Course on Numerical Methods in Heat Transfer and Fluid Dynamics

Numerical resolution of the generic convection-diffusion equation

Elliptic equations (v1.2c)

Escola Superior d'Enginyeries Industrial, Aeronàutica i Audiovisual de Terrassa (ESEIAAT)

Centre Tecnològic de Transferència de Calor (CTTC)

Universitat Politècnica de Catalunya (UPC)

Contents

Introduction (1/3)

- *Navier-Stokes equations* for perfect gases ($c_v = const$) can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = \nabla \cdot (\mu \nabla \vec{v}) + \{\nabla \cdot (\vec{\tau} - \mu \nabla \vec{v}) - \nabla p + \rho \vec{g}\}$$

$$\frac{\partial (\rho T)}{\partial t} + \nabla \cdot (\rho \vec{v} T) = \nabla \cdot \left(\frac{\lambda}{c_v} \nabla T\right) + \left\{\frac{-\nabla \cdot \vec{\dot{q}}^R - p \nabla \cdot \vec{v} + \vec{\tau} : \nabla \vec{v}}{c_v}\right\}$$

$$\frac{\partial (\rho Y_k)}{\partial t} + \nabla \cdot (\rho \vec{v} Y_k) = \nabla \cdot (\rho D_{km} \nabla Y_k) + \{\dot{\omega}_k\}$$

- As can be seen, all these transport equations (mass, momentum, energy, species *k*), and other transport equations (kinetic energy, entropy, etc.) have a common structure composed by unsteady terms, convective terms, diffusion terms and other terms.

Introduction (2/3)

- Be ϕ a generic variable (e.g. velocity, temperature, mass fractions of species, entropy, etc.). The generic convection-diffusion transport equation can then be written as:

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla \cdot (\rho\vec{v}\phi) = \nabla \cdot \left(\Gamma_\phi \nabla\phi\right) + \dot{s}_\phi$$

  where Γ_ϕ the diffusion coefficient and $\dot{s}_\phi$ the extra source/sink terms

- Using the mass conservation equation, $\partial\rho/\partial t + \nabla \cdot (\rho\vec{v}) = 0$, the previous generic convection-diffusion equation (conv-diff eq) can also be written equivalent convection-diffussion equation:

$$\rho\frac{\partial\phi}{\partial t} + \rho\vec{v} \cdot \nabla\phi = \nabla \cdot \left(\Gamma_\phi \nabla\phi\right) + \dot{s}_\phi$$

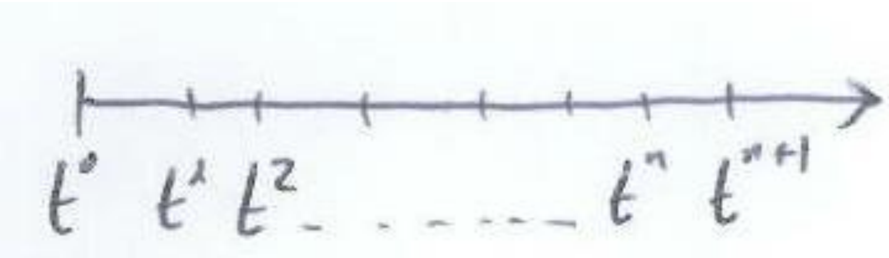
Introduction (3/3)

- Therefore, the above mentioned Navier-Stokes equations for perfect gases cast in the generic convection-diffusion form:

Equation	ϕ	Γ_ϕ	$\dot{s}_\phi$
Mass	1	0	0
Momentm	$\vec{v}$	μ	$\nabla \cdot (\vec{\tau} - \mu \nabla \vec{v}) - \nabla p + \rho \vec{g}$
Energy	T	$\frac{\lambda}{c_v}$	$\frac{1}{c_v}\left(-\nabla \cdot \dot{\vec{q}}^R - p\nabla \cdot \vec{v} + \vec{\tau} : \nabla \vec{v}\right)$
Species-k	Y_k	ρD_{km}	$\dot{\omega}_k$

Numerical method and discretization

- Finite volume approach
- The domain is completely discretized into non-overlapping finite volumes (CVs). A node is located at each one of the CVs (cell-centered nodes)
- Discretization meshes can be structured (not necessarily ortho-gonal) or unstructured
- Time discretization:



- Integration unsteady terms: implicit, explicit, Crank-Nicolson,...

