:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

In 3+1 splitting, we define vector tangent to 3-surfaces of consant (zero) curvature:

$$n_\mu = (-1, 0, 0, 0), \quad n_\mu n^\mu = -1.$$

Tensors are raised/lowered using the induced metric

$$\gamma_{\mu\nu} = \delta_{ij}, \qquad \gamma_{0\mu} = 0.$$

The fluctuation equations in Minkowski standard gravity are:

$$\delta G_{\mu\nu} = -\frac{1}{2}\partial_\alpha\partial^\alpha \delta g_{\mu\nu} + \frac{1}{2}\partial_\alpha\partial_\mu \delta g_\nu{}^\alpha + \frac{1}{2}\partial_\alpha\partial_\nu \delta g_\mu{}^\alpha + \frac{1}{2}g_{\mu\nu}\partial^\alpha\partial_\alpha \delta g^\gamma{}_\gamma - \frac{1}{2}g_{\mu\nu}\partial_\gamma\partial^\alpha \delta g_\alpha{}^\gamma - \frac{1}{2}\partial_\mu\partial_\nu \delta g^\alpha{}_\alpha$$

The fluctuation equations in Minkowski conformal gravity are:

$$\begin{aligned}\delta W_{\mu\nu} = \frac{1}{2}\partial_\beta\partial^\beta\partial_\alpha\partial^\alpha \delta g_{\mu\nu} + \frac{1}{6}g_{\mu\nu}\partial_\gamma\partial^\gamma\partial_\alpha\partial_\beta \delta g^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}\partial_\gamma\partial^\gamma\partial_\beta\partial^\beta \delta g^\alpha{}_\alpha - \frac{1}{2}\partial_\beta\partial^\beta\partial_\mu\partial_\alpha \delta g_\nu{}^\alpha \\ - \frac{1}{2}\partial_\beta\partial^\beta\partial_\nu\partial_\alpha \delta g_\mu{}^\alpha + \frac{1}{3}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta \delta g^{\alpha\beta} + \frac{1}{6}\partial_\mu\partial_\nu\partial_\beta\partial^\beta \delta g^\alpha{}_\alpha.\end{aligned}$$

Standard Gravity

Newton Gauge

Gauge:

$$h_{\mu\nu} = -2\phi n_\mu n_\nu - 2\psi\gamma_{\mu\nu}$$
$$ds^2 = -(1+2\phi)dt^2 + \delta_{ij}(1-2\psi)dx^i dx^j$$

Fluctuation Equations:

$$\begin{aligned}\delta G_{00} &= 2\nabla^2\psi \\ \delta G_{0i} &= 2\nabla_i\dot{\psi} \\ \delta G_{ij} &= \nabla_i\nabla_j(\psi - \phi) + \delta_{ij}(2\ddot{\psi} + \nabla^2\phi - \nabla^2\psi)\end{aligned}$$

Conformal Gravity

Newton Gauge

Gauge:

$$h_{\mu\nu} = -2\phi n_\mu n_\nu - 2\psi\gamma_{\mu\nu}$$
$$ds^2 = -(1+2\phi)dt^2 + \delta_{ij}(1-2\psi)dx^i dx^j$$

Fluctuation Equations:

$$\delta W_{00} = \frac{2}{3}\nabla^4(\phi+\psi)$$
$$\delta W_{0i} = \frac{2}{3}\nabla_i\nabla^2(\dot{\phi}+\dot{\psi})$$
$$\delta W_{ij} = \nabla_i\nabla_j(\ddot{\phi}+\ddot{\psi}-\frac{1}{3}\nabla^2\phi-\frac{1}{3}\nabla^2\psi)-\frac{1}{3}\delta_{ij}\nabla^2(\ddot{\phi}+\ddot{\psi}-\nabla^2\phi-\nabla^2\psi)$$

Standard Gravity

No Gauge

We now look at full perturbations without a choice of gauge, separating them into their spin parts.

Gauge:

$$h_{\mu\nu} = -2\phi n_\mu n_\nu - (B_\nu + \bar{\nabla}_\nu B)n_\mu - (B_\mu + \bar{\nabla}_\mu B)n_\nu - 2\gamma_{\mu\nu}\psi + \bar{\nabla}_\mu E_\nu + \bar{\nabla}_\nu E_\mu + 2\bar{\nabla}_\mu\bar{\nabla}_\nu E + 2E_{\mu\nu}$$
$$ds^2 = -(1+2\phi)dt^2 + 2(B_i+\nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i\nabla_j E + 2E_{ij}\right]dx^i dx^j$$

where

$$\bar{\nabla}_\mu = (0,\nabla_i), \qquad n^\mu B_\mu = 0, \qquad n^\mu n^\nu E_{\mu\nu} = 0, \qquad \bar{\nabla}^\mu B_\mu = 0, \qquad \bar{\nabla}^\mu E_\mu = 0, \qquad \bar{\nabla}^\mu E_{\mu\nu} = 0.$$

