1. a) Let $(X_n, Y_n)_{n>-\infty}$ be a stationary sequence given by

$$X_n = aX_{n-1} + V_n$$
$$Y_n = X_n + W_n,$$

where $(V_n)_{n>-\infty}$, $(W_n)_{n>-\infty}$ are orthogonal white noises, that is any linear combinations of (V_n) and (W_n) are uncorrelated and $\mathbf{E}V_n \equiv \mathbf{E}W_n = 0$, $\mathbf{E}V_n^2 \equiv \mathbf{E}W_n^2 \equiv 1$. It is also assumed that $|a| < 1$.

So, $X_n = \sum_{k=0}^{\infty} a^k V_{n-k}$. Find correlation functions $R_{xx}(k) = \mathbf{E}\left(X_{n+k}X_n\right), R_{xy}(k) = \mathbf{E}\left(X_{n+k}Y_n\right)$ and $R_{yy}(k) = \mathbf{E}\left(Y_{n+k}Y_n\right)$.

b) Let $\mathcal{M}_n$ be a linear subspace generated by $1, Y_n, Y_{n-1}, \ldots$. and $\hat{X}_n = \hat{E}(X_n|\mathcal{M}_n)$ be orthogonal projection of X_n to $\mathcal{M}_n$. Show that the error

$$P_n = \mathbf{E}\left[\left(\hat{X}_n - X_n\right)^2\right]$$

does not depend on n.

2. Assume X_n is a homogeneous Markov chain with transition kernel $p(r, u)$. Let

$$Y_n = c(X_n) + \sigma(X_n)W_n, n \geq 0,$$

where $W_0, W_1, \ldots$ are standard normal independent r.v. independent of $X_0, X_1, \ldots$, and $\sigma > 0$. Let

Denote $l(u, v) = \frac{1}{\sigma(u)}\rho\left(\frac{v-c(u)}{\sigma(u)}\right)$,where $\rho(z)$ is the density of a standard normal.

Show that the pair (X_n, Y_n) is homogeneous Markov chain with transition kernel

$$q^{r,u}(s, v) = p(r, u)l(u, v).$$

Note that in this case, the recursive formula for UFDF is

$$\phi^a_{n+1} = l(a, Y_{n+1})\int_{\mathbf{R}} p(r, a)\phi^r_n dr, a \in \mathbf{R}. \tag{1}$$

3. Assume $V_n, n \geq 1$,are independent atandard normal independent of $X_0 \sim N(7, 0.5^2)$ and

$$X_{n+1} = 1.004X_n + 0.06X_nV_n, n \geq 0.$$

So, X_n is a homogeneous Markov chain with transition kernel

$$p(r, u) = \frac{1}{0.06 \cdot r}\rho\left(\frac{u - 1.004r}{0.06 \cdot r}\right),$$

where ρ is the pdf of standard normal. Let the observation

$$Y_n = X_n + 2W_n, n \geq 0,$$

where W_n are independent standard normal independent of $V_n, n \geq 1$, and X_0. Use #2 and (1) to find the best mean square estimate $\hat{X}_n$ of X_n.

Simulate and plot on the same graph $X_n, Y_n, \hat{X}_n$, $0 \leq n \leq 250$.