Math508 Final Exam

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1 Problem 1

For Ornstein-Uhlenbeck stationary sequence, $EX_n = 0$, $R(k) = a^{|k|}$. The best linear estimate of X_m given $X_k, X_{k-1}, X_{k-2}, ...$ is

$$\hat{X_m} = k_0 + k_1 * X_k + k_2 * X_{k-1} + k_3 * X_{k-2} + ...k_0 = EX_m = 0 \tag{1}$$

The matrices in the normal equation (2) would be infinite although it could be easily be seen that $k_1 = a^{m-k}$ while other k's are all zero. Hence, $\hat{X_m} = a^{m-k} X_k$.

$$\begin{pmatrix} a^0 & a^1 & a^2 & \dots \\ a^1 & a0 & a^1 & \dots \\ \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a^{m-k} \\ a^{m-k+1} \\ \vdots \end{pmatrix} \tag{2}$$

Instead guessing, i use the sequential project to prove this result. M_n is a linear subspace generated by $\{1, X_k, X_{k-1}, \dots, Xk - n + 1\}$.

$$\hat{X_{m,n}} = E(X_m | M_n) \tag{3}$$

There's iteration that

$$\hat{X_{m,n}} = \hat{X_{m,n-1}} + \frac{c_{n,n-1}}{u_{n,n-1}} (X_{k-n} - X_{k-n+1})$$
$$\hat{X_{k-n,n-1}} = E(X_{k-n} | M_{n-1})$$
$$c_{n,n-1} = E[(X_m - \hat{X_{m,n-1}})(X_{k-n} - \hat{X_{k-n,n-1}})]$$

Now take $n = 2$, M_2 is a linear subspace generated by $\{1, X_k, X_{k-1}\}$ and M_1 is a linear subspace generated by $\{1, X_k\}$. We can get following results easily.

$$\hat{X_{m,n-1}} = \hat{X_{m,1}} = a^{m-k} * X_k \tag{4}$$
$$\hat{X_{k-n,n-1}} = \hat{X_{k-1,1}} = aX_k \tag{5}$$

We can prove $c_{n,n-1} = 0$.

$$\begin{aligned} c_{n,n-1} = c_{2,1} &= E[(X_m - a^{m-k} * X_k)(X_{k-1} - aX_k)] \\ = E(X_m X_{k-1} - a X_m X_k &- a^{m-k} X_k X_{k-1} + a^{m-k+1} X_k X_k) \\ &= a^{m-k+1} - a * a^{m-k} - a^{m-k} * a + a^{m-k+1} = 0 \end{aligned} \tag{6}$$
$$\hat{X_{m,2}} = \hat{X_{m,1}} = a^{m-k} * X_k \tag{7}$$

By induction,

$$\hat{X_m} = a^{m-k} X_k \tag{8}$$

2 Problem 2

2.1 part a

$$
\begin{aligned}
& P(X_{n+1} \le u, Y_{n+1} \le v | X_n = r, Y_n = s) \\
= & \int_{-\infty}^{u} P(Y_{n+1} \le v | X_{n+1} = x) f(X_{n+1} = x | X_n = r, Y_n = s) dx \\
= & \int_{-\infty}^{u} \int_{-\infty}^{v} l(X_{n+1} = x, Y_{n+1} = y) p(X_{n+1} = x | X_n = r) dy dx
\end{aligned}
$$

Differentiate both sides by $\frac{\partial^2}{\partial u \partial v}$ to get the transition kernel

$$q^{r,u}(s, v) = p(r, u) l(u, v) \tag{9}$$

2.2 part b

Given $X_n = r$, if $W_n \ge -r$, Y_n is distributed as $N(r, 1)$. If $W_n < -r$, Y_n is distributed as $N(-r, 1)$. Graphically, Y_n's distribution is a combination of $N(r, 1)$ and $N(-r, 1)$ with the only positive domain. So,

$$P(Y_n < u | X_n = r) = \int_0^u \rho(y - r) dy + \int_0^u \rho(y + r) dy \tag{10}$$

$\rho(x)$ is the standard normal density function. Differentiate both sides by $\frac{\partial}{\partial u}$ to get

$$l(r, u) = \rho(u - r) + \rho(u + r) \tag{11}$$

3 Problem 3

Professor, in this problem, i found a little problem with hint you gave us. Basically, the filtering procedure (i'll call it procedure 1) is more like a one step prediction. Following dependency structure will help illustrate the problem.

