DISCO

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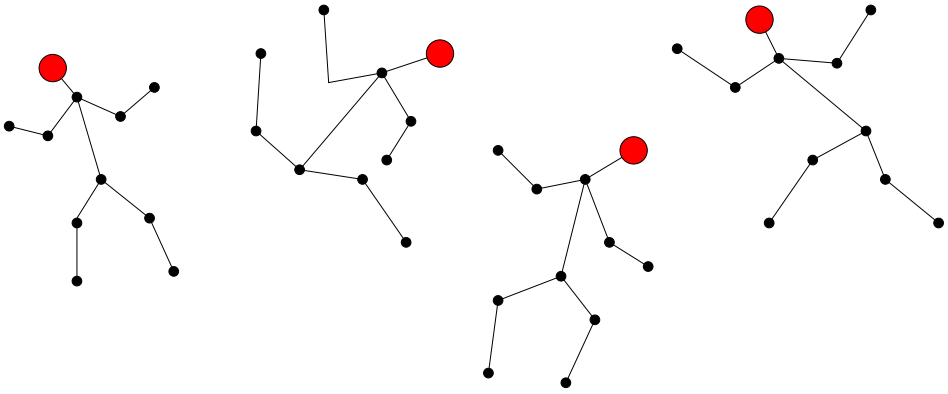

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Chapter 1

Introduction

To analyze the behavior of continuous materials and complex structures or systems, it is generally necessary to apply some sort of discretisation, both in space and in time.

Spatial discretisation implies that the behavior of the system is described by the time-varying state of a finite number of points, the *nodes* of the system. The state of the nodes is characterized by the value of one or more variables. These values have to satisfy a set of differential equations, resulting from the physics of the system.

As analytical solutions of the set of coupled differential equations seldom exist, approximate solutions must be determined following numerical integration procedures, which are based on a discretisation of time. This implies that solutions of the differential equations are determined at a finite number of moments in time.

Some numerical integration procedures are described in this report. They can be applied to study the dynamic behavior of mechanical, electrical and fluidial systems, which are analogous as far as their behavior is described by the same type of differential equations.

Chapter 2

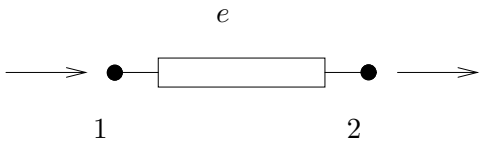
Discrete systems

The behavior of a discrete system can be described by the time-varying value of a *point variable* p in a number of discrete points, the *nodes*. These nodes are connected by line *elements*. Through the line elements flows the *flow variable* f. Point variables are generally called *degrees of freedom* (dof's), flow variables are referred to as (generalized) *internal forces*.

Flow and point variables may be scalars or vectors. In the latter case we consider their components in a global Cartesian coordinate system and a sign convention is necessary. For the scalar quantities we define a positive direction for the flow variable with relation to a single element. The element flow variable is generally referred to as the *internal* flow variable f_i. The two internal flows through the element nodes are stored in a column $\underset{\sim}{f}{}^e_i$. The two nodal point values are stored in the column $\underset{\sim}{p}{}^e$. [1]

The value of the flow variable f and/or its time derivative $\dot{f}$ in an element is related to the nodal differences of the point variable p or its time derivative $\dot{p}$ or its second-order time derivative $\ddot{p}$. Derivatives of f can be integrated over time.

Internal flow variables are denoted with indices a, b and c to indicate the order-difference of the time derivative of $\underset{\sim}{f}_i$ and $\underset{\sim}{p}$.



$$\underset{\sim}{f}{}^e_i = \begin{bmatrix} f^e_{i1} \\ f^e_{i2} \end{bmatrix} \quad ; \quad \underset{\sim}{p}{}^e = \begin{bmatrix} p^e_1 \\ p^e_2 \end{bmatrix}$$

Fig. 2.1 : *One-dimensional element with 2 nodes.*

first-order systems

$$\underset{\sim}{f}{}^e_{ia} = \underset{\sim}{f}{}^e_{ia}(\underset{\sim}{p}{}^e, \dot{\underset{\sim}{p}}{}^e) \quad ; \quad \underset{\sim}{f}{}^e_{ib} = \underset{\sim}{f}{}^e_{ib}(\underset{\sim}{p}{}^e) \quad ; \quad \dot{\underset{\sim}{f}}{}^e_{ic} = \dot{\underset{\sim}{f}}{}^e_{ic}(\underset{\sim}{p}{}^e) \quad \rightarrow \quad \underset{\sim}{f}{}^e_{ic} = \int_0^t \dot{\underset{\sim}{f}}{}^e_{ic}(\underset{\sim}{p}{}^e)\, d\tau$$

second-order systems

$$\underset{\sim}{f}{}^e_{ia} = \underset{\sim}{f}{}^e_{ia}(\underset{\sim}{p}{}^e, \dot{\underset{\sim}{p}}{}^e, \ddot{\underset{\sim}{p}}{}^e) \quad ; \quad \underset{\sim}{f}{}^e_{ib} = \underset{\sim}{f}{}^e_{ib}(\underset{\sim}{p}{}^e, \dot{\underset{\sim}{p}}{}^e) \quad ; \quad \underset{\sim}{f}{}^e_{ic} = \underset{\sim}{f}{}^e_{ic}(\underset{\sim}{p}{}^e)$$

[1] The upper-right index e indicates an element variable.

Assembling and equilibrium

The system behavior is described by a set of differential equations resulting from an assembling process which implies that the elements are connected. This procedure follows the following rules :

1 In every node the nodal point variables of the connected elements must be equal.
2 In every node the internal flow variables must be added.

The *external* nodal flows are stored in the column $\underset{\sim}{f}_e$ and may be zero, prescribed or unknown. The internal flow variables in each node must equal the external applied flow : $\underset{\sim}{f}_i = \underset{\sim}{f}_e$. The above equations represent *balance laws*.

Solving the system of differential equations implies that initial conditions are prescribed. In most cases the system will be at rest initially.

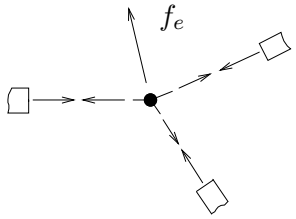
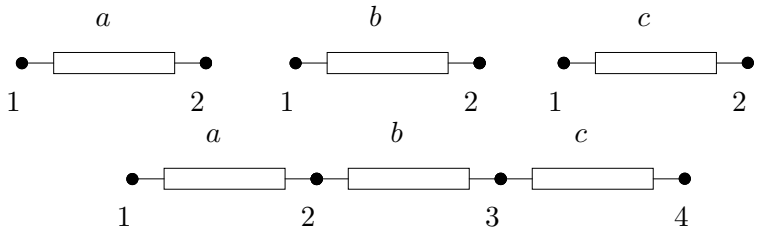


Fig. 2.2 : *Three elements connected in a node.*

$$\underset{\sim}{f}_i = \underset{\sim}{f}_{ia} + \underset{\sim}{f}_{ib} + \underset{\sim}{f}_{ic} = \underset{\sim}{f}_e \qquad \text{or} \qquad \dot{\underset{\sim}{f}}_i = \dot{\underset{\sim}{f}}_{ia} + \dot{\underset{\sim}{f}}_{ib} + \dot{\underset{\sim}{f}}_{ic} = \dot{\underset{\sim}{f}}_e$$

$$\underset{\sim}{p}(t=0) \quad ; \quad \dot{\underset{\sim}{p}}(t=0) \quad ; \quad \ddot{\underset{\sim}{p}}(t=0)$$

As an example we consider a one-dimensional system with three elements a, b and c. The element equations relate element nodal flow variables $\underset{\sim}{f}_i$ and element nodal point variables $\underset{\sim}{p}$ or their derivatives $\dot{\underset{\sim}{p}}$, respectively. The element nodes 1 and 2 are connected to the system nodes 1, 2, 3 and 4.



$$\begin{bmatrix} f_{i1} \\ f_{i2} \\ f_{i3} \\ f_{i4} \end{bmatrix} = \begin{bmatrix} f^a_{i1} \\ f^a_{i2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f^b_{i1} \\ f^b_{i2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f^c_{i1} \\ f^c_{i2} \end{bmatrix} = \begin{bmatrix} f^a_{i1} \\ f^a_{i2} + f^b_{i1} \\ f^b_{i2} + f^c_{i1} \\ f^c_{i2} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}_i = \underset{\sim}{f}_{ia} + \underset{\sim}{f}_{ib} + \underset{\sim}{f}_{ic}$$

2.1 Linear systems

If the relation between internal flow variables and point variables is linear for each element, assembling and applying nodal equilibrium results in a set of linear differential equations. The linear properties are denoted with the constant matrices $\underline{A}^e$, $\underline{B}^e$ and $\underline{C}^e$.

first-order systems

$$\left.\begin{aligned} \underset{\sim}{f}^e_{ia} &= \underline{A}^e \dot{\underset{\sim}{p}}^e \\ \underset{\sim}{f}^e_{ib} &= \underline{B}^e \underset{\sim}{p}^e \\ \dot{\underset{\sim}{f}}^e_{ic} &= \underline{C}^e \underset{\sim}{p}^e \quad \rightarrow \quad \underset{\sim}{f}^e_{ic} = \underline{C}^e \int_0^t \underset{\sim}{p}^e \, d\tau \end{aligned}\right\} \quad \Rightarrow$$

$$\underline{A}\,\dot{\underset{\sim}{p}} + \underline{B}\,\underset{\sim}{p} + \underline{C} \int \underset{\sim}{p} = \underset{\sim}{f}_e \qquad \rightarrow \qquad \underline{A}\,\ddot{\underset{\sim}{p}} + \underline{B}\,\dot{\underset{\sim}{p}} + \underline{C}\,\underset{\sim}{p} = \dot{\underset{\sim}{f}}_e$$

second-order systems

$$\left.\begin{aligned} \underset{\sim}{f}^e_{ia} &= \underline{A}^e \ddot{\underset{\sim}{p}}^e \\ \underset{\sim}{f}^e_{ib} &= \underline{B}^e \dot{\underset{\sim}{p}}^e \\ \underset{\sim}{f}^e_{ic} &= \underline{C}^e \underset{\sim}{p}^e \end{aligned}\right\} \quad \Rightarrow \quad \underline{A}\,\ddot{\underset{\sim}{p}} + \underline{B}\,\dot{\underset{\sim}{p}} + \underline{C}\,\underset{\sim}{p} = \underset{\sim}{f}_e$$

2.1.1 Time discretisation

The system behavior is described by a set of first- or second-order differential equations. As analytical solutions only exist for very simple problems, numerical integration methods are used to determine the solution. The total time interval is subdivided into a number (nic) of time increments $t \le \tau \le t + \Delta t$ or $t_n \le \tau \le t_{n+1}$; $n = 0, 1, .., nic - 1$, which are assumed to be equally spaced, although this is by no means necessary and often even not desirable. We only describe the second-order systems.

At the begin increment time $t = t_n$ all relevant variables $\{\underset{\sim}{p}_t, \dot{\underset{\sim}{p}}_t, \ddot{\underset{\sim}{p}}_t\}$ are known. A number of integration procedures can than be used to determine $\{\underset{\sim}{p}_{t+\Delta t}, \dot{\underset{\sim}{p}}_{t+\Delta t}, \ddot{\underset{\sim}{p}}_{t+\Delta t}\}$ such that the system of differential equations is satisfied on time $t + \theta\Delta t$ with $0 \le \theta \le 1$. For $\theta = 0$ we have an *explicit* and for $\theta > 0$ an *implicit* integration method.

$$\underline{A}\,\ddot{\underset{\sim}{p}}(\tau) + \underline{B}\,\dot{\underset{\sim}{p}}(\tau) + \underline{C}\,\underset{\sim}{p}(\tau) = \underset{\sim}{f}_e(\tau) \qquad \text{for} \qquad t \le \tau \le t + \Delta t$$

or

$$\underline{A}\,\ddot{\underset{\sim}{p}}_{t+\theta\Delta t} + \underline{B}\,\dot{\underset{\sim}{p}}_{t+\theta\Delta t} + \underline{C}\,\underset{\sim}{p}_{t+\theta\Delta t} = \underset{\sim}{f}_e(t + \theta\Delta t) \qquad \text{for} \qquad 0 \le \theta \le 1$$

2.1.2 Difference formulas

Using difference formulas, the derivatives of the point variables can be expressed in $\underset{\sim}{p}_{t+\Delta t}$ and values of $\underset{\sim}{p}$ at earlier discrete times. In a *one-step scheme* only values of variables on time t

occur in the difference formulas. In a *multi-step scheme* values on earlier times are also used in the formulas. In that case a special startup procedure is necessary.

$$\underline{A}\,\underset{\sim}{\ddot{p}}_{t+\theta\Delta t} + \underline{B}\,\underset{\sim}{\dot{p}}_{t+\theta\Delta t} + \underline{C}\,\underset{\sim}{p}_{t+\theta\Delta t} = \underset{\sim}{f}_e(t+\theta\Delta t)$$

$$\begin{aligned}
\underset{\sim}{\ddot{p}}_{t+\theta\Delta t} &= \alpha_1 \underset{\sim}{p}_{t+\Delta t} + \alpha_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\alpha}_j \underset{\sim}{\dot{p}}_{t+j\Delta t} + \bar{\bar{\alpha}}_j \underset{\sim}{\ddot{p}}_{t+j\Delta t} \\
\underset{\sim}{\dot{p}}_{t+\theta\Delta t} &= \beta_1 \underset{\sim}{p}_{t+\Delta t} + \beta_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\beta}_j \underset{\sim}{\dot{p}}_{t+j\Delta t} + \bar{\bar{\beta}}_j \underset{\sim}{\ddot{p}}_{t+j\Delta t} \\
\underset{\sim}{p}_{t+\theta\Delta t} &= \gamma_1 \underset{\sim}{p}_{t+\Delta t} + \gamma_j \underset{\sim}{p}_{t+j\Delta t}
\end{aligned}$$

$$\text{with} \qquad j = 0, -1, -2, ...$$

2.1.3 Algebraic equation system

Substituting the difference formulas in the differential equation results in a system of algebraic equations. After solving $\underset{\sim}{p}_{t+\Delta t}$, the derivatives $\underset{\sim}{\dot{p}}_{t+\Delta t}$ and $\underset{\sim}{\ddot{p}}_{t+\Delta t}$ are determined, using the difference formulas again.

$$\begin{aligned}
&\underline{A}\alpha_1 \underset{\sim}{p}_{t+\Delta t} + \underline{A}\alpha_j \underset{\sim}{p}_{t+j\Delta t} + \underline{A}\bar{\alpha}_j \underset{\sim}{\dot{p}}_{t+j\Delta t} + \underline{A}\bar{\bar{\alpha}}_j \underset{\sim}{\ddot{p}}_{t+j\Delta t} + \\
&\qquad \underline{B}\beta_1 \underset{\sim}{p}_{t+\Delta t} + \underline{B}\beta_j \underset{\sim}{p}_{t+j\Delta t} + \underline{B}\bar{\beta}_j \underset{\sim}{\dot{p}}_{t+j\Delta t} + \underline{B}\bar{\bar{\beta}}_j \underset{\sim}{\ddot{p}}_{t+j\Delta t} \\
&\qquad \underline{C}\gamma_1 \underset{\sim}{p}_{t+\Delta t} + \underline{C}\gamma_j \underset{\sim}{p}_{t+j\Delta t} = \underset{\sim}{f}_e(t+\theta\Delta t) \\
&[\underline{A}\alpha_1 + \underline{B}\beta_1 + \underline{C}\gamma_1]\, \underset{\sim}{p}_{t+\Delta t} = \\
&\qquad \underset{\sim}{f}_e(t+\theta\Delta t) - [\underline{A}\alpha_j + \underline{B}\beta_j + \underline{C}\gamma_j]\, \underset{\sim}{p}_{t+j\Delta t} - \\
&\qquad \left[\underline{A}\,\bar{\alpha}_j + \underline{B}\,\bar{\beta}_j\right] \underset{\sim}{\dot{p}}_{t+j\Delta t} - \left[\underline{A}\,\bar{\bar{\alpha}}_j + \underline{B}\,\bar{\bar{\beta}}_j\right] \underset{\sim}{\ddot{p}}_{t+j\Delta t}
\end{aligned}$$

$$\text{with} \qquad j = 0, -1, -2, ...$$

$$\text{short} \qquad \underline{S}\,\underset{\sim}{v} = \underset{\sim}{r} \qquad \text{with} \qquad \underset{\sim}{v} = \underset{\sim}{p}_{t+\Delta t}$$

2.2 Nonlinear systems

If the system element characteristics are a function of flow and/or point variables, the system is indicated to be nonlinear. Internal forces are nonlinear functions of point variables, their integrated values or their derivatives.

first-order systems

$$\left.\begin{array}{l} \underset{\sim}{f}^e_{ia} = \underset{\sim}{f}^e_{ia}(\underset{\sim}{p}^e, \dot{\underset{\sim}{p}}^e) \\ \underset{\sim}{f}^e_{ib} = \underset{\sim}{f}^e_{ib}(\underset{\sim}{p}^e) \\ \dot{\underset{\sim}{f}}^e_{ic} = \dot{\underset{\sim}{f}}^e_{ic}(\underset{\sim}{p}^e) \quad \rightarrow \quad \underset{\sim}{f}^e_{ic} = \int_0^t \dot{\underset{\sim}{f}}^e_{ic}(\underset{\sim}{p}^e)\, d\tau \end{array}\right\} \quad \Rightarrow$$

$$\underset{\sim}{f}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}) + \underset{\sim}{f}_{ib}(\underset{\sim}{p}) + \int \dot{\underset{\sim}{f}}_{ic}(\underset{\sim}{p}) = \underset{\sim}{f}_e \quad \rightarrow \quad \dot{\underset{\sim}{f}}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}) + \dot{\underset{\sim}{f}}_{ib}(\underset{\sim}{p}) + \dot{\underset{\sim}{f}}_{ic}(\underset{\sim}{p}) = \dot{\underset{\sim}{f}}_e$$

second-order systems

$$\left.\begin{array}{l} \underset{\sim}{f}^e_{ia} = \underset{\sim}{f}^e_{ia}(\underset{\sim}{p}^e, \dot{\underset{\sim}{p}}^e, \ddot{\underset{\sim}{p}}^e) \\ \underset{\sim}{f}^e_{ib} = \underset{\sim}{f}^e_{ib}(\underset{\sim}{p}^e, \dot{\underset{\sim}{p}}^e) \\ \underset{\sim}{f}^e_{ic} = \underset{\sim}{f}^e_{ic}(\underset{\sim}{p}^e) \end{array}\right\} \quad \Rightarrow \quad \underset{\sim}{f}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}, \ddot{\underset{\sim}{p}}) + \underset{\sim}{f}_{ib}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}) + \underset{\sim}{f}_{ic}(\underset{\sim}{p}) = \underset{\sim}{f}_e$$

2.2.1 Time discretisation

The time is discretised and an explicit or implicit integration scheme per increment can be used. For an implicit scheme ($\theta > 0$) the resulting set of algebraic equations is nonlinear and cannot be solved directly. An iterative solution procedure has to be used.

$$\underset{\sim}{f}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}, \ddot{\underset{\sim}{p}}) + \underset{\sim}{f}_{ib}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}) + \underset{\sim}{f}_{ic}(\underset{\sim}{p}) = \underset{\sim}{f}_e \qquad \text{for} \qquad t \le \tau \le t + \Delta t$$

or

$$\underset{\sim}{f}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}, \ddot{\underset{\sim}{p}})_{t+\theta\Delta t} + \underset{\sim}{f}_{ib}(\underset{\sim}{p}, \dot{\underset{\sim}{p}})_{t+\theta\Delta t} + \underset{\sim}{f}_{ic}(\underset{\sim}{p})_{t+\theta\Delta t} = \underset{\sim}{f}_e(t + \theta\Delta t) \qquad \text{for} \qquad 0 \le \theta \le 1$$

2.2.2 Iterative solution

In the iterative solution procedure the unknown $\underset{\sim}{p}_{t+\Delta t}$ is written as the sum of a known approximation $\underset{\sim}{p}^*_{t+\Delta t}$ and an unknown iterative change $\delta\underset{\sim}{p}$. For an implicit integration scheme the system matrices also have to be updated after each iteration step.

$$\underset{\sim}{f}_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}, \ddot{\underset{\sim}{p}})_{t+\theta\Delta t} + \underset{\sim}{f}_{ib}(\underset{\sim}{p}, \dot{\underset{\sim}{p}})_{t+\theta\Delta t} + \underset{\sim}{f}_{ic}(\underset{\sim}{p})_{t+\theta\Delta t} = \underset{\sim}{f}_e(t + \theta\Delta t)$$

$$\underset{\sim}{f}^*_{ia} + \frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta + \frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \dot{\underset{\sim}{p}}}\,\delta\dot{\underset{\sim}{p}}_\theta + \frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \ddot{\underset{\sim}{p}}}\,\delta\ddot{\underset{\sim}{p}}_\theta + \underset{\sim}{f}^*_{ib} + \frac{\partial \underset{\sim}{f}^*_{ib}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta + \frac{\partial \underset{\sim}{f}^*_{ib}}{\partial \dot{\underset{\sim}{p}}}\,\delta\dot{\underset{\sim}{p}}_\theta + \underset{\sim}{f}^*_{ic} + \frac{\partial \underset{\sim}{f}^*_{ic}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta = \underset{\sim}{f}_e$$

$$\frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \ddot{\underset{\sim}{p}}}\,\delta\ddot{\underset{\sim}{p}}_\theta + \frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \dot{\underset{\sim}{p}}}\,\delta\dot{\underset{\sim}{p}}_\theta + \frac{\partial \underset{\sim}{f}^*_{ia}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta + \frac{\partial \underset{\sim}{f}^*_{ib}}{\partial \dot{\underset{\sim}{p}}}\,\delta\dot{\underset{\sim}{p}}_\theta + \frac{\partial \underset{\sim}{f}^*_{ib}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta + \frac{\partial \underset{\sim}{f}^*_{ic}}{\partial \underset{\sim}{p}}\,\delta\underset{\sim}{p}_\theta =$$

$$\underset{\sim}{f}_e(t + \theta\Delta t) - \left[\underset{\sim}{f}^*_{ia} + \underset{\sim}{f}^*_{ib} + \underset{\sim}{f}^*_{ic}\right]_{t+\theta\Delta t}$$

$$\underline{A}_2\,\delta\ddot{\underset{\sim}{p}}_\theta + \underline{A}_1\,\delta\dot{\underset{\sim}{p}}_\theta + \underline{A}_0\,\delta\underset{\sim}{p}_\theta + \underline{B}_1\,\delta\dot{\underset{\sim}{p}}_\theta + \underline{B}_0\,\delta\underset{\sim}{p}_\theta + \underline{C}_0\,\delta\underset{\sim}{p}_\theta = \underset{\sim}{f}_e(t + \theta\Delta t) - \underset{\sim}{f}^*_i(t + \theta\Delta t)$$

Because at earlier time moments ($j = 0, -1, -2, \cdots$) the equilibrium equations have been satisfied, we can state that :

$$\underset{\sim}{p}_{t+j\Delta t} = \underset{\sim}{p}^*_{t+j\Delta t} \qquad ; \qquad \dot{\underset{\sim}{p}}_{t+j\Delta t} = \dot{\underset{\sim}{p}}^*_{t+j\Delta t} \qquad ; \qquad \ddot{\underset{\sim}{p}}_{t+j\Delta t} = \ddot{\underset{\sim}{p}}^*_{t+j\Delta t}$$

$$\begin{aligned}
\delta\ddot{\underset{\sim}{p}}_\theta &= \ddot{\underset{\sim}{p}}_{t+\theta\Delta t} - \ddot{\underset{\sim}{p}}^*_{t+\theta\Delta t} \\
&= \left[\alpha_1 \underset{\sim}{p}_{t+\Delta t} + \alpha_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\alpha}_j \dot{\underset{\sim}{p}}_{t+j\Delta t} + \bar{\bar{\alpha}}_j \ddot{\underset{\sim}{p}}_{t+j\Delta t}\right] - \\
&\qquad \left[\alpha_1 \underset{\sim}{p}^*_{t+\Delta t} + \alpha_j \underset{\sim}{p}^*_{t+j\Delta t} + \bar{\alpha}_j \dot{\underset{\sim}{p}}^*_{t+j\Delta t} + \bar{\bar{\alpha}}_j \ddot{\underset{\sim}{p}}^*_{t+j\Delta t}\right] = \alpha_1 \delta\underset{\sim}{p} \\
\delta\dot{\underset{\sim}{p}}_\theta &= \dot{\underset{\sim}{p}}_{t+\theta\Delta t} - \dot{\underset{\sim}{p}}^*_{t+\theta\Delta t} \\
&= \left[\beta_1 \underset{\sim}{p}_{t+\Delta t} + \beta_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\beta}_j \dot{\underset{\sim}{p}}_{t+j\Delta t} + \bar{\bar{\beta}}_j \ddot{\underset{\sim}{p}}_{t+j\Delta t}\right] - \\
&\qquad \left[\beta_1 \underset{\sim}{p}^*_{t+\Delta t} + \beta_j \underset{\sim}{p}^*_{t+j\Delta t} + \bar{\beta}_j \dot{\underset{\sim}{p}}^*_{t+j\Delta t} + \bar{\bar{\beta}}_j \ddot{\underset{\sim}{p}}^*_{t+j\Delta t}\right] = \beta_1 \delta\underset{\sim}{p} \\
\delta\underset{\sim}{p}_\theta &= \underset{\sim}{p}_{t+\theta\Delta t} - \underset{\sim}{p}^*_{t+\theta\Delta t} \\
&= \left[\gamma_1 \underset{\sim}{p}_{t+\Delta t} + \gamma_j \underset{\sim}{p}_{t+j\Delta t}\right] - \left[\gamma_1 \underset{\sim}{p}^*_{t+\Delta t} + \gamma_j \underset{\sim}{p}^*_{t+j\Delta t}\right] = \gamma_1\, \delta\underset{\sim}{p}
\end{aligned}$$

$$\underline{A}_2\, \alpha_1 \delta\underset{\sim}{p} + \underline{A}_1\, \beta_1 \delta\underset{\sim}{p} + \underline{A}_0\, \gamma_1 \delta\underset{\sim}{p} + \underline{B}_1\, \beta_1 \delta\underset{\sim}{p} + \underline{B}_0\, \gamma_1 \delta\underset{\sim}{p} + \underline{C}_0\, \gamma_1 \delta\underset{\sim}{p} = \underset{\sim}{f}_e(t+\theta\Delta t) - \underset{\sim}{f}^*_i(t+\theta\Delta t)$$
$$[(\underline{A}_2\alpha_1 + \underline{A}_1\beta_1 + \underline{A}_0\gamma_1) + (\underline{B}_1\beta_1 + \underline{B}_0\gamma_1) + \underline{C}_0\gamma_1]\, \delta\underset{\sim}{p} = \underset{\sim}{f}_e(t+\theta\Delta t) - \underset{\sim}{f}^*_i(t+\theta\Delta t)$$

short $\qquad \underline{S}\underset{\sim}{v} = \underset{\sim}{r} \quad$ with $\quad \underset{\sim}{v} = \delta\underset{\sim}{p}$

Resuming :

$$[(\underline{A}_2\alpha_1 + \underline{A}_1\beta_1 + \underline{A}_0\gamma_1) + (\underline{B}_1\beta_1 + \underline{B}_0\gamma_1) + \underline{C}_0\gamma_1]\, \delta\underset{\sim}{p} = \underset{\sim}{f}_e(t+\theta\Delta t) - \underset{\sim}{f}^*_i(t+\theta\Delta t)$$

with $\qquad \underline{A}_2 = \dfrac{\underset{\sim}{f}_{ia}}{\partial\ddot{\underset{\sim}{p}}} \qquad \underline{A}_1 = \dfrac{\underset{\sim}{f}_{ia}}{\partial\dot{\underset{\sim}{p}}} \qquad \underline{A}_0 = \dfrac{\underset{\sim}{f}_{ia}}{\partial\underset{\sim}{p}}$

$$\underline{B}_1 = \frac{\underset{\sim}{f}_{ib}}{\partial\dot{\underset{\sim}{p}}} \qquad \underline{B}_0 = \frac{\underset{\sim}{f}_{ib}}{\partial\underset{\sim}{p}}$$

$$\underline{C}_0 = \frac{\underset{\sim}{f}_{ic}}{\partial\underset{\sim}{p}}$$

$$\underset{\sim}{f}^*_i(t+\theta\Delta t) = \left[\underset{\sim}{f}^*_{ia}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}, \ddot{\underset{\sim}{p}}) + \underset{\sim}{f}^*_{ib}(\underset{\sim}{p}, \dot{\underset{\sim}{p}}) + \underset{\sim}{f}^*_{ic}(\underset{\sim}{p})\right]_{t+\theta\Delta t}$$

For a linear system, the derivatives are zero and the equation system of the earlier section results.

$$\underline{A}_2 = \underline{A} \quad ; \quad \underline{B}_1 = \underline{B} \quad ; \quad \underline{C}_0 = \underline{C} \qquad ; \qquad \underline{A}_1 = \underline{A}_0 = \underline{B}_0 = \underline{0}$$
$$\delta\underset{\sim}{p} = \underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}^*_{t+\Delta t}$$
$$\begin{aligned}
\underset{\sim}{f}^*_i(t+\theta\Delta t) &= \underline{A}\ddot{\underset{\sim}{p}}_{t+\theta\Delta t} + \underline{B}\dot{\underset{\sim}{p}}_{t+\theta\Delta t} + \underline{C}\underset{\sim}{p}_{t+\theta\Delta t} = \\
&\underline{A}\left\{\alpha_1 \underset{\sim}{p}_{t+\Delta t} + \alpha_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\alpha}_j \dot{\underset{\sim}{p}}_{t+j\Delta t} + \bar{\bar{\alpha}}_j \ddot{\underset{\sim}{p}}_{t+j\Delta t}\right\} + \\
&\underline{B}\left\{\beta_1 \underset{\sim}{p}_{t+\Delta t} + \beta_j \underset{\sim}{p}_{t+j\Delta t} + \bar{\beta}_j \dot{\underset{\sim}{p}}_{t+j\Delta t} + \bar{\bar{\beta}}_j \ddot{\underset{\sim}{p}}_{t+j\Delta t}\right\} + \\
&\underline{C}\left\{\gamma_1 \underset{\sim}{p}_{t+\delta t} + \gamma_j \underset{\sim}{p}_{t+\Delta t}\right\}
\end{aligned}$$

2.3 Links

Sometimes we want to implement relationships between degrees of freedom of a system. Such a relationship is called a *link* and is modeled as a (set of) equation(s). The *linked* degrees of freedom – indicated with an index l – are no longer free and can be eliminated from the equation system. The degrees of freedom to which they are linked, are *retained* – index r – in the system. All other degrees of freedom are *free* – index f –.

Associated with the linked degrees of freedom are nodal flow variables $\underset{\sim}{\bar{r}}_r$ and $\underset{\sim}{\bar{r}}_l$, which ensure the relationship. They are calculated by requiring that the total virtual energy associated with the links is zero.

The columns $\underset{\sim}{v}$ and $\underset{\sim}{r}$ and consequently the matrix $\underline{S}$ are reorganized, which is called *partitioning*.

$$\begin{bmatrix} \underline{S}_{ff} & \underline{S}_{fr} & \underline{S}_{fl} \\ \underline{S}_{rf} & \underline{S}_{rr} & \underline{S}_{rl} \\ \underline{S}_{lf} & \underline{S}_{lr} & \underline{S}_{ll} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_f \\ \underset{\sim}{v}_r \\ \underset{\sim}{v}_l \end{bmatrix} = \begin{bmatrix} \underset{\sim}{r}_f \\ \underset{\sim}{r}_r + \underset{\sim}{\bar{r}}_r \\ \underset{\sim}{r}_l + \underset{\sim}{\bar{r}}_l \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \underline{S}_{ff}\underset{\sim}{v}_f + \underline{S}_{fr}\underset{\sim}{v}_r + \underline{S}_{fl}\underset{\sim}{v}_l = \underset{\sim}{r}_f \\ \underline{S}_{rf}\underset{\sim}{v}_f + \underline{S}_{rr}\underset{\sim}{v}_r + \underline{S}_{rl}\underset{\sim}{v}_l = \underset{\sim}{r}_r + \underset{\sim}{\bar{r}}_r \\ \underline{S}_{lf}\underset{\sim}{v}_f + \underline{S}_{lr}\underset{\sim}{v}_r + \underline{S}_{ll}\underset{\sim}{v}_l = \underset{\sim}{r}_l + \underset{\sim}{\bar{r}}_l \end{array}$$

The link relations relate the linked dof's to the retained dof's. The associated forces must obey relations which state that they cannot do any virtual work. The latter relation results in a relation between the forces associated with the linked and retained dof's.

$$\begin{array}{l} \underset{\sim}{v}_l = \underline{L}_{lr}\underset{\sim}{v}_r \\ \underset{\sim}{\bar{r}}_l\delta\underset{\sim}{v}_l + \underset{\sim}{\bar{r}}_r\delta\underset{\sim}{v}_r = 0 \quad \forall \quad \{\delta\underset{\sim}{v}_l, \delta\underset{\sim}{v}_r\} \qquad \rightarrow \qquad \underset{\sim}{\bar{r}}_r = -\underset{\sim}{\bar{r}}_l\underline{L}_{lr} \end{array}$$

The link equations are substituted into the total set of system equations. The associated forces $\underset{\sim}{\bar{r}}_l$ can be eliminated from the system.

$$\begin{array}{l} \underline{S}_{ff}\underset{\sim}{v}_f + (\underline{S}_{fr} + \underline{S}_{fl}\underline{L}_{lr})\underset{\sim}{v}_r = \underset{\sim}{r}_f \\ \underline{S}_{rf}\underset{\sim}{v}_f + (\underline{S}_{rr} + \underline{S}_{rl}\underline{L}_{lr})\underset{\sim}{v}_r = \underset{\sim}{r}_r - \underset{\sim}{\bar{r}}_l\underline{L}_{lr} \\ \underline{S}_{lf}\underset{\sim}{v}_f + (\underline{S}_{lr} + \underline{S}_{ll}\underline{L}_{lr})\underset{\sim}{v}_r = \underset{\sim}{r}_l + \underset{\sim}{\bar{r}}_l \end{array}$$

elimination of $\underset{\sim}{\bar{r}}_l$

$$\begin{array}{l} \underline{S}_{ff}\underset{\sim}{v}_f + (\underline{S}_{fr} + \underline{S}_{fl}\underline{L}_{lr})\underset{\sim}{v}_r = \underset{\sim}{r}_f \\ (\underline{S}_{rf} + \underline{L}_{lr}\underline{S}_{lf})\underset{\sim}{v}_f + \\ \qquad (\underline{S}_{rr} + \underline{S}_{rl}\underline{L}_{lr} + \underline{L}_{lr}\underline{S}_{lr} + \underline{L}_{lr}\underline{S}_{ll}\underline{L}_{lr})\underset{\sim}{v}_r = \underset{\sim}{r}_r + \underline{L}_{lr}\underset{\sim}{r}_l \end{array}$$

short $\qquad \underline{S}\,\underset{\sim}{v} = \underset{\sim}{r} \qquad$ with $\qquad \underset{\sim}{v}^T = \begin{bmatrix} \underset{\sim}{v}_f^T & \underset{\sim}{v}_r^T \end{bmatrix}$

2.4 Boundary conditions

Boundary conditions are always given for point variables and may be given for flow variables. In one nodal point *either* the point variable *or* the flow variable is prescribed, the other

remaining unknown and to be solved.

