$S = 2\pi \int_a^b f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx$

DŁUGOŚĆ ŁUKU krzywej

$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

POLE POWIERZCHNI BOCZNEJ BRYŁY (obrót OX)

$S = 2\pi \int_a^b f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx$

OBJĘTOŚĆ (OX)

$V = \pi \int_a^b f^2(x)\,dx$

(OY) $S = 2\pi \int_a^b x\sqrt{1+[f'(x)]^2}\,dx$

(OY) $V = 2\pi \int_a^b x f(x)\,dx$

ANALIZA MATEMATYCZNA

LISTA ZADAŃ 13

16.01.2023

1. Obliczyć pole figury ograniczonej krzywymi:
   (a) $y = x^2$ i $y = 2x + 5$, (b) $y = e^x$ i prostą przechodzącą przez punkty $(0, 1)$ i $(1, e)$,
   (c) $y = \sin(x)$ i $y = \frac{2x}{\pi}$, (d) $y = x^4$ i $y = x^3$,
   (e) $y = \frac{1}{x}$ i $y = \frac{5}{2} - x$, (f) $y = \frac{1}{x^2}$, $y = \frac{1}{x}$ i $x = 2$.
2. Obliczyć długość łuku krzywej $y = f(x)$, $a \le x \le b$ dla podanych $f(x)$ i $[a, b]$:
   (a) x, $[1, 2]$, (b) $2x-3$, $[-7, 12]$, (c) e^x, $[1, 2]$, (d) $\sqrt{x^3}$, $[6, 10]$, (e) $\frac{e^x+e^{-x}}{2}$, $[0, 1]$.
3. Dla danych $f(x)$ i $[a, b]$ obliczyć pole powierzchni bocznej bryły powstałej przez obrót krzywej $y = f(x)$, $a \le x \le b$ wokół osi OX:
   (a) x^3, $[0, 5]$, (b) e^{-x}, $[0, 10]$, (c) $\sqrt{x}$, $[0, 4]$, (d) $\sin(x)$, $[0, \pi]$, (e) $\cos(7x)$, $[0, 2\pi]$.
4. Dla danych $f(x)$ i $[a, b]$ obliczyć objętość bryły powstałej przez obrót obszaru $0 \le y \le f(x)$, $a \le x \le b$ wokół osi OX:
   (a) $\sqrt{x}$, $[0, 1]$, (b) x, $[1, 5]$, (c) x^2, $[0, 10]$, (d) e^x, $[-3, 0]$, (e) $\sin(x)$, $[0, \frac{\pi}{2}]$.
5. Obliczyć długość łuku krzywej $y = \sqrt{(x+5)^3}$, $0 \le x \le 8$.
6. Obliczyć objętość bryły powstałej przez obrót obszaru $0 \le y \le xe^x$, $0 \le x \le 1$ wokół osi OX.
7. Obliczyć długość łuku krzywej $y = \log(x)$, $1 \le x \le \sqrt{3}$.
8. Obliczyć objętość bryły powstałej przez obrót obszaru $\arctan(x) \le y \le \sqrt{\arctan^2(x) + 1 + \sin(x)}$, $0 \le x \le 2\pi$ wokół osi OX.
9. Od pomarańczy o grubej skórce odcięto końce, tak, że ukazał się miąższ. Pomarańczę następnie pokrojono w równe plastry. Pokaż, że każdy plaster zawiera tyle samo skórki.
10. Zbadaj zbieżność całek niewłaściwych i oblicz te, które są zbieżne:
   (a) $\int_0^\infty \frac{dx}{x^2+1}$, (b) $\int_0^4 \frac{dx}{\sqrt{x}}$, (c) $\int_1^\infty \frac{dx}{\sqrt{x}}$,
   (d) $\int_{-1}^1 \frac{x-1}{x^2-1}\,dx$, (e) $\int_2^\infty \frac{dx}{x\log(x)}$, (f) $\int_0^\infty \frac{dx}{e^{\sqrt{x}}}$,
   (g) $\int_0^\infty \cos(x)\,dx$, (h) $\int_1^\infty x^{\frac{1}{2}}\,dx$, (i) $\int_0^\infty e^x\,dx$,
   (j) $\int_0^1 e^{\frac{1}{x}}\,dx$, (k) $\int_1^\infty \frac{e^{-x}}{x^2}\,dx$, (l) $\int_2^\infty \frac{dx}{x\log^2(x)}$,
   (m) $\int_0^\infty x^3\sin(x^4)\,dx$.

