Lecture Notes on Category Theory

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Chapter 1

Category

1.1 Small Category

Before we define a category in full generality, we shall focus our attention on the notion of *small category*. This notion is interesting to us because while it essentially describes the notion of *category* itself, it remains simple enough to be compared with various other algebraic structures. For example, consider the case of a monoid: a monoid is essentially a set M together with a binary relation $\circ$ defined on M which is associative, and an element e of M which acts as an identity element for $\circ$. In short a monoid is a tuple $(M, \circ, e)$ containing some data, and which satisfies certain properties. The same is true of a *small category*: it is also a tuple containing some data, and which satisfies certain properties:

Definition 1 *We call* small category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *with:*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} \textit{ is a set} \\
(2) & \mathrm{Arr} \textit{ is a set} \\
(3) & \mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a function} \\
(4) & \mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob} \textit{ is a function} \\
(5) & \mathrm{id} : \mathrm{Ob} \to \mathrm{Arr} \textit{ is a function} \\
(6) & \circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr} \textit{ is a partial function} \\
(7) & g \circ f \textit{ is defined } \Leftrightarrow \ \mathrm{cod}(f) = \mathrm{dom}(g) \\
(8) & \mathrm{cod}(f) = \mathrm{dom}(g) \ \Rightarrow \ \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\
(9) & \mathrm{cod}(f) = \mathrm{dom}(g) \ \Rightarrow \ \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\
(10) & \mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h) \ \Rightarrow \ (h \circ g) \circ f = h \circ (g \circ f) \\
(11) & \mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,) \\
(12) & \mathrm{dom}(f) = a \ \Rightarrow \ f \circ \mathrm{id}(a) = f \\
(13) & \mathrm{cod}(f) = a \ \Rightarrow \ \mathrm{id}(a) \circ f = f
\end{array}
$$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

So if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a small category, we have two sets Ob and Arr together with some structure defined on these sets. This feels very much like a monoid, except that we have two sets instead of one and it all looks more complicated. The set Ob is called the *set of objects* of the small category $\mathcal{C}$ and is denoted $\mathrm{Ob}\ \mathcal{C}$, while the set Arr is called the *set of arrows* of the small category $\mathcal{C}$ and is denoted $\mathrm{Arr}\ \mathcal{C}$. An element $x \in \mathrm{Ob}\ \mathcal{C}$ is called an *object* of $\mathcal{C}$, while an element $f \in \mathrm{Arr}\ \mathcal{C}$ is called an *arrow* of $\mathcal{C}$.

As part of the structure defined on the small category $\mathcal{C}$, we have two functions $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$ and $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$. Hence, given an arrow f of the small category $\mathcal{C}$, we have two objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ of the small category $\mathcal{C}$. The object $\mathrm{dom}(f)$ is called the *domain* of f. The object $\mathrm{cod}(f)$ is called the *codomain* of f. Note that an arrow f of the small category $\mathcal{C}$ is simply an element of the set $\mathrm{Arr}\ \mathcal{C}$. So it is itself a set but it may not be a function. The words *domain* and *codomain* are therefore overloaded as we are using them in relation to a set f which is possibly not a function. Whenever f is an arrow of the small category $\mathcal{C}$ and a, b are objets, it is common to use the notation $f : a \to b$ as a notational shortcut for the equations $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Once again, it is important to guard against the possible confusion induced by the notation $f : a \to b$ which does not mean that f is function. It simply means that f is an arrow with domain a and codomain b in the small category $\mathcal{C}$.

One of the main ingredients of the structure defining a small category $\mathcal{C}$ is the partial function $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$, called the *composition operator* in the small category $\mathcal{C}$. The domain of this partial function is made of all ordered pairs (g, f) of arrows in $\mathcal{C}$ for which $\mathrm{cod}(f) = \mathrm{dom}(g)$. As already indicated in definition (1), we use the infix notation $g \circ f$ rather than $\circ(g, f)$ and the arrow $g \circ f$ is called the *composition* of g and f. Once again, we should remember that the notation $g \circ f$ does not mean that g or f are functions. They are simply arrows in the small category $\mathcal{C}$. One key property of the composition operator $\circ$ is the associativity postulated by (10) of definition (1). Note that if $f : a \to b$ and $g : b \to c$, then from properties (8) and (9) of definition (1) we obtain $g \circ f : a \to c$. Furthermore, if $h : c \to d$ we have $h \circ g : b \to d$ and the arrows $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are therefore well-defined arrows with domain a and codomain d. This shows that the expression involved in the associativity condition (10) of definition (1) is always meaningful, involving terms which are well-defined provided $g \circ f$ and $h \circ g$ are themselves well-defined, i.e. provided $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$.

Finally, as part of the structure defining the small category $\mathcal{C}$, we have a function $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ called the *identity operator* on the small category $\mathcal{C}$. Hence, for every object a of $\mathcal{C}$ we have an arrow $\mathrm{id}(a)$ called the *identity at* a. Looking at property (11) of definition (1) we see that $\mathrm{id}(a) : a \to a$. In other words, the arrow $\mathrm{id}(a)$ has domain a and codomain a. Furthermore, looking at properties (12) and (13) of definition (1), for every arrow $f : a \to b$, the composition arrows $\mathrm{id}(b) \circ f$ and $f \circ \mathrm{id}(a)$ are well-defined and both equal to f.

1.2 Category

The notion of *small category* defined in definition (1) is similar to that of any other algebraic structure the reader may be familiar with. It can safely be encoded in set theory as a tuple (which is a set) containing data (which are other sets) which satisfies certain properties. In set theory, everything is a set. A small category $\mathcal{C}$ is a set, its collection of objects $\mathrm{Ob}\ \mathcal{C}$ is a set, its arrows $\mathrm{Arr}\ \mathcal{C}$ form a set, the functions dom, cod, id and the composition operator $\circ$ are all sets (functions are typically encoded as sets of ordered pairs).

Category theory falls outside of set theory. While the definition of a *category* we provide below is formally identical to that of a small category, the object we are defining can no longer be encoded in general as an object of set theory. For example, say we want to speak about the *universe of all sets* or the *universe of all monoids*. These *universes* which are known as *classes* cannot be represented as sets. They are not objects of set theory. Or say we are working within the formal framework of a proof assistant such as Coq, Agda or Lean. These tools are based on type theory and do not fall within the scope of set theory. When defining a *category*, we assume some form of meta-theoretic context, some form of logic, some way of reasoning about objects which may not be sets, where some meaning is attached to the words *tuple*, *collection*, *equality* and *map*. This may sound all very fuzzy, yet we cannot be more formal at this stage.

