

3-Space Projectors v1

1 Flat δG_{ij}

$$\begin{aligned}\delta G_{00} &= \frac{1}{2}\nabla_a\nabla^a h_{00} + \frac{1}{2}\nabla_a\nabla^a h - \frac{1}{2}\nabla_b\nabla_a h^{ab} \\ &= -2\nabla_a\nabla^a\psi\end{aligned}$$

$$\begin{aligned}\delta G_{0i} &= -\frac{1}{2}\nabla_a\dot{h}_i{}^a + \frac{1}{2}\nabla_a\nabla^a h_{0i} + \frac{1}{2}\nabla_i\dot{h}_{00} + \frac{1}{2}\nabla_i\dot{h} - \frac{1}{2}\nabla_i\nabla_a h_0{}^a \\ &= -2\nabla_i\dot{\psi} + \frac{1}{2}\nabla_a\nabla^a B_i - \frac{1}{2}\nabla_a\nabla^a \dot{E}_i\end{aligned}$$

$$\begin{aligned}\delta G_{ij} &= -\frac{1}{2}\ddot{h}_{ij} + \frac{1}{2}\ddot{h}_{00}g_{ij} + \frac{1}{2}\ddot{h}g_{ij} - g_{ij}\nabla_a\dot{h}_0{}^a + \frac{1}{2}\nabla_a\nabla^a h_{ij} - \frac{1}{2}g_{ij}\nabla_a\nabla^a h + \frac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} + \frac{1}{2}\nabla_i\dot{h}_{j0} \\ &\quad - \frac{1}{2}\nabla_i\nabla_a h_j{}^a + \frac{1}{2}\nabla_j\dot{h}_{i0} - \frac{1}{2}\nabla_j\nabla_a h_i{}^a + \frac{1}{2}\nabla_j\nabla_i h \\ &= -2\ddot{\psi}g_{ij} - g_{ij}\nabla_a\nabla^a \dot{B} + g_{ij}\nabla_a\nabla^a \ddot{E} - g_{ij}\nabla_a\nabla^a \phi + g_{ij}\nabla_a\nabla^a \psi + \nabla_j\nabla_i \dot{B} - \nabla_j\nabla_i \ddot{E} + \nabla_j\nabla_i \phi - \nabla_j\nabla_i \psi \\ &\quad + \frac{1}{2}\nabla_i \dot{B}_j - \frac{1}{2}\nabla_i \ddot{E}_j + \frac{1}{2}\nabla_j \dot{B}_i - \frac{1}{2}\nabla_j \ddot{E}_i - \ddot{E}_{ij} + \nabla_a\nabla^a E_{ij}\end{aligned}\tag{1.1}$$

2 3+1 Projectors

Recall the flat 3+1 projector

$$P_{\mu\nu} = \eta_{\mu\nu} + U_\mu U_\nu, \quad U_\mu = -\delta_\mu^0, \quad U^\mu = \delta_0^\mu.\tag{2.1}$$

In terms of the the flat space projectors, the splitting of the 3+1 components goes as

$$\rho = U^\sigma U^\tau T_{\sigma\tau} = T_{00}, \quad q_i = -P_i{}^\sigma U^\tau T_{\sigma\tau} = -T_{0i}, \quad \pi_{\mu\nu} = \left[\frac{1}{2}P_\mu{}^\sigma P_\nu{}^\tau + \frac{1}{2}P_\nu{}^\sigma P_\mu{}^\tau - \frac{1}{3}P_{\mu\nu}P^{\sigma\tau} \right] T_{\sigma\tau},\tag{2.2}$$

in which it follows

$$\pi_{\mu\nu} = \pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}.\tag{2.3}$$

We recall the definition of Q_i as

$$Q_i = q_i - \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^i q_i.\tag{2.4}$$

This may be alternatively expressed as

$$Q_i = -T_{0i} + \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^j T_{0j}\tag{2.5}$$

Noting that π_{ij} is already traceless by construction, we may project out its transverse part and define $\pi_{ij}^{T\theta}$ as

$$\begin{aligned}\pi_{ij}^{T\theta} &= \pi_{ij} - \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^k \pi_{jk} - \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}^k \pi_{ik} \\ &\quad + \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}_k \int d^3z D(y-z) \tilde{\nabla}_l \pi^{kl}.\end{aligned}\tag{2.6}$$

Substituting in $\pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}$, we have

$$\begin{aligned}\pi_{ij}^{T\theta} &= \left(T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}\right) - \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^k \left(T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}T_{mn}\right) \\ &\quad - \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}^k \left(T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}T_{mn}\right) \\ &\quad + \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}_k \int d^3z D(y-z) \tilde{\nabla}_l \left(T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}T_{mn}\right).\end{aligned}\quad (2.7)$$

In total, we may express relations (19) explicitly in terms of the components of the tensors as the following:

$$\bar{\rho} - \rho = \delta W_{00} - \delta T_{00} \quad (2.8)$$

$$\bar{Q}_i - Q_i = -(\delta W_{0i} - \delta T_{0i}) + \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^j (\delta W_{0j} - \delta T_{0j}) \quad (2.9)$$

$$\begin{aligned}\bar{\pi}_{ij}^{T\theta} - \pi_{ij}^{T\theta} &= \left[\delta W_{ij} - \delta T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}(\delta W_{kl} - \delta T_{kl})\right] \\ &\quad - \tilde{\nabla}_i \int d^3y D(x-y) \tilde{\nabla}^k \left[\delta W_{jk} - \delta T_{jk} - \frac{1}{3}\delta_{jk}\delta^{mn}(\delta W_{mn} - \delta T_{mn})\right] \\ &\quad - \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}^k \left[\delta W_{ik} - \delta T_{ik} - \frac{1}{3}\delta_{ik}\delta^{mn}(\delta W_{mn} - \delta T_{mn})\right] \\ &\quad + \tilde{\nabla}_i \tilde{\nabla}_j \int d^3y D(x-y) \tilde{\nabla}_k \int d^3z D(y-z) \tilde{\nabla}_l \left[\delta W_{kl} - \delta T^{kl} - \frac{1}{3}\delta^{kl}\delta^{mn}(\delta W_{mn} - \delta T_{mn})\right].\end{aligned}\quad (2.10)$$

