

# Cosmological Fluctuations in Standard and Conformal Gravity

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Doctoral Degree Final Examination



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- Introduction and Formalism
- Three Dimensional Scalar, Vector, Tensor Decomposition (SVT3)
- Four Dimensional Scalar, Vector, Tensor Decomposition (SVT4)
- Conformal Gravity (SVT and Conformal to Flat Backgrounds)
- Conformal Gravity Robertson-Walker Radiation Era Solution
- Computational Methods
- Conclusions

- Introduction and Formalism
  - Cosmological Geometries
  - Einstein Gravity
  - Perturbation Theory
  - Gauge Transformations
  - Solution Methods

- Cosmological Principle: Structure of spacetime is homoegenous and isotropic at large scales
- Geometries: Robertson Walker (flat, spherical, hyperbolic), de Sitter ( $dS_4 \subset RW$ )
- All background geometries relevant to cosmology can be expressed as conformal to flat

$$ds^2 = \Omega(x)^2 (-dt^2 + dx^2 + dy^2 + dz^2)$$



Hubble Ultra-Deep Field. NASA and the European Space Agency.

Comoving Robertson Walker geometry:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j \\ &= -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \end{aligned}$$

3-Space Curvature Tensors,

$$R_{ijkl} = k(\tilde{g}_{jk}\tilde{g}_{il} - \tilde{g}_{ik}\tilde{g}_{jl}), \quad R_{ij} = -3k\tilde{g}_{ij}, \quad R = -6k$$

with  $k \in \{-1, 0, 1\}$ . Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

$$ds^2 = a(\tau)^2 \left[ -d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

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with  $k \in \{-1, 0, 1\}$ . Define the conformal time

$$\tau = \int \frac{dt}{a(t)},$$

set  $k = 0$  (flat), simple conformal to flat form

$$ds^2 = a(\tau)^2 \left[ -d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

$k = 1$  (spherical)

$$ds^2 = a(\tau)^2 \left[ -d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

Set  $\sin \chi = r$ ,  $p = \tau$ ,

$$ds^2 = a(p)^2 \left[ -dp^2 + d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tan[(p + \chi)/2], & p' - r' &= \tan[(p - \chi)/2] \\ p' &= \frac{\sin p}{\cos p + \cos \chi}, & r' &= \frac{\sin \chi}{\cos p + \cos \chi} \end{aligned}$$

$$\implies \boxed{ds^2 = \frac{4a^2(p)}{[1 + (p' + r')^2][1 + (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2]}$$

$k = -1$  (hyperbolic)

$$ds^2 = a(\tau)^2 \left[ -d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

Set  $\sinh \chi = r$ ,  $p = \tau$ ,

$$ds^2 = a(p)^2 \left[ -dp^2 + d\chi^2 + \sinh^2 \chi d\theta^2 + \sinh^2 \chi \sin^2 \theta d\phi^2 \right]$$

Introduce coordinates

$$\begin{aligned} p' + r' &= \tanh[(p + \chi)/2], & p' - r' &= \tanh[(p - \chi)/2] \\ p' &= \frac{\sinh p}{\cosh p + \cosh \chi}, & r' &= \frac{\sinh \chi}{\cosh p + \cosh \chi} \end{aligned}$$

$$\implies \boxed{ds^2 = \frac{4a^2(p)}{[1 - (p' + r')^2][1 - (p' - r')^2]} [-dp'^2 + dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\phi^2]}$$

Einstein Hilbert action

$$I_{\text{EH}} = -\frac{1}{16\pi G} \int d^4x (-g)^{1/2} g^{\mu\nu} R_{\mu\nu}.$$

Functional variation w.r.t  $g_{\mu\nu}$  yields Einstein tensor,

$$\frac{16\pi G}{(-g)^{1/2}} \frac{\delta I_{\text{EH}}}{\delta g_{\mu\nu}} = G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha,$$

likewise, variation of matter action  $I_M$  w.r.t  $g_{\mu\nu}$  yields Energy Momentum tensor

$$\frac{2}{(-g)^{1/2}} \frac{\delta I_M}{\delta g_{\mu\nu}} = T_{\mu\nu}.$$

Requiring sum of actions to be stationary gives us Einstein field equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha = -8\pi G T^{\mu\nu},$$

subject to Bianchi identity

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R^\mu{}_\mu \implies \nabla_\mu G^{\mu\nu} = 0.$$

