1. Assume (α, β) is a 2D Gaussian r.v. (non-degenerated: neither α nor β is a constant). Find $\mathbf{E}(\alpha|\beta)$ and the best mean square linear estimator of α, given β.

Hint. The joint pdf of α and β is

$$f(x, y(= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{q(x,y}{2}\right\},$$

where $\sigma_\alpha, \sigma_\beta$ are standard deviations, $\rho = \mathbf{cov}(\alpha, \beta)/(\sigma_\alpha\sigma_\beta)$ is the correlation and

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_\alpha}{\sigma_\alpha}\right)^2 - 2\rho\left(\frac{x-\mu_\alpha}{\sigma_\alpha}\right)\left(\frac{y-\mu_\beta}{\sigma_\beta}\right) + \left(\frac{y-\mu_\beta}{\sigma_\beta}\right)^2\right]$$

$$= \frac{1}{1-\rho^2}[\left(\frac{x-\mu}{\sigma_\alpha}\right)^2 - 2\rho\left(\frac{x-\mu_a}{\sigma_\alpha}\right)\left(\frac{y-\mu_\beta}{\sigma_\beta}\right) + \rho^2\left(\frac{y-\mu_\beta}{\sigma_\beta}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_\beta}{\sigma_\beta}\right)^2]$$

$$= \frac{1}{1-\rho^2}[\left(\frac{x-\mu_\alpha}{\sigma_\alpha} - \rho\frac{y-\mu_\beta}{\sigma_\beta}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_\beta}{\sigma_\beta}\right)^2]$$

$$= \frac{1}{1-\rho^2}[\left(\frac{x-(\mu_\alpha+\rho y-\rho\mu_{\beta)}}{\sigma_\alpha}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_\beta}{\sigma_\beta}\right)^2].$$

Note that for all $\sigma > 0$ and $m \in \mathbf{R}$,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x\exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}dx = m,$$
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}dx = 1.$$

2. Let $\mathcal{M}$ be a linear subspace generated by $\{1, \beta_1, \ldots, \beta_n\}$. Assume

$$\hat{\alpha} = m + c_1(\beta_1 - \mathbf{E}\beta_1) + \ldots + c_n(\beta_n - \mathbf{E}\beta_n)$$

is the best linear estimate of a r.v. α. Show that the mean square error

$$\mathcal{E} = \mathbf{E}\left[(\alpha - \hat{\alpha})^2\right] = Var(\alpha) - \sum_{i,j} cov(\beta_i, \beta_j)c_ic_j.$$