2.8

8. Oblicz objętość bryły powstałej przez obrót obszaru $\arctan(x) \le y \le \sqrt{\arctan^2(x) + 1 + \sin(x)}$, $0 \le x \le 2\pi$ wokół osi OX.

$V_x = \pi \int_a^b f^2(x)\,dx$

$V = \pi \int_0^{2\pi} f_1^2(x) - f_2^2(x) = \frac{2\pi}{\pi}\int_0^{2\pi}\left[\cancel{\arctan^2(x)} + 1 + \sin(x) - \cancel{\arctan^2(x)}\right]dx =$

$\frac{1}{\pi}\int_0^{2\pi} 1\,dx + \int_0^{2\pi}\sin(x)\,dx = \frac{1}{\pi}\left[x - \cos x\right]_0^{2\pi} = (2\pi - 1) - (0 - 1) = 2\pi - 2$

2.10

a) $\int_0^\infty \frac{dx}{x^2+1} = \lim_{T\to\infty}\left[\arctan x\right]_0^T = \lim_{T\to\infty}\left[\arctan T - \arctan 0\right] = \frac{\pi}{2}$

b) $\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^4 2\sqrt{x}'\,dx = 2\lim_{\varepsilon\to 0^+}\left[2 - 2\sqrt{\varepsilon}\right] = 4$

$\int \frac{1}{\sqrt{x}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

f) $\int_0^\infty \frac{dx}{e^{\sqrt[3]{x}}} = \lim_{T\to\infty}\int_0^T f(x)\,dx = \lim_{T\to\infty}\left[3\cdot e^{\sqrt[3]{x}}(\sqrt[3]{x} - 1)\right]_0^\infty = \infty$, rozbieżna

$\int e^{\sqrt[3]{x}}\,dx = \left|\begin{array}{l} x = t^3 \\ 1\,dx = 3t^2\,dt \\ t = \sqrt[3]{x}\end{array}\right| = 3\int e^t\cdot t^2\,dt = \left|\begin{array}{ll} g = t^2 & f' = e^t \\ g' = 2t & f = e^t\end{array}\right| =$

$= 3\left[t^2\cdot e^t - 2\int t\cdot e^t\,dt\right] = \left|\begin{array}{ll} g = t & f' = e^t \\ g' = 1 & f = e^t\end{array}\right| =$

$= 3t^2\cdot e^t - 6\left[t\cdot e^t - e^t\right] = 3t^2\cdot e^t - 6te^t + 6e^t =$

$= 3e^t\left[t^2 - 2t + 1\right] = 3e^t\left[t - 1\right]^2 = 3\cdot e^{\sqrt[3]{x}}\left[\sqrt[3]{x} - 1\right]$

d) $\int_{-1}^1 \frac{x-1}{x^2-1}\,dx = +\infty - \frac{1}{2} = +\infty$, rozbieżna

$\int_{-1}^0 \frac{\cancel{x-1}\,dx}{(\cancel{x-1})(x+1)} \overset{\text{def}}{=} \lim_{\varepsilon\to 0^-}\int_{-1}^\varepsilon \frac{dx}{x+1} = \lim_{\varepsilon\to 0^-}\left[\frac{1}{\varepsilon+1} + \frac{1}{0}\right] = +\infty$