Definition 2 *We call* category *any tuple* $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *such that:*

(1) Ob *is a collection with equality*

(2) Arr *is a collection with equality*

(3) $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$ *is a map*

(4) $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$ *is a map*

(5) $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ *is a map*

(6) $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$ *is a partial map*

(7) $g \circ f$ *is defined* $\Leftrightarrow\ \mathrm{cod}(f) = \mathrm{dom}(g)$

(8) $\mathrm{cod}(f) = \mathrm{dom}(g)\ \Rightarrow\ \mathrm{dom}(g \circ f) = \mathrm{dom}(f)$

(9) $\mathrm{cod}(f) = \mathrm{dom}(g)\ \Rightarrow\ \mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

(10) $\mathrm{cod}(f) = \mathrm{dom}(g) \wedge \mathrm{cod}(g) = \mathrm{dom}(h)\ \Rightarrow\ (h \circ g) \circ f = h \circ (g \circ f)$

(11) $\mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,)$

(12) $\mathrm{dom}(f) = a\ \Rightarrow\ f \circ \mathrm{id}(a) = f$

(13) $\mathrm{cod}(f) = a\ \Rightarrow\ \mathrm{id}(a) \circ f = f$

where $(7) - (13)$ *hold for all* $f, g, h \in \mathrm{Arr}$ *and* $a \in \mathrm{Ob}$*:*

So let $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ be a category: then $\mathcal{C}$ is a *tuple* but it is no longer a tuple in a set-theoretic sense. We assume given some logical framework where the notion of *tuple* is clear, even if not formally defined. Furthermore, We are no longer imposing that Ob should be a set, but are instead using the phrase *collection with equality*, whatever this may mean in our given logical context. So we shall still make use of the notation Ob $\mathcal{C}$ but this will now refer to the *collection* of all *objects* of the category $\mathcal{C}$.

Notation 1 *Let $\mathcal{C}$ be a category. Its collection of* objects *is denoted* Ob $\mathcal{C}$.

In fact, if a is an object of the category $\mathcal{C}$, we shall abuse notations somewhat by writing '$a \in \mathrm{Ob}\ \mathcal{C}$' or even simply '$a \in \mathcal{C}$' to express the fact that a is an object of $\mathcal{C}$, being understood that this use of the set membership symbol '$\in$' does not mean anything is a set. Since we are stepping out of set theory, the objects of the category $\mathcal{C}$ may not be sets themselves. They are simply members of the *collection* Ob $\mathcal{C}$.

Notation 2 *Let $\mathcal{C}$ be a category. We write $a \in \mathcal{C}$ as a shortcut for $a \in \mathrm{Ob}\ \mathcal{C}$.*

However, properties $(7)-(13)$ of definition (2) are all referring to equalities between objects such that $\mathrm{cod}(f) = \mathrm{dom}(g)$. So it must be the case that the notion of *equality* be meaningful on the collection Ob $\mathcal{C}$. This explains our use of the phrase *collection with equality*: given $a, b \in \mathcal{C}$, the statement $a = b$ while not a set-theoretic equality is nonetheless assumed to be defined.

Similarly, the *collection* of *arrows* of the category $\mathcal{C}$ shall still be denoted Arr $\mathcal{C}$, but is no longer required to be a set. If f is an arrow of the category $\mathcal{C}$ then f itself may not be a set and we may still write '$f \in \mathrm{Arr}\ \mathcal{C}$' simply to indicate that f is a *member* of the *collection* Arr $\mathcal{C}$. Properties (10), (12) and (13) of definition (2) are all referring to equalities between arrows so the *collection* Arr $\mathcal{C}$ must have some notion of *equality* defined on it.

Notation 3 *Let $\mathcal{C}$ be a category. Its collection of* arrows *is denoted* Arr $\mathcal{C}$.

Since Ob and Arr are no longer sets in general, the *maps* $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$, $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$, $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ and the partial map $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$ cannot possibly be *functions* in the set-theoretic sense. So there must be some meaning to the word *map* (from one *collection* to another) in whatever logical framework we are working in. The *collection* $\mathrm{Arr} \times \mathrm{Arr}$ is not a set, and is simply the *collection* of all 2-dimensional *tuples* made from Arr. Our using the word *map* rather than *function* in definition (2) is simply an attempt at reminding ourselves of the fact these are not set-theoretic functions, eventhough the words *map* and *function* are perfectly interchangeable in standard (set-theoretic) mathematics. Given $f \in \mathrm{Arr}\ \mathcal{C}$, we shall still call the object $\mathrm{dom}(f)$ the *domain* of f and the object $\mathrm{cod}(f)$ the *codomain* of f.

Notation 4 *Let $\mathcal{C}$ be a category. The domain of an arrow $f \in \mathrm{Arr}\ \mathcal{C}$ is denoted* $\mathrm{dom}(f)$*, while its codomain is denoted* $\mathrm{cod}(f)$.

Remark: Notation (4) is potentially ambiguous as a mathematical object f could in principle be an arrow in several categories, and the designations $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ do not specfify which category is being referred to.

Given $a, b \in \mathcal{C}$, we shall still use the notation $f : a \to b$ as a notational shortcut for $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Hence we state:

Notation 5 *Let $\mathcal{C}$ be a category. We write $f : a \to b$ or $f : a \to b$ @ $\mathcal{C}$ as a shortcut for $f \in \mathrm{Arr}\ \mathcal{C}$ together with $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$.*

Remark: The qualification @ $\mathcal{C}$ in notation (5) may be useful to disambiguate between several categories in a given context.

The partial map $\circ$ is still the *composition operator* and the arrow $g \circ f$ shall still be called the *composition* of g and f, provided it is defined.

Notation 6 *Let $\mathcal{C}$ be a category. The composition operator on $\mathcal{C}$ is denoted $\circ$, and the composition of two arrows g and f is denoted $g \circ f$ or $g \circ f$ @ $\mathcal{C}$.*

Remark: $\circ$ may also be ambiguous in a context with several categories. It is also the common symbol to refer to standard function composition.

The map $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ is still the *identity operator* on the category $\mathcal{C}$, and for all $a \in \mathcal{C}$, the arrow $id(a) : a \to a$ is known as the *identity at a*.

Notation 7 *Let $\mathcal{C}$ be a category. We write* $\mathrm{id}(a)$ *or* $\mathrm{id}(a)$ @ $\mathcal{C}$ *to denote the* identity at a *in the category $\mathcal{C}$.*

For all arrows $f : a \to b$, it is still the case that the arrows $\mathrm{id}(b) \circ f$ and $f \circ \mathrm{id}(a)$ are well-defined and both equal to f. Just as in the case of a small category, whenever $f : a \to b$, $g : b \to c$ and $h : c \to d$, all the terms involved in the equation $(h \circ g) \circ f = h \circ (g \circ f)$ of definition (2) are well defined.

