Lecture Notes on Monte Carlo for Finance

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1 Ito isometry

In mathematics, the Itô isometry, named after Kiyoshi Itô, is a crucial fact about Itô stochastic integrals. One of its main applications is to enable the computation of variances for random variables that are given as Itô integrals.

Let$W : [0, T] \times \Omega \to \mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T > 0$, and let $X : [0, T] \times \Omega \to \mathbb{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then

$$\mathrm{E}\left[\left(\int_0^T X_t \, \mathrm{d}W_t\right)^2\right] = \mathrm{E}\left[\int_0^T X_t^2 \, \mathrm{d}t\right] \tag{1}$$

In other words, the Itô integral, as a function from the space $L^2_{\mathrm{ad}}([0, T] \times \Omega)$ of square-integrable adapted processes to the space $L^2(\Omega)$ of square-integrable random variables, is an isometry of normed vector spaces with respect to the norms induced by the inner products

$$(X, Y)_{L^2_{\mathrm{ad}}([0,T]\times\Omega)} := \mathrm{E}\left(\int_0^T X_t \, Y_t \, \mathrm{d}t\right) \tag{2}$$

and

$$(A, B)_{L^2(\Omega)} := \mathrm{E}(AB) \tag{3}$$

As a consequence, the Itô integral respects these inner products as well, i.e. we can write

$$\mathrm{E}\left[\left(\int_0^T X_t \, \mathrm{d}W_t\right)\left(\int_0^T Y_t \, \mathrm{d}W_t\right)\right] = \mathrm{E}\left[\int_0^T X_t Y_t \, \mathrm{d}t\right] \tag{4}$$

for $X, Y \in L^2_{\mathrm{ad}}([0, T] \times \Omega)$.

2 The Short-Rate Dynamics

We assume that the dynamics of the instantaneous short rate process under the risk-adjusted measure Q is given by:

$$r(t) = x(t) + \phi(t), \quad r(0) = r_0 \tag{5}$$

where the process $\{x(t) : \geq 0\}$ satisy Let's focus on the stochastic process

$$dx(t) = -ax(t) + \sigma dW(t), \quad x(0) = 0 \tag{6}$$

where a is a positive constant and W a one-dimensional brownian motion.The function ϕ is deterministic and well defined in the time interval $[0, T]$. In particular $\phi(0) = r_0$. We denote by $\mathcal{F}_t$ the sigma-field generated by the process x up to time t.

To integrate (6) for each $s < t$ we can start from

$$d[e^{av}x(v)] = e^{av}dx(v) + ae^{av}x(v)dt = -ae^{av}x(t) + e^{av}\sigma dW(t) + ae^{av}x(v)dt = e^{av}\sigma dW(t) \tag{7}$$

where we have used (6) to substitute $dx(v)$. Immediate integration give us

$$e^{at}x(t) - e^{as}x(s) = \sigma \int_s^t e^{au} \, dW(u) \Rightarrow x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} \, dW(u) \tag{8}$$

and

$$r(t) = x(s)e^{-a(t-s)} + \phi(t) + \sigma \int_s^t e^{-a(t-u)}\, dW(u) \tag{9}$$

meaning that $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance (remember Ito isometry!) given respectively by

$$E\{r(t)|\mathcal{F}_s\} = x(s)e^{-a(t-s)} + \phi(t) \tag{10}$$

$$Var\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right] \tag{11}$$

3 The Pricing of a Zero-Coupon Bond

We denote by $P(t,T)$ the price at time t of a zero-coupon bond maturing at T and with unit face value, so that

$$P(t,T) = E\left\{e^{-\int_t^T r_s ds}|\mathcal{F}_t\right\} \tag{12}$$

where E denotes the expectation under the risk-adjusted measure Q. In order to explicity compute this expectation we need the following

Lemma 3.1. *For each t, T the random variable*

$$I(t,T) = \int_t^T x(u)\, du$$

conditional to the sigma-field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \frac{1 - e^{-a(T-t)}}{a} x(t) \tag{13}$$

and

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \tag{14}$$

Proof. Stochastic integration by parts implies that

$$\int_t^T x(u)\, du = Tx(T) - tx(t) - \int_t^T u\, dx(u) = \int_t^T (T-u)\, dx(u) + (T-t)x(t) \tag{15}$$