$\int_0^1 \frac{1}{x+1}\,dx \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{1+x} = \lim_{\varepsilon\to 0^+}\left[\frac{1}{2} - \frac{1}{1+\varepsilon}\right] = -\frac{1}{2}$

k) $\int_1^\infty \frac{e^{-\frac{1}{x}}}{x^3}\,dx \overset{\text{def}}{=} \lim_{T\to\infty}\int_1^T \frac{e^{-\frac{1}{x}}}{x^3} = \int\left[e^{-\frac{1}{x}}\left(-\frac{1}{x} - 1\right)\right]_1^T = \left[+\frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}}\right]_1^T =$

$= \left[0 + 1 - \left(\frac{1}{e} + 1\right)\right] = 1 - \frac{2}{e}$

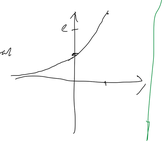
$\int \frac{e^{-\frac{1}{x}}}{x^3}\,dx = \int \frac{e^{-\frac{1}{x}}}{x^2\cdot x}\,dx = \left|\begin{array}{l} -\frac{1}{x} = t \\ \frac{1}{x^2}\,dx = 1\,dt \\ \frac{1}{x^3}\,dx = -t\,dt \\ x = -\frac{1}{t}\end{array}\right| = -\int \frac{e^t(-t)\,dt}{-\frac{1}{t}} = -\int t\cdot e^t\,dt =$

$= \left|\begin{array}{ll} g = t & f' = e^t \\ g' = 1 & f = e^t\end{array}\right| = -\left[t\cdot e^t - e^t\right] = -e^t(t-1) =$

$= e^{-\frac{1}{x}}\left(+\frac{1}{x} + 1\right)$

j) $\int_0^1 e^{\frac{1}{x}}\,dx = \ldots + \int_0^1 \frac{e^t}{t}\,dt$ = rozbieżne

$E_x(x)$
Exponencjał Eulera



$e^{\frac{1}{x}} = e$

$\lim_{x\to 0} e^{\frac{1}{x}} = \infty$

2.7

$x \in [1; \sqrt{3}]$

$y = \log(x)$

$f'(x) = \frac{1}{x}$

$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{1}{x^2}} = \sqrt{\frac{x^2+1}{x^2}}$

$L = \int_1^{\sqrt{3}} \sqrt{\frac{x^2+1}{x^2}}\,dx = \left[\sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|\right]_1^{\sqrt{3}} =$

$= \left(2 + \frac{1}{2}\log\frac{1}{3}\right) - \left(\sqrt{2} + \frac{1}{2}\log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right)$

$\int \frac{\sqrt{x^2+1}}{x}\,dx = \left|\begin{array}{l} t = \sqrt{x^2+1} \\ x^2+1 = t^2 \\ 2x\,dx = 2t\,dt \\ x^2 = t^2 - 1\end{array}\right| =$

$= \int \frac{t\cdot t}{t^2-1}\,dt = \int \frac{t^2-1}{t^2-1}\,dt + \int \frac{1}{t^2-1}\,dt = t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| =$

$\sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + c$

$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \quad /\cdot()$

$1 = At + A + Bt - B$

$1 = t(A+B) + A - B$

$\begin{cases} A + B = 0 \Rightarrow B = -A \\ A - B = 1 \\ 2A = 1 \Rightarrow A = \frac{1}{2} \\ B = -\frac{1}{2}\end{cases}$

$\frac{1}{2}\int \frac{1}{t-1}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt = \frac{1}{2}\left(\log|t-1| - \log|t+1|\right) = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$

2.9

$y = \log(x) \quad 1 \le x \le \sqrt{3}$

$L = \int_a^b \sqrt{1 + [f'(x)]^2}$

$f'(x) = \frac{1}{x}$

$L = \int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{x^2}}\,dx = \int_1^{\sqrt{3}} \sqrt{\frac{x^2+1}{x^2}}\,dx = \left[\sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|\right]_1^{\sqrt{3}} =$