$$X_0[d][r]X_1[d][r] \dots [r]X_n[d][r]X_{n+1}[d] \dots Y_0 Y_1 Y_2 \dots Y_{n+1} Y_{n+2} \dots W_1[u]W_2[u] \dots W_{n+1}[u]W_{n+2}[u] \dots \tag{12}$$

The recursion for **procedure 1** is

$$\gamma_0(H_0, X_0) = 0 \tag{13}$$

$$\gamma_0(X_0) = \frac{1}{0.1^2} \rho(x/0.1) \tag{14}$$

$$\gamma_{n+1}(H_{n+1}, X_{n+1}) = \int_R p(X_n, X_{n+1}) \lambda^{-1}(Y_{n+1}, X_n)[\gamma_n(H_n, X_n) + X_n \gamma_n(X_n)] dX_n \tag{15}$$

$$\gamma_{n+1}(X_{n+1}) = \int_R p(X_n, X_{n+1}) \lambda^{-1}(Y_{n+1}, X_n) \gamma_n(X_n) dX_n \tag{16}$$

$$E[H_n | Y_{[0,n]}] = \frac{\int_R \gamma_n(H_n, X_n) dX_n}{\int_R \gamma_n(X_n) dX_n} \tag{17}$$

$$E[f(X_n) | Y_{[0,n]}] = \frac{\int_R f(X_n) \gamma_n(X_n) dX_n}{\int_R \gamma_n(X_n) dX_n} \tag{18}$$

Basically, to get the signal of X_{n+1}, we are Y_{n+1}, one step prediction from X_n and all previous information. However, looking at diagram (12), we should incorporate Y_{n+2} to do a full filtering. So after some calculation, i propose **procedure 2**

$$\gamma_0(H_0, X_{-1}) = 0 \quad (19)$$

$$\gamma_0(X_0) = \lambda^{-1}(Y_1, X_0)\frac{1}{0.1^2}\rho(x/0.1) \quad (20)$$

$$\gamma_{n+1}(H_{n+1}, X_n) = \lambda^{-1}(Y_{n+1}, X_n)\int_R p(X_{n-1}, X_n)[\gamma_n(H_n, X_{n-1})dX_{n-1} + X_n\gamma_n(X_n) \quad (21)$$

$$\gamma_{n+1}(X_{n+1}) = \lambda^{-1}(Y_{n+2}, X_{n+1})\int_R p(X_n, X_{n+1})\gamma_n(X_n)dX_n \quad (22)$$

$$E[H_n|Y_{[0,n]}] = \frac{E[Z_n^{-1}H_n|Y_{[0,n]}}{E[Z_n^{-1}|Y_{[0,n]}} = \frac{\int_R \gamma_n(H_n, X_{n-1})dX_{n-1}}{\int_R \gamma_{n-1}(X_{n-1})dX_{n-1}} \quad (23)$$

$$E[f(X_n)|Y_{[0,n+1]}] = \frac{E[f(X_n)Z_{n+1}^{-1}|Y_{[0,n+1]}}{E[Z_{n+1}^{-1}|Y_{[0,n+1]}} = \frac{\int_R f(X_n)\gamma_n(X_n)dX_n}{\int_R \gamma_n(X_n)dX_n} \quad (24)$$

The idea is to get $\hat{H_n}$, only need information up to Y_n because $H_n = \sum_{k=1}^{k=n} X_{k-1}$ and the last included X is X_{n-1}. Looking at diagram (12), X_{n-1} is depended by Y_n. To get $\hat{X_n}$, we need information up to Y_{n+1}. It created a little bit hardship in programming as two estimates are not synchronized in terms of sequential computing.

Figure (1, 2, 3, 4) are due to *procedure 1.* Figure (5, 6, 7, 8) are due to *procedure 2.* Figure 5 from *procedure 2* has smaller MSE than Figure 1 from *procedure 1.* By checking closely, you can actually see $\hat{X_n}$ in Figure 1 is a little bit lagging behind $\hat{X_n}$ in Figure 5. This lagging phenomenon is due to the one-step-prediction in *procedure 1.* It's evident in other figures as well.

