Lecture Notes in Category Theory

Paul Ossientis

December 12, 2019

Contents

Chapter 1

Category

1.1 Definition

Before we define a category in full generality, we shall focus our attention on the notion of *small category*. This notion is interesting to us because while it essentially describes the notion of *category* itself, it remains simple enough to be compared with various other algebraic structures. So let us look at a monoid: a monoid is essentially a set M together with a binary relation $\circ$ defined on M which is associative, and an element e of M which acts as an identity element for $\circ$. In short a monoid is a tuple $(M, \circ, e)$ containing some data, and which satisfy certain properties. The same is true of a *small category*: it is also a tuple containing some data, and which satisfy certain properties:

Definition 1 *We call* small category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *with:*

(1)	Ob *is a set*
(2)	Arr *is a set*
(3)	$\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$ *is a function*
(4)	$\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$ *is a function*
(5)	$\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ *is a function*
(6)	$\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$ *is a partial function*
(7)	$g \circ f$ *is defined* $\Leftrightarrow$ $\mathrm{cod}(f) = \mathrm{dom}(g)$
(8)	$\mathrm{cod}(f) = \mathrm{dom}(g)$ $\Rightarrow$ $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$
(9)	$\mathrm{cod}(f) = \mathrm{dom}(g)$ $\Rightarrow$ $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$
(10)	$\mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h)$ $\Rightarrow$ $(h \circ g) \circ f = h \circ (g \circ f)$
(11)	$\mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,)$
(12)	$\mathrm{dom}(f) = a$ $\Rightarrow$ $f \circ \mathrm{id}(a) = f$
(13)	$\mathrm{cod}(f) = a$ $\Rightarrow$ $\mathrm{id}(a) \circ f = f$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

So if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a small category, we have two sets Ob and Arr together with some structure defined on these sets. This feels very much like a monoid, except that we have two sets instead of one and it all looks more complicated. The set Ob is called the *set of objects* of the small category $\mathcal{C}$ and is denoted Ob $\mathcal{C}$, while the set Arr is called the *set of arrows* of the small category $\mathcal{C}$ and is denoted Arr $\mathcal{C}$. An element $x \in \mathrm{Ob}\ \mathcal{C}$ is called an *object* of $\mathcal{C}$, while an element $f \in \mathrm{Arr}\ \mathcal{C}$ is called an *arrow* of $\mathcal{C}$. As part of the structure defined on the small category $\mathcal{C}$, we have two functions dom : Arr $\rightarrow$ Ob and cod : Arr $\rightarrow$ Ob. Hence, given an arrow f of the small category $\mathcal{C}$, we have two objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ of the small category $\mathcal{C}$. The object $\mathrm{dom}(f)$ is called the *domain* of f or the *source* of f. The object $\mathrm{cod}(f)$ is called the *codomain* of f or the *target* of f. Note that an arrow f of the small category $\mathcal{C}$ is simply an element of the set Arr $\mathcal{C}$. So it is itself a set but it may not be a function. The words *domain* and *codomain* are therefore overloaded as we are using them in relation to a set f which is possibly not a function. Whenever f is an arrow of the small category $\mathcal{C}$ and a, b are objets, it is common to use the notation $f : a \rightarrow b$ as a notational shortcut for the equations $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Once again, it is important to guard against the possible confusion induced by the notation $f : a \rightarrow b$ which does not mean that f is function. It simply means that f is an arrow with domain a and codomain b.

Definition 2 *We call* category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *such that:*

(1)   Ob *is a collection with equality*

(2)   Arr *is a collection with equality*

(3) $\mathrm{dom} : \mathrm{Arr} \rightarrow \mathrm{Ob}$ *is a map*

(4) $\mathrm{cod} : \mathrm{Arr} \rightarrow \mathrm{Ob}$ *is a map*

(5) $\mathrm{id} : \mathrm{Ob} \rightarrow \mathrm{Arr}$ *is a map*

(6) $\circ : \mathrm{Arr} \times \mathrm{Arr} \rightarrow \mathrm{Arr}$ *is a partial map*

(7) $g \circ f$ *is defined* $\Leftrightarrow$ $\mathrm{cod}(f) = \mathrm{dom}(g)$

(8) $\mathrm{cod}(f) = \mathrm{dom}(g)$ $\Rightarrow$ $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$

(9) $\mathrm{cod}(f) = \mathrm{dom}(g)$ $\Rightarrow$ $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

(10) $\mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h)$ $\Rightarrow$ $(h \circ g) \circ f = h \circ (g \circ f)$

(11) $\mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,)$

(12) $\mathrm{dom}(f) = a$ $\Rightarrow$ $f \circ \mathrm{id}(a) = f$

(13) $\mathrm{cod}(f) = a$ $\Rightarrow$ $\mathrm{id}(a) \circ f = f$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

Chapter 2

Functor

Chapter 3

Natural Transformation

Chapter 4

Adjunction

4.1 Definition

Definition 3 *We call* adjunction *an ordered pair* (F, G) *where* F *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *and* G *is a functor* $G : \mathcal{D} \to \mathcal{C}$ *while* $\mathcal{C}$ *and* $\mathcal{D}$ *are two locally-small categories for which there exists a natural isomorphism:*

$$\alpha \; : \; \mathcal{D} \circ (F \times I_{\mathcal{D}}) \; \Rightarrow \; \mathcal{C} \circ (I_{\mathcal{C}^{op}} \times G)$$

in the functor category $[\, \mathcal{C}^{op} \times \mathcal{D} \,, \, \mathcal{S}et \,]$, *where* F *also denotes* $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$.