$$h_{\mu\nu}^T = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \nabla_\mu \nabla_\nu \int D \nabla^\sigma W_\sigma \quad (2.11)$$

$$W_\mu = \int D \nabla^\sigma h_{\sigma\mu} \quad (2.12)$$

$$h_{\mu\nu}^{T\theta} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu \nabla_\nu \int D(h + \nabla^\sigma W_\sigma) \quad (2.13)$$

$$\begin{aligned}h_{\mu\nu} &= \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu \nabla_\nu \int D(h + \nabla^\sigma W_\sigma)\right]}_{h_{\mu\nu}^{T\theta}} \\ &\quad + \nabla_\mu W_\nu + \nabla_\nu W_\mu + \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) - \frac{1}{2}\nabla_\mu \nabla_\nu \int D(h + \nabla^\sigma W_\sigma)\end{aligned}\quad (2.14)$$

if $h = 0$ then

$$\begin{aligned}h_{\mu\nu} &= \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu + \frac{1}{2}g_{\mu\nu}\nabla^\sigma W_\sigma + \frac{1}{2}\nabla_\mu \nabla_\nu \int D \nabla^\sigma W_\sigma\right]}_{h_{\mu\nu}^{T\theta}} \\ &\quad + \underbrace{\nabla_\mu \left(W_\nu - \nabla_\nu \int D \nabla^\sigma W_\sigma\right)}_{V_\nu^T} + \underbrace{\nabla_\nu \left(W_\mu - \nabla_\mu \int D \nabla^\sigma W_\sigma\right)}_{V_\mu^T} \\ &\quad + \underbrace{\nabla_\mu \nabla_\nu \left(\frac{3}{2} \int D \nabla^\sigma W_\sigma\right)}_{2V} - \underbrace{g_{\mu\nu} \left(\frac{1}{2} \nabla^\sigma W_\sigma\right)}_{\frac{2}{3} \nabla^\alpha \nabla_\alpha V}\end{aligned}\quad (2.15)$$

$$h_{\mu\nu} = g_{\mu\nu}p + h_{\mu\nu}^{T\theta} + \nabla_\mu V_\nu^T + \nabla_\nu V_\mu^T + 2\nabla_\mu \nabla_\nu V - \frac{2}{3}g_{\mu\nu}\nabla_\alpha \nabla^\alpha V \quad (2.16)$$

$$V_\mu = W_\mu - \frac{1}{4}\nabla_\mu \int D\nabla^\sigma W_\sigma \quad (2.17)$$

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{3}\nabla^\sigma V_\sigma = \nabla_\mu W_\nu + \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}\nabla^\sigma W_\sigma - \frac{1}{2}\nabla_\mu \nabla_\nu \int D\nabla^\sigma W_\sigma \quad (2.18)$$

$$g_{\mu\nu} = P_{\mu\nu} - U_\mu U_\nu, \quad P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \quad (2.19)$$

For a little more simplicity, we assume a $T_{\mu\nu}$ that is symmetric.

$$\begin{aligned} T_{\mu\nu} &= g_\mu{}^\rho g_\nu{}^\sigma T_{\rho\sigma} \\ &= (P_\mu{}^\rho - U_\mu U^\rho)(P_\nu{}^\sigma - U_\nu U^\sigma)T_{\rho\sigma} \\ &= (P_\mu{}^\rho P_\nu{}^\sigma - P_\mu{}^\rho U_\nu U^\sigma - P_\nu{}^\sigma U_\mu U^\rho + U_\mu U_\nu U^\rho U^\sigma)T_{\rho\sigma} \\ &= U_\mu U_\nu \underbrace{U^\rho U^\sigma T_{\rho\sigma}}_\rho + U_\mu U_\nu \underbrace{\left(\frac{1}{3}P^{\rho\sigma}T_{\rho\sigma}\right)}_p + \underbrace{(P_{\mu\nu} - U_\mu U_\nu)}_{g_{\mu\nu}} \underbrace{\left(\frac{1}{3}P^{\rho\sigma}T_{\rho\sigma}\right)}_p \\ &\quad - U_\mu \underbrace{P_\nu{}^\sigma U^\rho T_{\rho\sigma}}_{-q_\nu} - U_\nu \underbrace{P_\mu{}^\rho U^\sigma T_{\rho\sigma}}_{-q_\mu} \\ &\quad \underbrace{\left(P_\mu{}^\rho P_\nu{}^\sigma - \frac{1}{3}P_{\mu\nu}P^{\rho\sigma}\right)T_{\rho\sigma}}_{\pi_{\mu\nu}} \end{aligned} \quad (2.20)$$

$$\begin{aligned} U^\rho U^\sigma T_{\rho\sigma} &= T_{00} = \rho \\ \frac{1}{3}P^{\rho\sigma}T_{\rho\sigma} &= \frac{1}{3}g^{ij}T_{ij} = p \\ -U_\mu P_\nu{}^\sigma U^\rho T_{\rho\sigma} &= S_{\mu\nu} = T_{0i} = U_\mu q_\nu \\ -U_\nu P_\mu{}^\sigma U^\rho T_{\rho\sigma} &= S_{\mu\nu} = T_{i0} = U_\nu q_\mu \\ (P_\mu{}^\rho P_\nu{}^\sigma - \frac{1}{3}P_{\mu\nu}P^{\rho\sigma})T_{\rho\sigma} &= T_{ij} - \frac{1}{3}g_{ij}g^{kl}T_{kl} = \pi_{ij} \end{aligned} \quad (2.21)$$