$= \left[2 + \frac{1}{2}\log\left|\frac{1}{3}\right| - 1 - \frac{1}{2}\log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right]$

$\int \sqrt{\frac{x^2+1}{x^2}}\,dx = \int \frac{\sqrt{x^2+1}}{x}\cdot\frac{x}{x}\,dx = \left|\begin{array}{l} t = \sqrt{x^2+1} \\ x^2+1 = t^2 \\ 2x\,dx = 2t\,dt \\ x^2 = t^2 - 1\end{array}\right| =$

$= \int \frac{t\cdot t}{t^2-1}\,dt = \int \frac{t^2}{t^2-1}\,dt = \int \frac{t^2-1}{t^2-1}\,dt + \int \frac{1}{t^2-1}\,dt =$

$= t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| = \sqrt{x^2+1} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|$

$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{A}{(t-1)} + \frac{B}{(t+1)} \quad /\cdot()$

$1 = At + A + Bt - B = t(A+B) + (A-B)$

$\begin{cases} A - B = 1 \Rightarrow A + A = 1 \Rightarrow A = \frac{1}{2} \\ A + B = 0 \quad B = -\frac{1}{2}\end{cases}$

$\int \frac{1}{t^2-1}\,dt = \frac{1}{2}\int \frac{1}{t-1}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt = \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| =$

$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$

2.1

a) $y = x^2 \quad y = 2x + 5$

$x^2 - 2x - 5 = 0$

$\Delta = 4 + 20 = 24 = \sqrt{4\cdot 6} = 2\sqrt{6}$

$x_1 = \frac{2 - 2\sqrt{6}}{2} = 1 - \sqrt{6}$

$x_2 = 1 + \sqrt{6}$

$\int_{x_1}^{x_2} 2x + 5 - x^2 = \left[\cancel{2}\frac{x^2}{\cancel{2}} + 5x - \frac{x^3}{3}\right]_{x_1}^{x_2}$ POD i CAŁ

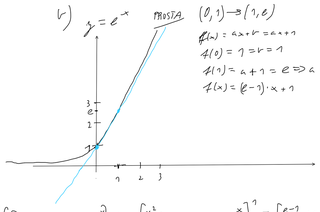
b) $y = e^x$ PROSTA $(0,1) \to (1,e)$

$f(x) = ax + b = ax + 1$

$f(0) = 1 = b = 1$

$f(1) = a + 1 = e \Rightarrow a = e - 1$

$f(x) = (e-1)\cdot x + 1$



$\int_0^1 \left[(e-1)\cdot x + 1 - e^x\right]dx = \left[\frac{x^2}{2}(e-1) + x - e^x\right]_0^1 = \left[\frac{e-1}{2} + 1 - e + 1\right] =$

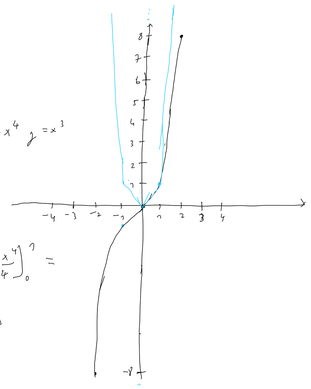
$= \frac{e - 1 + 2 - 2e + 2}{2} = \frac{-e+3}{2}$

d) $y = x^4 \quad y = x^3$

$x^4 = x^3$

$x^3(x - 1) = 0$

$x = 0 \vee x = 1$



$\int_0^1 x^3 - x^4 = \left[\frac{x^5}{5} + \frac{x^4}{4}\right]_0^1 =$

$= \left[\frac{1}{5} + \frac{1}{4}\right]$

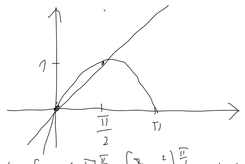
2.1

c) $y = \sin(x) \quad y = \frac{2x}{\pi}$

$x \in [0; \pi]$

$f'(x) = \cos(x)$

$g'(x) = \frac{2}{\pi}$



$\int_0^{\frac{\pi}{2}} \sin(x) - \frac{2x}{\pi}\,dx = \int_0^{\frac{\pi}{2}} \sin(x)\,dx - \int_0^{\frac{\pi}{2}} \frac{2}{\pi}x\,dx = \left[-\cos(x)\right]_0^{\frac{\pi}{2}} - \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} = +1 - \frac{\pi}{4}$

