1. Consider an HMM representation of coin tossing experiment. Assume a three-state model (corresponding to three different coins) with probabilities

	State 1	State 2	State 3
$P(H)$	0.75	0.5	0.25
$P(T)$	0.25	0.5	0.75

and with all state-transition probabilities equal to 1/3. Assume initial state probabilities of 1/3.The sequence $Y_{[0,9]} = HHHHTHTTTT$ was observed. Use Viterbi algorithm to find the optimal state path.

2. Assume the signal X_n is a simple reflected r.w. on $A = [0, 20] \cap \mathbf{Z}$ starting at 10. Let $W_0, W_1, \ldots$ be a sequence of i.i.d. taking two values with prob. 1/2:

$$\mathbf{P}\left(W_1 = \pm L\right) = 1/2,$$

The observation Y_n is defined by

$$Y_n = \min\left\{\max\left\{X_n + W_n, 0\right\}, 20\right\}.$$

(It is hidden Markov model with $\gamma(x, w) = \min\left\{\max\left\{x + w, 0\right\}, 20\right\}$).

Simulate $X_n, Y_n, 0 \le n \le 400$, for $L = 10, 14, 16, 17$, and find two optimal path estimates $X_n^{*\max}$ and $X_n^{*\min}$ of X_n by choosing maximal and minimal solution of the maximization problem in every Viterbi algorithm step.

Plot on the same graph $X_n, X_n^{*\max}$, $X_n^{*\min}$, $\frac{1}{2}\left(X_n^{*\max} + X_n^{*\min}\right)$ and the smoothing estimate $\hat{X}_{n,400}, 0 \le n \le 400$.

Hint. The transition prob. function is

$$p(r, u) = \begin{cases} 1/2, & \text{if } u = \min\{r+1, 20\}, 0 \le r \le 20, \\ 1/2, & \text{if } u = \max\{r-1, 0\}, 0 \le r \le 20, \\ 0, & \text{otherwise.} \end{cases}$$

The function $l(u, v) = \mathbf{P}\left(\gamma(u, W_0) = v\right)$, is given by

$$l(u, v) = \begin{cases} 1/2, & \text{if } v = \max\{u - L, 0\}, 0 \le u \le 20, \\ 1/2, & \text{if } v = \min\{u + L, 20\}, 0 \le u \le 20, \\ 0, & \text{otherwise.} \end{cases}$$

We construct a sequence of Viterbi value functions: $\delta_n(a) = \delta_n(a, Y_{[0,n]})$:

1. Initialization: $\delta_0(a) = 1_{10=a} l(10, Y_0), a \in [0, 20]$;

2. Recursion:

$$\begin{aligned} \delta_{n+1}(a) &= l(a, Y_{n+1}) \max_r p(r, a)\delta_n(r) \\ &= \frac{1}{4}\left(1_{Y_{n+1}=\max\{a-L,0\}} + 1_{Y_{n+1}=\min\{a+L,20\}}\right) \max\left\{\delta_n(\min\{a+1, 20\}), \delta_n(\max\{a-1, 0\})\right\}, \end{aligned}$$

$a \in [0, 20]$.

Find $V_n^{\min}(a)$, the minimal r at which $\max_r p(r,a)\delta_n(r) = \frac{1}{2}\max\left\{\delta_n(\min\{a+1, 20\}), \delta_n(\max\{a-1, 0\})\right.$ is achieved, $a \in [0, 20]$;

Find $V_n^{\max}(a)$, the maximal r at which $\max_r p(r,a)\delta_n(r) = \frac{1}{2}\max\left\{\delta_n(\min\{a+1, 20\}), \delta_n(\max\{a-1, 0\}\right.$ is achieved, $a \in [0, 20]$;

Note $V_n^{\min}(a)$ and $V_n^{\max}(a), a \in [0, 20]$, are not sets but functions/vectors.

3. Termination. Find $\delta_{400}(a), a \in A$;

Find $X_{400}^{*\min}$ the minimal r at which $\max_r \delta_{400}(r)$ is achieved;

Find $X_{400}^{*\max}$ the maximal r at which $\max_r \delta_{400}(r)$ is achieved;

4. Backtracking of two optimal trajectories:

$X_{399}^{*\min} = V_{399}^{\min}(X_{400}^{*\min}), \ldots, X_n^{*\min} = V_n^{\min}(X_{n+1}^{*\min})$;

$X_{399}^{*\max} = V_{399}^{\max}(X_{400}^{*\max}), \ldots, X_n^{*\max} = V_n^{\max}(X_{n+1}^{*\max})$;