We can place the *u*nknown point variables in a column $\underset{\sim}{v}_u$, the *p*rescribed variables in a column $\underset{\sim}{v}_p$. The flow variables associated with $\underset{\sim}{v}_u$ are prescribed and placed in column $\underset{\sim}{r}_u$. The unknown variables, associated with $\underset{\sim}{v}_p$ reside in column $\underset{\sim}{r}_p$. The columns $\underset{\sim}{v}$ and $\underset{\sim}{r}$ and consequently the matrix $\underline{S}$ are reorganized (partitioning).

$$\left[\begin{array}{cc} \underline{S}_{uu} & \underline{S}_{up} \\ \underline{S}_{pu} & \underline{S}_{pp} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{v}_u \\ \underset{\sim}{v}_p \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{r}_u \\ \underset{\sim}{r}_p \end{array}\right] \quad \rightarrow \quad \left.\begin{array}{l} \underline{S}_{uu}\underset{\sim}{v}_u + \underline{S}_{up}\underset{\sim}{v}_p = \underset{\sim}{r}_u \\ \underline{S}_{pu}\underset{\sim}{v}_u + \underline{S}_{pp}\underset{\sim}{v}_p = \underset{\sim}{r}_p \end{array}\right\}$$

2.5 Solve unknowns

Only the first equation system is needed to solve $\underset{\sim}{v}_u$. With $\underset{\sim}{v}_u$ being solved, the unknown $\underset{\sim}{r}_p$ can be easily calculated. For interpretation purposes the original $\underset{\sim}{v}$ and $\underset{\sim}{r}$ are restored.

solve $\underset{\sim}{v}_u$	$\underline{S}_{uu}\underset{\sim}{v}_u = \underset{\sim}{r}_u - \underline{S}_{up}\underset{\sim}{v}_p$	$\rightarrow$	$\underset{\sim}{v}_u = \underline{S}_{uu}^{-1}(\underset{\sim}{r}_u - \underline{S}_{up}\underset{\sim}{v}_p)$
calculate unknown $\underset{\sim}{r}_p$	$\underset{\sim}{r}_p = \underline{S}_{pu}\underset{\sim}{v}_u + \underline{S}_{pp}\underset{\sim}{v}_p$		

Chapter 3

Integration of differential equations

For linear systems, some integration procedures are described in the following.

first-order systems	$\underline{A}\,\dot{\underset{\sim}{p}} + \underline{B}\,\underset{\sim}{p} + \underline{C}\int \underset{\sim}{p} = \underset{\sim}{f}_e \quad \rightarrow \quad \underline{A}\,\ddot{\underset{\sim}{p}} + \underline{B}\,\dot{\underset{\sim}{p}} + \underline{C}\,\underset{\sim}{p} = \dot{\underset{\sim}{f}}_e$
second-order systems	$\underline{A}\,\ddot{\underset{\sim}{p}} + \underline{B}\,\dot{\underset{\sim}{p}} + \underline{C}\,\underset{\sim}{p} = \underset{\sim}{f}_e$

3.1 First-order systems

We use an implicit scheme : $\theta = 1$. When it is assumed that the point variable changes linearly during an increment, the integral $\underset{\sim}{f}_{ic} = \underline{C}\int \underset{\sim}{p}$ can be calculated. The first-order differential equation is integrated using the trapezium rule for $\dot{\underset{\sim}{p}}_{t+\Delta t}$.

$$\begin{aligned}\underset{\sim}{f}_{ic} &= \underline{C}\int_{\tau=0}^{\tau=t+\Delta t} \underset{\sim}{p}(\tau)\,d\tau = \underline{C}\int_{\tau=0}^{\tau=t} \underset{\sim}{p}(\tau)\,d\tau + \underline{C}\,\Delta t\,\tfrac{1}{2}\{\underset{\sim}{p}_{t+\Delta t} + \underset{\sim}{p}_t\} = \underline{C}\,\underset{\sim}{h}_{t+\Delta t} \\ &= \underline{C}\,\underset{\sim}{h}_t + \tfrac{1}{2}\Delta t\,\underline{C}\,\{\underset{\sim}{p}_{t+\Delta t} + \underset{\sim}{p}_t\}\end{aligned}$$

differential equation

$$\underline{A}\,\dot{\underset{\sim}{p}}_{t+\Delta t} + \underline{B}\,\underset{\sim}{p}_{t+\Delta t} + \underline{C}\,\underset{\sim}{h}_{t+\Delta t} = \dot{\underset{\sim}{f}}_{t+\Delta t}$$

difference formula for $\dot{\underset{\sim}{p}}_{t+\Delta t}$

$$\tfrac{1}{2}\left\{\dot{\underset{\sim}{p}}_{t+\Delta t} + \dot{\underset{\sim}{p}}_t\right\} = \frac{1}{\Delta t}\,\left\{\underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}_t\right\} \quad \rightarrow \quad \dot{\underset{\sim}{p}}_{t+\Delta t} = \frac{2}{\Delta t}\left\{\underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}_t\right\} - \dot{\underset{\sim}{p}}_t$$

set of algebraic equations

$$\left[\frac{2}{\Delta t}\,\underline{A} + \underline{B} + \tfrac{1}{2}\,\Delta t\,\underline{C}\right]\underset{\sim}{p}_{t+\Delta t} = \dot{\underset{\sim}{f}}_{t+\Delta t} + \frac{2}{\Delta t}\,\underline{A}\underset{\sim}{p}_t + \underline{A}\dot{\underset{\sim}{p}}_t - \underline{C}\,\underset{\sim}{h}_t - \tfrac{1}{2}\,\Delta t\,\underline{C}\underset{\sim}{p}_t$$

coefficients

$\beta_1 = \dfrac{2}{\Delta t}$	$\beta_0 = -\dfrac{2}{\Delta t}$	$\bar{\beta}_0 = -1$

3.2 Second-order systems

3.2.1 Central difference method

The central difference method is an explicit method : $\theta = 0$. Difference formulas for $\underset{\sim}{\ddot{p}}_t$ and $\underset{\sim}{\dot{p}}_t$ are substituted into the differential equations, resulting in a set of algebraic equations. The unknown $\underset{\sim}{p}_{t+\Delta t}$ can be calculated directly.

Because the central difference method is explicit, there is a stability requirement for the time step :

$$\Delta t \leq \Delta t_c = \frac{T_{min}}{\pi} = \frac{2}{\omega_{max}}$$
$$\omega_{max} = \text{max. eigen frequency from } [\underline{C} - \omega^2 \underline{A}]\underset{\sim}{p} = \underset{\sim}{0}$$

The central difference method has 2nd-order accuracy.

differential equation $$\underline{A}\,\underset{\sim}{\ddot{p}}_t + \underline{B}\,\underset{\sim}{\dot{p}}_t + \underline{C}\,\underset{\sim}{p}_t = \underset{\sim}{f}_t$$

difference formulas for $\underset{\sim}{\ddot{p}}_t$ and $\underset{\sim}{\dot{p}}_t$

$$\underset{\sim}{\ddot{p}}_t = \frac{1}{\Delta t^2}\left\{\underset{\sim}{p}_{t+\Delta t} - 2\underset{\sim}{p}_t + \underset{\sim}{p}_{t-\Delta t}\right\} \quad ; \quad \underset{\sim}{\dot{p}}_t = \frac{1}{2\Delta t}\left\{\underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}_{t-\Delta t}\right\}$$

set of algebraic equations

$$\left[\frac{1}{\Delta t^2}\underline{A} + \frac{1}{2\Delta t}\underline{B}\right]\underset{\sim}{p}_{t+\Delta t} = \underset{\sim}{f}_t + \left[\frac{2}{\Delta t^2}\underline{A} - \underline{C}\right]\underset{\sim}{p}_t + \left[-\frac{1}{\Delta t^2}\underline{A} + \frac{1}{2\Delta t}\underline{B}\right]\underset{\sim}{p}_{t-\Delta t}$$

coefficients

$\alpha_1 = \frac{1}{\Delta t^2}$	$\alpha_0 = -\frac{2}{\Delta t^2}$	$\alpha_{-1} = \frac{1}{\Delta t^2}$
$\beta_1 = \frac{1}{2\Delta t}$	$\beta_{-1} = -\frac{1}{2\Delta t}$	

Startup procedure

The central difference method is a two-step method. A special startup procedure is necessary at time $t = 0$. Because $\underset{\sim}{p}_0$ and $\underset{\sim}{\dot{p}}_0$ are known as initial conditions $\underset{\sim}{p}_{-1}$ can be solved and used to determine $\underset{\sim}{p}_1$.

$$\underset{\sim}{\dot{p}}_0 = \frac{1}{2\Delta t}(\underset{\sim}{p}_1 - \underset{\sim}{p}_{-1}) \quad \rightarrow \quad \underset{\sim}{p}_{-1} = \underset{\sim}{p}_1 - 2\Delta t\underset{\sim}{\dot{p}}_0 \qquad \rightarrow$$
$$\frac{2}{\Delta t^2}\underline{A}\,\underset{\sim}{p}_1 = \underset{\sim}{f}_0 + \left[\frac{2}{\Delta t^2}\underline{A} - \underline{C}\right]\underset{\sim}{p}_0 + \left[\frac{2}{\Delta t}\underline{A} - \underline{B}\right]\underset{\sim}{\dot{p}}_0$$

3.2.2 Implicit Euler method

For the implicit Euler method we have $\theta = 1$, Difference formulas for $\ddot{\underset{\sim}{p}}_{t+\Delta t}$ and $\dot{\underset{\sim}{p}}_{t+\Delta t}$ are substituted into the differential equations. From the resulting set of algebraic equations $\underset{\sim}{p}_{t+\Delta t}$ can be solved.

differential equation

$$\underline{A}\,\ddot{\underset{\sim}{p}}_{t+\Delta t} + \underline{B}\,\dot{\underset{\sim}{p}}_{t+\Delta t} + \underline{C}\,\underset{\sim}{p}_{t+\Delta t} = \underset{\sim}{f}_{t+\Delta t}$$

difference formulas for $\ddot{\underset{\sim}{p}}_{t+\Delta t}$ and $\dot{\underset{\sim}{p}}_{t+\Delta t}$

$$\ddot{\underset{\sim}{p}}_{t+\Delta t} = \frac{1}{\Delta t}\left\{\dot{\underset{\sim}{p}}_{t+\Delta t} - \dot{\underset{\sim}{p}}_{t}\right\} \quad ; \quad \dot{\underset{\sim}{p}}_{t+\Delta t} = \frac{1}{\Delta t}\left\{\underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}_{t}\right\}$$

set of algebraic equations

$$\left[\frac{1}{\Delta t^2}\underline{A} + \frac{1}{\Delta t}\underline{B} + \underline{C}\right]\underset{\sim}{p}_{t+\Delta t} = \underset{\sim}{f}_{t+\Delta t} + \left[\frac{1}{\Delta t^2}\underline{A} + \frac{1}{\Delta t}\underline{B}\right]\underset{\sim}{p}_{t} + \left[\frac{1}{\Delta t}\underline{A}\right]\dot{\underset{\sim}{p}}_{t}$$

coefficients

$\alpha_1 = \frac{1}{\Delta t^2}$	$\alpha_0 = -\frac{1}{\Delta t^2}$	$\bar{\alpha}_0 = -\frac{1}{\Delta t}$
$\beta_1 = \frac{1}{\Delta t}$	$\beta_0 = -\frac{1}{\Delta t}$	$\gamma_1 = 1$

3.2.3 Newmark method

The Newmark method is an implicit method : $\theta = 1$. A difference formula for $\ddot{\underset{\sim}{p}}_{t+\Delta t}$ is combined with a relation for $\dot{\underset{\sim}{p}}_{t+\Delta t}$ to result in a difference formula for $\dot{\underset{\sim}{p}}_{t+\Delta t}$.

The time step selection is based on the highest relevant eigen frequency of the system, ω_{max} :

$$\Delta t \approx \frac{1.0}{\omega_{max}} \approx 0.15 T_{min}$$

To avoid strong numerical damping, the time step can be chosen smaller :

$$\Delta t \approx \frac{0.5}{\omega_{max}} \approx 0.08 T_{min}$$

This maximum time step will result in phase errors of about 6%.

Substitution of the difference formulas for $\dot{\underset{\sim}{p}}_{t+\Delta t}$ and $\ddot{\underset{\sim}{p}}_{t+\Delta t}$ in the set of differential equations results in a set of algebraic equations.

differential equation

$$\underline{A}\,\ddot{\underset{\sim}{p}}_{t+\Delta t} + \underline{B}\,\dot{\underset{\sim}{p}}_{t+\Delta t} + \underline{C}\,\underset{\sim}{p}_{t+\Delta t} = \underset{\sim}{f}_{t+\Delta t}$$

difference formulas for $\ddot{\underset{\sim}{p}}_{t+\Delta t}$ and $\dot{\underset{\sim}{p}}_{t+\Delta t}$

$$
\left.
\begin{aligned}
\underset{\sim}{p}_{t+\Delta t} &= \underset{\sim}{p}_t + \dot{\underset{\sim}{p}}_t \Delta t + \{(\tfrac{1}{2}-\alpha)\ddot{\underset{\sim}{p}}_t + \alpha \ddot{\underset{\sim}{p}}_{t+\Delta t}\}\Delta t^2 \quad \rightarrow \\
\ddot{\underset{\sim}{p}}_{t+\Delta t} &= \frac{1}{\alpha \Delta t^2}(\underset{\sim}{p}_{t+\Delta t} - \underset{\sim}{p}_t - \dot{\underset{\sim}{p}}_t \Delta t) - \tfrac{1}{\alpha}(\tfrac{1}{2}-\alpha)\ddot{\underset{\sim}{p}}_t \\
\dot{\underset{\sim}{p}}_{t+\Delta t} &= \dot{\underset{\sim}{p}}_t + \{(1-\delta)\ddot{\underset{\sim}{p}}_t + \delta \ddot{\underset{\sim}{p}}_{t+\Delta t}\}\Delta t \\
\dot{\underset{\sim}{p}}_{t+\Delta t} &= \frac{\delta}{\alpha \Delta t}\underset{\sim}{p}_{t+\Delta t} - \frac{\delta}{\alpha \Delta t}\underset{\sim}{p}_t + (1-\frac{\delta}{\alpha})\dot{\underset{\sim}{p}}_t + \Delta t(1-\frac{\delta}{2\alpha})\ddot{\underset{\sim}{p}}_t
\end{aligned}
\right\} \rightarrow
$$

set of algebraic equations

$$
\begin{aligned}
&\left[\frac{1}{\alpha \Delta t^2}\underline{A} + \frac{\delta}{\alpha \Delta t}\underline{B} + \underline{C}\right]\underset{\sim}{p}_{t+\Delta t} = \\
&\quad \underset{\sim}{f}_{t+\Delta t} + \left[\frac{1}{\alpha \Delta t^2}\underline{A} + \frac{\delta}{\alpha \Delta t}\underline{B}\right]\underset{\sim}{p}_t + \left[\frac{1}{\alpha \Delta t}\underline{A} - (1-\frac{\delta}{\alpha})\underline{B}\right]\dot{\underset{\sim}{p}}_t + \\
&\quad \left[\tfrac{1}{\alpha}(\tfrac{1}{2}-\alpha)\underline{A} - \Delta t(1-\frac{\delta}{2\alpha})\underline{B}\right]\ddot{\underset{\sim}{p}}_t
\end{aligned}
$$

coefficients

$\alpha_1 = \dfrac{1}{\alpha \Delta t^2}$	$\alpha_0 = -\dfrac{1}{\alpha \Delta t^2}$	$\bar{\alpha}_0 = -\dfrac{1}{\alpha \Delta t}$	$\bar{\bar{\alpha}}_0 = -\dfrac{1}{\alpha}\left(\dfrac{1}{2}-\alpha\right)$
$\beta_1 = \dfrac{\delta}{\alpha \Delta t}$	$\beta_0 = -\dfrac{\delta}{\alpha \Delta t}$	$\bar{\beta}_0 = 1 - \dfrac{\delta}{\alpha}$	$\bar{\bar{\beta}}_0 = \Delta t\left(1 - \dfrac{\delta}{2\alpha}\right)$

Special cases of Newmark's method result for specific choices of δ and α. It can be proved that Newmark's method is unconditionally stable for certain values of δ and α.

With $\delta = \frac{1}{2}$ the accuracy is of 2nd-order. With $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ (trapezium rule) there is no artificial damping. With $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$ (Houbolt) there is strong artificial damping.

$$
\begin{aligned}
&\delta = \tfrac{1}{2} \quad ; \quad \alpha = \tfrac{1}{6} && \longrightarrow && \text{Wilson-}\theta \text{ with } \theta = 1 = \text{Houbolt} \\
&\delta = \tfrac{1}{2} \quad ; \quad \alpha = \tfrac{1}{4} && \longrightarrow && \text{trapezium rule} \\
&\delta \geq \tfrac{1}{2} \quad \& \quad \alpha \geq \tfrac{1}{4}(\tfrac{1}{2}+\delta)^2 && \longrightarrow && \text{unconditionally stable}
\end{aligned}
$$

Integration schemes (summary)

The next table summarizes the integration schemes discussed in the previous pages.

first order implicit ($\theta = 1$)			
$\beta_1 = \dfrac{2}{\Delta t}$	$\beta_0 = -\dfrac{2}{\Delta t}$	$\bar{\beta}_0 = -1$	
central difference ($\theta = 0$)			
$\alpha_1 = \dfrac{1}{\Delta t^2}$	$\alpha_0 = -\dfrac{2}{\Delta t^2}$	$\alpha_{-1} = \dfrac{1}{\Delta t^2}$	
$\beta_1 = \dfrac{1}{2\Delta t}$	$\beta_{-1} = -\dfrac{1}{2\Delta t}$		
implicit Euler ($\theta = 1$)			
$\alpha_1 = \dfrac{1}{\Delta t^2}$	$\alpha_0 = -\dfrac{1}{\Delta t^2}$	$\bar{\alpha}_0 = -\dfrac{1}{\Delta t}$	
$\beta_1 = \dfrac{1}{\Delta t}$	$\beta_0 = -\dfrac{1}{\Delta t}$	$\gamma_1 = 1$	
Newmark ($\theta = 1$)			
$\alpha_1 = \dfrac{1}{\alpha \Delta t^2}$	$\alpha_0 = -\dfrac{1}{\alpha \Delta t^2}$	$\bar{\alpha}_0 = -\dfrac{1}{\alpha \Delta t}$	$\bar{\bar{\alpha}}_0 = -\dfrac{1}{\alpha}\left(\dfrac{1}{2} - \alpha\right)$
$\beta_1 = \dfrac{\delta}{\alpha \Delta t}$	$\beta_0 = -\dfrac{\delta}{\alpha \Delta t}$	$\bar{\beta}_0 = 1 - \dfrac{\delta}{\alpha}$	$\bar{\bar{\beta}}_0 = \Delta t\left(1 - \dfrac{\delta}{2\alpha}\right)$
Newmark ($\theta = 1$) $\alpha = \frac{1}{4}, \delta = \frac{1}{2}$ $\Rightarrow$ trapezium rule			
Newmark ($\theta = 1$) $\alpha = \frac{1}{6}, \delta = \frac{1}{2}$ $\Rightarrow$ Wilson-θ			
Newmark ($\theta = 1$) $\alpha = \frac{1}{6}, \delta = \frac{1}{2}$ $\Rightarrow$ Houbolt			

Chapter 4

Harmonic loading and response

The response of a system to a load, which is a periodic function of time, can be calculated by direct integration. When the response is needed for a sequence of frequencies, a complete analysis in the time domain is needed for each frequency, which may become time consuming and even not feasible. For linear systems subjected to an harmonic load, an harmonic analysis can be made, resulting in the response in the frequency domain, so the response as a function of the frequency.

Because this procedure is applied to linear systems, superposition can be used to determine the response for multiply loadings with different frequencies. Also non-harmonic periodic excitations can be analyzed, approximating the excitation with a Fourier series.

4.1 Harmonic load

The general harmonic load can be written as the imaginary part of a complex number $\hat{\underset{\sim}{f}}_o e^{i\omega t}$, where ω is the radial frequency and $\hat{\underset{\sim}{f}}_o$ the complex amplitude, with real part $\underset{\sim}{f}_{or}$ and imaginary part $\underset{\sim}{f}_{oi}$. The response is also the imaginary part of a complex number. It has complex amplitude $\hat{\underset{\sim}{p}}_o$ with real part $\underset{\sim}{p}_{or}$ and imaginary part $\underset{\sim}{p}_{oi}$.

Substitution of the excitation and the response and its derivatives in the differential equation leads to a linear set of equations from which $\underset{\sim}{p}_{or}$ and $\underset{\sim}{p}_{oi}$ can be solved.

differential equation	$\underline{A}\,\ddot{\underset{\sim}{p}} + \underline{B}\,\dot{\underset{\sim}{p}} + \underline{C}\,\underset{\sim}{p} = \underset{\sim}{f}_e$
harmonic load	$\underset{\sim}{f}_e = \mathrm{Im}\{\hat{\underset{\sim}{f}}_o e^{i\omega t}\} = \mathrm{Im}\{(\underset{\sim}{f}_{or} + i\underset{\sim}{f}_{oi})e^{i\omega t}\}$
harmonic response	$\underset{\sim}{p} = \mathrm{Im}\{\hat{\underset{\sim}{p}}_o e^{i\omega t}\} = \mathrm{Im}\{(\underset{\sim}{p}_{or} + i\underset{\sim}{p}_{oi})e^{i\omega t}\}$
	$\dot{\underset{\sim}{p}} = \mathrm{Im}\{\hat{\underset{\sim}{p}}_o i\omega e^{i\omega t}\}$
	$\ddot{\underset{\sim}{p}} = \mathrm{Im}\{-\hat{\underset{\sim}{p}}_o \omega^2 e^{i\omega t}\}$

$$\underline{A}(-\omega^2\hat{\underset{\sim}{p}}_o) + \underline{B}(i\omega\,\hat{\underset{\sim}{p}}_o) + \underline{C}(\hat{\underset{\sim}{p}}_o) = \hat{\underset{\sim}{f}}_o$$

$$\left[-\omega^2\underline{A} + i\omega\underline{B} + \underline{C}\right]\hat{\underset{\sim}{p}}_o = \hat{\underset{\sim}{f}}_o$$

$$\left[-\omega^2\underline{A} + i\omega\underline{B} + \underline{C}\right](\underset{\sim}{p}_{or} + i\underset{\sim}{p}_{oi}) = \underset{\sim}{f}_{or} + i\underset{\sim}{f}_{oi}$$

$$\begin{bmatrix} -\omega^2\underline{A} + \underline{C} & -\omega\underline{B} \\ \omega\underline{B} & -\omega^2\underline{A} + \underline{C} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_{or} \\ \underset{\sim}{p}_{oi} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_{or} \\ \underset{\sim}{f}_{oi} \end{bmatrix}$$

For a simplified harmonic excitation $\underset{\sim}{f}_e(t) = \underset{\sim}{f}_o \sin(\omega t)$, the real and imaginary parts of the response amplitude can be calculated subsequently.

harmonic load $\quad \underset{\sim}{f}_e = \underset{\sim}{f}_o \sin(\omega t) \quad \rightarrow \quad \underset{\sim}{f}_{or} = \underset{\sim}{f}_o \quad ; \quad \underset{\sim}{f}_{oi} = \underset{\sim}{0}$

$$\begin{bmatrix} -\omega^2\underline{A} + \underline{C} & -\omega\underline{B} \\ \omega\underline{B} & -\omega^2\underline{A} + \underline{C} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_{or} \\ \underset{\sim}{p}_{oi} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_{or} \\ \underset{\sim}{0} \end{bmatrix}$$

$$(-\omega^2\underline{A} + \underline{C}) \quad \underset{\sim}{p}_{or} \quad + \quad (-\omega\underline{B}) \quad \underset{\sim}{p}_{oi} \quad = \quad \underset{\sim}{f}_o \qquad (1)$$

$$(\omega\underline{B}) \quad \underset{\sim}{p}_{or} \quad + \quad (-\omega^2\underline{A} + \underline{C}) \quad \underset{\sim}{p}_{oi} \quad = \quad \underset{\sim}{0} \qquad (2)$$

$$(2) \quad \rightarrow \quad \underset{\sim}{p}_{oi} = -\,(-\omega^2\underline{A} + \underline{C})^{-1}(\omega\underline{B})\,\underset{\sim}{p}_{or} \quad \rightarrow \quad (1) \quad \rightarrow$$

$$(-\omega^2\underline{A} + \underline{C})\,\underset{\sim}{p}_{or} - (-\omega\underline{B})(-\omega^2\underline{A} + \underline{C})^{-1}(\omega\underline{B})\,\underset{\sim}{p}_{or} = \underset{\sim}{f}_o \quad \rightarrow$$

$$\underset{\sim}{p}_{or} = \left[(-\omega^2\underline{A} + \underline{C}) + (\omega\underline{B})(-\omega^2\underline{A} + \underline{C})^{-1}(\omega\underline{B})\right]^{-1} \underset{\sim}{f}_o$$

$$\underset{\sim}{p}_{oi} = -\,(-\omega^2\underline{A} + \underline{C})^{-1}(\omega\underline{B})\,\underset{\sim}{p}_{or}$$

amplitude $\quad \underset{\sim}{\bar{p}}_o = \sqrt{\underset{\sim}{p}_{or}^2 + \underset{\sim}{p}_{oi}^2} \quad \rightarrow$ transfer function $\quad \underset{\sim}{H} = \dfrac{\underset{\sim}{\bar{p}}_o}{\underset{\sim}{f}_o}$

phase shift $\quad \underset{\sim}{\Phi} = \arctan\left(\dfrac{\underset{\sim}{p}_{or}}{\underset{\sim}{p}_{oi}}\right)$

Chapter 5

One-dimensional mechanical systems

5.1 Translational mechanical systems

The flow variable in translational mechanical systems is the *force* f. The point variable is the *displacement* u. There are three elements : a spring, a dashpot and a mass.

Linear springs and dashpots are characterized by the constant stiffness and the constant damping, respectively. The behavior of the linear elements is described in the next section.

5.2 Linear translational elements

5.2.1 Spring

For a linear spring the axial force f is proportional to the elongation Δu. The proportionality coefficient is the stiffness k.

The elongation can be written as the difference of the nodal displacements. The nodal forces can be expressed in the nodal displacements with a stiffness matrix $\underline{K}$.

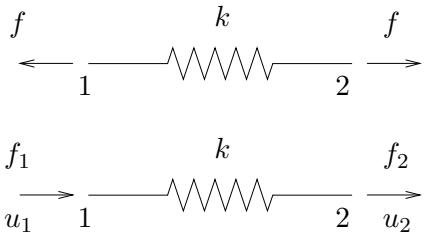


Fig. 5.1 : *One-dimensional linear spring element.*

$$f = k\Delta u = k(u_2 - u_1) \quad \rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}(t) = \underline{K}\underset{\sim}{u}(t)$$

5.2.2 Dashpot

For a linear dashpot the axial force f is proportional to the elongation rate $\Delta\dot{u}$. The proportionality coefficient is the damping b.

The elongation rate can be written as the difference of the nodal velocities. The nodal forces can be expressed in the nodal velocities with a damping matrix $\underline{B}$.

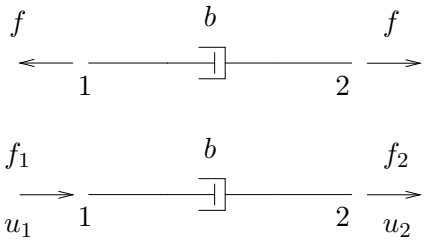
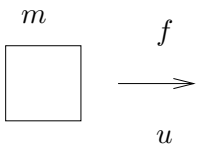


Fig. 5.2 : *One-dimensional linear dashpot element.*

$$f = b\Delta\dot{u} = b(\dot{u}_2 - \dot{u}_1) \quad \rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}(t) = \underline{B}\underset{\sim}{\dot{u}}(t)$$

5.2.3 Mass

A mass can be considered as a nodal variable and is then placed in a node of the system. According to Newton's second law, the resulting force on the mass is proportional to its acceleration.



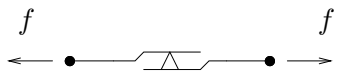
$$f(t) = m\ddot{u}(t)$$

Fig. 5.3 : *Mass*

5.3 Special elements

5.3.1 Friction element

A friction element is rigid, so without any possibility for elongation, when the absolute value of the force f is below a limit value f_y. When $|f|$ exceeds this threshold value, the stiffness is zero : there is no resistnce against elongation.



$$f = k\,\Delta u$$
$$|f| < f_y \quad \rightarrow \quad k = \infty$$
$$|f| \geq f_y \quad \rightarrow \quad k = 0$$

Fig. 5.4 : *Friction element.*

5.3.2 Rigid bar / chain

A rigid bar is a spring with infinitely large stiffness. A chain has infinite stiffness, when it is loaded in tension. Subjected to a compressive force, the stiffness is zero.

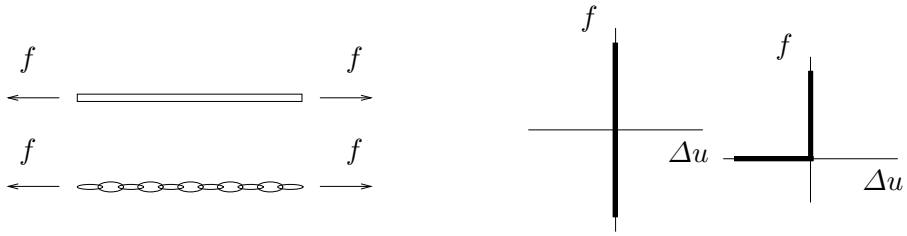
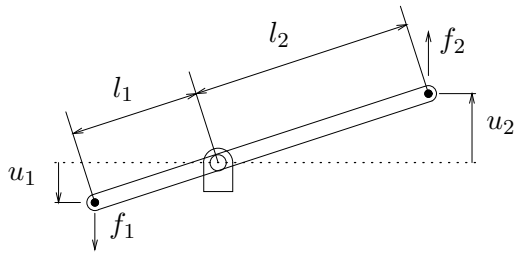


Fig. 5.5 : *Chain element and force-elongation behavior*

5.3.3 Transformer

A *transformer* relates the displacement in one node directly to the displacement in another node, thereby changing its value. It is modeled as a lever. When the displacement is transformed, the forces are also transformed, in such a way that energy is conserved. In the program it is modeled with a link relation.



$$\left.\begin{array}{l} u_2 = -\,n u_1 \\ f_1 u_1 = f_2 u_2 \end{array}\right\} \rightarrow$$
$$f_2 = -\,\frac{1}{n}\,f_1 \quad ; \quad n = \frac{l_2}{l_1}$$

Fig. 5.6 : *Mechanical transformer.*

5.4 Rotational mechanical systems

The flow variable in rotational mechanical systems is the *torque* T. The point variable is the *rotation angle* ϕ. There are three elements : a spring, a dashpot and an inertia.

Linear springs and dashpots are characterized by the constant stiffness and the constant damping, respectively. The behavior of the linear elements is described in the next section.

5.5 Linear rotational elements

5.5.1 Spring

For a linear spring the torque T is proportional to the relative rotation $\Delta\phi$. The proportionality coefficient is the stiffness k.

The relative rotation can be written as the difference of the nodal rotations. The nodal torques can be expressed in the nodal rotations with a stiffness matrix $\underline{K}$.

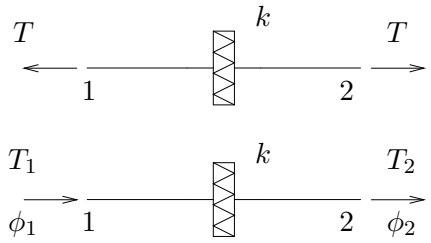


Fig. 5.7 : *Rotational linear spring element.*

$$T = k\Delta\phi = k(\phi_2 - \phi_1) \quad \rightarrow \quad \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{T}(t) = \underline{K}\underset{\sim}{\phi}(t)$$

5.5.2 Dashpot

For a linear dashpot the torque T is proportional to the difference in rotation rate $\Delta\dot{\phi}$. The proportionality coefficient is the damping b.

The rotation rate difference can be written as the difference of the nodal rotation rates. The nodal torques can be expressed in the nodal rotation rates with a damping matrix $\underline{B}$.

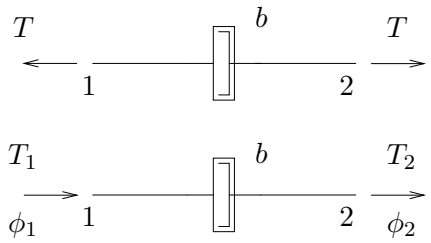
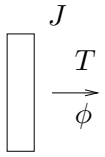


Fig. 5.8 : *Rotational linear dashpot element.*

$$T = b\Delta\dot{\phi} = b(\dot{\phi}_2 - \dot{\phi}_1) \quad \rightarrow \quad \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{T}(t) = \underline{B}\dot{\underset{\sim}{\phi}}(t)$$

5.5.3 Inertia

An inertia can be considered as a nodal variable and is then placed in a node of the system. According to Newton's second law, the resulting torque on the inertia is proportional to its angular acceleration.



