Coordinate Transformation RW SVT3 v2

1 RW $\Omega(\tau)$

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(\tau)(\tilde{g}_{\mu\nu} + f_{\mu\nu})dx^\mu dx^\nu \tag{1.1}$$

$$\tilde{g}_{\mu\nu} = \mathrm{diag}\left(-1, \frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right) \qquad \tilde{\Gamma}^\lambda_{\alpha\beta} = \delta^\lambda_i \delta^j_\alpha \delta^k_\beta \tilde{\Gamma}^i_{jk} \tag{1.2}$$

1.1 $f_{\mu\nu}(SVT3)$

$$\begin{aligned}
f_{00} &= -2\phi \\
f_{0i} &= B_i + \tilde{\nabla}_i B \\
f_{ij} &= -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \\
\tilde{g}^{ij} f_{ij} &= -6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E \\
\tilde{g}^{\mu\nu} f_{\mu\nu} &= 2\phi - 6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E
\end{aligned} \tag{1.3}$$

$$-\tilde{\nabla}^a\tilde{\nabla}^\alpha\Omega f_{a\alpha} = \dot{\Omega}\tilde{\nabla}_a\tilde{\nabla}^a B \tag{1.4}$$

1.2 $SVT3(f_{\mu\nu})$

$$\phi = -\tfrac{1}{2} f_{00} \tag{1.5}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a B = \tilde{\nabla}^a f_{0a} \tag{1.6}$$

$$(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)B_i = (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)f_{0i} - \tilde{\nabla}_i\tilde{\nabla}^a f_{0a} \tag{1.7}$$

$$(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\psi = \frac{1}{4}\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} - (\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{g}^{bc} f_{bc})\right] \tag{1.8}$$

$$(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b E = \frac{3}{4}\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} - \frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{g}^{bc} f_{bc})\right] \tag{1.9}$$

$$(\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 2k)E_i = (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)\tilde{\nabla}^b f_{ib} - \tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} \tag{1.10}$$

$$\begin{aligned}
(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)(2E_{ij}) &= (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)f_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\tilde{\nabla}_j\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} + (\tilde{\nabla}_a\tilde{\nabla}^a + 4k)(\tilde{g}^{bc} f_{bc})\right] \\
&\quad + \tfrac{1}{2}\tilde{g}_{ij}\left[(\tilde{\nabla}_a\tilde{\nabla}^a - 4k)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - (\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b - 2k\tilde{\nabla}_a\tilde{\nabla}^a + 4k^2)(\tilde{g}^{bc} f_{bc})\right] \\
&\quad - (\tilde{\nabla}_a\tilde{\nabla}^a - 3k)(\tilde{\nabla}_i\tilde{\nabla}^b f_{jb} + \tilde{\nabla}_j\tilde{\nabla}^b f_{ib})
\end{aligned} \tag{1.11}$$

1.3 $\Delta_\epsilon[SVT3]$

$$\bar{x}^\mu = x^\mu - \epsilon^\mu(x) \quad \Longrightarrow \quad \bar{h}_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \tag{1.12}$$

$$\Delta_\epsilon\left[\phi\right] = \dot{\Omega}\Omega^{-1}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)f_0 \tag{1.13}$$

$$\Delta_\epsilon\left[\tilde{\nabla}_a\tilde{\nabla}^a B\right] = \tilde{\nabla}_a\dot{f}^a + \tilde{\nabla}_a\tilde{\nabla}^a f_0 \tag{1.14}$$

$$\Delta_\epsilon\left[(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)B_i\right] = (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)\dot{f}_i - \tilde{\nabla}_i\tilde{\nabla}_a\dot{f}^a \tag{1.15}$$

$$\Delta_\epsilon\left[(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\psi\right] = -\dot{f}_0 - \dot{\Omega}f_0\Omega^{-1} \tag{1.16}$$

$$\Delta_\epsilon\left[(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b E\right] = (\tilde{\nabla}_b\tilde{\nabla}^b + 3k)\tilde{\nabla}_a f^a \tag{1.17}$$

$$\Delta_\epsilon\left[(\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 2k)E_i\right] = (\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 2k)f_i - \tilde{\nabla}_i(\tilde{\nabla}_b\tilde{\nabla}^b + 4k)\tilde{\nabla}_a f^a \tag{1.18}$$

$$\Delta_\epsilon\left[(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)(2E_{ij})\right] = 0 \tag{1.19}$$