Flat Einstein

$$\begin{aligned}
\delta G_{\mu\nu} &= \frac{1}{2}\nabla_\beta\nabla^\beta h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla_\beta\nabla^\beta h - \frac{1}{2}\nabla_\beta\nabla_\mu h_\nu{}^\beta - \frac{1}{2}\nabla_\beta\nabla_\nu h_\mu{}^\beta + \frac{1}{2}g_{\mu\nu}\nabla_\zeta\nabla_\beta h^{\beta\zeta} + \frac{1}{2}\nabla_\nu\nabla_\mu h \\
&= \left[\frac{1}{2}g^\sigma{}_\mu g^\rho{}_\nu \nabla_\alpha \nabla^\alpha - \frac{1}{2}g_{\mu\nu}g^{\sigma\rho}\nabla_\alpha \nabla^\alpha - \frac{1}{2}g^\rho{}_\nu \nabla^\sigma \nabla_\mu - \frac{1}{2}g^\sigma{}_\mu \nabla^\rho \nabla_\nu \right. \\
&\quad \left. + \frac{1}{2}g_{\mu\nu}\nabla^\sigma \nabla^\rho + \frac{1}{2}g^{\sigma\rho}\nabla_\mu \nabla_\nu \right] h_{\sigma\rho} \\
&= \hat{\mathcal{L}}^{\sigma\rho}{}_{\mu\nu} h_{\sigma\rho}
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\delta G_{00} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{00} h_{\sigma\rho} \\
&= \frac{1}{2}\nabla_a \nabla^a (h_{00} + h) - \frac{1}{2}\nabla^a \nabla^b h_{ab} \\
&= \nabla^2 p - \frac{2}{3}\nabla^4 V \\
\delta G_{0i} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{0i} h_{\sigma\rho} \\
&= -\frac{1}{2}\nabla^a \dot{h}_{ia} + \frac{1}{2}\nabla^2 h_{0i} + \frac{1}{2}\nabla_i \dot{h}_{00} + \frac{1}{2}\nabla_i \dot{h} - \frac{1}{2}\nabla_i \nabla^a h_{0a} \\
&= \nabla_i \left(\dot{p} - \frac{2}{3}\nabla^2 \dot{V} \right) + \frac{1}{2}\nabla^2 (Q_i - \dot{V}_i) \\
\delta G_{ij} &= \hat{\mathcal{L}}^{\sigma\rho}{}_{ij} h_{\sigma\rho} \\
&= \frac{1}{2}(\nabla^2 - \partial_t^2)h_{ij} + \frac{1}{2}g_{ij}(\ddot{h}_{00} + \ddot{h}) + \frac{1}{2}g_{ij}\nabla^2 h + \frac{1}{2}\nabla_i \nabla_j h - g_{ij}\nabla^a \dot{h}_{0a} \\
&\quad + \frac{1}{2}g_{ij}\nabla^a \nabla^b h_{ab} + \frac{1}{2}(\nabla_i \dot{h}_{j0} + \nabla_j \dot{h}_{i0}) - \frac{1}{2}(\nabla_i \nabla^a h_{ja} + \nabla_j \nabla^a h_{ia}) \\
&= g_{ij}(\ddot{p} - \frac{2}{3}\nabla^2 \ddot{V}) - \frac{1}{2}g_{ij}\nabla^2(p - \frac{2}{3}\nabla^2 V) + \frac{1}{2}g_{ij}\nabla^2(\rho - 2\dot{Q} + 2\ddot{V}) \\
&\quad + \frac{1}{2}\nabla_i \nabla_j(p - \frac{2}{3}\nabla^2 V) - \frac{1}{2}\nabla_i \nabla_j(\rho - 2\dot{Q} + 2\ddot{V}) \\
&\quad + \frac{1}{2}\nabla_i(\dot{Q}_j - \ddot{V}_j) + \frac{1}{2}\nabla_j(\dot{Q}_i - \ddot{V}_i) - \ddot{V}_{ij} + \nabla^2 V_{ij}
\end{aligned} \tag{2.23}$$

$$\ddot{p}g_{ij} + \frac{1}{3}g_{ij}\nabla_a \nabla^a \ddot{V} - g_{ij}\nabla_a \nabla^a \dot{Q} - \frac{1}{2}g_{ij}\nabla_a \nabla^a p + \frac{1}{2}g_{ij}\nabla_a \nabla^a \rho \tag{2.24}$$

$$+ \frac{1}{3}g_{ij}\nabla_b \nabla^b \nabla_a \nabla^a V - \nabla_j \nabla_i \ddot{V} + \nabla_j \nabla_i \dot{Q} + \frac{1}{2}\nabla_j \nabla_i p - \frac{1}{2}\nabla_j \nabla_i \rho - \frac{1}{3}\nabla_j \nabla_i \nabla_a \nabla^a V \tag{2.25}$$

Bianchi Identity $\nabla^\mu \delta G_{\mu\nu} = 0$

$$\begin{aligned}
\dot{\rho} &= \nabla^2 Q \\
\dot{Q}_i + \nabla_i \dot{Q} &= \nabla_i p + \frac{4}{3}\nabla^2 \nabla_i V + \nabla^2 V_i
\end{aligned} \tag{2.26}$$