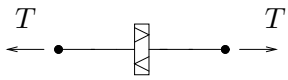
$$T(t) = J\ddot{\phi}(t)$$

Fig. 5.9 : *Inertia.*

5.6 Special elements

5.6.1 Friction element

A rotational friction element is rigid, so without any possibility for rotation, when the absolute value of the torque T is below a limit value T_y. When $|T|$ exceeds this threshold value, the stiffness is zero : there is no resistnce against rotation.

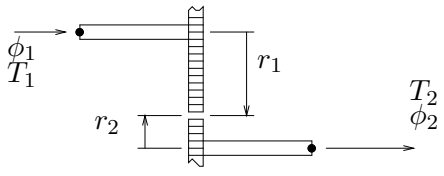


$$\begin{aligned} &T = k\,\Delta\phi \\ &|T| < T_y \quad \rightarrow \quad k = \infty \\ &|T| \geq T_y \quad \rightarrow \quad k = 0 \end{aligned}$$

Fig. 5.10 : *Rotational friction element.*

5.6.2 Transformer

A *transformer* relates the rotation in one node directly to the rotation in another node, thereby changing its value. It is modeled as a gear-set. When the rotation is transformed, the torques are also transformed, in such a way that energy is conserved. In the program it is modeled with a link relation.



$$\left.\begin{aligned} &\phi_2 = -\,n\phi_1 \\ &T_1\phi_1 = T_2\phi_2 \end{aligned}\right\} \rightarrow$$

$$T_2 = -\,\frac{1}{n}\,T_1 \quad ; \quad n = \frac{r_1}{r_2}$$

Fig. 5.11 : *Rotational transformer.*

Chapter 6

Two-dimensional mechanical systems

The mechanical elements can be used to model systems in two or three dimensions. Here we only consider two-dimensional systems. In two dimensions the nodal point and flow variables of a mechanical system have two components, one in each coordinate direction. For an element with two nodes we have four point variable components (displacements) and four flow variable components (forces). Element matrices for spring and dashpot are four-by-four matrices and can be derived rather straightforwardly.

6.1 Linear mechanical systems

The linear two-dimensional systems are systems of linear springs and linear dashpots. Point masses are located in the system nodes. There is a special element, which combines an axial stiffness with a transversal stiffness.

6.1.1 Two-dimensional spring element

The relation between the nodal forces and the nodal displacements is given by a 2×2 stiffness matrix, when forces and displacements are orientated along the spring axis.

To derive the stiffness matrix for a two-dimensional spring, forces and displacements perpendicular to the axis are introduced. The four nodal force components are related to the four nodal displacement components with a 4×4 stiffness matrix.

$$\begin{bmatrix} f_{i1} \\ f_{i2} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

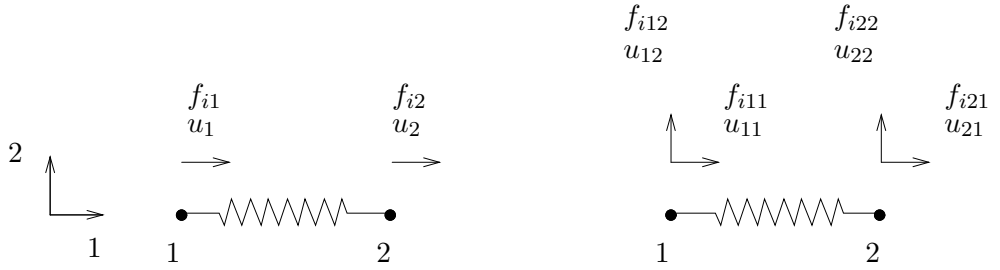


Fig. 6.1 : *One-dimensional spring element with expanded nodal forces and displacements*

$$\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \end{bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix}$$

The spring is rotated over an angle α with respect to the global 1-direction. The components of the internal nodal forces and nodal displacements are still related to the axial direction and the direction perpendicular to the element axis. We denote these components with the upper index $()^L$ (= local).

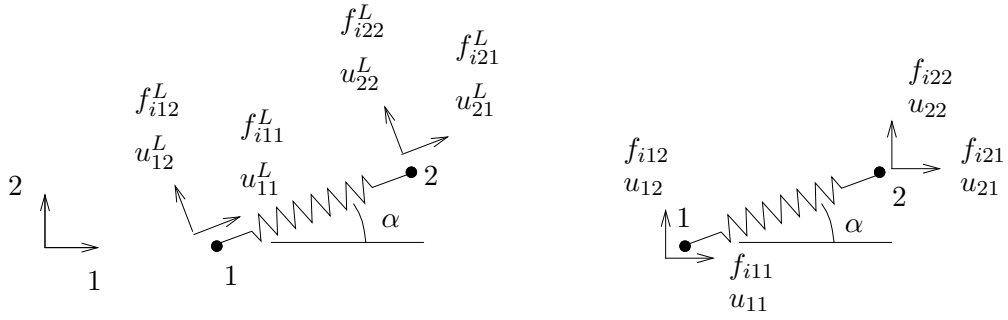


Fig. 6.2 : *Two-dimensional spring element with local force and displacement components (index $()^L$ = local).*

$$\begin{bmatrix} f^L_{i11} \\ f^L_{i12} \\ f^L_{i21} \\ f^L_{i22} \end{bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^L_{11} \\ u^L_{12} \\ u^L_{21} \\ u^L_{22} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}^L_i = \underline{K}^L \underset{\sim}{u}^L$$

The local nodal point displacements are linear combinations of the components in the global coordinate directions. With $s = \sin(\alpha)$ and $c = \cos(\alpha)$ we can derive the transformation matrix $\underline{T}$. This matrix is orthonormal which implies : $\underline{T}^{-1} = \underline{T}^T$. For the internal nodal forces an analogous relation holds.

nodal displacement components

$$\begin{bmatrix} u_{11}^L \\ u_{12}^L \\ u_{21}^L \\ u_{22}^L \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{u}^L = \underline{T}\underset{\sim}{u}$$

nodal internal force components

$$\begin{bmatrix} f_{i11}^L \\ f_{i12}^L \\ f_{i21}^L \\ f_{i22}^L \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}_i^L = \underline{T}\,\underset{\sim}{f}_i \qquad \overset{invert}{\longrightarrow}$$

$$\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} f_{i11}^L \\ f_{i12}^L \\ f_{i21}^L \\ f_{i22}^L \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f}_i = \underline{T}^{-1}\underset{\sim}{f}_i^L$$

The element stiffness matrix for the two-dimensional spring is derived by transformation. The stiffness matrix is symmetric and singular.

$$\underset{\sim}{f}_i = \underline{T}^T\underline{K}^L\underline{T}\,\underset{\sim}{u} \quad \rightarrow \quad \underset{\sim}{f}_i = \underline{K}\underset{\sim}{u} \quad \text{with} \quad \underline{K} = \underline{T}^T\underline{K}^L\underline{T} \quad \rightarrow$$

$$\underline{K} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$= \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} k \begin{bmatrix} c & s & -c & -s \\ 0 & 0 & 0 & 0 \\ -c & -s & c & s \\ 0 & 0 & 0 & 0 \end{bmatrix} = k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

6.1.2 Two-dimensional dashpot element

The damping matrix of a two-dimensional dashpot is derived by transformation in the same way as was done for the stiffness matrix.

$$\underset{\sim}{f} = \underline{B}\dot{\underset{\sim}{u}} \qquad ; \qquad \underline{B} = b \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

6.1.3 System equations

A two-dimensional mechanical system generally consists of a number of springs, dashpots and nodal masses. Springs and dashpots are mutually connected, which is also referred to as *assembling*. Newton's second law must be obeyed. The result is a set of second-order differential equations in the nodal displacements $\underset{\sim}{u}$. The nodal masses are located on the diagonal of the mass matrix $\underline{M}$.

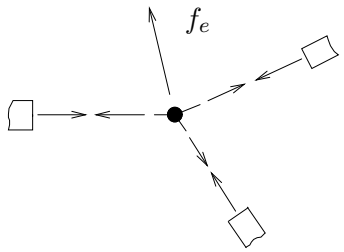


Fig. 6.3 : *Node with three attached elements.*

$$\underset{\sim}{f}_i = \underset{\sim}{f}_e \quad \rightarrow \quad \underline{M}\ddot{\underset{\sim}{u}}(t) + \underline{B}\dot{\underset{\sim}{u}}(t) + \underline{K}\underset{\sim}{u}(t) = \underset{\sim}{f}_e(t)$$

Local coordinate system

In a node of a two-dimensional mechanical system displacement and force components are defined in two directions. In every direction either the displacement or the force is unknown.

Normally the displacement and force components are defined in the global coordinate directions : 1 and 2. It may be necessary to prescribe components in a different direction. In that case we have to define a local coordinate system in the node with the angle α between the local 1^L-axis and the global 1-axis. Angles are positive when rotating anti-clockwise, as is shown in the figure.

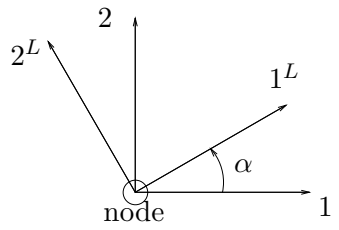


Fig. 6.4 : *Local nodal coordinate system.*

The given information is used to transform the nodal displacements and forces to their local directions, indicated with the upper index $(\ )^l$. This implies the transformation of the relevant components of the left-hand matrix of the system. Indicating this matrix with $\underline{S}$, the displacement column with $\underset{\sim}{u}$ and the force column with $\underset{\sim}{f}$, this transformation for one node can be elaborated ($c = \cos(\alpha); s = \sin(\alpha)$).

After calculating $[u_1^l u_2^l]$ and $[f_1^l f_2^l]$ the components are transformed back to their global equivalents.

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \rightarrow$$

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_1^l \\ u_2^l \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} f_1^l \\ f_2^l \end{bmatrix} \qquad \rightarrow$$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_1^l \\ u_2^l \end{bmatrix} = \begin{bmatrix} f_1^l \\ f_2^l \end{bmatrix} \qquad \rightarrow$$

$$\begin{bmatrix} S_{11}c^2 + S_{12}cs + S_{21}cs + S_{22}s^2 & \\ & -S_{11}cs + S_{12}c^2 - S_{21}s^2 + S_{22}cs \\ -S_{11}cs - S_{12}s^2 + S_{21}c^2 + S_{22}cs & \\ & S_{11}s^2 - S_{12}cs - S_{21}cs + S_{22}c^2 \end{bmatrix} \begin{bmatrix} u_1^l \\ u_2^l \end{bmatrix} = \begin{bmatrix} f_1^l \\ f_2^l \end{bmatrix}$$

6.2 Nonlinear mechanical systems

For nonlinear two-dimensional systems, the nodal internal forces of an element are a function of the element properties, geometry and spatial orientation. The elongation is characterised by the elongation ratio λ.

Spring

The derivative of $\underset{\sim}{f}^e_{ic}$ w.r.t. the nodal displacements $\underset{\sim}{u}^e$ results in a (4×4)-matrix, the tangential stiffness matrix $\underline{K}$. The component K_{11} is elaborated.

$$\underset{\sim}{f}_{ic}(\underset{\sim}{u}) = N(\lambda) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix} \quad \rightarrow \quad \underline{K}^* = \left. \frac{d\underset{\sim}{f}_{ic}}{d\underset{\sim}{u}} \right|_{\underset{\sim}{u}^*} = \begin{bmatrix} \frac{\partial f_{i11}}{\partial u_{11}} & \frac{\partial f_{i11}}{\partial u_{12}} & \frac{\partial f_{i11}}{\partial u_{21}} & \frac{\partial f_{i11}}{\partial u_{22}} \\ \frac{\partial f_{i12}}{\partial u_{11}} & \frac{\partial f_{i12}}{\partial u_{12}} & \frac{\partial f_{i12}}{\partial u_{21}} & \frac{\partial f_{i12}}{\partial u_{22}} \\ \frac{\partial f_{i21}}{\partial u_{11}} & \frac{\partial f_{i21}}{\partial u_{12}} & \frac{\partial f_{i21}}{\partial u_{21}} & \frac{\partial f_{i21}}{\partial u_{22}} \\ \frac{\partial f_{i22}}{\partial u_{11}} & \frac{\partial f_{i22}}{\partial u_{12}} & \frac{\partial f_{i22}}{\partial u_{21}} & \frac{\partial f_{i22}}{\partial u_{22}} \end{bmatrix}^*$$

$$\begin{aligned} K_{11}^* &= \left. \frac{\partial f_{i11}}{\partial u_{11}} \right|^* = \left. \frac{\partial}{\partial u_{11}} \{N(\lambda)(-c)\} \right|^* = -\frac{\partial N(\lambda)}{\partial u_{11}} c^* + N(\lambda^*) \left(-\frac{\partial c}{\partial u_{11}} \right) \\ &= \left(-\left. \frac{dN}{d\lambda} \right|_{\lambda^*} \frac{\partial \lambda}{\partial u_{11}} \right) c^* + N(\lambda^*) \left(-\frac{\partial c}{\partial u_{11}} \right) \\ &\qquad \frac{\partial \lambda}{\partial u_{11}} = -\frac{c^*}{l_0} \qquad \text{and} \qquad \frac{\partial c}{\partial u_{11}} = -\frac{s^{*2}}{l^*} \\ &= \left. \frac{dN}{d\lambda} \right|_{\lambda^*} \frac{c^{*2}}{l_0} + N(\lambda^*) \frac{s^{*2}}{l^*} \end{aligned}$$

Similarly the other components of the stiffness matrix can be derived.

$$\underline{K}^* = \left[\left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{1}{l_0}\right] \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* + \left[N(\lambda^*)\frac{1}{l^*}\right] \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^*$$
$$= \left[\left.\frac{dN}{d\lambda}\right|^* \frac{1}{l_0}\right] \underline{M}_L^* + \left[\frac{N^*}{l^*}\right] \underline{M}_N^*$$

6.3 Spring systems

6.3.1 Spring structure and local coordinates

Two linear springs (stiffness 100 kN/mm, length 100 mm) are connected in node 2, which is allowd to move along a slope with an angle of 60^o w.r.t. global x-axis. In node 2 a local coordinate system is introduced. The displacement in node 3 is prescribed (0.1 mm) in the global y-direction.

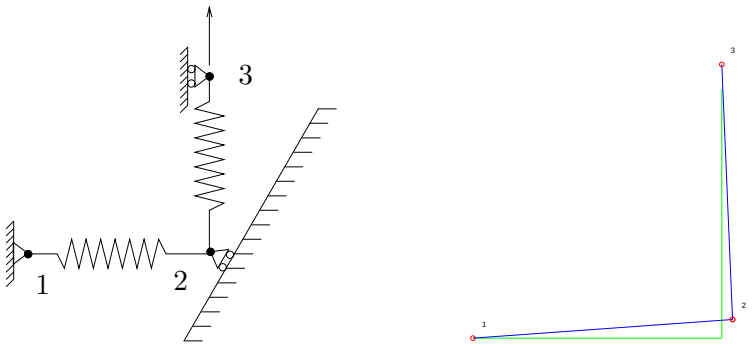


Fig. 6.5 : *Two springs and local dofs in node 2.*

6.4 Spring systems

6.4.1 Spring structure with large deformation

The static deformation of spring structures requires the correct modelling of large rotations. As an example the shown 5-element structure is subjected to a large prescribed $y-$deformation of node 4. Nodes 1 and 3 are fixed completely and node 2 is fixed in $x-$direction. Deformed

structure and displacement history of node 4 and 2 are shown in the figures. Exactly the same results came out of an analysis with MSC.Marc.

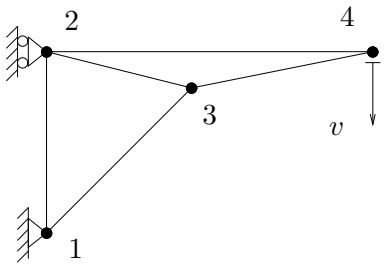


Fig. 6.6 : *Truss structure with 4 trusses.*

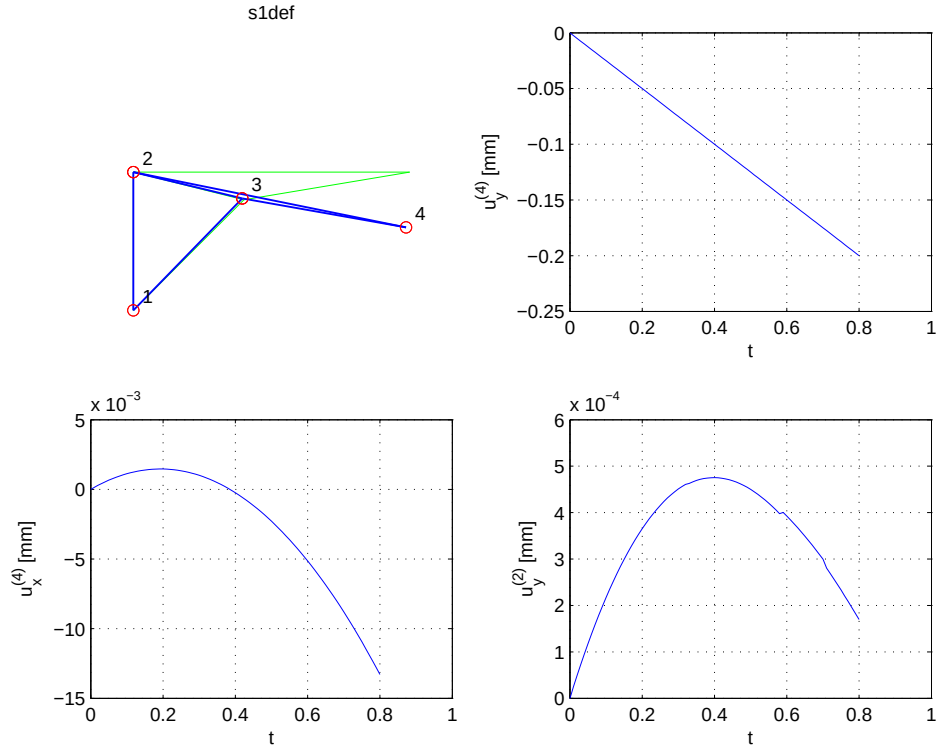
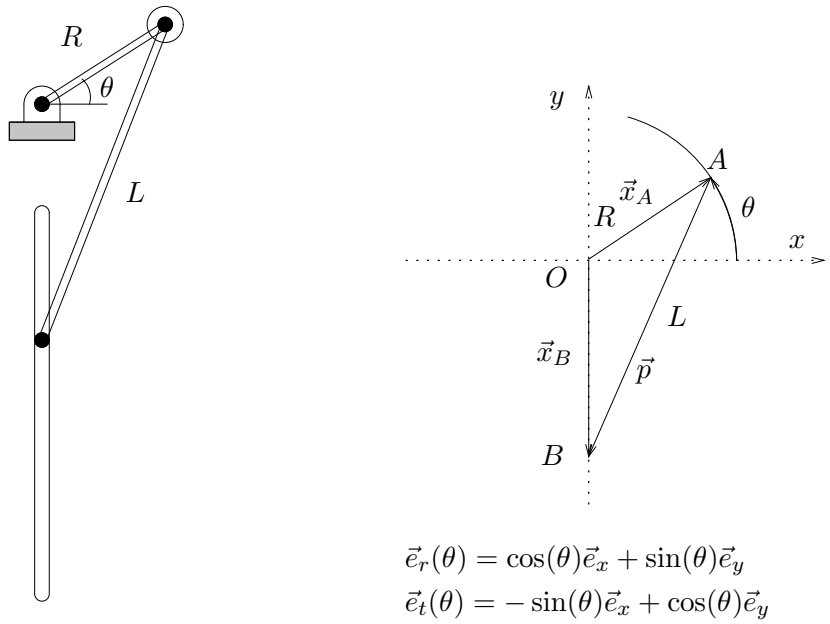


Fig. 6.7 : *Deformation of the truss structure (a); Displacement of node 4 (b and c); displaceement of node 2.*

6.4.2 Crank mechanism

A crank rotates around the fixed hinge O with a constant angular velocity $\dot{\theta}$. The crank length is R. At its end A, the truss AB is connected by a hinge. The endpoint B can only move in vertical direction.

The position and velocity of endpoint B can be calculated analytically.



$$\vec{e}_r(\theta) = \cos(\theta)\vec{e}_x + \sin(\theta)\vec{e}_y$$
$$\vec{e}_t(\theta) = -\sin(\theta)\vec{e}_x + \cos(\theta)\vec{e}_y$$

Fig. 6.8 : *A crack mechanism with slot-point B.*

$$\left.\begin{aligned} &\vec{x}_A = R\,\vec{e}_r(\theta) \\ &\vec{p} = \vec{x}_B - \vec{x}_A = -R\cos(\theta)\vec{e}_x - \sqrt{L^2 - R^2\cos^2(\theta)}\,\vec{e}_y \end{aligned}\right\} \quad \rightarrow$$

$$\begin{aligned} \vec{x}_B = \vec{p} + \vec{x}_A &= -R\cos(\theta)\vec{e}_x - \sqrt{L^2 - R^2\cos^2(\theta)}\,\vec{e}_y + R\vec{e}_r(\theta) \\ &= \left\{R\sin(\theta) - \sqrt{L^2 - R^2\cos^2(\theta)}\right\}\vec{e}_y \end{aligned}$$

$$\dot{\vec{x}}_B = R\dot{\theta}\cos(\theta)\left\{1 - \frac{R\sin(\theta)}{\sqrt{L^2 - R^2\cos^2(\theta)}}\right\}\vec{e}_y$$

When the crank and connected truss are modelled with 'rigid" springs, the position and velocity of point B can be calculated, when the position of point A is prescribed as a function of time, to follow a circular trajectory with constant velocity.

The force in point A is also calculated and is almost zero, which should be so, as no masses are involved.

6.5 Spring-dashpot systems

6.5.1 Maxwell element

A Maxwell element is a series arrangement of a spring and a dashpot. Only displacements and forces in axial direction are considered.

To validate the numerical results, we need some analytical solutions of the system behavior. We are interested in the relation between the force and the displacement of node 3. This relation is given by a differential equation which can be derived very easily.

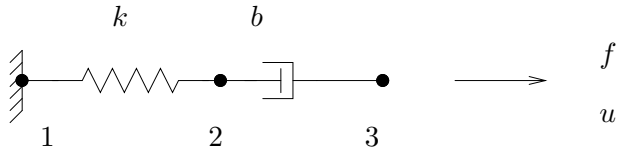


Fig. 6.9 : *Spring-dashpot system (Maxwell element).*

differential equation $\qquad f + \frac{b}{k}\dot{f} = b\dot{u}$

Maxwell : prescribed force

When we prescribe the force $f(t)$, the displacement can be determined by direct integration. It is assumed that $u(t = 0^-) = 0$.

differential equation $\qquad b\dot{u} = f + \frac{b}{k}\dot{f}$

displacement (direct integration) $\qquad u(t) - u(0^-) = \frac{1}{b}\int\limits_{\tau=0^-}^{t} \left\{ f(\tau) + \frac{b}{k}\dot{f}(\tau) \right\} d\tau$

IC : $u(t = 0^-) = 0 \quad \rightarrow \qquad u(t) = \frac{1}{b}\int\limits_{\tau=0^-}^{t} f(\tau)\, d\tau + \frac{1}{k}\int\limits_{\tau=0^-}^{t} \dot{f}(\tau)\, d\tau$

partial integration first term $\rightarrow \qquad u(t) = \frac{1}{b}\int\limits_{\tau=0^-}^{t} \left\{ (t-\tau) + \frac{b}{k} \right\} \dot{f}(\tau)\, d\tau$

Maxwell : force step

A prescribed force step at time $t = 0$ is mathematically denoted with a Heaviside function : $f(t) = f_0 H(t, 0)$. The displacement response can be determined by substitution in the general solution.

force step $\qquad f(t) = f_0 H(t,0) \quad ; \quad H(\ ,\ )$: Heaviside function
$\dot{f}(t) = f_0 \delta(t,0) \quad ; \quad \delta(\ ,\ )$: Dirac function

displacement $\qquad u(t) = \frac{1}{b}\int\limits_{\tau=0^-}^{t} \left\{ (t-\tau) + \frac{b}{k} \right\} f_0 \delta(\tau, 0)\, d\tau = f_0 \frac{1}{b}\left(t + \frac{b}{k} \right)$

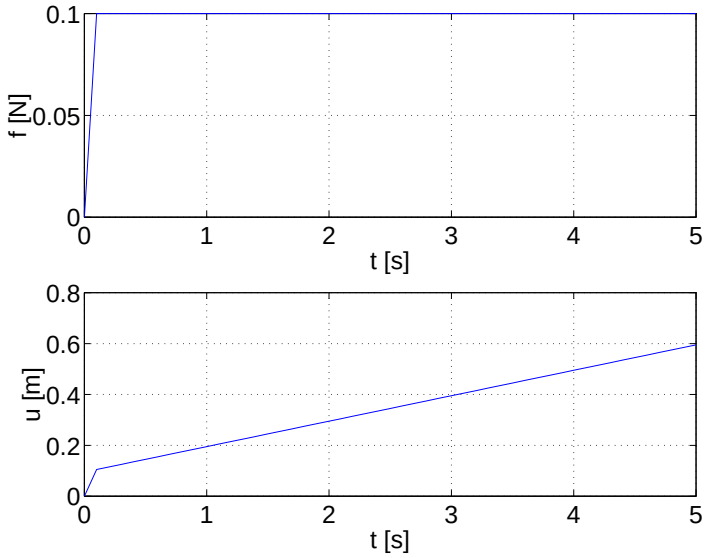


Fig. 6.10 : *Displacement response to a step excitation of the force.*

Maxwell : prescribed displacement

Solving the Maxwell differential equation to determine the force response with the displacement prescribed as a function of time, gives a relation for $f(t)$, where we use $u(t=0) = \dot{u}(t=0) = f(t=0) = 0$.

differential equation $$f + \frac{b}{k}\dot{f} = b\dot{u}$$

ICs $$u(t=0) = \dot{u}(t=0) = f(t=0) = 0$$

force $$f(t) = \int_{\tau=0^-}^{t} k e^{-\frac{k}{b}(t-\tau)} \dot{u}(\tau)\, d\tau$$

Maxwell : displacement step

A displacement step $u(t) = u_0 H(t,0)$ and its derivative $\dot{u}(t) = u_0 \delta(t,0)$, where $\delta(t,\tau)$ is a Dirac function, results in the step response.

displacement step $$u(t) = u_0 H(t,0) \quad \rightarrow \quad \dot{u}(t) = u_0 \delta(t,0)$$

force $$f(t) = \int_{\tau=0^-}^{t} k e^{-\frac{k}{b}(t-\tau)} u_0 \delta(\tau,0)\, d\tau = k u_0 e^{-\frac{k}{b}t}$$

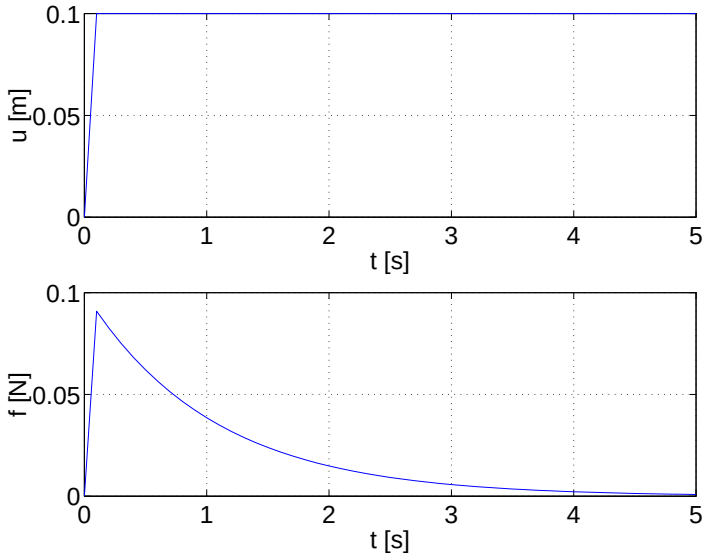


Fig. 6.11 : *Force response to a step excitation of the displacement.*

6.5.2 Kelvin-Voigt element

A Kelvin-Voigt element is a parallel arrangement of a spring and a dashpot. The differential equation relating the axial displacement $u(t)$ and the force $f(t)$ in node 2 is easily derived.

When the displacement is prescribed, the force response can be calculated directly.

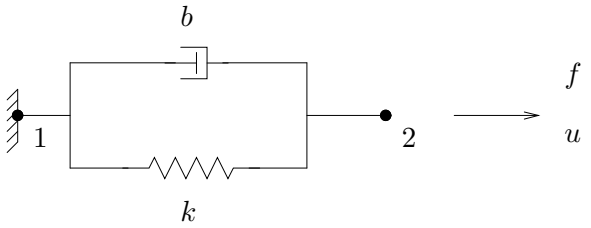


Fig. 6.12 : *Kelvin-Voigt element.*

differential equation $\qquad f = ku + b\dot{u}$

Kelvin-Voigt : prescribed force

Solving the differential equation to determine $u(t)$ when the force is prescribed, leads, with $u(t=0) = 0$ and $f(t=0) = 0$, to the general solution.

differential equation $\qquad ku + b\dot{u} = f$

ICs $\qquad u(t=0) = 0 \quad ; \quad f(t=0) = 0$

displacement $\qquad u(t) = \dfrac{1}{k} \displaystyle\int_{\tau=0^-}^{t} \left\{ 1 - e^{-\frac{k}{b}(t-\tau)} \right\} \dot{f}(\tau)\, d\tau$

Kelvin-Voigt : force step

Applying a step $f(t) = f_0 H(t,0)$ results in the step response.

force step $\qquad f(t) = f_0 H(t,0)$

displacement $\qquad u(t) = f_0 \frac{1}{k}\left(1 - e^{-\frac{k}{b}t}\right)$

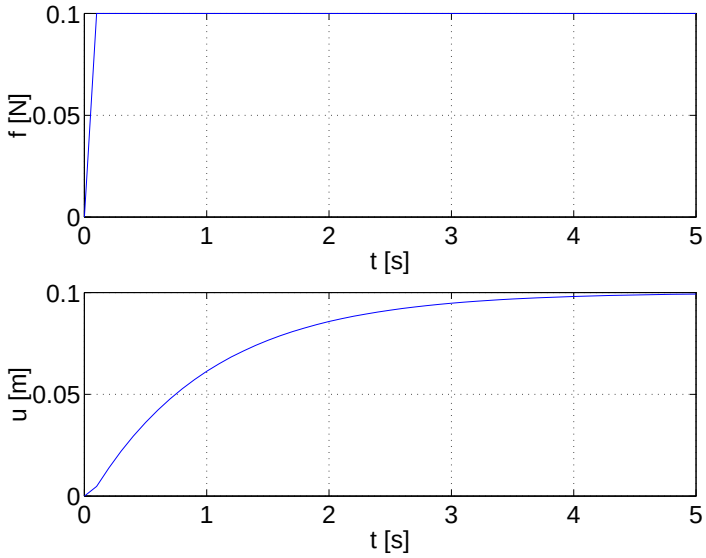


Fig. 6.13 : *Displacement response to a step excitation of the force.*

6.5.3 Standard linear element

The so-called standard linear element is a parallel arrangement of a spring and a Maxwell element. The differential equation describing the relation between the axial displacement and force in node 3, can be derived by assembling element matrices and eliminating variables in node 2.

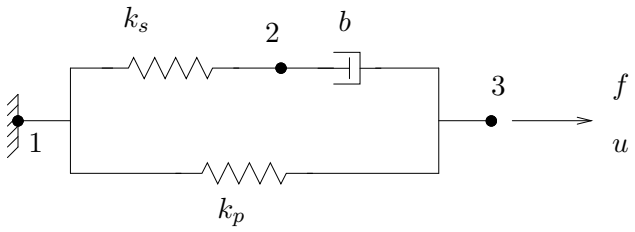


Fig. 6.14 : *Standard solid element.*

system equations

$$\begin{bmatrix} b & -b & 0 \\ -b & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_{11} \\ \dot{u}_{21} \\ \dot{u}_{31} \end{bmatrix} + \begin{bmatrix} p & 0 & -p \\ 0 & s & -s \\ -p & -s & p+s \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{bmatrix}$$

BCs/ICs : $u_{11} = 0 \quad ; \quad \dot{u}_{11} = 0 \quad ; \quad f_{21} = 0 \quad ; \quad f_{31} = f \quad \rightarrow$

$$\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_{21} \\ \dot{u}_{31} \end{bmatrix} + \begin{bmatrix} s & -s \\ -s & p+s \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad \rightarrow$$

$$su_{21} - su_{31} + b\dot{u}_{21} = 0 \quad (*)$$
$$-su_{21} + (p+s)u_{31} = f (**)$$

After elimination of u_{21} and $\dot{u}_{21}$, $u_{31} = u$ is substituted.

elimination of u_{21} and $\dot{u}_{21}$

$$(**) \quad \rightarrow \quad u_{21} = -\frac{f}{s} + \frac{p+s}{s} u_{31} \quad \rightarrow \quad \dot{u}_{21} = -\frac{\dot{f}}{s} + \frac{p+s}{s} \dot{u}_{31} \quad \rightarrow \quad (*) \quad \rightarrow$$
$$-pu_{31} + \frac{b}{s}\dot{f} - \frac{b(p+s)}{s}\dot{u}_{31} = -f$$

differential equation ($u_{31} = u$) $$f + \frac{b}{s}\dot{f} = pu + \frac{b}{s}(p+s)\dot{u}$$

Standard linear : prescribed force

For a prescribed force $f(t)$ the general solution for the displacement $u(t)$ can be derived, where the initial conditions $u(t=0) = \dot{u}(t=0) = f(t=0) = 0$ are used.

general solution $$u(t) = \int_{\tau=0^-}^{t} \frac{1}{p}\{1 - \frac{s}{p+s} e^{-\frac{ps}{b(p+s)}(t-\tau)}\}\dot{f}(\tau)\, d\tau$$

ICs $$u(t=0) = \dot{u}(t=0) = f(t=0) = 0$$

displacement $$u(t) = \int\limits_{\tau=0^-}^{t} \frac{1}{p}\left\{1 - \frac{s}{p+s}\, e^{-\frac{ps}{b(p+s)}(t-\tau)}\right\}\dot{f}(\tau)\, d\tau$$

Standard linear : force step

The response to a step $f(t) = f_0 H(t,0)$ can be calculated.

