Estimating total magnetization direction using equivalent-layer technique

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ABSTRACT

We have developed a new method for estimating the total magnetization direction of magnetic sources based on equivalent layer technique using total field anomaly data. In this approach, we do not have to impose a strong information about the shape and the depth of the sources, and do not require a regularly spaced data. Usually, this technique is used for processing potential data estimating a 2D magnetic moment distribution over a fictitious layer composed by dipoles below the observation plane. In certain conditions, when the magnetization direction of equivalent sources is almost the same of true body, the estimated magnetic property over the layer is all positve. The methodology uses a positivity constraint to estimate a set of magnetic moment over the layer and a magnetization direction of the layer through a iterative process. Mathematically, the algorithm solve a least squares problem in two steps: the first one solve a linear problem for estimating a magnetic moment and the second solve a non-linear problem for magnetization direction of the layer. We test the methodology applying to synthetic data for different geometries and

magnetization types of sources. Moreover, we applied this method to field data from Goias Alkaline Province (GAP), center of Brazil (FALAR DO RESULTADO DAS APLICAES NAS ANOMALIAS)

METHODOLOGY

Fundamentals of magnetic equivalent layer and the positive magnetic-moment distribution

Considering a Cartesian coordinate system with x-, y- and z-axis being oriented to north, east and downward, respectively. Let $\Delta T_i \equiv \Delta T(x_i, y_i, z_i)$ be the total field anomaly, at the ith position (x_i, y_i, z_i), produced by a continuos layer located below the observation plane on the depth z_c, where $z_c > z_i$, and $p(x', y', z_c)$ is the distribution of magnetic dipoles per unit area over the layer surface. In this case, the total-field anomaly produced by a continuous layer is given by equation

$$\Delta T_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x', y', z_c)[\gamma_m \hat{\mathbf{F}}_0^T \mathbf{H}(x_i, y_i, z_i, x', y', z_c)\, \hat{\mathbf{h}}(\mathbf{q})] dx'\, dy', \tag{1}$$

where γ_m is a constant proportional to the vacuum permeability, $\hat{\mathbf{F}}_0$ is a unit vector with the same direction of the main geomagnetic field given by

$$\hat{\mathbf{F}}_0 = \begin{bmatrix} \cos I \cos D \\ \cos I \sin D \\ \sin I \end{bmatrix}, \tag{2}$$

where I and D are the inclination and declination, respectively, and $\mathbf{H}(x_i, y_i, z_i, x', y', z_c)$ is a 3×3 dimensional matrix equal to

$$\mathbf{H}(x_i, y_i, z_i, x', y', z_c) = \begin{bmatrix} \partial_{xx}\phi & \partial_{xy}\phi & \partial_{xz}\phi \\ \partial_{yx}\phi & \partial_{yy}\phi & \partial_{yz}\phi \\ \partial_{zx}\phi & \partial_{zy}\phi & \partial_{zz}\phi \end{bmatrix}, \tag{3}$$

where $\partial_{\alpha\beta}\phi$, $\alpha = x, y, z$ and $\beta = x, y, z$, is the second derivative of the scalar function

$$\phi(x_i, y_i, z_i, x', y', z_c) = \frac{1}{[(x_i - x')^2 + (y_i - y')^2 + (z_i - z_c)^2]^{\frac{1}{2}}}. \tag{4}$$

with respect to the Cartesian coordinates x_i, y_i and z_i of the observation points. The $\hat{\mathbf{h}}(\mathbf{q})$ is a unit vector with the magnetization direction of the layer given by

$$\hat{\mathbf{h}}(\mathbf{q}) = \begin{bmatrix} \cos\tilde{\imath}\cos\tilde{d} \\ \cos\tilde{\imath}\sin\tilde{d} \\ \sin\tilde{\imath} \end{bmatrix} \tag{5}$$

and $\mathbf{q}$ is a 2×1 vector with components given by

$$\mathbf{q} = \begin{bmatrix} \tilde{\imath} \\ \tilde{d} \end{bmatrix}, \tag{6}$$

where $\tilde{\imath}$ and $\tilde{d}$ is the inclination and declination of magnetization of the layer, respectively. We can also notice that the vector defined in equation 5 represents the uniform magnetization direction on the layer. For convenience, this unit vector can be rewritten as follows