Fluctuation Equations:

$$\delta G_{00} = 2\nabla^2\psi$$
$$\delta G_{0i} = 2\nabla_i\dot{\psi} - \frac{1}{2}\nabla^2 B_i + \frac{1}{2}\nabla^2\dot{E}_i$$
$$\delta G_{ij} = 2\delta_{ij}\ddot{\psi} + (\delta_{ij}\nabla^2 - \nabla_i\nabla_j)(\phi-\psi+\dot{B}-\ddot{E}) - \frac{1}{2}(\nabla_i\dot{B}_j + \nabla_j\dot{B}_i) + \frac{1}{2}(\nabla_i\ddot{E} + \nabla_j\ddot{E}_i) + \ddot{E}_{ij} - \nabla^2 E_{ij}$$

We see that the Einstein tensor itself separates into linear combinations of different spin tensors:

$$\delta G^{(S)}_{00} = \delta G_{00} = 2\nabla^2\psi$$
$$\delta G^{(S)}_{0i} = 2\nabla_i\dot{\psi}$$
$$\delta G^{(V)}_{0i} = -\frac{1}{2}\nabla^2 B_i + \frac{1}{2}\nabla^2\dot{E}_i$$
$$\delta G^{(S)}_{ij} = 2\delta_{ij}\ddot{\psi} + (\delta_{ij}\nabla^2 - \nabla_i\nabla_j)(\phi-\psi+\dot{B}-\ddot{E})$$
$$\delta G^{(V)}_{ij} = -\frac{1}{2}(\nabla_i\dot{B}_j + \nabla_j\dot{B}_i) + \frac{1}{2}(\nabla_i\ddot{E} + \nabla_j\ddot{E}_i)$$
$$\delta G^{(T)}_{ij} = \ddot{E}_{ij} - \nabla^2 E_{ij}$$

Conformal Gravity

No Gauge

Gauge:

$$h_{\mu\nu} = -2\phi n_\mu n_\nu - (B_\nu + \bar{\nabla}_\nu B)n_\mu - (B_\mu + \bar{\nabla}_\mu B)n_\nu - 2\gamma_{\mu\nu}\psi + \bar{\nabla}_\mu E_\nu + \bar{\nabla}_\nu E_\mu + 2\bar{\nabla}_\mu\bar{\nabla}_\nu E + 2E_{\mu\nu}$$
$$ds^2 = -(1+2\phi)dt^2 + 2(B_i+\nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i\nabla_j E + 2E_{ij}\right]dx^i dx^j$$

where

$$\bar{\nabla}_\mu = (0,\nabla_i), \qquad n^\mu B_\mu = 0, \qquad n^\mu n^\nu E_{\mu\nu} = 0, \qquad \bar{\nabla}^\mu B_\mu = 0, \qquad \bar{\nabla}^\mu E_\mu = 0, \qquad \bar{\nabla}^\mu E_{\mu\nu} = 0.$$

Fluctuation Equations:

$$
\begin{aligned}
\delta W_{00} &= \frac{2}{3}\nabla^4(\phi + \psi + \dot{B} - \ddot{E}) \\
\delta W_{0i} &= \frac{2}{3}\nabla_i\nabla^2(\dot{\phi} + \dot{\psi} + \ddot{B} - \dddot{E}) + \frac{1}{2}\nabla^2(\ddot{B}_i - \dddot{E}_i - \nabla^2 B_i + \nabla^2 \dot{E}_i) \\
\delta W_{ij} &= \frac{1}{3}\delta_{ij}\nabla^2\left(-\ddot{\phi} - \ddot{\psi} - \dddot{B} + \ddddot{E} + \nabla^2(\phi + \psi + \dot{B} - \ddot{E})\right) + \nabla_i\nabla_j\left(\ddot{\phi} + \ddot{\psi} + \dddot{B} - \ddddot{E} - \frac{1}{3}\nabla^2(\phi + \psi + \dot{B} - \ddot{E})\right) \\
&\qquad + \frac{1}{2}\nabla_i\left(\dddot{B}_j - \dddot{E}_j - \nabla^2(\dot{B}_j - \ddot{E}_j)\right) + \frac{1}{2}\nabla_j\left(\dddot{B}_i - \ddddot{E}_i - \nabla^2(\dot{B}_i - \ddot{E}_i)\right) \\
&\qquad - \ddddot{E}_{ij} + 2\nabla^2\ddot{E}_{ij} - \nabla^4 E_{ij}
\end{aligned}
$$