$f(t) = f_0 H(t,0) \quad \rightarrow \quad \dot{f}(t) = f_0 \delta(t,0)$

$$
\begin{aligned}
u(t) &= \int_{\tau=0^-}^{t} \frac{1}{p}\left\{1 - \frac{s}{p+s}\, e^{-\frac{ps}{b(p+s)}(t-\tau)}\right\} \dot{f}(\tau)\, d\tau \\
&= \int_{\tau=0^-}^{t} \left\{\frac{1}{p} - \frac{s}{p(p+s)}\, e^{-\frac{ps}{b(p+s)}(t-\tau)}\right\} f_0 \delta(\tau, 0)\, d\tau \\
&= f_0 \frac{1}{p}\left(1 - \frac{s}{p+s}\, e^{-\frac{ps}{b(p+s)}t}\right)
\end{aligned}
$$

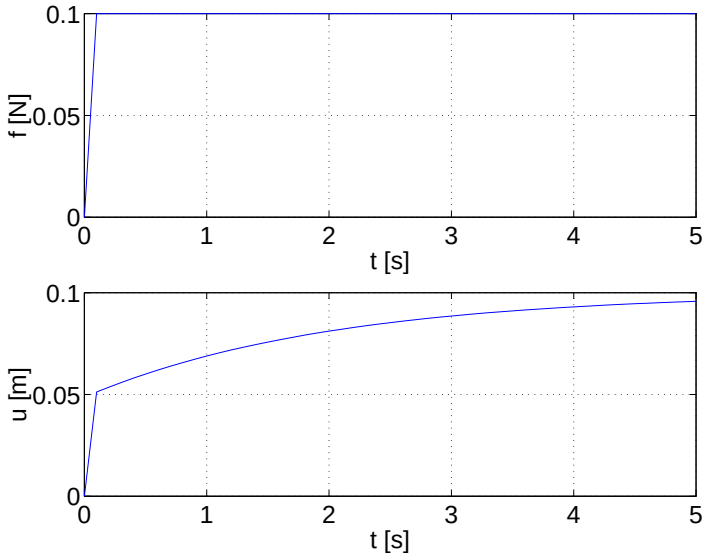


Fig. 6.15 : *Displacement response to a step excitation of the force.*

Standard linear : prescribed displacement

For a prescribed displacement $u(t)$ and velocity $\dot{u}(t)$, the general solution for the force response $f(t)$ can be derived, where the initial conditions $u(t=0) = \dot{u}(t=0) = f(t=0) = 0$ are used.

force $$f(t) = \int_{\tau=0^-}^{t} \left\{p + se^{-\frac{s}{b}(t-\tau)}\right\} \dot{u}(\tau)\, d\tau$$

ICs $$u(t=0) = \dot{u}(t=0) = f(t=0) = 0$$

Standard linear : displacement step

The response to a step $u(t) = u_0 H(t, 0)$ is calculated.

displacement step $$u(t) = u_0 H(t, 0) \qquad \rightarrow$$

force $$f(t) = \int_{\tau=0^-}^{t} \left\{ p + s e^{-\frac{s}{b}(t-\tau)} \right\} u_0 \delta(\tau, 0)\, d\tau = u_0 \left(p + s e^{-\frac{s}{b} t} \right)$$

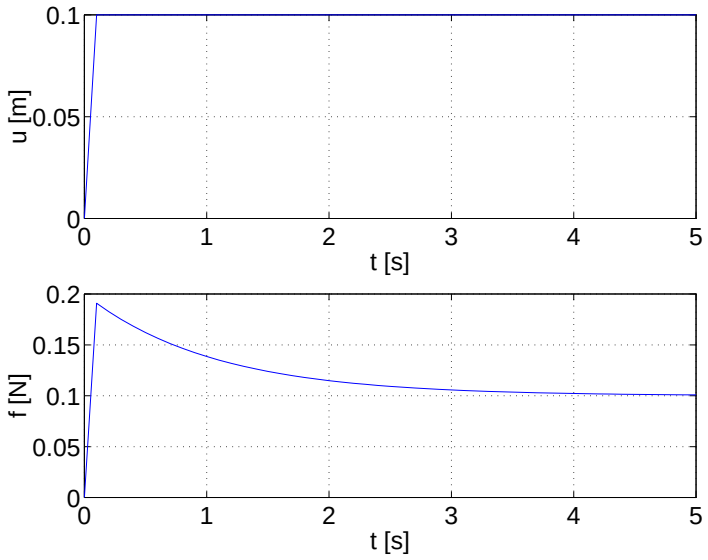


Fig. 6.16 : *Force response to a step excitation of the displacement.*

6.6 Spring-mass systems

We consider the simple spring-mass system, which is loaded by an axial force f. The differential equation for the displacement u is easily derived.

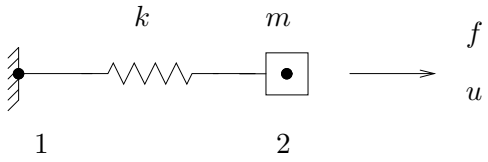


Fig. 6.17 : *Spring-mass system.*

differential equation $$m\ddot{u} + ku = f$$

ICs $$\dot{u}(t=0) = 0 \quad ; \quad u(t=0) = u_0$$

6.6.1 Free vibration

First we study the free vibration, when $f = 0$. Solving the homogeneous differential equation gives the solution for $u(t)$. This solution is taken from the class $u = Ce^{\lambda t}$ and substitution leads to the characteristic equation, from which λ is solved.

The so-called natural solution is an harmonic vibration with *angular frequency* $\omega_r = \sqrt{\frac{k}{m}}$. The frequency $f_r = \omega_r/2\pi$ is the *eigen frequency* and $T = 2\pi/\omega_r$ is the *period time*.

$$m\lambda^2 + k = 0 \quad \rightarrow \quad \lambda^2 = -\frac{k}{m} \quad \rightarrow \quad \lambda_{1,2} = \pm i\sqrt{k/m}$$

solution ($\{c_1, c_2\}$: complex ; $\{a_1,a_2\}$: real)

displacement

$$\begin{aligned} u(t) &= c_1 e^{i(\sqrt{k/m})t} + c_2 e^{-i(\sqrt{k/m})t} u(t) \\ &= c_1 \left[\cos\left(\sqrt{\frac{k}{m}}\, t\right) + i \sin\left(\sqrt{\frac{k}{m}}\, t\right)\right] + c_2 \left[\cos\left(\sqrt{\frac{k}{m}}\, t\right) - i \sin\left(\sqrt{\frac{k}{m}}\, t\right)\right] \\ &= a_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + a_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) \end{aligned}$$

velocity $$\dot{u}(t) = -a_1\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) + a_2\sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

As an example we analyse the free vibration of a mass $m = 1$ [kg] on a spring with stiffness $k = 100$ [N/m]. The resulting eigenfrquency $\omega_r = 10$ [rad/s] is equivalent to a period time $T_r = (2\pi)/\omega_r = 0.62$ [s].

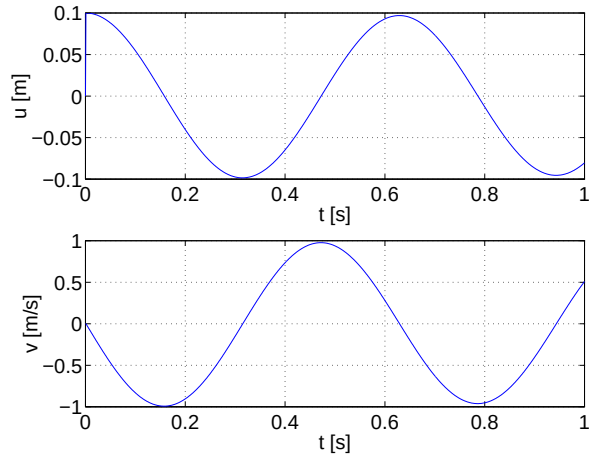


Fig. 6.18 : *Displacement and velocity of the mass as result of a free vibration.*

A free vibration results also, when a pulse force (10 N) is applied at time $t = 0$ [s].

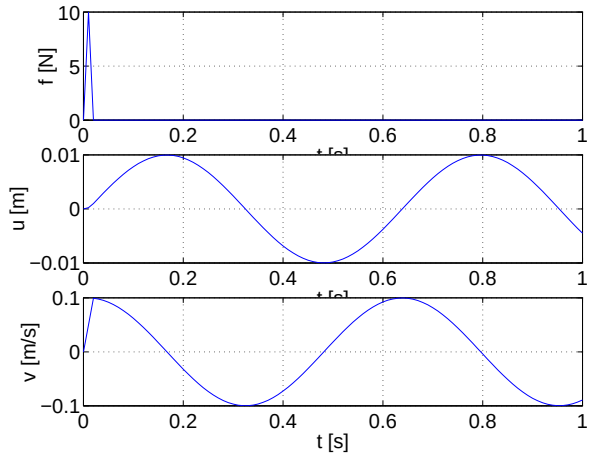


Fig. 6.19 : *Displacement and velocity of the mass as result of a pulse force excitation.*

6.6.2 Harmonic force

We now consider the same spring-mass system but now with a force $f(t) = A\sin(\omega t)$ working on the mass for $t \geq 0$. Initial conditions are : $\dot{u}(t=0) = 0$ and $u(t=0) = 0$.

The displacement of the mass m is the sum of the natural solution (of the homogeneous equation) $u_H(t)$, and a forced solution $u_P(t)$, which has to be found such that the differential equation is satisfied.

differential equation $\qquad m\ddot{u} + ku = A\sin(\omega t)$

harmonic force $\qquad f(t) = A\sin(\omega t) \quad \text{for} \quad t \geq 0$

ICs $\qquad \dot{u}(t=0) = 0 \quad ; \quad u(t=0) = 0$

particulate solution $\qquad u_P(t) = B\sin(\omega t) \quad \rightarrow \quad \ddot{u}_P(t) = -B\omega^2\sin(\omega t)$

substitution in differential equation $\rightarrow$

$$-mB\omega^2\sin(\omega t) + kB\sin(\omega t) = A\sin(\omega t)$$

equation must be satisfied for every time t $\rightarrow$

$$B = \frac{A}{-m\omega^2 + k} = \frac{\frac{A}{m}}{-\omega^2 + \frac{k}{m}} = \frac{\frac{A}{m}}{\omega_r^2 - \omega^2} \quad \rightarrow$$

particulate solution $\quad (\omega \neq \omega_r) \qquad u_P(t) = \dfrac{\frac{A}{m}}{\omega_r^2 - \omega^2}\sin(\omega t)$

Assuming that $\omega \neq \omega_r$ the total solution is derived. For $\omega \rightarrow \omega_r$ strange things happen : The amplitude of the displacement $u(t)$ becomes infinite, which is called *resonance*.

displacement

$$\begin{aligned} u(t) &= u_H(t) + u_P(t) \\ &= a_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + a_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{\frac{A}{m}}{\omega_r^2 - \omega^2} \sin(\omega t) \end{aligned}$$

velocity

$$\dot{u}(t) = -a_1 \sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) + a_2 \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{\frac{A}{m}}{\omega_r^2 - \omega^2} \omega \cos(\omega t)$$

ICs $\rightarrow$ integration constants

$$a_1 = 0 \qquad ; \qquad a_2 = -\sqrt{\frac{m}{k}}\, \frac{\frac{A}{m}}{\omega_r^2 - \omega^2} \omega$$

A numerical solution is determined for the next parameter values :

$$f = 10 \sin(10t) \text{ and } f = 10 \sin(20t) \; ; \; m = 1 \text{ [kg]} \; ; \; k = 100 \text{ [N/m]}$$

The solution coincides with the analytical solution for $\omega = 20$ rad/s. Fpr $\omega = 10$ rad/s the analytical solution immedeatly results in an infinite displacement amplitude.

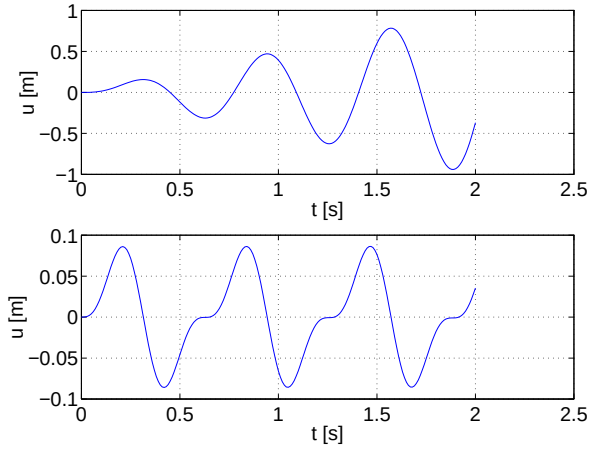


Fig. 6.20 : *Displacement of the mass resulting from an harmonic force with frequencies 10 and 20 rad/s.*

6.6.3 Spring-mass-spring

A two-spring system with a mass located in the connection point is loaded with a step force of 10 N at one end of the system. The mass is 1 kg and the spring stiffnesses are both 100 N/m.

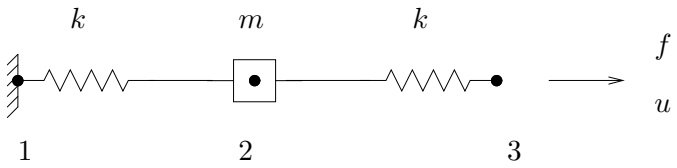


Fig. 6.21 : *Mass between two springs.*

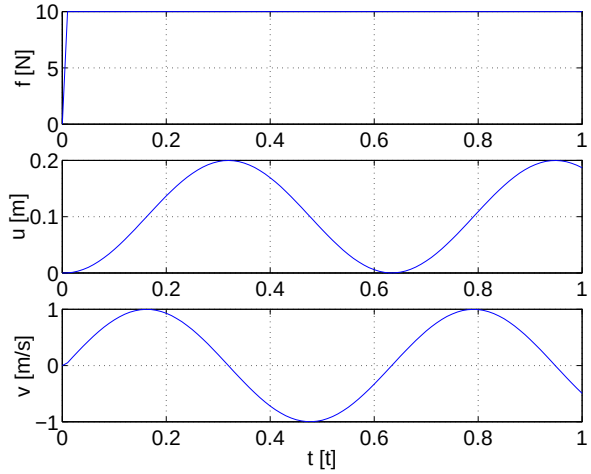


Fig. 6.22 : *Displacement and velocity of the mass resulting from a step excitation of the force.*

6.6.4 Ballistic mass

A mass m is launched with initial velocity vector $\vec{v}_0 = v_{x0}\vec{e}_x + v_{y0}\vec{e}_y$. Ignoring air resistance, the only external force experienced by the mass is the gravity weight $-mg\vec{e}_y$, where g is the gravitational acceleration.

$$m\ddot{x}\vec{e}_x + m\ddot{y}\vec{e}_y = -mg\vec{e}_y \quad \rightarrow \quad (1) \quad \ddot{x} = 0 \qquad ; \qquad (2) \quad \ddot{y} = -g$$

$$\begin{aligned}
&(1) \quad \rightarrow \quad \dot{x} = C = \dot{x}(t=0) = v_{x0} = v_0\cos(\theta_0) \quad \rightarrow \quad x(t) = v_0 t\cos(\theta_0) \\
&(2) \quad \rightarrow \quad \dot{y} = -gt + C = -gt + \dot{y}(t=0) = -gt + v_{y0} = -gt + v_0\sin(\theta_0) \quad \rightarrow \\
&\qquad\quad y(t) = -\tfrac{1}{2}gt^2 + v_0 t\sin(\theta_0)
\end{aligned}$$

$$y = 0 \quad \rightarrow \quad t_e = \frac{2v_0}{g}\sin(\theta_0) \quad \rightarrow \quad x(t=t_e) = \frac{v_0^2}{g}\sin(2\theta_0)$$

A mass $m = 0.1$ kg is launched with an initial velocity $v_0 = 20$ m/s at angles $\{20^o, 40^o, 60^o\}$ with the horizontal axis. The gravitational velocity is $g = 9.81$ m/s^2. Both numerical and analytical solution is plotted and can be seen to coincide. For the middle velocity, the mass just reaches the ground at $t_e = 2.62$ s.

Although the mass is located in the second node of a spring, whose other node is fixed at $x = 0$, the spring has zero stiffness and is irrelevant.

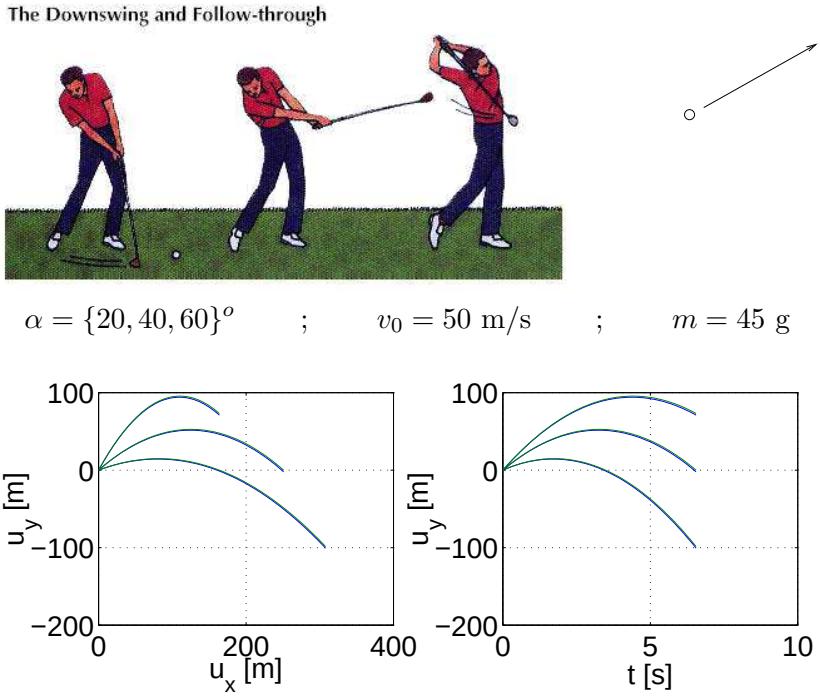


Fig. 6.23 : *Displacement of a mass when launched at three different angles.*

6.6.5 Pendulum

A pendulum consists of a mass m connected to frictionless hinge by means of a cord of length l. We consider the pendulum at an angle θ with the horizontal x-axis. The position, velocity and acceleration of the mass can be calculated in the polar coordinate system, with base vectors $\vec{e}_r(\theta)$ and $\vec{e}_t(\theta)$. The tension force $\vec{F}$ in the pendulum's cord and the gravitational force $\vec{G} = -mg\vec{e}_y$ provoke the acceleration of the mass. Two equations of motion can be derived

$$
\begin{aligned}
\vec{x} &= l\vec{e}_r(\theta) \\
\dot{\vec{x}} &= \dot{l}\vec{e}_r(\theta) + l\dot{\vec{e}}_r = \dot{l}\vec{e}_r(\theta) + l\frac{d\vec{e}_r}{d\theta}\,\dot{\theta} = \dot{l}\vec{e}_r(\theta) + l\dot{\theta}\vec{e}_t(\theta) \\
\ddot{\vec{x}} &= \ddot{l}\vec{e}_r(\theta) + \dot{l}\dot{\vec{e}}_r(\theta) + \dot{l}\dot{\theta}\vec{e}_t(\theta) + l\ddot{\theta}\vec{e}_t(\theta) + l\dot{\theta}\dot{\vec{e}}_t(\theta) \\
&= \ddot{l}\vec{e}_r(\theta) + \dot{l}\dot{\theta}\vec{e}_t(\theta) + \dot{l}\dot{\theta}\vec{e}_t(\theta) + l\ddot{\theta}\vec{e}_t(\theta) - l\dot{\theta}^2\vec{e}_r(\theta) \\
&= \left(\ddot{l} - l\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{l}\dot{\theta} + l\ddot{\theta}\right)\vec{e}_t
\end{aligned}
$$

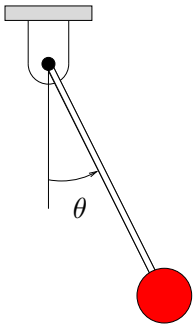
$$
m\ddot{\vec{x}} = \vec{F} + \vec{G} \quad ; \quad \vec{F} = -F\vec{e}_r \quad ; \quad \vec{G} = mg\cos(\theta)\vec{e}_r - mg\sin(\theta)\vec{e}_t \quad \rightarrow
$$

$$(1) \quad -F + mg\cos(\theta) = m\left(\ddot{l} - l\dot{\theta}^2\right)$$
$$(2) \quad -mg\sin(\theta) = m\left(2\dot{l}\dot{\theta} + l\ddot{\theta}\right)$$

It is now assumed that the cord is inextensible : $\dot{l} = 0$. This allows us to derive a non-linear differential equation for θ and from there to derive an expression for the tensile force F in the cord.

$$(2) \quad \rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \quad \rightarrow \quad \frac{d}{dt}\left\{\tfrac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos(\theta)\right\} = 0 \quad \rightarrow$$
$$\left.\begin{array}{l} \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos(\theta) = C \\ \theta = \theta_0 \quad \rightarrow \quad \dot{\theta} = 0 \end{array}\right\} \quad \rightarrow \quad C = -\frac{g}{l}\cos(\theta_0) \quad \rightarrow$$
$$\dot{\theta}^2 = 2\,\frac{g}{l}\left\{\cos(\theta) - \cos(\theta_0)\right\} \quad \Rightarrow \quad (1) \quad \rightarrow$$
$$F = mg\left\{3\cos(\theta) - 2\cos(\theta_0)\right\}$$

A pendulum with mass 1 kg and cord-length 4 m starts to swing from the angle $\theta_0 = 0^o$. The solution is shown in the plots. The force in the cord slightly deviates from the exact solution, which is due to numerical errors.



$$F = mg\left\{3\cos(\theta) - 2\cos(\theta_0)\right\}$$
$$\theta_0 = 90^o$$

Fig. 6.24 : *Pendulum.*

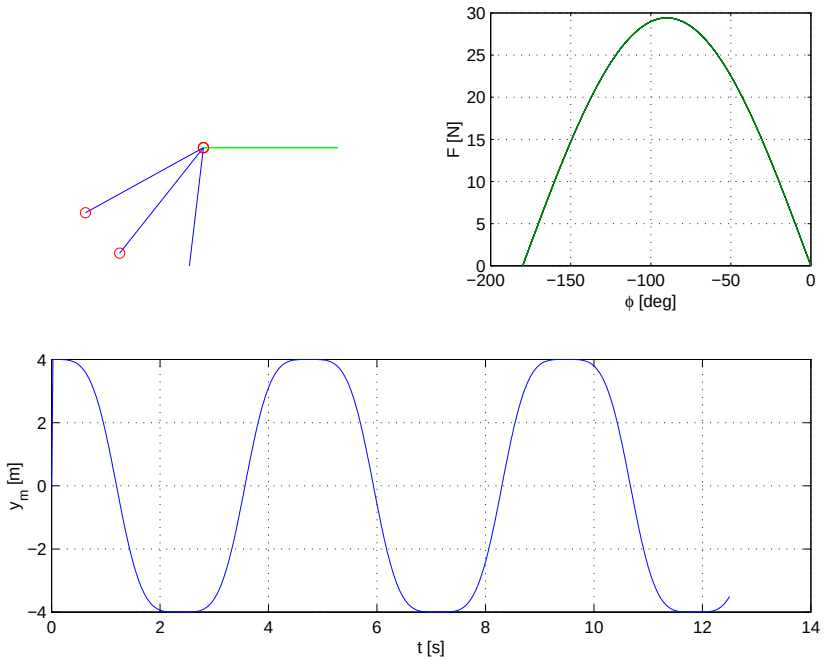


Fig. 6.25 : *Deformation (ul), force in the swing-arm (ur) and vertical displacement of the mass (b).*

6.6.6 Multi-segment pendulums

A multi-segment pendulum consists of a number of rigid bars, mutually connected in frictionless hinges. The first segment is connected to the world. In the hinges masses can be located.

The first example shows a two-segment pendulum, with masses of 1 kg in the free hinges. The length of the segments is about 2.5 m. The plot shows the length of the displacement vector of the end-point as a function of time.

The second example shows a 10-segment pendulum of total length $10\sqrt{2}$ m. Masses are placed in each free hinge. Mass values range between 10 and 50 kg. The plot shows the reaction force in the fixed point as a function of time.

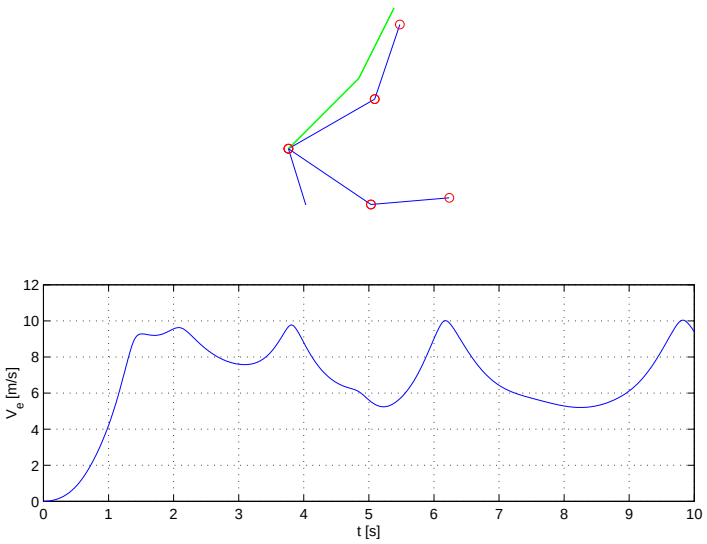


Fig. 6.26 : *Two-segment pendulum : vertical displacement of the mass.*

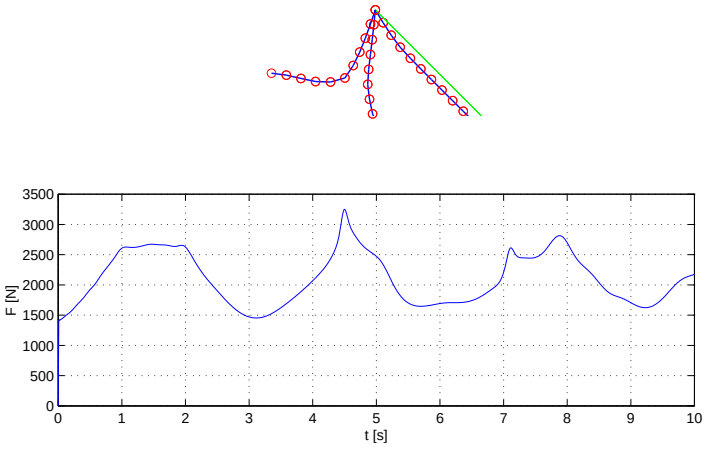


Fig. 6.27 : *10-segment pendulum; reaction force at the fixation.*

6.6.7 Chains

The difference between a chain and a pendulum is that the elements in a chain are nonlinear. Their stiffness is very large when the chain element is stretched and very low when it is compressed. This chain element can be seen as a mechanical diod. In the following ezamples, masses of 1 kg were placed in the hinges between the chain segments.

The first example shows a chain, which falls from a uplifted initial position. The plot shows the force in one of the fixaton points as a function of time.

the second example shows a falling chain. The plot is again the reaction force in the

fixation point as a function of time.

The third example shows a the same chain, of which one point is fixed, while the other is allowed to move horizontally. The plot shows the reaction force in the fixation point as a function of time.

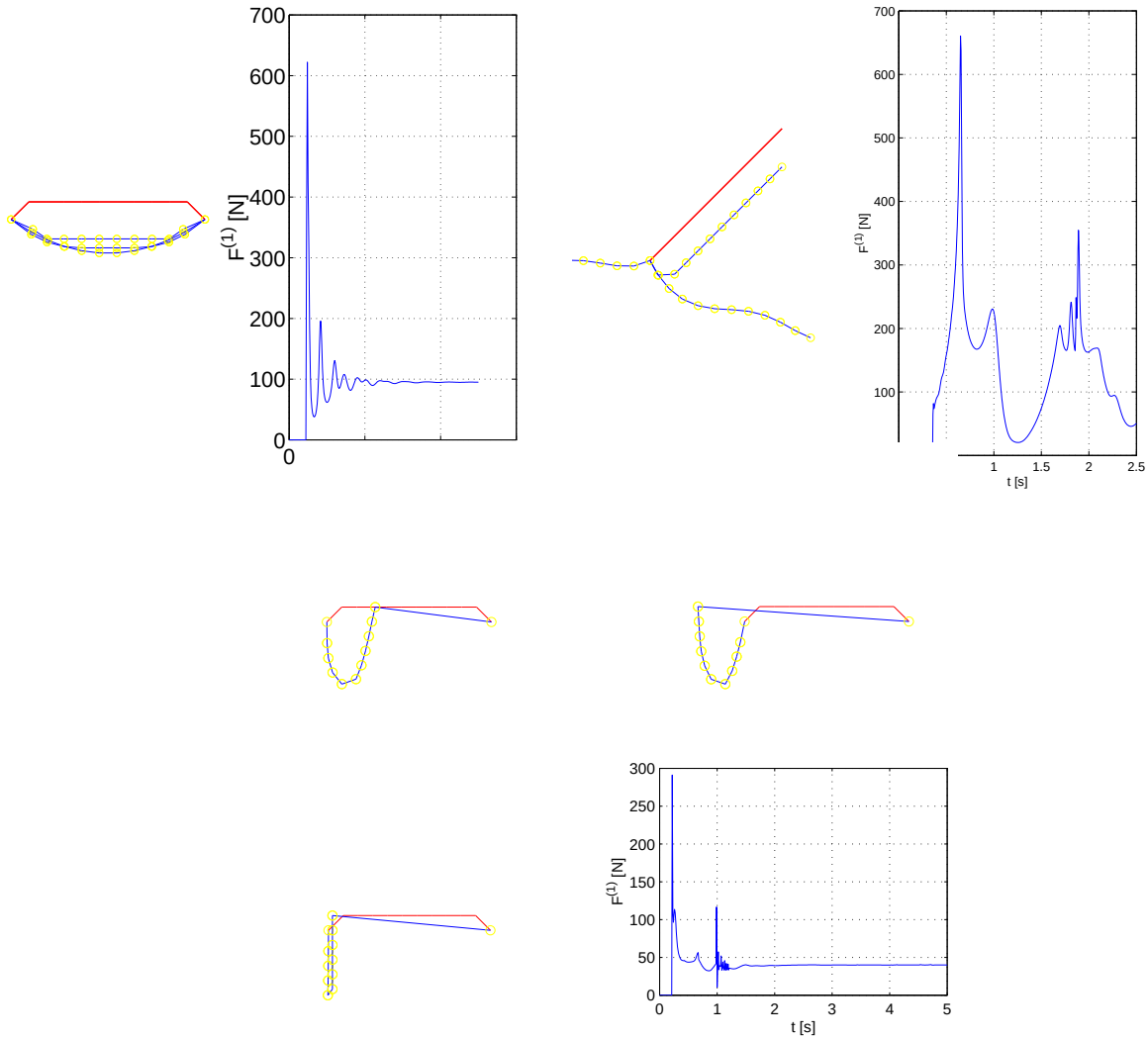


Fig. 6.28 : *Falling chains with masses.*

6.6.8 Crank with mass

The crank mechanism, which motion was analyzed before, is now provided with a mass of 100 kg in the end-point. The plots show the vertical position and velocity of and the reaction force in the end-point as a function of time.

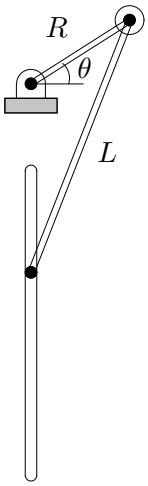


Fig. 6.29 : *Crack mechanism moving a mass in a slit.*

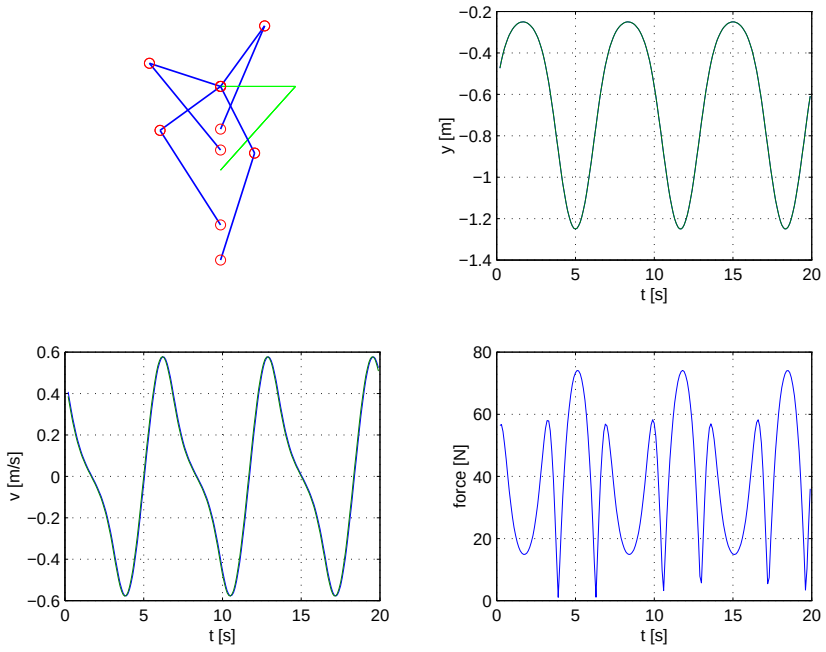


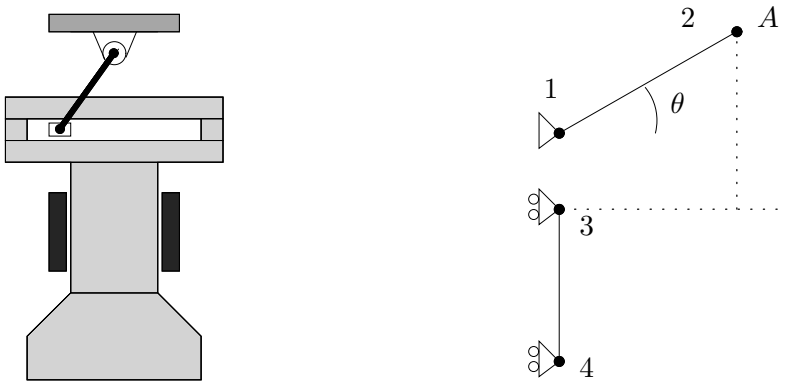
Fig. 6.30 : *Deformation (ul), displacement (ur), velocity (bl) and force (br).*

6.6.9 Crank-slider mechanism

The mechanism shown in the figure is used to raise and lower a forging hammer. The crank has a length R and turns anti-clockwise at the constant rate of $\omega = 30/60 \times 2\pi$ rad/sec. It is pinned to a block, which slides without friction in slot of the hammer substructure. The slot

and hammer have an offset such that they do not interfere with the crank and the pin. The weight of the hammer is 500 kg.

