Lecture Notes in Category Theory

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Chapter 1

Category

1.1 Small Category

Before we define a category in full generality, we shall focus our attention on the notion of *small category*. This notion is interesting to us because while it essentially describes the notion of *category* itself, it remains simple enough to be compared with various other algebraic structures. For example, consider the case of a monoid: a monoid is essentially a set M together with a binary relation $\circ$ defined on M which is associative, and an element e of M which acts as an identity element for $\circ$. In short a monoid is a tuple $(M, \circ, e)$ containing some data, and which satisfy certain properties. The same is true of a *small category*: it is also a tuple containing some data, and which satisfy certain properties:

Definition 1 *We call* small category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *with:*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} \textit{ is a set} \\
(2) & \mathrm{Arr} \textit{ is a set} \\
(3) & \mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a function} \\
(4) & \mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a function} \\
(5) & \mathrm{id} : \mathrm{Ob} \to \mathrm{Arr} \textit{ is a function} \\
(6) & \circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr} \textit{ is a partial function} \\
(7) & g \circ f \textit{ is defined} \ \Leftrightarrow \ \mathrm{cod}(f) = \mathrm{dom}(g) \\
(8) & \mathrm{cod}(f) = \mathrm{dom}(g) \ \Rightarrow \ \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\
(9) & \mathrm{cod}(f) = \mathrm{dom}(g) \ \Rightarrow \ \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\
(10) & \mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h) \ \Rightarrow \ (h \circ g) \circ f = h \circ (g \circ f) \\
(11) & \mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,) \\
(12) & \mathrm{dom}(f) = a \ \Rightarrow \ f \circ \mathrm{id}(a) = f \\
(13) & \mathrm{cod}(f) = a \ \Rightarrow \ \mathrm{id}(a) \circ f = f
\end{array}
$$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

So if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a small category, we have two sets Ob and Arr together with some structure defined on these sets. This feels very much like a monoid, except that we have two sets instead of one and it all looks more complicated. The set Ob is called the *set of objects* of the small category $\mathcal{C}$ and is denoted $\mathrm{Ob}\,\mathcal{C}$, while the set Arr is called the *set of arrows* of the small category $\mathcal{C}$ and is denoted $\mathrm{Arr}\,\mathcal{C}$. An element $x \in \mathrm{Ob}\,\mathcal{C}$ is called an *object* of $\mathcal{C}$, while an element $f \in \mathrm{Arr}\,\mathcal{C}$ is called an *arrow* of $\mathcal{C}$.

As part of the structure defined on the small category $\mathcal{C}$, we have two functions $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$ and $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$. Hence, given an arrow f of the small category $\mathcal{C}$, we have two objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ of the small category $\mathcal{C}$. The object $\mathrm{dom}(f)$ is called the *domain* of f. The object $\mathrm{cod}(f)$ is called the *codomain* of f. Note that an arrow f of the small category $\mathcal{C}$ is simply an element of the set $\mathrm{Arr}\,\mathcal{C}$. So it is itself a set but it may not be a function. The words *domain* and *codomain* are therefore overloaded as we are using them in relation to a set f which is possibly not a function. Whenever f is an arrow of the small category $\mathcal{C}$ and a, b are objets, it is common to use the notation $f : a \to b$ as a notational shortcut for the equations $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Once again, it is important to guard against the possible confusion induced by the notation $f : a \to b$ which does not mean that f is function. It simply means that f is an arrow with domain a and codomain b in the small category $\mathcal{C}$.

One of the main ingredients of the structure defining a small category $\mathcal{C}$ is the partial function $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$, called the *composition operator* in the small category $\mathcal{C}$. The domain of this partial function is made of all ordered pairs (g, f) of arrows in $\mathcal{C}$ for which $\mathrm{cod}(f) = \mathrm{dom}(g)$. As already indicated in definition (1), we use the infix notation $g \circ f$ rather than $\circ(g, f)$ and the arrow $g \circ f$ is called the *composition* of g and f. Once again, we should remember that the notation $g \circ f$ does not mean that g or f are functions. They are simply arrows in the small category $\mathcal{C}$. One key property of the composition operator $\circ$ is the associativity postulated by (10) of definition (1). Note that if $f : a \to b$ and $g : b \to c$, then from properties (8) and (9) of definition (1) we obtain $g \circ f : a \to c$. Furthermore, if $h : c \to d$ we have $h \circ g : b \to d$ and the arrows $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are therefore well-defined arrows with domain a and codomain d. This shows that the expression involved in the associativity condition (10) of definition (1) is always meaningful, involving terms which are well-defined provided $g \circ f$ and $h \circ g$ are themselves well-defined, i.e. provided $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$.