Discretization of the continuity equation (1/3)

- The continuity (or mass conservation) equation can be written in integral form as:

$$\frac{\partial}{\partial t}\int_{V_P}\rho dV+\int_{S_f}\rho\vec{v}\cdot\vec{n}dS=0$$

- Consider the CV-P in a 2D Cartesian mesh. The previous equation in semi-discretized form is

$$V_P\frac{\partial\bar{\rho}_P}{\partial t}+\dot{m}_e-\dot{m}_w+\dot{m}_n-\dot{m}_s=0$$

where $\bar{\rho}_P=\frac{1}{V_P}\int_{V_P}\rho dV$ and, for example, $\dot{m}_e=\int_{S_e}\rho\vec{v}\cdot\vec{n}dS$ and $\dot{m}_w=-\int_{S_w}\rho\vec{v}\cdot\vec{n}dS$ ($\dot{m}_f$ is positive in the positive coordinate direction)

Discretization of the continuity equation (2/3)

- Integrating between the instants t^n and t^{n+1} and using a second-order approach for the volume integral ($\bar{\rho}_P \approx \rho_P$)

$$V_P \int_{t^n}^{t^{n+1}} \frac{\partial \rho_P}{\partial t} dt + \int_{t^n}^{t^{n+1}} (\dot{m}_e - \dot{m}_w + \dot{m}_n - \dot{m}_s)\, dt = 0$$

- An explicit discretization is written as:

$$(\rho_P^{n+1} - \rho_P^n) V_P + (\dot{m}_e^n - \dot{m}_w^n + \dot{m}_n^n - \dot{m}_s^n)\Delta t = 0$$

- While an implicit discretization has the form

$$(\rho_P^{n+1} - \rho_P^n) V_P + (\dot{m}_e^{n+1} - \dot{m}_w^{n+1} + \dot{m}_n^{n+1} - \dot{m}_s^{n+1})\Delta t = 0$$

Discretization of the continuity equation (3/3)

- From now on we are going to use the implicit discretization.
- For convenience, superindex n, corresponding to the previous time step, is substituted by o, while superindex $n+1$ corresponding to the current time step is simply dropped.
- Then, the already written implicit form of mass conservation

$$\frac{\rho_P^{n+1}-\rho_P^n}{\Delta t}V_P+\dot{m}_e^{n+1}-\dot{m}_w^{n+1}+\dot{m}_n^{n+1}-\dot{m}_s^{n+1}=0$$

now is written in the following more convenient form:

$$\frac{\rho_P-\rho_P^o}{\Delta t}V_P+\dot{m}_e-\dot{m}_w+\dot{m}_n-\dot{m}_s=0$$

- Extension to 3D flows is straightforward:

$$\frac{\rho_P-\rho_P^o}{\Delta t}V_P+\dot{m}_e-\dot{m}_w+\dot{m}_n-\dot{m}_s+\dot{m}_t-\dot{m}_b=0$$

Discretization of the generic conv-diff eq (1/2)

- Numerical implicit approximation of the different terms:

$$\int_{t^n}^{t^{n+1}} \int_{V_P} \frac{\partial(\rho\phi)}{\partial t} dV dt \approx V_P \int_{t^n}^{t^{n+1}} \frac{\partial(\rho_P \phi_P)}{\partial t} dt = V_P(\rho_P \phi_P - \rho_P^o \phi_P^o)$$

$$\int_{t^n}^{t^{n+1}} \int_{V_P} \nabla \cdot (\rho \vec{v} \phi) dV dt = \int_{t^n}^{t^{n+1}} \int_{S_f} \rho \vec{v} \phi \cdot \vec{n} dS dt \approx (\dot{m}_e \phi_e - \dot{m}_w \phi_w + \dot{m}_n \phi_n - \dot{m}_s \phi_s)\Delta t$$

$$\int_{t^n}^{t^{n+1}} \int_{V_P} \nabla \cdot (\Gamma_\phi \nabla \phi) dV dt = \int_{t^n}^{t^{n+1}} \int_{S_f} \Gamma_\phi \nabla \phi \cdot \vec{n} dS dt$$

$$\approx \left(\Gamma_e \frac{\phi_E - \phi_P}{d_{PE}} S_e - \Gamma_w \frac{\phi_P - \phi_W}{d_{PW}} S_w + \Gamma_n \frac{\phi_N - \phi_P}{d_{PN}} S_n - \Gamma_s \frac{\phi_P - \phi_S}{d_{PS}} S_s \right) \Delta t$$

$$\int_{t^n}^{t^{n+1}} \int_{V_P} \dot{s}_\phi dV dt \approx \bar{\dot{s}}_{\phi P} V_P \Delta t = \left(S_C^\phi + S_P^\phi \phi_P \right) V_P \Delta t$$

