1 Random Walks

Definition 1 *Assume $U_1, U_2, \ldots$ are $\mathbf{Z}^d$-valued i.i.d. and X_0 is a $\mathbf{Z}^d$-valued r.v. that is independent of $(U_1, U_2, \ldots)$.*

A random walk (r.w.) is a sequence X_n defined by

$$X_{n+1} = X_n + U_{n+1}, n = 0, 1, \ldots.$$

The random variables U_n are called the steps of random walk, and X_n is the state of r.w. at time n. A r.w. X_n is called symmetric if the step distribution is symmetric.

A r.w. X_n is simple if

$$\mathbf{P}\left(U_i = X_{i+1} - X_i = \pm e_i\right) = \frac{1}{2d},$$

where $e_i, i = 1, \ldots, d$ are the standard basis vectors: the ith coordinate is 1 and the remaining coordinates are zeros (this means that a simple r.w. moves only parallel to the coordinate axis to the closest grid point).

Theorem 2 *A r.w. X_n is a Markov chain: for all $c \in \mathbf{Z}^d, n \geq 1$,*

$$\mathbf{P}(X_{n+1} = c | (X_0, X_1, \ldots, X_n)) = \mathbf{P}(X_{n+1} = c | X_n).$$

Proof. Since $X_{n+1} = X_n + U_{n+1}$ and U_{n+1} is independent of $X_0, \ldots, X_n$,

$$\begin{aligned} \mathbf{P}(X_{n+1} &= c | (X_0, X_1, \ldots, X_n)) = \mathbf{P}(X_n + U_{n+1} = c | (X_0, X_1, \ldots, X_n)) \\ &= \mathbf{P}\left(x + U_{n+1} = c\right)|_{x = X_n}, \end{aligned}$$

and, similarly

$$\begin{aligned} \mathbf{P}(X_{n+1} &= c | X_n) = \mathbf{P}(X_n + U_{n+1} = c | X_n) \\ &= \mathbf{P}\left(x + U_{n+1} = c\right)|_{x = X_n}. \end{aligned}$$

The statement follows. ■

Remark 3 *The moment generating functions of a step $U_1 = X_1$ of a simple r.w. starting at 0:*

a) in 1D:

$$\mathbf{E}^{\lambda_1 U_1} = \frac{1}{2}\left(e^{\lambda_1} + e^{-\lambda_1}\right);$$

b) in 2D: (in this case $U_1 = X_1 = (X_1^1, X_2^2)$)

$$\mathbf{E}^{\lambda_1 X_1^1 + \lambda_2 X_1^2} = \frac{1}{4}\left(e^{\lambda_1} + e^{-\lambda_1} + e^{\lambda_2} + e^{-\lambda_2}\right);$$

Some properties of r.w.

1. *Consider a simple r.w. X_n on Z starting at 0. Then*

$$\mathbf{P}\left(U_n = X_n - X_{n-1} = \pm 1\right) = \frac{1}{2},$$

and

$$\mathbf{P}\left(X_n = k\right) = \begin{cases} 2^{-n}\binom{n}{(n+k)/2}, \text{ if } n+k \text{ is even,} \\ \quad 0, \text{ if } n+k \text{ is odd.} \end{cases} \tag{1}$$

In particular, $P\left(X_n = 0\right) = 2^{-n}\binom{n}{n/2}$, if n is even.

Proof. Indeed, we have $X_n = U_1 + \ldots + U_n = M_n - (n - M_n) = 2M_n - n$, where M_n is the number of "1" and $n - M_n$ is the number of "-1" resulting in n independent trials $U_i, \ldots, U_n$. So, M_n is binomial$(n, \frac{1}{2})$ and (1) follows. ■

2. *a) Assume that X_n and Y_n are two independent simple r.w. on Z starting at 0. Then $Z_n = (X_n, Y_n)$ is a symmetric r.w. (not simple). But $\tilde{Z}_n = ((X_n - Y_n)/2, (X_n + Y_n)/2)$ is simple r.w. on Z^2.*

b) On the other hand, if $X_n = (X_n^1, X_n^2)$ is a simple r.w. on Z^2 starting at 0, then $Y_n = X_n^1 - X_n^2$ and $Z_n = X_n^1 + X_n^2$ are two independent r.w. on Z. The probability

$$\mathbf{P}(X_n = 0) = 2^{-2n}\binom{n}{n/2}^2,$$

if n is even, and zero otherwise.