Proposition 1 *A small category is a category.*

Proof
When considering a small category $\mathcal{C}$, we are implicitely working within a set theoretic framework in which equality between any two sets is always meaningful and elements of sets are themselves sets. So any set can be viewed as a *collection with equality* and hence a small category satisfies definition (2). $\diamond$

1.3 Equality of Categories

Whichever logical framework we are working from, we saw that when defining a category $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$, some notion of equality had to be defined on the collections Ob and Arr. Now if $\mathcal{C}' = (\mathrm{Ob}', \mathrm{Arr}', \mathrm{dom}', \mathrm{cod}', \mathrm{id}', \circ')$ is another category, the question may arise as to whether $\mathcal{C} = \mathcal{C}'$. Or indeed, we may simply be asking whether the collections Ob and Ob' are the same, or whether $\mathrm{dom} = \mathrm{dom}'$ etc. It is very difficult for us to carry out any sort of formal reasoning on things without equality. So having equality defined on Ob and Arr is neccessary for definition (2) to even make sense, but it is not enough for us to formally prove anything about categories. Hence we shall assume:

Axiom 1 *A notion of equality exists for* collections*.*

It is implicit in the statement of axiom (1) that the notion of equality between *collections* should be reflexive, symmetric and transitive. Furthermore:

Axiom 2 *Two collections with identical members are equal and conversely.*

In particular, if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a category and we have another partial map $\circ' : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$ such that $g \circ' f$ is defined if and only if $g \circ f$ is defined, then axiom (2) allows us to argue that the domain of $\circ'$ is the same collection as the domain of $\circ$.

Axiom 3 *Let A be a collection and B be a collection with equality. Then two maps $F, G : A \to B$ are equal if and only if $F(x) = G(x)$ for all x in A.*

In particular, if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a category and $\mathrm{dom}' : \mathrm{Arr} \to \mathrm{Ob}$ is another map such that $\mathrm{dom}'(x) = \mathrm{dom}(x)$ for every object $x \in \mathcal{C}$, then $\mathrm{dom}' = \mathrm{dom}$. Or if $\circ' : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$ is another partial map with the same domain as that of $\circ$ and such that $g \circ' f = g \circ f$ when defined, then $\circ' = \circ$.

Axiom 4 *Two tuples with identical entries are equal.*

So if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ and $\mathcal{C}' = (\mathrm{Ob}', \mathrm{Arr}', \mathrm{dom}', \mathrm{cod}', \mathrm{id}', \circ')$ are two catgeories such that $\mathrm{Ob} = \mathrm{Ob}'$, $\mathrm{Arr} = \mathrm{Arr}'$, $\mathrm{dom} = \mathrm{dom}'$, $\mathrm{cod} = \mathrm{cod}'$, $\mathrm{id} = \mathrm{id}'$ and $\circ = \circ'$ then we have the equality $\mathcal{C} = \mathcal{C}'$.

Notation 8 *If $f : A \to B$ and $g : B \to C$ are maps between collections, we denote $g \circ f$ the map $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.*

We shall need the following criterium to determine if a collection is a set:

Axiom 5 *Let $\mathcal{C}$ be a collection for which there exists a set A such that every member of $\mathcal{C}$ is an element of A. Then $\mathcal{C}$ is itself a set.*

Remark: We do not attempt to provide a formal proof of axiom (5), in part because we do not have a formal notion of *collection*. However, if our collection $\mathcal{C}$ is a collection of sets, it is probably specified as a predicate $\mathcal{C}(x)$ with free variable x (and possibly others), the assumption $\mathcal{C} \subseteq A$ for some set A, allows us to write $\mathcal{C} = \{ \ x \in A \mid \mathcal{C}(x) \ \}$ and the existence of $\mathcal{C}$ as a set is guaranteed by the *axiom schema of specification*, which is a common axiom schema of Zermelo-Fraenkel set theory.

Axiom 6 *Let $\mathcal{C}$ be a collection of sets for which there exists a injective map $f : \mathcal{U} \to \mathcal{C}$ defined on the collection of all sets $\mathcal{U}$. Then $\mathcal{C}$ is not a set.*

Remark: Heuristically, a proof of axiom (6) goes as follows: Assume $\mathcal{C}$ is indeed a set. Then the collection $\{ \ f(x) \mid x \in \mathcal{U} \ \}$ being a sub-collection of $\mathcal{C}$ is also a set by virtue of axiom (5). The injectivity of f allows us to define a map $f^{-1} : \{ \ f(x) \mid x \in \mathcal{U} \ \} \to \mathcal{U}$, whose range is the whole of $\mathcal{U}$. The *axiom schema of substitution*, (another common axiom schema of ZF set theory), allows us to conclude that $\mathcal{U}$ is itself a set, which is a contradiction.

1.4 Category of Sets

Definition 3 *We call* **Set** *the category* **Set** $= (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *where*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} = \{\ x \mid x \textit{ is a set}\ \} \\
(2) & \mathrm{Arr} = \{\ (a, b, f) \mid a, b \in \mathrm{Ob}\ ,\ f \textit{ is a function } f : a \to b\ \} \\
(3) & \mathrm{dom}\,(a, b, f) = a \\
(4) & \mathrm{cod}\,(a, b, f) = b \\
(5) & \mathrm{id}(a) = (a, a, i(a)) \\
(6) & (b, c, g) \circ (a, b, f) = (a, c, g \circ f)
\end{array}
$$

where $(3) - (6)$ *hold for all sets* a, b, c *and functions* $f : a \to b$*,* $g : b \to c$*,* $i(a) : a \to a$ *denotes the usual identity function on* a*, and* $g \circ f$ *denotes the usual function composition defined by* $(g \circ f)(x) = g(f(x))$*, for all* $x \in a$*.*

The collection of objects of the category **Set** is defined to be the class of all sets. We are using the set comprehension notation $\{\ x \mid x \text{ is a set}\ \}$ to denote this class, but this is an abuse of notation as Ob is not a set but a proper class. One could think of a class as a precicate $P(x)$ of first order logic with one free variable. From this point of view Ob becomes the predicate $\mathrm{Ob}(x) = \top$, i.e. the predicate which returns true for all x. Every set satisfies the predicate Ob, so every set is a member of the class Ob. The class Ob is not a set because the set-theoretic statement $\exists y, \forall z, z \in y \ \Leftrightarrow\ \mathrm{Ob}(z)$ can be proven false. In other words, there exists no set y whose elements z are exactly the sets which satisfy the predicate Ob. There exists no set which contains all sets.

The collection of arrows of the category **Set** is defined to be the class of triples (a, b, f) where a, b are sets and f is a function $f : a \to b$. This last notation is a common set-theoretic shortcut to express the fact that f is a *function* with *domain* a and *range* **which is a subset of** b. A *function* is any set f whose elements are ordered pairs (x, y) and which is functional, i.e. for which the following implication holds for all sets x, y, y':

$$(x, y) \in f \ \wedge\ (x, y') \in f \ \Rightarrow\ y = y'$$

The *domain* of a function f is the set of all sets x for which there exists a set y with $(x, y) \in f$. The *range* of a function f is the set of all sets y for which there exists a set x with $(x, y) \in f$. If x belongs to the domain of a function f, the notation '$f(x)$' commonly refers to the unique set y with $(x, y) \in f$.

