Fitting smiles with SABR

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April 30, 2020

Overview

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   - Practical problems
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Background

- Define S_t as time t stock price. A *European call option* on S_t, with *maturity date* T and *strike price* K, has payoff at T

$$\max\{S_T - K, 0\}$$

  At time T, gain $S_T - K$ in value if $S_T > K$, else get nothing.
- "Ramp" payoff = **no downside**. What's the "right" (unique) price?

Background

- *Martingale pricing.* Allows us to write option time t value V_t as

$$V_t = B_t \mathbb{E}\left[B_T^{-1} \max\{S_T - K, 0\} \,\middle|\, \mathcal{F}_t\right] \tag{1}$$

- B_t is a *numeraire*, in this case the *bank account*, where given r_t, the time t *risk-free short rate*, $B_t \triangleq e^{\int_0^t r_\tau \, d\tau}$. $\mathcal{F}_t$ is time t "information".
- Interpretation: The option value V_t is a **conditional expectation** of the **discounted payoff** under *risk-neutral probability measure* $\mathbb{Q}$.
  - Informally, if $\mathbb{P}$ gives "real-world" probabilities, $\mathbb{Q}$ gives probabilities from a world adjusted for risk; i.e. all **traded** assets earn r_t.
- We discuss T'-maturity forward prices $F_{t,T'}$, and modify (1) into

$$V_t = Z_{t,T} \widetilde{\mathbb{E}}\left[\max\{F_{T,T'} - K, 0\} \,\middle|\, \mathcal{F}_t\right] \tag{2}$$

Numeraire is $Z_{t,T}$, T-maturity *zero coupon bond*, where $Z_{T,T} \triangleq 1$.

Black's formula

- Variation of Black-Scholes formula for futures and forward[1] prices.

$$
\begin{aligned}
V_t &= Z_{t,T}\left[F\Phi(d_1) - K\Phi(d_2)\right] \\
d_{1,2} &\triangleq \frac{\log(F/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
\tag{3}
$$

  $F \triangleq F_{t,T}$, Φ is standard normal cdf, $\sigma > 0$ is volatility parameter.
- Black's model essentially assumes that $F_{t,T'}$ follows

$$
dF_{t,T'} = \sigma F_{t,T'} dW_t \tag{4}
$$

  W_t is Wiener process under *forward measure* $\mathbb{Q}^{T'}$.
- Interpretation: **Lognormal**, **martingale** forward under $\mathbb{Q}^{T'}$.
  - I.e. $F_{t,T'} = \widetilde{\mathbb{E}}[F_{T,T'} \mid \mathcal{F}_t]$, $\forall T \in (t, T']$.

[1] Application to forwards is more recent and based on (2).

Practical problems

- Simple, intuitive, easy to use model. But empirically **wrong**.
- Backing out σ over varying strikes K and maturities T gives a **non-flat** surface = **different** Black-Scholes σ at each K, T pair!
  - Surface would be **flat** if Black-Scholes is $\approx$ correct.
- Define $\hat{\sigma}(K, T)$ as *implied volatility* for strike K, maturity T.
  - Black-Scholes σ **implied** by option price for K, T.

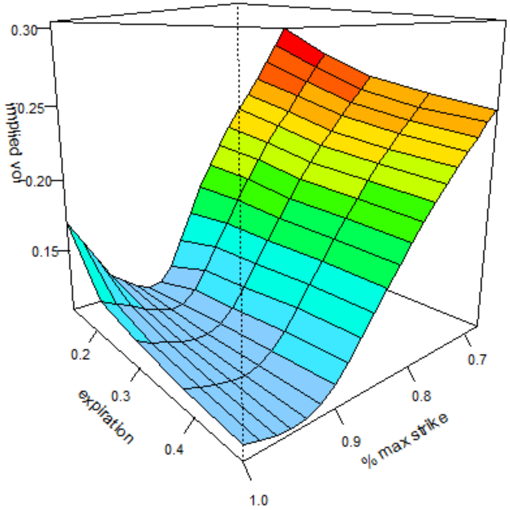

Figure 1: E-Mini S&P 500 futures options surface for $\hat{\sigma}(K, T)$ dated 08/13/2019.

