Special Gauge v4 Matthew

The perturbed Einstein tensor $\delta G_{\mu\nu}(h_{\mu\nu})$ evaluated in the metric decomposed to first order

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} = \Omega^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{1}$$

is calculated as

$$\begin{aligned}\delta G_{\mu\nu} = {} & \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}h_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}h_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha h_{\mu\nu} - 2\eta^{\alpha\beta}h_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\mu h_{\nu\alpha} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\nu h_{\mu\alpha} \\ & + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta h_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta h_{\alpha\gamma} + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}h_{\alpha\gamma}\Omega^{-1}\partial_\zeta\partial_\beta\Omega \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu h_{\nu\beta} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu h_{\mu\beta} - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + \tfrac{1}{2}\partial_\nu\partial_\mu h.\end{aligned} \tag{2}$$

Now we split $h_{\mu\nu}$ into its traceless and trace components, i.e.

$$h_{\mu\nu} = K_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h \tag{3}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. With this substitution, (2) takes the form

$$\begin{aligned}\delta G_{\mu\nu} = {} & -2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\alpha\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta K_{\mu\nu} - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \eta^{\alpha\beta}K_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha K_{\mu\nu} + 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ & - 2\eta^{\alpha\beta}K_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega + 2\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\zeta K_{\beta\gamma} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\zeta}\eta_{\mu\nu}\partial_\zeta\partial_\beta K_{\alpha\gamma} \\ & - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\mu\partial_\beta K_{\nu\alpha} - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} + 2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega \\ & - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\nu\partial_\beta K_{\mu\alpha} + \tfrac{3}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha\Omega\partial_\beta h - \eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & - \tfrac{1}{8}\partial_\mu\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu h - \tfrac{1}{4}\Omega^{-1}\partial_\mu h\partial_\nu\Omega + \tfrac{3}{8}\partial_\nu\partial_\mu h.\end{aligned} \tag{4}$$

Now we impose a generalized gauge of the form

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = \Omega^{-1}J\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + P\partial_\nu h + R\Omega^{-1}h\partial_\nu\Omega. \tag{5}$$

Within this gauge, (4) is evaluated as

$$\begin{aligned}\delta G_{\mu\nu} = {} & -2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\alpha\partial_\nu\Omega + \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\beta K_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega \\ & + 2P\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \tfrac{1}{2}JP\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega + \tfrac{1}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}\Omega^{-1}\partial_\alpha h\partial_\beta\Omega \\ & - \eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{3}{2}J\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}J^2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \eta^{\alpha\beta}K_{\mu\nu}\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{3}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega \\ & + \tfrac{1}{2}JR\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-2}\partial_\alpha\Omega\partial_\beta\Omega + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha K_{\mu\nu} - \tfrac{1}{4}\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h + \tfrac{1}{2}P\eta^{\alpha\beta}\eta_{\mu\nu}\partial_\beta\partial_\alpha h \\ & + 2\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega + \tfrac{1}{2}J\eta^{\alpha\gamma}\eta^{\beta\zeta}\eta_{\mu\nu}K_{\gamma\zeta}\Omega^{-1}\partial_\beta\partial_\alpha\Omega - 2\eta^{\alpha\beta}K_{\mu\nu}\Omega^{-1}\partial_\beta\partial_\alpha\Omega \\ & + \tfrac{1}{2}R\eta^{\alpha\beta}\eta_{\mu\nu}h\Omega^{-1}\partial_\beta\partial_\alpha\Omega - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} - \tfrac{1}{2}J\eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\mu K_{\nu\beta} \\ & + \tfrac{1}{2}J\eta^{\alpha\beta}K_{\nu\beta}\Omega^{-2}\partial_\alpha\Omega\partial_\mu\Omega - \tfrac{1}{2}J\eta^{\alpha\beta}K_{\nu\beta}\Omega^{-1}\partial_\mu\partial_\alpha\Omega - \tfrac{1}{8}\partial_\mu\partial_\nu h - \tfrac{1}{2}P\partial_\mu\partial_\nu h \\ & - \tfrac{1}{2}Rh\Omega^{-1}\partial_\mu\partial_\nu\Omega - \eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} - \tfrac{1}{2}J\eta^{\alpha\beta}\Omega^{-1}\partial_\alpha\Omega\partial_\nu K_{\mu\beta} - \tfrac{1}{4}\Omega^{-1}\partial_\mu\Omega\partial_\nu h \\ & - \tfrac{1}{2}R\Omega^{-1}\partial_\mu\Omega\partial_\nu h + \tfrac{1}{2}J\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-2}\partial_\alpha\Omega\partial_\nu\Omega - \tfrac{1}{4}\Omega^{-1}\partial_\mu h\partial_\nu\Omega - \tfrac{1}{2}R\Omega^{-1}\partial_\mu h\partial_\nu\Omega \\ & + Rh\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega + 2\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega - \tfrac{1}{2}J\eta^{\alpha\beta}K_{\mu\beta}\Omega^{-1}\partial_\nu\partial_\alpha\Omega + \tfrac{3}{8}\partial_\nu\partial_\mu h \\ & - \tfrac{1}{2}P\partial_\nu\partial_\mu h - \tfrac{1}{2}Rh\Omega^{-1}\partial_\nu\partial_\mu\Omega.\end{aligned} \tag{6}$$

