SUMA RIEMANNA

$\Delta x = \frac{b-a}{n}$

$$\sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n} \xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx$$

całkowanie f. nieparzystej w przedziale symetrycznym podlega

$f(-x) = -f(x)$ $\quad t=-x$ $\quad dt = -dx$

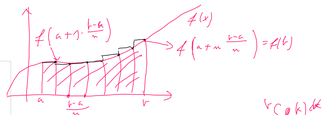
$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(-t)(-dt) + \int_0^a f(x)\,dx =$$

$$= -\int_a^0 f(t)\,dt + \int_0^a f(x)\,dx = \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 0$$

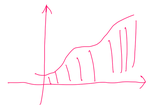
NEWTONA-LEIBNIZA

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

wyniki całkowania nieoznaczonego (bez stałych)



$$S_n = \sum_{k=1}^{n} \frac{b-a}{n}\cdot f\left(a + k\cdot\frac{b-a}{n}\right) \quad \xrightarrow[n\to\infty]{} \quad S_n = \int_a^b f(x)\,dx$$



$I_1 = \{a, b\}$

$I_2 = \{a, t_1, b\}$

$I_3 = \{a, c_1, c_2, c_3\}$

$\vdots$

$\Delta(I_n) \to 0$

$n(I_n) \to \infty$

$L(I_n) \to L$

$I = \{x_0, \dots, x_n\}$

$\Delta(I) = \max_{1\le i\le n-1} \{x_{i+1} - x_i\}$

ANALIZA MATEMATYCZNA

LISTA ZADAŃ 12

9.01.2023

1. Podaj wzór na $C_n = \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$, a następnie oblicz $\lim_{n\to\infty} C_n$

   (a) $f(x) = 1$, $a = 5$, $b = 8$. (b) $f(x) = x$, $a = 0$, $b = 1$,
   (c) $f(x) = x$, $a = 1$, $b = 5$. (d) $f(x) = x^2$, $a = 0$, $b = 5$,
   (e) $f(x) = x^3$, $a = 0$, $b = 1$. (f) $f(x) = 2x + 5$, $a = -3$, $b = 4$,
   (g) $f(x) = x^2 + 1$, $a = -1$, $b = 2$, (h) $f(x) = x^2 + x$, $a = 0$, $b = 4$.
   (i) $f(x) = e^x$, $a = 0$, $b = 1$.

2. Oblicz następujące całki oznaczone poprzez konstrukcję ciągu podziałów przedziału, odpowiadającego mu ciągu sum Riemanna, oraz jego granicy

   (a) $\int_{20}^{3} x^{10}\,dx$, $(x_i = 2\cdot 2^{i/n})$, (b) $\int_1^e \frac{\log(x)}{x}\,dx$, $(x_i = e^{i/n})$,
   (c) $\int_{10}^{5} x\,dx$, (d) $\int_{-5}^{10} e^{2x}\,dx$,
   (e) $\int_0^1 \sqrt[n]{x}\,dx$, $(x_i = \frac{i}{n+1})$, (f) $\int_{-1}^{1} |x|\,dx$,
   (g) $\int_1^3 \frac{dx}{x}\,dx$, $(x_i = 2^{i/n})$, (h) $\int_0^1 \sqrt{x}\,dx$, $(x_i = \frac{4i^2}{n^2})$.

