Quantum Mechanics II
HW 9

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1. Review of Lorentz transformation in relativistic notation

A Lorentz boost from the unprimed inertial frame to the primed inertial frame, moving with velocity **v** relative to the original frame, can be written as

$$x_0' = \gamma\left(x_0 - \vec{\beta}\cdot\vec{x}\right), \qquad \vec{x}' = \vec{x} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\vec{x})\vec{\beta} - \gamma\vec{\beta}x_0$$

where $\beta = \frac{\vec{v}}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

(a) Express this Lorentz transformation as a matrix equation

$$x' = \Lambda x$$

where x is the 4-component column vector (and similarly for x'):

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Hence confirm simply that $x_\mu x^\mu$ is invariant under the Lorentz transformation.

First we compose the Lorentz transformation matrix - this can be done more or less by inspection. Denoting $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{(\gamma-1)}{\beta^2}\beta_1^2 & \frac{(\gamma-1)}{\beta^2}\beta_1\beta_2 & \frac{(\gamma-1)}{\beta^2}\beta_1\beta_3 \\ -\gamma\beta_2 & \frac{(\gamma-1)}{\beta^2}\beta_1\beta_2 & 1+\frac{(\gamma-1)}{\beta^2}\beta_2^2 & \frac{(\gamma-1)}{\beta^2}\beta_2\beta_3 \\ -\gamma\beta_3 & \frac{(\gamma-1)}{\beta^2}\beta_1\beta_3 & \frac{(\gamma-1)}{\beta^2}\beta_2\beta_3 & 1+\frac{(\gamma-1)}{\beta^2}\beta_3^2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In tensor notation, this corresponds to

$$x'^\mu = \Lambda^\mu_\nu x^\nu.$$

To show that the inner product is invariant under the Lorentz boost we need to show that

$$x'_\mu x'^\mu = x_\mu x^\mu.$$

This can be formulated as

$$g_{\mu\nu}x'^\nu x'^\mu = g_{\mu\nu}x^\nu x^\mu$$
$$g_{\mu\nu}\Lambda^\nu_\alpha x^\alpha \Lambda^\mu_\beta x^\beta = g_{\mu\nu}x^\nu x^\mu$$

$$\left(g_{\mu\nu}\Lambda^{\nu}_{\alpha}\Lambda^{\mu}_{\beta}\right)x^{\alpha}x^{\beta} = g_{\mu\nu}x^{\nu}x^{\mu}.$$

This implies

$$\begin{aligned}
g_{\mu\nu}\Lambda^{\nu}_{\alpha}\Lambda^{\mu}_{\beta} &= g_{\alpha\beta} \\
\Lambda^{\nu}_{\alpha}\Lambda_{\nu\beta} &= g_{\alpha\beta} \\
\left(\Lambda^{\nu}_{\alpha}\Lambda_{\nu\beta}\right)g^{\beta\mu} &= g_{\alpha\beta}g^{\beta\mu} \\
\Lambda^{\nu}_{\alpha}\Lambda^{\mu}_{\nu} &= \delta^{\mu}_{\alpha}
\end{aligned}$$

Explicitly,

$$\sum_{\nu=0}^{3}\Lambda^{\nu}_{\alpha}\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\alpha} = \Lambda^{0}_{\alpha}\Lambda^{\mu}_{0} + \Lambda^{1}_{\alpha}\Lambda^{\mu}_{1} + \Lambda^{2}_{\alpha}\Lambda^{\mu}_{2} + \Lambda^{3}_{\alpha}\Lambda^{\mu}_{3}.$$

Lets test for $\alpha = \mu$. Since our Lorentz transformation matrix is symmetric, we are essentially summing the squares of each element in a column to see if they each equal unity

$$\left(\Lambda^{0}_{\alpha}\right)^2 + \left(\Lambda^{1}_{\alpha}\right)^2 + \left(\Lambda^{2}_{\alpha}\right)^2 + \left(\Lambda^{3}_{\alpha}\right)^2 = 1.$$