Proof. a) By Remark 3, the mgf of $\tilde{Z}_n$ step,

$$\begin{aligned}
\mathbf{E}\exp\left\{\lambda_1\frac{X_1 - Y_1}{2} + \lambda_2\frac{X_1 + Y_1}{2}\right\} &= \mathbf{E}\exp\{\frac{\lambda_1 + \lambda_2}{2}X_1 + \frac{\lambda_2 - \lambda_1}{2}Y_1\} \\
&= \mathbf{E}\exp\{\frac{\lambda_1 + \lambda_2}{2}X_1\}\mathbf{E}\exp\{\frac{\lambda_2 - \lambda_1}{2}Y_1\} \\
&= \frac{1}{4}\left(e^{(\lambda_1+\lambda_2)/2} + e{-}^{(\lambda_1+\lambda_2)/2})(e^{(\lambda_2-\lambda_1)/2} + e{-}^{(\lambda_2-\lambda_1)/2}\right) \\
&= \frac{1}{4}\left(e^{\lambda_1} + e^{-\lambda_1} + e^{\lambda_2} + e^{-\lambda_2}\right);
\end{aligned}$$

b) By Remark 3, the mgf of (Y_n, Z_n) step

$$\begin{aligned}
\mathbf{E}\exp\left\{\lambda_1 Y_1 + \lambda_2 Z_1\right\} &= \mathbf{E}\exp\left\{\lambda_1(X_1^1 - X_1^2) + \lambda_2(X_1^1 + X_1^2\right\} \\
&= \mathbf{E}\exp\left\{(\lambda_1 + \lambda_2)X_1^1 + (\lambda_2 - \lambda_1)X_1^2\right\} \\
&= \frac{1}{4}\left(e^{(\lambda_1+\lambda_2)} + e^{-(\lambda_1+\lambda_2)} + e^{(\lambda_2-\lambda_1)} + e^{-(\lambda_2-\lambda_1)}\right) \\
&= \frac{1}{2}\left(e^{\lambda_1} + e^{-\lambda_1}\right)\frac{1}{2}\left(e^{\lambda_2} + e^{-\lambda_2}\right).
\end{aligned}$$

Obviously, $X_n = 0$ if and only if $Y_n = Z_n = 0$. Therefore,

$$\mathbf{P}\left(X_n = 0\right) = \mathbf{P}(Y_n = 0)\mathbf{P}(Z_n = 0) = 2^{-2n}\binom{n}{n/2}^2,$$

if n is even, and zero otherwise. ■

3. *(Recurrent random walk). Consider a r.w.* X_n *on* Z^d *starting at* 0*. Let* $T = \inf\{n \geq 1 : X_n = 0\}$ *(assuming* $\inf \varnothing = +\infty$*). Then*

$$\mathbf{P}(T < \infty) = 1 \text{ if and only if } \sum_n \mathbf{P}(X_n = 0) = +\infty.$$

Markov property implies that $P(T < \infty) = 1$ *if and only if* X_n *returns to zero infinitely many times a.s. (such a r.w. is called recurrent).*

Proof. Consider pieces of a trajectory of X_n between returns to 0. They can be considered as the results of independent trials with probability of the success

$$p = P(T = \infty).$$

Then $N =$"number of trials needed to reach a success (no return)" has a geometric distribution with parameter p. This distribution can be degenerated if $p = 0$ (In this case $N = \infty$ a.s. and X_n visits zero infinitely many times) or $p = 1$ (in this case $N = 1$ a.s. and X_n never returns to zero a.s.). In all cases, $\mathbf{E}N = \frac{1}{p}$. On the other hand,