By definition of x, the integral in the right-hand side can be written as

$$\int_t^T (T-u)\, dx(u) = -a \int_t^T (T-u)x(u)\, du + \sigma \int_t^T (T-u)\, dW(u) \tag{16}$$

by substituting the expression fro $dx(u)$, and

$$\int_t^T (T-u)x(u)\, du = x(t) \int_t^T (T-u)e^{-a(u-t)}\, du + \sigma \int_t^T (T-u) \int_t^u e^{-a(u-s)}\, dW(s)\, du \tag{17}$$

Calculating separately the last two integrals (multiplied by $-a$), we have

$$-ax(t) \int_t^T (T-u)e^{-a(u-t)}\, du = -x(t)(T-t) - \frac{e^{-a(T-t)} - 1}{a} x(t) \tag{18}$$

let's take the second integral

$$-a\sigma \int_t^T (T-u) \int_t^u e^{-a(u-s)}\, dW(s)\, du = -a\sigma \int_t^T \left(\int_t^u e^{as}\, dW(s)\right) [(T-u)e^{-au}]\, du \tag{19}$$

Recalling the Leibniz Integral Rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x,b(x)\big)\cdot\frac{d}{dx}b(x) - f\big(x,a(x)\big)\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt \tag{20}$$

we can write

$$[(T-u)e^{-au}]\,du = d_u\left(\int_t^u (T-v)e^{-av}\,dv\right)$$

and

$$-a\sigma\int_t^T (T-u)\int_t^u e^{-a(u-s)}\,dW(s)\,du = -a\sigma\int_t^T\left(\int_t^u e^{as}\,dW(s)\right)d_u\left(\int_t^u (T-v)e^{-av}\,dv\right) \tag{21}$$

and again by integration by parts

$$\begin{aligned}
-a\sigma\int_t^T (T-u)\int_t^u e^{-a(u-s)}\,dW(s)\,du &= -a\sigma\Bigg[\left(\int_t^T e^{au}\,dW(u)\right)\left(\int_t^T (T-v)e^{-av}\,dv\right)\\
&-\int_t^T\left(\int_t^u (T-v)e^{-av}\,dv\right)e^{au}\,dW(u)\Bigg]\\
&= -a\sigma\left[\int_t^T\left(\int_t^T (T-v)e^{-av}\,dv\right)e^{au}\,dW(u) - \int_t^T\left(\int_t^u (T-v)e^{-av}\,dv\right)e^{au}\,dW(u)\right]\\
&= -a\sigma\left[\int_t^T\left(\int_t^T (T-v)e^{-av}\,dv\right)e^{au}\,dW(u) + \int_t^T\left(\int_u^t (T-v)e^{-av}\,dv\right)e^{au}\,dW(u)\right]\\
&= -a\sigma\int_t^T\left(\int_u^T (T-v)e^{-av}\,dv\right)e^{au}\,dW(u)\\
&= -\sigma\int_t^T\left[(T-u)+\frac{e^{-a(T-u)}-1}{a}\right]dW(u)
\end{aligned}$$

where in the last step we have used the fact that

$$\int_u^T (T-v)e^{-av}\,dv = \frac{(T-u)e^{-au}}{a} + \frac{e^{-aT}-e^{-au}}{a^2}$$

adding up the previous terms, we obtain

$$\begin{aligned}
\int_t^T (T-u)dx(u) &= -a\int_t^T (T-u)x(u)\,du + \sigma\int_t^T (T-u)\,dW(u)\\
&= -ax(t)\int_t^T (T-u)e^{-a(u-t)}\,du - a\sigma\int_t^T (T-u)\int_t^u e^{-a(u-s)}\,dW(s)\,du + \sigma\int_t^T (T-u)\,dW(u)\\
&= -x(t)(T-t) - \frac{e^{-a(T-t)}-1}{a}x(t) - \sigma\int_t^T\left[(T-u)+\frac{e^{-a(T-u)}-1}{a}\right]dW(u) + \sigma\int_t^T (T-u)\,dW(u)
\end{aligned}$$

and finally

$$\int_t^T x(u)\,du = -x(t)(T-t) - \frac{e^{-a(T-t)}-1}{a}x(t) - \sigma\int_t^T \left[(T-u) + \frac{e^{-a(T-u)}-1}{a}\right] dW(u) + \sigma\int_t^T (T-u)\,dW(u)$$
$$+ (T-t)x(t)$$
$$= \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{\sigma}{a}\int_t^T \left[1-e^{-a(T-u)}\right] dW(u)$$

so that (13) is immediately verified. As to the calculation of the conditional variance, we have

$$Var[I(t,T)|\mathcal{F}_t] = Var\left(\frac{\sigma}{a}\int_t^T \left[1-e^{-a(T-u)}\right] dW(u)\right) = \frac{\sigma^2}{a^2}\int_t^T \left[1-e^{-a(T-u)}\right]^2 du$$

Simple integration then leads to (14).

□

Theorem 3.2. *The price at time t of a zero-coupon bond maturing at time T and with unit face value is*

$$P(t,T) = \exp\left\{-\int_t^T \phi(u)\,du - \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{1}{2}V(t,T)\right\} \tag{22}$$

Proof. Being ϕ a deterministic function, the theorem follows from straightforward application of Lemma (3.1) and the fact that if Z is a normal random variable with mean m_Z and variance σ_Z^2, then $E\{\exp(Z)\} = \exp(m_Z + \frac{1}{2}\sigma_Z^2)$. □

Let's now assume that the term structure of discount factors that is currently observed in the market is given by the sufficiently smooth function $T \to P^M(0,T)$. If we denote by $f^M(0,T)$ the instantaneous forward rate at time 0 for a maturity T implied by the term structure $T \to P^M(0,T)$, i.e.,

$$f^M(0,T) = -\frac{\partial \log P^M(0,T)}{\partial T}$$

we then have the following

Corollary 3.2.1. *The model* (5) *fits the currently observed term structure of discount factors if and only if, for each T*

$$\phi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 \tag{23}$$

i.e., if and only if

$$\exp\left\{-\int_t^T \phi(u)\,du\right\} = \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{-\frac{1}{2}\left[V(0,T) - V(0,t)\right]\right\} \tag{24}$$

so that the corresponding zero-coupon bond prices at time t are given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\mathcal{A}(t,T)\right\} \tag{25}$$

with

$$\mathcal{A}(t,T) = \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] - \frac{1-e^{-a(T-t)}}{a}x(t) \tag{26}$$

Proof. The model (5) fits the currently observed term structure of discount factors if and only if for each maturiy $T \le T^\star$ the discount factor $P(0,T)$ produced by the model coincides with the one observed in the market, i.e., if and only if

$$P^M(0,T) = \exp\left\{-\int_0^T \phi(u)\,du + \frac{1}{2}V(0,T)\right\}$$

Now let's take logs of both sides and differentiate with respect to T

$$\log P^M(0,T) = -\int_0^T \phi(u)\, du + \frac{1}{2}V(0,T) \Rightarrow \frac{\partial}{\partial T}\log P^M(0,T) = -\phi(T) + \frac{1}{2}\frac{\partial}{\partial T}V(0,T)$$

remember that

$$V(0,T) = \frac{\sigma^2}{a^2}\int_0^T \left[1 - e^{-a(T-u)}\right]^2 \Rightarrow \frac{\partial V}{\partial T} = \frac{\sigma^2}{a^2}\left[1 - e^{-aT}\right]^2$$

which gives us

$$\frac{\partial}{\partial T}\log P^M(0,T) = -f^M(0,T) = -\phi(T) + \frac{\sigma^2}{a2^2}\left[1 - e^{-aT}\right]^2$$

finally note that

$$\exp\left\{-\int_t^T \phi(u)\, du\right\} = \exp\left\{-\int_0^T \phi(u)\, du\right\}\exp\left\{-\int_0^t \phi(u)\, du\right\} = \frac{P^M(0,T)\exp\left[-\frac{1}{2}V(0,T)\right]}{P^M(0,t)\exp\left[-\frac{1}{2}V(0,t)\right]}$$

$\square$