Now, as already pointed out the notation $f : a \to b$ only requires that the range of f should be a subset of b. There is no requirement that the range of f should be equal to b. So if $f : a \to b$ and $b \subseteq c$ then $f : a \to c$. This explains why the collection of arrows Arr is defined as a class of triples (a, b, f) rather than a class of functions f. Knowing the function f does not tell you which *codomain* it should have. Any set b which is a superset of its range is a possible codomain. So we keep the set b together with the function f in the triple (a, b, f) so as to remember which codomain is intended for this particular arrow of the category **Set**. Incidentally, we also keep the range a of the function f in the triple (a,b,f) but this is not necessary, as the knowledge of f does allow us to recover its domain a. However, the triple (a,b,f) is convenient, allowing us to treat *domain* and *codomain* uniformly. In fact, it is worth pausing for a second and emphasize the difference between f and (a,b,f):

Definition 4 *Given a function $f : a \to b$ between two sets a and b we say that f is the* untyped function *while the triple (a,b,f) is called the* typed *function.*

Once again, it should be remembered that the collection of arrows Arr is not a set but a proper class, corresponding to the predicate $\mathrm{Arr}(x)$:

$$\mathrm{Arr}(x) = \exists a\, \exists b\, \exists f\ ,\ x = (a,b,f)\ \wedge\ f : a \to b$$

Informally, this predicates expresses the fact that x is a typed function. The maps $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$ and $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$ for the category **Set** are defined respectively by $\mathrm{dom}(a,b,f) = a$ and $\mathrm{cod}(a,b,f) = b$. This looks simple enough, but for those who worry about foundational issues, we should just note that these are also proper classes which can be encoded as predicates. For example:

$$\mathrm{dom}(x) = \exists u\, \exists v\ ,\ x = (u,v)\ \wedge\ \mathrm{Arr}(u)\ \wedge\ (\ \exists a\, \exists b\, \exists f\ ,\ u = (a,b,f)\ \wedge\ v = a\ )$$

In other words, any set x satisfies the predicate $\mathrm{dom}(x)$ if and only if it is an ordered pair (u,v) where u satisfies the predicate $\mathrm{Arr}(u)$ and for which there exist sets a,b,f with $u = (a,b,f)$ and $v = a$. In short, (u,v) satisfies the predicate dom if and only if u is an arrow $u = (a,b,f)$ and $v = a$.

We defined the identity operator id by $\mathrm{id}(a) = (a,a,i(a))$ and the composition operator $\circ$ by $(b,c,g)\circ(a,b,f) = (a,c,g\circ f)$ where $g\circ f$ is the usual function composition and $i(a) : a \to a$ is the usual identity function. As before, these defined maps are not functional sets of ordered pairs but rather proper classes which we could also encode as precicates of first order logic. One important point to note is the fact that (6) of definition (3) only defines the composition arrow $(b,c,g)\circ(a,b,f)$ where $f : a \to b$ and $g : b \to c$. In other words, the composition $(d,c,g)\circ(a,b,f)$ with $f : a \to b$ and $g : d \to c$ is only defined when $b = d$. Furthermore, the usual function composition $g \circ f$ is a function $g \circ f : a \to c$ which from (2) of definition (3) means that the composed arrow $(b,c,g)\circ(a,b,f) = (a,c,g\circ f)$ is indeed a member of the collection Arr, and the partial map $\circ$ thus defined is indeed a partial map $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$.

Proposition 2 *The category* **Set** *of definition (3) is a category.*

Proof
Now that we have defined the data $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ of the category **Set**, it is time to check this data actually forms a category. We need to check that conditions $(7) - (13)$ of definition (2) are satisfied.

(7): suppose f^* and g^* are two members of the collection Arr. We need to check that $g^* \circ f^*$ is defined if and only if $\mathrm{cod}(f^*) = \mathrm{dom}(g^*)$. Using (2) of definition (3), f^* can be written $f^* = (a,b,f)$ for some function $f : a \to b$ and

g^* can be written $g^* = (d, c, g)$ for some function $g : d \to c$. However, from (6) of definition (3), the arrrow $(d, c, g) \circ (a, b, f)$ is only defined in the case when $b = d$. Furthermore, from (4) of definition (3) we have $\mathrm{cod}(f^*) = b$ and from (3) of definition (3) we have $\mathrm{dom}(g^*) = d$. We conclude that $g^* \circ f^*$ is defined if and only if $\mathrm{cod}(f^*) = \mathrm{dom}(g^*)$ as required.

(8): Let $f^*, g^* \in \mathrm{Arr}$ such that $\mathrm{cod}(f^*) = \mathrm{dom}(g^*)$. We need to show that $\mathrm{dom}(g^* \circ f^*) = \mathrm{dom}(f^*)$. As before, f^* and g^* can be written as $f^* = (a, b, f)$ and $g^* = (b, c, g)$ where $f : a \to b$ and $g : b \to c$. We have $g^* \circ f^* = (a, c, g \circ f)$. Using (3) of definition (3) we obtain $\mathrm{dom}(g^* \circ f^*) = a = \mathrm{dom}(f^*)$.

(9): Let $f^*, g^* \in \mathrm{Arr}$ such that $\mathrm{cod}(f^*) = \mathrm{dom}(g^*)$. We need to show that $\mathrm{cod}(g^* \circ f^*) = \mathrm{cod}(g^*)$. As before, we have $g^* \circ f^* = (a, c, g \circ f)$ and $g^* = (b, c, g)$. Using (4) of definition (3) we obtain $\mathrm{cod}(g^* \circ f^*) = c = \mathrm{cod}(g^*)$.

(10): Let $f^*, g^*, h^* \in \mathrm{Arr}$ with $\mathrm{cod}(f^*) = \mathrm{dom}(g^*)$ and $\mathrm{cod}(g^*) = \mathrm{dom}(h^*)$. We need to show the equality: $(h^* \circ g^*) \circ f^* = h^* \circ (g^* \circ f^*)$. However, f^*, g^*, h^* can be decomposed as $f^* = (a, b, f)$, $g^* = (b, c, g)$ and $h^* = (c, d, h)$ with $f : a \to b$, $g : b \to c$, and $h : c \to d$. We have:

$$\begin{array}{rcl}
(h^* \circ g^*) \circ f^* & = & ((c, d, h) \circ (b, c, g)) \circ (a, b, f) \\
\text{(6) of Def (3)} \to & = & (b, d, h \circ g) \circ (a, b, f) \\
\text{(6) of Def (3)} \to & = & (a, d, (h \circ g) \circ f) \\
\text{assoc of usual composition} \to & = & (a, d, h \circ (g \circ f)) \\
\text{(6) of Def (3)} \to & = & (c, d, h) \circ (a, c, g \circ f) \\
\text{(6) of Def (3)} \to & = & (c, d, h) \circ ((b, c, g) \circ (a, b, f)) \\
 & = & h^* \circ (g^* \circ f^*)
\end{array}$$

(11): Let a be a set. We need to show that $\mathrm{dom}(\mathrm{id}(a)) = a = \mathrm{cod}(\mathrm{id}(a))$. This follows immediately from $\mathrm{id}(a) = (a, a, i(a))$ which is (5) of definition (3).