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \mathbf{1}_{\{X_n=0\}}, \\ \mathbf{E}N &= \frac{1}{p} = \sum_{n=1}^{\infty} \mathbf{P}(X_n = 0). \end{aligned}$$

So, $\mathbf{P}(T < \infty) = 1 \iff p = 0 \iff \mathbf{E}N = 0 \iff \sum_{n=1}^{\infty} \mathbf{P}(X_n = 0) = \infty$. ■

4. *A simple random walk* X_n *on* Z^d *(*$d = 1, 2$*) starting at 0 is recurrent.*

Proof. By the properties 1, 2,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}(X_n = 0) &= \sum_{k=1}^{\infty} \mathbf{P}(X_{2k} = 0) = \sum_k 2^{-2dk} \binom{2k}{k}^d \\ &= \sum_k 2^{-2dk} \frac{((2k)!)^d}{(k!)^{2d}} \end{aligned}$$

Using Stirling formula ($n! \sim n^n e^{-n} \sqrt{2\pi n}$), for large k,

$$2^{-2dk} \frac{((2k)!)^d}{(k!)^{2d}} \geq \frac{1}{3} 2^{-2dk} \frac{(2k)^{2dk} e^{-2dk} (\sqrt{2\pi 2k})^d}{k^{2kd} e^{-2kd} (2\pi)^d k^d} = \frac{1}{3} \frac{1}{\pi^{d/2} k^{d/2}}.$$

Therefore for $d = 1, 2$,

$$\sum_{k=1}^{\infty} \mathbf{P}(X_{2k} = 0) = \infty,$$

and the r.w. is recurrent. ■

5. *(Average exit time). Assume* X_n *is a simple r.w. on* Z *starting at zero. Let*

$$\tau_{ab} = \inf\{n \geq 0 : X_n = a \text{ or } X_n = b\},$$

where $a, b \in Z$ and $a < 0, b > 0$. Then $E\tau_{ab} = |a|b$.
Proof. For $x \in [a,b] \cap \mathbf{Z}$ define $\tau(x) = \inf\{n \geq 0 : x + X_n = a \text{ or } x + X_n = b\}$ and $u(x) = \mathbf{E}\tau(x)$. Obviously, $u(a) = u(b) = 0$. For $x \in (a,b)$, (by total probability formula),

$$\begin{aligned} u(x) &= \frac{1}{2}\mathbf{E}\left[\tau(x)|X_1 = 1\right] + \frac{1}{2}\mathbf{E}\left[\tau(x)|X_1 = -1\right] \\ &= \frac{1}{2}\{1 + \mathbf{E}\left[\tau(x+1)\right]\} + \frac{1}{2}\{\mathbf{E}\left[\tau(x-1)\right] + 1\} = \\ &= 1 + \frac{1}{2}[u(x+1) + u(x-1)], \end{aligned}$$

or $u(x+1) + u(x-1) - 2u(x) = -2, a < x < b$. The left hand side is the second difference of $u(x)$ (think about the ODE $u''(x) = -2, u(a) = u(b) = 0$, whose solution is a quadratic function $u(x) = (x-a)(b-x)$). ■

Note a) Assume X_n is a simple r.w. on Z starting at zero and $\tau_c = \inf\{n \geq 0 : X_n = c\}$. Then for every $a < 0$ $(a \in Z)$ and every $b > 0$ $(b \in Z)$, the expected values $E\tau_a = E\tau_b = \infty$.

b) If X_n is a simple r.w. on Z starting at zero and $T = \inf\{n \geq 1 : Z_n = 0\}$, then $ET = \infty$.
Proof. a) Indeed, by **5.,** $E\tau_a = \lim_{b\to\infty} \mathbf{E}\tau_{ab} = \lim_{b\to\infty}(-ab) = \infty$. Similarly, $E\tau_b = \lim_{a\to-\infty} \mathbf{E}\tau_{ab} = \lim_{a\to-\infty}(-ab) = \infty$.