For the time column ($\alpha = 0$)

$$\Lambda^{\nu}_{0}\Lambda^{0}_{\nu} = \gamma^2 + \gamma^2(1+\beta^2) = 1$$

For the other columns we also have

$$\Lambda^{\nu}_{i}\Lambda^{i}_{\nu} = 1.$$

We can continue to show that the product between any two columns in the transformation matrix must be zero unless they are the same column - this is due to the fact that a Lorentz boost is unitary (and thus orthogonal). Suffice to say, Lorentz boosts preserve the inner product and space-time interval, which is what we expect between two inertial frames of reference.

(b) Construct the Lorentz transformation matrix for two successive boosts, of velocities $\vec{v}$ and $\vec{u}$, each aligned along the x^1 axis. Hence derive the relativistic velocity addition formula:

$$w = \frac{u+v}{1+\frac{uv}{c^2}}$$

For a boost along the x^1 direction, we have $\vec{\beta} = (\beta, 0, 0)$ and our transformation matrix goes as

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As expected, only the x^0 and x^1 coordinates mix. We focus only on the transformation between these two components from hereon. With two successive boosts we have

$$\Lambda^{\alpha}_{\beta}\Lambda^{\beta}_{\nu}x^{\nu} = x''^{\mu} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v \\ -\gamma_v\beta_v & \gamma_v \end{pmatrix}\begin{pmatrix} \gamma_u & -\gamma_u\beta_u \\ -\gamma_u\beta_u & \gamma_u \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

From here we make the substitutions

$$\sinh\theta_v = \gamma_v\beta_v, \qquad \cosh\theta_v = \gamma_v$$

and similarly for velocity u. This becomes

$$x''^{\mu} = \begin{pmatrix} \cosh\theta_v & -\sinh\theta_v \\ -\sinh\theta_v & \cosh\theta_v \end{pmatrix}\begin{pmatrix} \cosh\theta_u & -\sinh\theta_u \\ -\sinh\theta_u & \cosh\theta_u \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\theta_v + \theta_u) & -\sinh(\theta_v + \theta_u) \\ -\sinh(\theta_v + \theta_u) & \cos(\theta_v + \theta_u) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

We see that this is equivalent to a single boost of "hyperbolic" velocity

$$\theta_w = \theta_v + \theta_u.$$

We can convert this to the actual relation between velocities by inverting our hyperbolic substitutions

$$\begin{aligned} \tanh\theta_w &= \frac{\tanh\theta_v + \tanh\theta_u}{1 + \tanh\theta_v \tanh\theta_u} \\ &= \frac{\beta_v + \beta_u}{1 + \beta_v\beta_u} = \beta_w \end{aligned}$$

hence

$$w = \frac{v + u}{1 + \frac{vu}{c^2}}.$$

(c) Recalling that the field strength $F^{\mu\nu}$ is a tensor, compute the effect on the electric and magnetic fields of a Lorentz transformation corresponding to a boost of velocity $\vec{v}$ along the x^1 axis.

Under a Lorentz boost, the tensor transforms according to

$$F'^{\mu\nu} = \Lambda^\mu_\alpha F^{\alpha\beta} \Lambda^\nu_\beta$$

Using the boost found earlier, this becomes, in matrix form

$$\begin{aligned} F'^{\mu\nu} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_1 & \gamma(E_2 - \beta B_3) & \gamma(E_3 + \beta B_2) \\ -E_1 & 0 & \gamma(B_3 - \beta E_2) & -\gamma(\beta E_3 + B_2) \\ \gamma(\beta B_3 - E_2) & \gamma(\beta E_2 - B_3) & 0 & B_1 \\ -\gamma(E_3 + \beta B_2) & \gamma(\beta E_3 + B_2) & -B_1 & 0 \end{pmatrix} \end{aligned}$$

We find that the EM fields in the x^1 direction are unaffected - but components in the plane orthogonal to the x^1-axis are mixed.