(12): Let $f^* = (a, b, f)$ be an arrow with $\mathrm{dom}(f^*) = a$. We need to show that $f^* \circ \mathrm{id}(a) = f^*$, which follows from:

$$\begin{array}{rcl}
f^* \circ \mathrm{id}(a) & = & (a, b, f) \circ \mathrm{id}(a) \\
\text{(5) of Def (3)} \to & = & (a, b, f) \circ (a, a, i(a)) \\
\text{(6) of Def (3)} \to & = & (a, b, f \circ i(a)) \\
\text{usual right-identity} \to & = & (a, b, f) \\
 & = & f^*
\end{array}$$

(13): Let $f^* = (b, a, f)$ be an arrow with $\mathrm{cod}(f^*) = a$. We need to show that $\mathrm{id}(a) \circ f^* = f^*$, which follows from:

$$\begin{array}{rcl}
\mathrm{id}(a) \circ f^* & = & \mathrm{id}(a) \circ (b, a, f) \\
\text{(5) of Def (3)} \to & = & (a, a, i(a)) \circ (b, a, f) \\
\text{(6) of Def (3)} \to & = & (b, a, i(a) \circ f) \\
\text{usual left-identity} \to & = & (b, a, f) \\
 & = & f^*
\end{array}$$

This completes our proof of properties (7) − (13). $\diamond$

Notation 9 *We shall often refer to an arrow* (a, b, f) *of the category* **Set** *i.e. a typed function, simply as its untyped counterpart* f*. The context should make it clear that* f *actually refers to a typed function.*

Remark So on top of its usual set-theoretic meaning for untyped functions, the notation $f : a \to b$ may also have its categorical meaning for typed functions, expressing the fact that f is an arrow of the category **Set** with domain a and codomain b, i.e. that f is really the typed function (a, b, f).

Whenever $f : a \to b$ is an arrow of the category **Set** and $x \in a$, the notation $f(x)$ is not strictly speaking meaningful since f is not a function but a typed function, i.e a tuple (a, b, f). However, it is natural enough to set:

Notation 10 *If* $f : a \to b$ *is an arrow of the category* **Set***, for all* $x \in a$ *we shall write* $f(x)$ *as a shortcut for* $f(x)$ *where* f *is the underlying untyped function.*

Functions in set theory are just sets, and equality between functions is simply the standard equality between sets. As it turns out, if f and g are two functions with the same domain a, then the set equality $f = g$ is equivalent to the *extensional* equality $\forall x \in a\ ,\ f(x) = g(x)$:

Proposition 3 *Let* f, g *be two functions with identical domain* a*. We have:*

$$f = g \ \Leftrightarrow\ \forall x \in a\ ,\ f(x) = g(x)$$

Proof
($\Rightarrow$): We assume that $f = g$ and $x \in a$. We need to show that $f(x) = g(x)$. However $f(x)$ is defined as the unique set y such that $(x, y) \in f$, while $g(x)$ is the unique set y such that $(x, y) \in g$. Having assumed that $f = g$, both $f(x)$ and $g(x)$ are the unique set y such that $(x, y) \in f$. So we must have $f(x) = g(x)$ by uniqueness.
($\Leftarrow$): We assume that $f(x) = g(x)$ for all $x \in a$. We need to show that $f = g$. Hence we need to show that $f \subseteq g$ and $g \subseteq f$. By symmetry, we can focus on proving $f \subseteq g$ as the same proof will carry over for $g \subseteq f$. So suppose $z \in f$. We need to show that $z \in g$. Having assumed that f is a function, the element z must be an ordered pair $z = (x, y)$. Hence we have $(x, y) \in f$. Having assumed that the domain of f is a, this shows that $x \in a$. Furthermore, from $(x, y) \in f$ we obtain $f(x) = y$. Hence, by assumption we obtain $g(x) = y$. However $g(x)$ is the unique set y' with $(x, y') \in g$. Hence, we have $(x, y) \in g$ and finally $z \in g$. $\diamond$

An important question which will invariably arise is deciding when two arrows of the category **Set** are equal. Although we have defined an arrow to be a typed function (a, b, f), as indicated in notation (9) we will often refer to such arrow simply as f. However, we should not forget that the equality $f = g$ between underlying untyped functions is not enough for two arrows (a, b, f) and (c, d, g) to be equal. The equality between untyped functions will ensure that they have the same domain and the same range, but it does not tell us anything about the intended codomains of their respective typed functions.

Proposition 4 *If two functions f, g are equal, they have the same domain.*

Proof
We assume f, g are functions and $f = g$. We need to show f and g have the same domain. By symmetry, it is sufficient to show that the domain of f is a subset of that of g. So let x be an element of the domain of f. There exists some y with $(x, y) \in f$. From $f = g$ we obtain $(x, y) \in g$ and consequently x is also an element of the domain of g. $\diamond$

Proposition 5 *If two functions f, g are equal, they have the same range.*

Proof
We assume f, g are functions and $f = g$. We need to show f and g have the same range. By symmetry, it is sufficient to show that the range of f is a subset of that of g. So let y be an element of the range of f. There exists some x with $(x, y) \in f$. From $f = g$ we obtain $(x, y) \in g$ and consequently y is also an element of the range of g. $\diamond$

Proposition 6 *Let $f : a \to b$ and $g : c \to d$ be two arrows of the category* **Set**. *Then $f = g$ if and only if $a = c$, $b = d$ and $f(x) = g(x)$ for all $x \in a$.*

Proof
Let $f^* : a \to b$ and $g^* : c \to d$, that is $f^* = (a, b, f)$ (for some f) and $g^* = (c, d, g)$ (for some g) be two arrows of the category **Set**. We call these arrows f^* and g^* in this proof so as to be very precise on the distinction between an arrow (a typed function) and its underlying function (an untyped function). Looking at definition (3), f and g are functions $f : a \to b$ and $g : c \to d$. So the domain of f is the set a while the domain of g is the set c, and the range of f is a subset of b while the range of g is a subset of d. First we assume that $f^* = g^*$. Then we have the equality between triples $(a, b, f) = (c, d, g)$. Hence we obtain immediately $a = c$ and $b = d$ as requested. However, we also obtain the equality between underlying functions $f = g$. Using proposition (3), we see that $f(x) = g(x)$ for all $x \in a$. By virtue of notation (10), $f^*(x)$ and $g^*(x)$ are notational shortcuts for $f(x)$ and $g(x)$ respectively. Hence we have $f^*(x) = g^*(x)$ for all $x \in a$ as requested. We now assume that $a = c$, $b = d$ and $f^*(x) = g^*(x)$ for all $x \in a$. We need to show that $f^* = g^*$, or in other words $(a, b, f) = (c, d, g)$. Hence it remains to show that $f = g$. This follows from proposition (3) and the fact that $f(x) = g(x)$ for all $x \in a$. $\diamond$

Proposition 7 *The category* **Set** *is not small.*

Proof
Assume **Set** is a small category. Then **Set** $= (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ where the entries of the tuple satify conditions $(1)-(13)$ of definition (1). In particular Ob is a set. Using axiom (4), from the equality **Set** $= (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ we obtain in particular Ob **Set** $=$ Ob. This is not a standard equality between sets since Ob **Set** was defined in (3) as the collection $\{ \ x \mid \text{x is a set} \ \}$ and not

as a set. However, it is an equality between collections and using axiom (2), it follows that the set Ob and the collection $\{ \ x \mid \text{x is a set} \ \}$ have identical members. So we have found a set Ob whose members are all possible sets. This is a contradiction as no such set exists. $\diamond$

Remark: These notes do not assume any specific formal foundations and the proof of proposition (7) is therefore standing on shaky grounds. However, we feel some form of formal reasoning is better than nothing at all.