3. Obliczcałki oznaczone

   (a) $\int_{-\pi}^{\pi} \sin(x^{2007})\,dx$, (b) $\int_0^5 \arctan(\lfloor x \rfloor)\,dx$,
   (c) $\int_{-3}^{3} \lfloor \cos(x^2) \rfloor\,dx$, (d) $\int_{-5}^{5} \sqrt{1+x}\,dx$,
   (e) $\int_{-1}^{-1} \frac{1}{(11+5x)^3}\,dx$, (f) $\int_{-15}^{0} \frac{1}{\sqrt[3]{(3-x)^4}}\,dx$,
   (g) $\int_{-1}^{1} \frac{x}{(x^2+1)^2}\,dx$, (h) $\int_{-5}^{5} \operatorname{sgn}(x^3 - x)\,dx$,
   (i) $\int_0^1 xe^{-x}\,dx$, (j) $\int_0^{\pi/2} x\cos(x)\,dx$,
   (k) $\int_0^{e-1} \log(x+1)\,dx$, (l) $\int_0^{\pi} x^2\sin(x)\,dx$,
   (m) $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\,dx$, (n) $\int_1^e \frac{1}{x\sqrt{1+\log(x)}}\,dx$,
   (o) $\int_1^2 \frac{1}{x+x^2}\,dx$, (p) $\int_0^1 \frac{1}{\sqrt{x+1}+\sqrt{(x+1)^3}}\,dx$,
   (q) $\int_0^5 |x^2 - 5x + 6|\,dx$, (r) $\int_0^1 \frac{e^x}{e^x - e^{-x}}\,dx$,

   ! BŁĄD

Lista zadań 12

   (s) $\int_2^3 x\log_2(x)\,dx$, (t) $\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{1+x^2}}\,dx$,
   (u) $\int_0^{\pi^2} |\sin(x)|\,dx$, (v) $\int_{-\pi}^{\pi/2} \cos(x)\sin^{11}(x)\,dx$,
   (w) $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+5}\,dx$, (y) $\int_{-\pi}^{\pi} x^{2007}\cos(x)\,dx$,
   (x) $\int_0^{2\pi} (x-\pi)^{2007}\cos(x)\,dx$.

4. Udowodnij następujące oszacowania

   (a) $\int_0^{\pi/2} \frac{\sin(x)}{x}\,dx < 2$, (b) $\frac{1}{5} < \int_0^2 \frac{1}{x^2+1}\,dx < \frac{1}{2}$,
   (c) $\frac{1}{11} < \int_0^{10} \frac{1}{x+\sin(x)}\,dx < \frac{1}{8}$, (d) $\int_{-1}^{2} \frac{|x|}{x^2+1}\,dx < \frac{3}{2}$,
   (e) $\int_0^1 x(1-x^{99+x})\,dx < \frac{1}{2}$, (f) $2\sqrt{2} < \int_0^1 x^{1/x}\,dx$,
   (g) $5 < \int_1^3 x^x\,dx < 31$, (h) $\int_1^2 \frac{1}{x}\,dx < \frac{3}{4}$.

5. Oblicz następujące granice

   (a) $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right)$,
   (b) $\lim_{n\to\infty} \left(\frac{1^{99}+2^{99}+3^{99}+\cdots+n^{99}}{n^{100}}\right)$,
   (c) $\lim_{n\to\infty} \left(\frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n}\right)\cdot n$,
   (d) $\lim_{n\to\infty} \left(\frac{n}{n^2+n} + \frac{n}{n^2+n-1} + \frac{n}{n^2+n-2} + \cdots + \frac{n}{n^2+n}\right)$,
   (e) $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + \cdots + \sin\left(\frac{n}{n}\right)\right)\cdot\frac{1}{n}$,
   (f) $\lim_{n\to\infty} \left(\sqrt{4n} + \sqrt{4n+1} + \sqrt{4n+2} + \cdots + \sqrt{5n}\right)\cdot\frac{1}{n\sqrt{n}}$,
   (g) $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{5n^2}\right)\cdot\frac{1}{\sqrt{n}}$,
   (h) $\lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}} + \frac{\sqrt{n}}{\sqrt{n+2}+\sqrt{n+4}} + \cdots + \frac{\sqrt{n}}{\sqrt{n+2n}+\sqrt{n+3n}}\right)$,
   (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \cdots + \frac{n}{n^2+n}\right)$,
   (j) $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right)$,
   (k) $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+2} + \frac{1}{n+4} + \frac{1}{n+6} + \cdots + \frac{1}{2n}\right)$,
   (l) $\lim_{n\to\infty} \left(\frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n}\right)$,
   (m) $\lim_{n\to\infty} \frac{1}{n}\left(e^{\sqrt{2}} + e^{\sqrt{2}} + e^{\sqrt{2}} + \cdots + e^{\sqrt{2}}\right)$,
   (n) $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right)\cdot\frac{1}{n}$,
   (o) $\lim_{n\to\infty} \left(\frac{n+1}{n^2+1} + \frac{n+1}{n^2+2} + \frac{n+1}{n^2+3} + \cdots + \frac{n+1}{n^2+n}\right)$,
   (p) $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n}\right)$,
   (r) $\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n^2+1+2} + \frac{1}{n^2+1+4} + \frac{1}{n^2+1+6} + \cdots + \frac{1}{n^2+n}\right)$.

