Introduction to Monte Carlo in Finance

2 - Dealing with Early Exercise Options

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Outline

Valuation of American Option by Simulation

- As we have seen Monte Carlo simulation is a flexible and powerful numerical method to value financial derivatives of any kind.
- However being a forward evolving technique, it is per se not suited to address the valuation of American or Bermudan options which are valued in general by backwards induction.
- Longstaff and Schwartz provide a numerically efficient method to resolve this problem by what they call Least-Squares Monte Carlo.
- The problem with Monte Carlo is that the decision to exercise an American option or not is dependent on the continuation value.

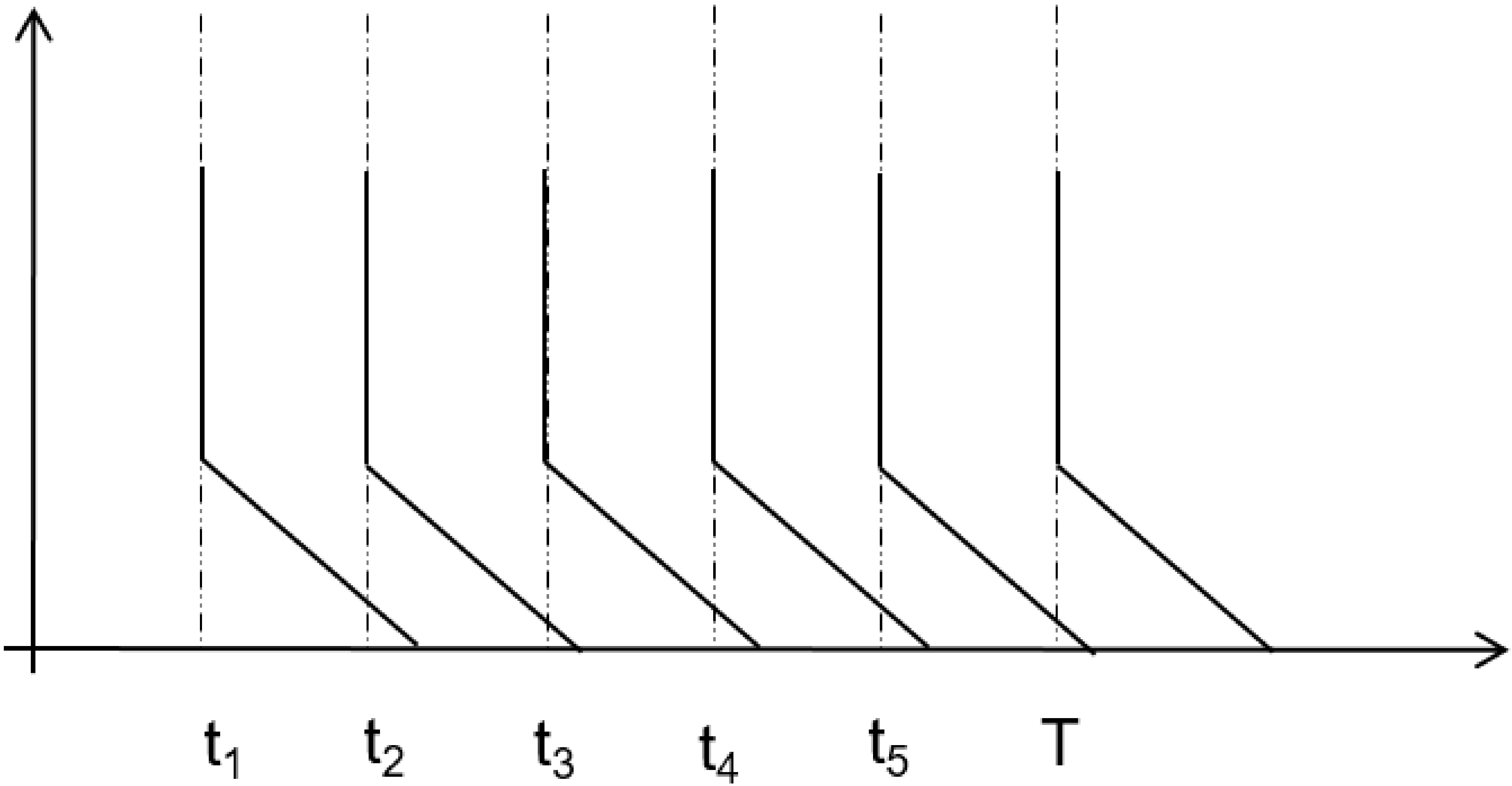
Valuation of American Option by Simulation