Fitting implied vol

- Account for negative return skew and investor risk aversion. Then,

$$dF_{t,T'} = \sigma F_{t,T'}^{\beta} dW_t \tag{5}$$

The *constant elasticity of variance*, or *CEV*, forward. $\beta \geq 0$.

- What if σ is **stochastic** α_t? Simple **two-factor** model, where

$$\begin{aligned} dF_{t,T'} &= \alpha_t F_{t,T'}^{\beta} dW_t \\ d\alpha_t &= \nu \alpha_t d\hat{W}_t \\ \rho dt &= \mathbb{E}[dW_t d\hat{W}_t] \end{aligned} \tag{6}$$

Stochastic α, β, ρ, or SABR, model. $\alpha_0 \triangleq \alpha > 0$, $\rho \in [0, 1)$.

- Interpretation: α is initial vol level, β controls forward skew, ρ is **correlation** between $F_{t,T}$ and α_t volatility process.

Fitting implied vol

- Due to [2] and [1], both CEV and SABR models have interpolation formulas to directly fit implied volatilities at a particular T.
- Then, send model implied volatilities to Black's model to get prices.
- CEV interpolation formula for T maturity implied volatilities is

$$\sigma_B(F, K) = \frac{\sigma}{F_{av}^{1-\beta}} \left\{ 1 + \frac{(1-\beta)(2+\beta)}{24} \left(\frac{F-K}{F_{av}} \right)^2 + \frac{(1-\beta)^2}{24} \frac{\sigma^2}{F_{av}^{2-2\beta}} (T-t) + \dots \right. \tag{7}$$

Here $F_{av} \triangleq (F+K)/2$, midpoint of F, K.

- SABR interpolation formula much more complicated.

Fitting implied vol

$$\sigma_B(F,K) = \frac{\alpha}{(FK)^{\frac{1-\beta}{2}}\left\{1+\frac{(1-\beta)^2}{24}\log(F/K)^2+\frac{(1-\beta)^2}{1920}\log(F/K)^4+\dots\right\}}$$
$$\cdot\left(\frac{z}{\chi(z)}\right)\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(FK)^{(1-\beta)/2}}\right.\right.$$
$$\left.\left.+\ \frac{2-3\rho^2}{24}\nu^2\right](T-t)+\dots\right\} \tag{8}$$

$$z = \frac{\nu}{\alpha}(FK)^{(1-\beta)/2}\log(F/K) \tag{9}$$

$$\chi(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\} \tag{10}$$

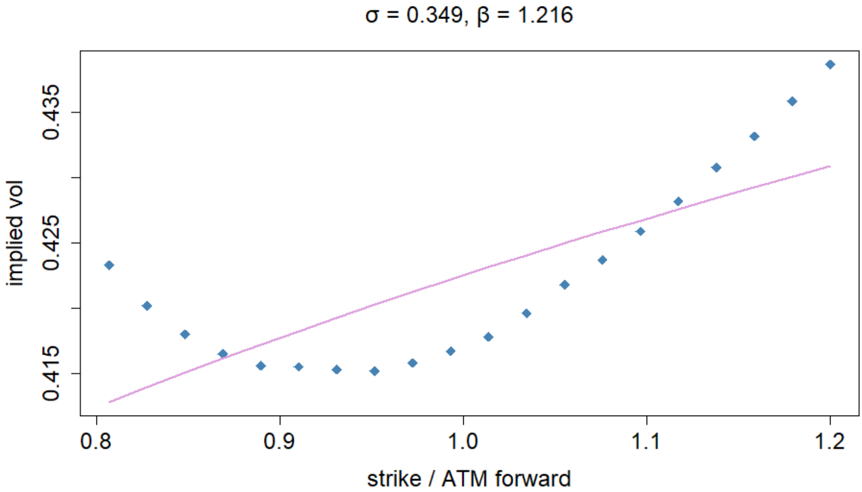

Figure 2: CEV fit for Henry Hub natural gas options expiring in 104 days.