2. Dirac matrices, Spin Matrices and Relativistic Total Angular Momentum

(a) Use the anti-commutation properties of the Pauli matrices to verify that the Dirac matrices really do satisfy the anti-commutation relations:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

Let's separate the space and time indices. With $\mu = \nu = 0$ we have

$$\{\gamma_0, \gamma_0\} = 2\gamma_0^2 = 2\mathbb{1}.$$

For $\nu = 0$ we note that

$$\gamma_i\gamma_0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix} = -\gamma_0\gamma_i$$

Therefore

$$\{\gamma_i, \gamma_0\} = 2\delta_{i0}\mathbb{1}.$$

Now for the space indices i, j,

$$\gamma_i\gamma_j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix}$$

hence

$$\{\gamma_i, \gamma_j\} = -\begin{pmatrix} \{\sigma_i, \sigma_j\} & 0 \\ 0 & \{\sigma_i, \sigma_j\} \end{pmatrix}.$$

Using the anti-commutation property of the Pauli matrices,

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$$

we have

$$\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbb{1}.$$

With our results,

$$\{\gamma_i, \gamma_0\} = 2\delta_{i0}\mathbb{1}; \qquad \{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbb{1},$$

we see that we get back our Minkowski metric with a factor of 2

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}.$$

(b) The relativistic spin matrices are defined as $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. Treat separately the cases where the space-time index is temporal or spatial, and compute σ_{0i} and σ_{ij}.

Most of the previous results apply here. Using $\gamma_0\gamma_i = -\gamma_i\gamma_0$,

$$\sigma_{0i} = \frac{i}{2}[\gamma_0, \gamma_i] = i\gamma_0\gamma_i = \sigma_i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

For σ_{ij},

$$\sigma_{ij} = \frac{i}{2}\left[-\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} + \begin{pmatrix} \sigma_j\sigma_i & 0 \\ 0 & \sigma_j\sigma_i \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix}.$$

With the Pauli commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

this becomes

$$\sigma_{ij} = \begin{pmatrix} \epsilon_{ijk}\sigma_k & 0 \\ 0 & \epsilon_{ijk}\sigma_k \end{pmatrix}.$$

(c) The total relativistic angular momentum is defined to be

$$\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma}$$

where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Show using the Dirac equation for an electron in a radially symmetric scalar potential, $\phi(r)$, that $\vec{J}$ is conserved.

The Dirac equation for a particle in a spherically symmetric potential $\phi(r)$ can be expressed as

$$i\hbar\frac{\partial\psi}{\partial t} = (c\vec{\alpha}\cdot\vec{p} + \vec{\beta}mc^2 + \phi(r))\psi$$

or as

$$i\hbar\frac{\partial\psi}{\partial t} = (c\gamma^0\gamma^i p_i + \gamma^0 mc^2 + \phi(r))\psi.$$

In this form, the Schrodinger-like Hamiltonian can be seen to be

$$H = c\gamma^0\gamma^i p_i + \gamma^0 mc^2 + \phi(r).$$

The Hamiltonian is a 4×4 matrix. To determine if $\vec{J}$ is conserved, it must commute with the Hamiltonian. We note that the Dirac matrices belong to a group of tensor operators and therefore only act on the spinor components; similar reasoning applies the momentum and position operators.