1.4 Gauge Invariants

We mix time derivative notation a bit, using ∂_0 upon $f_{\mu\nu}$ and dot upon Ω and SVT3 quantities.

$$\begin{aligned}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\phi + \psi + \dot{B} - \ddot{E}] &= (\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}^b(\partial_0 f_{0b}) - \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 2k - \partial_0^2)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}^{cd}f_{cd}) \\ &\quad + \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_0^2)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - \tfrac{1}{2}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b f_{00}\end{aligned} \tag{1.20}$$

$$\begin{aligned}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}_b\tilde{\nabla}^b[\psi - \dot{\Omega}\Omega^{-1}(B - \dot{E})] &= -\dot{\Omega}\Omega^{-1}(\tilde{\nabla}_a\tilde{\nabla}^a + 3k)\tilde{\nabla}^b f_{0b} + \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 3\dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} \\ &\quad - \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a + 2k + \dot{\Omega}\Omega^{-1}\partial_0)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}^{cd}f_{cd})\end{aligned} \tag{1.21}$$

$$\begin{aligned}(\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 2k)[B_i - \dot{E}_i] &= (\tilde{\nabla}_a\tilde{\nabla}^a + 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 2k)f_{0i} - (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)\tilde{\nabla}^b(\partial_0 f_{ib}) \\ &\quad - \tilde{\nabla}_i(\tilde{\nabla}_a\tilde{\nabla}^a + 4k)\tilde{\nabla}^b f_{0b} + \tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}^b(\partial_0 f_{ab})\end{aligned} \tag{1.22}$$

$$\begin{aligned}(\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)[2E_{ij}] &= (\tilde{\nabla}_a\tilde{\nabla}^a - 2k)(\tilde{\nabla}_b\tilde{\nabla}^b - 3k)f_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\tilde{\nabla}_j\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} + (\tilde{\nabla}_a\tilde{\nabla}^a + 4k)(\tilde{g}^{bc}f_{bc})\right] \\ &\quad + \tfrac{1}{2}\tilde{g}_{ij}\left[(\tilde{\nabla}_a\tilde{\nabla}^a - 4k)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - (\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b - 2k\tilde{\nabla}_a\tilde{\nabla}^a + 4k^2)(\tilde{g}^{bc}f_{bc})\right] \\ &\quad - (\tilde{\nabla}_a\tilde{\nabla}^a - 3k)(\tilde{\nabla}_i\tilde{\nabla}^b f_{jb} + \tilde{\nabla}_j\tilde{\nabla}^b f_{ib})\end{aligned} \tag{1.23}$$

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2 RW $\Omega(T,R)$

$$ds^2 = (g'_{\mu\nu} + h'_{\mu\nu})dx'^\mu dx'^\nu = \Omega^2(T,R)(\tilde{g}'_{\mu\nu} + f_{\mu\nu})dx'^\mu dx'^\nu \tag{2.1}$$

$$\tilde{g}'_{\mu\nu} = \mathrm{diag}\left(-1, 1, R^2, R^2\sin^2\theta\right) \tag{2.2}$$

2.1 $f_{\mu\nu}(SVT3)$

$$
\begin{aligned}
f_{00} &= -2\phi \\
f_{0i} &= B_i + \tilde{\nabla}_i B \\
f_{ij} &= -2\tilde{g}'_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \\
\tilde{g}'^{ij} f_{ij} &= -6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E \\
\tilde{g}'^{\mu\nu} f_{\mu\nu} &= 2\phi - 6\psi + 2\tilde{\nabla}^k\tilde{\nabla}_k E
\end{aligned} \tag{2.3}
$$

2.2 $SVT3(f_{\mu\nu})$

These quantities mimic (1.5)-(1.11) with $k = 0$.

$$\phi = -\tfrac{1}{2} f_{00} \tag{2.4}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a B = \tilde{\nabla}^a f_{0a} \tag{2.5}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a B_i = \tilde{\nabla}_a\tilde{\nabla}^a f_{0i} - \tilde{\nabla}_i\tilde{\nabla}^a f_{0a} \tag{2.6}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a \psi = \frac{1}{4}\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} - \tilde{\nabla}_a\tilde{\nabla}^a(\tilde{g}'^{bc} f_{bc})\right] \tag{2.7}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b E = \frac{3}{4}\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} - \frac{1}{3}\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{g}'^{bc} f_{bc})\right] \tag{2.8}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b E_i = \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b f_{ib} - \tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} \tag{2.9}$$

$$
\begin{aligned}
\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(2E_{ij}) &= \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b f_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\tilde{\nabla}_j\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} + \tilde{\nabla}_a\tilde{\nabla}^a(\tilde{g}'^{bc} f_{bc})\right] \\
&\quad + \tfrac{1}{2}\tilde{g}'_{ij}\left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}'^{bc} f_{bc})\right] \\
&\quad - \tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_i\tilde{\nabla}^b f_{jb} + \tilde{\nabla}_j\tilde{\nabla}^b f_{ib})
\end{aligned} \tag{2.10}
$$

2.3 $\Delta_\epsilon[f_{\mu\nu}]$

$$\bar{x}^\mu = x'^\mu - \epsilon^\mu(x) \implies \Delta_\epsilon\left[h_{\mu\nu}\right] = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu \tag{2.11}$$

$$f_\mu = \Omega^2\epsilon_\mu, \qquad f^\mu = \epsilon^\mu \tag{2.12}$$

$$f_0 = -T, \qquad f_a = \tilde{\nabla}_a L + L_a, \qquad \tilde{\nabla}^a L_a = 0 \tag{2.13}$$

$$\Delta_\epsilon\left[f_{\mu\nu}\right] = \tilde{\nabla}_\mu f_\nu + \tilde{\nabla}_\nu f_\mu + 2f^\gamma\tilde{g}'_{\mu\nu}\Omega^{-1}\tilde{\nabla}_\gamma\Omega \tag{2.14}$$

$$\Delta_\epsilon\left[\tilde{f}_{00}\right] = -2\dot{T} - 2\Omega^{-1}\left[T\dot{\Omega} + (\tilde{\nabla}^a L + L^a)\tilde{\nabla}_a\Omega\right] \tag{2.15}$$

$$\Delta_\epsilon\left[\tilde{f}_{0i}\right] = \tilde{\nabla}_i\dot{L} + \dot{L}_i - \tilde{\nabla}_i T \tag{2.16}$$

$$\Delta_\epsilon\left[\tilde{f}_{ij}\right] = 2\tilde{\nabla}_i\tilde{\nabla}_j L + \tilde{\nabla}_i L_j + \tilde{\nabla}_j L_i + 2\Omega^{-1}\tilde{g}_{ij}\left[T\dot{\Omega} + (\tilde{\nabla}^a L + L^a)\tilde{\nabla}_a\Omega\right] \tag{2.17}$$

$$\Delta_\epsilon\left[\tilde{g}'^{ab} f_{ab}\right] = 2\tilde{\nabla}_a\tilde{\nabla}^a L + 6\Omega^{-1}\left[T\dot{\Omega} + (\tilde{\nabla}^a L + L^a)\tilde{\nabla}_a\Omega\right] \tag{2.18}$$

$$\Delta_\epsilon\left[\tilde{g}'^{\alpha\beta}f_{\alpha\beta}\right] = 2\dot{T}+2\tilde{\nabla}_a\tilde{\nabla}^a L+8\Omega^{-1}\left[T\dot{\Omega}+(\tilde{\nabla}^a L+L^a)\tilde{\nabla}_a\Omega\right] \tag{2.19}$$

$$\Delta_\epsilon\left[\tilde{\nabla}^a f_{0a}\right] = \tilde{\nabla}_a\tilde{\nabla}^a(\dot{L}-T) \tag{2.20}$$

$$\begin{aligned}\Delta_\epsilon\left[\tilde{\nabla}^b f_{ab}\right] &= 2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a T+T(2\Omega^{-1}\tilde{\nabla}_a\dot{\Omega}-2\dot{\Omega}\Omega^{-2}\tilde{\nabla}_a\Omega)+2\Omega^{-1}\tilde{\nabla}_a L^b\tilde{\nabla}_b\Omega\\ &\quad +L^b(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+2\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)+\tilde{\nabla}_b\tilde{\nabla}^b L_a+2\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a L\\ &\quad +(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+2\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)\tilde{\nabla}^b L+2\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a L\tilde{\nabla}^b\Omega\end{aligned} \tag{2.21}$$

