n03_mcs

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1 Introduction to Monte Carlo Simulation in Finance

1.1 Example: A Crude Simulation of a Call Option Price under the Black-Scholes Model

Ref.: *Don L. McLeish* **Monte Carlo Simulation and Finance** Wiley

```
In [79]: %matplotlib inline

         import numpy as np
         import scipy as sc

         from math        import sqrt
         from scipy.stats import norm as scnorm
         from pylab       import *
         from matplotlib  import pyplot as pl
```

It's worth to recast the pricing problem into a simple integral formulation in order to gain some insight into the general problem;

So let's consider again the payoff of a simple plain vanilla option

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}[h(S_T)] = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[h\left(S_0 e^{\log(S_T/S_0)}\right)\right]$$

By a simple application of Ito's lemma is easy to demonstrate that the variable $X = \log(S_T/S_0)$ has a normal distribution with mean $m = (r - \frac{1}{2}\sigma^2)T$ and variance $s = \sigma^2 T$.

So we can write

$$C(S,t) = e^{-rT}\int\limits_{-\infty}^{+\infty} \max[S_0 e^X - K, 0] e^{-\frac{(X-m)^2}{2s^2}} dX$$

It is possible to generate a normally distributed random variable $X = \Phi^{-1}(U; (r - \frac{1}{2}\sigma^2)T; \sigma^2 T)$ using the inverse transform method, where $\Phi^{-1}(U; (r - \frac{1}{2}\sigma^2)T; \sigma^2 T)$ is the inverse of the normal cumulative distribution function evaluated at U, a uniform $[0, 1]$ random variable.

$$U = \Phi[X; m, u], \quad u \to 1\, when\, X \to +\infty, \quad u \to 0\, when\, X \to -\infty$$

has a uniform distribution;

From the previous relation we find (within a normalization factor)

$$du = \frac{d\Phi[X; m, u]}{dX} dX \Rightarrow dX = \frac{1}{e^{-\frac{(X-m)^2}{2s^2}}} du$$

Then the value of the option can be written as an expectation over the distribution of the uniform random variabile U:

$$C(S,t) = \int_0^1 f(u)du$$

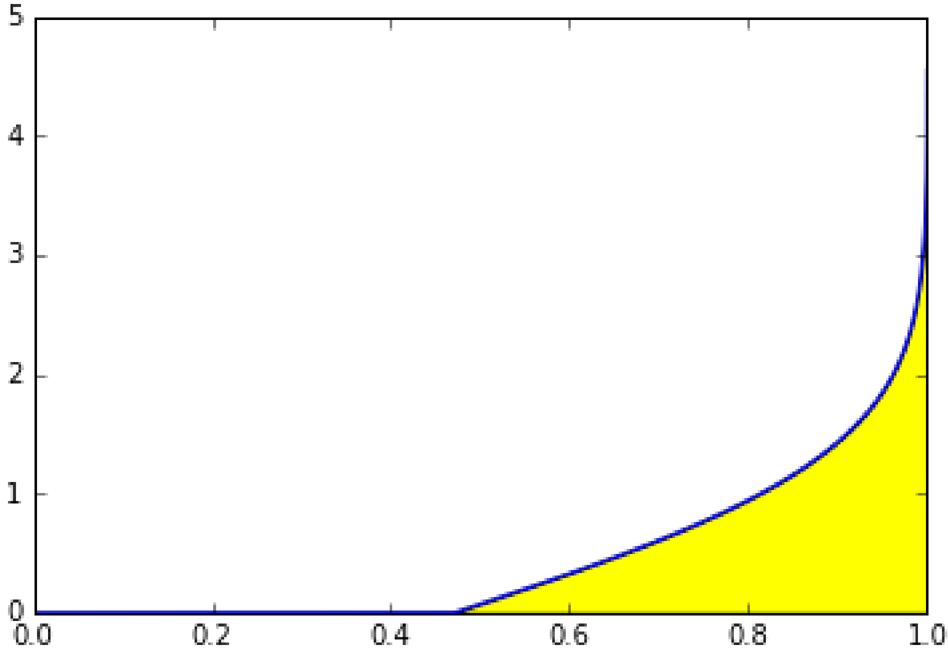
where \$f(u) = e^{-rT} max $[S\_0 \exp(\Phi\{-1\}(u; m, s)) - K, 0]$\$

```
In [80]: def f(u, S0, K, r, sigma, T):
             m      = (r - .5*sigma*sigma)*T
             s      = sigma*sqrt(T)
             f_u    = exp(-r*T)*np.maximum(S0*exp(scnorm.ppf(u, m, s))-K,0)
             return f_u
             #
             # this is the same code in R language
             #
             #x = S0*exp(qnorm(u,mean=r*T-sigma^2*T/2,sd=sigma*sqrt(T)))
             #v = exp(-r*T) * pmax((x-K),0)
```