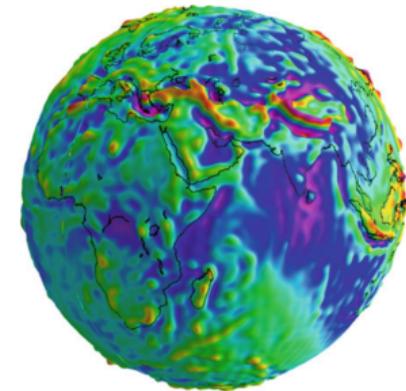
Decompose metric into background and fluctuation, truncating at linear order

$$g_{\mu\nu}(x) = g_{(0)}^{(0)}(x) + h_{\mu\nu}(x), \quad g_{(0)}^{\mu\nu} h_{\mu\nu} \equiv h$$

$$G_{\mu\nu} = G_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta G_{\mu\nu}(h_{\mu\nu})$$

$$G_{\mu\nu}^{(0)} = R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R_{\alpha}^{(0)\alpha}$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R_{\alpha}^{(0)\alpha} - \frac{1}{2} g_{\mu\nu} \delta R^{\alpha}_{\alpha}.$$



Likewise perturb  $T_{\mu\nu}$  around background

$$T_{\mu\nu} = T_{\mu\nu}(g_{\mu\nu}^{(0)}) + \delta T_{\mu\nu}(h_{\mu\nu})$$

Form background and first order equations of motion (upon setting  $8\pi G = 1$ )

$$\Delta_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} + T_{\mu\nu}^{(0)} = 0$$

$$\Delta_{\mu\nu} = \delta G_{\mu\nu}^{(0)} + \delta T_{\mu\nu}^{(0)} = 0$$

- Under coordinate transformation  $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$ , with  $\epsilon^\mu \sim \mathcal{O}(h)$ , the perturbed metric transforms as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$$

- For every solution  $h_{\mu\nu}$  to  $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$ , a transformed  $h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$  will also serve as a solution
- Set of four  $\epsilon^\mu(x)$  define gauge freedom under coordinate transformation
- 10 components in  $h_{\mu\nu}$ , 4 coordinate transformations, leads to 6 independent degrees of freedom
- Under  $x^\mu \rightarrow x^\mu - \epsilon^\mu(x)$ , the perturbed tensors transform as

$$\begin{aligned}\delta G_{\mu\nu} &\rightarrow \delta G_{\mu\nu} + {}^{(0)}G^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}G^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda G_{\mu\nu}^{(0)} \epsilon^\lambda \\ \delta T_{\mu\nu} &\rightarrow \delta T_{\mu\nu} + {}^{(0)}T^\lambda{}_\mu \nabla_\nu \epsilon_\lambda + {}^{(0)}T^\lambda{}_\nu \nabla_\mu \epsilon_\lambda + \nabla_\lambda T_{\mu\nu}^{(0)} \epsilon^\lambda.\end{aligned}$$

- If background  $G_{\mu\nu}^{(0)} = 0$ , then  $\delta G_{\mu\nu}$  separately gauge invariant; likewise for vanishing background energy momentum tensor
- If  $G_{\mu\nu}^{(0)} \neq 0$ , then only the entire  $\Delta_{\mu\nu} = \delta G_{\mu\nu} + T_{\mu\nu}$  is gauge invariant