Upon taking $J=-2$, $P=\frac{1}{2}$, and $R=0$, viz.

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + \tfrac{1}{2}\partial_\nu h, \tag{7}$$

for a strictly time dependent conformal factor $\Omega(\tau)$, we find the flucuations take the form (computer output)

$$\delta G_{00} = (3\Omega^{-2}\dot{\Omega}^2 - \Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0 - \tfrac{1}{4}\partial_0\partial_0)h \tag{8}$$

$$\delta G_{0i} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{0i} + (-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_i - \tfrac{1}{4}\partial_i\partial_0)h \tag{9}$$

$$\begin{aligned}\delta G_{ij} = {}& \delta_{ij}(-2\Omega^{-2}\dot{\Omega}^2 K_{00} + \Omega^{-1}\ddot{\Omega}K_{00}) + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{ij} \\ & + (-\delta_{ij}\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0 - \tfrac{1}{4}\partial_i\partial_j)h \end{aligned} \tag{10}$$

In the deSitter background, we take $\Omega(\tau) = \frac{1}{H\tau}$, in which $\delta G_{\mu\nu}$ reduces to

$$\delta G_{00} = (\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{00} + (\tfrac{1}{4}\tau^{-1}\partial_0 - \tfrac{1}{4}\partial_0\partial_0)h \tag{11}$$

$$\delta G_{0i} = (2\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{0i} + (\tfrac{1}{4}\tau^{-1}\partial_i - \tfrac{1}{4}\partial_i\partial_0)h \tag{12}$$

$$\delta G_{ij} = (3\tau^{-2} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tau^{-1}\partial_0)K_{ij} + (\delta_{ij}\tfrac{1}{4}\tau^{-1}\partial_0 - \tfrac{1}{4}\partial_i\partial_j)h \tag{13}$$

Notes

If we express the harmonic gauge in terms of $K_{\mu\nu}$, this is

$$\begin{aligned} \eta^{\alpha\beta}\partial_\alpha h_{\beta\nu} ={}& \frac{1}{2}\partial_\nu h \\ \eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} + \frac{1}{4}\partial_\nu h ={}& \frac{1}{2}\partial_\nu h \\ \eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} ={}& \frac{1}{4}\partial_\nu h. \end{aligned} \tag{14}$$

Now evaluate the above in the metric of (1), such that the gauge becomes

$$\eta^{\alpha\beta}\partial_\alpha K_{\beta\nu} = -2\Omega^{-1}\eta^{\alpha\beta}K_{\nu\alpha}\partial_\beta\Omega + \frac{1}{4}\partial_\nu h. \tag{15}$$

This corresponds to $J=-2$, $P=\frac{1}{4}$, $R=0$.