$$\hat{\mathbf{h}}(\mathbf{q}) = \mathbf{R}\hat{\mathbf{m}}\,, \tag{7}$$

where $\hat{\mathbf{m}}$ defines the uniform magnetization direction of an abitrary magnetic source and $\mathbf{R}$ is a 3×3 matrix obtained from Euler's rotation theorem. This theorem states that any rotation can be parametrized by using three parameters called Euler angles (CITAR GOLDSTEIN). That is, if the unit vector $\hat{\mathbf{h}}(\mathbf{q})$ (equation 5) has the same direction as unit vector $\hat{\mathbf{m}}$ in the direction of the magnetic source, the matrix $\mathbf{R}$ is equal to identity (equation 7). For this reason, the total-field anomaly produced by equivalent layer at the ith position (x_i, y_i, z_i) (equation 1) can be rewritten as

$$\Delta T_i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x', y', z_c)[\gamma_m \hat{\mathbf{F}}_0^T \mathbf{H}(x_i, y_i, z_i, x', y', z_c)\, \hat{\mathbf{m}}]dx'\, dy', \tag{8}$$

which represents the total-field anomaly produced by continuous layer with the same direction of the arbitrary magnetic source. Thus, the RTP field ΔT_i^{PL} produced by equivalent layer at the point (x_i, y_i, z_i) is equal to

$$\Delta T_i^{PL} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x', y', z_c)[\gamma_m \partial_{zz}\phi(x_i, y_i, z_i, x', y', z_c)]dx'\, dy', \tag{9}$$

where $\partial_{zz}\phi(x_i, y_i, z_i, x', y', z_c)$ is the second derivative of the inverse of distance (equation 4) with respect of z, evaluated at the point (x_i, y_i, z_i). However, considering the RTP field ΔT_i^{PS} produced by an arbitrary uniformly magnetized source equal to

$$\Delta T_i^{PS} = \gamma_m \partial_{zz}\Gamma(x_i, y_i, z_i)\, m, \tag{10}$$

which represents the total-field anomaly produced at the pole and m is the magnetization intensity of the magnetic source. The $\partial_{zz}\Gamma(x_i, y_i, z_i)$ is the second derivative in relation to z of a scalar function $\Gamma(x_i, y_i, z_i)$ that depends on source geometry and the observation point (x_i, y_i, z_i). By comparing equation 9 and 10, we obtain

$$m\, \partial_{zz}\Gamma(x_i, y_i, z_i) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x', y', z_c)\partial_{zz}\phi(x_i, y_i, z_i, x', y', z_c)dx'\, dy'. \tag{11}$$

We can notice that equation 11 can be calculated differentiating the following equation

$$m\, \partial_{z}\Gamma(x_i, y_i, z_i) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{p(x', y', z_c)(z_c - z_i)}{[(x_i - x')^2 + (y_i - y')^2 + (z_i - z_c)^2]^{\frac{3}{2}}} dx'\, dy', \tag{12}$$

where $z_c > z_i$, with respect to the vertical component z. From potential field theory, we can highlight the *classical upward continuation integral*

$$U(x_i, y_i, z_i) = \frac{(z_c - z_i)}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{U(x', y', z_c)}{[(x_i - x')^2 + (y_i - y')^2 + (z_i - z_c)^2]^{\frac{3}{2}}} dx'\, dy', \qquad (13)$$

where the function $U(x_i, y_i, z_i$ is an hamornic function at all (x_i, y_i, z_i) (CITAR BLAKELY). In this case, this function represents the total-field anomaly at the point (x_i, y_i, z_i) and also can be reproduced as the convolution between its values $U(x', y', z_c)$ and the vertical derivative in relation to z of the equation 4, evaluated on the horizontal plane $z = z_c$. Therefore, according the classical upward continuation function (equation 13), we can notice that the magnetic-moment distribution $p(x', y', z_c)$ in equation 12 assumes the form

$$p(x', y', z_c) = \frac{m}{2\pi}\, \partial_z \Gamma(x', y', z_c), \qquad (14)$$

where $\partial_z \Gamma(x', y', z_c)$ is the derivative of the scalar function $\partial_z \Gamma(x_i, y_i, z_i)$ in relation to z evaluated over the equivalent layer. The most interesting aspect of equation 14 is that the magnetic-moment distribution is defined as the product of a positive constant $\frac{m}{2\pi}$ and the function $\partial_z \Gamma(x', y', z_c)$, which is all positive at all points (x', y', z_c) over the equivalent layer. This relation is simular to that presented by (CITAR PEDERSEN, LI, LIMA and BARATCHART). By following different appproches, they proved the existence of all-positive magnetic moment distribution within a continuous layer. However, until this moment, their approach is valid only for the case in which the observed total-field anomaly is produced by magnetic sources having a purely and vertical induced magnetization. In the case of LIMA AND BARATCHART, their approach uses magnetic microscopy data for the aplication of this property and not relating this property to an analytical mathematical approach to this problem. Our approach is very important owing to some aspects, (1) does not impose an induced magnetization within the layer in any moment of mathematical development, (2) holds true for all cases in which the magnetization of the planar equivalent layer has the same direction of the true magnetic source, whenever is is purely magnetized or not, and (3) does not require that the observed total-field anomay be on a plane.