- Perturbed field equations  $\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$  form a rather complex and extensive set of coupled tensor PDE's
- Much effort involved in simplifying, decoupling, and solving them

$$\begin{aligned}
 \delta G_{ij} = & -\frac{1}{2}\ddot{h}_{ij} + \frac{1}{2}\ddot{h}_{00}\tilde{g}_{ij} + \frac{1}{2}\ddot{h}\tilde{g}_{ij} - k\tilde{g}^{ba}\tilde{g}_{ij}h_{ab} + 3kh_{ij} - \dot{\Omega}^2 h_{ij}\Omega^{-2} - \dot{\Omega}^2 \tilde{g}_{ij}h_{00}\Omega^{-2} \\
 & - \dot{h}_{ij}\dot{\Omega}\Omega^{-1} + 2\dot{h}_{00}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + \dot{h}\dot{\Omega}\tilde{g}_{ij}\Omega^{-1} + 2\ddot{\Omega}h_{ij}\Omega^{-1} + 2\ddot{\Omega}\tilde{g}_{ij}h_{00}\Omega^{-1} \\
 & + 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}h_{0b}\Omega^{-2}\tilde{\nabla}_a\Omega - 2\dot{h}_{0b}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega - \tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\dot{h}_{0a} \\
 & - 4\tilde{g}^{ba}\tilde{g}_{ij}h_{0a}\Omega^{-1}\tilde{\nabla}_b\dot{\Omega} + \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_b h_{ij} - 2\dot{\Omega}\tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_b h_{0a} \\
 & - \tilde{g}^{ba}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a h\tilde{\nabla}_b\Omega - \tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{cd}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \tilde{g}^{ba}h_{ij}\Omega^{-2}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega \\
 & + \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_a h_{ij} - \frac{1}{2}\tilde{g}^{ba}\tilde{g}_{ij}\tilde{\nabla}_b\tilde{\nabla}_a h - 2\tilde{g}^{ba}h_{ij}\Omega^{-1}\tilde{\nabla}_b\tilde{\nabla}_a\Omega \\
 & - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_i h_{ja} - \frac{1}{2}\tilde{g}^{ba}\tilde{\nabla}_b\tilde{\nabla}_j h_{ia} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_d h_{cb} \\
 & + \frac{1}{2}\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}\tilde{\nabla}_d\tilde{\nabla}_c h_{ab} + 2\tilde{g}^{ca}\tilde{g}^{db}\tilde{g}_{ij}h_{ab}\Omega^{-1}\tilde{\nabla}_d\tilde{\nabla}_c\Omega + \frac{1}{2}\tilde{\nabla}_i\dot{h}_{0j} \\
 & - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_i h_{jb} + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_i h_{0j} + \frac{1}{2}\tilde{\nabla}_j\dot{h}_{0i} - \tilde{g}^{ba}\Omega^{-1}\tilde{\nabla}_a\Omega\tilde{\nabla}_j h_{ib} \\
 & + \dot{\Omega}\Omega^{-1}\tilde{\nabla}_j h_{0i} + \frac{1}{2}\tilde{\nabla}_j\tilde{\nabla}_i h,
 \end{aligned}$$

- Three-dimensional Scalar, Vector, Tensor Basis (SVT3)
  - SVT3 Decomposition
  - Decouple Einstein Fluctuations in a de Sitter Background
  - Integral Formalism

Decompose the metric perturbation  $h_{\mu\nu}$  into a set of scalars, vectors, and tensors according to their transformation behavior under 3D rotations

- Define  $h_{\mu\nu} = \Omega^2(x) f_{\mu\nu}$ , perform 3 + 1 decomposition

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = (g_{\mu\nu}^{(0)} + h_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) (\tilde{g}_{\mu\nu}^{(0)} + f_{\mu\nu}) dx^\mu dx^\nu \\ &= \Omega^2(x) [(-1 + f_{00}) dt^2 + 2f_{0i} dt dx^i + (\tilde{g}_{ij} + f_{ij})] dx^i dx^j \end{aligned}$$

- Decompose  $f_{00}$ ,  $f_{0i}$ , and  $f_{ij}$  in terms of 3-dimensional scalars, vectors, and tensors

$$\begin{aligned} f_{00} &= -2\phi, \quad f_{0i} = B_i + \tilde{\nabla}_i B \\ f_{ij} &= -2\psi \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned}$$

with vectors and tensors obeying

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0.$$

$$\begin{aligned} ds^2 &= \Omega^2(x) \left[ - (1 + 2\phi) dt^2 + 2(B_i + \tilde{\nabla}_i B) dt dx^i \right. \\ &\quad \left. + [(1 - 2\psi) \tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}] dx^i dx^j \right], \end{aligned}$$