First lets analyze the potential part of the Hamiltonian. We know, either from other problems like the hydrogen atom or by explicit calculation, that the orbital angular momentum operator commutes with any function of r

$$[L_i, r] \sim [L_i, (x^2 + y^2 + z^2)] = 0.$$

In addition, the spin angular momentum components are the identity in position space, so they too commute:

$$[\Sigma_i, \phi(r)] = 0.$$

Therefore the total angular momentum commutes with the potential

$$[J_i, \phi(r)] = 0.$$

Next we look at γ^0 with $\vec{J}$. The relevant commutator is

$$\frac{\hbar}{2}mc^2[\gamma^0, \vec{\Sigma}] = \frac{\hbar}{2}mc^2\left[\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} - \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\right] = 0.$$

Lastly, we must look at the $c\gamma^0\gamma^i p_i$ (or $c\vec{\alpha}\cdot\vec{p}$) term with $\vec{J}$. Lets commute with the z component of spin angular momentum

$$\begin{aligned}[c\vec{\alpha}\cdot\vec{p}, \frac{\hbar}{2}\Sigma_3] &= \frac{\hbar c}{2}[\alpha_i p^i, \Sigma_3]\\ &= \frac{\hbar c}{2}\left(p_1[\alpha_1, \Sigma_3] + p_2[\alpha_2, \Sigma_3] + p_3[\alpha_3, \Sigma_3]\right).\end{aligned}$$

Individually, the commutators are

$$\begin{aligned}[\alpha_1, \Sigma_3] &= \left[\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} - \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\right]\\ &= \begin{pmatrix}0 & [\sigma_1, \sigma_3]\\ [\sigma_1, \sigma_3] & 0\end{pmatrix}\\ &= -2i\alpha_2.\end{aligned}$$

Hence

$$[\alpha_2, \Sigma_3] = 2i\alpha_1$$

and

$$[\alpha_3, \Sigma_3] = 0.$$

In general we then have

$$[\alpha_i, \Sigma_j] = \epsilon_{ijk}2i\alpha_k.$$

Now we can evaluate

$$
\begin{aligned}
[c\vec{\alpha}\cdot\vec{p}, \frac{\hbar}{2}\Sigma_3] =& i\hbar c\left(p_1[\alpha_1,\Sigma_3] + p_2[\alpha_2,\Sigma_3] + p_3[\alpha_3,\Sigma_3]\right) \\
&= i\hbar c\left(\alpha_1 p_2 - \alpha_2 p_1\right) \\
&= i\hbar c(\vec{\alpha}\times\vec{p})_3
\end{aligned}
$$

Therefore the more general result is

$$
[c\vec{\alpha}\cdot\vec{p}, \frac{\hbar}{2}\Sigma_i] = i\hbar c(\vec{\alpha}\times\vec{p})_i. \tag{1}
$$

Lastly we evaluate

$$
[c\vec{\alpha}\cdot\vec{p}, L_i] = c[\alpha_i p^i, L_j].
$$

Looking at the case for L_3

$$
\begin{aligned}
[\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3, L_3] &= [\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3, (x_3 p_2 - x_2 p_3)] \\
&= \alpha_1 p_2 [p_1, x_1] - \alpha_2 p_1 [p_2, x_2] \\
&= -i\hbar(\alpha_1 p_2 - \alpha_2 p_1) \\
&= -i\hbar(\vec{\alpha}\times\vec{p})_3
\end{aligned}
$$

So in general we have

$$
[c\vec{\alpha}\cdot\vec{p}, L_i] = -i\hbar c(\vec{\alpha}\times\vec{p})_i. \tag{2}
$$

Since (1) and (2) are the only non-zero contributions to the commutator with the Hamiltonian, and because they are exactly equal and opposite, we conclude that the total angular momentum commutes with the Hamiltonian

$$
[H, \mathbf{J}] = 0
$$

and thus is a conserved quantity.

3. 4-momentum in an electromagnetic field

Defining the kinetic 4-momentum

$$
\pi_\mu \equiv c\left(p_\mu + \frac{e}{c}A_\mu\right)
$$

use the quantum mechanical relation $p_\mu \to i\hbar\frac{\partial}{\partial x^\mu}$ to commute the operator

$$
(\gamma^\mu \pi_\mu)^2
$$

Express your answer in terms of the Klein-Gordan part, $\pi_\mu\pi^\mu$, and parts linear in the electric and magnetic field.