Conformal Part

$$\begin{aligned}
\delta G_{\mu\nu}^{\Omega(x)} &= -2h_{\mu\nu}\Omega^{-1}\nabla_\alpha \nabla^\alpha \Omega - g_{\mu\nu}\Omega^{-1}\nabla_\alpha \Omega \nabla^\alpha h + \Omega^{-1}\nabla_\alpha h_{\mu\nu} \nabla^\alpha \Omega + h_{\mu\nu}\Omega^{-2}\nabla_\alpha \Omega \nabla^\alpha \Omega + 2g_{\mu\nu}\Omega^{-1}\nabla^\alpha \Omega \nabla_\beta h_\alpha{}^\beta \\
&\quad - g_{\mu\nu}h_{\alpha\beta}\Omega^{-2}\nabla^\alpha \Omega \nabla^\beta \Omega + 2g_{\mu\nu}h_{\alpha\beta}\Omega^{-1}\nabla^\beta \nabla^\alpha \Omega - \Omega^{-1}\nabla^\alpha \Omega \nabla_\mu h_{\nu\alpha} - \Omega^{-1}\nabla^\alpha \Omega \nabla_\nu h_{\mu\alpha} \\
&= \Omega^{-1} \left[-2g^\sigma{}_\mu g^\rho{}_\nu \nabla_\alpha \nabla^\alpha \Omega - g_{\mu\nu}g^{\sigma\rho}\nabla_\alpha \Omega \nabla^\alpha + g^\sigma{}_\mu g^\rho{}_\nu \nabla_\alpha \Omega \nabla^\alpha + 2g_{\mu\nu}\nabla^\sigma \Omega \nabla^\rho + 2g_{\mu\nu}\nabla^\sigma \nabla^\rho \right. \\
&\quad \left. - g^\rho{}_\nu \nabla^\sigma \Omega \nabla_\mu - g^\sigma{}_\mu \nabla^\rho \Omega \nabla_\nu \right] + \Omega^{-2} \left[g^\sigma{}_\mu g^\rho{}_\nu \nabla_\alpha \Omega \nabla^\alpha \Omega - g_{\mu\nu}\nabla^\rho \Omega \nabla^\sigma \Omega \right] h_{\sigma\rho} \\
&= \hat{\mathcal{J}}^{\sigma\rho}{}_{\mu\nu} h_{\sigma\rho}
\end{aligned} \tag{2.27}$$

3 Gauge Invariance

Under $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$

$$\Delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \quad (3.1)$$

where

$$\epsilon_0 = -T, \quad \epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j \epsilon_j}_{L_i} + \underbrace{\nabla_i \int D\nabla^j \epsilon_j}_L \quad (3.2)$$

$$\begin{aligned} \Delta_\epsilon g_{00} &= -2\dot{T} \\ \Delta_\epsilon g_{0i} &= -\nabla_i T + \dot{L}_i + \nabla_i \dot{L} \\ \Delta_\epsilon g_{ij} &= 2\nabla_i \nabla_j L + \nabla_i L_j + \nabla_j L_i \end{aligned} \quad (3.3)$$

$$\begin{aligned} \bar{\rho} &= \rho - 2\dot{T} \\ \bar{Q}_i + \nabla_i \bar{Q} &= Q_i + \nabla_i Q + \nabla_i (\dot{L} - T) + \dot{L}_i \\ g_{ij} \bar{p} + 2\nabla_i \nabla_j \bar{V} - \frac{2}{3} g_{ij} \nabla^2 \bar{V} + \nabla_i \bar{V}_j + \nabla_j \bar{V}_i + \bar{V}_{ij} &= g_{ij} p + 2\nabla_i \nabla_j V - \frac{2}{3} g_{ij} \nabla^2 V + \nabla_i V_j + \nabla_j V_i + V_{ij} \\ &\quad + 2\nabla_i \nabla_j L + \nabla_i L_j + \nabla_j L_i \end{aligned} \quad (3.4)$$

$$\begin{aligned} \bar{\rho} &= \rho - 2\dot{T} \\ \bar{Q} &= Q + \dot{L} - T \\ \bar{Q}_i &= Q_i + \dot{L}_i \\ \bar{V}_i &= V_i + L_i \\ \bar{p} &= p + \frac{2}{3} \nabla^2 L \\ \bar{V} &= V + L \\ \bar{V}_{ij} &= V_{ij} \end{aligned} \quad (3.5)$$

$$\rho - 2\dot{Q} + 2\ddot{V}, \quad p - \frac{2}{3} \nabla^2 V, \quad Q_i - \dot{V}_i, \quad V_{ij} \quad (3.6)$$

$$\begin{aligned} \underbrace{\bar{h}_{00}}_{\bar{\rho}} &= \underbrace{h_{00}}_{\rho} - 2\dot{T} \\ \underbrace{\int D\nabla^j \bar{h}_{0j}}_{\bar{Q}} &= \underbrace{\int D\nabla^j h_{0j}}_Q + \underbrace{\int D\nabla^2 (\dot{L} - T)}_{(\dot{L}-T)^L} \\ \underbrace{\bar{h}_{0i} - \nabla_i \int D\nabla^j \bar{h}_{0j}}_{\bar{Q}_i} &= \underbrace{h_{0i} - \nabla_i \int D\nabla^j h_{0j}}_{Q_i} + \underbrace{\dot{L}_i + \nabla_i (\dot{L} - T) - \nabla_i \int D\nabla^2 (\dot{L} - T)}_{\nabla_i (\dot{L}-T)^T} \\ \underbrace{g^{ij} \bar{h}_{ij}}_{3\bar{p}} &= \underbrace{g^{ij} h_{ij}}_{3p} + 2\nabla^2 L \\ \underbrace{\frac{3}{4} \int D\nabla^k \bar{W}_k}_{\bar{V}} &= \underbrace{\frac{3}{4} \int D\nabla^k W_k}_V + \frac{3}{4} \int D\nabla^k \int D\nabla^2 (\frac{4}{3} \nabla_k L + L_k) \\ \underbrace{\bar{W}_i - \nabla_i \int D\nabla^j \bar{W}_j}_{\bar{V}_i} &= \underbrace{W_i - \nabla_i \int D\nabla^j W_j}_{V_i} + \int D\nabla^2 (\frac{4}{3} \nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2 (\frac{4}{3} \nabla_j L + L_j) \end{aligned} \quad (3.7)$$