- Consider a simulation with $M + 1$ points in time and I paths.
- Given a simulated index level $S_{t,i}, t \in \{0, ..., T\}, i \in \{1, ..., I\}$, what is the continuation value $C_{t,i}(S_{t,i})$, i.e. the expected payoff of not exercising the option?
- The approach of Longstaff-Schwartz approximates continuation values for American options in the backwards steps by an ordinary least-squares regression.
- Equipped with such approximations, the option is exercised if the approximate continuation value is lower than the value of immediate exercise. Otherwise it is not exercised.

Valuation of American Option by Simulation

- In order to explain the metodology, let's start from a simpler problem.
- Consider a bermudan option which is similar to an american option, except that it can be early exercised once only on a specific set of dates.
- In the next figure, we can represent the schedule of a put bermudan option with strike K and maturity in 6 years. Each year you can choose whether to exercise or not ...

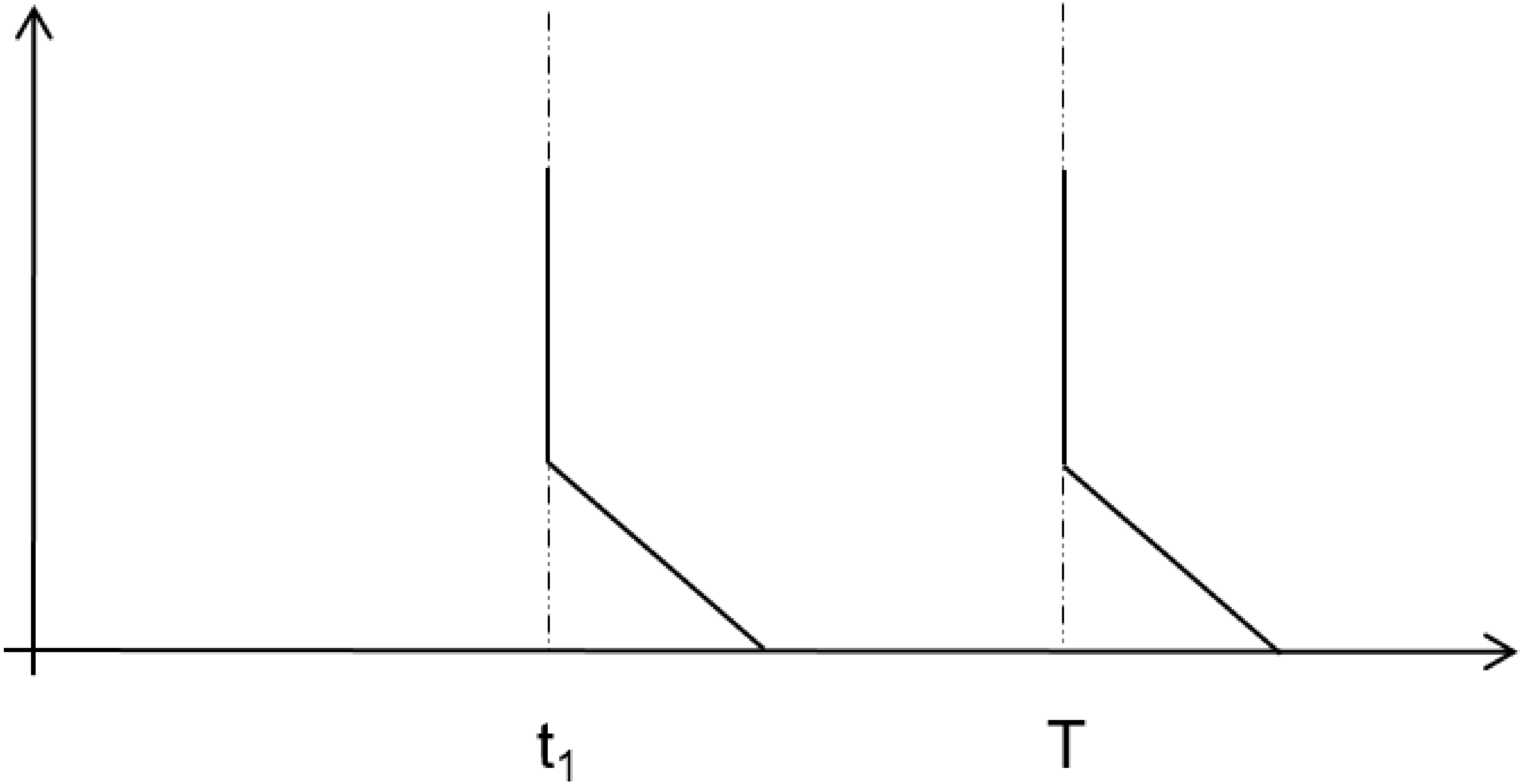
Valuation of American Option by Simulation



Valuation of American Option by Simulation

- Let's consider a simpler example: a put option which can be exercised early only once ...

Valuation of American Option by Simulation



Valuation of American Option by Simulation

- Can we price this product by means of a Monte Carlo? Yes we can! Let's see how.
- Let's implement a MC which actually simulates, besides the evolution of the market, what an investor holding this option would do (clearly an investor who lives in the risk neutral world). In the following example we will assume the following data, $S(T) =$, $K =$, $r =$, $\sigma =$, $t_1 = 1y$, $T = 2y$.
- We simulate that 1y has passed, computing the new value of the asset and the new value of the money market account

$$S(t_1 = 1y) = S(t_0)e^{(r-\frac{1}{2}\sigma^2)(t_1-t_0)+\sigma\sqrt{t_1-t_0}N(0,1)}$$

$$B(t_1 = 1y) = B(t_0)e^{r(t_1-t_0)}$$

Valuation of American Option by Simulation

- At this point the investor could exercise. How does he know if it is convenient?
- In case of exercise he knows exactly the payoff he's getting.
- In case he continues, he knows that it is the same of having a European Put Option.
- So, in mathematical terms we have the following payoff in t_1

$$\max\left[K - S(t_1), P(t_1, T; S(t_1), K)\right]$$

  where $P(t_1, T; S(t_1), K)$ is the price of a Put which we compute analytically!
- In the jargon of american products, P is called the continuation value, i.e. the value of holding the option instead of early exercising it.

Valuation of American Option by Simulation

- So the premium of the option is the average of this discounted payoff calculated in each iteration of the Monte Carlo procedure.

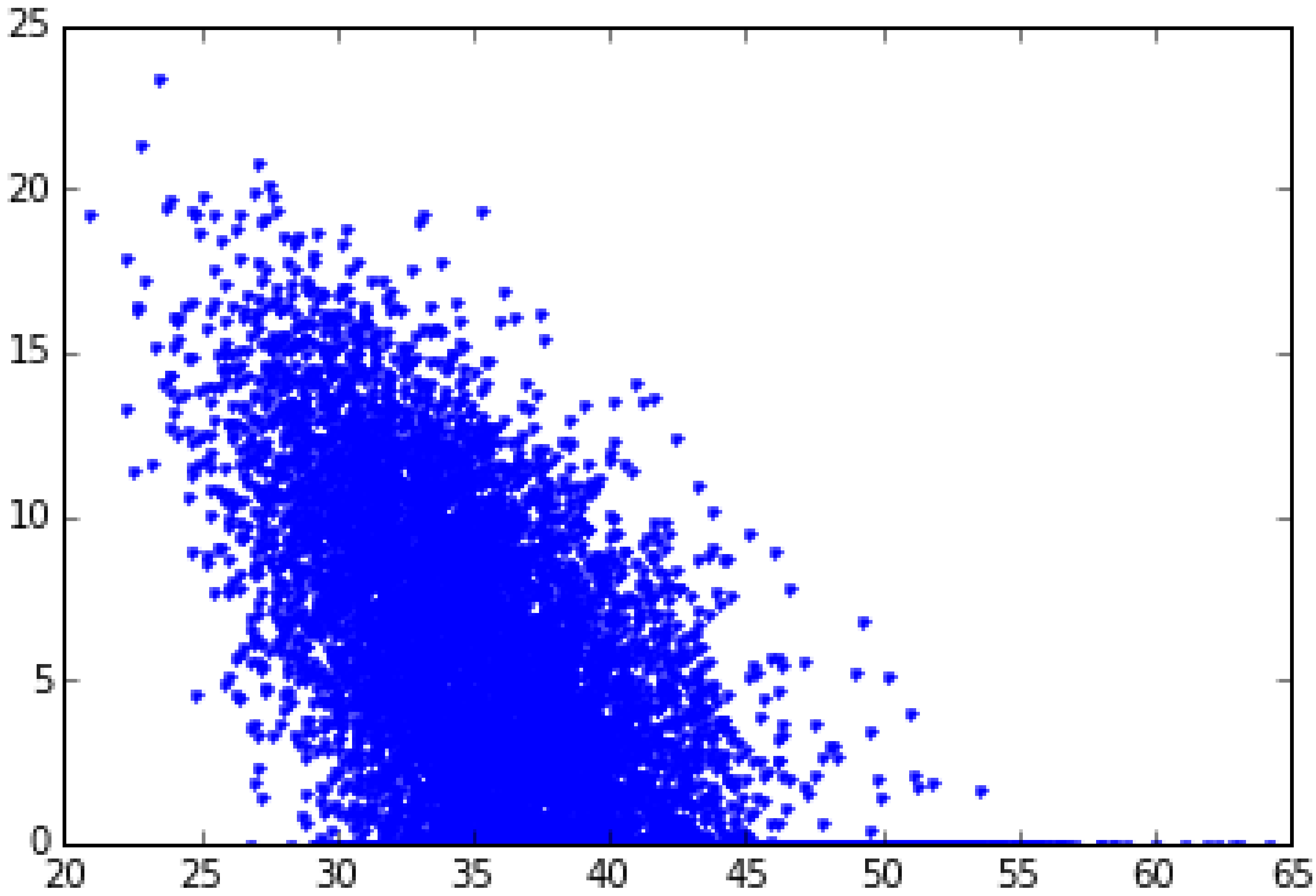
$$\frac{1}{N}\sum_{i}\max\left[K - S_i(t_1), P(t_1, T; S_i(t_1), K)\right]$$

- Some considerations are in order.
- We could have priced this product because we have an analytical pricing formula for the put. What if we didn't have it?

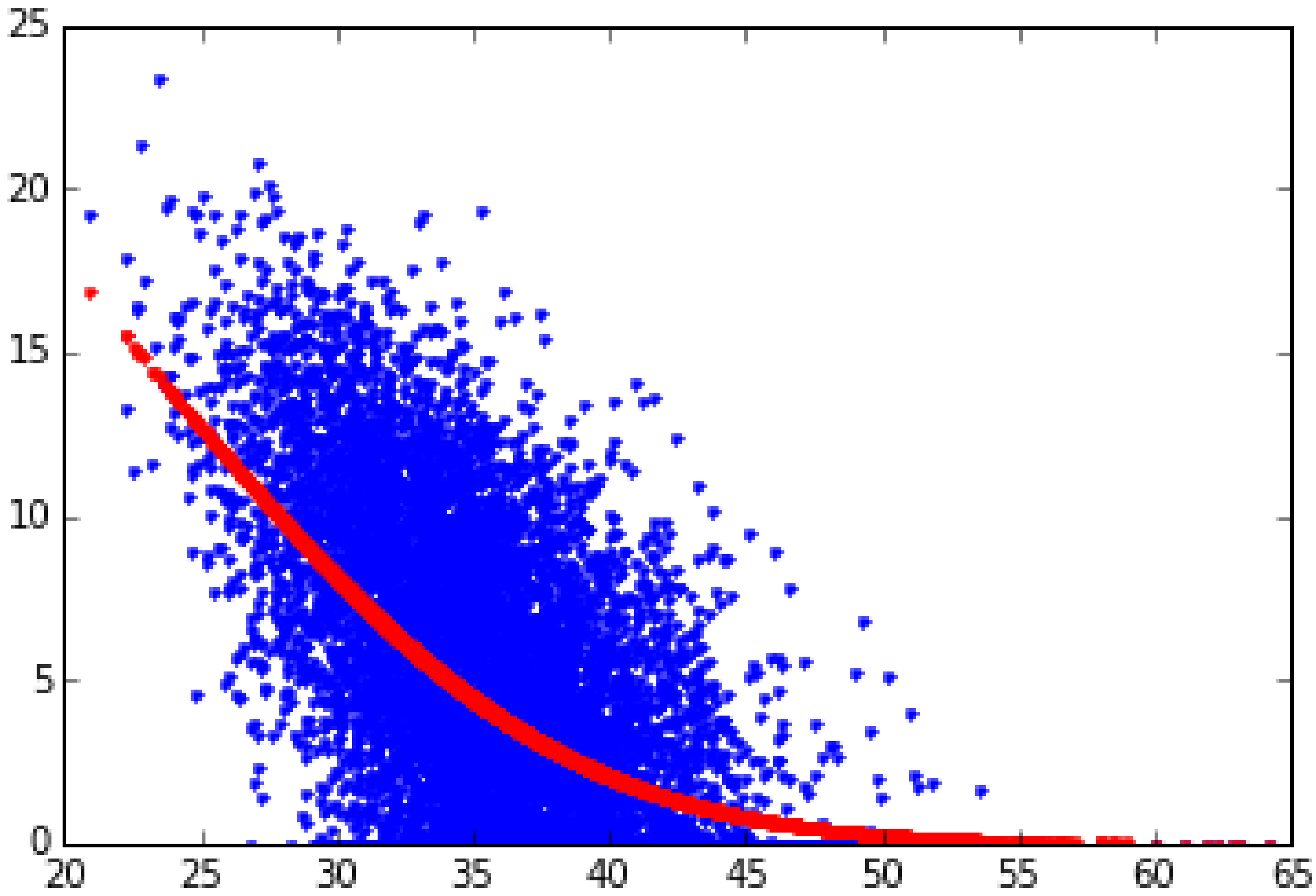
Valuation of American Option by Simulation

- Brute force solution: for each realization of $S(t_1)$ we run another Monte Carlo to price the put.
- This method (called Nested Monte Carlo) is very time consuming. For this very simple case it's time of execution grows as N^2, which becomes prohibitive when you deal with more than one exercise date!
- Let's search for a finer solution analyzing the relationship between the continuation value (in this very simple example) and the simulated realization of S at step t_1.
- let's plot the discounted payoff at maturity, P_i, versus $S_i(t_1)$ ...

Valuation of American Option by Simulation

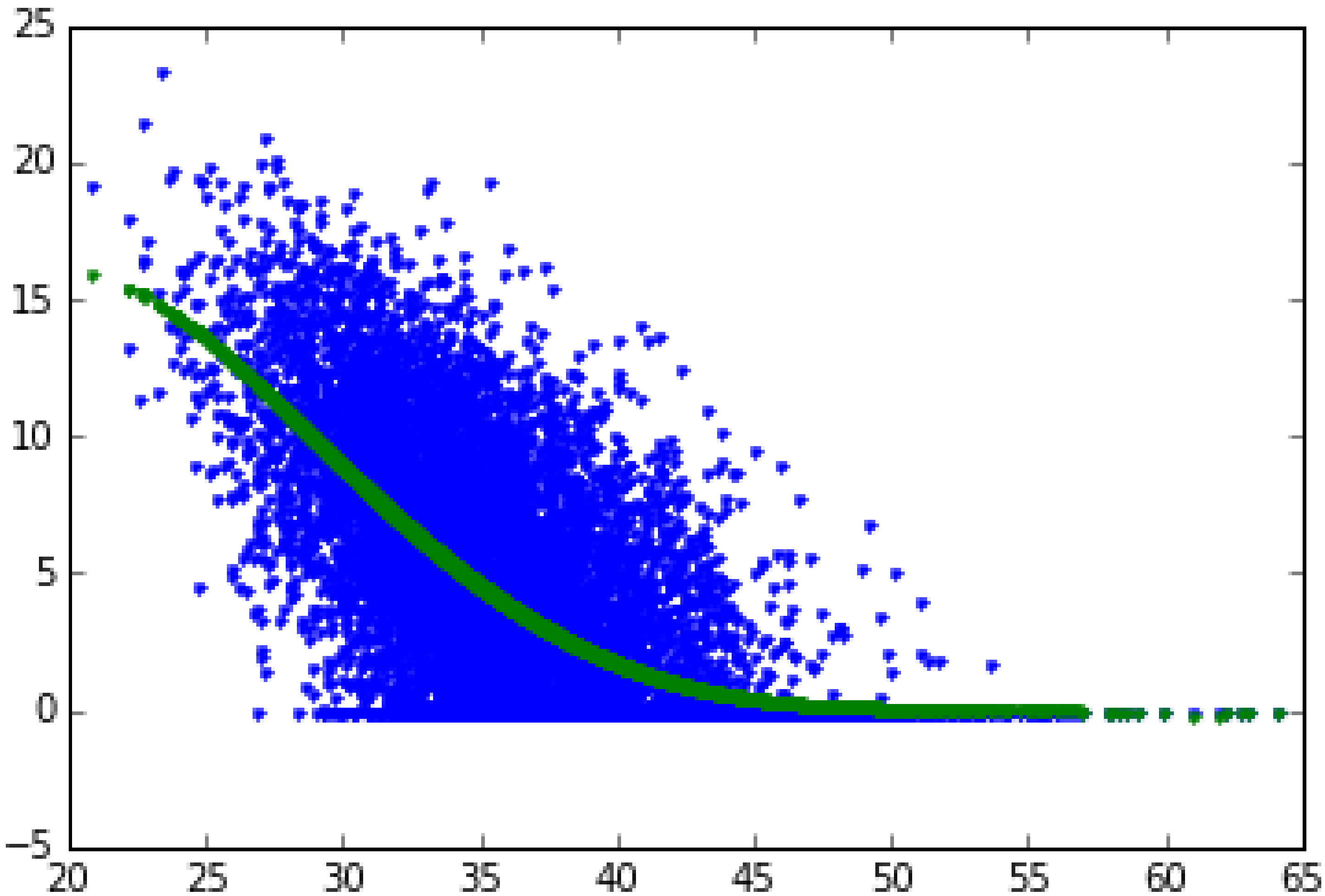


Valuation of American Option by Simulation

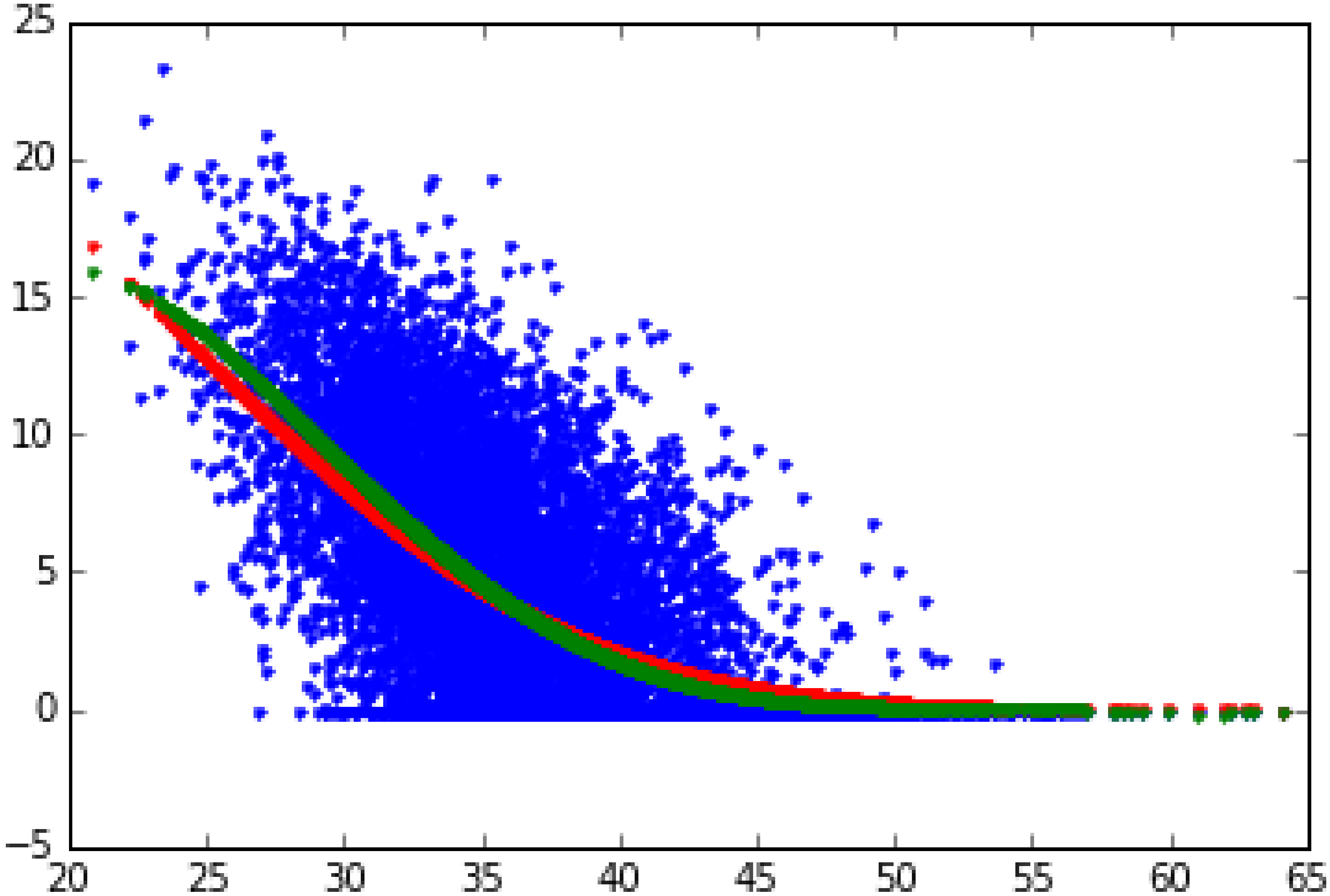


- As you can see, the analytical price of the put is a curve which kinds of interpolate the cloud of Monte Carlo points.
- This suggest us that the price at time t_1 can be computed by means of an average on all discounted payoff (i.e. the barycentre of the cloud made of discounted payoff)
- So maybe... the future value of an option can be seen as the problem of finding the curve that best fits the cloud of discounted payoff (up to date of interest)!!!
- In the next slide, for example, there is a curve found by means of a linear regression on a polynomial of 5th order...