Notes: i) Volume (V_P) and time (t^n to t^{n+1}) integration of each term; ii) Γ_ϕ at face f is simply written as Γ_f, instead of $\Gamma_{\phi f}$; iii) source term is linearized (for numerical reasons)

Discretization of the generic conv-diff eq (2/2)

- Introducing all these terms into the conv-diff eq:

$$\frac{\rho_P\phi_P - \rho_P^o\phi_P^o}{\Delta t}V_P + \dot{m}_e\phi_e - \dot{m}_w\phi_w + \dot{m}_n\phi_n - \dot{m}_s\phi_s$$
$$= \mathrm{D_e}(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) + D_n(\phi_N - \phi_P) - D_s(\phi_P - \phi_S) + \left(S_C^\phi + S_P^\phi\,\phi_P\right)V_P$$

where, $\mathrm{D_e} = \Gamma_e S_e/\mathrm{d_{PE}}$, $\mathrm{D_w} = \Gamma_w S_w/\mathrm{d}_{\mathrm{P}W}$, etc.

- An equivalent form of the above equation can be found using the discretized mass conservation equation*:

$$\rho_P^o\frac{\phi_P - \phi_P^o}{\Delta t}V_P + \dot{m}_e(\phi_e - \phi_P) - \dot{m}_w(\phi_w - \phi_P) + \dot{m}_n(\phi_n - \phi_P) - \dot{m}_s(\phi_s - \phi_P)$$
$$= \mathrm{D_e}(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) + D_n(\phi_N - \phi_P) - D_s(\phi_P - \phi_S) + \left(S_C^\phi + S_P^\phi\,\phi_P\right)V_P$$

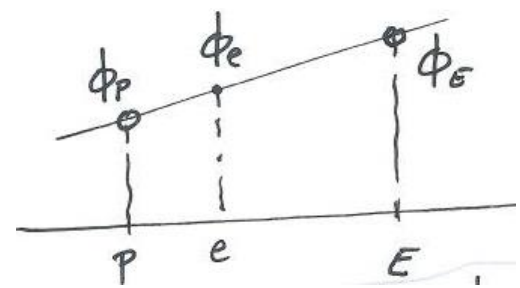
- This implicit form of the conv-diff eq is second-order accurate in the diffusion and source terms. The main question now is how convective contribution is expressed in terms of nodal values

(*) Previous conv-diff eq. minus ϕ_P times the mass conservation equation ($\frac{\rho_P - \rho_P^o}{\Delta t}V_P + \dot{m}_e - \dot{m}_w + \dot{m}_n - \dot{m}_s = 0$)

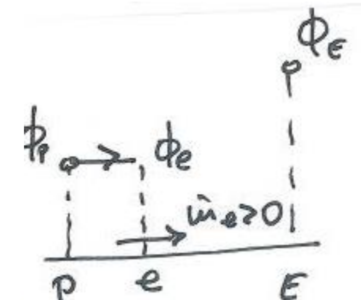
Evaluation of the convective terms (1/3)

- The simplest method is to assume a linear ϕ distribution. This is a central-difference scheme (CDS). E.g. for the east face:

$$\phi_e - \phi_P = f_e(\phi_E - \phi_P), \text{ where } f_e = d_{Pe}/d_{PE}$$



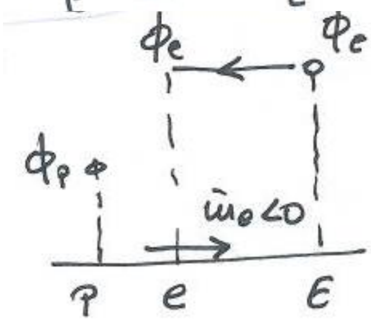
- For incompressible flows, or gases at low Mach, convective terms are more influenced by upstream than downstream conditions. This behaviour leads to the upwind-difference scheme (UDS):



$$\phi_e - \phi_P = f_e(\phi_E - \phi_P)$$

but now, $f_e = 0$ if $\dot m_e > 0$, and $f_e = 1$ if $\dot m_e < 0$.