$$\begin{aligned}
\Delta_\epsilon k_{ij} &= 2\nabla_i \nabla_j L - \frac{2}{3} g_{ij} \nabla^2 L + \nabla_i L_j + \nabla_j L_i \\
\Delta_\epsilon \nabla^j k_{ij} &= \nabla^2 \left(\frac{4}{3} \nabla_i L + L_i \right) \\
\Delta_\epsilon W_i &= \int D\nabla^j k_{ij} = \int D\nabla^2 \left(\frac{4}{3} \nabla_i L + L_i \right)
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\bar{V}_{ij} - V_{ij} &= \Delta_\epsilon \left[k_{ij} - \nabla_i W_j - \nabla_j W_i + \frac{1}{2} g_{ij} \nabla^k W_k + \frac{1}{2} \nabla_i \nabla_j \int D\nabla^k W_k \right] \\
&= 2\nabla_i \nabla_j L - \frac{2}{3} g_{ij} \nabla^2 L + \nabla_i L_j + \nabla_j L_i - \nabla_i \int D\nabla^2 \left(\frac{4}{3} \nabla_j L + L_j \right) - \nabla_j \int D\nabla^2 \left(\frac{4}{3} \nabla_i L + L_i \right) \\
&\quad + \frac{1}{2} g_{ij} \nabla^k \int D\nabla^2 \left(\frac{4}{3} \nabla_k L + L_k \right) + \frac{1}{2} \nabla_i \nabla_j \int D\nabla^k \int D\nabla^2 \left(\frac{4}{3} \nabla_k L + L_k \right)
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\bar{\rho} &= \rho - 2\dot{T} \\
\bar{p} &= p + \frac{2}{3} \nabla^2 L \\
\bar{Q} &= Q + \int D\nabla^2 (\dot{L} - T) \\
\bar{Q}_i &= Q_i + \dot{L}_i + \nabla_i (\dot{L} - T) - \nabla_i \int D\nabla^2 (\dot{L} - T) \\
\bar{V} &= V + \frac{3}{4} \int D\nabla^k \int D\nabla^2 \left(\frac{4}{3} \nabla_k L + L_k \right) \\
\bar{V}_i &= V_i + \int D\nabla^2 \left(\frac{4}{3} \nabla_i L + L_i \right) - \nabla_i \int D\nabla^j \int D\nabla^2 \left(\frac{4}{3} \nabla_j L + L_j \right)
\end{aligned} \tag{3.10}$$

Appendix A Bach Tensor

$$\begin{aligned}
W_{\mu\nu} &= -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R_\mu{}^\alpha R_{\nu\alpha} \\
&\quad -\frac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \nabla_\alpha\nabla^\alpha R_{\mu\nu} + 2\nabla_\beta\nabla_\alpha R_\mu{}^\alpha{}_\nu{}^\beta + \frac{2}{3}\nabla_\nu\nabla_\mu R \\
&= -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \frac{1}{3}\nabla_\nu\nabla_\mu R \\
&= -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R_\mu{}^\alpha R_{\nu\alpha} \\
&\quad -\frac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \nabla_\alpha\nabla_\mu R_\nu{}^\alpha - \nabla_\alpha\nabla_\nu R_\mu{}^\alpha + \frac{2}{3}\nabla_\nu\nabla_\mu R \\
\nabla_\alpha\nabla^\alpha G_{\mu\nu}^{T\theta} &= \nabla_\alpha\nabla^\alpha G_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\nabla_\rho\nabla^\rho g^{\alpha\beta}G_{\alpha\beta} + \frac{1}{3}\nabla_\mu\nabla_\nu g^{\alpha\beta}G_{\alpha\beta} \\
&= \nabla_\alpha\nabla^\alpha R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^\alpha R - \frac{1}{3}\nabla_\mu\nabla_\nu R \\
\text{remaining} &= -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\
&= -\frac{1}{6}g_{\mu\nu}R^2 + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}RR_{\mu\nu} - 2R_\mu{}^\alpha R_{\nu\alpha} \\
&\quad -\nabla_\alpha\nabla_\mu R_\nu{}^\alpha - \nabla_\alpha\nabla_\nu R_\mu{}^\alpha + \nabla_\mu\nabla_\nu R
\end{aligned} \tag{A.1}$$

Appendix B Einstein Related to Weyl

$$\begin{aligned}
I_G &= \int d^4x \sqrt{g} (G_{\mu\nu}G^{\mu\nu} - \frac{1}{3}(g^{\alpha\beta}G_{\alpha\beta})^2) \\
&= I_{G_2} - \frac{1}{3}I_{G_1} = \int d^4x \sqrt{g} G_{\mu\nu}G^{\mu\nu} - \frac{1}{3} \int d^4x \sqrt{g} (g^{\alpha\beta}G_{\alpha\beta})^2
\end{aligned} \tag{B.1}$$

Recalling

$$\delta(\sqrt{g}) = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \tag{B.2}$$