Finally, as part of the structure defining the small category $\mathcal{C}$, we have a function $id : \mathrm{Ob} \to \mathrm{Arr}$ called the *identity operator* on the small category $\mathcal{C}$. Hence, for every object a of $\mathcal{C}$ we have an arrow $id(a)$ called the *identity at a*. Looking at property (11) of definition (1) we see that $id(a) : a \to a$. In other words, the arrow $id(a)$ has domain a and codomain a. Furthermore, looking at properties (12) and (13) of definition (1), for every arrow $f : a \to b$, the composition arrows $id(b) \circ f$ and $f \circ id(a)$ are well-defined and both equal to f.

1.2 Category

The notion of *small category* defined in definition (1) is similar to that of any other algebraic structure the reader may be familiar with. It can safely be encoded in set theory as a tuple (which is a set) containing data (which are other sets) which satisfies certain properties. In set theory, everything is a set. A small category $\mathcal{C}$ is a set, its collection of objects $\mathrm{Ob}\ \mathcal{C}$ is a set, its arrows $\mathrm{Arr}\ \mathcal{C}$ form a set, the functions dom, cod, id and the composition operator $\circ$ are all sets (functions are typically encoded as sets of ordered pairs).

Category theory falls outside of set theory. While the definition of a *category* we provide below is formally identical to that of a small category, the object we are defining can no longer be encoded in general as an object of set theory. For example, say we want to speak about the *universe of all sets* or the *universe of all monoids*. These *universes* which are known as *classes* cannot be represented as sets. They are not objects of set theory. Or say we are working within the formal framework of a proof assistant such as Coq, Agda or Lean. These tools are based on type theory and do not fall within the scope of set theory. When defining a *category*, we assume some form of meta-theoretic context, some form of logic, some way of reasoning about objects which may not be sets, where some meaning is attached to the words *tuple*, *collection*, *equality* and *map*. This may sound all very fuzzy, yet we cannot be more formal at this stage.

Definition 2 *We call* category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *such that:*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} \textit{ is a collection with equality} \\
(2) & \mathrm{Arr} \textit{ is a collection with equality} \\
(3) & \mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(4) & \mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a map} \\
(5) & \mathrm{id} : \mathrm{Ob} \to \mathrm{Arr} \textit{ is a map} \\
(6) & \circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr} \textit{ is a partial map} \\
(7) & g \circ f \textit{ is defined } \Leftrightarrow\ \mathrm{cod}(f) = \mathrm{dom}(g) \\
(8) & \mathrm{cod}(f) = \mathrm{dom}(g)\ \Rightarrow\ \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\
(9) & \mathrm{cod}(f) = \mathrm{dom}(g)\ \Rightarrow\ \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\
(10) & \mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h)\ \Rightarrow\ (h \circ g) \circ f = h \circ (g \circ f) \\
(11) & \mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,) \\
(12) & \mathrm{dom}(f) = a\ \Rightarrow\ f \circ \mathrm{id}(a) = f \\
(13) & \mathrm{cod}(f) = a\ \Rightarrow\ \mathrm{id}(a) \circ f = f
\end{array}
$$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

So here we are.

Chapter 2

Functor

Chapter 3

Natural Transformation

Chapter 4

Adjunction

4.1 Definition

Definition 3 *We call* adjunction *an ordered pair* (F, G) *where* F *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *and* G *is a functor* $G : \mathcal{D} \to \mathcal{C}$ *while* $\mathcal{C}$ *and* $\mathcal{D}$ *are two locally-small categories for which there exists a natural isomorphism:*

$$\alpha \; : \; \mathcal{D} \circ (F \times I_{\mathcal{D}}) \; \Rightarrow \; \mathcal{C} \circ (I_{\mathcal{C}^{op}} \times G)$$

in the functor category $[\,\mathcal{C}^{op} \times \mathcal{D}\,,\, \mathcal{S}et\,]$*, where* F *also denotes* $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$*.*