多元统计分析课程作业 1

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题目 1. 考虑点 (x_1, x_2) 的集合，对于 $c^2 = 1$ 和 $c^2 = 4$ 从原点到该点的距离由下式给出：

$$c^2 = 4x_1^2 + 3x_2^2 - 2\sqrt{2}x_1x_2$$

确定常数距离椭圆的长短轴及其相应的长度，画其草图并注明位置。当 c^2 增大时会怎么样？

解答.

改变上式形式，

$$\because c^2 = x'Ax \therefore a_{11} = 4, a_{22} = 3, a_{12} = a_{21} = -\sqrt{2} \therefore A = \begin{bmatrix} 4 & -\sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$$

计算 A 的特征值：

$$|A - \lambda I| = \begin{bmatrix} 4-\lambda & -\sqrt{2} \\ -\sqrt{2} & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 = 0 \therefore \lambda_1 = 5, \lambda_2 = 2$$

计算矩阵 A 的特征值归一化特征向量对：

当 $Ax = \lambda_1 x$ 时，

$$\therefore \begin{bmatrix} 4-\lambda & -\sqrt{2} \\ -\sqrt{2} & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 = -\sqrt{2}x_2 \therefore e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$$

当 $Ax = \lambda_2 x$ 时，

$$\therefore \begin{bmatrix} 4-\lambda & -\sqrt{2} \\ -\sqrt{2} & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_2 = \sqrt{2}x_1 \therefore e_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

当 $c^2 = 1$ 时，沿向量 e_1, e_2 的常距椭圆的半长度（长轴和短轴）为：

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{5}} = 0.447, \frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{2}} = 0.707$$

当 $c^2 = 4$ 时，沿向量 e_1, e_2 的常距椭圆的半长度（长轴和短轴）为：

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{5}} = 0.894, \frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{2}} = 1.414$$

结论是：当 c 增大时，常距椭圆的半长度（长轴和短轴）以 c 的比例同时增大。

题目 1 的注记. 第一题的图就不画了。

$$e = \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

题目 2. 已知:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

请计算平方根矩阵 $A^{\frac{1}{2}}$ 以及 $A^{-\frac{1}{2}}$。然后证明下式：

$$A^{\frac{1}{2}} A^{-\frac{1}{2}} = A^{-\frac{1}{2}} A^{\frac{1}{2}} = I$$

解答. 计算 A 的特征值：

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 1 = 0$$

$$\therefore \lambda_1 = \frac{5+\sqrt{5}}{2} = 3.618, \lambda_2 = \frac{5-\sqrt{5}}{2} = 1.382$$

计算矩阵 A 的特征值归一化特征向量对，当 $Ax = \lambda x$ 时

$$\therefore \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3.618 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow (x_1, x_2) = (1, 1.618) \therefore e_1 = (0.5257, 0.8507)^T$$

$$\text{同理:}\lambda_2 = 1.382, e_2 = (0.8507, -0.5257)^T$$

$$A^{\frac{1}{2}} = \sqrt{\lambda_1} e_1 e_1^T + \sqrt{\lambda_2} e_2 e_2^T = \begin{bmatrix} 1.376 & 0.325 \\ 0.325 & 1.701 \end{bmatrix}$$

$$A^{-\frac{1}{2}} = \frac{1}{\sqrt{\lambda_1}} e_1 e_1^T + \frac{1}{\sqrt{\lambda_2}} e_2 e_2^T = \begin{bmatrix} 0.760 & -0.145 \\ -0.145 & 0.615 \end{bmatrix}$$

验证则可得证:

$$A^{\frac{1}{2}} A^{-\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-\frac{1}{2}} A^{\frac{1}{2}}$$

题目 3. 已知:

$$A = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

(a) 计算 AA' 及其特征值和特征向量

(b) 计算 $A'A$ 及其特征值和特征向量，(b) 的结果和 (a) 的结果是否一致?

(c) 求出 A 的奇异值分解。

解答.

$$AA^T = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$$

$$0 = |A - \lambda I| = (144 - \lambda)(126 - \lambda) - (12)^2 = (150 - \lambda)(120 - \lambda)$$

$$\therefore \lambda_1 = 150, \lambda_2 = 120$$

当 $Ax = \lambda_1 x$ 时,

$$\begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 150 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{cases} 144x_1 - 12x_2 = 150x_1 \\ -12x_1 + 126x_2 = 150x_2 \end{cases}, x_1 = -2x_2$$

可得 (a)

$$\therefore e_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{同理}\lambda_2 = 120\text{时}, e_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{bmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{bmatrix} = 0$$

$$\therefore \text{解得}\lambda_1 = 150, \lambda_2 = 120, \lambda_3 = 0$$

当 $Ax = \lambda_1 x$ 时,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 150 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{cases} -120x_1 + 60x_2 = 0 \\ -25x_1 + 5x_3 = 0 \end{cases}, e_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

同理可得 (b)

$$e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, e_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(c) 根据 $A = \sqrt{\lambda_1} e_1 e_1^T + \sqrt{\lambda_2} e_2 e_2^T$

$$A = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \sqrt{150} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \sqrt{120} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

题目 4. 设 X 有协方差矩阵

$$A = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

(a) 确定 ρ 和 $V^{\frac{1}{2}}$

(b) 用矩阵乘法验证关系式:

$$V^{\frac{1}{2}} \rho V^{\frac{1}{2}} = \Sigma$$

解答.

$$\Sigma = Cov(x) = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

$$\therefore \sigma_{11} = 25, \sigma_{22} = 4, \sigma_{33} = 9, \sqrt{\sigma_{11}} = 5, \sqrt{\sigma_{22}} = 2, \sqrt{\sigma_{33}} = 3$$

已知总体相关系数 $\rho_{ik} = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{kk}}}$

$$\rho = \begin{bmatrix} \frac{25}{5 \cdot 5} & \frac{-2}{2 \cdot 5} & \frac{4}{3 \cdot 5} \\ \frac{-2}{2 \cdot 5} & \frac{4}{2 \cdot 2} & \frac{1}{3 \cdot 2} \\ \frac{4}{5 \cdot 3} & \frac{1}{2 \cdot 3} & \frac{9}{3 \cdot 3} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix}$$

已知 $p \times p$ 标准差矩阵 $V^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}$

$$V^{\frac{1}{2}} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$V^{\frac{1}{2}}\rho V^{\frac{1}{2}}=\begin{bmatrix}5&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&-\frac{1}{5}&\frac{4}{15}\\-\frac{1}{5}&1&\frac{1}{6}\\\frac{4}{15}&\frac{1}{6}&1\end{bmatrix}\begin{bmatrix}5&0&0\\0&2&0\\0&0&3\end{bmatrix}=\begin{bmatrix}25&-2&4\\-2&4&1\\4&1&9\end{bmatrix}=\Sigma\text{得证}$$

题目 4 的注记. 同理 $\rho=(V^{\frac{1}{2}})^{-1}\Sigma(V^{\frac{1}{2}})^{-1}$