$$\min_{c\le x\le b} f(x)(b-a) \le \int_a^b \min_{a\le x\le b} f(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \max_{a\le x\le b} f(x)\,dx = \max_{c\le x\le b} f(x)(b-a)$$

2.3

e) $\int_0^1 x\left(1 - x^{99+x}\right)dx \quad x \in [0;1]$

$f(x) = x - x^{100+x} \le x - x^{100+1} = x - x^{101} = g(x)$

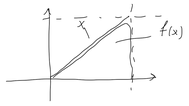
$g'(x) = 1 - 101x^{100}$

$g'(x)$ zmienia $x^{100} = \frac{1}{101}$

$x = \left(\frac{1}{101}\right)^{\frac{1}{100}}$

$g(x) = \left(\frac{1}{101}\right)^{\frac{1}{100}}\left(1 - \frac{1}{101}\right)$

$x\left(1 - x^{99+x}\right) < x$



$101\sqrt[101]{\frac{1}{101}} = \frac{1}{\sqrt[101]{101^{100}}} < \frac{1}{\sqrt[101]{100^{100}}}$

2.1

b) $f(x) = 1$

$$c_n = \sum_{k=1}^{n} \frac{1-0}{n}\left(0 + k\cdot\frac{1-0}{n}\right) = \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{k}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}$$

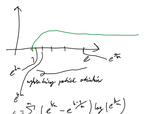
2.2

b) $$\int_1^e \frac{\log(x)}{x}\,dx = \sum_{k=1}^{n} \frac{1}{k}\left(1 - e^{-\frac{1}{k}}\right) =$$

$$= \frac{1 - e^{-\frac{1}{n}}}{n}\sum_{k=1}^{n} k = \frac{1 - e^{-\frac{1}{n}}}{\frac{1}{n}}\cdot\frac{n(n+1)}{2} =$$

$$= \frac{1 - e^{-\frac{1}{n}}}{\frac{1}{n}} + \frac{1 - e^{-\frac{1}{n}}}{2}$$

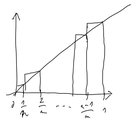
$\frac{1}{n} \xrightarrow[n\to\infty]{} 0$ $\quad c_n = \frac{1 - e^{-x}}{x} \xrightarrow[x\to 0]{H} \frac{e^{-x}}{1} = 1$



$$c_n = \frac{\sum_{k=1}^{n}\left(e^k - e^{k-1}\right)\log\left(e^k\right)}{e^n}$$

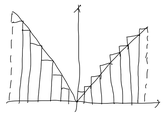
f) $$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx = 2\int_0^1 x\,dx =$$

$I_n = \{0; \frac{1}{n}; \frac{2}{n}; \dots; \frac{n-1}{n}; 1\}$



$$S_n = \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{k}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{2}$$

$\uparrow$ suma prostokątów



$$D_n = \sum_{k=n+1}^{n} \frac{1}{n}\cdot\left|\frac{k}{n}\right| = \sum_{k=n+1}^{0} \frac{k}{n^2}$$

2.1

g) $f(x) = x^2 + 1 \qquad a = 1 \qquad b = 2$

$$\sum_{k=0}^{n}\left(\frac{b-a}{n}\cdot f\left(a + k\cdot\frac{b-a}{n}\right)\right)$$