In de numerical model, the vertical displacement of the hammer is linked to the vertical displacement of the crank point A. The numerical solution for displacement and velocity equal the analytically calculated values. The vertical force on the hammer is also shown.



$$\vec{x}_A = R\vec{e}_r(\theta) = R\cos(\theta)\vec{e}_x + R\sin(\theta)\vec{e}_y$$
$$\dot{\vec{x}}_A = R\dot{\theta}\vec{e}_t(\theta) = -R\dot{\theta}\sin(\theta)\vec{e}_x + R\dot{\theta}\cos(\theta)\vec{e}_y$$

Fig. 6.31 : *Crank-slider mechanism and model. Position and velocity vectors of the crank point A (= 4).*

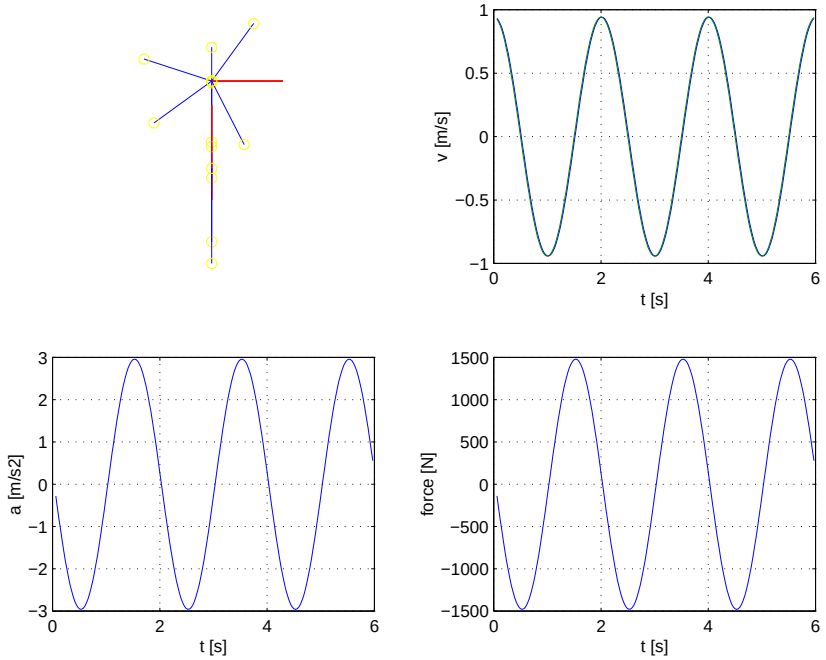


Fig. 6.32 : *Deformation (ul), velocity (ur), acceleration (bl) and vertical force (br).*

6.6.10 Multi-mass system

Mutual interaction of multiple masses can be modelled by connecting all masses with springs. With M the number of masses, the number n of interactions can be calculated.

The interaction between two masses can be described by a potential $U(r)$, where r represents the distance between the masses. The interaction force $F(r)$ and the bond stiffness $k(r)$ can be calculated as the first and second derivative of U w.r.t. r, respectively. The stiffness can be modelled as a non-linear spring between the masses. Here we use a *Lennard-Jones* potential, which represents the interaction between atoms in a material. The equilibrium distance is referred to as a_0. The potential contains three material constants : U_0, m and n.

In the example six masses of 1 kg are placed at horizontal and vertical disances of 1 m. At time $t = 0$ a pulse force of 10 N is applied to the upper-left mass. As a result the masses start to move and interaction between some of them is lost completely.

$$\text{number of interactions} \quad :n = \sum_{i=1}^{M} i * (M - i)$$

$$U(r) = \frac{m}{n-m} U_0 \left[\left(\frac{a_0}{r}\right)^n - \frac{n}{m} \left(\frac{a_0}{r}\right)^m \right]$$

$$F(r) = \frac{dU}{dr} = \frac{mn}{n-m} \frac{U_0}{a_0} \left[\left(\frac{a_0}{r}\right)^{m+1} - \left(\frac{a_0}{r}\right)^{n+1} \right]$$

$$k(r) = \frac{d^2U}{dr^2} = \frac{mn}{n-m} \frac{U_0}{a_0^2} \left[(n+1) \left(\frac{a_0}{r}\right)^{n+2} - (m+1) \left(\frac{a_0}{r}\right)^{m+2} \right]$$

$$k(a_0) = mn \frac{U_0}{a_0^2} \qquad \text{(equilibrium stiffness)}$$

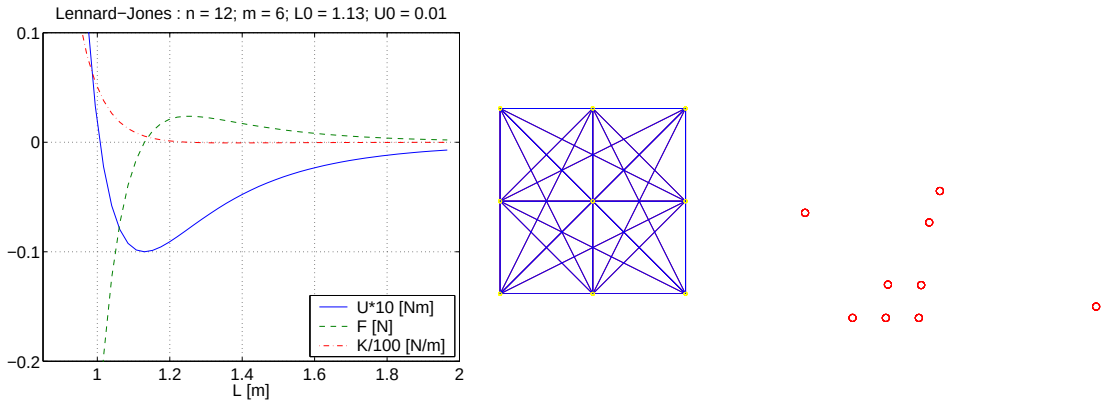


Fig. 6.33 : *Interconnected masses (potential, force and stiffnes of the bonds are shown) and their motion when subjected to a pulse force at* $t = 0$.

6.7 Spring-dashpot-mass systems

6.7.1 Spring-dashpot-mass system

We consider a simple spring-dashpot-mass system consisting of a parallel arrangement of a spring and a dashpot. The displacement $u(t)$ of the mass m is written as the sum of the solution $u_H(t)$ of the homogeneous differential equation and the solution $u_P(t)$ of the total equation.

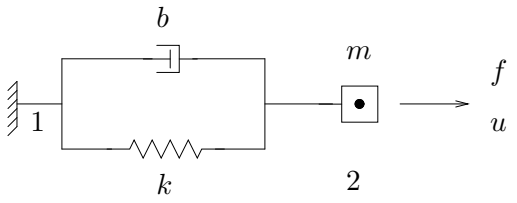


Fig. 6.34 : *Spring-dashpot-mass system.*

$$m\ddot{u} + b\dot{u} + ku = f$$

6.7.2 Free damped vibration

The solution of the homogeneous equation is the *free damped vibration* of the mass m. Substitution of $u_H = Ce^{\lambda t}$ gives the characteristic equation which can easily be solved.

The solution $u_H(t)$ will damp out as time proceeds : it is the *initial transient response*. The value of $b^2 - 4mk$ determines the character of this damping. A much used parameter is the *damping ratio* $\zeta = b/(2\sqrt{mk})$. Three cases have to be considered : an over-damped system ($\zeta^2 > 1$), a critically damped system ($\zeta^2 = 1$) and an under-damped system ($\zeta^2 < 1$).

differential equation $\qquad m\ddot{u} + b\dot{u} + ku = 0$

substitution of $u_H = Ce^{\lambda t}$ $\quad \rightarrow$

$$m\lambda^2 + b\lambda + k = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{1}{2m}\left\{-b \pm \sqrt{b^2 - 4mk}\right\} \qquad \rightarrow$$

initial transient response

$$u_H(t) = c_1 e^{(-\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2-4mk})t} + c_2 e^{(-\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2-4mk})t}$$

$b^2 - 4mk \quad \rightarrow$ character of damping

$\zeta = b/(2\sqrt{mk}) \quad :$ damping ratio

Over-damped system $\qquad b^2 - 4mk > 0 \quad \rightarrow \quad \zeta^2 > 1$

$$u_H(t) = \text{damped monotonously}$$

Critically damped system $\qquad b^2 - 4mk = 0 \quad \rightarrow \quad \zeta^2 = 1$

equal solutions characteristic equation $\quad \rightarrow$

$$u_H(t) = c_1 e^{(-\frac{b}{2m})t} + c_2 t e^{(-\frac{b}{2m})t}$$
$$= \text{damped not monotonously}$$

(damping depends on value c_2)

Under-damped system $\qquad b^2 - 4mk < 0 \quad \rightarrow \quad \zeta^2 < 1$

$$u_H(t) = c_1 e^{-\frac{b}{2m}t} e^{i\frac{1}{2m}\sqrt{4mk-b^2}\,t} + c_2 e^{-\frac{b}{2m}t} e^{-i\frac{1}{2m}\sqrt{4mk-b^2}\,t}$$
$$= e^{-\frac{b}{2m}t}\left[c_1 e^{i\frac{1}{2m}\sqrt{4mk-b^2}\,t} + c_2 e^{-i\frac{1}{2m}\sqrt{4mk-b^2}\,t}\right]$$
$$= \text{damped harmonic function}$$

As an example a system is analyzed for three different values of b. The system is loaded with a 20 N step force

m	1.6	kg	
k	1000	N/m	
b	80	Ns/m	critically damped
b	20	Ns/m	under-damped
b	200	Ns/m	over-damped

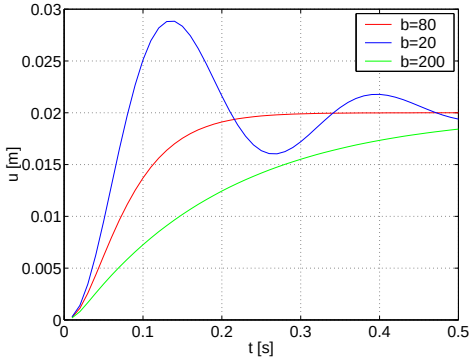


Fig. 6.35 : *Free vibration of the mass of the spring-dashpot-mass system.*

6.7.3 Harmonic force

Now we have to determine the solution $u_P(t)$, belonging to $f(t) \neq 0$. Assuming that the excitation force is harmonic : $f(t) = A\sin(\omega t)$, we choose $u_P(t)$ to have the form $u_P(t) = B\sin(\omega t + \phi)$, where ϕ is the phase difference of $f(t)$ with respect to $u_P(t)$. Both $u_P(t)$, $\dot{u}_P(t)$ and $\ddot{u}_P(t)$ are substituted in the differential equation.

$$-mB\omega^2\sin(\omega t+\phi) + bB\omega\cos(\omega t+\phi) + kB\sin(\omega t+\phi) = A\sin(\omega t) \qquad \rightarrow$$
$$[-mB\omega^2\cos(\phi) - bB\omega\sin(\phi) + kB\cos(\phi)]\sin(\omega t) +$$
$$[-mB\omega^2\sin(\phi) + bB\omega\cos(\phi) + kB\sin(\phi)]\cos(\omega t) = A\sin(\omega t)$$

relations must hold for every $t \quad \rightarrow$

$$-mB\omega^2\cos(\phi) - bB\omega\sin(\phi) + kB\cos(\phi) = A$$
$$-m\omega^2\sin(\phi) + b\omega\cos(\phi) + k\sin(\phi) = 0$$

The first equation gives us a relation for the amplitude B, as a function of ω and ϕ. The second relation leads, with $B \neq 0$ to an equation for the phase difference ϕ as a function of the angular frequency ω. This is back-substituted in the relation for $B(\omega, \phi)$.

first equation $\rightarrow$

$$\begin{aligned} B(\omega,\phi) &= \frac{A}{-m\omega^2\cos(\phi) - b\omega\sin(\phi) + k\cos(\phi)} \\ &= \frac{A}{(-m\omega^2+k)\cos(\phi) - b\omega\sin(\phi)} \end{aligned}$$

second equation (with $B \neq 0$) $\rightarrow$

$$\tan(\phi) = \frac{b\omega}{m\omega^2 - k} \qquad \rightarrow \qquad \phi = \arctan\left(\frac{b\omega}{m\omega^2-k}\right)$$

substitution in the equation for $B(\omega, \phi)$ $\rightarrow$

$$B(\omega) = \frac{A(-m\omega^2+k)\cos^{-1}(\phi)}{(-m\omega^2+k)^2 + b^2\omega^2} = -\frac{A}{\sqrt{b^2\omega^2 + (m\omega^2-k)^2}}$$

$$\text{with} \quad \cos(\phi) = \frac{m\omega^2 - k}{\sqrt{b^2\omega^2 + (m\omega^2-k)^2}}$$

From the forced (particulate) solution the amplitude ratio of the response and excitation and the phase difference can be derived, using the *undamped angular (eigen)frequency* $\omega_r = \sqrt{k/m}$ and the *damping ratio* $\zeta = b/(2\sqrt{mk})$.

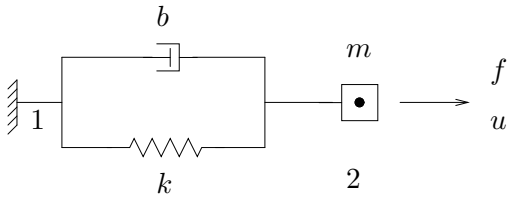
particulate solution

$$\begin{aligned} u_P &= B\cos(\phi)\sin(\omega t) + B\sin(\phi)\cos(\omega t) \\ &= \frac{A(-m\omega^2+k)}{(-m\omega^2+k)^2+b^2\omega^2}\sin(\omega t) + \frac{A(-m\omega^2+k)}{(-m\omega^2+k)^2+b^2\omega^2}\frac{b\omega}{m\omega^2-k}\cos(\omega t) \\ &= \frac{A(-m\omega^2+k)}{(-m\omega^2+k)^2+b^2\omega^2}\sin(\omega t) - \frac{Ab\omega}{(-m\omega^2+k)^2+b^2\omega^2}\cos(\omega t) \end{aligned}$$

undamped angular (eigen)frequency : $\omega_r = \sqrt{k/m}$

$$\frac{B}{A} = \frac{-1/k}{\sqrt{\left(1-\frac{\omega^2}{\omega_r^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_r}\right)^2}} \quad ; \quad \phi = -\arctan\left(\frac{2\zeta\frac{\omega}{\omega_r}}{1-\frac{\omega^2}{\omega_r^2}}\right)$$

The numerical example is a spring-dashpot-mass system, loaded with an harmonic force with amplitude 10 N and radial frequence $\omega = 20$ rad/s. The plot shows the force and the displacement of node 2.



$k = 1000$ [N/m] ; $b = 3$ [Ns/m] ; $m = 1.6$ [kg]
$f(t) = f_0 \sin(\omega t)$; $f_0 = 10$ [N] ; $\omega = 20$ [rad/s]

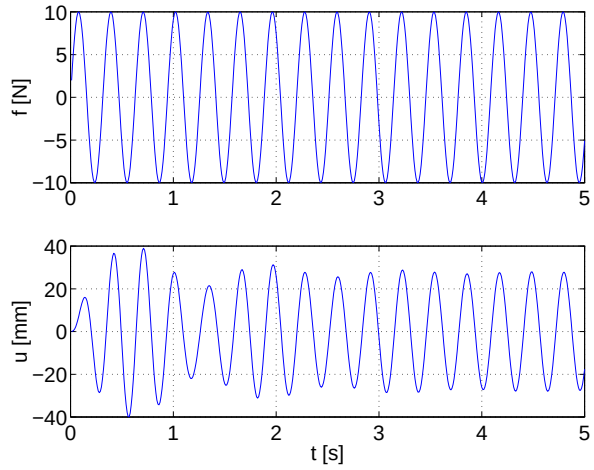


Fig. 6.36 : *Harmonic force excitation and displacement response of the mass in a spring-dashpot-mass system.*

6.7.4 Transfer and phase functions

A nanoindenter has several operation modes to apply load or displacement to a material specimen and to analyse the data. A DC current is applied to move the indenter continuously into the material specimen. Using AC modulation of the DC current, an harmonic force loading with a small amplitude and (variable) frequency can be superimposed onto the constant (or monotonically increasing) force. The resulting indentation displacement is also harmonic. Comparing excitation and response allows for determination of elastic and viscoelastic material parameters.

The mechanical model of the indenter is a Kelvin model with stiffness k, damping b and mass m. From the response to various load frequencies, these parameters can be fitted.

The system is loaded with an harmonic force $f(t) = f_0 \sin(\omega t)$ and the resulting steady state displacement will also be harmonic, with an amplitude u_0 and a phase shift ϕ with respect to the force. The stiffness and the phase shift can be derived.

$$k = 116 \text{ N/m} \ , \ b = 0.008 \text{ Ns/m} \ , \ m = 256 \text{ mg}$$
$$f_0 = 1 \text{ N} \ , \ \omega_r = \sqrt{\frac{k}{m}} = 673 \text{ rad/s} 0 < \omega < 1200 \text{ rad/s}$$

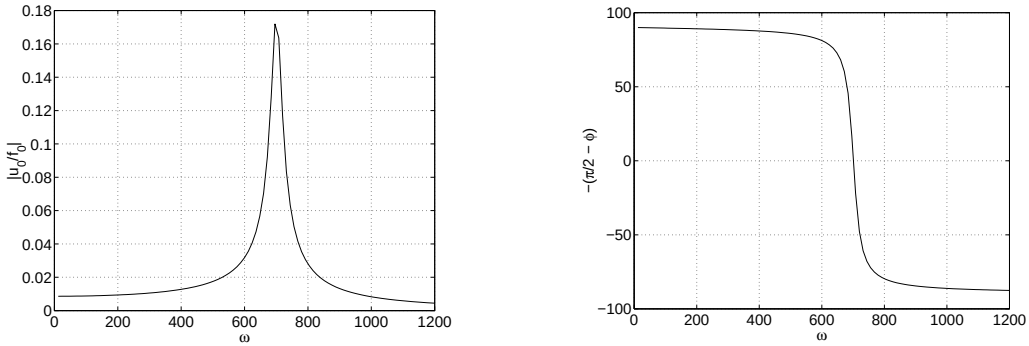


Fig. 6.37 : *Transfer function and phase shift as a function of the radial frequency.*

For other damping values the system response can also be calculated.

$$b \in \{0.05, 0.01, 0.1, 1\}$$

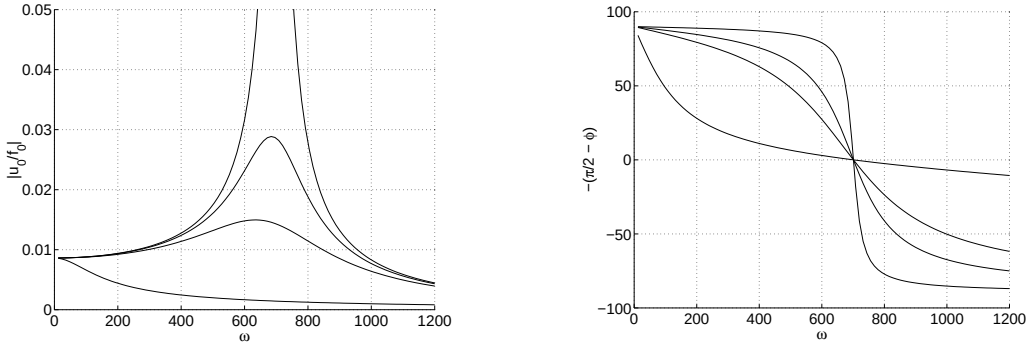
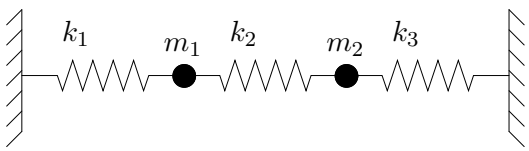


Fig. 6.38 : *Transfer function and phase shift as a function of the radial frequency.*

6.7.5 Coupled undamped oscillators

An series arrangement of three springs is fixed between rigid walls. Masses of 1 kg are located in the two connection points. In the first connection point a force is applied; a pulse force and an harmonic force, respectively. The plots show the displacement of the two connection points as a function of time.



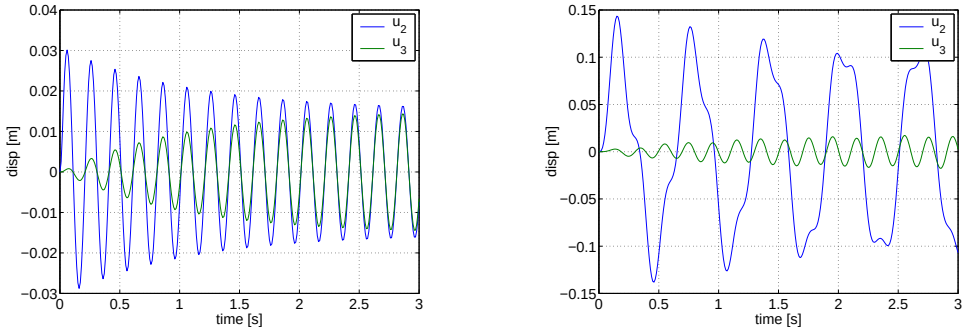


Fig. 6.39 : *Coupled undamped oscillator and the displacement of the masses resulting from a step force and an harmonic force on mass m_1.*

6.7.6 Coupled damped oscillators

An series arrangement of two Kelvin systems connected with a spring is fixed between rigid walls. Masses of 1 kg are located in the two connection points. In the first connection point a force is applied; a pulse force and an harmonic force, respectively. The plots show the displacement of the two connection points as a function of time.

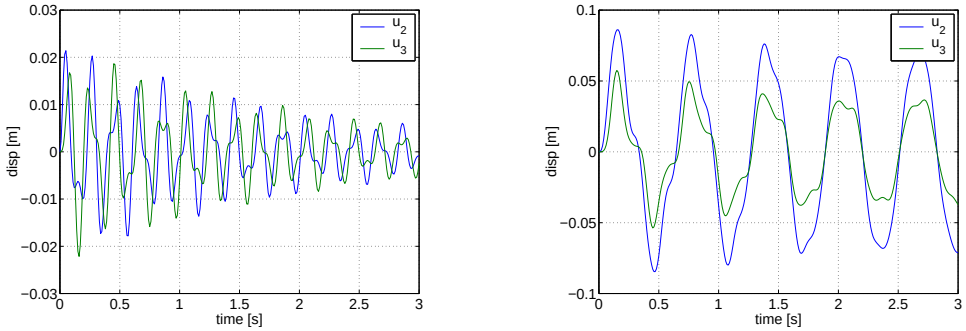


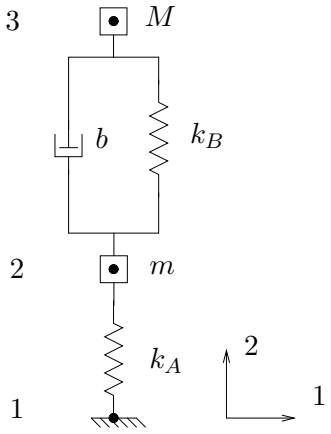
Fig. 6.40 : *Coupled damped oscillator and the displacement of the masses resulting from a step force and an harmonic force on mass m_1.*

6.7.7 Car suspension model

A car suspension is modeled with a dashpot (damping constant b), a spring (spring constant k_B), the mass m of the wheel and a spring representing the stiffness of the tire (spring constant k_A). The mass of the car, carried by the suspension, is M.

This spring-dashpot-mass system is excited by the displacement of node 1 (the road) in 2-direction. All displacements in 1-direction are suppressed. First we prescribe a step displacement of node 1 : $u_{12}(t < 0) = 0; u_{12}(t \geq 0) = 0.1$ [m]. The displacement of node 3, u_{32}, as a function of time is calculated for two values of the mass M : $M = 250$ [kg] and $M = 400$ [kg]. The trapezium integration rule is used with a time step of 0.05 s.

For a block displacement of node 1 the response is also calculated again for $M = 250$ [kg] and $M = 400$ [kg]. The trapezium rule is used for integration with a time step of 0.05 [s].



$b = 800$ [Ns/m] ; $k_B = 12500$ [N/m]
$k_A = 100000$ [N/m] ; $m = 30$ [kg]
$M = 250$ [kg] and $M = 400$ [kg]

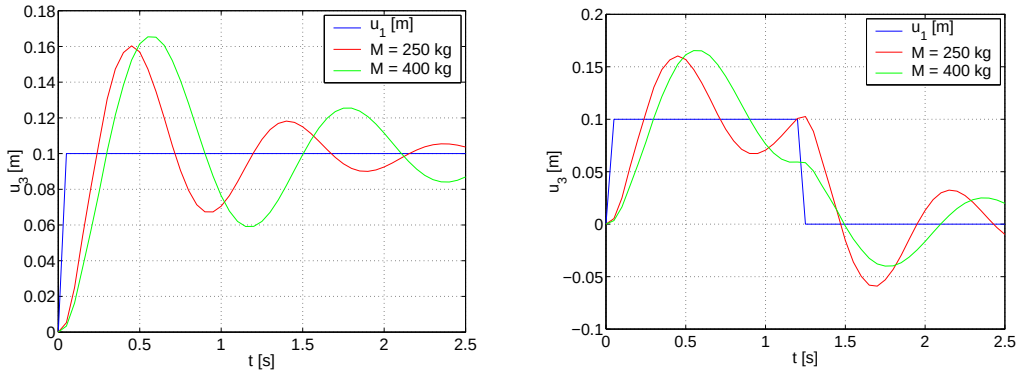


Fig. 6.41 : *Model of a car suspension, its parameters and the displacement of mass M resulting from a single and a double step displacement in point 1, respectively.*

6.7.8 Human body

It is important to know how the human body is responding to vibrations in their environment. A model of the body is made as shown in the figure.

Although the dynamic properties of the human body are amplitude-dependent, this is not considered here. Parameters are taken to be constant and values are based on literature, where is was found that 1) the compressive stiffness of the adult spinal column functional unit is about 250 N/mm and 2) the damping of the human body is 30-50 % critical. This is achieved with a single spring-dashpot-mass model with $k = 10^6$ [N/m], $b = 10^4$ [Ns/m] and m = 55 [kg] :

$$\zeta = \frac{b}{2\sqrt{mk}} \approx \frac{2}{3} \quad \rightarrow \quad \zeta^2 = \frac{4}{9} = 0.44$$

The load is a prescribed displacement of the floor, so applied at the feet. Both a step displacement and a harmonic vibration is considered.

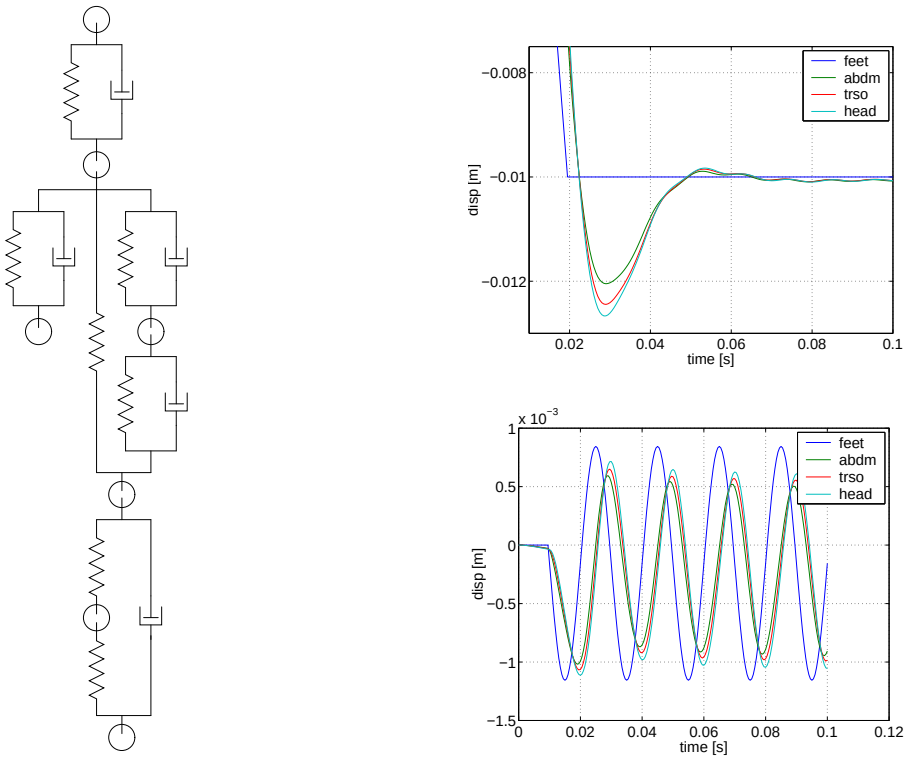
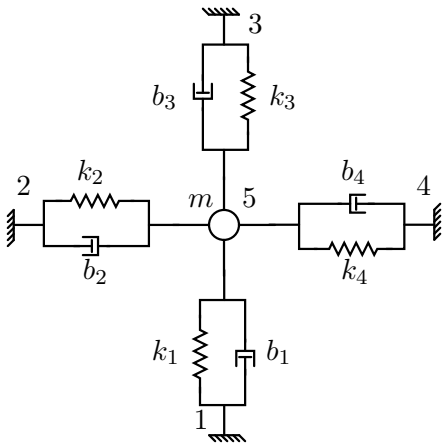


Fig. 6.42 : *Model of the human body and displacement of certain body parts as a result of an excitation of the feet.*

6.7.9 Two-dimensional spring-dashpot-mass system

A mass m is suspended by four spring-dashpot systems. At $t < 0$ the mass displacement is $(u_{51} = 0.1,\ u_{52} = 0.1)$. The mass position in the time interval $[0, 50]$ [s] is calculated.



m	k_1	k_2	k_3	k_4	$b_{1:4}$
kg	N/m	N/m	N/m	N/m	Ns/m
0.1	1	2	3	4	{0, 0.1, 1}

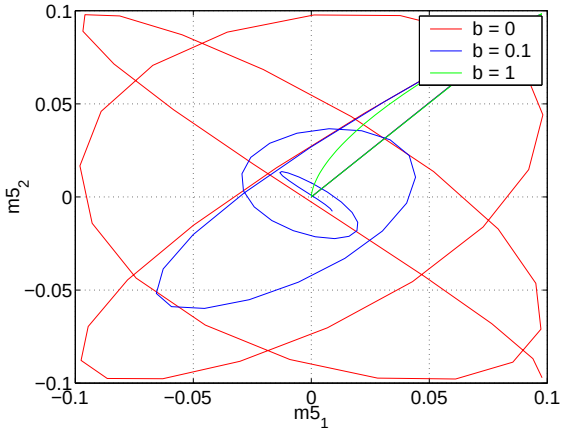


Fig. 6.43 : *Two-dimensional spring-dashpot-mass system, its parameters and the motion of the central mass for various dashpot damping values.*

6.8 Material modelling

The behavior of materials can be simulated with discrete mechanical models. Note however that a material is not build from springs, dashpots and friction elements.

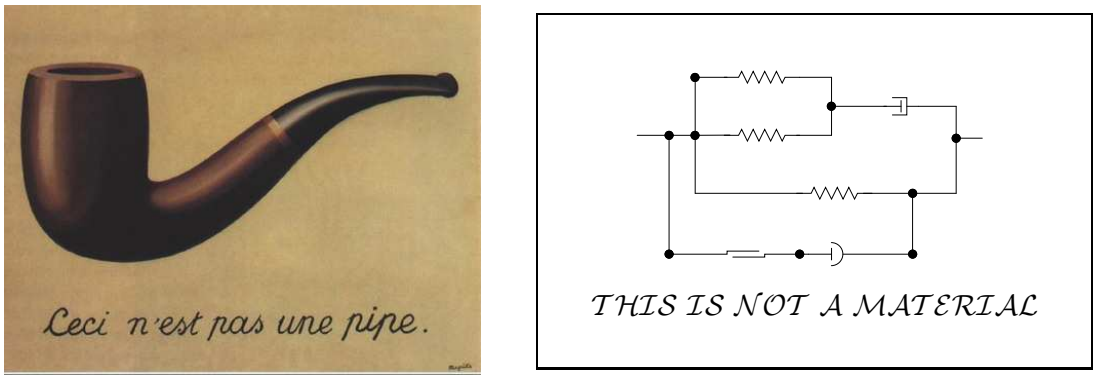


Fig. 6.44 : *A material model is a MODEL and not the representation of the material structure.*

6.8.1 Elastic material behaviour

The discrete mechanical element for the description of elastic material behavior is a spring, which can be linear or nonlinear.

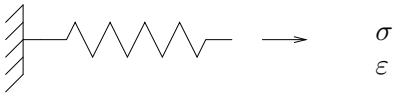


Fig. 6.45 : *Spring.*

6.8.2 Elasto-plastic material behaviour

Elasto-plastic material behaviour can be represented by a combination of three elements: two linear springs and a friction slider. When the stress is below the yield limit, the friction slider is rigid. The deformation is purely elastic: $\dot{\varepsilon} = \dot{\varepsilon}^e$. The elastic stiffness E of the system is represented by one spring element.