1.5 Opposite Category

Definition 5 *Let* $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *be a category. We call* opposite category of $\mathcal{C}$*, the category denoted* $\mathcal{C}^{op}$ *and defined by:*

$$\mathcal{C}^{op} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{cod}, \mathrm{dom}, \mathrm{id}, \circ')$$

where the composition operator $\circ'$ *is defined by* $f \circ' g = g \circ f$*, for all* $f, g \in \mathrm{Arr}$.

So if $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a category, the opposite category $\mathcal{C}^{op}$ is almost identical, except for the composition operator $\circ'$ which is a flipped version of $\circ$, and for dom and cod which have been swapped with each other. The collection of objects of $\mathcal{C}^{op}$ is the same as that of $\mathcal{C}$, giving us the equality $\mathrm{Ob}\ \mathcal{C}^{op} = \mathrm{Ob}\ \mathcal{C}$. Likewise, the collection of arrows of $\mathcal{C}^{op}$ is the same as that of $\mathcal{C}$, giving us this other equality $\mathrm{Arr}\ \mathcal{C}^{op} = \mathrm{Arr}\ \mathcal{C}$. If we denote $\mathrm{dom}' : \mathrm{Arr} \to \mathrm{Ob}$ and $\mathrm{cod}' : \mathrm{Arr} \to \mathrm{Ob}$ the domain and codomain maps on $\mathcal{C}^{op}$, then $\mathrm{dom}' = \mathrm{cod}$ and $\mathrm{cod}' = \mathrm{dom}$. The identity operator $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ is the same for both $\mathcal{C}$ and $\mathcal{C}^{op}$, and the composition arrow $f \circ' g$ in $\mathcal{C}^{op}$ is defined whenever the composition arrow $g \circ f$ in $\mathcal{C}$ is defined, and we have $f \circ' g = g \circ f$.

Proposition 8 *Let* $\mathcal{C}$ *be a category. Then* $\mathcal{C}^{op}$ *of definition (5) is a category.*

Proof
We need to check that the data $\mathcal{C}^{op} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{cod}, \mathrm{dom}, \mathrm{id}, \circ')$ of definition (5) forms a category, having assumed that the underlying data for $\mathcal{C}$ does. We have indeed two collections Ob and Arr with maps between them $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$, $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$, $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$ and partial map $\circ' : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$. So it remains to show that conditions $(7) - (13)$ of definition (2) are satisfied. For the purpose of this proof, we shall denote $\mathrm{dom}' = \mathrm{cod}$ and $\mathrm{cod}' = \mathrm{dom}$.

(7): We need to check that $f \circ' g$ is defined if and only if $\mathrm{cod}'(g) = \mathrm{dom}'(f)$ which is $\mathrm{dom}(g) = \mathrm{cod}(f)$. However by definition, we have set $f \circ' g$ to be defined whenever $g \circ f$ is itself defined, and since $\mathcal{C}$ is a category, this is in turn equivalent to $\mathrm{cod}(f) = \mathrm{dom}(g)$. Hence, we are done.

(8): We need to check that $\mathrm{dom}'(f \circ' g) = \mathrm{dom}'(g)$ which can be written as $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$ and which is true since $\mathcal{C}$ is a category.

(9): We need to check that $\mathrm{cod}'(f \circ' g) = \mathrm{cod}'(f)$ which can be written as $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and which is true since $\mathcal{C}$ is a category.

(10): Given arrows h, g, f with $\mathrm{cod}'(h) = \mathrm{dom}'(g)$ and $\mathrm{cod}'(g) = \mathrm{dom}'(f)$, we need to check that $(f \circ' g) \circ' h = f \circ' (g \circ' h)$. However, our assumption can be written as $\mathrm{dom}(h) = \mathrm{cod}(g)$ and $\mathrm{dom}(g) = \mathrm{cod}(f)$ and having assumed that $\mathcal{C}$ is a category, by property (10) of definition (2) we have:

$$\begin{aligned}(f \circ' g) \circ' h &= h \circ (f \circ' g)\\ &= h \circ (g \circ f)\\ \mathcal{C} \text{ is a category} \rightarrow \quad &= (h \circ g) \circ f\\ &= f \circ' (h \circ g)\\ &= f \circ' (g \circ' h)\end{aligned}$$

(11): We need to check that $\mathrm{dom}'(\,\mathrm{id}(a)\,) = a = \mathrm{cod}'(\,\mathrm{id}(a)\,)$ for all $a \in \mathcal{C}$, which follows from $\mathrm{dom}' = \mathrm{cod}$, $\mathrm{cod}' = \mathrm{dom}$ and the fact that $\mathcal{C}$ is a category.

(12): We need to check that $f \circ' \mathrm{id}(a) = f$ whenever $\mathrm{dom}'(f) = a$, that is $\mathrm{id}(a) \circ f = f$ whenever $\mathrm{cod}(f) = a$, which follows from $\mathcal{C}$ being a category.

(13): We need to check that $\mathrm{id}(a) \circ' f = f$ whenever $\mathrm{cod}'(f) = a$, that is $f \circ \mathrm{id}(a) = f$ whenever $\mathrm{dom}(f) = a$, which follows from $\mathcal{C}$ being a category. This completes our proof of properties (7) − (13). $\diamond$

Proposition 9 *Let $\mathcal{C}$ be a category. Then the opposite category of $\mathcal{C}^{op}$ is $\mathcal{C}$, i.e.*

$$(\mathcal{C}^{op})^{op} = \mathcal{C}$$

Proof
Let $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ be a category. From definition (5), we have $\mathcal{C}^{op} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{cod}, \mathrm{dom}, \mathrm{id}, \circ')$ and consequently:

$$(\mathcal{C}^{op})^{op} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ'')$$

In order to show that $(\mathcal{C}^{op})^{op} = \mathcal{C}$, by virtue of axiom (4) we simply need to show that the partial maps $\circ, \circ'' : \mathrm{Arr} \times \mathrm{Arr} \rightarrow \mathrm{Arr}$ are equal. Given two arrows f and g, the composition arrow $g \circ'' f$ is defined if and only if $f \circ' g$ is defined, which is itself equivalent to $g \circ f$ being defined. By virtue of axiom (2), both $\circ$ and $\circ''$ are therefore defined on the same collection of arrow tuples (g, f). Furthemore, whenever $g \circ f$ is defined, we have $g \circ'' f = f \circ' g = g \circ f$. Using axiom (3) we conclude that $\circ'' = \circ$ as requested. $\diamond$

