1. Assume the signal X_n is a simple reflected r.w. on $A = [0, 20] \cap \mathbf{Z}$ starting at 10. Let $W_0, W_1, \ldots$ be a sequence of i.i.d. taking two values with prob. 1/2:

$$\mathbf{P}\left(W_1 = \pm L\right) = 1/2,$$

The observation Y_n is defined by

$$Y_n = \min\left\{\max\left\{X_n + W_n, 0\right\}, 20\right\}.$$

(It is hidden Markov model with $\gamma(x, w) = \min\left\{\max\left\{x + w, 0\right\}, 20\right\}$). Choose $L = 5$.

a) (smoothing) Using forward and backward Baum-Welch equations, simulate $X_n, Y_n, 0 \le n \le 200$ and the smoothing estimates of X_{100} :

$$\hat{X}_{100,T} = \frac{\sum_{a \in A} a \phi^a_{100,T}}{\sum_{r \in A} \phi^r_T},$$

for $T = 100, \ldots, 200$. Plot on the same graph X_{100} (as a constant) and $\hat{X}_{100,T}, T = 100, \ldots, 200$.

b) (prediction) Simulate $X_n, Y_n, 0 \le n \le 200$ and the prediction estimates of X_{200} :

$$\hat{X}_{200,n} = \frac{\sum_{a \in A} a \phi^a_{200,n}}{\sum_{r \in A} \phi^r_n}$$

for $n = 100, \ldots, 200$. Plot on the same graph X_{200} (as a constant) and $\hat{X}_{200,n}, n = 100, \ldots, 200$.in time interval $[0, 200]$, for $L = 1$; $L = 3$; $L = 5$; $L = 10$; $L = 15$; $L = 18$; $L = 20$.

2. Assume we have three coins $\{1, 2, 3\}$ with the following probabilities

	1	2	3
$P(H)$	0.3	0.5	0.7
$P(T)$	0.7	0.5	0.3

At $n = 0$ a coin is randomly selected and tossed, then the tossing continues possibly changing the coin with probability 1/3 (the chosen coin is the signal $X_n \in \{1, 2, 3\}$. Assume we observed $Y_{[0,10]} = HHTHHTHTTTT$.

a) Find $\mathbf{P}\left(X_5 = i | Y_{[0,10]}\right), i = 1, 2, 3$;

and $\mathbf{P}\left(X_5 = i | Y_{[0,10]}\right), i = 1, 2, 3$;

b) Find $\mathbf{P}(X_{10} = i | Y_{[0,5]})$ and $\mathbf{P}(X_{10} = i | Y_{[0,6]}), i = 1, 2, 3$.