Alternatively, we can write:



$$\dot m_e(\phi_e^{UDS} - \phi_P) = \frac{\dot m_e - |\dot m_e|}{2}(\phi_E - \phi_P)$$

Evaluation of the convective terms (2/3)

- CDS is prone to stability problems but is second-order accurate. UDS is much more stable but is first-order accurate (too diffusive). Many convective schemes have been proposed in the technical literature.
- For instance, the bases of the exponential-difference scheme (EDS) is to assume a ϕ distribution between nodal points based on a simplified form of the conv-diff eq., i.e. steady 2D without source term: $\frac{d}{dx}(\rho v_x \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)$.

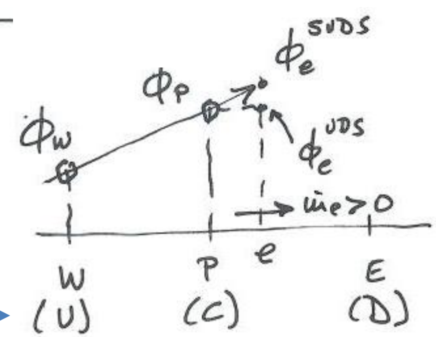
  Considering ρ, v_x and Γ constants between nodal values and equal to the ones at the face, the equation can be easily integrated. E.g. for the east face: $\phi_e - \phi_P = f_e(\phi_E - \phi_P)$, where

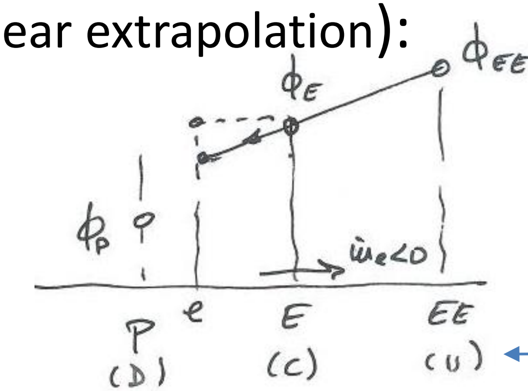
$$f_e = \frac{e^{Pe d_{Pe}/d_{PE}} - 1}{e^{Pe} - 1}, \qquad Pe = \frac{\rho_e v_{xe} d_{PE}}{\Gamma_e}$$

Evaluation of the convective terms (3/3)

- EDS is still first-order accurate. More accurate schemes are the second-order upwind scheme (SUDS) or the second/third-order QUICK scheme.
- SUDS (second-order upwind linear extrapolation):

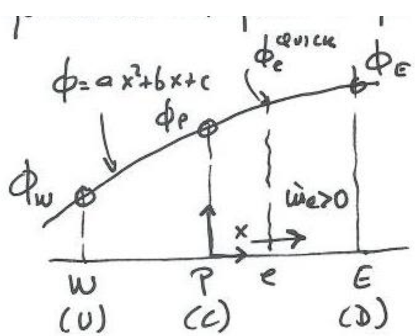
U – C – D

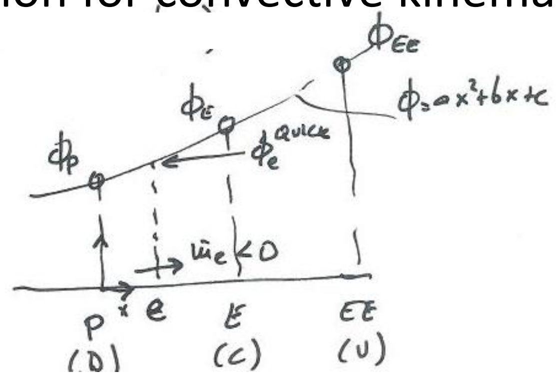




D – C – U

- QUICK (quadratic upwind interpolation for convective kinematics):