b) We have $ET = \frac{1}{2}E\left[T|X_1 = 1\right] + \frac{1}{2}E\left[T|X_1 = -1\right] = \frac{1}{2}\mathbf{E}\tau(1)_0 + \frac{1}{2}\mathbf{E}\tau(-1)_0 = \infty$, where $\tau(x)_c$ is the first moment to reach c starting at x. ■

Remark 4 *If X_n is a r.w. on $\mathbf{Z}$ starting at 0 and*

$$\mathbf{P}(U_1 = X_1 = 1) = p, \mathbf{P}(U_1 = X_1 = -1) = q = 1 - p,$$

then $X_n/n \to p - q$ a.s.: with probability 1 it never returns if $p \neq q$ and n is large.

2 Discrete time and space Markov chains

Denote $\mathbf{S}$ a finite or infinite but countable set.

Definition 5 *A sequence of $\mathbf{S}$-valued r. variables $Z_n, n \geq 0$, is called Markov chain, if the Markov property holds: for all $z_{n+1} \in \mathbf{S}$ and n,*

$$\mathbf{P}\left(Z_{n+1} = z_{n+1}|Z_{[0,n]}\right) = \mathbf{P}\left(Z_{n+1} = z_{n+1}|Z_n\right)$$

,where $Z_{[0,n]} = (Z_0, \dots Z_n)$. Equivalently,

$$\mathbf{P}\left(Z_{n+1} = z_{n+1}|Z_{[0,n]} = z_{[0,n]}\right) = \mathbf{P}\left(Z_{n+1} = z_{n+1}|Z_n = z_n, \dots, Z_0 = z_0\right) = \mathbf{P}\left(Z_{n+1} = z_{n+1}|Z_n = z_n\right)$$

for all $z_0, \dots, z_{n+1} \in \mathbf{S}$, where a notation $z_{[0,n]} = (z_0, \dots, z_n)$ is used.

A Markov chain is completely described by its initial distribution $\pi^0(z) = \mathbf{P}\left(Z_0 = z\right), z \in \mathbf{S}$, and the functions defined in the following statement.

Theorem 6 *For every $n = 0, 1, \ldots$ there exists a function $p_{n+1}(c, z)$, $c, z \in S$, taking values in $[0, 1]$ such that*

$$\begin{aligned} \sum_{z \in \mathbf{S}} p_{n+1}(c, z) &= 1 \text{ for all } c \in \mathbf{S}, \\ & \quad \text{and} \\ \mathbf{P}\left(Z_{n+1} = z, Z_n = c\right) &= p_{n+1}(c, z)\mathbf{P}\left(Z_n = c\right) \text{ for all } c, z \in \mathbf{S}. \end{aligned} \tag{2}$$

If $\mathbf{P}\left(Z_n = c\right) > 0$, then ***necessarily*** $p_{n+1}(c, z) = \mathbf{P}\left(Z_{n+1} = z | Z_n = c\right), z \in \mathbf{S}$.

Proof. If $\mathbf{P}\left(Z_n = c\right) > 0$, we set

$$p_{n+1}(c, z) = \mathbf{P}\left(Z_{n+1} = z, | Z_n = c\right), \; z \in \mathbf{S}.$$

Obviously, $\sum_{z \in \mathbf{S}} p_{n+1}(c, z) = 1$ and

$$\mathbf{P}\left(Z_{n+1} = z | Z_n = c\right) = p_{n+1}(c, z)\mathbf{P}\left(Z_n = c\right) \text{ for all } z \in \mathbf{S}, \tag{3}$$

in this case.

If $\mathbf{P}\left(Z_n = c\right) = 0$, there infinitely many $p_{n+1}(c, z), z \in \mathbf{S}$, satisfying (3) and, for example, we could go with $p_{n+1}(c, z) = \pi^0(z) = \mathbf{P}\left(Z_0 = z\right), z \in \mathbf{S}$. ∎

Definition 7 *Any function $p_{n+1}(c, z), c, z \in \mathbf{S}$, satisfying (2) is called an nth transition function of the Markov chain Z_n.*

Denote by $\pi^n(z) = \mathbf{P}\left(Z_n = z\right), z \in \mathbf{S}$, the distribution of Z_n, $n = 0, 1, 2, \ldots$. All one dimensional and multidimensional distributions of Z_n are completely determined by $\pi^0(z)$ and transition functions $p_{n+1}(c, z), c, z \in \mathbf{S}, n = 0, 1, \ldots$. The following statement holds.