we have

$$\begin{aligned}
W_{(2)}^{\mu\nu} &= \frac{1}{\sqrt{g}} \frac{\delta I_{G_2}}{\delta g_{\mu\nu}} = \frac{1}{2}g^{\mu\nu}G_{\alpha\beta}G^{\alpha\beta} + \frac{\delta}{\delta g_{\mu\nu}} (G_{\alpha\beta}G^{\alpha\beta}) \\
\frac{\delta}{\delta g_{\mu\nu}} (G_{\alpha\beta}G^{\alpha\beta}) &= -2\delta g_{\mu\nu}G^\mu{}_\alpha G^{\nu\alpha} + 2\delta G_{\mu\nu}G^{\mu\nu}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
2G^{\mu\nu}\delta G_{\mu\nu} &= 2G^{\mu\nu}(\delta R_{\mu\nu} - \frac{1}{2}\delta g_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}\delta g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\delta R_{\alpha\beta}) \\
&= 2\underbrace{(G^{\mu\nu} - \frac{1}{2}g_{\alpha\beta}G^{\alpha\beta}g^{\mu\nu})}_{\bar{G}^{\mu\nu}}\delta R_{\mu\nu} + (g_{\alpha\beta}G^{\alpha\beta}R^{\mu\nu} - G^{\mu\nu}R)\delta g_{\mu\nu}
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\delta R_{\mu\nu} &= \nabla_\nu\delta\Gamma_{\mu\lambda}^\lambda - \nabla_\lambda\delta\Gamma_{\mu\nu}^\lambda \\
\delta\Gamma_{\mu\nu}^\lambda &= \frac{1}{2}g^{\lambda\rho}[\nabla_\nu\delta g_{\mu\rho} + \nabla_\mu\delta g_{\nu\rho} - \nabla_\rho\delta g_{\mu\nu}]
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
2\bar{G}^{\mu\nu}\delta R_{\mu\nu} &= 2[\nabla_\nu(\bar{G}^{\mu\nu}\delta\Gamma_{\mu\lambda}^\lambda) - \nabla_\nu\bar{G}^{\mu\nu}\delta\Gamma_{\mu\lambda}^\lambda - \nabla_\lambda(\bar{G}^{\mu\nu}\delta\Gamma_{\mu\nu}^\lambda) + \nabla_\lambda\bar{G}^{\mu\nu}\delta\Gamma_{\mu\nu}^\lambda] \\
&= \nabla_\alpha\nabla^\mu\bar{G}^{\alpha\nu} - \nabla_\alpha\nabla^\alpha\bar{G}^{\mu\nu} - \nabla^\nu\nabla_\alpha\bar{G}^{\mu\alpha} + \nabla^\mu\nabla_\alpha\bar{G}^{\nu\alpha} \\
&= (g^{\mu\nu}\nabla_\alpha\nabla_\beta\bar{G}^{\alpha\beta} - \nabla_\alpha\nabla^\nu\bar{G}^{\mu\alpha} - \nabla_\alpha\nabla^\mu\bar{G}^{\alpha\nu} + \nabla_\alpha\nabla^\alpha\bar{G}^{\mu\nu})\delta g_{\mu\nu}
\end{aligned} \tag{B.6}$$

Hence we have for $W_{(2)}^{\mu\nu}$

$$\begin{aligned}
W_{(2)}^{\mu\nu} &= \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{g} G^{\mu\nu} G_{\mu\nu} \\
&= \int d^4x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta} - 2G^\mu{}_\alpha G^{\nu\alpha} + g_{\alpha\beta} G^{\alpha\beta} R^{\mu\nu} - G^{\mu\nu} R \right. \\
&\quad \left. + g^{\mu\nu} \nabla_\alpha \nabla^\beta \bar{G}^{\alpha\beta} - \nabla_\alpha \nabla^\nu \bar{G}^{\mu\alpha} - \nabla_\alpha \nabla^\mu \bar{G}^{\alpha\nu} + \nabla_\alpha \nabla^\alpha \bar{G}^{\mu\nu} \right) \delta g_{\mu\nu} \\
&= \frac{1}{2} g^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta} - 2G^\mu{}_\alpha G^{\nu\alpha} + g_{\alpha\beta} G^{\alpha\beta} R^{\mu\nu} - G^{\mu\nu} R - g^{\mu\nu} \nabla_\alpha \nabla^\alpha G + \nabla^\mu \nabla^\nu G \\
&\quad + \nabla_\alpha \nabla^\alpha G^{\mu\nu} - \nabla_\alpha \nabla^\mu G^{\alpha\nu} - \nabla_\alpha \nabla^\nu G^{\mu\alpha}
\end{aligned} \tag{B.7}$$

$$\delta(g^{\alpha\beta} G_{\alpha\beta})^2 = -2g^{\alpha\beta} G_{\alpha\beta} G^{\mu\nu} \delta g_{\mu\nu} + 2g^{\alpha\beta} G_{\alpha\beta} g^{\mu\nu} \delta G_{\mu\nu} \tag{B.8}$$

$$2Gg^{\mu\nu} \delta G_{\mu\nu} = -2Gg^{\mu\nu} \delta R_{\mu\nu} + (4GR^{\mu\nu} - Gg^{\mu\nu} R) \delta g_{\mu\nu} \tag{B.9}$$

$$-2Gg^{\mu\nu} \delta R_{\mu\nu} = (-2\nabla_\alpha \nabla^\alpha G + 2\nabla^\mu \nabla^\nu G) \delta g_{\mu\nu} \tag{B.10}$$

And for $W_{(1)}^{\mu\nu}$

$$\begin{aligned}
W_{(1)}^{\mu\nu} &= \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{g} (g^{\alpha\beta} G_{\alpha\beta})^2 \\
&= \int d^4x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} G^2 - 2GG^{\mu\nu} + 4GR^{\mu\nu} - g^{\mu\nu} GR - 2\nabla_\alpha \nabla^\alpha G + 2\nabla^\mu \nabla^\nu G \right) \delta g_{\mu\nu}
\end{aligned} \tag{B.11}$$

$$W_{(1)}^{\mu\nu} = \frac{1}{2} g^{\mu\nu} G^2 - 2GG^{\mu\nu} + 4GR^{\mu\nu} - g^{\mu\nu} GR - 2\nabla_\alpha \nabla^\alpha G + 2\nabla^\mu \nabla^\nu G \tag{B.12}$$