We require to take $J=-2$ to cancel the appearance of K_{0i} terms within δG_{ij}. There are other plausible choices of P and R, however (7) provides the most overall simplification. Here are some other examples.

$J=-2$, $P=\frac{1}{4}$, $R=\frac{1}{2}$

$$\delta G_{00} = (3\Omega^{-2}\dot{\Omega}^2 - \Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{00} + (-\tfrac{3}{4}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} + \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu - \tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0)h \tag{16}$$

$$\delta G_{0i} = (\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{0i} \tag{17}$$

$$\begin{aligned}\delta G_{ij} = {}& \delta_{ij}(-2\Omega^{-2}\dot{\Omega}^2 K_{00} + \Omega^{-1}\ddot{\Omega}K_{00}) + (-\Omega^{-2}\dot{\Omega}^2 + 2\Omega^{-1}\ddot{\Omega} + \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu - \Omega^{-1}\dot{\Omega}\partial_0)K_{ij} \\ & + \delta_{ij}(\tfrac{1}{4}\Omega^{-2}\dot{\Omega}^2 + \tfrac{1}{4}\Omega^{-1}\ddot{\Omega} - \tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_0)h. \end{aligned} \tag{18}$$

$J=-2,\ P=\frac{1}{4},\ R=0$

$$\delta G_{00}=(3\Omega^{-2}\dot{\Omega}^2-\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{00}+(\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu-\tfrac{1}{2}\Omega^{-1}\dot{\Omega}\partial_0)h \tag{19}$$

$$\delta G_{0i}=(\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{0i}-\tfrac{1}{4}\Omega^{-1}\dot{\Omega}\partial_i h \tag{20}$$

$$\delta G_{ij}=\delta_{ij}(-2\Omega^{-2}\dot{\Omega}^2K_{00}+\Omega^{-1}\ddot{\Omega}K_{00})+(-\Omega^{-2}\dot{\Omega}^2+2\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{ij}-\delta_{ij}\tfrac{1}{8}\eta^{\mu\nu}\partial_\mu\partial_\nu h. \tag{21}$$

$J=-2,\ P=\frac{1}{2},\ R=1$

$$\delta G_{00}=(3\Omega^{-2}\dot{\Omega}^2-\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{00}+(\tfrac{3}{2}\Omega^{-2}\dot{\Omega}^2-\tfrac{1}{2}\Omega^{-1}\ddot{\Omega}-\tfrac{3}{4}\Omega^{-1}\dot{\Omega}\partial_0-\tfrac{1}{4}\partial_0\partial_0)h \tag{22}$$

$$\delta G_{0i}=(\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{0i}+(-\tfrac{3}{4}\Omega^{-1}\dot{\Omega}\partial_i-\tfrac{1}{4}\partial_i\partial_0)h \tag{23}$$

$$\begin{aligned}\delta G_{ij}=&\ \delta_{ij}(-2\Omega^{-2}\dot{\Omega}^2K_{00}+\Omega^{-1}\ddot{\Omega}K_{00})+(-\Omega^{-2}\dot{\Omega}^2+2\Omega^{-1}\ddot{\Omega}+\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu-\Omega^{-1}\dot{\Omega}\partial_0)K_{ij}\\&+(-\delta_{ij}\tfrac{1}{2}\Omega^{-2}\dot{\Omega}^2-\delta_{ij}\tfrac{1}{2}\Omega^{-1}\ddot{\Omega}-\delta_{ij}\tfrac{3}{4}\Omega^{-1}\dot{\Omega}\partial_0-\tfrac{1}{4}\partial_i\partial_j)h\end{aligned} \tag{24}$$