Theorem 8 *a) (Kolmogorov equation) For every $n = 0, 1, \ldots$,*

$$\pi^{n+1}(z) = \sum_{c \in \mathbf{S}} p_{n+1}(c, z)\pi^n(c), z \in \mathbf{S};$$

b) For every n and $z_n, \ldots, z_0 \in \mathbf{S}$,

$$\mathbf{P}\left(Z_n = z_n, \ldots, Z_0 = z_0\right) = p_n(z_{n-1}, z_n) \ldots p_1(z_0, z_1)\pi^0(z_0);$$

More general, for $n > m \geq 0$ and $z_n, z_{n-1}, \ldots, z_m \in \mathbf{S}$,

$$\mathbf{P}\left(Z_n = z_n, \ldots, Z_m = z_m\right) = p_n(z_{n-1}, z_n) \ldots p_{m+1}(z_m, z_{m+1})\pi^m(z_m).$$

Proof. a) By the formula of total probability,

$$\begin{aligned} \pi^{n+1}(z) &= \mathbf{P}\left(Z_{n+1} = z\right) = \sum_{c \in \mathbf{S}} \mathbf{P}\left(Z_{n+1} = z, Z_n = c\right) \\ &= \sum_{c \in s} p_{n+1}(c, z)\mathbf{P}(Z_n = c). \end{aligned}$$

b) Joint probability by conditioning combined with Markov property,

$$\begin{aligned}
& \mathbf{P}(Z_n = z_n, \ldots, Z_m = z_m) \\
= \;& \mathbf{P}(Z_n = z_n | Z_{n-1} = z_{n-1}, \ldots, Z_m = z_m) \ldots \mathbf{P}(Z_{m+1} = z_{m+1} | Z_m = z_m) \mathbf{P}(Z_m = z_m) \\
= \;& p_n(z_{n-1}, z_n) p_{n-1}(z_{n-2}, z_{n-1}) \ldots p_{m+1}(z_m, z_{m+1}) \pi^m(z_m)
\end{aligned}$$

■

Definition 9 *Markov chain Z_n is called time homogeneous if $p_n(c,z) = p(c,z)$ for $n \geq 1, c, z \in \mathbf{S}$ (transition probabilities do not depend on n).*

We can rewrite Theorem 8 for time-homogeneous walks.

Theorem 10 *Assume Z_n is time homogeneous with transition probability $p(c,z)$. Then*

a) (Kolmogorov equation) for every $n = 0, 1, \ldots,$

$$\pi^{n+1}(z) = \sum_{c \in \mathbf{S}} p(c,z) \pi^n(c), z \in \mathbf{S}; \tag{4}$$

b) For every n and $z_n, \ldots, z_0 \in \mathbf{S}$,

$$\mathbf{P}(Z_n = z_n, \ldots, Z_0 = z_0) = p(z_{n-1}, z_n) \ldots p(z_0, z_1) \pi^0(z_0);$$

More general, for $n > m \geq 0$ and $z_n, z_{n-1}, \ldots, z_m \in \mathbf{S}$,

$$\mathbf{P}(Z_n = z_n, \ldots, Z_m = z_m) = p(z_{n-1}, z_n) \ldots p(z_m, z_{m+1}) \pi^m(z_m) \tag{5}$$

