External Projection v1

δG_{ij}

Within the geometry of

$$ds^2 = -(g_{ij} + h_{ij})dx^i dx^j \tag{1}$$

with maximally symmetric background

$$g_{ij} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{2}$$

assume the metric perturbation can be (covariant) SVT decomposed as

$$h_{ij} = -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}, \tag{3}$$

with 3-trace

$$h = -6\psi + 2\nabla^a\nabla_a E. \tag{4}$$

The three dimensional Einstein background field equations take the form $G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$. Since the background is maximally symmetric, the solution to the zeroth order Einstein equations yields energy momentum tensor $T^{(0)}_{\mu\nu} = \Lambda g^{(0)}_{\mu\nu} = kg^{(0)}_{\mu\nu}$.

The perturbed Einstein equations then take the form,

$$\delta G_{ij} = \delta T_{ij} \tag{5}$$
$$= -kh_{ij} \tag{6}$$

Evaluate the Einstein tensor in terms of (3) yields

$$\delta G_{ij} = \tfrac{1}{2}\nabla_a\nabla^a h_{ij} - \tfrac{1}{2}g_{ij}\nabla_a\nabla^a h + \tfrac{1}{2}g_{ij}\nabla_b\nabla_a h^{ab} - \tfrac{1}{2}\nabla_i\nabla_a h_j{}^a - \tfrac{1}{2}\nabla_j\nabla_a h_i{}^a + \tfrac{1}{2}\nabla_j\nabla_i h, \tag{7}$$

which takes the SVT form

$$\delta G_{ij} = \nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi. \tag{8}$$

Composing the field equation $\delta G_{\mu\nu} = \delta T_{\mu\nu}$ yields

$$\nabla_a\nabla^a E_{ij} + g_{ij}\nabla_a\nabla^a\psi + k\nabla_i E_j + k\nabla_j E_i + 2k\nabla_j\nabla_i E - \nabla_j\nabla_i\psi = \tag{9}$$
$$k(-2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij}), \tag{10}$$

which may be simplified as

$$(\nabla_a\nabla^a - 2k)E_{ij} + g_{ij}\nabla_a\nabla^a\psi - \nabla_j\nabla_i\psi + 2kg_{ij}\psi = 0. \tag{11}$$

Taking the trace gives the solution for ψ

$$(\nabla_a \nabla^a + 3k)\psi = 0 \tag{12}$$

Under gauge transformation

$$h_{ij} \to \bar{h}_{ij} = h_{ij} + \nabla_i \epsilon_j + \nabla_j \epsilon_i \tag{13}$$

with $\epsilon_i = \nabla_i L + L_i$ and $\nabla^i L_i = 0$, we find that h_{ij} transforms as

$$
\begin{aligned}
&-2g_{ij}\bar{\psi} + 2\nabla_i\nabla_j\bar{E} + \nabla_i\bar{E}_j + \nabla_j\bar{E}_i + 2\bar{E}_{ij} = &\quad(14)\\
&\quad -2g_{ij}\psi + 2\nabla_i\nabla_j E + \nabla_i E_j + \nabla_j E_i + 2E_{ij} + 2\nabla_i\nabla_j L + \nabla_i L_j + \nabla_j L_i. &\quad(15)
\end{aligned}
$$

Taking the trace of the above, we have

$$-6\bar{\psi} + 2\nabla^i\nabla_i\bar{E} = -6\psi + 2\nabla^i\nabla_i E + 2\nabla^i\nabla_i L \tag{16}$$

$$
\begin{aligned}
\bar{\psi} &= \psi &\quad(17)\\
\bar{E} &= E - L &\quad(18)\\
\bar{E}_i &= E_i - L_i &\quad(19)\\
\bar{E}_{ij} &= E_{ij} &\quad(20)
\end{aligned}
$$

$$
\begin{aligned}
\nabla^i\nabla^j h_{ij} &= -2\nabla^i\nabla_i\psi + 2\nabla^i\nabla_i\nabla^j\nabla_j E + 2k\nabla_i\nabla^i E &\quad(21)\\
\nabla^j\delta G_{ij} &= -2k\nabla_i\psi + k(\nabla^a\nabla_a + 2k)E_i + 2k\nabla^a\nabla_a\nabla_i E &\quad(22)\\
\nabla^i\nabla^j\delta G_{ij} &= -2k\nabla^a\nabla_a\psi + 2k\nabla^a\nabla_a(\nabla^b\nabla_b + 2k)E &\quad(23)
\end{aligned}
$$

Appendix

$$[\nabla_i, \nabla_j]V_k = V_m R^m{}_{kij} = k(g_{ki}g^m{}_j - g^m{}_i g_{kj})V_m = \tag{24}$$

Christoffels for

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = \left(dt^2 - \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{25}$$

$$
\begin{aligned}
&\Gamma^r_{rr} = \frac{kr}{1-kr^2}, \quad \Gamma^r_{\theta\theta} = -r(1-kr^2), \quad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta\\
&\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta,
\end{aligned}
$$

with all others zero.