多元统计分析期中考试

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题目 1. 2.20

已知:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

请计算平方根矩阵 $A^{\frac{1}{2}}$ 以及 $A^{-\frac{1}{2}}$。然后证明下式:

$$A^{\frac{1}{2}}A^{-\frac{1}{2}} = A^{-\frac{1}{2}}A^{\frac{1}{2}} = I$$

解答. 计算 A 的特征值:

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 1 = 0$$

$$\therefore \lambda_1 = \frac{5+\sqrt{5}}{2} = 3.618, \lambda_2 = \frac{5-\sqrt{5}}{2} = 1.382$$

计算矩阵 A 的特征值归一化特征向量对，当$Ax = \lambda x$时

$$\therefore \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3.618 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow (x_1, x_2) = (1, 1.618)$$

$$\therefore e_1 = (0.5257, 0.8507)^\top$$

同理:$\lambda_2 = 1.382, e_2 = (0.8507, -0.5257)^\top$

$$A^{\frac{1}{2}} = \sqrt{\lambda_1} e_1 e_1^\top + \sqrt{\lambda_2} e_2 e_2^\top = \begin{bmatrix} 1.376 & 0.325 \\ 0.325 & 1.701 \end{bmatrix}$$

$$A^{-\frac{1}{2}} = \frac{1}{\sqrt{\lambda_1}} e_1 e_1^\top + \frac{1}{\sqrt{\lambda_2}} e_2 e_2^\top = \begin{bmatrix} 0.760 & -0.145 \\ -0.145 & 0.615 \end{bmatrix}$$

验证则可得证：$A^{\frac{1}{2}}A^{-\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-\frac{1}{2}}A^{\frac{1}{2}}$

题目 2. 4.8.

已知 $X_1 \sim N(0,1)$

$$X_2 = \begin{cases} -X_1, & -1 \leq x_1 \leq 1 \\ X_1, & x_1 > 1 \text{或} x_1 < -1 \end{cases}$$

(a) 证明 $X_1 \sim N(0,1)$

(b) 证明 X_1, X_2 不符合二元正态分布

解答.

(a)

$$-1 \leq x_1 \leq 1 \Rightarrow x_2 = -x_1 \sim N(0,1)$$

$$x_1 > 1 \text{或} x_1 < -1 \Rightarrow x_2 = x_1 \sim N(0,1)$$

$$\therefore X_2 \sim N(0,1)$$

(b)

$$\text{对于二元正态分布，} x_1, x_2 \text{的联合分布应该是二元正态分布}$$

$$\text{而} x_1, x_2 \text{的联合分布不是二元正态分布，因为} x_1, x_2 \text{不是线性相关}$$

题目 3. 2.32

给定了随机向量 $\mathbf{X}^\top = (X_1, X_2, X_3, X_4, X_5)$，均值向量 $\mu^\top = (2, 4, -1, 3, 0)$，以及方差-协方差矩阵 $\mathbf{\Sigma}_{\boldsymbol{X}}$

$$\mathbf{\Sigma}_{\boldsymbol{X}} = \begin{bmatrix} 4 & -1 & 0.5 & -0.5 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 0.5 & 1 & 6 & 1 & -1 \\ -0.5 & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

X 划分为

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(2)} \end{bmatrix}$$

又

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

考虑 $\mathbf{AX}^{(1)}$ 和 $\mathbf{BX}^{(2)}$ 的线性组合，求：

(a) $E(\mathbf{X}^{(1)})$

(b) $E(\mathbf{AX}^{(1)})$

(c) $Cov(\mathbf{X}^{(1)})$

(d) $Cov(\mathbf{AX}^{(1)})$

(e) $E(\mathbf{X}^{(2)})$

(f) $E(\mathbf{B}\mathbf{X}^{(2)})$

(g) $Cov(\mathbf{X}^{(2)})$

(h) $Cov(\mathbf{B}\mathbf{X}^{(2)})$

(i) $Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$

(j) $Cov(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)})$

解答.

(a)

$$E(\mathbf{X}^{(1)}) = \mu^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(b)

$$\mathbf{A}\mu^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(c)

$$Cov(\mathbf{X}^{(1)}) = \mathbf{\Sigma_{11}} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

(d)

$$Cov(\mathbf{A}\mathbf{X}^{(1)}) = \mathbf{A}\mathbf{\Sigma_{11}}\mathbf{A}^\top = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

(e)

$$E(\mathbf{X}^{(2)}) = \mu^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

(f)

$$\mathbf{B}\mu^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

(g)

$$Cov(\mathbf{X}^{(2)}) = \mathbf{\Sigma_{22}} = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(h)

$$Cov(\mathbf{B}\mathbf{X}^{(2)}) = \mathbf{B}\mathbf{\Sigma_{22}}\mathbf{B}^{\top} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$$

(i)

$$Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(j)

$$Cov(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}) = \mathbf{A}\mathbf{\Sigma_{12}}\mathbf{B}^{\top} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

题目 4. 4.18.

试根据来自二维正态总体的随机样本

$$X=\begin{bmatrix}3&6\\4&4\\5&7\\4&7\end{bmatrix}$$

求 2×1 均值向量 μ 和 2×2 协方差矩阵 Σ 的极大似然估计

解答. 根据 4.11 结果，μ 和 Σ 的最大似然估计分别为 $\hat{\mu}=\bar{x}=[4,6]'$

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n}\left(x_j-\bar{x}\right)\left(x_j-\bar{x}\right)'\\
&=\frac{1}{4}\left\{\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}3\\6\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\4\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right.\\
&\left.+\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'+\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)\left(\begin{bmatrix}4\\7\end{bmatrix}-\begin{bmatrix}4\\6\end{bmatrix}\right)'\right\}\\
&=\frac{1}{4}\left(\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1&0\end{bmatrix}+\begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}0&-2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}\right)\\
&=\frac{1}{4}\begin{bmatrix}2&1\\1&6\end{bmatrix}
\end{aligned}$$

题目 5. 5.1

(a) 使用给定的数据计算检验统计量 T^2 以检验原假设 $H_0\colon \mu' = [7, 11]$,

$$\mathbf{X} = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$$

(b) 描述（a）部分中 T^2 统计量的分布情况。

(c) 结合（a）和（b）的结果，在 $\alpha = 0.05$ 的显著性水平下对原假设 H_0 进行检验。你会得出什么结论?

解答.

(a)

$$\text{变量数} p = 2, \text{样本大小} n = 4$$

$$\bar{x} = \begin{bmatrix} \frac{2+8+6+8}{n} \\ \frac{12+9+9+10}{n} \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix},$$

$$S_{ij} = \frac{1}{n-1} \sum_{k=1}^{n} (x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j), \mathbf{S} = \begin{bmatrix} 8 & -\frac{10}{3} \\ -\frac{10}{3} & 2 \end{bmatrix},$$

$$T^2 = n(\bar{x} - \mu')^\top \mathbf{S}^{-1} (\bar{x} - \mu') = \frac{150}{11} = 13.64$$

(b)

$$\because T^2 \sim \frac{(n-1)p}{(n-p)} F_{p,n-p},$$

$$\therefore T^2 \sim 3F_{2,2}$$

(c)

$$H_0\colon \mu' = [7, 11]$$
$$\because \alpha = 0.05 \therefore F_{2,2}(0.05) = 19$$
$$\because T^2 = 13.64 < 3F_{2,2}(0.05) = 57$$
$$\therefore \alpha = 0.05\text{时接受原假设}H_0$$

题目 6. 4.11.

A 是方阵，请证明：

$$\begin{aligned}|A| &= |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}| \quad (|A_{22}| \neq 0)\\ &= |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}| \quad (|A_{11}| \neq 0)\end{aligned}$$

解答.

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$
$$\text{两边取行列式}|A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|$$
$$\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$
$$\text{两边取行列式}|A| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|$$

题目 7. 设样本 $X_1, X_2, \cdots, X_n$ 是从正态总体 $N_p(\mu, \Sigma)$ 抽取的样本, 对于一切可能的 μ, Σ, 多元正态似然函数的最大值是:

$$\begin{aligned}\max_{\mu,\Sigma} L(\mu, \Sigma) &= \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma_\mu|} e^{-\frac{np}{2}} \\ &= \frac{1}{n}\sum_{i=1}^{n}(x_j - \bar{x})(x_j - \bar{x})^\top\end{aligned}$$

再假设 $H_0 : \mu = \mu_0$ 下正态似然函数的最大值为:

$$\max_{\Sigma} L(\mu_0, \Sigma) = \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma_0|} e^{-\frac{np}{2}}$$

$$\Sigma_0 = \frac{1}{n}\sum_{i=1}^{n}(x_j - \mu_0)(x_j - \mu_0)^\top$$

$$\text{验证:} \quad \Lambda^{\frac{2}{n}} = \frac{|\Sigma_\mu|}{|\Sigma_0|} = \left(1 + \frac{T^2}{n-1}\right)^{-1}$$

解答.

$$令(p+1)\times(p+1)矩阵\ \mathrm{A}\ 为$$

$$\begin{aligned}
A &= \begin{bmatrix} \sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^\top & \sqrt{n}(\bar{x}-\mu_0) \\ \sqrt{n}(\bar{x}-\mu_0)^\top & -1 \end{bmatrix} \\
&= \left|\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^\top\right| \left|-1-\sqrt{n}(\bar{x}-\mu_0)^\top A_{11}^{-1}\sqrt{n}(\bar{x}-\mu_0)\right| \\
&= (-1)\left|\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^\top+\sqrt{n}(\bar{x}-\mu_0)\sqrt{n}(\bar{x}-\mu_0)\right|
\end{aligned}$$

$$\boxed{\begin{aligned}
\sum_{i=1}^{n}(x_i-\mu_0)(x_i-\mu_0)^\top &= \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu_0)(x_i-\bar{x}+\bar{x}-\mu_0)^\top \\
&= \sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^\top+n(\bar{x}-\mu_0)(\bar{x}-\mu_0)^\top
\end{aligned}}$$

$$两边取行列式并乘\text{-}1:$$

$$(-1)\left|\sum_{i=1}^{n}(x_i-\mu_0)(x_i-\mu_0)^\top\right| = (-1)\left|\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^\top+n(\bar{x}-\mu_0)(\bar{x}-\mu_0)^\top\right|$$

$$\Rightarrow (-1)\left|\sum_{i=1}^{n}(x_i-\mu_0)(x_i-\mu_0)^\top\right| = \left|\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^\top\right|\left|-1-n(\bar{x}-\mu_0)^\top A_{11}^{-1}(\bar{x}-\mu_0)\right|$$

$$\Rightarrow (-1)\left|n\Sigma_0\right| = \left|n\Sigma\right|\left|-1-n(\bar{x}-\mu_0)^\top\frac{S^{-1}}{n-1}(\bar{x}-\mu_0)\right|$$

$$\boxed{\begin{aligned}
&\because S_{p\times p} = \frac{1}{n-1}\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^\top \\
&\therefore S^{-1} = \frac{n-1}{\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^\top} \\
&\therefore \left[\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^\top\right]^{-1} = \frac{S^{-1}}{n-1}
\end{aligned}}$$

$$\Rightarrow \left|n\Sigma_0\right| = \left|n\Sigma\right|\left|1+\frac{T^2}{n-1}\right|$$

$$\Rightarrow \frac{\Sigma_0}{\Sigma} = 1+\frac{T^2}{n-1}$$