$$\begin{aligned}
W_{(2)}^{\mu\nu} &= \frac{1}{2} g^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta} - 2G^\mu{}_\alpha G^{\nu\alpha} + g_{\alpha\beta} G^{\alpha\beta} R^{\mu\nu} - G^{\mu\nu} R - g^{\mu\nu} \nabla_\alpha \nabla^\alpha G + \nabla^\mu \nabla^\nu G \\
&\quad + \nabla_\alpha \nabla^\alpha G^{\mu\nu} - \nabla_\alpha \nabla^\mu G^{\alpha\nu} - \nabla_\alpha \nabla^\nu G^{\mu\alpha}
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
W_{\mu\nu}^{(2)} - \frac{1}{3} W_{\mu\nu}^{(1)} &= \frac{1}{2} g_{\mu\nu} G^{\alpha\beta} G_{\alpha\beta} - 2G_{\mu\alpha} G_\nu{}^\alpha - \frac{1}{3} GR_{\mu\nu} - RG_{\mu\nu} + \frac{2}{3} GG_{\mu\nu} - \frac{1}{6} g_{\mu\nu} G^2 \\
&\quad - \frac{1}{3} g_{\mu\nu} \nabla_\alpha \nabla^\alpha G + \frac{1}{3} \nabla_\mu \nabla_\nu G + \nabla_\alpha \nabla^\alpha G_{\mu\nu} - \nabla^\alpha \nabla_\mu G_{\alpha\nu} - \nabla^\alpha \nabla_\nu G_{\mu\alpha}
\end{aligned} \tag{B.14}$$

$$h_{\mu\nu}^{T\theta} = h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2} g_{\mu\nu} (h - \nabla^\sigma W_\sigma) + \frac{1}{2} \left[\nabla_\mu \nabla_\nu - \frac{1}{6} R g_{\mu\nu} \right] \int D(h + \nabla^\sigma W_\sigma) \tag{B.15}$$

$$\nabla^2 G_{\mu\nu}^{T\theta} = \nabla^2 G_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \nabla^2 G + \frac{1}{3} \nabla^2 \left[\nabla_\mu \nabla_\nu - \frac{1}{12} R g_{\mu\nu} \right] \int DG \tag{B.16}$$

$$\begin{aligned}
\nabla^\mu W_{\mu\nu} &= G^{\alpha\beta} \nabla_\nu G_{\alpha\beta} - 2G_{\mu\alpha} \nabla^\mu G_\nu{}^\alpha - \frac{1}{3} \nabla^\mu GR_{\mu\nu} - \frac{1}{6} G \nabla_\nu R - \nabla^\mu RG_{\mu\nu} + \frac{2}{3} \nabla^\mu GG_{\mu\nu} \\
&\quad - \frac{1}{3} G \nabla_\nu G - \frac{1}{3} \nabla^\mu \nabla^2 G + \frac{1}{3} \nabla^2 \nabla_\nu G + \nabla^\mu \nabla^2 G_{\mu\nu} - \nabla^\mu \nabla^\alpha \nabla_\mu G_{\alpha\nu} - \nabla^\mu \nabla^\alpha \nabla_\nu G_{\mu\alpha}
\end{aligned} \tag{B.17}$$

If $R = \text{const}$

$$\nabla^\mu W_{\mu\nu} = G^{\alpha\beta} \nabla_\nu G_{\alpha\beta} - 2G_{\mu\alpha} \nabla^\mu G_\nu{}^\alpha + \nabla^\mu \nabla^2 G_{\mu\nu} - \nabla^\mu \nabla^\alpha \nabla_\mu G_{\alpha\nu} - \nabla^\mu \nabla^\alpha \nabla_\nu G_{\mu\alpha} \tag{B.18}$$

Appendix C SVT Traditional Form

Applied to 3 dimensions.

$$\begin{aligned}
h_{\mu\nu} = & \underbrace{\left[h_{\mu\nu} - \nabla_\mu W_\nu - \nabla_\nu W_\mu - \frac{1}{2}g_{\mu\nu}(h - \nabla^\sigma W_\sigma) + \frac{1}{2}\nabla_\mu \nabla_\nu \int D(h + \nabla^\sigma W_\sigma) \right]}_{2E_{\mu\nu}^{\mathcal{T}\theta}} \\
& + \underbrace{\nabla_\mu \left(W_\nu - \nabla_\nu \int D\nabla^\sigma W_\sigma \right)}_{E_\nu} + \underbrace{\nabla_\nu \left(W_\mu - \nabla_\mu \int D\nabla^\sigma W_\sigma \right)}_{E_\mu} \\
& - 2g_{\mu\nu} \underbrace{\left(\frac{1}{4}\nabla^\sigma W_\sigma - \frac{1}{4}h \right)}_{\psi} + 2\underbrace{\nabla_\mu \nabla_\nu \int D\left(\frac{3}{4}\nabla^\sigma W_\sigma - \frac{1}{4}h \right)}_E
\end{aligned} \tag{C.1}$$

C.1 Gauge Invariance

Under $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$

$$\Delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \tag{C.2}$$

where

$$\epsilon_0 = -T, \quad \epsilon_i = \underbrace{\epsilon_i - \nabla_i \int D\nabla^j \epsilon_j}_{L_i} + \underbrace{\nabla_i \int D\nabla^j \epsilon_j}_L \tag{C.3}$$

$$\begin{aligned}
\Delta_\epsilon h_{00} &= -2\dot{T} \\
\Delta_\epsilon h_{0i} &= -\nabla_i T + \dot{L}_i + \nabla_i \dot{L} \\
\Delta_\epsilon h_{ij} &= 2\nabla_i \nabla_j L + \nabla_i L_j + \nabla_j L_i \\
\Delta_\epsilon (\nabla^j h_{ij}) &= 2\nabla^2 \nabla_i L + \nabla^2 L_i \\
\Delta_\epsilon W_i &= \int D\nabla^2 (2\nabla_i L + L_i) \\
\Delta_\epsilon (g^{ij} h_{ij}) &= 2\nabla^2 L
\end{aligned} \tag{C.4}$$