When the stress exceeds the yield limit, the friction element 'opens' and the deformation is elasto-plastic : $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$. The stiffness of the system is $C = \frac{EH}{E+H}$. The evolution of the yield limit is given by a hardening relation, which relates the yield limit to the effective plastic strain.

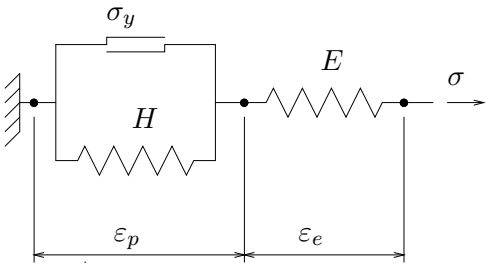


Fig. 6.46 : *Discrete model for elastoplastic material behavior.*

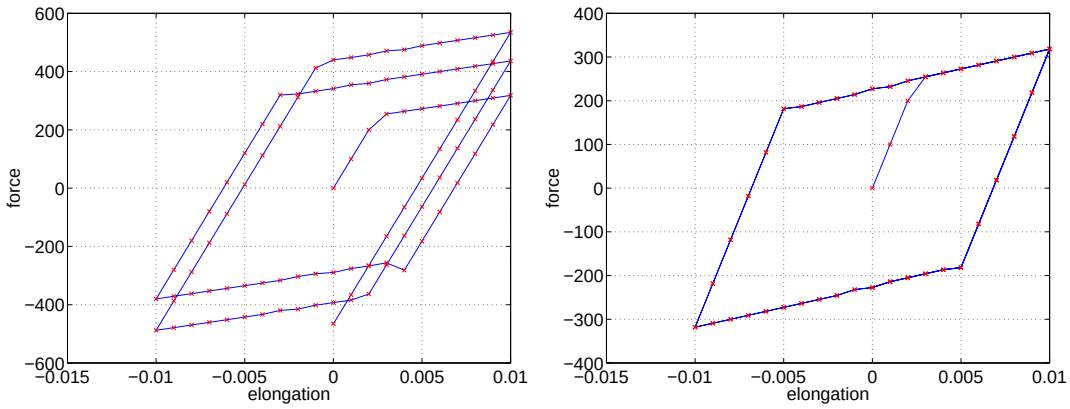


Fig. 6.47 : *Force-displacement curve for isotropic (l) and kinematic (r) hardening behavior.*

6.8.3 Linear viscoelastic behaviour

Linear viscoelastic material behaviour can be modelled with spring-dashpot systems, like Maxwell, Kelvin-Voigt and Standard Linear. Spring constants are indicated as elasticity constants (E) and damping constants as viscosities (η). Point variables are now strains (ε) instead of displacements and flow variables are stresses (σ) instead of forces. The respons of a certain system to a prescribed stress or strain can be calculated and compared to experimental data. The system parameters, being the material parameters, can than be determined such that both results coincide. The simple systems contain only two or three parameters, which make a good fit of the parameters difficult or impossible.

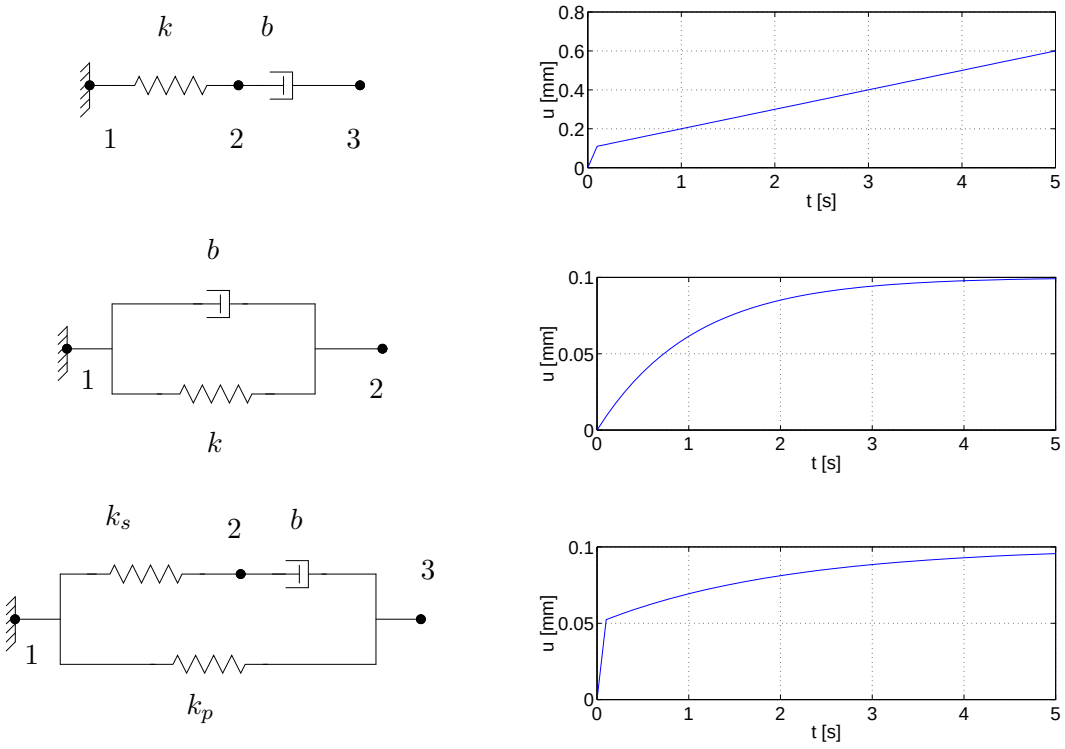


Fig. 6.48 : *Maxwell, Kelvin-Voigt and Standard Solid models and their displacement response to a step-force.*

Multi-mode Maxwell model

Real material behaviour is generally described by a multi-mode or generalized Maxwell model. Its relaxation function is :

$$E(t) = E_\infty + \sum_{i=1}^{n} E_i e^{-\frac{t}{\tau_i}}$$

Experimental data can be used to fit the parameters.

	oil ; $E_\infty = 0$		gel ; $E_\infty = 9.6\ 10^2$		rubber ; $E_\infty = 2.5\ 10^5$	
n	E_i	τ_i	E_i	τ_i	E_i	τ_i
1	$2.4\ 10^{-7}$	$1.0\ 10^4$	$1.0\ 10^3$	$1.0\ 10^4$	$1.2\ 10^4$	$2.2\ 10^3$
2	$2.7\ 10^{-5}$	$7.2\ 10^2$	$1.8\ 10^2$	$1.0\ 10^3$	$8.0\ 10^3$	$2.3\ 10^2$
3	$4.5\ 10^{-3}$	$5.1\ 10^1$	$3.3\ 10^1$	$1.0\ 10^2$	$1.7\ 10^4$	$2.4\ 10^1$
4	$9.0\ 10^{-1}$	$2.3\ 10^0$	$1.0\ 10^1$	$9.9\ 10^0$	$3.3\ 10^4$	$2.5\ 10^0$
5	$1.7\ 10^3$	$9.8\ 10^{-2}$	$2.8\ 10^2$	$8.1\ 10^{-1}$	$3.7\ 10^4$	$2.5\ 10^{-1}$
6	$1.9\ 10^4$	$1.3\ 10^{-2}$	$5.3\ 10^2$	$8.2\ 10^{-2}$	$7.6\ 10^4$	$2.6\ 10^{-2}$
7	$5.3\ 10^4$	$1.2\ 10^{-3}$	$1.1\ 10^3$	$4.9\ 10^{-3}$	$2.3\ 10^5$	$2.7\ 10^{-3}$
8	$7.6\ 10^4$	$9.8\ 10^{-5}$	$2.9\ 10^3$	$3.3\ 10^{-4}$	$1.3\ 10^6$	$2.8\ 10^{-4}$
9			$1.9\ 10^4$	$6.8\ 10^{-5}$	$5.4\ 10^6$	$2.9\ 10^{-5}$
10					$3.9\ 10^6$	$3.0\ 10^{-6}$
11					$1.4\ 10^6$	$3.0\ 10^{-7}$
12					$3.0\ 10^6$	$3.1\ 10^{-8}$

For these three materials the stress response for a strain step with amplitude $\varepsilon_0 = 0.1$ is calculated.

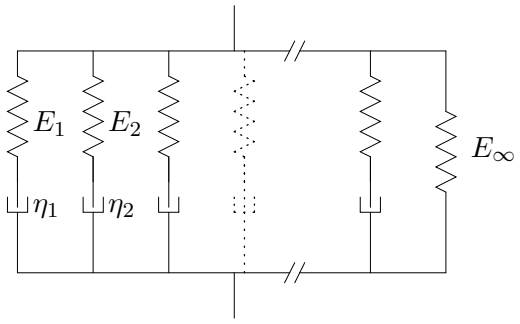


Fig. 6.49 : *Multi-mode Maxwell model.*

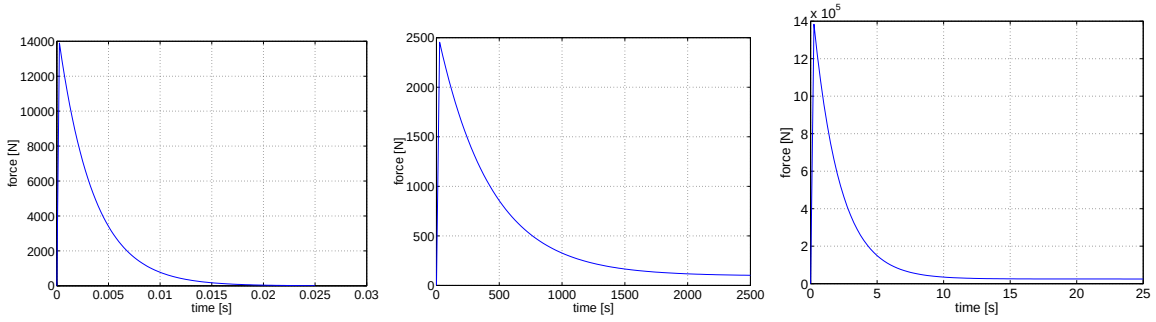


Fig. 6.50 : *Three different displacement responses (for different parameter values) to a step force.*

6.8.4 Creep behaviour

Creep is modelled with a nonlinear dashpot in series with a linear spring.

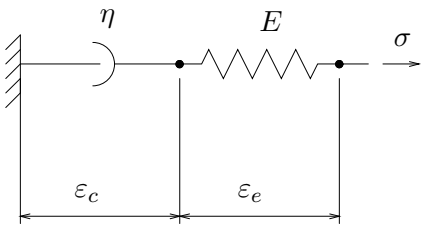


Fig. 6.51 : *Creep model.*

6.8.5 Viscoplastic behaviour

Viscoplastic material behavior is modelled with a linear spring in series with a parallel arrangment of a friction slider, a linear dashpot and a linear spring.

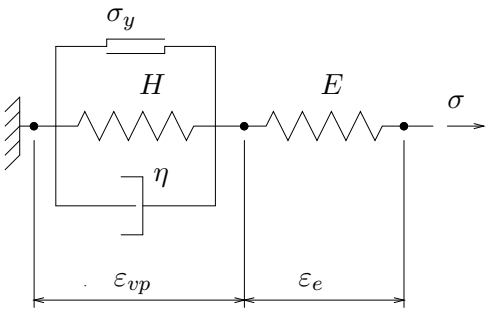


Fig. 6.52 : *Discrete model for viscoplastic material behavior.*

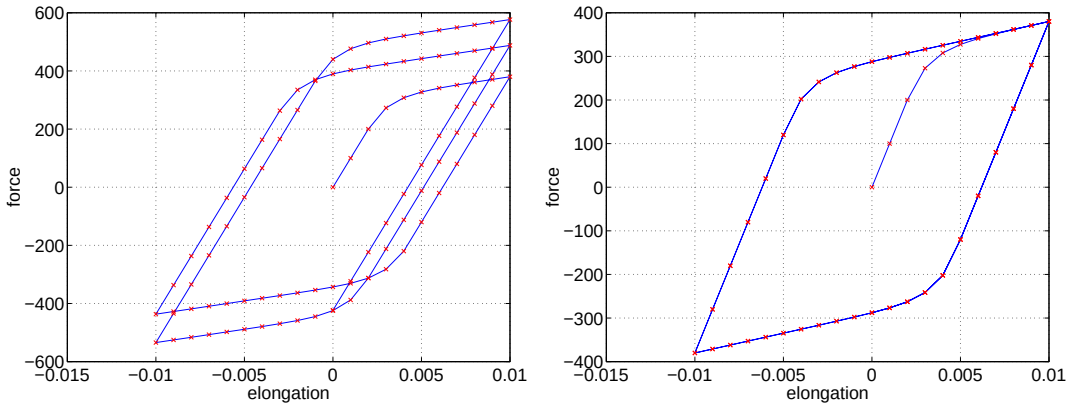


Fig. 6.53 : *Force-displacement curve for isotropic and kinematic hardening of a viscoplastic material model.*

6.8.6 Compressible Leonov model

The compressible Leonov model is used to describe non-linear viscoelastic behaviour of polymers and metals. A one-dimensional version can be modelled with a spring-dashpot system, which contains a non-linear dashpot, the viscosity of which decreases very sharply over many decades as a function of stress but also as a function of a nonrecoverable damage variable.

The compression modulus κ and the shear modulus G are used as elastic parameters :

$$\kappa = \frac{E}{3(1-2\nu)} \quad ; \quad G = \frac{E}{2(1+\nu)}$$

The softening behaviour is described by a damage variable D, which has to be solved from an evolution equation.

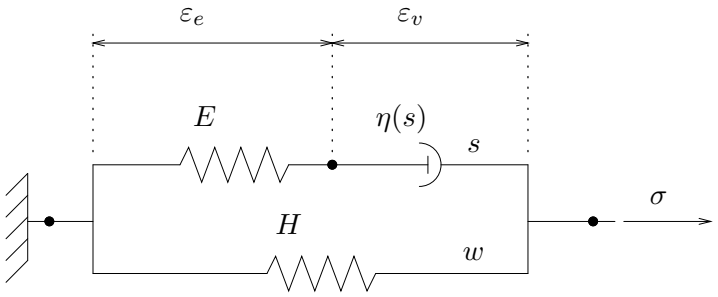


Fig. 6.54 : *Model for nonlinear viscoelastic behavior.*

- $\sigma = s + w = s^h + s^d + w = \kappa(J-1) + E\varepsilon_e^d + H\varepsilon^d$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\dot{\varepsilon}_v = \dfrac{1}{2\eta(\bar{s}, D)}\, s^d$
- $\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v$

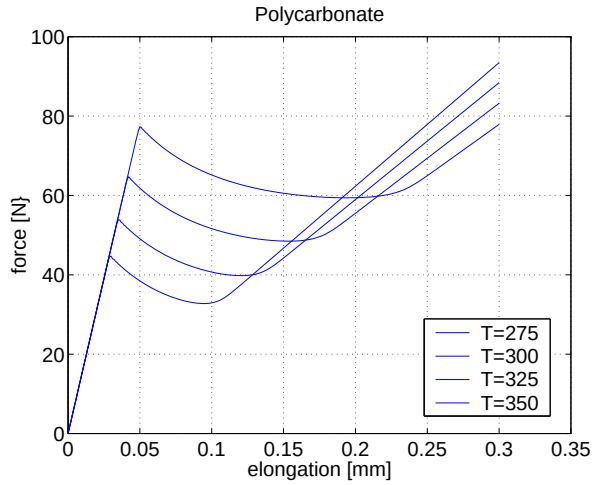


Fig. 6.55 : *Force-elongation curve for Polycarbonate at various temperatures.*

Viscosities

The Leonov model can be used successfully to describe the elastoviscoplastic behaviour of many materials, just by choosing the proper relation for the viscosity. For polymers the Eyring viscosity function is used and for metals the Bodner-Partom viscosity function. The universal gas constant is $R = 8.314$ J/(mol.K). Parameters for different materials are determined experimentally :

	PET	PC	PS	PP	
E	2400	2305	3300	1092	MPa
ν	0.35	0.37	0.37	0.4	-
H	15	29	13	3	MPa
h	13	270	100	0	-
D_∞	11	19	14	-	-
A_0	3.8568E-27	9.7573E-27	4.2619E-34	2.0319E-29	s
ΔH	2.617E+05	2.9E+05	2.6E+5	2.2E+5	J/mol
μ	0.0625	0.06984	0.294	0.23	-
τ_0	0.927	0.72	2.1	1.0	MPa

Bodner-Partom parameters for steel and aluminium :

	steel	Al	
G	7.8E4	2.6E4	MPa
K	1.52E5	7.8E4	MPa
Γ_0	1.0E8	1.0E8	s^{-2}
n	1.82	3.4	-
m	20	13.8	-
Z_0	810	81.4	MPa
Z_1	930	170	MPa

Eyring $$\eta = \frac{A\bar{\sigma}}{\sqrt{3}\,\sinh\left(\frac{\bar{\sigma}}{\sqrt{3}\tau_0}\right)} \quad ; \quad A = A_0 \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right] \quad ; \quad \tau_0 = \frac{RT}{V}$$

Bodner-Partom $$\eta = \frac{\bar{\sigma}}{\sqrt{12\Gamma_0}} \exp\left[\frac{1}{2}\left(\frac{Z}{\bar{\sigma}}\right)^{2n}\right] \quad ; \quad Z = Z_1 + (Z_0 - Z_1)\exp\left[-m\bar{\varepsilon}_p\right]$$

6.8.7 Damage

Damage propagation in a material can be simulated with a lattice model for that material, where it is considered to be a lattice of trusses (or beams). During loading, the stress in each truss is calculated and its stiffness is reduced, when the stress exceeds a threshold value.

The example shows a lattice in its initial state and in a state where damage has been propagated.

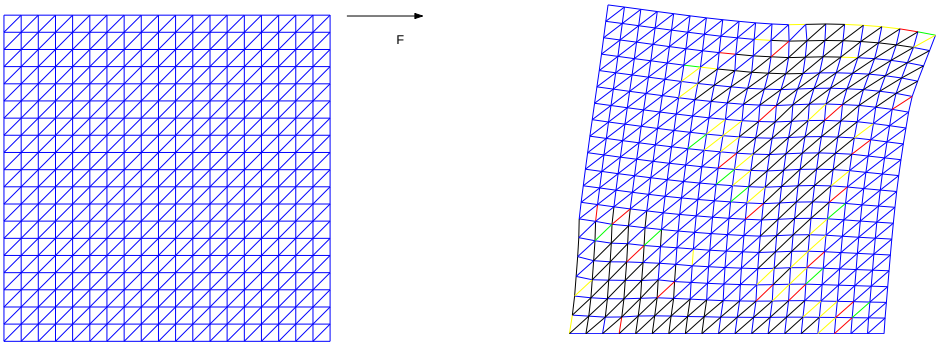


Fig. 6.56 : *Evolution of damage in a lattice structure.*

6.8.8 Cohezive zone models

Interface failure is a local failure mechanism, which can be modeled by cohesive zones. Two associated points on the two faces of an interface are connected by a spring, which models a softening traction-opening law. Parameters have been determined experimentally for the interface between a polymer coating and the underlying metal substrate.

$$T = \frac{\phi}{\delta}\left(\frac{\Delta}{\delta}\right)\exp\left(-\frac{\Delta}{\delta}\right)$$
$$k = \frac{\partial T}{\partial \Delta} = \left(\frac{\phi}{\delta}\right)\left(\frac{1}{\delta}\right)\left(1-\frac{\Delta}{\delta}\right)\exp\left(-\frac{\Delta}{\delta}\right)$$
$$\phi = 194\ [\mathrm{J/m^2}] \quad ; \quad \delta = 1\ [\mu\mathrm{m}] \quad ; \quad T_{max} = \frac{\phi}{\delta}\exp(1)$$

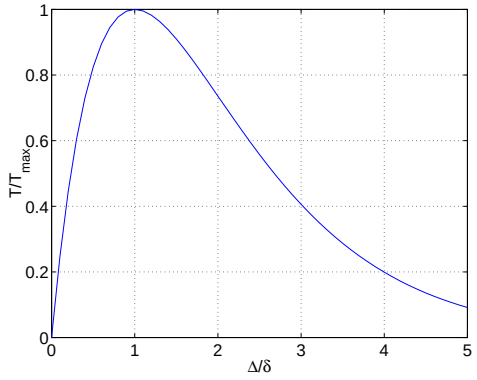


Fig. 6.57 : *Traction-opening relation for a cohesive zone model.*

Interface behaviour

Delamination of a rigid and a compliant layer is illustrated. The compliant layer is modelled as an elastic truss structure. The plots show the force-elongation curves for the cohesive zone springs.

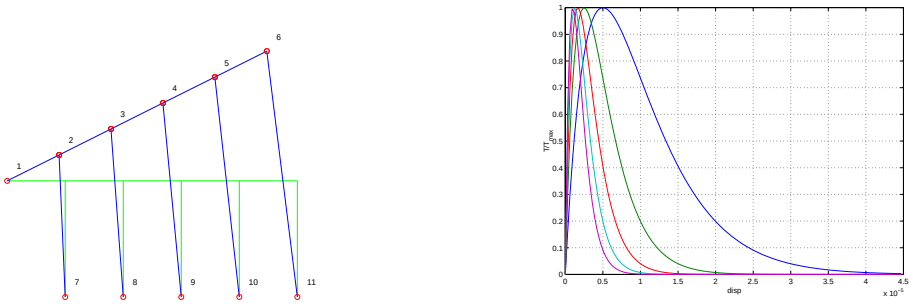


Fig. 6.58 : *Deformation and force-elongation curves for the individual cohesive bonds for a rigid layer.*

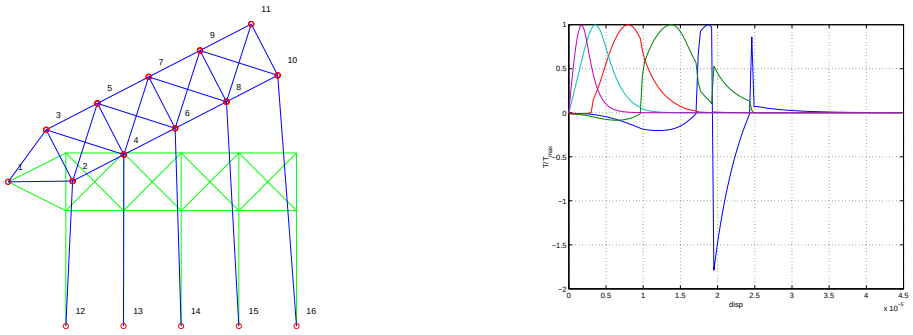


Fig. 6.59 : *Deformation and force-elongation curves for the individual cohesive bonds for a deformable layer.*

6.8.9 Cohesive zone for fatigue

For fatigue damage simulation, the cohesive stiffness is linear, but gradually loses its stiffness due to the increase of damage, described by a damage variable. The value of the damage variable increases from zero (no damage) to one (full failure) as a function of loading and deformation. Its growth is described by an evolution equation. Parameters have been determined for the interface between solder and solder pad in electronic joints in BGA's (Ball Grid Arrays).

The example shows the damage as the result of a two-cycle opening loading. Different damage evolution curves arise due to different algorithms for damage propagation. The fastest damage increase occurs when damage is always propagating. The least damage increase occurs when it is assumed that only opning with a positive opening rate introduces damage.

$$T = k(1-D)\Delta$$
$$\dot{D} = c\,|\dot{\Delta}|\,(1-D+r)^r \left\langle \frac{|T|}{(1-D)} - \sigma_f \right\rangle$$

k	c	m	r	σ_f
10^9	0.00025	3	0	0

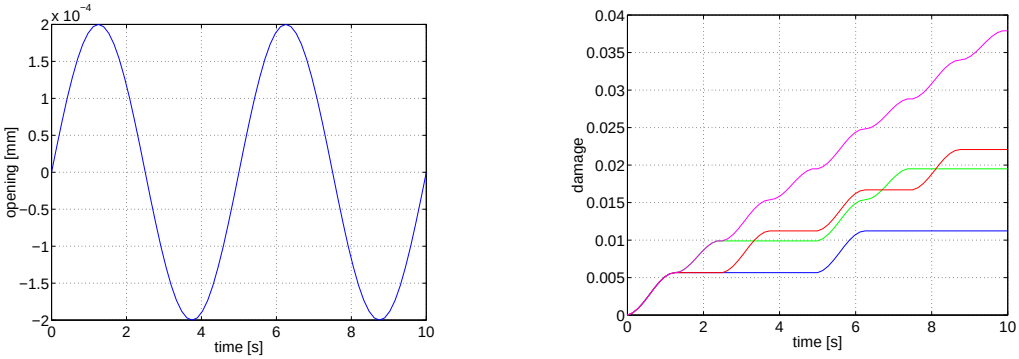


Fig. 6.60 : *Harmonic opening and resulting damage for different settings.*

6.8.10 Cohesive zone for fatigue

The opening behavior of an interface, which is subjected to cyclic loading, is illustrated. Values of the cohesive zone parameters are listed in the table.

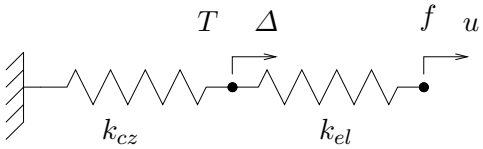


Fig. 6.61 : *Rigid substrate (fixed world), cohesive zone and elastic layer loaded with an harmonic displacement u.*

k_{cz}	c	m	r	σ_f
10^9	5000	3	0	0

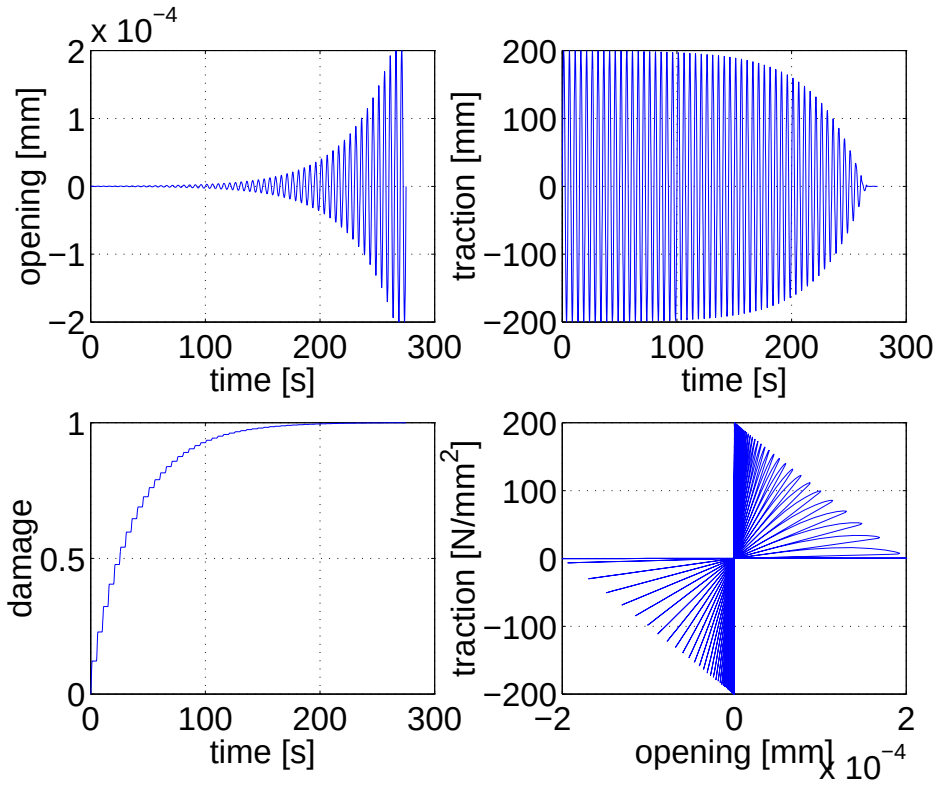


Fig. 6.62 : *Opening (ul), traction (ur), damage (bl) and stiffness (br) of the cohesive zone.*

Chapter 7

Electrical systems

In electrical systems the point variable is *voltage* V and the flow variable is *electrical current* I. Currents are *positive* when flowing *into* an element. Electrical elements are the inductance (value L), the resistance (value R) and the capacitance (value C).

7.1 Electrical elements

7.1.1 Inductance

The *inductance* is an element which relates the time derivative of the current to the voltage difference. Because time derivatives of a flow variable are not well suitable for our formulation, the inductance relation is integrated. This results in a relation between the current and the time-integral of the voltage difference.

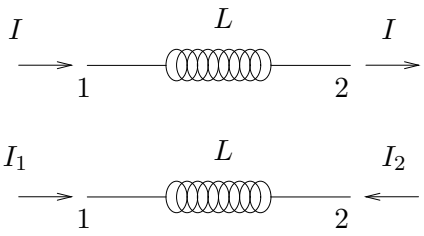


Fig. 7.1 : *Inductance with element current and nodal currents.*

$$\dot{I} = \frac{1}{L}\Delta V = \frac{1}{L}(V_1 - V_2) \quad \rightarrow \quad \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{\dot{I}}(t) = \underline{\bar{L}}\,\underset{\sim}{V}(t) \quad \rightarrow \quad \underset{\sim}{I}(t) = \underline{\bar{L}} \int_{\tau=0}^{t} \underset{\sim}{V}(\tau)\, d\tau + \underset{\sim}{I}(0)$$

7.1.2 Resistance

Ohm's law for electrical resistance was discovered in 1826 by Georg Simon Ohm : the current through a wire is inversely proportional to the voltage over the wire. An electrical *resistance* relates the current to the voltage difference.

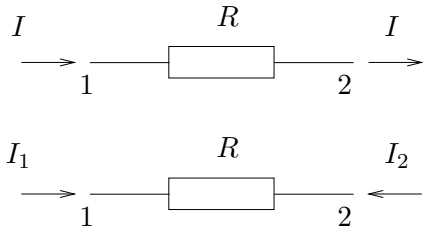


Fig. 7.2 : *Resistance with element current and nodal currents.*

$$I = \frac{1}{R}\Delta V = \frac{1}{R}(V_1 - V_2) \quad \rightarrow \quad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{R}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{I}(t) = \underline{\bar{R}}\underset{\sim}{V}(t)$$

7.1.3 Capacitance

A *capacitance* is an element between two nodes, which relates the current to the time derivative of the voltage difference. The capacitance stores electrical charge.

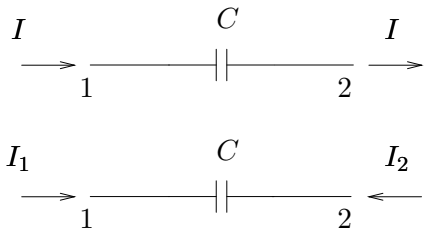


Fig. 7.3 : *Capacitance with element current and nodal currents.*

$$I = C\Delta\dot{V} = C(\dot{V}_1 - \dot{V}_2) \quad \rightarrow \quad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = C\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{I}(t) = \underline{C}\dot{\underset{\sim}{V}}(t)$$

7.2 System equations

The system behavior is described by a set of differential equations resulting from an assembling process which implies that the elements are connected. This procedure follows the next rules :

1 In every node the sum of the *outgoing* currents to the connected elements must equal the total external *incoming* current I_e.

2 In every node the potential (voltage) of the connected element nodes must be equal.

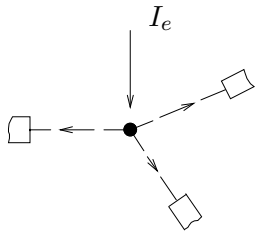


Fig. 7.4 : *Node with three attached elements.*

Assembling results in a set of first order differential equations. We can differentiate this to get a second order system, which can be solved by numerical integration.

$$\underline{C}\underset{\sim}{\dot{V}}(t) + \underline{\bar{R}}\underset{\sim}{V}(t) + \underline{\bar{L}} \int_{\tau=0}^{t} \underset{\sim}{V}(\tau)\, d\tau + \underset{\sim}{I}(0) = \underset{\sim}{I}_e(t) \quad \rightarrow$$
$$\underline{C}\underset{\sim}{\dot{V}}(t) + \underline{\bar{R}}\underset{\sim}{V}(t) + \underset{\sim}{I}_{ind}(t) = \underset{\sim}{I}_e(t)$$

second-order system

$$\underline{C}\underset{\sim}{\ddot{V}}(t) + \underline{\bar{R}}\underset{\sim}{\dot{V}}(t) + \underset{\sim}{\dot{I}}_{ind}(t) = \underset{\sim}{\dot{I}}_e(t) \quad \rightarrow \quad \underline{C}\underset{\sim}{\ddot{V}}(t) + \underline{\bar{R}}\underset{\sim}{\dot{V}}(t) + \underline{\bar{L}}\underset{\sim}{V}(t) = \underset{\sim}{\dot{I}}_e(t)$$

7.3 Special elements

7.3.1 Diode

A *diode* is a device with a non-linear relation between current and voltage difference. In a tube diod the so-called *cathode* (terminal 2) is heated with an external current. When the opposite terminal, the *anode* (terminal 1) is at a high voltage, electrons will be pulled out from the cathode and into the anode. A current – positive charge – is flowing from anode to cathode as a result. In a semiconductor diod, the flow of electrons is allowed by the fact that there are deficiancies of electrons in the anode material and an abundance of electrons in the cathode material.

The diode is modeled as a resistance, having a certain resistance value when $\Delta V = V_1 - V_2 > 0$ and a much larger (infinite) value when $\Delta V < 0$. For the semi-conductor device this is shown in the figure as two straight lines. A diod acts as a valve for electrical current. In a tube-diod the current becomes zero at a certain negative value for ΔV. When $\Delta V = 0$ it will not be completely zero because electrons are released from the heated cathode and some of them will reach the anode. When $\Delta V = 0$, the current in a semi-conductor diod is zero. However, for negative values, a *leakage current* occurs. whic is very small : $\pm 10^{-14} - 10^{-9}$ A for a Si-diod and $\pm 10^{-7}$ A for a Ge-diod.