$\frac{\pi^2}{4\cdot 2}$

2.2

b) $f(x) = 2x - 3 \quad x \in [-7; 12]$

$f'(x) = 2$

$L = \int_{-7}^{12} \sqrt{1 + 4}\,dx = \left[\sqrt{5}x\right]_{-7}^{12} = \left[\sqrt{5}\cdot 12 + 7\sqrt{5}\right] = 19\sqrt{5}$

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

2.3

c) $f(x) = \sqrt{x} \quad x \in [0, 4]$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\sqrt{1 + \frac{1}{2x}}$

$\int \sqrt{x}\cdot\sqrt{\frac{2x+1}{2x}}\,dx = \int \sqrt{\frac{2x^2 + x}{2x}}\,dx = \int \sqrt{\frac{2x+1}{2}}\,dx =$

$= \left|\begin{array}{l} \frac{2x+1}{2} = t \\ x + \frac{1}{2} = t \\ 1\,dx = 2\,dt\end{array}\right| = \int t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}\left(\sqrt{x + \frac{1}{2}}\right)^3$

$S = 2\pi\frac{2}{3}\left[\left(\sqrt{x + \frac{1}{2}}\right)^3\right]_0^4 = \frac{4}{3}\pi\left[\left(\frac{9}{2}\right)^{\frac{3}{2}} - \left(\frac{1}{2}\right)^{\frac{3}{2}}\right]$

2.4

c) $x^2 \quad x \in [0, 10]$

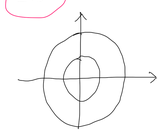
$V = \pi\int_a^b f^2(x)\,dx$

$\frac{10}{\pi}\int_0^{10} x^{24}\,dx = \frac{10}{\pi}\left[\frac{x^{25}}{25}\right]_0^{10} = \frac{10^{26}}{25}\cdot\frac{1}{\pi}$

e) $\sin(x) \quad \left[0, \frac{3\pi}{2}\right]$

$\int_0^{\frac{3\pi}{2}} \sin^2(x) =$

2.9



2.10

h) $\int_1^\infty x^{\frac{1}{x}}\,dx \overset{\text{def}}{=}$

2.5

$y = \sqrt{(x+5)^3} \quad x \in [0; 8]$

$L = \int_c^b \sqrt{1 + [f'(x)]^2}\,dx$

$f'(x) = \left(\sqrt{(x+5)^3}\right)' = \frac{1}{2\sqrt{(x+5)^3}}\cdot 3(x+5)^2 = \frac{3}{2}\frac{(x+5)^2}{\sqrt{(x+5)^3}}$

$\int \sqrt{1 + \left(\frac{3}{2}\right)^2\cdot\frac{(x+5)^4}{(x+5)^3}}\,dx = \int \sqrt{1 + \frac{9}{4}(x+5)}\,dx =$

$\int \sqrt{1 + \frac{9}{4}x + \frac{45}{4}}\,dx = \int \sqrt{\frac{49}{4} + \frac{9}{4}x}\,dx = \frac{1}{4}\int \sqrt{49 + 9x}\,dx =$

$= \left|\begin{array}{l} 49 + 9x = t^2 \\ 9\,dx = 2t\,dt \\ 1\,dx = \frac{2}{9}t\,dt\end{array}\right| = \frac{1}{4}\int t\cdot\frac{2}{9}\,dt = \frac{2}{36}\frac{t^2}{2} = \frac{t^2}{36} =$