As the kinetic momentum operator is separate from the gamma spin matrices, we of course have

$$
[\gamma^\mu, \pi_\nu] = 0
$$

and so we may write the product as

$$
\begin{aligned}
(\gamma^\mu \pi_\mu)^2 &= \gamma^\mu\gamma^\nu\pi_\mu\pi_\nu \\
&= \left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\pi_\mu\pi_\nu
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2}(2g^{\mu\nu}\mathbb{1}) + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right)\pi_\mu\pi_\nu \\
&= \frac{1}{2}[\gamma^\mu, \gamma^\nu]\,(\pi_\mu\pi_\nu) + \pi^\nu\pi_\nu \\
&= \frac{1}{4}[\gamma^\mu, \gamma^\nu]\,(\{\pi_\mu, \pi_\nu\} + [\pi_\mu, \pi_\nu]) + \pi^\nu\pi_\nu \qquad (3)
\end{aligned}
$$

Lets first take the anti-commutator

$$\frac{1}{4}[\gamma^\mu, \gamma^\nu]\{\pi_\mu, \pi_\nu\}.$$

Naturally, the commutator is antisymmetric while the anti-commutator is symmetric. Thus as we sum over all μ, ν, it will vanish (in the case where $\mu = \nu$, the commutator of course also vanishes). For the kinetic momentum commutator

$$
\begin{aligned}
[\pi_\mu, \pi_\nu] &= i\hbar ce\left(\left[\frac{\partial}{\partial x^\mu}, A_\nu\right] + \left[\frac{\partial}{\partial x^\nu}, A_\mu\right]\right) \\
&= i\hbar ce\left(\frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu\right) \\
&= i\hbar ce F_{\mu\nu}.
\end{aligned}
$$

We may now use this result to form

$$\frac{1}{4}[\gamma^\mu, \gamma^\nu]\{\pi_\mu, \pi_\nu\} = \frac{i\hbar ce}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}.$$

In the case where $\mu = 0$, $\nu \neq 0$ we have

$$[\gamma^0, \gamma^\nu] = 2\sigma_\nu \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

so

$$\frac{i\hbar ce}{4}[\gamma^0, \gamma^\nu]F_{0\nu} = \frac{i\hbar ce}{2}\sigma_\nu F_{0\nu}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

The components of $F_{0\nu}$ are just the components of $\vec{E}$ so we may rewrite this as

$$\frac{i\hbar ce}{4}[\gamma^0, \gamma^\nu]F_{0\nu} = \frac{i\hbar ce}{2}\vec{\sigma} \cdot \vec{E}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad (4)$$

Similarly, for $\mu \neq 0$, $\nu = 0$, we have the same result due to antisymmetry in the commutator canceling antisymmetry in the EM tensor. Lastly, for μ, $\nu = 1, 2, 3$ we know that

$$[\gamma^i, \gamma^j] = 2i\epsilon_{ijk}\sigma_k \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

so

$$\frac{i\hbar ce}{4}[\gamma^i, \gamma^j]F_{ij} = -\frac{\hbar ce}{2}\epsilon_{ijk}\sigma_k F_{ij}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

The off diagonal elements of F_{ij} for i, $j = 1, 2, 3$ are precisely the components of the magnetic field so we have

$$\frac{i\hbar ce}{4}[\gamma^i, \gamma^j]F_{ij} = \hbar ce(\vec{\sigma} \cdot \vec{B})\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad (5)$$

Combining (3), (4), and (5) we finally have

$$(\gamma^\mu\pi_\mu)^2 = \pi_\mu\pi^\mu + \hbar ec\begin{pmatrix} \vec{\sigma} \cdot \vec{B} & i\vec{\sigma} \cdot \vec{E} \\ i\vec{\sigma} \cdot \vec{E} & \vec{\sigma} \cdot \vec{B} \end{pmatrix}.$$