$-1 + k\cdot\frac{3}{n}$

$f(k) = 1 - \frac{6k}{n} + \frac{9k^2}{n^2}$

$$c_n = \sum_{k=0}^{n} \frac{3}{n}\cdot\left(\frac{9k^2}{n^2} - \frac{6k}{n} + 1\right) =$$

$$= \sum_{k=0}^{n} \frac{27k^2}{n^3} - \sum_{k=0}^{n} \frac{18k}{n^2} + \sum_{k=0}^{n} \frac{3}{n} =$$

$$= \frac{27}{n^3}\underbrace{\sum_{k=0}^{n} k^2}_{\frac{n(n+1)(2n+1)}{6}} - \frac{18}{n^2}\sum_{k=0}^{n} k + \frac{3}{n} = \frac{27}{n^3}\cdot\frac{n(1+\frac{1}{n})(2+\frac{1}{n})}{6} - \frac{9}{n^2}\cdot\frac{n(1+\frac{1}{n})}{2} + \frac{3}{n}$$

$$\lim_{n\to\infty} C_n = 9 - 9 = 0$$

2.3

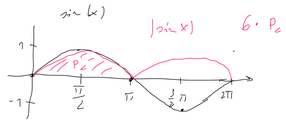
m) $$\int_4^9 \frac{\sqrt{x}}{\sqrt{x} - 1}\,dx = \left\{\begin{array}{l} x = t^2 \\ dx = 2t\,dt \\ \text{dla } x = 4 \quad t^2 = 4 \Rightarrow t = 2 \\ \text{dla } x = 9 \quad t^2 = 9 \Rightarrow t = 3 \end{array}\right\} =$$

$$= \int_2^3 \frac{t}{t-1}\cdot 2t\,dt = 2\int_2^3 \frac{t^2}{t-1}\,dt = 2\left(\int_2^3 \frac{t^2-1}{t-1}\,dt + \int_2^3 \frac{1}{t-1}\,dt\right) =$$

$$= 2\left(\left[\frac{t^3}{3} + \frac{t^2}{2} + t\right]_2^3 + \left[\log|t-1|\right]_2^3\right) + \text{POLICZYĆ RÓŻNICĘ}$$

$$\int \frac{(t-1)(t^2+t+1)}{(t-1)}\,dt = \frac{t^3}{3} + \frac{t^2}{2} + t$$

$$\int \frac{1}{t-1}\,dt = \log|t-1|$$



n) $$\int_0^{6\pi} |\sin(x)|\,dx = 6\cdot 2 = 12$$

$$\int_0^{\pi} |\sin(x)|\,dx = \int_0^{\pi} \sin(x)\,dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2$$

t) $$\int_0^{\sqrt{3}} \frac{x^3}{\sqrt[3]{1+x^2}}\,dx = \frac{3}{2}\left[\frac{\left(\sqrt[3]{x^2+1}\right)^4}{4} - \sqrt[3]{x^2+1}\right]_0^{\sqrt{3}} \quad + \text{POLICZYĆ}$$

$$\int \frac{x^3}{\sqrt[3]{1+x^2}}\,dx = \left\{\begin{array}{l} x^2 + 1 = t^3 \\ 2x\,dx = 3t^2\,dt \\ 1x\,dx = \frac{3}{2}t^2\,dt \\ x^2 = t^3 - 1 \\ \sqrt[3]{x^2+1} = t \end{array}\right\} = \int \frac{x^2}{t}\cdot\frac{3}{2}\frac{t^2}{x}\,dt = \frac{3}{2}\int \frac{(t^3-1)\cdot t}{t}\,dt =$$

$$= \frac{3}{2}\left(\frac{t^4}{4} - t\right)$$

n) $$\int_1^{e^3} \frac{1}{x\sqrt{1+\log(x)}}\,dx = 2\left[\sqrt{1+\log(x)}\right]_1^{e^3} = 2\left[2 - 1\right] = 2$$