$$\Delta_\epsilon \tag{C.5}$$

$$\begin{aligned}
\frac{\bar{h}_{00}}{-2\bar{\phi}} &= \frac{h_{00}}{-2\phi} - 2\dot{T} \\
\underbrace{\int D\nabla^j \bar{h}_{0j}}_{\bar{B}} &= \underbrace{\int D\nabla^j h_{0j}}_B + \underbrace{\int D\nabla^2(\dot{L} - T)}_{(\dot{L}-T)^L} \\
\underbrace{\bar{h}_{0i} - \nabla_i \int D\nabla^j \bar{h}_{0j}}_{\bar{B}_i} &= \underbrace{h_{0i} - \nabla_i \int D\nabla^j h_{0j}}_{B_i} + \underbrace{\dot{L}_i + \nabla_i(\dot{L} - T) - \nabla_i \int D\nabla^2(\dot{L} - T)}_{\nabla_i(\dot{L}-T)^T} \\
\underbrace{\frac{1}{4}\nabla^i \bar{W}_i - \frac{1}{4}g^{ij}\bar{h}_{ij}}_{\bar{\psi}} &= \underbrace{\frac{1}{4}\nabla^i W_i - \frac{1}{4}g^{ij}h_{ij}}_{\psi} - \frac{1}{2}\nabla^2 L + \frac{1}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) \\
\underbrace{\int D(\frac{3}{4}\nabla^i \bar{W}_i - \frac{1}{4}g^{ij}\bar{h}_{ij})}_{\bar{E}} &= \underbrace{\int D(\frac{3}{4}\nabla^i W_i - \frac{1}{4}g^{ij}h_{ij})}_E + \int D\left(\frac{3}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) - \frac{1}{2}\nabla^2 L\right) \\
\underbrace{\bar{W}_i - \nabla_i \int D\nabla^j \bar{W}_j}_{\bar{E}_i} &= \underbrace{W_i - \nabla_i \int D\nabla^j W_j}_{E_i} + \int D\nabla^2(\frac{4}{3}\nabla_i L + L_i) - \nabla_i \int D\nabla^j \int D\nabla^2(\frac{4}{3}\nabla_j L + L_j) \\
2\bar{E}_{ij} - 2E_{ij} &= 2\nabla_i \nabla_j L + \nabla_i L_j + \nabla_j L_i - \nabla_i \int D\nabla^2(2\nabla_j L + L_j) - \nabla_j \int D\nabla^2(2\nabla_i L + L_i) \\
&\quad - \frac{1}{2}g_{ij}\left(2\nabla^2 L - \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right) \\
&\quad + \frac{1}{2}\nabla_i \nabla_j \int D\left(2\nabla^2 L + \nabla^k \int D\nabla^2(2\nabla_k L + L_k)\right)
\end{aligned} \tag{C.6}$$

From integrating the identity

$$\nabla^2 D\phi = D\nabla^2\phi + \nabla_i (\nabla^i \phi D - \nabla^i D\phi), \tag{C.7}$$

we may decompose a general scalar ϕ into its harmonic (T) and non-harmonic (L) pieces viz

$$\phi = \underbrace{\int_V D\nabla^2\phi}_{\phi^L} + \underbrace{\oint_{\partial V} dS_i (D\nabla^i \phi - \nabla^i D\phi)}_{\phi^T}. \tag{C.8}$$

The harmonic ϕ^T is defined only upon the boundary surface with $\nabla^2\phi^T$ vanishing identically for any ϕ and with $\nabla^2\phi^L = 0$ only vanishing for $\phi^L = 0$. From (C.8) we see that if we require

1. $\phi(x) = 0$ for $x \in \partial V$
2. $\nabla_i D(x, y) = 0$ for $x \in \partial V$

then we may always use $\phi = \int D\nabla^2\phi$. By definition of the Green's function equation

$$\nabla^2 D(x, y) = \delta(x - y) \tag{C.9}$$

we may add to $D(x, y)$ a two-point function $F(x, y)$ that satisfies $\nabla^2 F(x, y) = 0$ (i.e. a harmonic F). Such an F must also be entirely defined on the boundary and thus we may use this freedom to construct a $D(x, y)$ such that $\nabla_i D(x, y) = 0$ for $x \in \partial V$.

The above conditions correspond to Dirichlet boundary conditions, however we may instead impose Neumann boundary conditions and use F to construct a Green's function that vanishes on the boundary itself. As expected from a

PDE, the solution of the general $\nabla^2 \phi = \rho$ has to include boundary conditions.

Rexpressing (C.6)

$$\begin{aligned}
\bar{\phi} &= \phi + \dot{T} \\
\bar{B} &= B + \int D\nabla^2(\dot{L} - T) \\
\bar{B}_i &= B_i + \dot{L}_i + \nabla_i(\dot{L} - T) - \nabla_i \int D\nabla^2(\dot{L} - T) \\
\bar{\psi} &= \psi - \frac{1}{2}\nabla^2 L + \frac{1}{4}\nabla^i \int D\nabla^2(2\nabla_i L + L_i) \\
\bar{E} &= E + \int D \left(\nabla^i \int D\nabla^2(2\nabla_i L + L_i) - \frac{1}{2}\nabla^2 L \right) \\
\bar{E}_i &= E_i + \int D\nabla^2\left(\frac{4}{3}\nabla_i L + L_i\right) - \nabla_i \int D\nabla^j \int D\nabla^2\left(\frac{4}{3}\nabla_j L + L_j\right) \\
\bar{E}_{ij} &= E_{ij} + \nabla_i \nabla_j L + \frac{1}{2}\nabla_i L_j + \frac{1}{2}\nabla_j L_i - \frac{1}{2}\nabla_i \int D\nabla^2(2\nabla_j L + L_j) - \frac{1}{2}\nabla_j \int D\nabla^2(2\nabla_i L + L_i) \\
&\quad - \frac{1}{4}g_{ij} \left(2\nabla^2 L - \nabla^k \int D\nabla^2(2\nabla_k L + L_k) \right) \\
&\quad + \frac{1}{4}\nabla_i \nabla_j \int D \left(2\nabla^2 L + \nabla^k \int D\nabla^2(2\nabla_k L + L_k) \right)
\end{aligned} \tag{C.10}$$