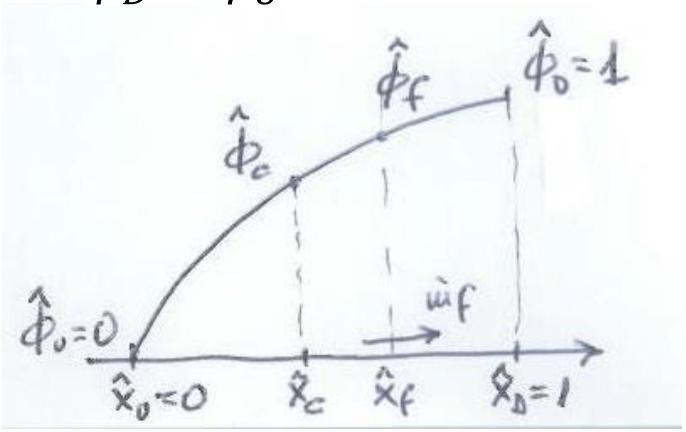


Normalization variables (1/3)

- Observed in the previous slide how the nodal points are identified according to the mass flow direction. Located at the face position, D refers to the downstream node, C is the first upstream node and U is the most upstream node.
- Once these three nodes are identified (D, C and U), the dependent variable and position are normalized in this way:

$$\hat{x} = \frac{x - x_U}{x_D - x_U}, \qquad \hat{\phi} = \frac{\phi - \phi_U}{\phi_D - \phi_U}$$

- Of course, $\left(\hat{x}_U = 0, \hat{\phi}_U = 0\right)$ and $\left(\hat{x}_D = 1, \hat{\phi}_D = 1\right)$. The figure on the right shows a normalized variables profile.



Normalization variables (2/3)

- In terms of the normalized variables, previous schemes can be written in an equivalent way as

Scheme	Face value
CDS	$\hat{\phi}_f = \frac{\hat{x}_f - \hat{x}_C}{1 - \hat{x}_C} + \frac{\hat{x}_f - 1}{\hat{x}_C - 1}\hat{\phi}_C$
UDS	$\hat{\phi}_f = \hat{\phi}_C$
SUDS	$\hat{\phi}_f = \frac{\hat{x}_f}{\hat{x}_C}\hat{\phi}_C$
QUICK	$\hat{\phi}_f = \hat{x}_f + \frac{\hat{x}_f(\hat{x}_f - 1)}{\hat{x}_C(\hat{x}_C - 1)}(\hat{\phi}_C - \hat{x}_C)$

- Many other schemes have been proposed in the technical literature (see paper [dar99]).

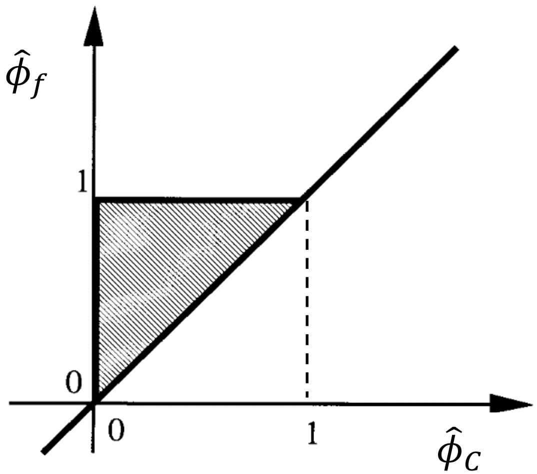
Normalization variables (3/3)

How to implement a subroutine for the evaluation of the dependent variable at any of the faces. E.g. east face *e*:

1. Input data: $\dot{m}_e, x_e,\quad x_P, \phi_P,\quad x_E, \phi_E,\quad x_W, \phi_W,\quad x_{EE}, \phi_{EE}$ and the selected convective scheme (UDS, CDS, SUDS, QUICK, etc.)
2. If $\dot{m}_e > 0 \rightarrow x_D = x_E, \phi_D = \phi_E, x_C = x_P, \phi_C = \phi_P, x_U = x_W, \phi_U = \phi_W$. If $\dot{m}_e < 0 \rightarrow x_D = x_P, \phi_D = \phi_P, x_C = x_E, \phi_C = \phi_E, x_U = x_{EE}, \phi_W = \phi_{EE}$
3. Normalization: $\hat{\phi}_C = \frac{\phi_C - \phi_U}{\phi_D - \phi_U},\quad \hat{x}_C = \frac{x_C - x_U}{x_D - x_U},\quad \hat{x}_e = \frac{x_e - x_U}{x_D - x_U}$
4. Evaluate $\hat{\phi}_e$ according to the selected scheme
5. Dimensional value: $\hat{\phi}_e = \frac{\phi_e - \phi_U}{\phi_D - \phi_U} \rightarrow \phi_e = \phi_U + (\phi_D - \phi_U)\hat{\phi}_e$
6. Return ϕ_e

