1. a) Find the spectral density of the Ornstein-Uhlenbeck stationary sequence with parameter $a \in (-1, 1)$;

b) Show that a stationary sequence such that $\sum_{m=-\infty}^{\infty} |R(m)| < \infty$ has a constant spectral density if and only if

$$R(m) = \begin{cases} c^2, & m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. (#7.3.5) (Filtering of a particular frequency) Assume we observe

$$X_n = \sum_{k=1}^{m} e^{i\lambda_k n} \eta_k, n \in \mathbf{Z},$$

where $\lambda_k \in (-\pi, \pi], \eta_k$ are uncorrelated r.v. To filter the frequency λ_1, for $a \in (0, 1)$ we define

$$\begin{aligned} Y_n &= (1-a^2)X_n + (1-a^2)ae^{i\lambda_1}X_{n-1} + \ldots + (1-a^2)\left(ae^{i\lambda_1}\right)^k X_{n-k} + \ldots \\ &= \sum_{k=0}^{\infty} (1-a^2)\left(ae^{i\lambda_1}\right)^k X_{n-k}, n \in \mathbf{Z}. \end{aligned}$$

Find

$$\lim_{a \to 1} Y_n.$$

Hint: collect terms with each $\eta_i, i = 1, \ldots, m$.

3. (Simulation of a frequency filtering). We observe

$$X_n = \eta_1 e^{i\lambda_1 n} + \eta_2 e^{i\lambda_2 n}, n \in \mathbf{Z},$$

where $\eta_i \sim N(0, \sigma_i^2), i = 1, 2$, are independent. For $a \in (0, 1)$ and $M > 0$, we will use

$$Y_n^M = (1-a^2)X_n + (1-a^2)ae^{i\lambda_1}X_{n-1} + \ldots + (1-a^2)\left(ae^{i\lambda_1}\right)^M X_{n-M}.$$

to filter $\theta_n = \eta_1 e^{i\lambda_1 n}$.

Do the following 3 times:

Simulate X_n, θ_n, Y_n^M, $0 \leq n \leq N$, for $N = 50, M = 10, a = 0.8, \sigma_1 = 1, \sigma_2 = 2, \lambda_1 = \pi/4, \lambda_2 = 2\pi/3$.

Plot on the same graph real parts of θ_n and $X_n, 0 \leq n \leq N$; plot on the same graph real parts of θ_n and Y_n^M $, 0 \leq n \leq N$.