3 Integral Relations

$$\begin{aligned}\Delta_\epsilon[\phi] &= \dot{T}+\Omega^{-1}\left[T\dot{\Omega}+(\tilde{\nabla}^a L+L^a)\tilde{\nabla}_a\Omega\right]\\ \Delta_\epsilon[B] &= \int D\nabla^2(\dot{L}-T)\\ \Delta_\epsilon[B_i] &= \dot{L}_i+\nabla_i(\dot{L}-T)-\nabla_i\int D\nabla^2(\dot{L}-T)\\ \Delta_\epsilon[\psi] &= -\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a L-\tfrac{3}{2}\Omega^{-1}\left[T\dot{\Omega}+(\tilde{\nabla}^a L+L^a)\tilde{\nabla}_a\Omega\right]\\ &\quad +\tilde{\nabla}^a\Bigg[\int D\Bigg(\tfrac{1}{2}\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a T+T(\tfrac{1}{2}\Omega^{-1}\tilde{\nabla}_a\dot{\Omega}-\tfrac{1}{2}\dot{\Omega}\Omega^{-2}\tilde{\nabla}_a\Omega)+\tfrac{1}{2}\Omega^{-1}\tilde{\nabla}_a L^b\tilde{\nabla}_b\Omega\\ &\quad +L^b(-\tfrac{1}{2}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+\tfrac{1}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)+\tfrac{1}{4}\tilde{\nabla}_b\tilde{\nabla}^b L_a+\tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a L\\ &\quad +(-\tfrac{1}{2}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+\tfrac{1}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)\tilde{\nabla}^b L+\tfrac{1}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a L\tilde{\nabla}^b\Omega\Bigg)\Bigg]\\ \Delta_\epsilon[E] &= \int D\Bigg[-\tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a L-\tfrac{3}{2}\Omega^{-1}\left[T\dot{\Omega}+(\tilde{\nabla}^a L+L^a)\tilde{\nabla}_a\Omega\right]\\ &\quad +\tilde{\nabla}^a\Bigg[\int D\Bigg(\tfrac{3}{2}\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a T+T(\tfrac{3}{2}\Omega^{-1}\tilde{\nabla}_a\dot{\Omega}-\tfrac{3}{2}\dot{\Omega}\Omega^{-2}\tilde{\nabla}_a\Omega)+\tfrac{3}{2}\Omega^{-1}\tilde{\nabla}_a L^b\tilde{\nabla}_b\Omega\\ &\quad +L^b(-\tfrac{3}{2}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+\tfrac{3}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)+\tfrac{3}{4}\tilde{\nabla}_b\tilde{\nabla}^b L_a+\tfrac{3}{2}\tilde{\nabla}_b\tilde{\nabla}^b\tilde{\nabla}_a L\\ &\quad +(-\tfrac{3}{2}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+\tfrac{3}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)\tilde{\nabla}^b L+\tfrac{3}{2}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a L\tilde{\nabla}^b\Omega\Bigg)\Bigg]\Bigg]\\ \Delta_\epsilon[E_i] &= \int D\Bigg[\tilde{\nabla}_a\tilde{\nabla}^a L_i+2\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i L^a+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i T+T(2\Omega^{-1}\tilde{\nabla}_i\dot{\Omega}-2\dot{\Omega}\Omega^{-2}\tilde{\nabla}_i\Omega)+2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_i\tilde{\nabla}_a L\\ &\quad +L^a(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_i\Omega+2\Omega^{-1}\tilde{\nabla}_i\tilde{\nabla}_a\Omega)+\tilde{\nabla}^a L(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_i\Omega+2\Omega^{-1}\tilde{\nabla}_i\tilde{\nabla}_a\Omega)\\ &\quad +2\tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a L\Bigg]-\tilde{\nabla}_i\int D\tilde{\nabla}^b\int D\Bigg[\tilde{\nabla}_a\tilde{\nabla}^a L_b+2\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b L^a+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_b T+T(2\Omega^{-1}\tilde{\nabla}_b\dot{\Omega}-2\dot{\Omega}\Omega^{-2}\tilde{\nabla}_b\Omega)\\ &\quad +2\Omega^{-1}\tilde{\nabla}^a\Omega\tilde{\nabla}_b\tilde{\nabla}_a L+L^a(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+2\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)+\tilde{\nabla}^a L(-2\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega+2\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega)\\ &\quad +2\tilde{\nabla}_b\tilde{\nabla}_a\tilde{\nabla}^a L\Bigg]\end{aligned} \tag{3.1}$$