4 Problem 4

By checking the notes, i realized there're some errors in both the notes and the hints although it won't hurt the result if $\frac{\alpha_{t_{i+1}}}{\alpha_{t_i}}$ is calculated approximately. Most errors are due to the swapping of λ_1 and λ_2 which depends on how to split the Ricotti equation. Ricotti equation is

$$\lambda_1 = \left(a^2B^4 + b^2B^2\right)^{1/2} + aB^2 \quad (25)$$

$$\lambda_2 = -\left(a^2B^4 + b^2B^2\right)^{1/2} + aB^2 \quad (26)$$

$$\frac{dP}{dt} = 2aP_t + b^2 - \frac{P_t^2}{B^2} \quad (27)$$

So assume Ricotti equation is split in the way following

$$\Rightarrow$$

$$\frac{dt}{dP} = \frac{C}{P-\lambda_1} - \frac{C}{P-\lambda_2} = \frac{-B^2}{(P^2 - 2aB^2P - b^2B^2)} \quad (28)$$

$$\Rightarrow C = \frac{B^2}{\lambda_2 - \lambda_1} \quad (29)$$

$$\Rightarrow Cln|\frac{P-\lambda_1}{P-\lambda_2}| = t + const \tag{30}$$

$$\Rightarrow |\frac{P-\lambda_1}{P-\lambda_2}| = Ke^{t/C} \tag{31}$$

$$\Rightarrow t = 0, K = |\frac{P_0-\lambda_1}{P_0-\lambda_2}| \tag{32}$$

With $P_0 = 0 < \lambda_1$, we'll get

$$K = \frac{\lambda_1}{-\lambda_2} \tag{33}$$

$$\frac{P-\lambda_1}{P-\lambda_2} = -Ke^{t/C} \tag{34}$$

$$\Rightarrow P_t = \frac{\lambda_2 Ke^{t/C} + \lambda_1}{Ke^{t/C}+1} = \lambda_2 + \frac{\lambda_1 - \lambda_2}{1 + Ke^{t/C}} \tag{35}$$

$$\alpha_t = exp\{\int_0^t (a - \frac{P_s}{B^2})ds\} = exp\{(a - B^{-2}\lambda_1)t\}\frac{(K+1)e^{t/C}}{Ke^{t/C}+1} \tag{36}$$

One thing about α_t, i'm not sure it should be λ_1 or λ_2 in the formula (i can't work out the complex integral.) So when calculating $\frac{\alpha_{t_{i+1}}}{\alpha_{t_i}}$, i tried both of them, calling them $exactratio_1$ and $exactratio_2$ and compared with the approximated version, which is

$$\frac{\alpha_{t_{i+1}}}{\alpha_{t_i}} \approx exp\{(a - \frac{P_{t_i}}{B^2})\delta t)\} \tag{37}$$

So in figure 9, the $exactratio_1$ is the correct formula as it's closer to the approximated ratio. So Figure (10, 11, 12, 13) are for different settings. The $\hat{X}_n$ is calculated using both the approximated $\frac{\alpha_{t_{i+1}}}{\alpha_{t_i}}$ and the exact one. Previously, due to the opposite sign of C, $\hat{X}_n$ based on the approximated ratio (red curve) has smaller MSE than $\hat{X}_n$ based on the exact ratio (cyan curve). Now with the correct formula for C (29), $\hat{X}_n$ based on the exact ratio (cyan curve) has smaller MSE, which is within expectation.

5 Figures

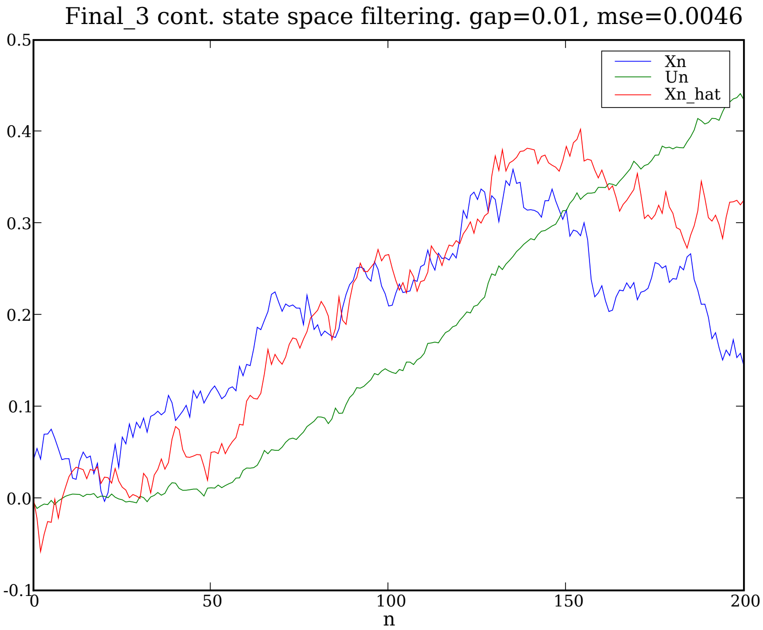


Figure 1:

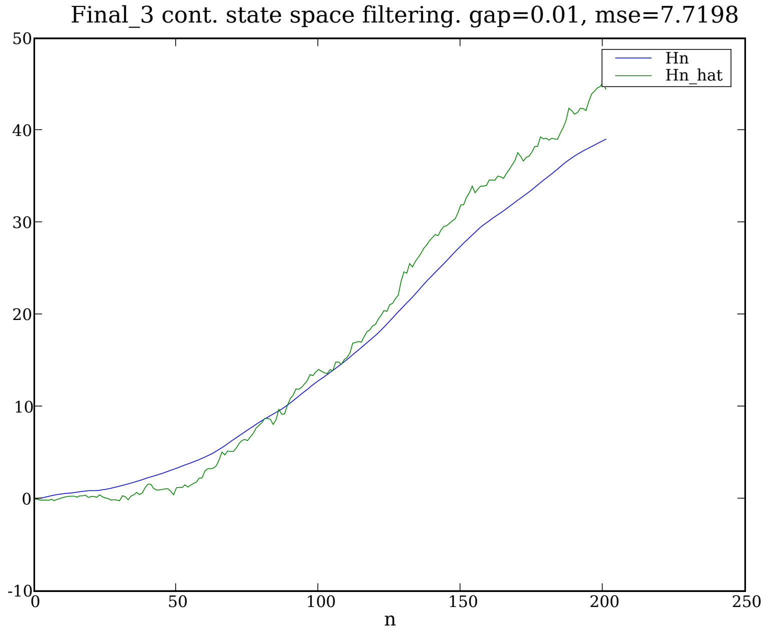


Figure 2:

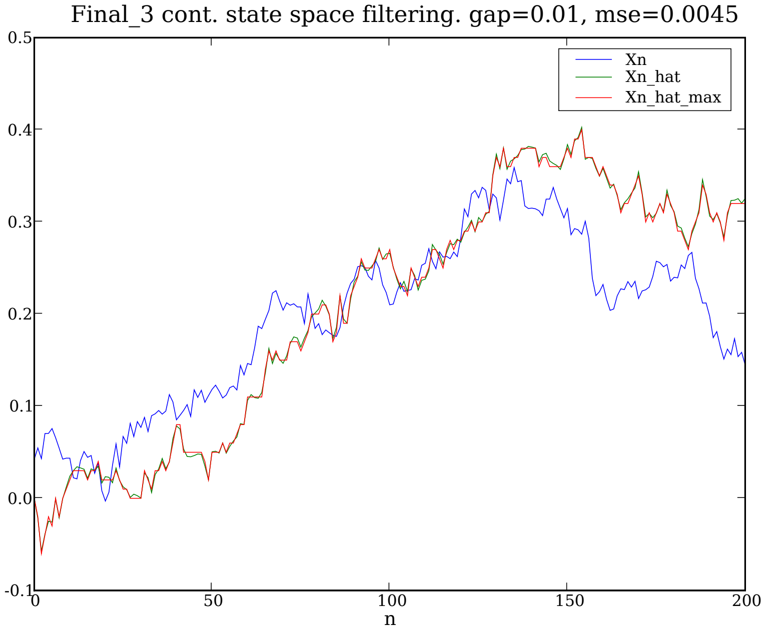


Figure 3:

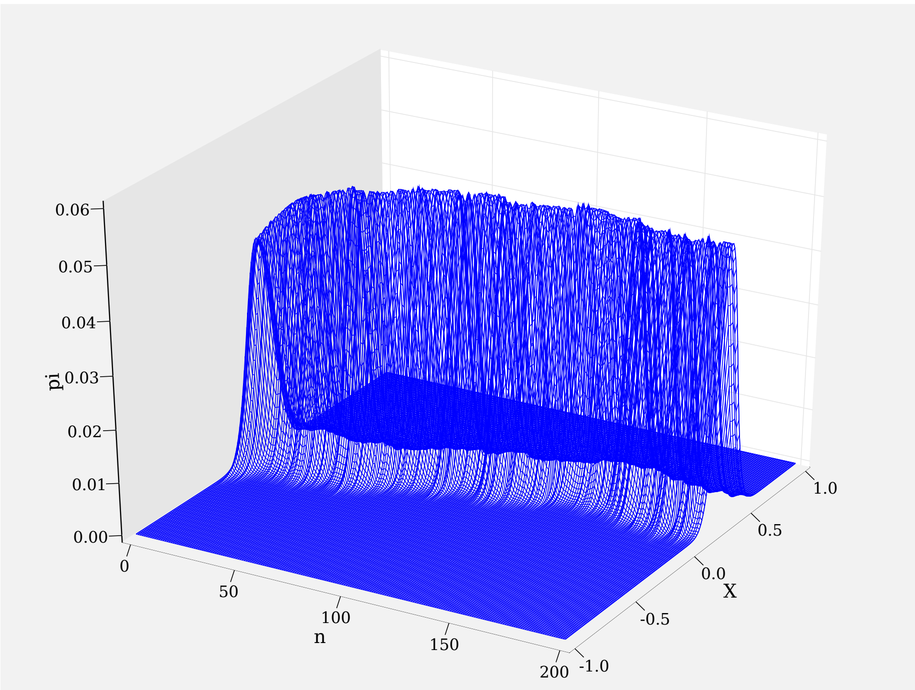


Figure 4:

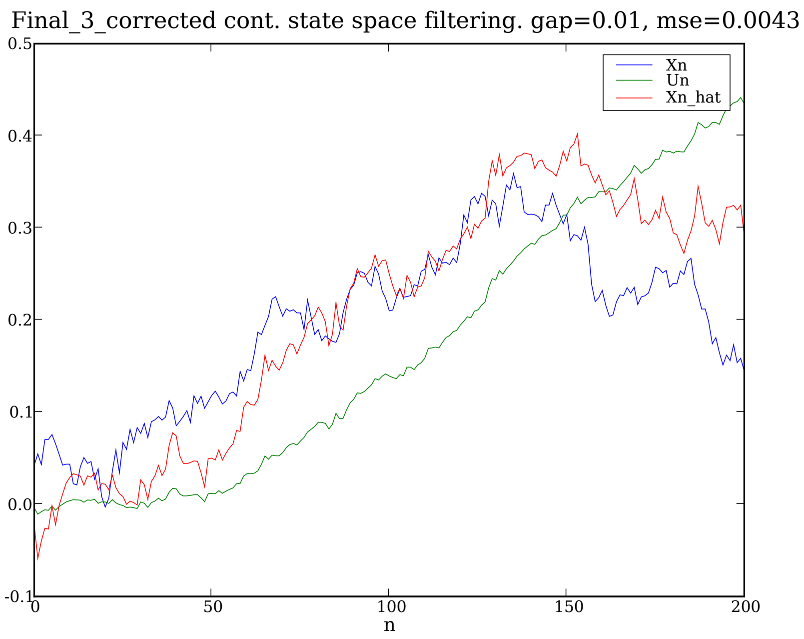


Figure 5:

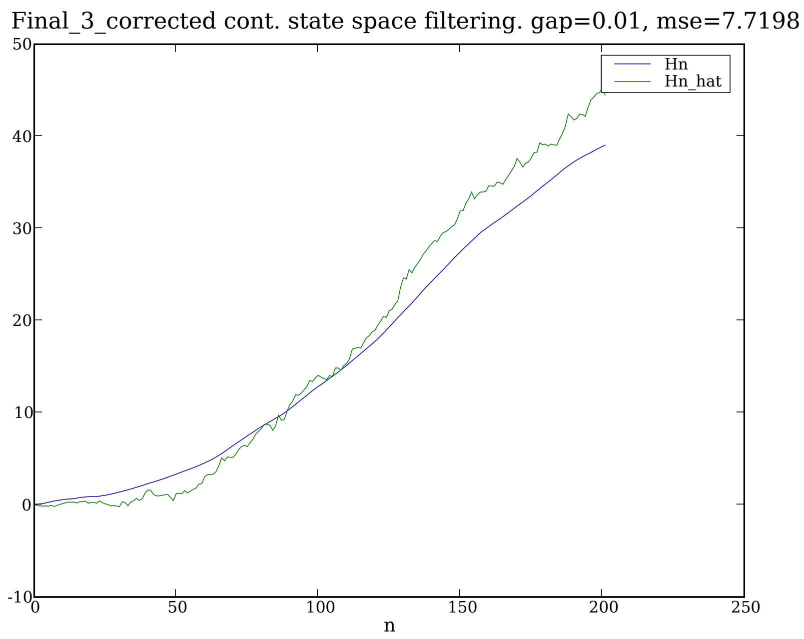


Figure 6:

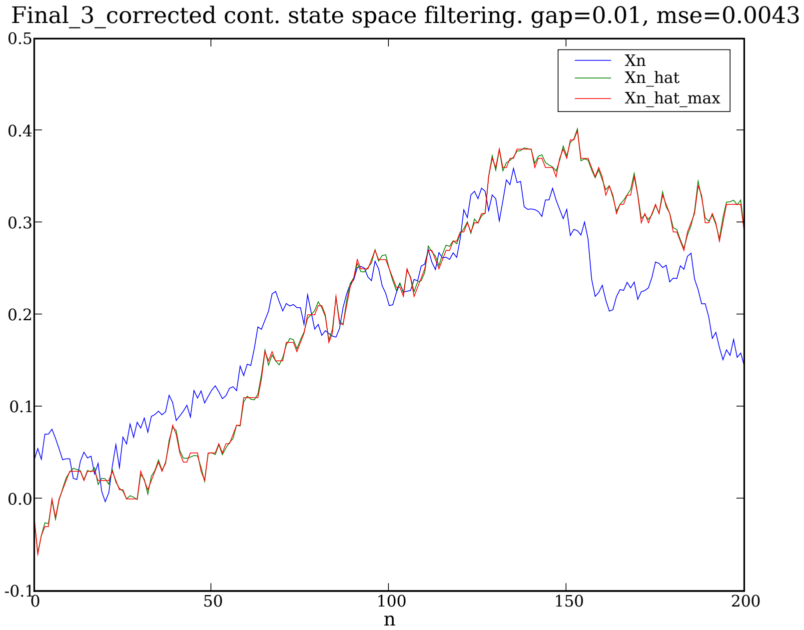


Figure 7:

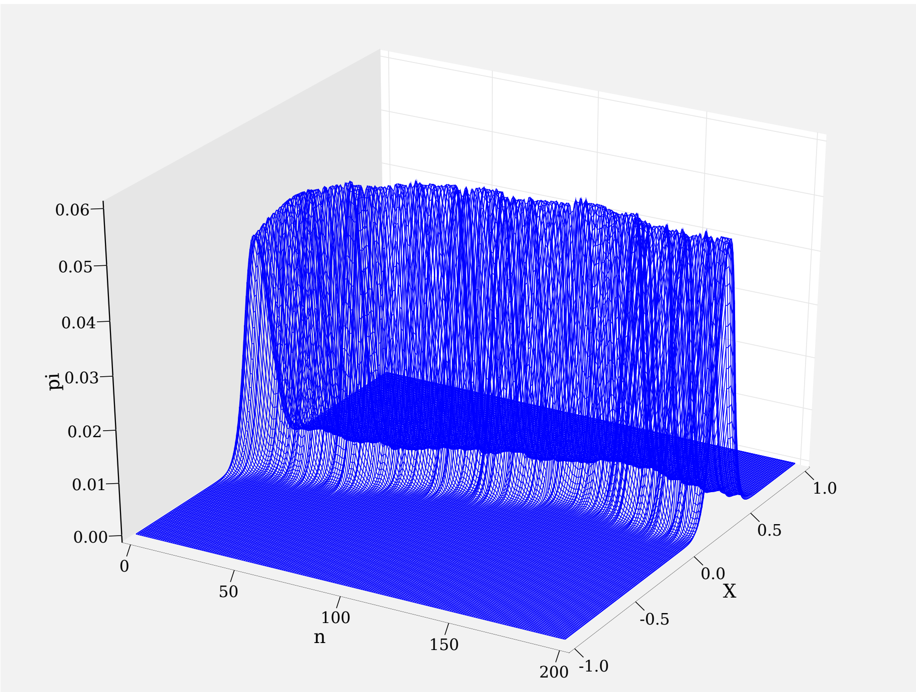


Figure 8:

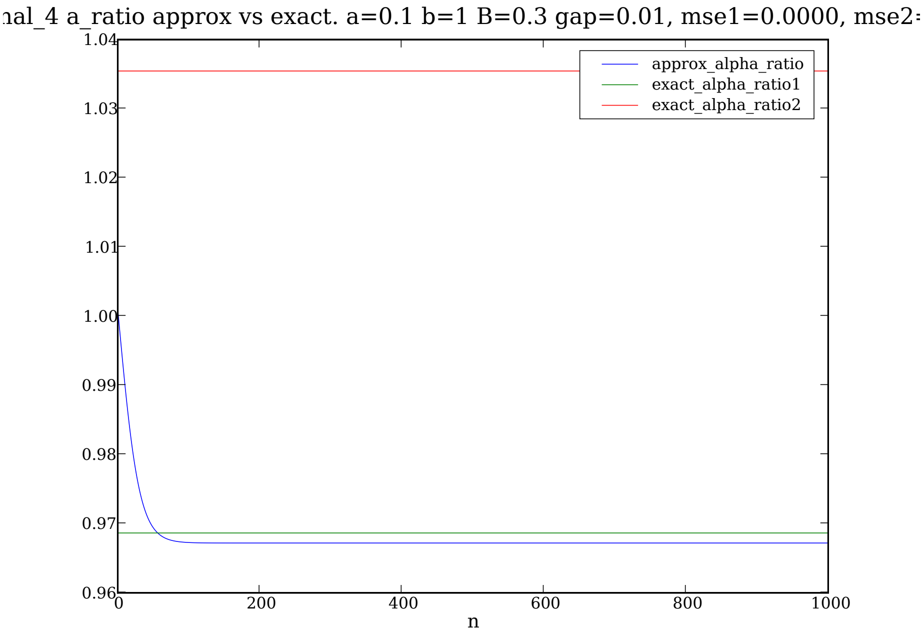


Figure 9:

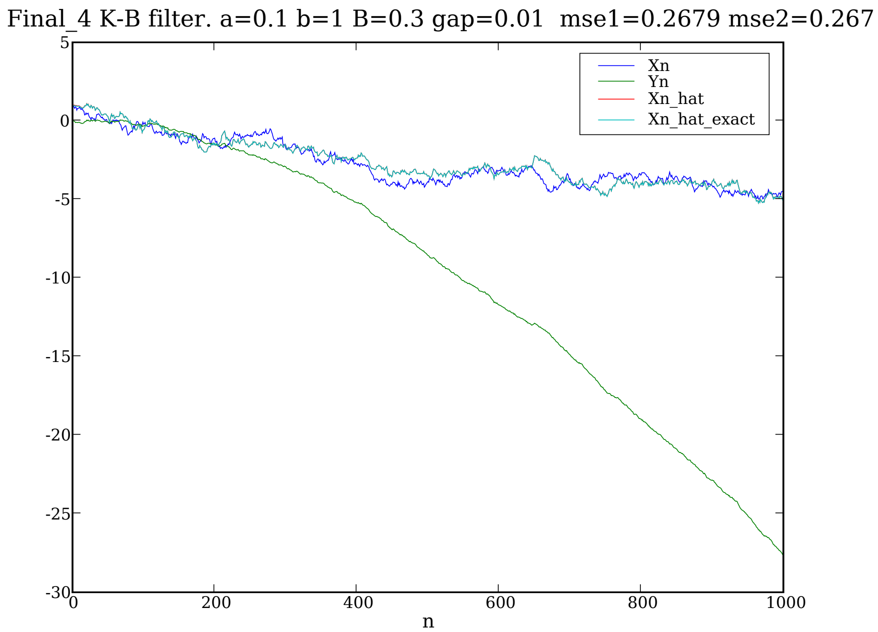


Figure 10:

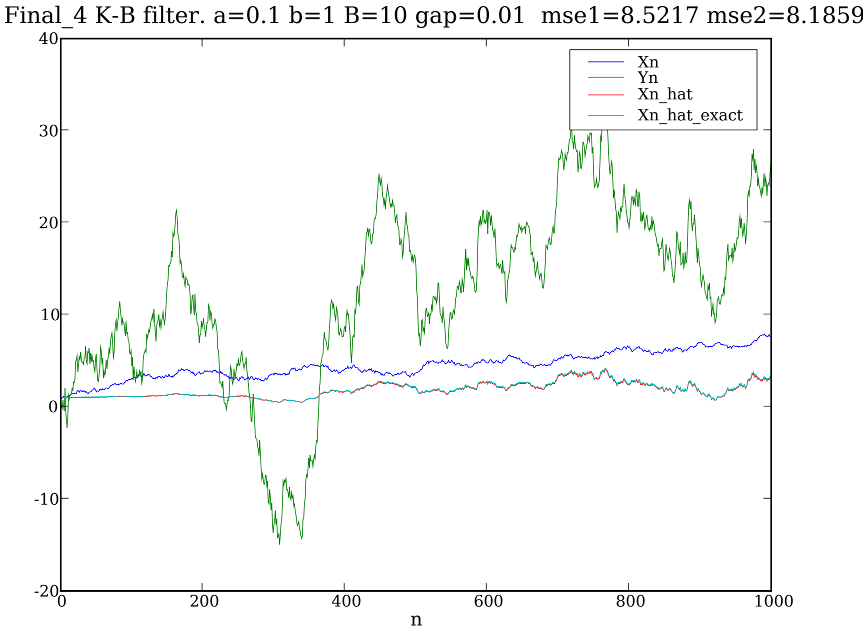


Figure 11:

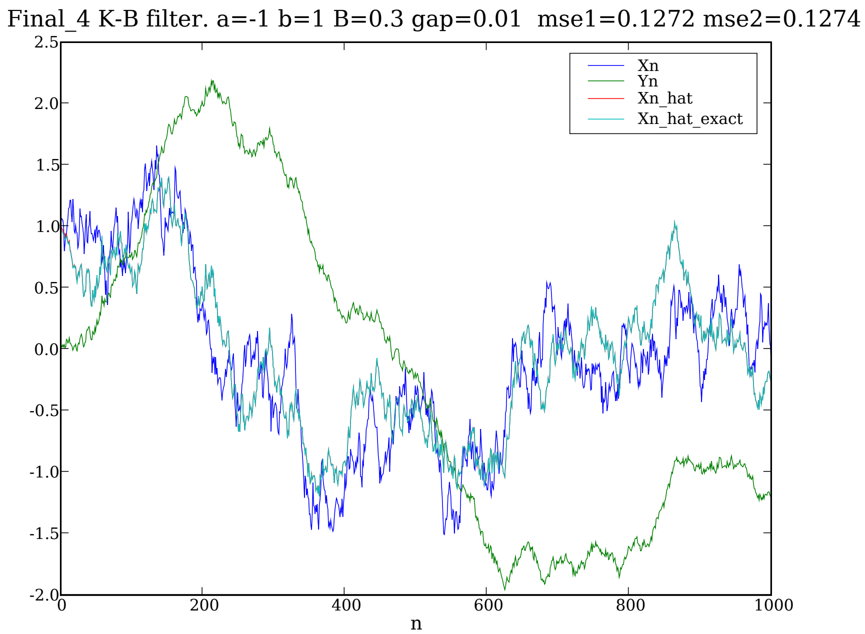
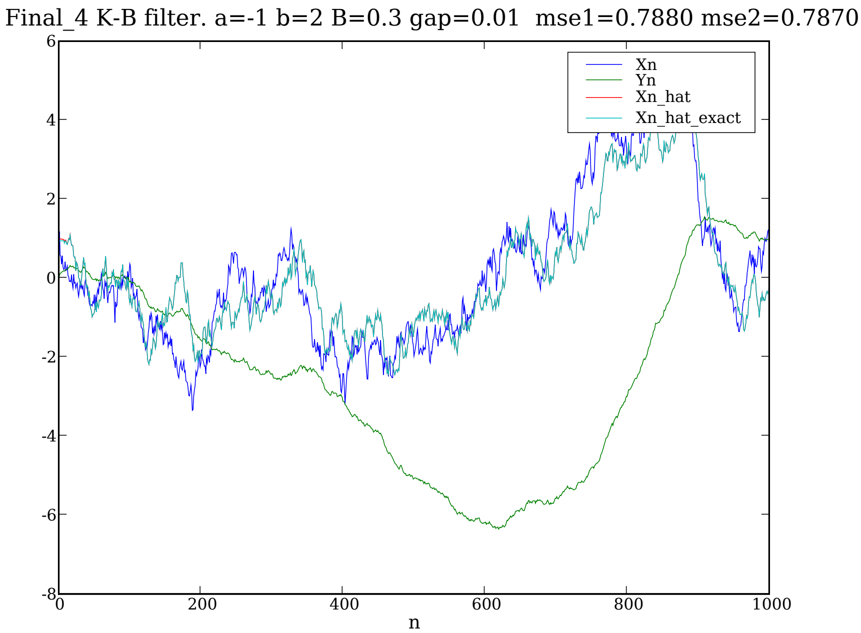


Figure 12: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Xn
Yn
Xn_hat
Xn_hat_exact
6
4
2
0
-2
-4
-6
-8
0
200
400
600
800
1000
n

Figure 13: