Lista 11
Monday, 2 January 2023 15:52

ANALIZA MATEMATYCZNA

LISTA ZADAŃ 11

2.01.2023

1. Oblicz całkę nieoznaczoną $\int f(x)\,dx$ gdzie f jest dana wzorem:

(a) $\frac{5x^2-12}{(x^2-6x+13)^2}$, (b) $\arctan(x)$, (c) $\arctan\sqrt{x}$,
(d) $\frac{1}{1+\sqrt{x+1}}$, (e) $x^2\log(x+1)$, (f) $\frac{x}{(x+1)(2x+1)}$,
(g) $\frac{x-2}{x^2-7x+10}$, (h) $\frac{x-2}{x^2-7x+12}$, (i) $\frac{x}{2x^2-3x-2}$,
(j) $\frac{4x+3}{(x-2)^3}$, (k) $\frac{x^3+1}{x^4-x^2}$, (l) $\frac{x^4}{x^2+1}$,
(m) $\frac{1}{(x^2+9)^3}$, (n) $\frac{x^3+x-1}{(x^2+2)^2}$, (o) $\frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}$,
(p) $\frac{1}{x\sqrt{x+1}}$, (q) $\frac{1}{1+\sqrt[3]{x+1}}$, (r) $\frac{e^x-1}{e^x+1}$ $(t=e^x)$,
(s) $\log(1+x^2)$, (t) $\frac{x^2}{1+x^2}$, (u) $x\cdot\log(x^2+1)$,
(v) $\frac{1}{x^2-x-1}$, (w) $\frac{7x^3+3x^2+4x}{x^2+x^3+2x^2+4}$, (x) $\sqrt{x}\cdot\log(x)$,
(y) $\frac{e^x}{e^{2x}+1}$, (z) $\frac{1}{e^{2x}+1}$, (aa) $\frac{e^x}{e^{3x}-1}$,
(ab) $\frac{1}{(x+1)\sqrt{x}}$, (ac) $\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$, (ad) $\frac{1}{x^6+x^4}$,
(ae) $\frac{1}{(x^2+2x+2)(x^2-4)}$, (af) $\frac{1}{\sqrt{1+\sqrt[3]{x+2}}}$, (ag) $\frac{x^4}{x^{15}-1}$,
(ah) $\frac{1}{x^4+1}$, (ai) $x^2\arctan(x)$, (aj) $\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}$.

2. Wyraź I_n przy pomocy I_{n-1} lub I_{n-2}

(a) $I_n(x)=\int\frac{1}{(x^2+4)^n}\,dx$, (b) $I_n(x)=\int x^n e^x\,dx$,
(c) $I_n(x)=\int\sin^n(x)\,dx$, (d) $I_n(x)=\int x^n\sin(x)\,dx$,
(e) $I_n(x)=\int\log^n(x)\,dx$, (f) $I_n(x)=\int x^n e^{x^2}\,dx$.

2.2

c) $\int \sin^n(x)\,dx = \int \underbrace{\sin^{n-1}(x)}_{u}\cdot\underbrace{\sin(x)}_{v'}\,dx =$

$= -\sin^{n-1}(x)\cdot\cos(x) - \int(\sin^{n-1}(x))'(-\cos(x))\,dx =$

$= -\sin^{n-1}(x)\cdot\cos(x) + \int(n-1)\sin^{n-2}(x)\cdot\underbrace{\cos x\cdot\cos(x)}_{1-\sin^2(x)}\,dx =$

$= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int\sin^{n-2}(x)\,dx - (n-1)\int\sin^n(x)\,dx$

$n\int\sin^n(x)\,dx = -\sin^{n-1}(x)\cdot\cos x + (n-1)I_{n-2}$

ŚWIĘTE WZORY

$\int\frac{f'(x)}{f(x)}\,dx = \left|\begin{array}{l} f(x)=s \\ f'(x)\,dx = 1\,ds\end{array}\right| = \int\frac{ds}{s} = \log|s| + C$

$\int\frac{dx}{1+x^2} = \arctan(x)+C$

• ROZKŁADANIE POTĘG
• DZIELENIE
• POSTAĆ KANONICZNA
• POSTAĆ 1 - ALE WYGODNIEJSZA

a) $\frac{5x^2-12}{(x^2-6x+13)^2} = \frac{Ax+B}{(x^2-6x+13)} + \frac{Cx+D}{(x^2-6x+13)^2}$ $\Big|(x^2-6x+13)$

$5x^2-12 = (Ax+B)(x^2-6x+13) + Cx + D$

$5x^2-12 = Ax^3 - 6Ax^2 + 13Ax + Bx^2 - 6Bx + 13B + Cx + D$

$5x^2-12 = Ax^3 + x^2(-6A+B) + x(13A-6B+C) + (13B+D)$

$\begin{cases} A=0 \\ -6A+B=5 \Rightarrow B=5 \\ \underbrace{13A}_{0} - \underbrace{6B}_{-30} + C = 0 \Rightarrow C=30 \\ 13B+D=-12 \\ D=-12-65=-77 \end{cases}$ $(x-3)^2 = x^2-6x+9$

$\int\frac{5x^2-12}{x^2-6x+13}\,dx = \int\frac{5}{(x^2-6x+13)}\,dx + \int\frac{30x-77}{(x^2-6x+13)^2}\,dx$

$\int\frac{5}{(x-3)^2+4}\,dx = \left|\begin{array}{l} x-3=2t \\ 1\,dx = 2\,dt\end{array}\right| = \frac{5}{4}\int\frac{dt}{t^2+1} =$

$= \frac{5}{2}\int\frac{dt}{t^2+1}\,dt = \frac{5}{2}\arctan\left(\frac{x-3}{2}\right)+C$

$\int\frac{30x-77}{[(x-3)^2+4]^2}\,dx = 30\int\frac{x}{[(x-3)^2+4]^2}\,dx - 77\int\frac{dx}{[(x-3)^2+4]^2}\,dx$

$30\int\frac{x}{[(x-3)^2+4]^2}\,dx = \left|\begin{array}{l} x-3=2t \\ 1\,dx = 2\,dt \\ x=2t+3\end{array}\right| = 30\int\frac{(2t+3)\cdot 2\,dt}{[4t^2+4]^2} = \frac{15}{4}\int\frac{2t+3}{[t^2+1]^2} = \frac{15}{4}\int\frac{2t+3}{t^4+2t^2+1}$

$\frac{2t+3}{t^4+2t^2+1}$

$\frac{1}{(t-1)(t+1)} = \frac{A}{(t-1)} + \frac{B}{(t+1)}$ $\Big|\cdot(t^2-1)$

$1 = A(t+1) + B(t-1)$

$1 = At + A + Bt - B = t(A+B) + (A-B)$

$\begin{cases} A+B=0 \Rightarrow B=-A=-\frac{1}{2} \\ A-B=1 \\ A+A=1 \Rightarrow A=\frac{1}{2}, \end{cases}$

$2\cdot\frac{15}{4}\int\frac{1}{(t-1)}\,dt - 3\cdot\frac{15}{4}\int\frac{1}{t+1}\,dt = \log|t-1| - \log|t+1| + C =$

$= \log|\sqrt{x+3}-1| - \log|\sqrt{x+3}+1| + C$

n) $\int\frac{x^3+x-1}{(x^2+2)^2} =$

$\frac{x^3+x-1}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$ $\Big|\cdot(x^2+2)^2$

$x^3+x-1 = (Ax+B)(x^2+2) + Cx + D$

$\text{—''—} = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$

$x^3+x-1 = x^3(A) + x^2(B) + x(2A+C) + (2B+D)$

$\begin{cases} A=1 \\ B=0 \\ 2A+C=1 \Rightarrow 2+C=1 \Rightarrow C=-1 \\ 2B+D=-1 \Rightarrow D=-1 \end{cases}$

$\int\frac{x}{(x^2+2)}\,dx = \int\frac{x+1}{(x^2+2)^2}\,dx = \frac{1}{2}\log|x^2+2| + \frac{1}{2}\cdot\frac{1}{(x^2+2)} - \frac{1}{4}\left[\frac{x}{x^2+2} + \frac{1}{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\right]$

$\frac{1}{2}\int\frac{2x}{x^2+2}\,dx = \frac{1}{2}\log|x^2+2|$

$\int\frac{x\,dx}{(x^2+2)^2} + \int\frac{1\,dx}{(x^2+2)^2}$

WZÓR: $\int\frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\int\frac{dx}{(1+x^2)^{n-1}}$

$\int\frac{x\,dx}{(x^2+2)^2} = \left|\begin{array}{l} x^2+2=t \\ 2x\,dx = 1\,dt \\ x\,dx = \frac{1}{2}\,dt\end{array}\right| = \frac{1}{2}\int\frac{dt}{t^2} = -\frac{1}{2}\cdot\frac{1}{t}$

$\int\frac{1\,dx}{(x^2+2)^2} = \left|\begin{array}{l} x=\sqrt{2}t \\ 1\,dx = \sqrt{2}\,dt\end{array}\right| = \int\frac{\sqrt{2}\,dt}{(2t^2+2)^2} = \frac{\sqrt{2}}{4}\int\frac{dt}{(t^2+1)^2} =$

$= \frac{\sqrt{2}}{4}\left[\frac{t}{2(2-1)(1+t^2)^{2-1}} + \frac{2\cdot 2-3}{2(2-1)}\int\frac{dt}{(1+t^2)^{2-1}}\right] =$

$= \frac{\sqrt{2}}{4}\left[\frac{t}{2(t^2+1)} + \frac{1}{2}\arctan(t)\right] = \frac{\sqrt{2}}{4}\left[\frac{\frac{x}{\sqrt{2}}}{2(\frac{x^2}{2}+1)} + \frac{1}{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\right] =$

$= \frac{1}{4}\left[\frac{x}{x^2+2} + \frac{1}{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\right]$

af) $\int\frac{1}{\sqrt{1+\sqrt[3]{x+2}}}\,dx = \left|\begin{array}{l} x+2=t^3 \\ 1\,dx = 3t^2\,dt \\ t=\sqrt[3]{x+2}\end{array}\right| = 3\int\frac{t^2\,dt}{\sqrt{1+t}}$

$= 3\int\frac{t^2\,dt}{\sqrt{1+t}} = \left|\begin{array}{l} 1+t=s^2 \\ 1\,dt = 2s\,ds \\ t=s^2-1\end{array}\right| = 3\int\frac{(s^2-1)^2\,2s\,ds}{s}$

$= 6\int s^4 - 2s^2 + 1\,ds =$

$= 6\left[\frac{s^5}{5} - 2\frac{s^3}{3} + s\right]$ + WSTAWIĆ

j) $\int\frac{4x+3}{(x-2)^3} = \left|\begin{array}{l} x-2=t \\ 1\,dx = 1\,dt \\ x=t+2\end{array}\right| = \int\frac{4(t+2)+3}{t^3}\,dt =$

$= \int\frac{4t+11}{t^3}\,dt = 4\int\frac{1t}{t^2}\,dt + 11\int\frac{1}{t^3}\,dt =$

$= 4\cdot\left(-\frac{1}{t}\right) + 11\cdot\left(-\frac{1}{2t^2}\right) = \frac{-4}{x-2} - \frac{11}{2(x-2)^2}$

ae) $\int\frac{1}{(x^2+2x+2)(x^2-4)}\,dx$

$\frac{1}{(\quad)} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-4}$ $\Big|\cdot(\quad)$

$1 = (Ax+B)(x^2-4) + (Cx+D)(x^2+2x+2)$

$1 = Ax^3 - 4Ax + Bx^2 - 4B + Cx^3 + 2Cx^2 + 2Cx + Dx^2 + 2Dx + 2D$

$1 = x^3(A+C) + x^2(B+2C+D) + x(-4A+2C+2D) + (-4B+2D)$

$\begin{cases} A+C=0 \Rightarrow C=-A \\ B-2A+D=0 \Rightarrow D=-B+2A=A \\ -4A-2A-2B+4A=0 \Rightarrow -2A=2B \\ 4A+2A=6A=1 \quad A=\frac{1}{6} \end{cases}$ $A=-B$

$B=-\frac{1}{6}$
$C=-\frac{1}{6}$
$D=\frac{1}{6}$

$\int\frac{1}{(\quad)}\,dx = -\frac{1}{6}\int\frac{2x-1}{(x^2+2x+2)}\,dx - \frac{1}{6}\int\frac{x-1}{x^2-4}\,dx$

$2x+2$

g) $\int\frac{1}{1+\sqrt[3]{x+1}}\,dx = \left|\begin{array}{l} x+1=t^3 \\ 1\,dx = 3t^2\,dt\end{array}\right| =$

$= 3\int\frac{t^2}{1+t}\,dt = \left|\begin{array}{l} 1+t=s \\ 1\,dt = 1\,ds \\ t=s-1\end{array}\right| = 3\int\frac{(s-1)^2}{s}\,ds =$

$= 3\int\frac{s^2-2s+1}{s}\,ds = 3\int\frac{s^2}{s} - 2\frac{s}{s} + \frac{1}{s}\,ds =$

$= 3\left[\frac{s^2}{2} - 2s + \log|s|\right] + C =$

$= 3\left[\frac{(1+t)^2}{2} - 2(1+t) + \log|1+t|\right] + C =$

$= 3\left[\frac{(1+\sqrt[3]{x})^2}{2} - 2(1+\sqrt[3]{x}) + \log|1+\sqrt[3]{x}|\right] + C$

h) $\int\frac{x^2}{x^2+1}\,dx = \int\frac{x^2+1-1}{x^2+1}\,dx = \int 1\,dx - \int\frac{1}{x^2+1}\,dx = x - \arctan(x) + C$

i) $\int\frac{x^2+1}{x^3-x^2}\,dx = \int\frac{x^3-x^2+x^2+1}{x^3-x^2}\,dx =$

$= \int 1\,dx + \int\frac{x^2+1}{x^2(x-1)}$

$\frac{x^2+1}{x^2(x-1)} = \frac{A^{x-1}}{x} + \frac{B^{x-1}}{x^2} + \frac{C^{x^2}}{x-1}$ $\Big|\cdot(\quad)$

$x^2+1 = A(x^2-x) + B(x-1) + Cx^2$

$x^2+1 = Ax^2 - Ax + Bx - B + Cx^2$

$x^2+1 = x^2(A+C) + x(-A+B) - B$

$\begin{cases} A+C=1 \Rightarrow C=2 \\ -A+B=0 \Rightarrow -A-1=0 \Rightarrow A=-1 \\ -B=1 \Rightarrow B=-1 \end{cases}$

$2\int\frac{1}{x}\,dx - \int\frac{1}{x^2} + 2\int\frac{x}{x-1} = -\log|x| + \frac{1}{x} + 2\log|x-1| + C$

aj) $\int\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx$

$\frac{2x^2+41x-91}{(\quad)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}$ $\Big|\cdot(\quad)$

$2x^2+41x-91 = A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x+3)$

$\text{—''—} = A(x^2-x-12) + B(x^2-5x+4) + C(x^2+2x-3)$

$2x^2+41x-91 = x^2(A+B+C) + x(-A-5B+2C) + (-12A+4B-3C)$

$\begin{cases} A+B+C=2 \Rightarrow B=2-A-C=2-A-\frac{-51-4A}{7} = \frac{14-7A-51+4A}{7} = \frac{-3A-37}{7} \\ -A-5B+2C=41 \\ -12A+4B-3C=-91 \end{cases}$

$-A-5(2-A-C)+2C=41$

$-A-10+5A+7C=41$

$4A+7C=51$

$C=\frac{51-4A}{7}$

$-12A+\frac{-12A-148}{7} - \frac{153-12A}{7} = -91 \quad \Big|\cdot 7$

$-12A+-12A-148-153+12A=-637$

$-12A=-336 \quad \Big|:(-12)$

$A=28$

$C=\frac{51-112}{7}=\frac{17}{7}$

$B=\frac{112}{7}$

$28\int\frac{1}{x-1} + \frac{112}{7}\int\frac{1}{x+3} + \frac{17}{7}\int\frac{1}{x-4} = 28\log|x-1| + \frac{112}{7}\log|x+3| + \frac{17}{7}\log|x-4|$

o) $\int\frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,dx = \left|\begin{array}{l} x=t^3 \\ 1\,dx = 3t^2\,dt \\ \sqrt{x}=\sqrt{t^3}=t\sqrt{t}\end{array}\right| =$

$= 3\int\frac{t\sqrt{t}}{t\sqrt{t}-t} = 3\int\frac{t^2\sqrt{t}}{\sqrt{t}-1}\,dt = \left|\begin{array}{l} \sqrt{t}-1=k \\ \sqrt{t}=k+1 \\ t=(k+1)^2 \\ 1\,dt = 2(k+1)\,dk\end{array}\right| =$

$= 6\int\frac{(k+1)^4(k+1)^2}{k}\,dk = 6\int\frac{(k+1)^6}{k} =$ $\quad t=\sqrt[3]{x}$, $k=\sqrt[6]{x}-1$

$= 6\int\frac{k^6+6k^5+15k^4+20k^3+15k^2+6k+1}{k}\,dk =$

$= 6\int(k^5+6k^4+15k^3+20k^2+15k+6+\frac{1}{k})\,dk =$

$= 6\left[\frac{k^6}{6} + 6\cdot\frac{k^5}{5} + 15\frac{k^4}{4} + 20\frac{k^3}{3} + 15\frac{k^2}{2} + 6k + \log|k|\right]$ + WSTAWIĆ

c) $\int\arctan(\sqrt{x})\,dx = \left|\begin{array}{l} x=t^2 \\ 1\,dx = 2t\,dt\end{array}\right| =$

$= 2\int t\arctan(t)\,dt = \left|\begin{array}{ll} g=\arctan(t) & f'=t \\ g'=\frac{1}{1+t^2} & f=\frac{t^2}{2}\end{array}\right| =$

$= 2\left[\frac{1}{2}t^2\cdot\arctan(t) - \frac{1}{2}\int\frac{t^2}{1+t^2}\,dt\right] =$

$= 2\left[\frac{1}{2}x^1\arctan(\sqrt{x}) - \frac{1}{2}[\sqrt{x} - \arctan\sqrt{x}]\right]$

$\int\frac{t^2}{1+t^2}\,dt = \int\frac{t^2+1-1}{t^2+1}\,dt =$

$= \int 1\,dt - \int\frac{1}{t^2+1} = t - \arctan(t)$

b) $\int\frac{x-2}{x^2-7x+12}\,dx = \int\frac{x-2}{(x-4)(x-3)}\,dx$

$(x-4)(x-3)$

$\frac{x-2}{(x-4)(x-3)} = \frac{A}{(x-4)} + \frac{B}{(x-3)}$ $\Big|\cdot(\quad)$

$x-2 = Ax-3A + Bx-4B$

$x-2 = x(A+B) + (-3A-4B)$

$\begin{cases} A+B=1 \Rightarrow B=1-A \\ -3A-4B=-2 \end{cases}$

$-3A-4+4A=-2$

$A=2$

$B=-1$

$2\int\frac{1}{x-4}\,dx - \int\frac{1}{x-3}\,dx = 2\log|x-4| - \log|x-3| + C$