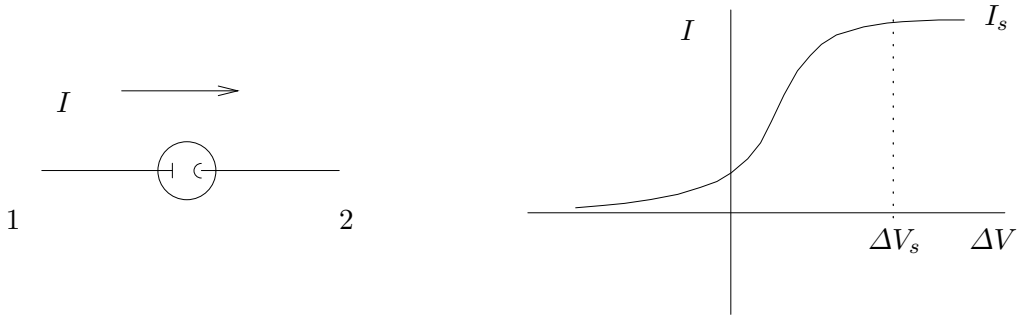


Fig. 7.5 : *Tube-iode and its current-voltage characteristic.*

$$
\begin{array}{lll}
I = Ke^{a\Delta V} & \text{for} & \Delta V < 0 \\
I = K'\Delta V^{\frac{3}{2}} & \text{for} & 0 < \Delta V < \Delta V_s \\
I = I_s & \text{for} & \Delta V > \Delta V_s
\end{array}
$$

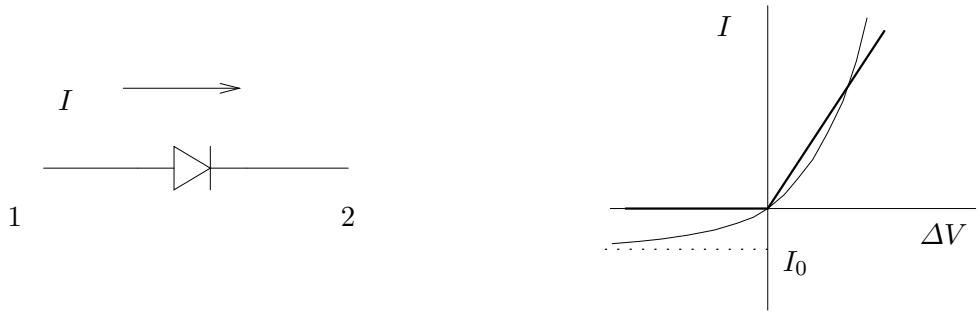


Fig. 7.6 : *Semiconductor-diode and its current-voltage characteristic.*

$$I = I_0 \left(e^{\frac{q}{kT}\Delta V} - 1 \right) \quad \text{with} \quad \frac{q}{kT} = 40 \text{ for } T = 300K$$

ΔV	= voltage difference	$[V]$
q	= electron charge $(1.6 \times 10^{-19}$ C)	
k	= Boltzmann's constant $(1.38 \times 10^{-23}$ J/^{o}K)	
T	= absolute temperature	$[K]$
I_0	= leakage current	$[A]$

7.3.2 Electrical transformer

An electrical *transformer* changes the voltage and thus the current between two nodes of the system. It is modeled with a link.

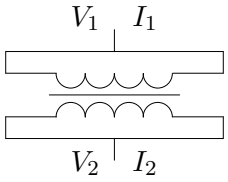


Fig. 7.7 : *Electrical transformer.*

$$V_2 = nV_1 \quad ; \quad V_1 I_1 = V_2 I_2 \quad \rightarrow \quad I_2 = \frac{1}{n} I_1$$

7.4 Examples

7.4.1 RC system

As an example we load an RC system with an harmonic current in node 1. The potential in this node can be calculated analytically and numerically.

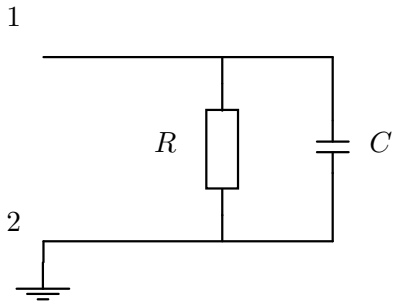


Fig. 7.8 : *RC system.*

differential equation ($V = V_1$)

$$I = I_1 = I_C + I_R = C\dot{V} + \frac{1}{R}V \qquad \rightarrow \qquad CR\,\dot{V}(t) + V(t) = R\,I(t)$$

harmonic current $I(t)$ $\qquad I(t) = I_0 \sin(\omega t)$

homogeneous solution $\qquad CR\dot{V} + V = 0 \qquad$ with $\quad V_H = Ae^{\lambda t} \qquad \rightarrow$
$CR\lambda + 1 = 0 \quad \rightarrow \quad V_H = Ae^{-t/(CR)}$

particulate solution $\qquad V_P = a_1 \sin(\omega t) + a_2 \cos(\omega t)$

substitution of V_P and $\dot{V}_P$ in differential equation

$$CRa_1\omega\cos(\omega t) - CRa_2\omega\sin(\omega t) + a_1\sin(\omega t) + a_2\cos(\omega t) = RI_0\sin(\omega t) \quad \Rightarrow$$

$$\left.\begin{array}{l} CRa_1\omega + a_2 = 0 \quad \rightarrow \quad a_2 = -CRa_1\omega \\ -CRa_2\omega + a_1 = RI_0 \end{array}\right\} \quad \rightarrow \quad a_1 = \frac{RI_0}{(CR)^2\omega^2 + 1}$$

general solution

$$V(t) = Ae^{-t/(CR)} + \left[\frac{RI_0}{(CR)^2\omega^2 + 1}\right]\sin(\omega t) - \left[\frac{CR^2I_0\omega}{(CR)^2\omega^2 + 1}\right]\cos(\omega t)$$

IC $\qquad V(t=0) = 0 \quad \rightarrow \quad A = \left[\frac{CR^2I_0\omega}{(CR)^2\omega^2 + 1}\right]$

solution

$$V(t) = \left[\frac{CR^2I_0\omega}{(CR)^2\omega^2 + 1}\right]e^{-t/(CR)} + \left[\frac{RI_0}{(CR)^2\omega^2 + 1}\right]\sin(\omega t) - \left[\frac{CR^2I_0\omega}{(CR)^2\omega^2 + 1}\right]\cos(\omega t)$$

The numerical and exact solution is plotted below and can be seen to be identical. Parameter values are given.

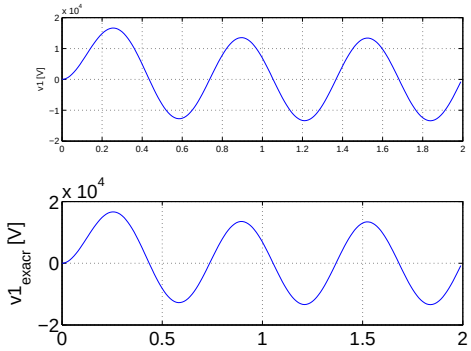


Fig. 7.9 : *Analytical and numerical solution for the potential in node 1, where an harmonic current is prescribed.*

$$C = 20 * 10^{-6} \quad [\text{f}] \quad ; \quad R = 10^4 \quad [\Omega] \quad ; \quad I_1 = 3\sin(10t) \quad [\text{A}]$$

The voltage over and currents through resistance and capacitance are plotted as a function of time.

$$C = 20 * 10^{-6} \quad [\text{f}] \quad ; \quad R = 10^4 \quad [\Omega] \quad ; \quad I_1 = 3\sin(10t) \quad [\text{A}]$$

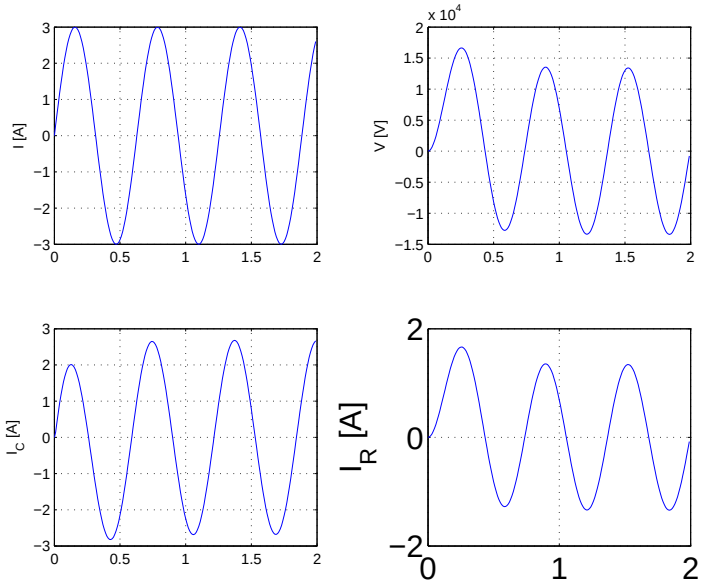
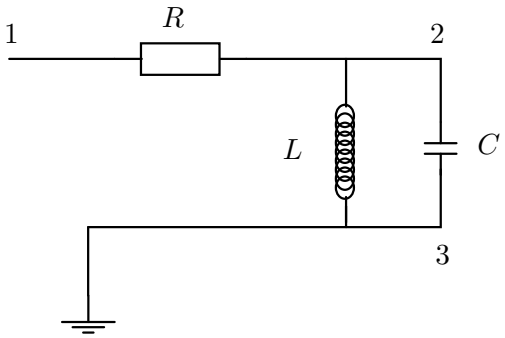


Fig. 7.10 : *Potential over and currents through resistance and capacitance.*

7.4.2 RCL system

A simple system with one resistance, inductance and capacitor is loaded with a voltage step of 0.1 V in node 1 at time $t = 0$. The response voltage in node 2 is calculated and it is obvious that the system is a damped oscillator.



$C = 10^{-13}$ [f]
$R = 10^{10}$ [Ω]
$L = 10^{6}$ [h]

Fig. 7.11 : *RCL system with parameter values.*

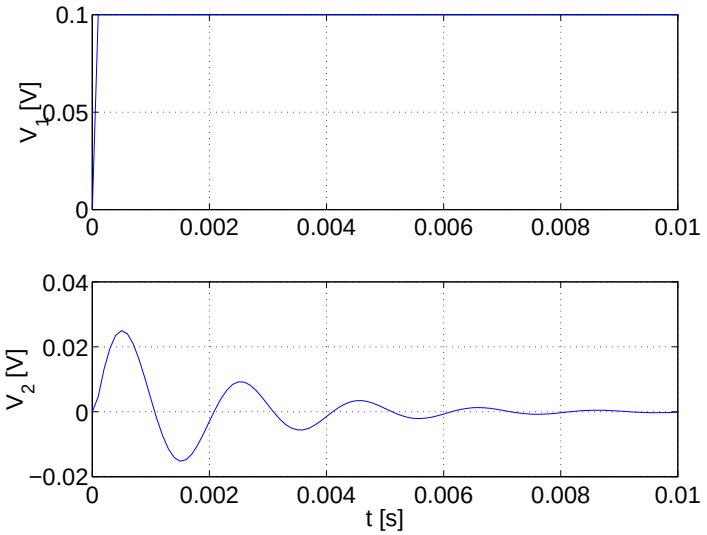
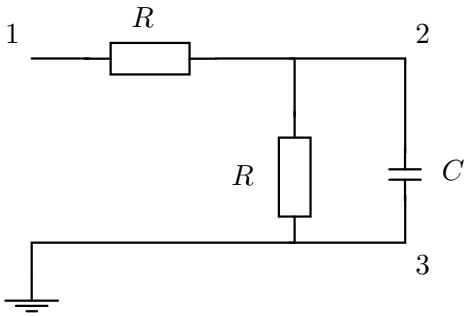


Fig. 7.12 : *Potential in node 1 (prescribed step) and node 2 (response).*

7.4.3 RRC system

An RRC system is loaded with a voltage (amplitude 10 [V]) in node 1, which varies as a block-function in time. The voltage in node 2 and the current through the capacitance is calculated numerically.



$C = 0.05 \quad [\mu\text{f}]$
$R = 20 \quad [\text{k}\Omega]$

Fig. 7.13 : *RRC system with parameter values.*

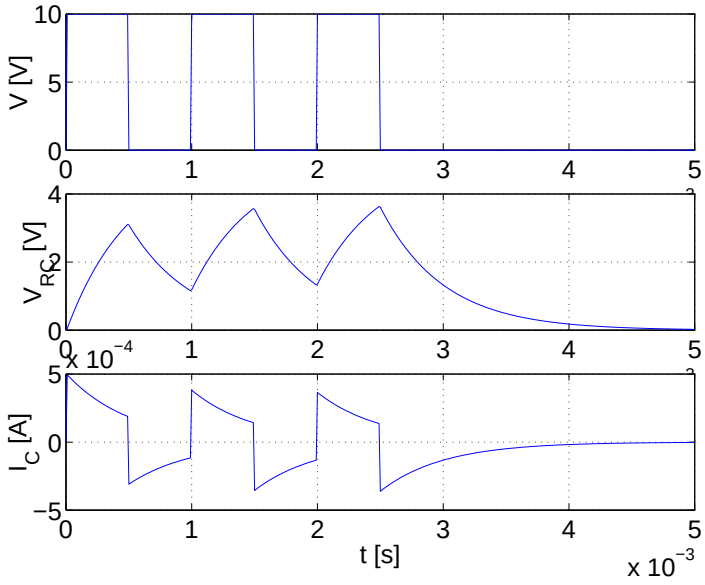
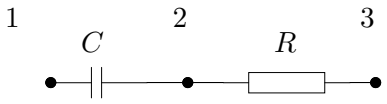
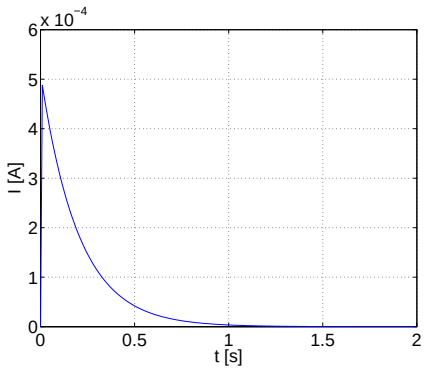


Fig. 7.14 : *Prescribed potential in node 1 (block function), resulting potential over the RC system and current through the capacitance.*

7.4.4 Resistance-capacitor/inductance

A series-arrangement of a resistance and a capacitor or an inductance, is loaded with a step voltage. The response of the two systems is antagonistic. The resistance-capacitor system shows a jump in current, which diminishes in time to become zero. The resistance-inductance system shows a gradual increase of the current towards a final steady value.

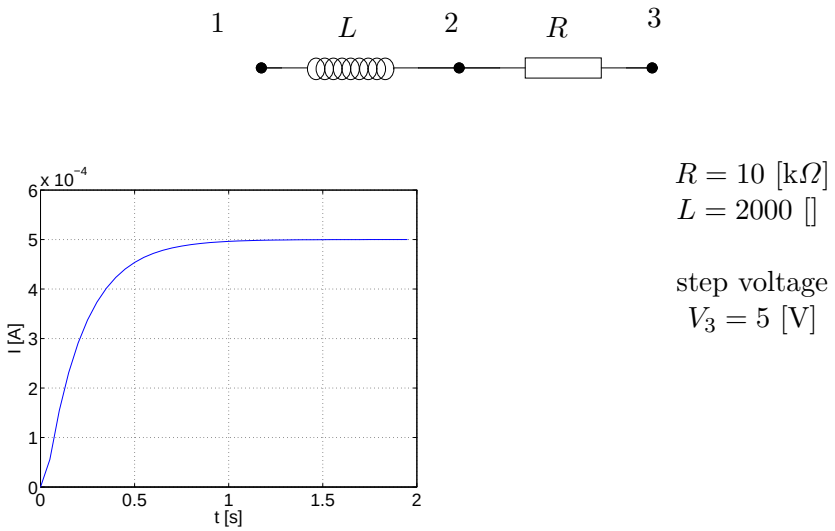




$R = 10$ [kΩ]
$C = 20$ [μF]

step voltage
$V_3 = 5$ [V]

Fig. 7.15 : *CR system with parameter values and response current resulting from a step voltage.*



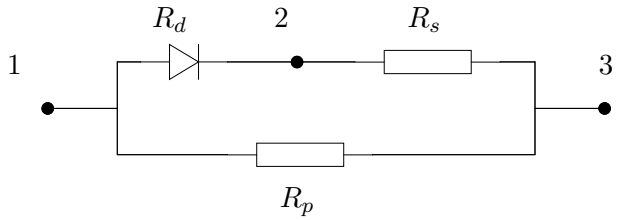
$R = 10$ [kΩ]
$L = 2000$ []

step voltage
$V_3 = 5$ [V]

Fig. 7.16 : *LR system with parameter values and response current resulting from a step voltage.*

7.4.5 Diode

As said before the diode is an electronic valve. This is illustrated with the next system. The voltage in node 1 is prescribed to be harmonic. Voltages and currents are calculated and show a typical pulsating response.



$R_s = 10$ [$k\Omega$] $\qquad$ $R_p = 10$ [$k\Omega$] $\qquad$ $R_d = 10$ [$k\Omega$]

Fig. 7.17 : *Electrical system with a diode and two resistances.*

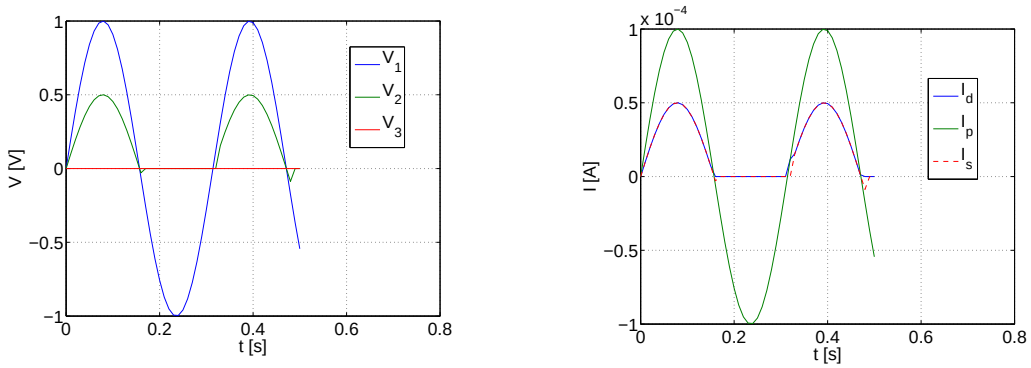


Fig. 7.18 : *Potential and currents.*

7.4.6 Bridge

To change alternating current (AC) into direct current (DC) a diode bridge or Graetz circuit can be used. It is loaded with a sine-voltage difference between points 1 and 2, having amplitude 1 V.

Without the capacitor C the voltage difference between points 3 and 4 is positive and pulsating. After adding a capacitor of 10^{-7} f(arad), the voltage difference is constant.

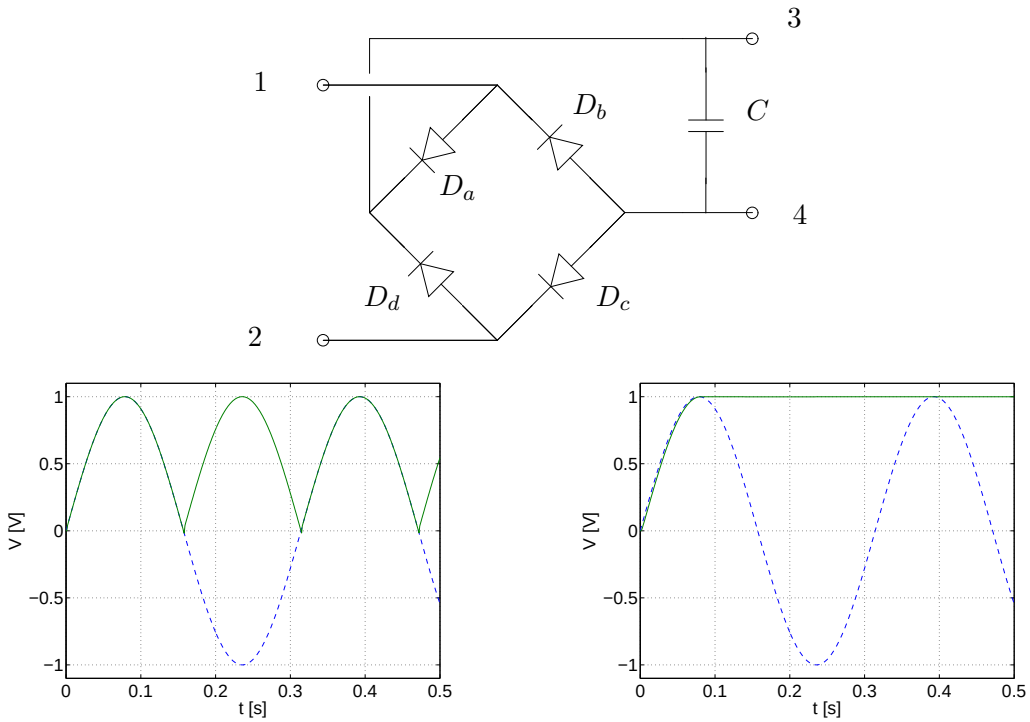


Fig. 7.19 : *Bridge of Graetz and resulting potentials between node 3 and 4.*

Chapter 8

Fluidial systems

In fluidial systems the point variable is *fluid pressure* p and the flow variable is *fluid flow* Q. Fluid flow is *positive* when flowing *into* an element. Fluid elements are the inertance L_f, the resistance R_f, and the capacitance C_f.

8.1 Fluidial elements

8.1.1 Inertance

The *inertance* is an element which relates the time derivative of the flow to the pressure difference. Because time derivatives of a flow variable is not well suitable for our formulation, the inertance relation is integrated. This results in a relation between the flow and the time-integral of the pressure difference.

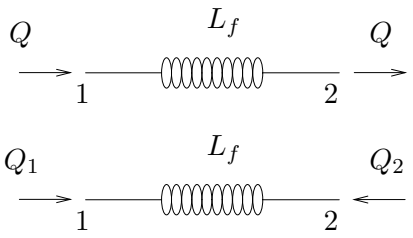


Fig. 8.1 : *Inertance.*

$$\dot{Q} = \frac{1}{L_f}\Delta p = \frac{1}{L_f}(p_1 - p_2) \quad \rightarrow \quad \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{bmatrix} = \frac{1}{L_f}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{\dot{Q}}(t) = \underline{\bar{L}}_f \underset{\sim}{p}(t) \quad \rightarrow \quad \underset{\sim}{Q}(t) = \underline{\bar{L}}_f \int_{\tau=0}^{t} \underset{\sim}{p}(\tau)\, d\tau + \underset{\sim}{Q}(0)$$

8.1.2 Resistance

A fluidial *resistance* relates the flow to the pressure difference.

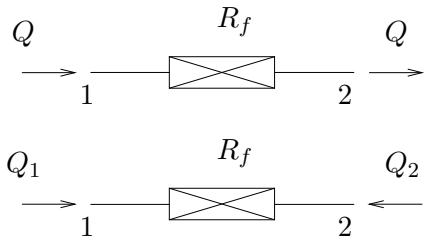


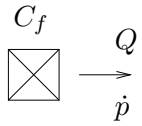
Fig. 8.2 : *Resistance.*

$$Q = \frac{1}{R_f}\Delta p = \frac{1}{R_f}(p_1 - p_2) \quad \rightarrow \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{1}{R_f}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{Q}(t) = \underline{\bar{R}}_f \underset{\sim}{p}(t)$$

8.1.3 Capacitance

A *capacitance* is a nodal variable, which relates the flow to the time derivative of the pressure difference. It generally represents a storage facility.



$$Q(t) = C_f \dot{p}(t)$$

Fig. 8.3 : *Capacitance.*

8.2 System equations

The system behavior is described by a set of differential equations resulting from an assembling process which implies that the elements are connected. This procedure follows the next rules :

1 In every node the difference of the *incoming* external flow Q_e and the total *outgoing* flow to the connected elements, must equal the product of capacitance and the derivative of the pressure change in that node.

2 In every node the pressures of the connected element nodes must be equal.

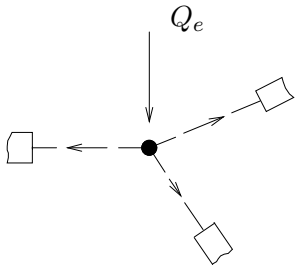


Fig. 8.4 : *Nde with three attached elements.*

Assembling results in a set of first order differential equations. Because of the nodal character of the capacitances, the matrix $\underline{C}_f$ is a diagonal matrix. We can rewrite the system and differentiate it to get a second order system. This system can be solved by numerical integration.

$$\underset{\sim}{Q}_e(t) - \left[\underline{\bar{R}}_f \underset{\sim}{p}(t) + \underline{\bar{L}}_f \int_{\tau=0}^{t} \underset{\sim}{p}(\tau)\, d\tau + \underset{\sim}{Q}(0)\right] = \underline{C}_f \dot{\underset{\sim}{p}}(t) \quad \rightarrow$$
$$\underline{C}_f \dot{\underset{\sim}{p}}(t) + \underline{\bar{R}}_f \underset{\sim}{p}(t) + \underset{\sim}{Q}_{ine}(t) = \underset{\sim}{Q}_e(t)$$

second-order system

$$\underline{C}_f \ddot{\underset{\sim}{p}}(t) + \underline{\bar{R}}_f \dot{\underset{\sim}{p}}(t) + \dot{\underset{\sim}{Q}}_{ine}(t) = \dot{\underset{\sim}{Q}}_e(t) \quad \rightarrow \quad \underline{C}_f \ddot{\underset{\sim}{p}}(t) + \underline{\bar{R}}_f \dot{\underset{\sim}{p}}(t) + \underline{\bar{L}}_f \underset{\sim}{p}(t) = \dot{\underset{\sim}{Q}}_e(t)$$

8.3 Element specification

Fluidial elements must be specified, because they depend on the character of the flow and other system properties.

8.3.1 Inertance

The *inertance* can be calculated from Newton's law, which says that force results in acceleration of mass. The fluid is considered to be in a pipe with cross-sectional area A and length l. Its density is ρ and acceleration $\dot{v}$. The pressure difference is Δp.

$$\begin{array}{ll} A\,\Delta p & = \Delta p = \rho l \dot{v} = \dfrac{\rho l}{A}\,\dot{Q} = L_f \dot{Q} \\ (\rho A l)\dot{v} \quad \rightarrow & \end{array}$$

8.3.2 Resistance : laminar flow

For stationary, uniform, laminar flow of an incompressible, viscous fluid through a circular pipe, the Hagen-Poiseuille law applies, which relates the pressure difference to the volume flow. The resistance depends on the viscosity η $[\frac{\mathrm{Ns}}{\mathrm{m}^2}]$ of the fluid, which is assumed to be constant (Newtonian fluid), the length l [m] and the diameter d [m] of the pipe.

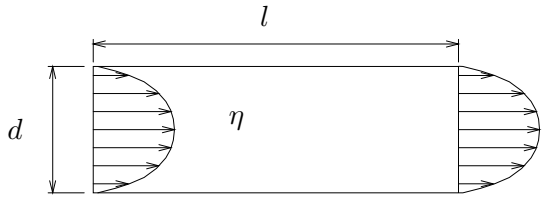


Fig. 8.5 : *Laminar flow.*

$$\Delta p = \frac{128\,\eta l}{\pi d^4}\,Q = R_f\,Q$$

8.3.3 Resistance : turbulent flow

When the Reynolds number $Re = \frac{4\rho Q}{\pi d \eta}$ exceeds a certain limit value (about 5000), the flow in the tube will most likely be turbulent. In that case the relation between Δp and Q is nonlinear and has to be determined experimentally.

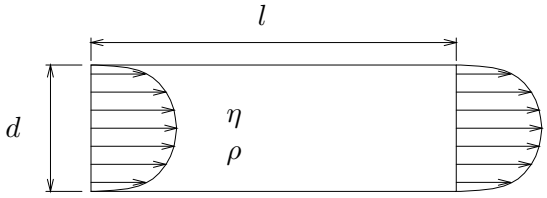


Fig. 8.6 : *Turbulent flow.*

$$\Delta p = a|Q|^{\frac{3}{4}}\,Q = R_f\,Q$$

8.3.4 Resistance : orifice

The pressure drop across an orifice is a quadratic function of the flow through its area A_o. The resistance depends on A_o, the fluid density ρ and a so-called *discharge coefficient* C_d, which is about 0.62 for a sharp-edged orifice.

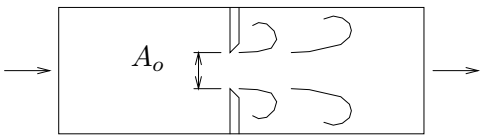


Fig. 8.7 : *Orifice.*

$$\Delta p = \left(\tfrac{1}{2}\rho C_d^2 A_o^2\right) Q^2 = R_f\,Q$$

8.3.5 Capacitance : fluid reservoir

An example of a fluid capacitance is a fluid reservoir : an open tank where fluid is stored. The pressure at the outflow opening depends on the height of the fluid column above the opening : $p = \rho g h$ [N/m^2], where ρ is the fluid density and g is the gravitational acceleration. Assuming that the fluid is incompressible and the reservoir is rigid, we have $Q = A\dot{h}$, where A is the surface area of the fluid.

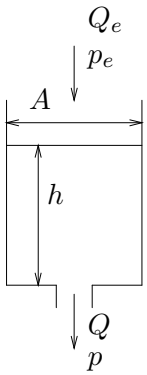


Fig. 8.8 : *Fluid reservoir.*

$$\left.\begin{array}{l} Q = C_f \dot{p} \\ Q = \dot{V} = A\dot{h} \\ p = \rho g h \qquad \rightarrow \qquad \dot{p} = \rho g \dot{h} \end{array}\right\} \quad \rightarrow \quad A\dot{h} = C_f \rho g \dot{h} \quad \rightarrow \quad C_f = \frac{A}{\rho g}$$

8.3.6 Capacitance : pressurized tank

A closed tank with constant volume V is completely filled with a fluid, which is held under a prescribed pressure p. The compressibility or bulk modulus of the fluid is K. Gravity, fluid inertia and frictional effects are neglected. When it is assumed that the fluid is nearly incompressible (very high value for K), the fluid flow can be related to the pressure rate, using the continuity equation $\rho J = \rho_0$, where J is the volume ratio, which is almost 1.

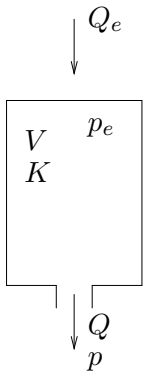


Fig. 8.9 : *Pressurized tank.*

$$\left.\begin{array}{lll} \rho J = \rho_0 & \rightarrow & \dfrac{\dot\rho}{\rho} = -\dfrac{\dot J}{J} = -\dfrac{\dot V}{V} \\ J - 1 = -\dfrac{1}{K}\,p & \rightarrow & \dot J = -\dfrac{1}{K}\,\dot p \end{array}\right\} \quad \rightarrow \quad \frac{\dot\rho}{\rho} = \frac{\dot p}{JK} \approx \frac{\dot p}{K}$$

$$\rho Q = \frac{d}{dt}(\rho V) = V\dot\rho \quad \rightarrow \quad Q = V\,\frac{\dot\rho}{\rho} = \frac{V}{K}\,\dot p = C_f\,\dot p$$

8.4 Special elements

8.4.1 Valve

A *valve* is a fluidial resistance, which has (almost) zero or infinite (= very high) value, dependent of the sign of the pressure difference.

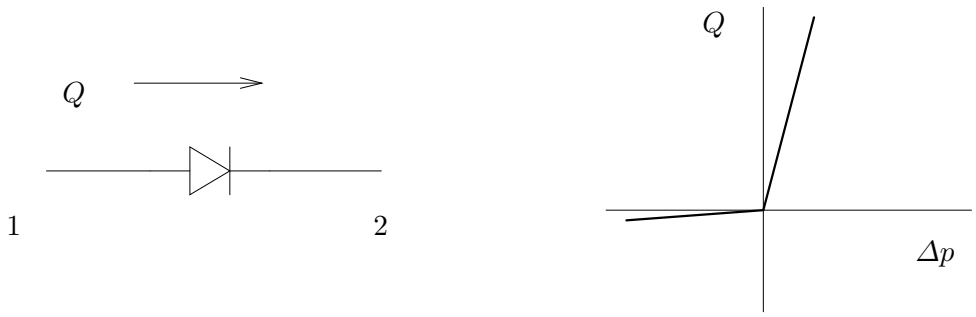


Fig. 8.10 : *A valve with its flow-pressure characteristic.*

8.4.2 Transformer

A *transformer* changes the pressure and reversely the flow without energy loss (in theory). It mostly consists of a connection of two cylinders with different cross-sectional areas. It is modeled as a link between two nodes.

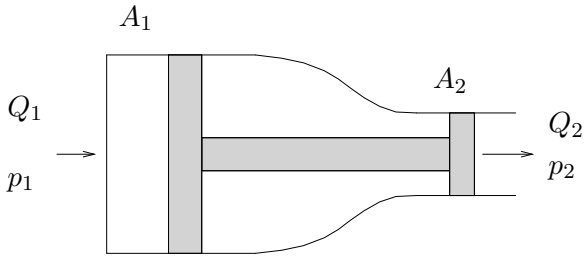


Fig. 8.11 : *Transformer.*

$$p_2 = np_1 \quad ; \quad Q_1 p_1 = Q_2 p_2 \quad \rightarrow \quad Q_2 = \frac{1}{n} Q_1$$
$$n = \frac{A_1}{A_2}$$

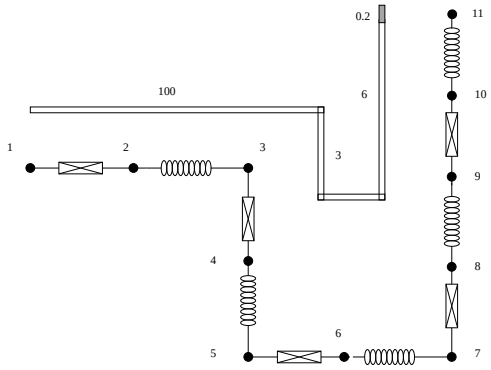
8.5 Examples

8.5.1 Pipe system

A simple pipe system consists of a 100 m long straight pipe and three shorter pipes (3, 2 and 6 m) connected at a 90 degree angle. The pipes have equal inner diameter of 1 cm and are assumed to be rigid. At the end a nozzle is fitted which has a diameter of 1 mm. Laminar pipe flow is assumed.