High-order bounded convection schemes

- Conditions for stability and accuracy are formulated in [gas88]:
  - $\hat{\phi}_f$ must continuous
  - If $\hat{\phi}_C = 0 \rightarrow \hat{\phi}_f = 0$
  - If $\hat{\phi}_C = 1 \rightarrow \hat{\phi}_f = 1$
  - If $0 < \hat{\phi}_C < 1 \rightarrow \hat{\phi}_C < \hat{\phi}_f < 1$



- These conditions are clearly indicated in the normalized variable diagram (NVD) on the right.
- A bounded convective scheme must lie in the diagonal line and within the shadow triangular area.
- Previous second and higher-order schemes (e.g. CDS, SUDS, QUICK) are not bounded. They present numerical instabilities in implicit calculations.

An example of a bounded convective scheme

- Many bounded convective schemes have been proposed in the technical literature (see e.g. [dar99]).
- The SMART scheme (Sharp and Monotonic Algorithm for Realistic Transport) is one of them [gas88]:

$$If \;\; 0 < \hat{\phi}_C < \frac{\hat{x}_C}{3} \rightarrow \hat{\phi}_f = -\frac{\hat{x}_f\left(1 - 3\hat{x}_C + 2\hat{x}_f\right)}{\hat{x}_C(\hat{x}_C - 1)}\hat{\phi}_C$$

$$If \;\; \frac{\hat{x}_C}{3} < \hat{\phi}_C < \frac{\hat{x}_C}{\hat{x}_f}\left(1 + \hat{x}_f - \hat{x}_C\right) \rightarrow \hat{\phi}_f = \frac{\hat{x}_f\left(\hat{x}_f - \hat{x}_C\right)}{1 - \hat{x}_C} + \frac{\hat{x}_f\left(\hat{x}_f - 1\right)}{\hat{x}_C(\hat{x}_C - 1)}\hat{\phi}_C$$

$$If \;\; \frac{\hat{x}_C}{\hat{x}_f}\left(1 + \hat{x}_f - \hat{x}_C\right) < \hat{\phi}_C < 1 \rightarrow \hat{\phi}_f = 1$$

$$otherwise \rightarrow \hat{\phi}_f = \hat{\phi}_C$$

Final form of the discretized conv-diff eq. (1/3)

- Some of the presented schemes (e.g. CDS, UDS, EDS) only involve adjacent nodes to the volume faces. Therefore, they are relatively easy to introduce. Patankar (see pat80) shows a clear and compact form of introducing these schemes.
- However, high-resolution schemes (from now on HRS), such as QUICK or SMART, involve a larger molecule. In this case, it is recommended to use the deferred correction approach. The basic idea for any of the faces f is the following:

$$\phi_f^{HRS} - \phi_P = \left(\phi_f^{UDS} - \phi_P\right) + \left(\phi_f^{HRS,*} - \phi_f^{UDS,*}\right)$$

Here, ϕ_f^{HRS} and ϕ_f^{UDS} are the current calculated values of the variable using the selected HRS and the UDS. However, $\phi_f^{HRS,*}$ and $\phi_f^{UDS,*}$ are the values calculated in the previous iteration. After convergence, $\phi_f^{UDS} = \phi_f^{UDS,*}$ and $\phi_f^{HRS} = \phi_f^{HRS,*}$.

Final form of the discretized conv-diff eq (2/3)

- The already presented UDS was written in this form:

$$\dot{m}_e(\phi_e^{UDS} - \phi_P) = \frac{\dot{m}_e - |\dot{m}_e|}{2}(\phi_E - \phi_P)$$