$= \frac{49 + 9x}{36}$

$L = \int_0^8 f(x)\,dx = \left[\frac{49 + 9x}{36}\right]_0^8 = \left[\frac{49 + 9\cdot 8}{36} - \frac{49}{36}\right] =$

$= \left[\frac{121}{36} - \frac{49}{36}\right] = \frac{72}{36} = 2$

2.2

e) $f(x) = \frac{e^x + e^{-x}}{2}$

$f'(x) = \frac{2(e^x - e^{-x})}{4\cdot 2} = \frac{e^x - e^{-x}}{2}$

$L = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} = \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right]_0^1 = \left[\frac{e}{2} - \frac{1}{2}\cdot\frac{1}{e} - \frac{1}{2} + \frac{1}{2}\right] = \frac{1}{2}\left(e - \frac{1}{e}\right)$

$e^x\cdot e^{-x} = e^0 = 1$

$\int \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} = \int \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} = \int \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2} = \int \frac{e^x}{2}\,dx + \int \frac{e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

2.7

a) $\int_0^\infty \frac{dx}{x^2+1} = \lim_{T\to\infty}\left[\arctan x\right]_0^T = \lim_{T\to\infty}\left[\arctan T - \arctan 0\right] = \frac{\pi}{2}$

b) $\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^4 2\sqrt{x}'\,dx = 2\lim_{\varepsilon\to 0^+}\left[2 - 2\sqrt{\varepsilon}\right] = 4$

$\int \frac{1}{\sqrt{x}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

f) $\int_0^\infty \frac{dx}{e^{\sqrt[3]{x}}} = \lim_{T\to\infty}\int_0^T f(x)\,dx = \lim_{T\to\infty}\left[3\cdot e^{\sqrt[3]{x}}(\sqrt[3]{x} - 1)\right]_0^\infty = \infty$, rozbieżna

$\int e^{\sqrt[3]{x}}\,dx = \left|\begin{array}{l} x = t^3 \\ 1\,dx = 3t^2\,dt \\ t = \sqrt[3]{x}\end{array}\right| = 3\int e^t\cdot t^2\,dt = \left|\begin{array}{ll} g = t^2 & f' = e^t \\ g' = 2t & f = e^t\end{array}\right| =$

$= 3\left[t^2\cdot e^t - 2\int t\cdot e^t\,dt\right] = \left|\begin{array}{ll} g = t & f' = e^t \\ g' = 1 & f = e^t\end{array}\right| =$

$= 3t^2\cdot e^t - 6\left[t\cdot e^t - e^t\right] = 3t^2\cdot e^t - 6te^t + 6e^t =$

$= 3e^t\left[t^2 - 2t + 1\right] = 3e^t\left[t - 1\right]^2 = 3\cdot e^{\sqrt[3]{x}}\left[\sqrt[3]{x} - 1\right]$

d) $\int_{-1}^1 \frac{x-1}{x^2-1}\,dx = +\infty - \frac{1}{2} = +\infty$, rozbieżna

$\int_{-1}^0 \frac{\cancel{x-1}\,dx}{(\cancel{x-1})(x+1)} \overset{\text{def}}{=} \lim_{\varepsilon\to 0^-}\int_{-1}^\varepsilon \frac{dx}{x+1} = \lim_{\varepsilon\to 0^-}\left[\frac{1}{\varepsilon+1} + \frac{1}{0}\right] = +\infty$

$\int_0^1 \frac{1}{x+1}\,dx \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{1+x} = \lim_{\varepsilon\to 0^+}\left[\frac{1}{2} - \frac{1}{1+\varepsilon}\right] = -\frac{1}{2}$

k) $\int_1^\infty \frac{e^{-\frac{1}{x}}}{x^3}\,dx \overset{\text{def}}{=} \lim_{T\to\infty}\int_1^T \frac{e^{-\frac{1}{x}}}{x^3} = \int\left[e^{-\frac{1}{x}}\left(-\frac{1}{x} - 1\right)\right]_1^T = \left[+\frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}}\right]_1^T =$