$$\int \frac{1}{x\cdot\sqrt{1+\log(x)}}\,dx = \left\{\begin{array}{l} 1 + \log(x) = t^2 \\ \frac{1}{x}\,dx = 2t\,dt \\ 1\,dx = 2xt\,dt \\ t = \sqrt{1+\log(x)} \end{array}\right\} = \int \frac{2\cancel{x}\cancel{t}\,dt}{\cancel{x}\cdot\cancel{t}} = 2t =$$

$$= 2\sqrt{1+\log(x)}$$

$$2\sqrt{1+3} = 4$$
$$2\sqrt{1} = 2$$

2.5

v) $$\lim_{n\to\infty}\left(\frac{n}{2n^2} + \frac{n}{n^2+(n+1)^2} + \frac{n}{n^2+(n+2)^2} + \dots + \frac{n}{50n^2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{n}{2n^2} + \frac{n}{n^2\left(1+\left(1+\frac{1}{n}\right)^2\right)} + \frac{n}{n^2\left(1+\left(1+\frac{2}{n}\right)^2\right)} + \dots + \frac{n}{50n^2}\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{2} + \frac{n}{n\left(1+\left(1+\frac{1}{n}\right)^2\right)} + \frac{n}{n\left(1+\left(1+\frac{2}{n}\right)^2\right)} + \dots + \frac{n}{50n}\right) =$$

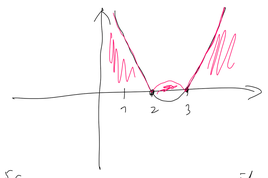
$$\left(\frac{1}{1+(1+\frac{1}{n})^2} + \frac{1}{1+(1+\frac{2}{n})^2} + \dots + \frac{1}{1+(1+\frac{48}{n})^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{49}\left(\frac{1}{1+\left(1+\frac{i}{n}\right)^2}\right) = \int_0^1 f(x) = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$\frac{b-a}{n} = \frac{1}{n}$ $\quad b - a = 1$ $\quad [1; 0] = [0, 1]$

$$f(x) = \frac{1}{1+(1+x)^2}$$

$$\int \frac{1}{1+(1+x)^2}\,dx = \left\{\begin{array}{l} 1 + x = t \\ 1\,dx = 1\,dt \end{array}\right\} = \int \frac{1\,dt}{1+t^2} = \arctan(1+x)$$



q) $$\int_0^5 |x^2 - 5x + 6|\,dx \quad \text{całka na 3 wzory liczyć}$$

$$|(x-3)(x-2)|\,dx$$

dla $x \in [0; 2]$

$$\int_0^2 |x^2 - 5x + 6| = \left[\frac{x^3}{3} - 5\frac{x^2}{2} + 6x\right]_0^2 = \frac{8}{3} - 10 + 12 = \frac{14}{3} \approx 2$$

CAŁKA

$$\int x^2\,dx - 5\int x\,dx + 6\int 1\,dx = \frac{x^3}{3} - 5\frac{x^2}{2} + 6x$$

dla $x \in [2, 3]$ $\quad -\int x^2\,dx + 5\int x\,dx - 6\int 1\,dx = -\frac{x^3}{3} + 5\frac{x^2}{2} - 6x$

$$\int_2^3 -x^2 + 5x - 6 = \left[-\frac{x^3}{3} + 5\frac{x^2}{2} - 6x\right]_2^3 \approx 0{,}17$$

$$= -\frac{27}{3} + \frac{45}{2} - 18 + \frac{8}{3} - \frac{10}{2} + 12$$

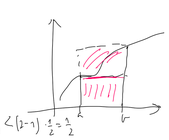
$-\frac{19}{3}$ $\quad \frac{25}{2}$

$\int_3^5$ take same for opposite

2.4

b) $f(x) = \frac{1}{x^2+1}$

$f'(x) = \frac{-2x}{(x^2+1)^2}$

$f(1) = \frac{1}{2}$ $\quad f(2) = \frac{1}{5}$

$\frac{1}{5} = (2x - 1)\cdot\frac{1}{5} < \int_1^2 \frac{1}{x^2+1} < (2-1)\cdot\frac{1}{2} = \frac{1}{2}$