- de Sitter geometry

$$\begin{aligned} ds^2 = & \frac{1}{H^2 \tau^2} \left[ -(1 + 2\phi)dt^2 + 2(B_i + \tilde{\nabla}_i B)dtdx^i \right. \\ & \left. + [(1 - 2\psi)\delta_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j \right], \end{aligned}$$

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2 \Omega(\tau)^2 f_{\mu\nu}$$

- Insert the SVT3 decomposed  $h_{\mu\nu}$  into a 3+1  $\delta G_{\mu\nu}$

# SVT3 $\delta G_{\mu\nu}$ in a de Sitter Background

- Energy momentum tensor

$$T_{\mu\nu} = -3H^2 g_{\mu\nu} \implies \delta T_{\mu\nu} = -3H^2 h_{\mu\nu} = -3H^2 \Omega(\tau)^2 f_{\mu\nu}$$

- Insert the SVT3 decomposed  $h_{\mu\nu}$  into a 3+1  $\delta G_{\mu\nu}$

$$\begin{aligned}\delta G_{00} &= -\frac{6}{\tau} \dot{\psi} - \frac{2}{\tau} \tilde{\nabla}^2 (\tau\psi + B - \dot{E}), \\ \delta G_{0i} &= \frac{1}{2} \tilde{\nabla}^2 (B_i - \dot{E}_i) + \frac{1}{\tau^2} \tilde{\nabla}_i (3B - 2\tau^2 \dot{\psi} + 2\tau\phi) + \frac{3}{\tau^2} B_i, \\ \delta G_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[ -2\tau^2 \ddot{\psi} + 2\tau\dot{\phi} + 4\tau\dot{\psi} - 6\phi - 6\psi \right. \\ &\quad \left. + \tilde{\nabla}^2 (2\tau B - \tau^2 \dot{B} + \tau^2 \ddot{E} - 2\tau\dot{E} - \tau^2 \phi + \tau^2 \psi) \right] \\ &\quad + \frac{1}{\tau^2} \tilde{\nabla}_i \tilde{\nabla}_j \left[ -2\tau B + \tau^2 \dot{B} - \tau^2 \ddot{E} + 2\tau\dot{E} + 6E + \tau^2 \phi - \tau^2 \psi \right] \\ &\quad + \frac{1}{2\tau^2} \tilde{\nabla}_i \left[ -2\tau B_j + 2\tau\dot{E}_j + \tau^2 \dot{B}_j - \tau^2 \ddot{E}_j + 6E_j \right] \\ &\quad + \frac{1}{2\tau^2} \tilde{\nabla}_j \left[ -2\tau B_i + 2\tau\dot{E}_i + \tau^2 \dot{B}_i - \tau^2 \ddot{E}_i + 6E_i \right] \\ &\quad - \ddot{E}_{ij} + \frac{6}{\tau^2} E_{ij} + \frac{2}{\tau} \dot{E}_{ij} + \tilde{\nabla}^2 E_{ij},\end{aligned}$$

- Compose  $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu}$

$$\begin{aligned}
 \Delta_{00} &= -\frac{6}{\tau^2}(\dot{\beta} - \alpha) - \frac{2}{\tau}\tilde{\nabla}^2\beta = 0, \\
 \Delta_{0i} &= \frac{1}{2}\tilde{\nabla}^2(B_i - \dot{E}_i) - \frac{2}{\tau}\tilde{\nabla}_i(\dot{\beta} - \alpha) = 0, \\
 \Delta_{ij} &= \frac{\delta_{ij}}{\tau^2} \left[ -2\tau(\ddot{\beta} - \dot{\alpha}) + 6(\dot{\beta} - \alpha) + \tau\tilde{\nabla}^2(2\beta - \tau\alpha) \right] + \frac{1}{\tau}\tilde{\nabla}_i\tilde{\nabla}_j(-2\beta + \tau\alpha) \\
 &\quad + \frac{1}{2\tau}\tilde{\nabla}_i[-2(B_j - \dot{E}_j) + \tau(\dot{B}_j - \ddot{E}_j)] + \frac{1}{2\tau}\tilde{\nabla}_j[-2(B_i - \dot{E}_i) + \tau(\dot{B}_i - \ddot{E}_i)] \\
 &\quad - \ddot{E}_{ij} + \frac{2}{\tau}\dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} = 0, \\
 g^{\mu\nu}\Delta_{\mu\nu} &= H^2[-6\tau(\ddot{\beta} - \dot{\alpha}) + 24(\dot{\beta} - \alpha) + 6\tau\tilde{\nabla}^2\beta - 2\tau^2\tilde{\nabla}^2\alpha] = 0,
 \end{aligned}$$