1.6 Canonical Product of Categories

Definition 6 *We call* canonical product *of categories $\mathcal{C}_1$ and $\mathcal{C}_2$ the category denoted $\mathcal{C}_1 \times \mathcal{C}_2$ and defined by $\mathcal{C}_1 \times \mathcal{C}_2 = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ where:*

$$\begin{array}{ll}
(1) & \mathrm{Ob} = \{\ (x_1, x_2) \mid x_1 \in \mathrm{Ob}\ \mathcal{C}_1\ ,\ x_2 \in \mathrm{Ob}\ \mathcal{C}_2\ \}\\
(2) & \mathrm{Arr} = \{\ (f_1, f_2) \mid f_1 \in \mathrm{Arr}\ \mathcal{C}_1\ ,\ f_2 \in \mathrm{Arr}\ \mathcal{C}_2\ \}\\
(3) & \mathrm{dom}(f_1, f_2) = (\,\mathrm{dom}(f_1)\,,\,\mathrm{dom}(f_2)\,)\\
(4) & \mathrm{cod}(f_1, f_2) = (\,\mathrm{cod}(f_1)\,,\,\mathrm{cod}(f_2)\,)\\
(5) & \mathrm{id}(x_1, x_2) = (\,\mathrm{id}(x_1)\,,\,\mathrm{id}(x_2)\,)\\
(6) & (g_1, g_2) \circ (f_1, f_2) = (\,g_1 \circ f_1\,,\,g_2 \circ f_2\,)
\end{array}$$

where (3) *and* (4) *hold for all* $f_1 \in \text{Arr } \mathcal{C}_1$ *and* $f_2 \in \text{Arr } \mathcal{C}_2$*,* (5) *holds for all* $x_1 \in \text{Ob } \mathcal{C}_1$ *and* $x_2 \in \text{Ob } \mathcal{C}_2$*, and* (6) *holds for all* $f_1, g_1 \in \text{Arr } \mathcal{C}_1$ *and* $f_2, g_2 \in \text{Arr } \mathcal{C}_2$ *for which* $g_1 \circ f_1$ *and* $g_2 \circ f_2$ *are defined.*

So if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two categories, the objects of $\mathcal{C}_1 \times \mathcal{C}_2$ are the collection of all tuples (x_1, x_2) where x_1 is an object of $\mathcal{C}_1$ and x_2 is an object of $\mathcal{C}_2$. The set comprehension notation $\{\ (x_1, x_2) \mid x_1 \in \text{Ob } \mathcal{C}_1\ ,\ x_2 \in \text{Ob } \mathcal{C}_2\ \}$ is of course an abuse of notation as it does not in general represent a set but a collection. We could also have denoted this collection $\text{Ob } \mathcal{C}_1 \times \text{Ob } \mathcal{C}_2$ using a cartesian product notation, keeping in mind that this is a product of two collections.

Similarly, the arrows of $\mathcal{C}_1 \times \mathcal{C}_2$ are the collection of all tuples (f_1, f_2) where f_1 is an arrow of $\mathcal{C}_1$ and f_2 is an arrow of $\mathcal{C}_2$, a collection which could reasonably be denoted $\text{Arr } \mathcal{C}_1 \times \text{Arr } \mathcal{C}_2$ instead of the set-comprehension notation.

It should be clear from definition (6) that the notations dom, cod, id and $\circ$ are overloaded, referring either to $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_1 \times \mathcal{C}_2$. Given our definitions of $\text{Ob } (\mathcal{C}_1 \times \mathcal{C}_2)$ and $\text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2)$, given that we have $\text{dom} : \text{Arr } \mathcal{C}_1 \to \text{Ob } \mathcal{C}_1$ and $\text{dom} : \text{Arr } \mathcal{C}_2 \to \text{Ob } \mathcal{C}_2$ it should be clear that (3) of definition (6) defines a map $\text{dom} : \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2) \to \text{Ob } (\mathcal{C}_1 \times \mathcal{C}_2)$, and $\text{cod} : \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2) \to \text{Ob } (\mathcal{C}_1 \times \mathcal{C}_2)$ follows from (4). Furthermore from $\text{id} : \text{Ob } \mathcal{C}_1 \to \text{Arr } \mathcal{C}_1$ and $\text{id} : \text{Ob } \mathcal{C}_2 \to \text{Arr } \mathcal{C}_2$ we obtain $\text{id} : \text{Ob } (\mathcal{C}_1 \times \mathcal{C}_2) \to \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2)$ using (5). Finally using (6), given the partial maps $\circ : \text{Arr } \mathcal{C}_1 \times \text{Arr } \mathcal{C}_1 \to \text{Arr } \mathcal{C}_1$ and $\circ : \text{Arr } \mathcal{C}_2 \times \text{Arr } \mathcal{C}_2 \to \text{Arr } \mathcal{C}_2$ we obtain a partial map $\circ : \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2) \times \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2) \to \text{Arr } \mathcal{C}_1 \times \mathcal{C}_2$.

Proposition 10 *The canonical product* $\mathcal{C}_1 \times \mathcal{C}_2$ *of definition (6) is a category.*

Proof
We need to check that the data $\mathcal{C}_1 \times \mathcal{C}_2 = (\text{Ob}, \text{Arr}, \text{dom}, \text{cod}, \text{id}, \circ)$ of definition (6) forms a category, having assumed $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories. We have established that Ob and Arr are collections, that dom, cod, id are maps with the appropriate signatures and $\circ$ is a partial map with the appropriate signature. It remains to check properties (7) $-$ (13) of definition (2).

(7): Let $f, g \in \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2)$. We need to show that $g \circ f$ is defined if and only if $\text{cod}(f) = \text{dom}(g)$. Let $f_1, g_1 \in \text{Arr } \mathcal{C}_1$ and $f_2, g_2 \in \text{Arr } \mathcal{C}_2$ such that $f = (f_1, f_2)$ and $g = (g_1, g_2)$. Then $\text{cod}(f) = (\text{cod}(f_1), \text{cod}(f_2))$ and $\text{dom}(g) = (\text{dom}(g_1), \text{dom}(g_2))$. So we need to show that $g \circ f$ is defined if and only if $\text{cod}(f_1) = \text{dom}(g_1)$ and $\text{cod}(f_2) = \text{dom}(g_2)$. However since $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories, $\text{cod}(f_1) = \text{dom}(g_1)$ is equivalent to $g_1 \circ f_1$ being defined, and $\text{cod}(f_2) = \text{dom}(g_2)$ is equivalent to $g_2 \circ f_2$ being defined. So we need to show that $g \circ f$ is defined if and only if both $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined which follows exactly from definition (6).