$= \left[0 + 1 - \left(\frac{1}{e} + 1\right)\right] = 1 - \frac{2}{e}$

$\int \frac{e^{-\frac{1}{x}}}{x^3}\,dx = \int \frac{e^{-\frac{1}{x}}}{x^2\cdot x}\,dx = \left|\begin{array}{l} -\frac{1}{x} = t \\ \frac{1}{x^2}\,dx = 1\,dt \\ \frac{1}{x^3}\,dx = -t\,dt \\ x = -\frac{1}{t}\end{array}\right| = -\int \frac{e^t(-t)\,dt}{-\frac{1}{t}} = -\int t\cdot e^t\,dt =$

$= \left|\begin{array}{ll} g = t & f' = e^t \\ g' = 1 & f = e^t\end{array}\right| = -\left[t\cdot e^t - e^t\right] = -e^t(t-1) =$

$= e^{-\frac{1}{x}}\left(+\frac{1}{x} + 1\right)$

j) $\int_0^1 e^{\frac{1}{x}}\,dx = \ldots + \int_0^1 \frac{e^t}{t}\,dt$ = rozbieżne

$E_x(x)$
Exponencjał Eulera

$e^{\frac{1}{x}} = e$

$\lim_{x\to 0} e^{\frac{1}{x}} = \infty$

l) $\int_1^\infty x^{\frac{1}{x}}\,dx \overset{\text{def}}{=} \lim_{T\to\infty}\int_1^T x^{\frac{1}{x}} = \infty$ rozbieżne

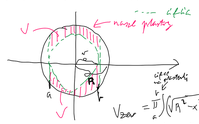
$\lim_{n\to\infty}\sqrt[n]{n} = 1$

i) $\int_{-\infty}^\infty e^x\,dx = \int_{-\infty}^0 e^x\,dx + \int_0^\infty e^x\,dx =$

$\lim_{T\to-\infty}\int_T^0 e^x\,dx + \lim_{w\to\infty}\int_0^w e^x\,dx = \lim_{T\to-\infty}\left[e^0 - e^T\right] + \lim_{w\to\infty}\left[e^w - e^0\right] =$

$= \{1 + \infty\}$ = rozbieżne

9. Od pomarańczy o grubej skórce odcięto końce, tak, że ukazał się miąższ. Pomarańczę następnie pokrojono w równe plastry. Pokaż, że każdy plaster zawiera tyle samo skórki.



$r^2 = x^2 + y^2$

$R^2 = x^2 + y^2$

$y = \sqrt{r^2 - x^2}$

$y = \sqrt{R^2 - x^2}$

$V_{zew} = \pi\int_a^b \left(\sqrt{R^2 - x^2}\right)^2 dx = \pi\int_a^b R^2 - x^2\,dx = \left[\pi\left(R^2x - \frac{x^3}{3}\right)\right]_a^b = \pi\left(R^2b - \frac{b^3}{3} - R^2a + \frac{a^3}{3}\right)$

$V_{wew} = \pi\int_a^b \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi\int_a^b r^2 - x^2\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_a^b = \pi\left[r^2b - \frac{b^3}{3} - r^2a + \frac{a^3}{3}\right]$

$V = \pi\left(R^2b - \frac{b^3}{3} - R^2a + \frac{a^3}{3} - r^2b + \frac{b^3}{3} + r^2a - \frac{a^3}{3}\right) =$

zewnętrzne - wewnętrzne

$= \pi\left[b\left(R^2 - r^2\right) - a\left(R^2 - r^2\right)\right] = \pi\left[(b-a)\left(R^2 - r^2\right)\right]$

$b - a$