where scnorm.ppf is the inverse of cumulative normal
We recall the Black and Scholes formulas from the pricing of Call and Put Options...
$C(S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$
$P(S(t)) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1)$
where
$d_1 = \frac{\log(S(t)/K)+(r+\sigma^2/2}{\sigma\sqrt{T}}$
$d_2 = d_1 - \sigma\sqrt{T}$

```
In [81]: import numpy as np
         import scipy.stats as ss
         import time

         #Black and Scholes
         def d1(S0, K, r, sigma, T):
             return (np.log(S0/float(K)) + (r + sigma**2 / 2.0) * T)/ float(sigma * np.sqrt(T))

         def d2(S0, K, r, sigma, T):
             return (np.log(S0/float(K)) + (r - sigma**2 / 2.0) * T) / float(sigma * np.sqrt(T))

         def BlackScholes(payoff, S0, K, r, sigma, T):
             if payoff == 1:
                 return S0 * ss.norm.cdf(d1(S0, K, r, sigma, T)) - K * np.exp(-r * T) * ss.norm.cdf(d2(S
             else:
                 return K * np.exp(-r * T) * ss.norm.cdf(-d2(S0, K, r, sigma, T)) - S0 * ss.norm.cdf(-d
```

```
In [82]: S0     = 10
         K      = 10
         r      = 0.05
         T      = 0.25
         sigma  = 0.2

         u      = np.linspace(0, 1, 10000)
         f_u    = f(u,S0,K,r,sigma,T)

         pl.plot(u, f_u)
         pl.fill_between(u, f_u, facecolor='yellow', interpolate=True)
         pl.show()
```



1.1.1 Crude Montecarlo

```
In [83]: u       = rand(500000)
         f_u     = f(u,S0,K,r,sigma,T)
         mc      = np.mean(f_u)
         sf      = sqrt(np.var(f_u))
         se      = sf/sqrt(len(u))
         bs      = BlackScholes(1,S0,K,r,sigma, T)
         delta   = abs(mc-bs)

         print 'Montecarlo estimate : %f'%mc
         print 'Analytical result   : %f'%bs
         print 'Difference          : %f'%delta
         print 'Sample Standard Dev : %f'%sf
         print 'Standard Error      : %f'%se
```

```
Montecarlo estimate : 0.461095
Analytical result   : 0.461500
Difference          : 0.000404
Sample Standard Dev : 0.660562
Standard Error      : 0.000934
```

1.1.2 Antithetic Random Numbers

```
In [84]: u       = rand(1000000)
         f_u     = f(u,S0,K,r,sigma,T)
         mc      = np.mean(f_u)
         sf      = sqrt(np.var(f_u))
         var_cm  = np.var(f_u)
         se_cm   = sf/sqrt(len(u))
```

```
         bs     = BlackScholes(1,S0,K,r,sigma, T)

         print 'Montecarlo estimate : %f'%mc
         print 'Analytical result   : %f'%bs
         print 'Sample Standard Dev : %f'%sf
         print 'Sample Variance     : %f'%var_cm
         print 'Standard Error      : %f'%se_cm
```

```
Montecarlo estimate : 0.462105
Analytical result   : 0.461500
Sample Standard Dev : 0.661123
Sample Variance     : 0.437084
Standard Error      : 0.000661
```

```
In [85]: u      = rand(500000)
         f_u    = .5*(f(u,S0,K,r,sigma,T) + f(1-u,S0,K,r,sigma,T))
         mc     = mean(f_u)
         sf     = sqrt(np.var(f_u))
         var_an = np.var(f_u)
         se     = sf/sqrt(len(u))
         bs     = BlackScholes(1,S0,K,r,sigma, T)

         print 'Montecarlo estimate : %f'%mc
         print 'Analytical result   : %f'%bs
         print 'Sample Standard Dev : %f'%sf
         print 'Sample Variance     : %f'%var_an
         print 'Standard Error      : %f'%se

         print 'Efficiency Gain     : %f'%(var_cm/var_an)
```

```
Montecarlo estimate : 0.462120
Analytical result   : 0.461500
Sample Standard Dev : 0.334394
Sample Variance     : 0.111820
Standard Error      : 0.000473
Efficiency Gain     : 3.908828
```