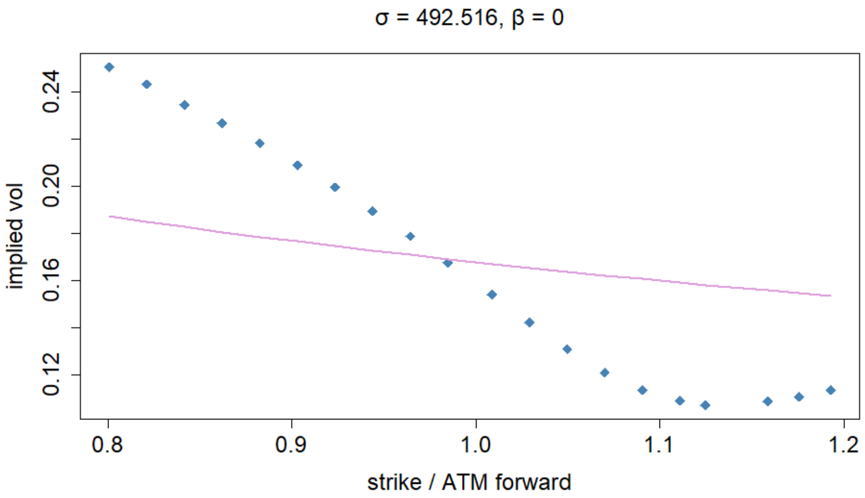

Figure 3: CEV fit for E-Mini S&P 500 futures options expiring in 171 days.

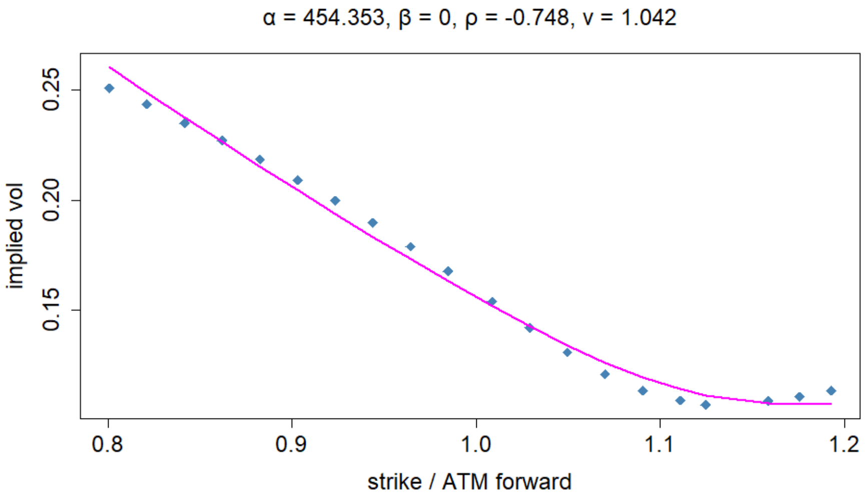

Figure 4: SABR fit for E-Mini S&P 500 futures options expiring in 171 days.

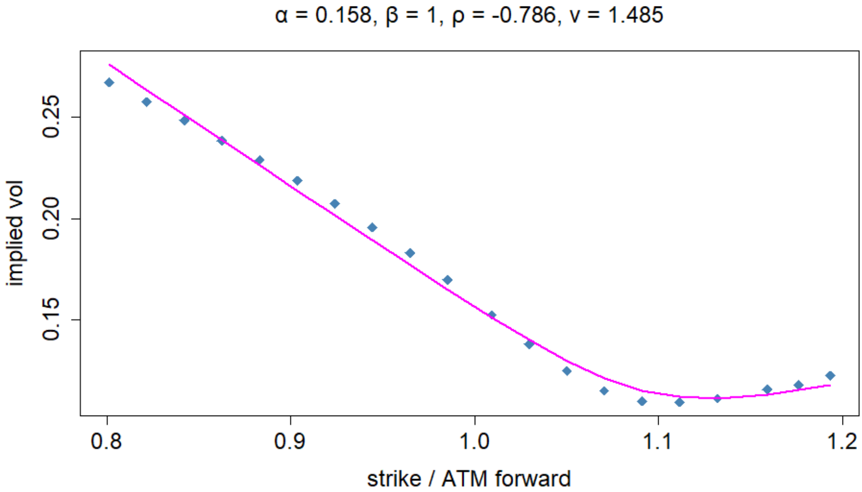

Figure 5: SABR fit for E-Mini S&P 500 futures options expiring in 108 days.