- If the deferred correction approach is introduced in the discretized conv-diff eq. using the above UDS:

$$\rho_P^o \frac{\phi_P - \phi_P^o}{\Delta t} V_P + \frac{\dot{m}_e - |\dot{m}_e|}{2}(\phi_E - \phi_P) - \frac{\dot{m}_w + |\dot{m}_w|}{2}(\phi_W - \phi_P) + \frac{\dot{m}_n - |\dot{m}_n|}{2}(\phi_N - \phi_P)$$
$$- \frac{\dot{m}_s + |\dot{m}_s|}{2}(\phi_S - \phi_P)$$
$$= \mathrm{D_e}(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) + D_n(\phi_N - \phi_P) - D_s(\phi_P - \phi_S) - \dot{m}_e\left(\phi_e^{HRS,*} - \phi_e^{UDS,*}\right)$$
$$+ \dot{m}_w\left(\phi_w^{HRS,*} - \phi_w^{UDS,*}\right) - \dot{m}_n\left(\phi_n^{HRS,*} - \phi_n^{UDS,*}\right) + \dot{m}_s\left(\phi_s^{HRS,*} - \phi_s^{UDS,*}\right) + \left(S_C^\phi + S_P^\phi\,\phi_P\right)V_P$$

- Rearrange terms:

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b_P$$

Final form of the discretized conv-diff eq (3/3)

where:

$$a_E = D_e - \frac{\dot{m}_e - |\dot{m}_e|}{2}; \qquad a_W = D_w + \frac{\dot{m}_w + |\dot{m}_w|}{2}$$

$$a_N = D_n - \frac{\dot{m}_n - |\dot{m}_n|}{2}; \qquad a_S = D_s + \frac{\dot{m}_s + |\dot{m}_s|}{2}$$

$$a_P = a_E + a_W + a_N + a_S + \frac{\rho_P^o V_P}{\Delta t} - S_P^\phi V_P$$

$$b_P = \frac{\rho_P^o V_P}{\Delta t}\phi_P^o + S_C^\phi V_P - \dot{m}_e\left(\phi_e^{HRS,*} - \phi_e^{UDS,*}\right) + \dot{m}_w\left(\phi_w^{HRS,*} - \phi_w^{UDS,*}\right)$$

$$-\dot{m}_n\left(\phi_n^{HRS,*} - \phi_n^{UDS,*}\right) + \dot{m}_s\left(\phi_s^{HRS,*} - \phi_s^{UDS,*}\right)$$

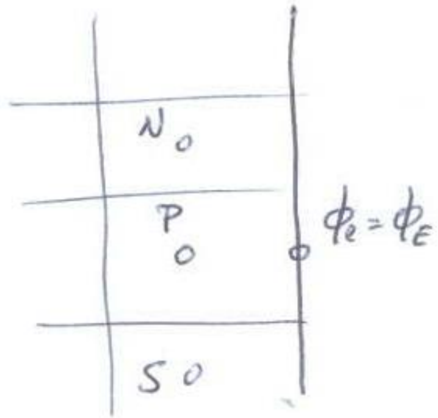
Remember: $\mathrm{D_e} = \Gamma_e S_e/\mathrm{d_{PE}}$, $\mathrm{D_w} = \Gamma_w S_w/\mathrm{d}_{PW}$, etc.

According to [pat80], “when the source term is linearized as $\bar{s}_{\phi P} = S_C^\phi + S_P^\phi\,\phi_P$, the coefficient S_P^ϕ must always be less than or equal to zero”. This rule assure that a_P is always positive provided that the coefficients of the neighbour nodes are always positive.

Boundary conditions

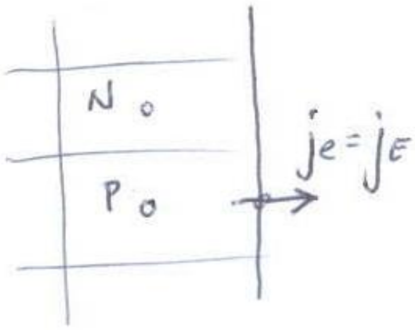
- Case of Dirichlet BC.
  - At boundary, the variable itself is known: ϕ_E
  - Fluxes can easily be calculated:

$$j_e = -\Gamma_\mathrm{P} \frac{\phi_E - \phi_P}{d_{PE}}$$



- Case of Neumann BC
  - Fluxes are known at the boundary: j_E
  - The variable can be calculated from the known flux:

$$j_E = -\Gamma_\mathrm{P} \frac{\phi_E - \phi_P}{d_{PE}} \rightarrow \phi_E = \phi_P - \frac{j_E d_{PE}}{\Gamma_\mathrm{P}}$$