Forward problem for magnetic equivalent-layer technique

However, in practical situations, its not possible to determine a continuous magnetic-moment distribution $p(x', y', z_c)$ over the layer as shown in equation 1. For this reason, the continuous equivalent layer have to be approximated by a discrete set of M dipoles with unit volume located at a constant depth $z = z_c$. Let $\mathbf{p}$ be an M-dimensional parameter vector, whose jth element p_j is the magnetic intensity of the jth dipole and $\mathbf{q}$ be a vector containing the inclination $\tilde{i}$ and declination $\tilde{d}$ of all dipole, analogously to equation 6. Mathematically, by discretizing the integrand of equation 1, the total-field anomaly produced by equivalent layer at the point (x_i, y_i, z_i) is given by

$$\Delta T_i(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{M} p_j g_{ij}(\mathbf{q}) \tag{15}$$

where

$$g_{ij}(\mathbf{q}) = \gamma_m \hat{\mathbf{F}}_0^T \mathbf{H}_{ij} \hat{\mathbf{h}}(\mathbf{q}) \tag{16}$$

is an harmonic function representing the total-field anomaly produced at the ith position (x_i, y_i, z_i) by a dipole located at (x_j, y_j, z_c) with unitary magnetic-moment intensity. The matrix $\mathbf{H}_{ij}$ is formed by the second derivatives of a function ϕ_{ij} that depends on the inverse of the scalar function $r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_c)^2]^{1/2}$, analogously to equation

3 and 4. In matrix notation, the equation 15 can be represented as

$$\mathbf{\Delta T}(\mathbf{p}, \mathbf{q}) = \mathbf{G}(\mathbf{q})\mathbf{p} \tag{17}$$

where $\mathbf{G}(\mathbf{q})$ is an $N \times M$ matrix whose ijth element is defined by the harmonic function $g_{ij}(||)$ (equation 16) and $\mathbf{\Delta T}(\mathbf{p}, \mathbf{q})$ is an $N \times 1$ vector whose the ith element is the predicted total-field anomaly $\Delta T_i(\mathbf{p}, \mathbf{q})$ (equation 15). As can be noticed from equation 15-17, the predicted total-field anomaly produced by equivalent layer has a linear relation with the magnetic moment $\mathbf{p}$ and a nonlinear relation with the magnetization direction $\mathbf{q}$.

Iterative process for magnetization estimation

Let $\mathbf{\Delta T}^o$ be an N-dimensional vector whose ith element ΔT_i^o is the total-field anomaly observation produced by a magnetic source at the point (x_i, y_i, z_i), $i = 1, \ldots, N$. The estimation of a set of magnetic moments $\mathbf{p}$ and the magnetization direction $\mathbf{q}$ consists to formulate an inverse problem by imposing a positivity constraint on the magnetic-moment distribution. It can be performed by minimizing the difference between the observed data $\mathbf{\Delta T}^o$ and the predicted data $\Delta T(\mathbf{p}, \mathbf{q})$ (equation 17). In other words, a stable estimates $\mathbf{p}^\sharp$ and $\mathbf{q}^\sharp$ can be obtained by minimizing the objective function given by

$$\Psi(\mathbf{p}, \mathbf{q}) = \| \mathbf{\Delta T}^o - \mathbf{\Delta T}(\mathbf{p}, \mathbf{q}) \|_2^2, \tag{18}$$

where $\Psi(\mathbf{p}, \mathbf{q})$ is the data misfit, which is the Euclidean norm of the difference between the $\mathbf{\Delta T}^o$ and $\mathbf{\Delta T}(\mathbf{p}, \mathbf{q})$.

The procedure of finding a set of magnetic moment $\mathbf{p}^\sharp$ and magnetization direction $\mathbf{q}^\sharp$ which minimize the equation 18 consists to solve an inverse problem for estimating a set of parameters in two steps. Therefore, we split the inverse problem in a mixed solution of two systems of equations. The first one solves a linear system for estimating the part of the magnetic moment. Secondly, the problem is solved through a non-linear process to calculate successive approximations for the part of the magnetization direction at each iteration along the process.

