**Parameter estimation as a filtering problem.** Consider the following model: $X = \theta$ is an unknown constant, $W_n, n \geq 1$, is a sequence of i.i.d. Assume that

$$Y_{n+1} = f\left(X\right) Y_n + \sigma W_{n+1}, n \geq 0,$$

where $\sigma$ is a constant and $f$ is a function, $Y_0 = 0$. We do not know $\theta$ exactly but we usually have some idea about the area where it might be. Let us interpret this knowledge by assuming that $X_n = X = \theta$, $n \geq 0$, is a homogeneous Markov chain independent of $W_n$, $n \geq 1$, and a given initial distribution $\pi_0$. For example, assume that $X$ takes values in $A = (0, 1, 2, 3, 4, 5)$ with probabilities $1/6$ (it is the so called prior distribution for $X$).

a) Assume $\sigma = 2, f(\theta) = \theta$, and $\mathbf{P}\left(W_1 = \pm 1\right) = 0.5$. Select any of fixed $a_0 \in S$. Simulate $Y_n, 0 \leq n \leq 10$. Compute, in a form of a table, the posterior distribution

$$\mathbf{P}(X_0 = a | Y_{[0,n]}) = \mathbf{P}(X_n = a | Y_{[0,n]}) = \pi_n^a, a \in A, 0 \leq n \leq 10.$$

b) Assume $\sigma = 1, f(\theta) = \theta, W_1 \sim N(0, 1)$. Select any of fixed $a_0 \in S$. Simulate $Y_n, 0 \leq n \leq 10$. Compute, in a form of a table, the posterior distribution

$$\mathbf{P}(X_0 = a | Y_{[0,n]}) = \mathbf{P}(X_n = a | Y_{[0,n]}) = \pi_n^a, a \in A, 0 \leq n \leq 10.$$

**Hint**. $\mathbf{P}\left(X_0 = \theta | Y_0\right) = P\left(X = \theta\right), \theta \in A$ : in other words, $\mathbf{P}\left(X = \theta | Y_0\right)$ coincides with the prior distribution of $X$.

In the case b), show that the joint density functions $\phi_n^a\left(b_{[0,n]}\right)$of $Y_{[0,n]}$ and $X_n$ satisfy the recursive equations

$$\phi_{n+1}^a\left(b_{[0,n+1]}\right) = \rho\left(b_{n+1} - f(a)\, Y_n\right) \phi_n^a\left(b_{[0,n]}\right),$$

where $\rho$ is the pdf of a standard normal.