Again, the Weyl tensor itself separates into linear combinations of different spin tensors:

$$
\begin{aligned}
\delta W^{(S)}_{00} &= \delta W_{00} = \frac{2}{3}\nabla^4(\phi + \psi + \dot{B} - \ddot{E}) \\
\delta W^{(S)}_{0i} &= \frac{2}{3}\nabla_i\nabla^2(\dot{\phi} + \dot{\psi} + \ddot{B} - \dddot{E}) \\
\delta W^{(V)}_{0i} &= \frac{1}{2}\nabla^2(\ddot{B}_i - \dddot{E}_i - \nabla^2 B_i + \nabla^2 \dot{E}_i) \\
\delta W^{(S)}_{ij} &= \frac{1}{3}\delta_{ij}\nabla^2\left(-\ddot{\phi} - \ddot{\psi} - \dddot{B} + \ddddot{E} + \nabla^2(\phi + \psi + \dot{B} - \ddot{E})\right) + \nabla_i\nabla_j\left(\ddot{\phi} + \ddot{\psi} + \dddot{B} - \ddddot{E} - \frac{1}{3}\nabla^2(\phi + \psi + \dot{B} - \ddot{E})\right) \\
\delta W^{(V)}_{ij} &= \frac{1}{2}\nabla_i\left(\dddot{B}_j - \dddot{E}_j - \nabla^2(\dot{B}_j - \ddot{E}_j)\right) + \frac{1}{2}\nabla_j\left(\dddot{B}_i - \ddddot{E}_i - \nabla^2(\dot{B}_i - \ddot{E}_i)\right) \\
\delta W^{(T)}_{ij} &= -\ddddot{E}_{ij} + 2\nabla^2\ddot{E}_{ij} - \nabla^4 E_{ij} = -\bar{\nabla}_\alpha\bar{\nabla}^\alpha\bar{\nabla}_\beta\bar{\nabla}^\beta E_{ij}.
\end{aligned}
$$

Supplementary

Ricci Tensor

Gauge:

$$
h_{\mu\nu} = -2\phi n_\mu n_\nu - (B_\nu + \bar{\nabla}_\nu B)n_\mu - (B_\mu + \bar{\nabla}_\mu B)n_\nu - 2\gamma_{\mu\nu}\psi + \bar{\nabla}_\mu E_\nu + \bar{\nabla}_\nu E_\mu + 2\bar{\nabla}_\mu\bar{\nabla}_\nu E + 2E_{\mu\nu}
$$

$$
ds^2 = -(1 + 2\phi)dt^2 + 2(B_i + \nabla_i B)dx^i dt + \left[-2\delta_{ij}\psi + (\nabla_i E_j + \nabla_j E_i) + 2\nabla_i\nabla_j E + 2E_{ij}\right] dx^i dx^j
$$

where

$$
\bar{\nabla}_\mu = (0, \nabla_i), \qquad n^\mu B_\mu = 0, \qquad n^\mu n^\nu E_{\mu\nu} = 0, \qquad \bar{\nabla}^\mu B_\mu = 0, \qquad \bar{\nabla}^\mu E_\mu = 0, \qquad \bar{\nabla}^\mu E_{\mu\nu} = 0.
$$

Fluctuation Equations:

$$
\begin{aligned}
\delta R_{00} &= 3\ddot{\psi} + \nabla^2(\phi + \dot{B} - \ddot{E}) \\
\delta R_{0i} &= -\frac{1}{2}\nabla^2(B_i + \dot{E}_i) + 2\nabla_i\dot{\psi} \\
\delta R_{ij} &= -\delta_{ij}(\ddot{\psi} - \nabla^2\psi) + \nabla_i\nabla_j(-\phi + \psi - \dot{B} + \ddot{E}) - \frac{1}{2}\nabla_i(\dot{B}_j - \ddot{E}_j) - \frac{1}{2}\nabla_j(\dot{B}_i - \ddot{E}_i) + \ddot{E}_{ij} - \nabla^2 E_{ij}
\end{aligned}
$$

Separation of the Riemann tensor into spin parts:

$$
\begin{aligned}
\delta R^{(S)}_{00} &= \delta R_{00} = 3\ddot{\psi} + \nabla^2(\phi + \dot{B} - \ddot{E}) \\
\delta R^{(S)}_{0i} &= 2\nabla_i\dot{\psi}
\end{aligned}
$$

$$\delta R_{0i}^{(V)} = -\frac{1}{2}\nabla^2(B_i + \dot{E}_i)$$
$$\delta R_{ij}^{(S)} = -\delta_{ij}(\ddot{\psi} - \nabla^2\psi) + \nabla_i\nabla_j(-\phi + \psi - \dot{B} + \ddot{E})$$
$$\delta R_{ij}^{(V)} = -\frac{1}{2}\nabla_i(\dot{B}_j - \ddot{E}_j) - \frac{1}{2}\nabla_j(\dot{B}_i - \ddot{E}_i)$$
$$\delta R_{ij}^{(T)} = \ddot{E}_{ij} - \nabla^2 E_{ij}.$$