We may also include the trace condition

$$-6\bar{\psi}+2\nabla^2\bar{E} = -6\psi+2\nabla^2 E+2\nabla^2 L. \tag{3.2}$$

3.1 $\Delta_\epsilon[SVT3]$

$$\Delta_\epsilon\left[\phi\right] = -\dot{f}_0-\dot{\Omega}f_0\Omega^{-1}+f^a\Omega^{-1}\tilde{\nabla}_a\Omega \tag{3.3}$$

$$\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a B\right] = \tilde{\nabla}'_a\dot{f}^a + \tilde{\nabla}'_a\tilde{\nabla}'^a f_0 \tag{3.4}$$

$$\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a B_i\right] = \tilde{\nabla}'_a\tilde{\nabla}'^a \dot{f}_i - \tilde{\nabla}'_a\tilde{\nabla}'_i\dot{f}^a \tag{3.5}$$

$$\begin{aligned}\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a \psi\right] = {} & f_0\Omega^{-1}\tilde{\nabla}'_a\tilde{\nabla}'^a\dot{\Omega} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}'_a\tilde{\nabla}'^a f_0 - \dot{\Omega}f_0\Omega^{-2}\tilde{\nabla}'_a\tilde{\nabla}'^a\Omega + 2\Omega^{-1}\tilde{\nabla}'_a f_0\tilde{\nabla}'^a\dot{\Omega} - 2f_0\Omega^{-2}\tilde{\nabla}'_a\Omega\tilde{\nabla}'^a\dot{\Omega} \\ & -2\dot{\Omega}\Omega^{-2}\tilde{\nabla}'_a\Omega\tilde{\nabla}'^a f_0 + 2\dot{\Omega}f_0\Omega^{-3}\tilde{\nabla}'_a\Omega\tilde{\nabla}'^a\Omega - \Omega^{-1}\tilde{\nabla}'^a\Omega\tilde{\nabla}'_b\tilde{\nabla}'^b f_a + f^a\Omega^{-2}\tilde{\nabla}'_a\Omega\tilde{\nabla}'_b\tilde{\nabla}'^b\Omega \\ & -f^a\Omega^{-1}\tilde{\nabla}'_b\tilde{\nabla}'^b\tilde{\nabla}'_a\Omega + 2\Omega^{-2}\tilde{\nabla}'_a\Omega\tilde{\nabla}'_b\Omega\tilde{\nabla}'^b f^a - 2\Omega^{-1}\tilde{\nabla}'_b\tilde{\nabla}'_a\Omega\tilde{\nabla}'^b f^a - 2f^a\Omega^{-3}\tilde{\nabla}'_a\Omega\tilde{\nabla}'_b\Omega\tilde{\nabla}'^b\Omega \\ & +2f^a\Omega^{-2}\tilde{\nabla}'_b\tilde{\nabla}'_a\Omega\tilde{\nabla}'^b\Omega\end{aligned} \tag{3.6}$$

$$\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b E\right] = \tilde{\nabla}'_b\tilde{\nabla}'^b\tilde{\nabla}'_a f^a \tag{3.7}$$

$$\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b E_i\right] = \tilde{\nabla}'_b\tilde{\nabla}'^b\tilde{\nabla}'_a\tilde{\nabla}'^a f_i - \tilde{\nabla}'_b\tilde{\nabla}'^b\tilde{\nabla}'_i\tilde{\nabla}'_a f^a \tag{3.8}$$

$$\Delta_\epsilon \left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b (2E_{ij})\right] = 0 \tag{3.9}$$