Global algorithm

1. Input data (geometry, velocity field, initial ϕ values, boundary conditions, ρ, Γ, s_ϕ, mesh distribution, convergence criteria δ)
2. Mesh generation:$x_{cv}[i], y_{cv}[j], x_P[i], y_P[j], V[i][j]$, etc.
3. Initial map: $\phi^o[i][j] = \phi(t = 0, x, y)$, $t = 0$
4. Evaluation of the new time step: $t = t + \Delta t$
5. Initial estimated values: $\phi^*[i][j] = \phi^o[i][j]$
6. Evaluation of the discretization coefficients: $a_E[i][j]$, $a_W[i][j]$, $a_N[i][j]$, …
7. Resolution of the set of discretized equations: $a_P[i][j]\phi[i][j] = a_E[i][j]\phi[i +$

Proposal of exercises (1/2)*

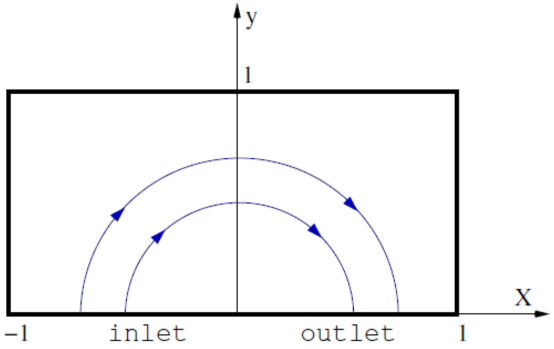
1. **Parallel flow.** Velocity field: $v_x = v_o, v_y = 0$. Inlet conditions $(x =$

2. **Diagonal flow.** Square domain $L \times L$. Velocity field: $v_x = v_o \cos(\alpha)$, $v_y = v_o \sin(\alpha)$, with $\alpha = 45°$. Boundary conditions: $(x, y = 0)$ and $(x = L, y)$: $\phi = \phi_{low}$; $(x = 0, y)$ and $(x, y = H)$: $\phi = \phi_{high}$. Test different Peclect numbers. See analytical solution when $Pe = \infty$.

(*) All three cases are steady and 2D. Velocity field is known. Density and diffusion coefficients are known constant values. Recommendation: write a code to solve a rectangular domain $L \times H$ with arbitrary boundary conditions. All the three proposed exercises can be solved in a single code (just changing BC).

Proposal of exercises (2/2)

3. **Smith-Hutton case.** Rectangular domain $2L \times L$. Velocity field: $v_x = 2y(1-x^2)$, $v_y = -2x(1-y^2)$. This velocity field verifies the incompressibility condition, $\nabla \cdot \vec{v} = 0$. The stream function can be easily calculated: $\psi = -(1-x^2)(1-y^2)$.



Inlet flow ($L = 1$ in the figure) ($-1 \leq x \leq 0, y = 0$): $\phi = 1 + tanh[10(2x+1)]$.

Outlet flow ($0 < x < 1, y = 0$): $\partial\phi/\partial y = 0$.

Rest of boundaries $(x = -1, y)(-1 \leq x \leq 1, y = 1)(x = 1, y)$: $\phi = 1 - tanh(10)$.

Table on the right: ϕ distribution at the outlet section ($0 < x < 1, y = 0$) and for different values of ρ/Γ, i.e. Pe if $L = v_o = 1$.

x-position	$\rho/\Gamma = 10$	$\rho/\Gamma = 10^3$	$\rho/\Gamma = 10^6$
0.0	1.989	2.0000	2.000
0.1	1.402	1.9990	2.000
0.2	1.146	1.9997	2.000
0.3	0.946	1.9850	1.999
0.4	0.775	1.8410	1.964
0.5	0.621	0.9510	1.000
0.6	0.480	0.1540	0.036
0.7	0.349	0.0010	0.001
0.8	0.227	0.0000	0.000
0.9	0.111	0.0000	0.000
1.0	0.000	0.0000	0.000

Summary

- Any transport equation can be cast into de generic convection-diffusion equation
- The discretization equation has been obtained, with special emphasis in the analysis of convection terms
- High-order convective schemes shows numerical instabilities problems. The use of the normalized variable methodology and the identification of the properties of bounded schemes have allow to develop a variety of stable and accurate schemes
- Examples are proposed, all of them in the field of liquids and gases at low Mach number
- It must be said that highly compressible flows of gases need special treatment. As will be see later in this course, upwind schemes must take into account the characteristic lines

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