At time $t = 0$ the inlet pressure is increased stepwise to 1 bar (10^5 Pa). The pressures at points 3, 6 and 10 are plotted as a function of time. A section of the plot is enlarged to show the pressure difference, due to friction loss.

The outflow from the nozzle is also plotted as a function of time. The steady state flow is 1.3×10^{-5} m^3/s = 0.013 l/s = 0.78 l/min. When the nozzle has the same diameter as the pipe, the steady state flow is 2.5×10^{-4} m^3/s = 0.25 l/s = 15 l/min, and is reached after 20 seconds.



$\rho = 1000$ kg/m^3
$\eta = 0.00089$ Pa.s
$D_p = 0.01$ m
$D_n = 0.001$ m

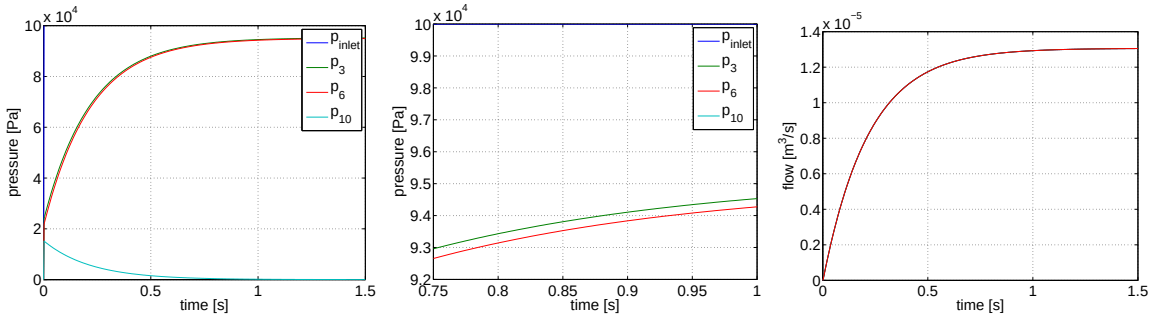


Fig. 8.12 : *Pressures and flow in a pipe system.*

8.5.2 Extruder

A picture and a schematic view of an extruder is shown in the figure. The extruder can be considered as a pump, which drives molten material through a die into a mold.

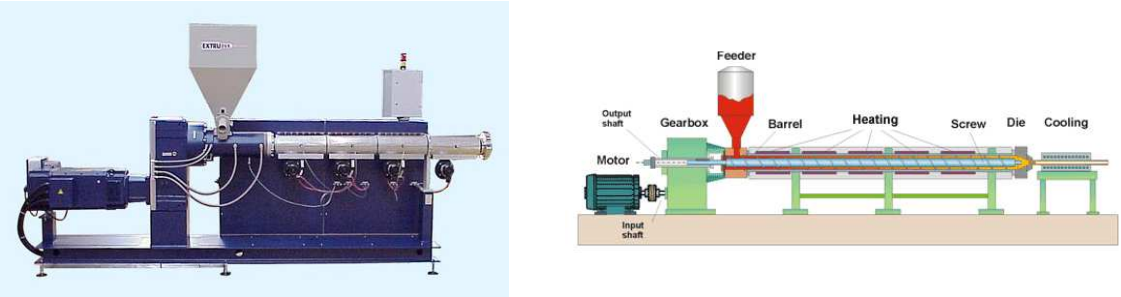


Fig. 8.13 : *Extruder.*

When the flow of the material meets no resistance from die or mould, the flow is maximum, Q_{max}, and depends only of characteristic dimensions (diameter D, channel depth d_c, length L, helix angle β) and rotation rate (N r/min) of the extruder. When the outflow is completely prohibited, the pressure is maximum : p_{max}. This maximum pressure can be calculated again from characteristic extruder dimensions, rotation rate and the viscosity η of the molten material. It is assumed that the flow-pressure characteristic is linear.

The discrete model of the extruder is an arrangement of two fluid resistances R_{pa} and R_{pb}. The value of R_{pa} equals the ratio of p_{max} and Q_{max}. The value of R_{pb} is much smaller then R_{pa}. When we take $R_{pb} = \alpha R_{pa}$ the value of α must be very small, typically 10^{-3}.

The maximum flow Q^*_{max} is slightly smaller than the theoretically calculated value.

$$Q_{max} = \tfrac{1}{2}\pi^2 D^2 d_c N \sin(\beta)\sin(\beta) \quad [\mathrm{m^2 s^{-1}}] \quad ; \quad p_{max} = \frac{6\pi D L N \eta}{d_c^2}\cot(\beta) \quad [\mathrm{Pa}]$$

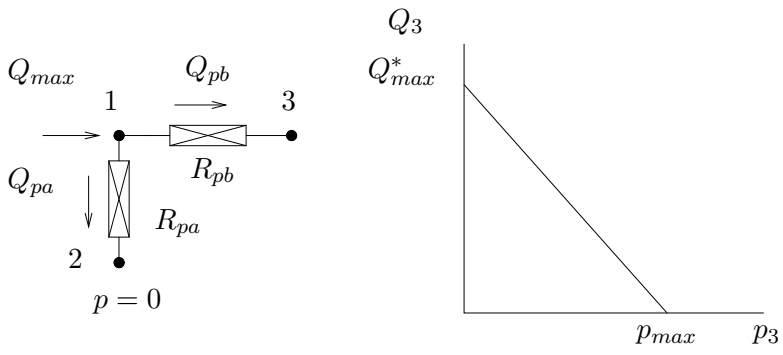


Fig. 8.14 : *Discrete model of the extruder.*

$$Q_{pb} = Q_{max} - Q_{pa} = Q_{max} - \frac{p_1}{R_{pa}} \quad \rightarrow \quad 0 = Q_{max} - \frac{p_{max}}{R_{pa}} \rightarrow \boxed{R_{pa} = \frac{p_{max}}{Q_{max}}}$$

$$\left.\begin{aligned} Q_3 = Q_{pb} &= Q_{max} - \frac{Q_{max}}{p_{max}}\, p_1 \\ Q_3 = Q_{pb} &= \frac{1}{R_{pb}}\,(p_1 - p_3) \quad \rightarrow \quad p_1 = R_{pb} Q_3 + p_3 \end{aligned}\right\} \quad \rightarrow$$

$$\left(1 + \frac{Q_{max}}{p_{max}}\, R_{pb}\right) Q_3 = Q_{max} - \frac{Q_{max}}{p_{max}}\, p_3 \quad \rightarrow$$

$$\left(1 + \frac{R_{pb}}{R_{pa}}\right) Q_3 = Q_{max} - \frac{1}{R_{pa}}\, p_3$$

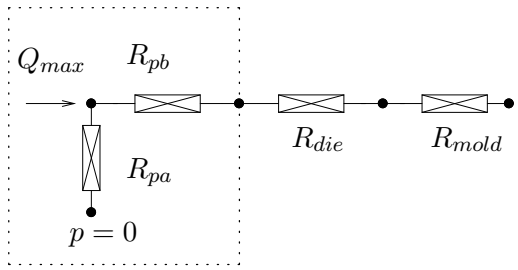
$$R_{pb} = \alpha R_{pa} \quad \rightarrow \quad Q^*_{max} = \frac{Q_{max}}{1+\alpha}$$

The resistance of die and mold depend of geometry parameters and the viscosity of the material. For circular dimensions they are :

$$R_{die} = \frac{128 \eta L_d}{\pi D_d^2} \qquad ; \qquad R_{mould} = \frac{128 \eta L_m}{\pi D_m^2}$$

where D and L indicate diameter and length.

The pressure at the die inlet is increased as a function of time. The flow into the die is plotted as a function of time. Also the flow is plotted as a function of the pressure.



$$R_{pa} = \frac{p_{max}}{Q_{max}}$$
$$R_{pb} = R_{pa}/1000$$

Fig. 8.15 : *Discrete model of the extruder, the die and the mold.*

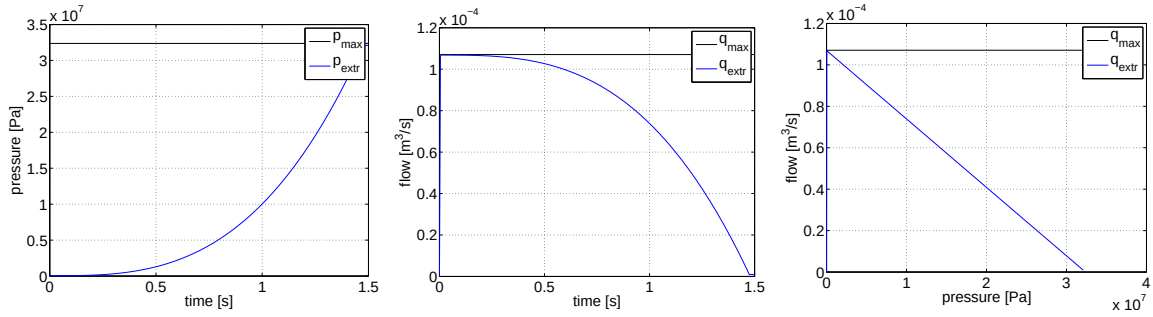
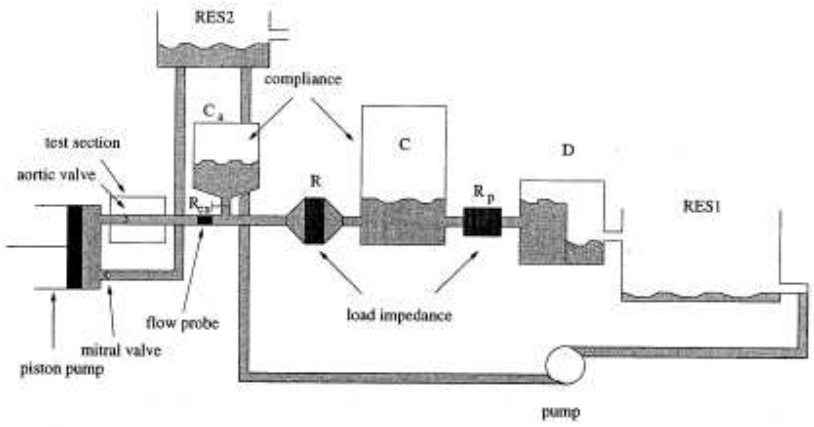


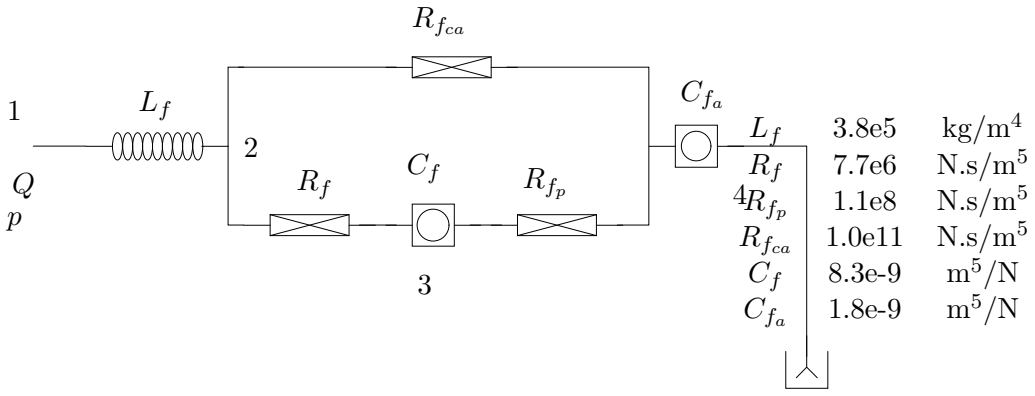
Fig. 8.16 : *Pressures and flow.*

8.5.3 Pulse duplicator system

About 15 years ago, artificial heart valves were designed, made and tested in the bio-laboratory of the TUE. Testing was done in a so-called pulse duplicator system. A pulsating flow was generated by a piston pump and in the channel behind the pump the valve was placed and could be observed opening and closing. Resistances were placed in the system to act as loads. Also storage tanks were used. A rotating pump transported the fluid back to the inlet storage tank.

The whole system can be modeled with resistances, inertances and capacitors.





L_f	3.8e5	kg/m^4
R_f	7.7e6	N.s/m^5
R_{f_p}	1.1e8	N.s/m^5
$R_{f_{ca}}$	1.0e11	N.s/m^5
C_f	8.3e-9	m^5/N
C_{f_a}	1.8e-9	m^5/N

Fig. 8.17 : *Pulse duplicator system and its electrical model with model parameter values.*

The system is first loaded first with a sinusoidal flow with amplitude of 50 [l/min] at the entrance. The required pressure [Pa] is calculated.

Next, the system is loaded with a sinusoidal pressure with amplitude 15000 [Pa] at the entrance and the resulting flow [l/min] is calculated.

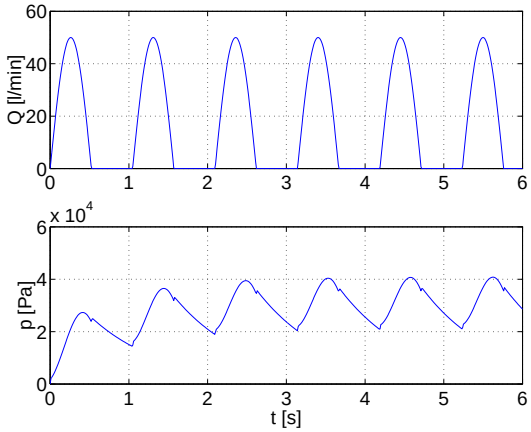


Fig. 8.18 : *Response of flow and pressure to an harmonic flow at the entrance.*

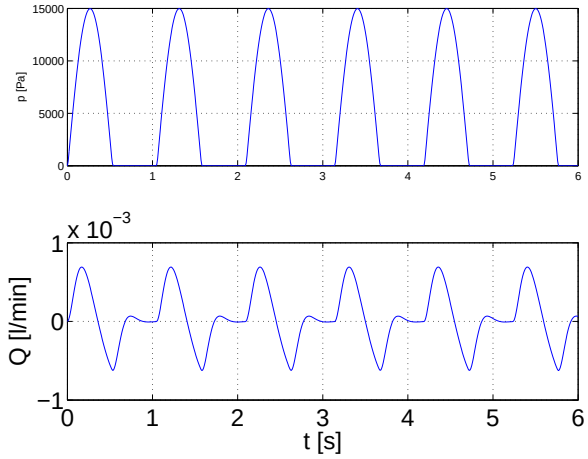


Fig. 8.19 : *Response of pressure and flow to an harmonic pressure at the entrance.*

8.5.4 Haert and circulation model

A circulation model is loaded by the haert. The haert model first updates the left ventricle blood volume V_{lv}, the arterial blood volume V_a and the venous blood volume V_v. Subsequently the left ventricle pressure p_{lv} is calculated. The arterial pressure p_a and the venous pressure p_v are calculated using the arterial and venous capacities.

$$\frac{dV_{lv}}{dt} = q_m(t) - q_a(t) \quad ; \quad \frac{dV_a}{dt} = q_a(t) - q_p(t)$$

$$V_{lv} = V_{lv}(t) + \frac{dV_{lv}}{dt}\,\Delta t \quad ; \quad V_a = V_a(t) + \frac{dV_a}{dt}\,\Delta t \quad ; \quad ; \quad V_v = V_{tot} - (V_{lv} + V_a)$$

$$\Delta t < T_{act} \quad \rightarrow \quad p_{lv} = \left(E_{pas} + (E_{max} - E_{pas})\sin^2(\frac{\pi \Delta t}{T_{act}})\right)(V_{lv} - V_0)$$

$$\Delta t \geq T_{act} \quad \rightarrow \quad p_{lv} = E_{pas}\,(V_{lv} - V_0)$$

$$p_a = \frac{1}{C_a}\,(V_a - V_{a0}) \quad ; \quad p_v = \frac{1}{C_v}\,(V_v - V_{v0})$$

The pressures p_a, p_v and p_{lv} are applied as loads on the circulation system.

The model was part of a more complicated mother/placenta/fetus model, made and studied by Beatrijs van de Hout for her MSc-project (2007).

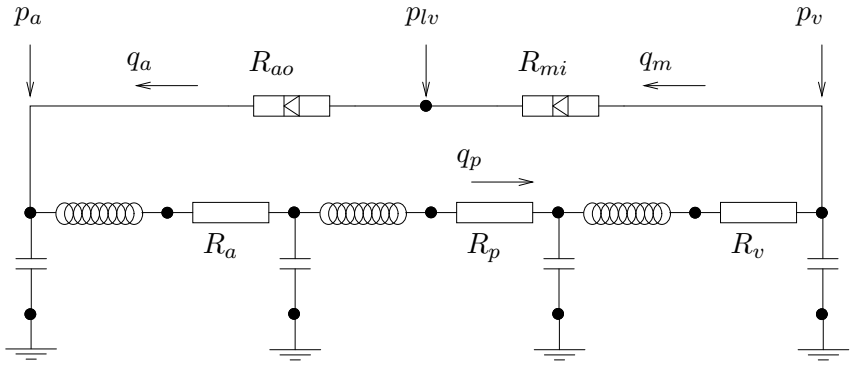


Fig. 8.20 : *Electrical model of the haert circulation.*

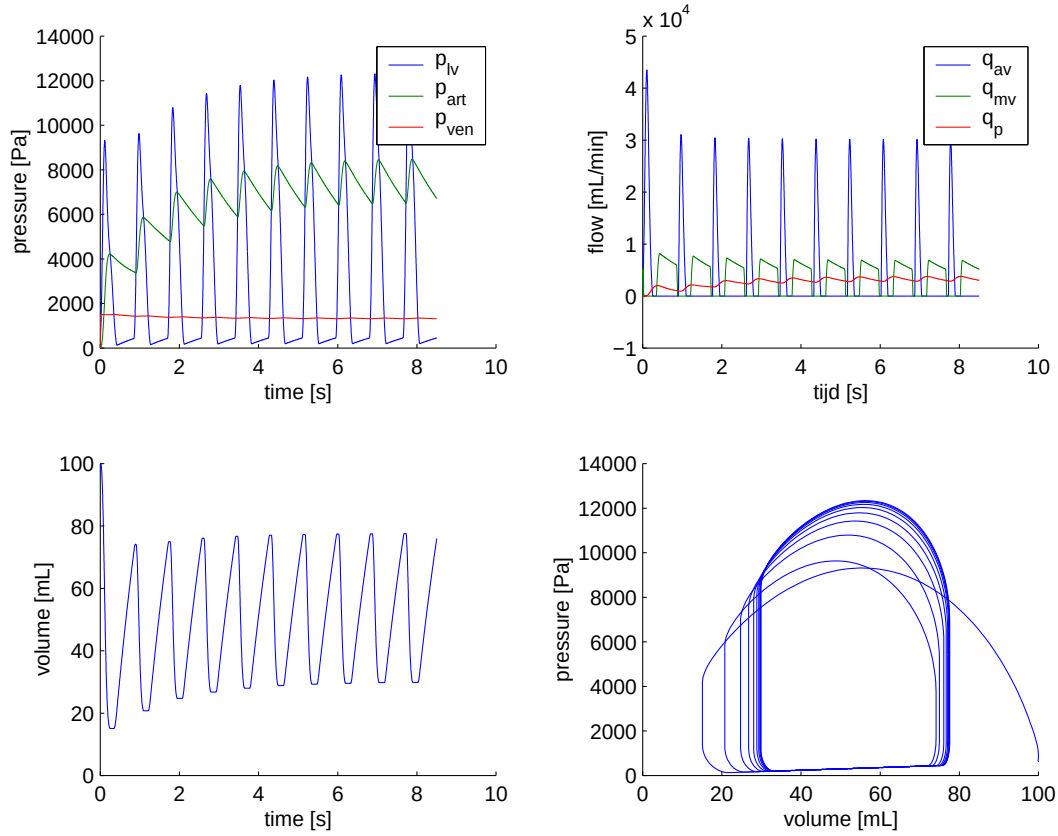


Fig. 8.21 : *Response of the haert circulation model.*

8.5.5 Circulation model

A large electrical model for the human circulation contains various inductance-resistance-capacitor elements for different vessels. Also end-impedances are used to replace not-modeled parts.

The model was used to study the consequences of the application of bypasses and grafts. It was the MSc-project of Boudewijn Sleutjes (2007).

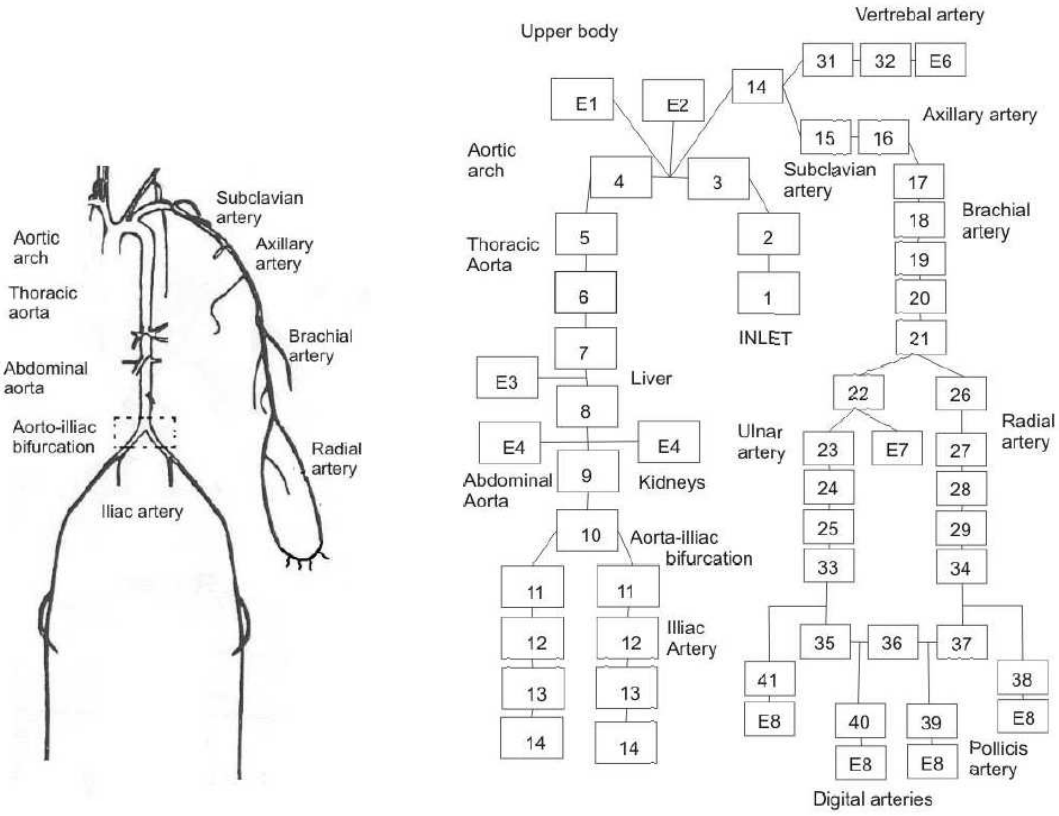


Fig. 8.22 : *Model for the body circulation.*

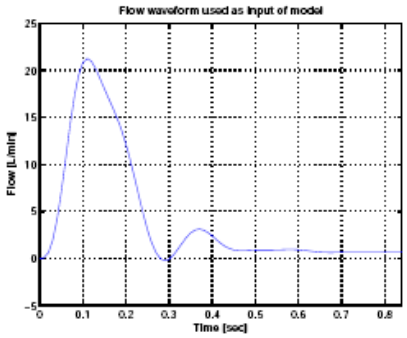


Fig. 8.23 : *Excitation of the model with heart pressure.*

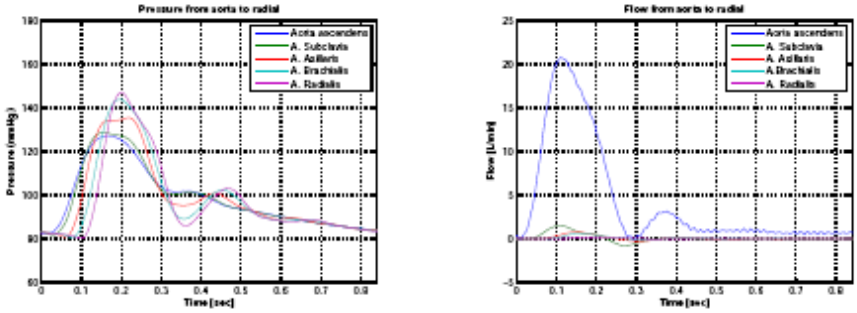


Fig. 8.24 : *Response of the model.*

Chapter 9

Thermal systems

The thermal flow variable is the *heat flow* q [J/s = W]. The point variable is the *temperature* T. Heat flow is *positive* when flowing *into* an element. The two thermal elements are the resistance R_t and the capacitance C_t. At ordinary temperatures there is no thermal inductance.

9.1 Thermal elements

For thermal systems there is no inertance. Also there is no transformer, because heat, a form of energy, cannot be transformed to a higher or lower value without energy addition or loss.

9.1.1 Resistance

A *thermal resistance* relates the heat flow to the temperature difference.

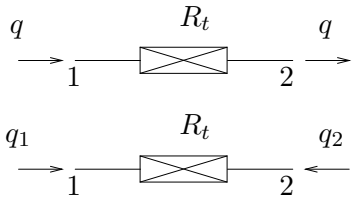


Fig. 9.1 : *Thermal resistance with element and nodal heat flow.*

$$q = \frac{1}{R_t}\Delta T = \frac{1}{R_t}(T_1 - T_2) \quad \rightarrow$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{R_t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{q}(t) = \underline{\bar{R}}_t \underset{\sim}{T}(t)$$

9.1.2 Capacitance

A *capacitance* is a nodal variable, which relates the heat flow to the time derivative of the temperature difference. The capacitance of a material to store thermal energy is determined by its specific heat C_p [J/kg.^{o}C]. A certain mass m [kg] of this material has a thermal capacitance $C_t = C_p\, m$ [J/^{o}C].

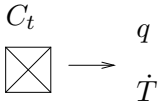


Fig. 9.2 : *Thermal capacitance.*

$$q(t) = C_t \dot{T}(t)$$

9.2 System equations

The behavior of thermal systems is described by a set of differential equations resulting from an assembling process which implies that the elements are connected. This procedure follows the next rules :

1 In every node the difference of the *incoming* external flow q_e and the total *outgoing* flow to the connected elements, must equal the product of capacitance and the derivative of the temperature change in that node.

2 In every node the temperatures of the connected element nodes must be equal.

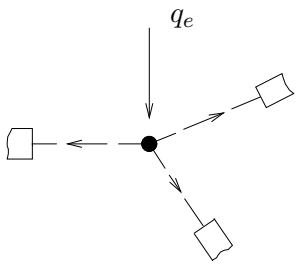


Fig. 9.3 : *Node with three attached elements.*

Assembling results in a set of first order differential equations. Because of the nodal character of the capacitances, the matrix $\underline{C_t}$ is a diagonal matrix. We can rewrite the system and differentiate it to get a second order system. This system can be solved by numerical integration.

$$\underset{\sim}{q}_e(t) - \underline{\bar{R}}_t \underset{\sim}{T}(t) = \underline{C}_t \dot{\underset{\sim}{T}}(t) \quad \rightarrow \quad \underline{C}_t \dot{\underset{\sim}{T}}(t) + \underline{\bar{R}}_t \underset{\sim}{T}(t) = \underset{\sim}{q}_e(t) \quad \Rightarrow$$
$$\underline{C}_t \ddot{\underset{\sim}{T}}(t) + \underline{\bar{R}}_t \dot{\underset{\sim}{T}}(t) = \dot{\underset{\sim}{q}}_e(t)$$

9.3 Element specification

9.3.1 Capacitance

The thermal capacitance of a material is the product of its volume, its density and its specific heat capacity at constant pressure.

$$C_t = C_p \rho V$$

9.3.2 Resistance : conduction

Heat flow through a material is referred to as *conduction*. It is related to the temperature difference by the thermal conductivity.

Conduction is the heat flow by the transfer of kinetic energy from atom to atom. This is described by *Fourier's law*. The conductive heat flow is a linear function of the temperature gradient. It is proportional to the area A [m^2] normal to the flow direction and the thermal conductivity λ [J/s.m.^{o}C]. It is inversely proportional to the length l [m] in flow direction.

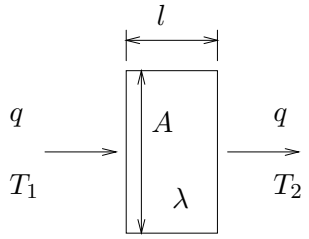


Fig. 9.4 : *Conductive resistance.*

$$q = \frac{\lambda A}{l} (T_1 - T_2) = \frac{1}{R_t} (T_1 - T_2) \quad \rightarrow$$
$$R_t = \frac{l}{\lambda A} \qquad [\ ^o\text{C.s/J}]$$

9.3.3 Resistance : convection

Heat transfer from one medium to another is referred to as *convection*. The convective heat flow is a linear function of the temperature difference. It is proportional to the area A [m^2] normal to the direction of heat flow and to the *film coefficient of heat transfer* α [J/^{o}C.s.m^2].

The resistance is described by the convection or film coefficient. Its value is not easy to determine as it depends on the flow of the fluid. It is often given as a function of dimensionless variables, e.g. the Nusselt, Prandtl or Reynolds number.

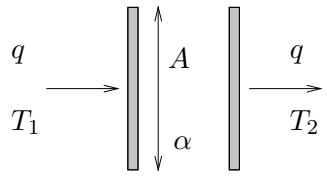


Fig. 9.5 : *Convective resistance.*

$$q = \alpha A \left(T_1 - T_2\right) = \frac{1}{R_t} \left(T_1 - T_2\right)$$

$$R_t = \frac{1}{\alpha A} \qquad [\ ^o\text{C.s/J}]$$

9.3.4 Resistance : radiation

Heat transfer by radiation is described by the *Stefan-Boltzmann law*. The heat flow between two points is proportional to the fourth power of the absolute temperature difference. The proportionality is expressed by a constant β, which depends on the shape, area and surface characteristics of the bodies emanating and receiving radiation, and on the wavelength of the radiation. Its unit is [J/s.^{o}K].

Obviously, the radiation heat resistance is nonlinear.

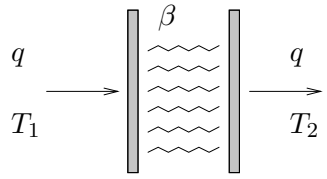


Fig. 9.6 : *Radiation resistance.*

$$q = \beta \left(T_1^4 - T_2^4\right)$$

9.4 Examples

9.4.1 Double glass window pane

Isolating double glass windows are made of two glass plates with an intermediate space, filled with air or another gas. There are various thicknesses of glass panes and inter-spacing distances. All variables result in a certain degree of isolation to prevent energy loss (cooling) or accumulation (heating) of the inner space.

Geometric and physical data are listed in the table. The model thermal system has three resistances – two for conduction and one for convection – and two capacities. Initially all materials have uniform temperature of 0^o C. At time $t = 0$ the outside temperature is decreased to -10^o C and the inside temperature is increased to 27^o C. Taking into account thermal conductivity and thermal capacitance, the heat flow and the temperature is calculated as a function of time. In the steady state situation there is a heat flow of about 4 W from inside to outside.

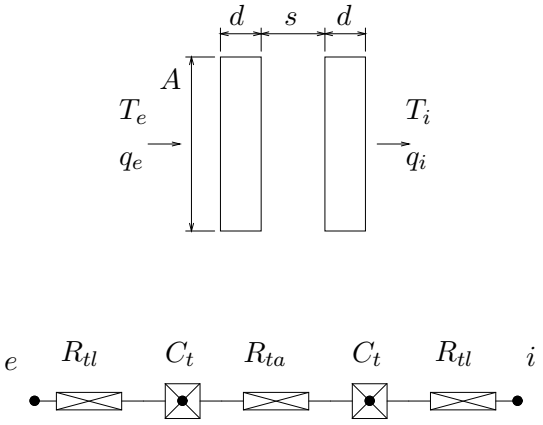


Fig. 9.7 : *Double glass window pane and discrete model with parameter values.*

A	d	s	λ	ρ	C_p	α
m^2	m	m	W/mK	kg/m^3	J/kgK	W/m^2K
1	0.005	0.015	0.96	2500	840	0.1

$$R_{tl} = \frac{d}{\lambda A} \quad ; \quad R_{ta} = \frac{1}{\alpha A} \quad ; \quad C_t = C_p A d \rho \quad ; \quad T_e = -10^o \quad ; \quad T_i = 27^o$$

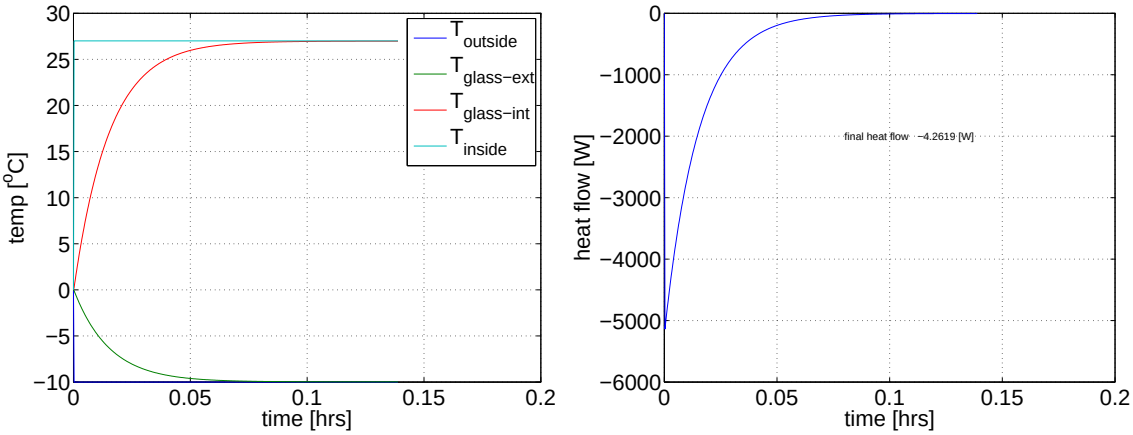


Fig. 9.8 : *Temperatures and heat flow as a function of time.*

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