3.2 Gauge Invariants

We mix time derivative notation a bit, using ∂_0 upon $f_{\mu\nu}$ and dot upon Ω and SVT3 quantities.

$$\begin{aligned}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b[\phi + \psi + \dot{B} - \ddot{E}] = {} & \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b(\partial_0 f_{0b}) - \tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a - \partial_0^2)\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}'^{cd}f_{cd}) \\ & +\tfrac{1}{4}(\tilde{\nabla}_a\tilde{\nabla}^a - 3\partial_0^2)\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - \tfrac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b f_{00}\end{aligned} \tag{3.10}$$

$$\begin{aligned}& \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b \times \\ & \left[\psi - \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega]\right] = ?\end{aligned} \tag{3.11}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b[B'_i - \dot{E}'_i] = \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b f_{0i} - \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b(\partial_0 f_{ib}) - \tilde{\nabla}_i\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b f_{0b} + \tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}^b(\partial_0 f_{ab}) \tag{3.12}$$

$$\begin{aligned}\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b[2E_{ij}] = {} & \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b f_{ij} + \tfrac{1}{2}\tilde{\nabla}_i\tilde{\nabla}_j\left[\tilde{\nabla}^a\tilde{\nabla}^b f_{ab} + \tilde{\nabla}_a\tilde{\nabla}^a(\tilde{g}'^{bc}f_{bc})\right] \\ & +\tfrac{1}{2}\tilde{g}'_{ij}\left[\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b\tilde{\nabla}^c f_{bc} - \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b(\tilde{g}'^{bc}f_{bc})\right] \\ & -\tilde{\nabla}_a\tilde{\nabla}^a(\tilde{\nabla}_i\tilde{\nabla}^b f_{jb} + \tilde{\nabla}_j\tilde{\nabla}^b f_{ib})\end{aligned} \tag{3.13}$$

3.3 On the G.I. of $\psi - \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a\Omega]$

In the conformal to flat decomposition, E_i is given by the integral

$$E_i = \int D\tilde{\nabla}^k f_{ik} - \tilde{\nabla}_i\int D\tilde{\nabla}^k\tilde{\nabla}^l f_{kl}, \qquad \tilde{\nabla}_a\tilde{\nabla}^a D(x,x') = \delta(x - x'). \tag{3.14}$$

As given in (2.9), the lowest derivative relation in terms of $f_{\mu\nu}$ for E_i is

$$\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}^b E_i = \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\nabla}^b f_{ib} - \tilde{\nabla}_i\tilde{\nabla}^a\tilde{\nabla}^b f_{ab}. \tag{3.15}$$

E_i can also be found as a single derivative within f_{ij}

$$f_{ij} = -2\tilde{g}_{ij}\psi + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij} \tag{3.16}$$

When we take any derivative upon the gauge invariant

$$\psi - \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a \Omega], \tag{3.17}$$

from the product rule we will necessarily generate terms that depend on E_a alone; i.e. terms that could only be expressed as integrals over f_{ij} and not derivatives of f_{ij}. Consequently, it would not seem possible to construct this gauge invariant based on any combination of $f_{\mu\nu}$ or derivatives thereof.

It would then seem puzzling how we were able to express $\Delta_{\mu\nu}$ in terms of the gauge invariant $\gamma = \psi - \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a \Omega]$. Looking at *RW_Radiation_SVT3_Conformal_Flat_-k_Cartesian_v2.pdf*, it turns out that neither $\delta G_{\mu\nu}$ nor $\delta T_{\mu\nu}$ have any terms that depend on E_i without derivatives. When forming the gauge invariant combinations, we made substitutions like

$$\psi = \gamma + \Omega^{-1}[(B - \dot{E})\dot{\Omega} - (\tilde{\nabla}_a E + E_a)\tilde{\nabla}^a \Omega]. \tag{3.18}$$

All contributions of E_a that we originally introduce end up canceling after simplifying all relevant terms.