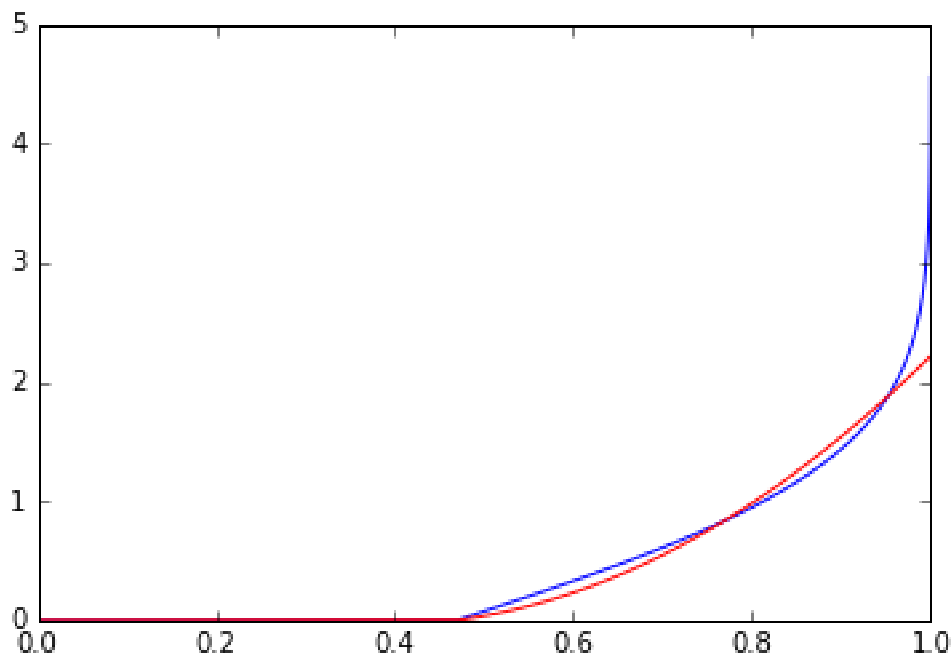
1.1.3 Stratified Sampling

```
In [86]: a      = 0.7
         f_u    = a*f(a*u,S0,K,r,sigma,T) + (1-a)*f(a+(1-a)*u,S0,K,r,sigma,T)
         mc     = mean(f_u)
         sf     = sqrt(np.var(f_u))
         var    = np.var(f_u)
         se     = sf/sqrt(len(u))
         bs     = BlackScholes(1,S0,K,r,sigma, T)

         print 'Montecarlo estimate : %f'%mc
         print 'Analytical result   : %f'%bs
         print 'Sample Standard Dev : %f'%sf
         print 'Sample Variance     : %f'%var
         print 'Standard Error      : %f'%se

         print 'Efficiency Gain     : %f'%(var_cm/var)
```

```
Montecarlo estimate : 0.461888
Analytical result   : 0.461500
Sample Standard Dev : 0.292259
Sample Variance     : 0.085415
Standard Error      : 0.000413
Efficiency Gain     : 5.117173
```

1.1.4 Control Variates

In [87]:
```
def g(u):
    return 6*(np.maximum(u-0.47,0))**2 + np.maximum(u-0.47,0)


S0     = 10
K      = 10
r      = 0.05
T      = 0.25
sigma  = 0.2

u      = np.linspace(0, 1, 10000)
f_u    = f(u,S0,K,r,sigma,T)
g_u    = g(u)
plot(u, f_u)
plot(u, g_u, color='red')
pl.show()
```



In [88]:
```
u      = rand(500000)
f_u    = f(u,S0,K,r,sigma,T)
g_u    = g(u)
```

```python
        intg   = 2*(0.53)**3 + 0.5*(0.53)**2
        mc     = intg + np.mean(f_u - g_u)
        var_1  = np.var(f_u)
        var_2  = np.var(f_u-g_u)
        bs     = BlackScholes(1,S0,K,r,sigma, T)

        print 'Montecarlo estimate : %f'%mc
        print 'Analytical result   : %f'%bs
        print 'Efficiency gain     : %f'%(var_1/var_2)
```

```
Montecarlo estimate : 0.461453
Analytical result   : 0.461500
Efficiency gain     : 30.301033
```