Applying Theorem 10 a) repeatedly, we obtain immediately

Corollary 11 *Assume Z_n is homogeneous with transition probability $p(c,z)$. Then for every $n \geq 1$,*

$$\pi^n(z) = \sum_{c \in \mathbf{S}} p(c,z) \pi^{n-1}(c) = \sum_{c \in \mathbf{S}} p^{*n}(c,z) \pi^0(c), z \in \mathbf{S},$$

where

$$\begin{aligned}
p^{*n}(c,z) &= \sum_{w \in \mathbf{S}} p(c,w) p^{*(n-1)}(w,z) = \sum_{z_1, \ldots, z_{n-1} \in \mathbf{S}} p(c,z_1) p(z_1,z_2) \ldots p(z_{n-2}, z_{n-1}) p(z_{n-1}, z) \\
&= \sum_{w \in \mathbf{S}} p^{*(n-1)}(c,w) p(w,z), c, z \in \mathbf{S}.
\end{aligned}$$

The function $p^{*n}(c,z)$ is called n-step transition function: for $m \geq 0$ and $\mathbf{P}(Z_m = c) > 0$, we have

$$\mathbf{P}(Z_{n+m} = z | Z_m = c) = p^{*n}(c,z)$$

and for every $z \in \mathbf{S}$,

$$\pi^{n+m}(z) = \sum_{c \in \mathbf{S}} p^{*n}(c,z) \pi^m(c)$$

(total probability formula).

Remark 12 *If* $\mathbf{S} = \{s_1, \ldots, s_m\}$ *is finite, then we can look at* $p(s_i, s_j) = p_{ij}, i = 1, \ldots, m,$ *as a* $m \times m$ *matrix* $P = (p_{ij})_{1 \le i,j \le m}$. *If* $\mathbf{S} = \{s_1, s_2, \ldots\}$ *is infinite countable set, then we can look at* $p(s_i, s_j) = p_{ij}, i = 1, \ldots$ *as an infinite dimensional matrix* $P = (p_{ij})_{i,j \ge 1}$. *The probability distribution* $\pi^n(s_i) = \pi_i^n$ *can be regarded as finite or infinite dimensional column vector* $\pi^n = (\pi_i^n)$. *Following this point of view, we can summarize Theorem 8 and Corollary 11 as follows:*

a) For all $n \ge 1$,

$$\pi^n = \pi^{n-1} P = \pi^0 P^n,$$

where $P^n = P \cdot \ldots \cdot P$ *is the nth power of the matrix* P*;*

b) For every $n > m \ge 0$ *and* $\mathbf{P}(Z_m = s_i) = \pi_i^m > 0$, *we have*

$$\mathbf{P}(Z_n = s_j | Z_m = s_i) = \left(P^{n-m}\right)_{ij},$$

where $(P^{n-m})_{ij}$ *is the "ij"th entry of the matrix* P^{n-m}*;*

c) For every $n > m \ge 0$, *we have*

$$\pi^n = \pi^m P^{n-m}.$$

2.1 Invariant (equilibrium) distribution

Sometimes it is more practical to approximate π^n for large n by its limit π as $n \to \infty$, provided such a limit exists. Since $\pi^n = \pi^{n-1}P$ such a limit must satisfy the equation

$$\pi = \pi P.$$

Definition 13 *A probability distribution $\pi(z), z \in \mathbf{S}$, is called invariant or equilibrium distribution for Z_n, if it satisfies the equation*

$$\pi(z) = \sum_{c \in \mathbf{S}} p(c, z)\pi(c).$$

Note, that if π^0 is invariant, then $\pi^n = \pi^0$ for all $n \geq 0$. In this case Z_n is discrete time stationary process. It turns out that the invariant distribution and the limit above exists in very many situations.

Theorem 14 *Assume $\mathbf{S} = \{s_1, \dots s_d\}$ and there is $m \geq 1$ such that all entries of the matrix P^m (see Remark 12) are strictly positive.*

Then there is a unique invariant distribution $\pi = (\pi_1, \dots \pi_d)$ and for all $i, j = 1, \dots, d$,

$$\lim_{n\to\infty} \pi_j^n = \pi_j, \lim_{n\to\infty} (P^n)_{ij} = \pi_j.$$

Proof. Let $Q = P^m = (q_{ij})$, and $q_{ij} \geq \delta > 0$ for all i, j. Then for an arbitrary vector $\nu = (\nu_1, \dots, \nu_d)$ such that $\sum_i \nu_i = 0$, we have

$$\begin{aligned} \sum_i |(\nu Q)_i| &= \sum_i |\sum_j \nu_j q_{ji}| \\ &= \sum_i |\sum_j \nu_j (q_{ji} - \delta)| \\ &\leq \sum_j \sum_i |\nu_j| (q_{ji} - \delta)| = (1 - d\delta) \sum_j |\nu_j|. \end{aligned} \tag{6}$$