However, at the kth iteration, we impose positivity constraint on the magnetic-moment distribution estimate $\mathbf{p}^k$ by solving the following constrained problem of

$$\begin{aligned} &\text{minimizing} \quad \|\Delta\mathbf{T}^o - \mathbf{G}(\mathbf{q}_{k-1})\mathbf{p}^k\|_2^2 \\ &\text{subject to} \quad \mathbf{p}^k \geqslant 0 \end{aligned} \tag{19}$$

where $\mathbf{G}(\mathbf{q}_{k-1})$ is the $N \times M$ matrix defined in equation 17, $\|\cdot\|_2^2$ represents the squared Euclidean norm and $\mathbf{p}^k \geqslant 0$ means that the magnetic moments of all equivalent sources are positive. This problem is solved by using the nonnegative least squares (NNLS) proposed by (CITAR LAWSON HANSON 1974). In other words, we solve a linear system with positivity constraint at each kth iteration given by the equation

$$\mathbf{p}^k = \left(\mathbf{G}_p^{(k)T}\mathbf{G}_p^{(k)}\right)^{-1}\mathbf{G}_p^{(k)T}\Delta\mathbf{T}^o \tag{20}$$

where $\mathbf{G}_p^{(k)}$ is the magnetic-moment sensitivity matrix at the kth iteration. The elements of this matrix are composed by derivative of equation 15 in relation of jth element of the vector $\mathbf{p}^k$.

After estimating the magnetic-moment distribution $\mathbf{p}^k$ at the kth iteration using the previous estimate $\mathbf{q}_{k-1}$ for the magnetization direction, we estimate a new vector $\mathbf{q}^k$ by solving an unconstrained nonlinear inverse problem of minimizing the squared Euclidean norm of the difference between the observed and predicted total-field anomalies. In this nonlinear inversion we use the Levenberg-Marquardt method (CITAR ASTER). That is, we calculate at each kth iteration the step $\mathbf{\Delta q}^k$ for the magnetization direction by using the equation

$$\Delta\mathbf{q}^k = (\mathbf{G}_q^{(k)T}\mathbf{G}_q^{(k)} + \lambda\mathbf{I})^{-1}\mathbf{G}_q^{(k)T}\mathbf{r}^k \tag{21}$$

where λ is the Marquardt parameter that is updated along the iterative process, $\mathbf{I}$ is a indentity matrix, and the residual at the kth iteration $r^k = \Delta\mathbf{T}^o - \mathbf{\Delta T}(\mathbf{p}^k, \mathbf{q}^{k-1})$. $\mathbf{G}_q^k$ is a sensitivity matrix of the magnetization direction part, whose elements are composed by derivative of equation 15 in relation of each component of the vector $\mathbf{q}^k$, that are the inclination and declination, respectively. The iterative process stops when the squared Euclidean norm of the difference between the observed data $\mathbf{\Delta T}^o$ and predicted data $\mathbf{\Delta T}(\mathbf{p}, \mathbf{q})$ (equation 17) is invariant along succesive iterations (Figure 2).

The choice of layer depth $\mathbf{z_c}$ and regularization parameter μ

The procedure for the use of our methodology for estimating the total magnetization require the choice of two main parameters. The first one is the layer depth z_c as shown in figure 1 and the second is the regularization parameter μ shown in equation 20.

The method of the choice of layer is based on a classical approach proposed by (CITAR DAMPNEY). The author pointed out that the layer depth should satisfy an interval from 2.5 to 6 times the grid spacing below the observation plane. It should be notice that the rule proposed by (CITAR DAMPNEY) was applied on a regularly spaced data grid. Analogously to (DAMPNEY), the choice for applying our method should correspond to an interval from

2 to 3 times to the lower grid spacing in the case of an airborne survey, for example. It is very important to mention that the upper bound in this case was based empirically.

To solve the equation 20 we have to choose a reliable regularization parameter μ. For this purpose, we use the L-curve method proposed by (CITAR HANSEN 1992). This approach is widely used in the literature to find a regularization parameter which filtering out enough noise whithout loosing to much information in the final solution. The procedure of finding the parameter is basically to plot a curve of optimal values between the solution norm and residual norm. The corner of the curve is the final result which gives a threshold between the regularization function and the data misfit. (INVESTIGAR MELHOR)

