Lecture Notes in Category Theory

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Contents

Chapter 1

Category

1.1 Definition

Before we define a category in full generality, we shall focus our attention on the notion of *small category*. This notion is interesting to us because while it essentially describes the notion of *category* itself, it remains simple enough to be compared with various other algebraic structures. So let us look at a monoid: a monoid is essentially a set M together with a binary relation $\circ$ defined on M which is associative, and an element e of M which acts as an identity element for $\circ$. In short a monoid is a tuple $(M, \circ, e)$ containing some data, and which satisfy certain properties. The same is true of a *small category*: it is also a tuple containing some data, and which satisfy certain properties:

Definition 1 *We call* small category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *with:*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} \textit{ is a set} \\
(2) & \mathrm{Arr} \textit{ is a set} \\
(3) & \mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(4) & \mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(5) & \mathrm{id} : \mathrm{Ob} \to \mathrm{Arr} \textit{ is a map} \\
(6) & \circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr} \textit{ is a partial map} \\
(7) & g \circ f \textit{ is defined} \Leftrightarrow \mathrm{cod}(f) = \mathrm{dom}(g) \\
(8) & \mathrm{cod}(f) = \mathrm{dom}(g) \Rightarrow \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\
(9) & \mathrm{cod}(f) = \mathrm{dom}(g) \Rightarrow \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\
(10) & \mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h) \Rightarrow (h \circ g) \circ f = h \circ (g \circ f) \\
(11) & \mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,) \\
(12) & \mathrm{dom}(f) = a \Rightarrow f \circ \mathrm{id}(a) = f \\
(13) & \mathrm{cod}(f) = a \Rightarrow \mathrm{id}(a) \circ f = f
\end{array}
$$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

Definition 2 *We call* category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *such that:*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} \textit{ is a collection with equality} \\
(2) & \mathrm{Arr} \textit{ is a collection with equality} \\
(3) & \mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(4) & \mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(5) & \mathrm{id} : \mathrm{Ob} \to \mathrm{Arr} \textit{ is a map} \\
(6) & \circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr} \textit{ is a partial map} \\
(7) & g \circ f \textit{ is defined } \Leftrightarrow \mathrm{cod}(f) = \mathrm{dom}(g) \\
(8) & \mathrm{cod}(f) = \mathrm{dom}(g) \Rightarrow \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\
(9) & \mathrm{cod}(f) = \mathrm{dom}(g) \Rightarrow \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\
(10) & \mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h) \Rightarrow (h \circ g) \circ f = h \circ (g \circ f) \\
(11) & \mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,) \\
(12) & \mathrm{dom}(f) = a \Rightarrow f \circ \mathrm{id}(a) = f \\
(13) & \mathrm{cod}(f) = a \Rightarrow \mathrm{id}(a) \circ f = f
\end{array}
$$

where (7) – (13) *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

Chapter 2

Functor

Chapter 3

Natural Transformation

Chapter 4

Adjunction

4.1 Definition

Definition 3 *We call* adjunction *an ordered pair* (F, G) *where* F *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *and* G *is a functor* $G : \mathcal{D} \to \mathcal{C}$ *while* $\mathcal{C}$ *and* $\mathcal{D}$ *are two locally-small categories for which there exists a natural isomorphism:*

$$\alpha \ : \ \mathcal{D} \circ (F \times I_{\mathcal{D}}) \ \Rightarrow \ \mathcal{C} \circ (I_{\mathcal{C}^{op}} \times G)$$

in the functor category $[\,\mathcal{C}^{op} \times \mathcal{D}\,,\, \mathcal{S}et\,]$*, where* F *also denotes* $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$.