Uniqueness. Assume π and μ are both invariant probability distributions on $\mathbf{S}$. Then $\pi - \mu = (\pi - \mu)\,Q$, and by (6) it follows

$$\sum_i |\mu_i - \pi_i| \leq (1 - \delta d) \sum_i |\mu_i - \pi_i|.$$

So, $\mu_i = \pi_i$ for all i.

Existence. For **any initial distribution** π^0 and $l = 0, \dots, m-1$, we have (Remark 12) for all $k \geq 1$,

$$\pi^{(k+1)m+l} = \pi^{km+l} Q. \tag{7}$$

By (6), for all $k \geq 1$,

$$\sum_i |\pi_i^{(k+1)m+l} - \pi_i^{km+l}| \leq (1 - \delta d) \sum_i |\pi_i^{km+l} - \pi_i^{(k-1)m+l}| \leq 2(1 - \delta d)^k.$$

Therefore the limit $\pi(n) = \lim_{k\to\infty} \pi^{km+l}$ exists and passing to the limit in (7) we see that $\pi(n)$ is invariant. In fact $\pi(n) = \pi$ does not depend on n because of the uniqueness of invariant distribution. Since $l = 0, \dots, m-1$ is arbitrary, $\lim_{n\to\infty} \pi^n = \pi$. Initial distribution can be arbitrary as well and taking $\pi^0 = (\delta_{ij})_{1\le i\le d}$, we obtain $\pi_j^n = (P^n)_{ij} \to \pi_j$ for every $i = 1, \dots, d$. ■

Remark 15 *Random walks Z_n on $\mathbf{Z}^d$ which are non trivial in the sense that $\mathbf{P}(Z_1 \ne Z_0) > 0$, do not have invariant distributions.*

Indeed, if an invariant distribution exists, then for the r.w. with invariant starting state we would have (using independence of increments),

$$\begin{aligned}\mathbf{E}\exp\{i\lambda Z_0\} &= \mathbf{E}\exp\{i\lambda Z_1\} = \mathbf{E}\exp\{i\lambda(Z_1 - Z_0) + i\lambda Z_0\} \\ &= \mathbf{E}\exp\{i\lambda Z_0\}\,\mathbf{E}\exp\{i\lambda(Z_1 - Z_0)\}\end{aligned}$$

which is a contradiction.

The following statement shows that finite state Markov chains always have an invariant distribution.

Proposition 16 *Let $\mathbf{S} = \{s_1, \dots s_d\}$ and Z_n be time homogeneous M. chain on $\mathbf{S}$. Then it has an invariant distribution.*

Proof. The matrix $A = P - I = (P_{ij} - \delta_{ij})$ has linearly dependent columns (their sum is 0). Therefore the rows are linearly independent as well. So, there is $\lambda = (\lambda_1, \dots \lambda_d) \ne 0$ such that

$$\sum_i \lambda_i a_{ij} = 0 \text{ for all } j$$

or, equivalently,

$$\sum_i \lambda_i p_{ij} = \lambda_j \text{ for all } j.$$

Let $\pi_i = |\lambda_i| / \sum_j |\lambda_j|$. Then

$$\pi_j \le \sum_i \pi_i P_{ij} \text{ for all } j,$$

and

$$1 = \sum_j \pi_j \le \sum_{i,j} \pi_i P_{ij} = 1,$$

which implies that

$$\sum_j \left(\sum_i \pi_i P_{ij} - \pi_j \right) = 0$$

or $\sum_i \pi_i P_{ij} - \pi_j = 0$ for all j : $\pi = (\pi_j)$ is invariant distribution. ■