APPENDIX A

CONSEQUENCES OF HIGH-LATITUDE ESTIMATION

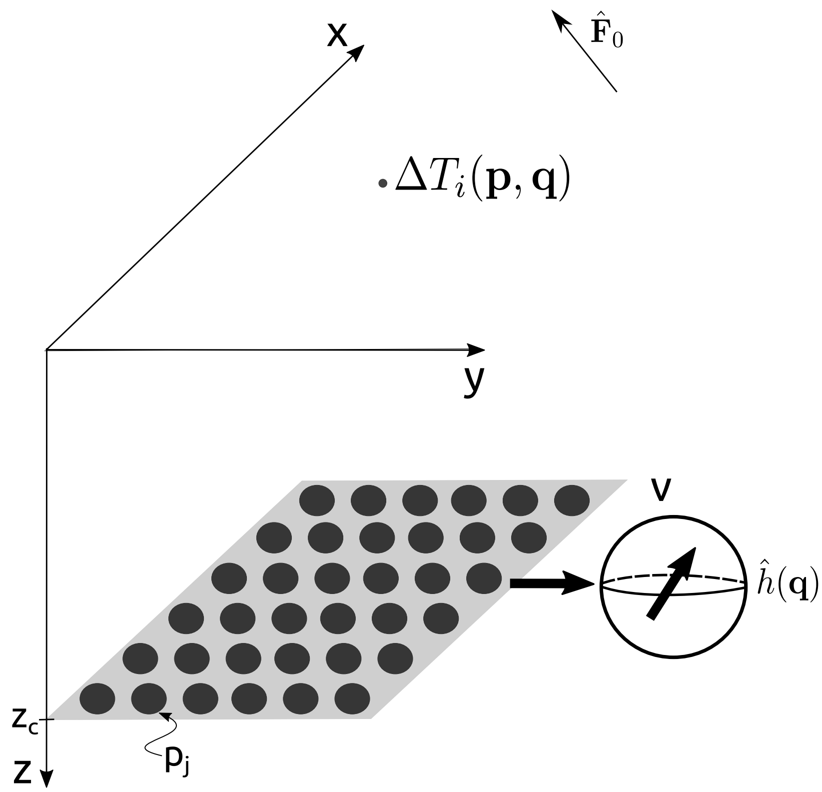

Figure 1: Schematic representation of an equivalent layer. The layer is positioned over the horizontal plane at a depth of $z = z_c$. $\Delta T_i(\mathbf{p}, \mathbf{q})$ is the predicted total-field anomaly at the point (x_i, y_i, z_i) produced by the set of M equivalent sources (black dots). Each source is located at the point (x_j, y_j, z_c), $j = 1, \ldots, M$, and represented by a dipole with unity volume υ with magnetization direction $\hat{\mathbf{h}}(\mathbf{q})$ and magnetic moment p_j.

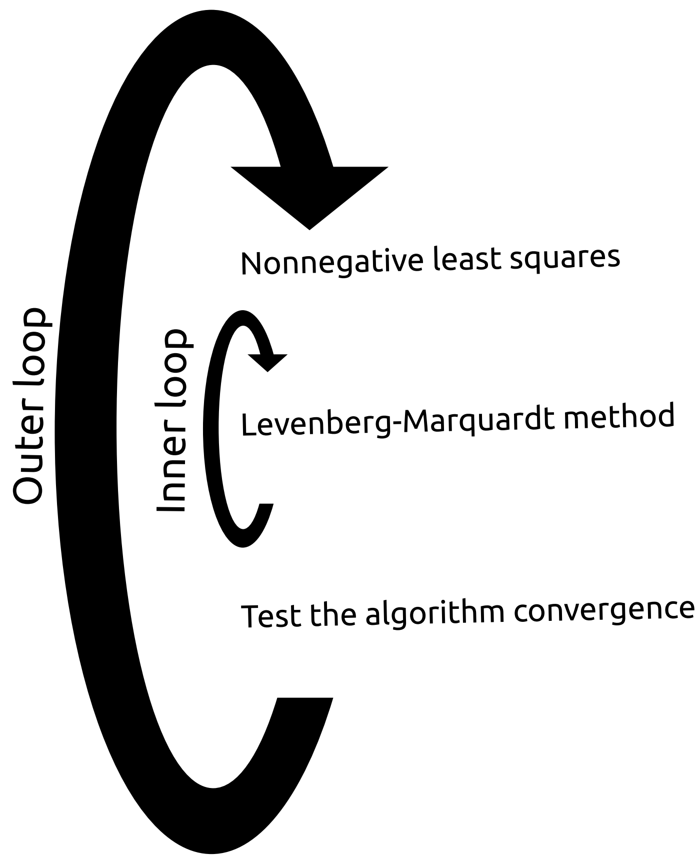

Figure 2: Iterative scheme overview for NNLS and Levenberg-Marquardt method for estimating magnetization direction. The outer loop is the nonnegative solution for magnetic-moment distribution and the inner loop calculates the magnetization direction using Levenberg-Marquardt method.