Shortcomings

- Possibility of extremely poor fits for $\beta = 0$

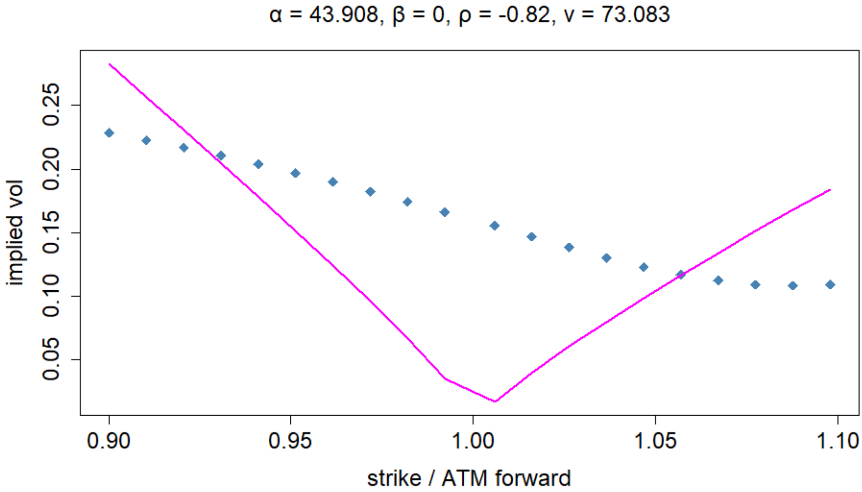

Figure 6: Terrible SABR fit to S&P E-Mini 500 futures options expiring in 79 days.

Shortcomings

- Sensitivity to initial parameter guesses for $\beta < 1$

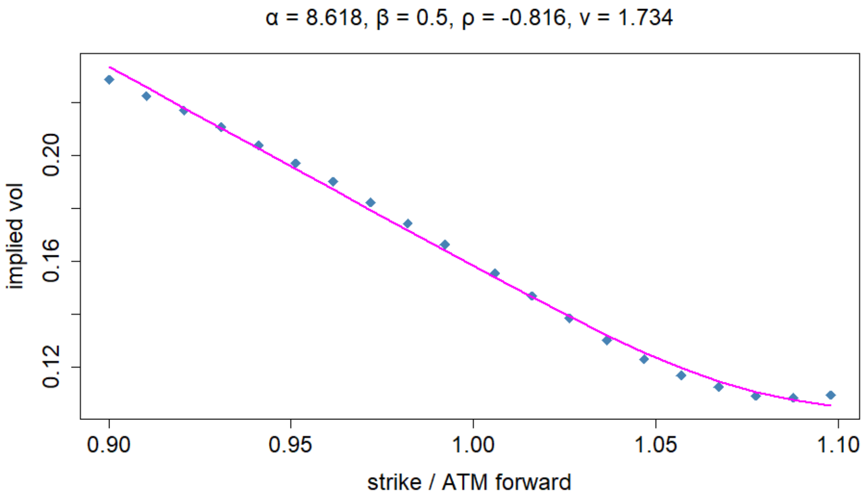

Figure 7: SABR fit to same data with initial guesses $\rho^* = 0.5$, $\nu^* = 0.9$