where

$$\alpha = \phi + \psi + \dot{B} - \ddot{E}, \quad \beta = \tau\psi + B - \dot{E}, \quad B_i - \dot{E}_i, \quad E_{ij}.$$

- Decouple scalar, vector, and tensor gauge invariants by applying higher derivatives

$$\tilde{\nabla}^4(\alpha + \dot{\beta}) = 0, \quad \tilde{\nabla}^4(\alpha - \dot{\beta}) = 0,$$

$$\tilde{\nabla}^4(B_i - \dot{E}_i) = 0,$$

$$\tilde{\nabla}^4 \left( -\ddot{E}_{ij} + \frac{2}{\tau} \dot{E}_{ij} + \tilde{\nabla}^2 E_{ij} \right) = 0.$$

- Recap:

- Perturb  $\delta G_{\mu\nu}$  and  $\delta T_{\mu\nu}$ , evaluating in de Sitter background
- Decompose  $h_{\mu\nu}$  into SVT3 components, inserting into fields equations
- Compose  $\Delta_{\mu\nu} = \delta G_{\mu\nu} + \delta T_{\mu\nu} = 0$  to form evolution equations consisting entirely of gauge invariant quantities
- Apply higher derivatives to decouple SVT3 representations, solve

- How can we ensure such an SVT3 decomposition exists for the general  $h_{\mu\nu}$ ? Let's recall

$$\begin{aligned} f_{00} &= -2\phi, & f_{0i} &= B_i + \tilde{\nabla}_i B \\ f_{ij} &= -2\psi\tilde{g}_{ij} + 2\tilde{\nabla}_i \tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}, \end{aligned}$$

$$\tilde{\nabla}^i B_i = \tilde{\nabla}^i E_i = 0, \quad E_{ij} = E_{ji}, \quad \tilde{\nabla}^i E_{ij} = 0, \quad \delta^{ij} E_{ij} = 0.$$

Decomposition of  $V_i = V_i^T + \partial_i V$

- Does not hold for any scalar.  $\partial^i V_i = \partial_i \partial^i V$
- Introduce Green's function  $\partial_i \partial^i D(x - x') = \delta^3(x - x')$  and Green's identity

$$V(x') \partial_i \partial^i D(x - x') = D(x - x') \partial_i \partial^i V(x') + \partial_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')]$$

- Integrate

$$V(x) = \underbrace{\int_V d^3x' D(x - x') \partial_i \partial^i V(x')}_{\text{Non-Harmonic}} + \underbrace{\oint_{\partial V} dS_i [V(x') \partial^i D(x - x') - D(x - x') \partial^i V(x')]}_{\text{Harmonic}}$$

$$\begin{aligned} I_W &= -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\ &\equiv -2\alpha_g \int d^4x (-g)^{1/2} [R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} (R^\alpha{}_\alpha)^2], \end{aligned}$$

$$\begin{aligned} C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} - \frac{1}{2} (g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}) \\ &\quad + \frac{1}{6} R^\alpha{}_\alpha (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) \end{aligned}$$

$$-\frac{2}{(-g)^{1/2}} \frac{\delta I_W}{\delta g_{\mu\nu}} = 4\alpha_g W^{\mu\nu} = 4\alpha_g [2\nabla_\kappa \nabla_\lambda C^{\mu\lambda\nu\kappa} - R_{\kappa\lambda} C^{\mu\lambda\nu\kappa}].$$





