h_{ij} Decomposition v1

1 $h^{T\theta}_{ij} = 2E_{ij}$

Evaluated in a maximally 3-symmetric background, $R_{ij} = -2kg_{ij}$, we may express the transverse-traceless component of h_{ij} as

$$\begin{aligned}(\nabla^2-2k)(\nabla^2-3k)h^{T\theta}_{ij} &= (\nabla^2-2k)(\nabla^2-3k)h_{ij} - \nabla^2\nabla_i\nabla^l h_{jl} - \nabla^2\nabla_j\nabla^l h_{il} + 3k\nabla_j\nabla^l h_{il} + 3k\nabla_i\nabla^l h_{jl}\\ &+\tfrac{1}{2}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl} + \tfrac{1}{2}g_{ij}\nabla^2\nabla^k\nabla^l h_{kl} - 2kg_{ij}\nabla^l\nabla^k h_{kl} + \tfrac{1}{2}\nabla_i\nabla_j(\nabla^2+4k)(g^{ab}h_{ab})\\ &-\tfrac{1}{2}g_{ij}\nabla^2(\nabla^2-3k)(g^{ab}h_{ab}) - \tfrac{1}{2}g_{ij}k(\nabla^2+4k)(g^{ab}h_{ab}).\end{aligned} \tag{1.1}$$

Substituting

$$\begin{aligned}h_{ij} &= -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij},\\ g^{ab}h_{ab} &= -6\psi + 2\nabla_a\nabla^a E,\end{aligned} \tag{1.2}$$

we find the right hand side of (1.1) evaluates to

$$(\nabla^2-2k)(\nabla^2-3k)h^{T\theta}_{ij} = (\nabla^2-2k)(\nabla^2-3k)(2E_{ij}). \tag{1.3}$$

To test gauge invariance, we take

$$h_{ij} = \nabla_i\epsilon_j + \nabla_j\epsilon_i, \qquad (g^{ab}h_{ab}) = 2\nabla_a\epsilon^a, \tag{1.4}$$

$$\epsilon_i = \nabla_i L + L_i, \qquad \nabla^i L_i = 0, \tag{1.5}$$

and substitute into the RHS of (1.1). The result vanishes, confirming gauge invariance.