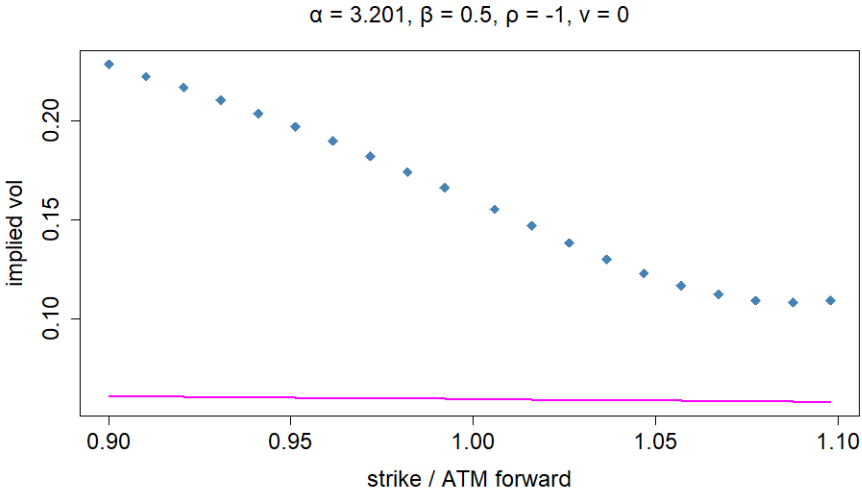

Figure 8: SABR fit to same data with initial guesses $\rho^* = \nu^* = 0.5$

Shortcomings

- Unable to closely fit noticeably concave skews

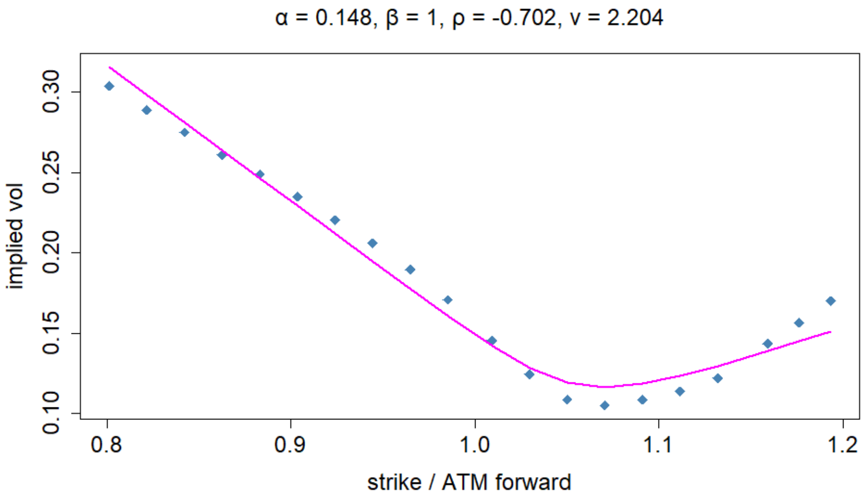

Figure 9: SABR fit to E-Mini S&P 500 futures options expiring in 43 days.

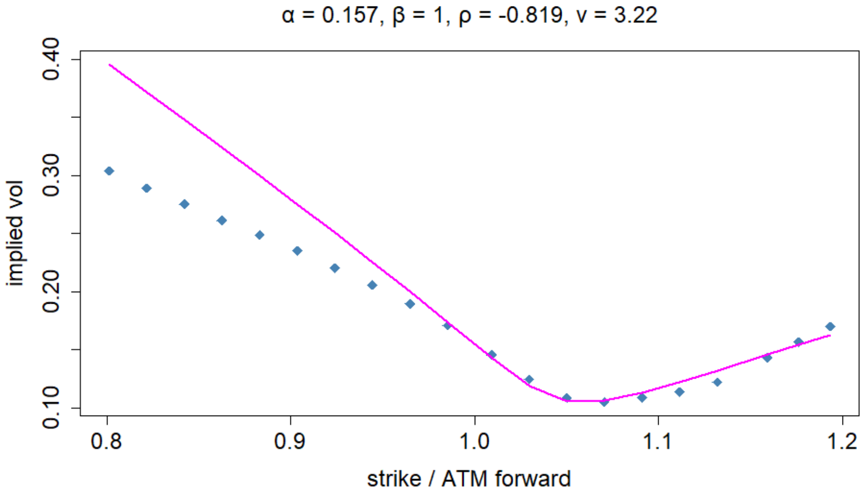

Figure 10: Weighted SABR fit to the same data, unable to fit the concave skew.

Conclusions

- More flexible and “realistic” fits compared to one-factor CEV model
- Interpolation formula = fast implied volatility calibration in real-time
- Complicated formula not perfect and may give poor fits

References

Hagan, P. S., Kumar, D., Lesniewski, A. S., & Woodward, D.E. (2002). Managing smile risk. *Wilmott*, *2002*(1), 84-108. http://web.math.ku.dk/~rolf/SABR.pdf

Hagan, P. S., & Woodward, D. E. (1999). Equivalent black volatilities. *Applied Mathematical Finance*, *6*(3), 147-157.