Valuation of American Option by Simulation



Valuation of American Option by Simulation



Valuation of American Option by Simulation

- We now have an empirical pricing formula for the put to be used in my MCS

$$P(t_1, T, S(t_1), K) = c_0 + c_1 S(t_1) + c_2 S(t_1)^2 + c_3 S(t_1)^3 + c_4 S(t_1)^4 + c_5 S(t_1)^5$$

- The formula is obviously fast, the cost of the algorithm being the best fit.
- Please note that we could have used any form for the curve (not only a polynomial).
- This method has the advantage that it can be solved as a linear regression, which is fast.

The Longstaff-Schwartz Algorithm

- The major insight of Longstaff-Schwartz is to estimate the continuation value $C_{t,i}$ by ordinary least-squares regression, therefore the name "Least Square Monte Carlo" for their algorithm;
- They propose to regress the I continuation values $Y_{t,i}$ against the I simulated index levels $S_{t,i}$.
- Given D basis functions b with $b_1, \dots, b_D : \mathbb{R}^D \to \mathbb{R}$ for the regression, the continuation value $C_{t,i}$ is according to their approach approximated by:

The Longstaff-Schwartz Algorithm

- Given D basis functions b with $b_1, \ldots, b_D : \mathbb{R}^D \to \mathbb{R}$ for the regression, the continuation value $C_{t,i}$ is according to their approach approximated by:

$$\hat{C}_{t,i} = \sum_{d=1}^{D} \alpha^{\star}_{d,t} b_d(S_{t,i}) \quad (1)$$

- The optimal regression parameters $\alpha^{\star}_{d,t}$ are the result of the minimization

$$\min_{\alpha_{1,t},\ldots,\alpha_{D,t}} \frac{1}{I} \sum_{i=1}^{I} \left(Y_{t,i} - \sum_{d=1}^{D} \alpha_{d,t} b_d(S_{t,i}) \right)^2$$

The Longstaff-Schwartz Algorithm

- Simulate I index level paths with $M+1$ points in time leading to index level values $S_{t,i}$, $t \in \{0, ..., T\}$, $i \in \{1, ..., I\}$;
- For $t = T$ the option value is $V_{T,i} = h_T(S_{T,i})$ by arbitrage
- Start iterating backwards $t = T - \Delta t, \ldots, \Delta t$:
  - regress the $T_{t,i}$ against the $S_{t,i}$, $i \in \{1, \ldots, I\}$, given D basis function b
  - approximate $C_{t,i}$ by $\hat{C}_{t,i}$ according to (1) given the optimal parameters $\alpha^{\star}_{d,t}$ from (2)
  - set

$$V_{t,i} = \begin{cases} h_t(S_{t,i}) & \text{if } h_t(S_{t,i}) > \hat{C}_{t,i} \quad \text{exercise takes place} \\ Y_{t,i} & \text{if } h_t(S_{t,i}) \leq \hat{C}_{t,i} \quad \text{no exercise takes place} \end{cases}$$

  repeat iteration steps until $t = \Delta t$;
- for $t = 0$ calculate the LSM estimator

$$\hat{V}_0^{LSM} = e^{-r\Delta t} \frac{1}{I} \sum_{i=1}^{I} V_{\Delta t,i}$$

Notebook

- **GitHub :** polyhedron-gdl;
- **Notebook :** mcs_american;