(8): Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be arrows in $\mathcal{C}_1 \times \mathcal{C}_2$ with the equality $\text{cod}(f) = \text{dom}(g)$, i.e. for which $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined. We need to check the equality $\text{dom}(g \circ f) = \text{dom}(f)$ which goes as follows:

$$\begin{array}{rcl} \text{dom}(g \circ f) & = & \text{dom}(\,(g_1, g_2) \circ (f_1, f_2)\,) \\ (6) \text{ of def. } (6) \to & = & \text{dom}(\,(g_1 \circ f_1\,,\, g_2 \circ f_2)\,) \end{array}$$

$$
\begin{aligned}
\text{(3) of def. (6)} \rightarrow \quad &= \ (\,\mathrm{dom}(g_1 \circ f_1)\,,\, \mathrm{dom}(g_2 \circ f_2)\,) \\
\mathcal{C}_1,\, \mathcal{C}_2 \text{ categories, (8) of def. (2)} \rightarrow \quad &= \ (\,\mathrm{dom}(f_1)\,,\, \mathrm{dom}(f_2)\,) \\
\text{(3) of def. (6)} \rightarrow \quad &= \ \mathrm{dom}(f_1, f_2) \\
&= \ \mathrm{dom}(f)
\end{aligned}
$$

(9): Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be arrows in $\mathcal{C}_1 \times \mathcal{C}_2$ with the equality $\mathrm{cod}(f) = \mathrm{dom}(g)$, i.e. for which $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined. We need to check the equality $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$ which goes as follows:

$$
\begin{aligned}
\mathrm{cod}(g \circ f) \quad &= \ \mathrm{cod}(\,(g_1, g_2) \,\circ\, (f_1, f_2)\,) \\
\text{(6) of def. (6)} \rightarrow \quad &= \ \mathrm{cod}(\,(g_1 \circ f_1 \,,\, g_2 \circ f_2)\,) \\
\text{(4) of def. (6)} \rightarrow \quad &= \ (\,\mathrm{cod}(g_1 \circ f_1)\,,\, \mathrm{cod}(g_2 \circ f_2)\,) \\
\mathcal{C}_1,\, \mathcal{C}_2 \text{ categories, (9) of def. (2)} \rightarrow \quad &= \ (\,\mathrm{cod}(g_1)\,,\, \mathrm{cod}(g_2)\,) \\
\text{(4) of def. (6)} \rightarrow \quad &= \ \mathrm{cod}(g_1, g_2) \\
&= \ \mathrm{cod}(g)
\end{aligned}
$$

(10): Let $f = (f_1, f_2)$, $g = (g_1, g_2)$ and $h = (h_1, h_2)$ be arrows in $\mathcal{C}_1 \times \mathcal{C}_2$ with the equalities $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$, i.e. for which the composition arrows $g_1 \circ f_1$, $g_2 \circ f_2$, $h_1 \circ g_1$ and $h_2 \circ g_2$ are defined. We need to check the equality $(h \circ g) \circ f = h \circ (g \circ f)$ which goes as follows:

$$
\begin{aligned}
(h \circ g) \circ f \quad &= \ (\,(h_1, h_2) \,\circ\, (g_1, g_2)\,) \,\circ\, (f_1, f_2) \\
\text{(6) of def. (6)} \rightarrow \quad &= \ (\,h_1 \circ g_1 \,,\, h_2 \circ g_2\,) \,\circ\, (f_1, f_2) \\
\text{(6) of def. (6)} \rightarrow \quad &= \ (\,(h_1 \circ g_1) \circ f_1 \,,\, (h_2 \circ g_2) \circ f_2\,) \\
\mathcal{C}_1,\, \mathcal{C}_2 \text{ categories, (10) of def. (2)} \rightarrow \quad &= \ (\,h_1 \circ (g_1 \circ f_1)\,,\, h_2 \circ (g_2 \circ f_2)\,) \\
\text{(6) of def. (6)} \rightarrow \quad &= \ (h_1, h_2) \circ (\,(g_1 \circ f_1)\,,\, (g_2 \circ f_2)\,) \\
\text{(6) of def. (6)} \rightarrow \quad &= \ (h_1, h_2) \circ (\,(g_1, g_2) \,\circ (f_1, f_2)\,) \\
&= \ h \circ (g \circ f)
\end{aligned}
$$

(11): Let $a = (a_1, a_2)$ be an object in $\mathcal{C}_1 \times \mathcal{C}_2$. We need to check that $\mathrm{dom}\,(\,\mathrm{id}(a)\,) = a = \mathrm{cod}\,(\,\mathrm{id}(a)\,)$ which goes as follows:

$$
\begin{aligned}
\mathrm{dom}\,(\,\mathrm{id}(a)\,) \quad &= \ \mathrm{dom}\,(\,\mathrm{id}(a_1, a_2)\,) \\
\text{(5) of def. (6)} \rightarrow \quad &= \ \mathrm{dom}\,(\,\mathrm{id}(a_1)\,,\, \mathrm{id}(a_2)\,) \\
\text{(3) of def. (6)} \rightarrow \quad &= \ (\,\mathrm{dom}\,(\,\mathrm{id}(a_1)\,)\,,\, \mathrm{dom}\,(\,\mathrm{id}(a_2)\,)\,) \\
\mathcal{C}_1,\, \mathcal{C}_2 \text{ categories, (11) of def. (2)} \rightarrow \quad &= \ (a_1, a_2) \\
&= \ a
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{cod}\,(\,\mathrm{id}(a)\,) \quad &= \ \mathrm{cod}\,(\,\mathrm{id}(a_1, a_2)\,) \\
\text{(5) of def. (6)} \rightarrow \quad &= \ \mathrm{cod}\,(\,\mathrm{id}(a_1)\,,\, \mathrm{id}(a_2)\,) \\
\text{(4) of def. (6)} \rightarrow \quad &= \ (\,\mathrm{cod}\,(\,\mathrm{id}(a_1)\,)\,,\, \mathrm{cod}\,(\,\mathrm{id}(a_2)\,)\,) \\
\mathcal{C}_1,\, \mathcal{C}_2 \text{ categories, (11) of def. (2)} \rightarrow \quad &= \ (a_1, a_2) \\
&= \ a
\end{aligned}
$$

(12): Let $f = (f_1, f_2)$ be an arrow and $a = (a_1, a_2)$ be an object in $\mathcal{C}_1 \times \mathcal{C}_2$ such that $\mathrm{dom}(f) = a$. We need to show that $f \circ \mathrm{id}(a) = f$ which goes as follows: Using (3) of definition (6) and the condition $\mathrm{dom}(f) = a$ we obtain the equation $(\,\mathrm{dom}(f_1)\,,\,\mathrm{dom}(f_2)\,) = (a_1, a_2)$. Hence, we have:

$$\begin{array}{rcl}
f \circ \mathrm{id}(a) & = & (f_1, f_2) \circ \mathrm{id}\,(a_1, a_2) \\
\text{(5) of def. (6)} \rightarrow & = & (f_1, f_2) \circ (\,\mathrm{id}(a_1)\,,\,\mathrm{id}(a_2)\,) \\
\text{(6) of def. (6)} \rightarrow & = & (\,f_1 \circ \mathrm{id}(a_1)\,,\, f_2 \circ \mathrm{id}(a_2)\,) \\
\mathcal{C}_1 \text{ category, } \mathrm{dom}(f_1) = a_1 \rightarrow & = & (f_1,\; f_2 \circ \mathrm{id}(a_2)\,) \\
\mathcal{C}_2 \text{ category, } \mathrm{dom}(f_2) = a_2 \rightarrow & = & (f_1, f_2) \\
 & = & f
\end{array}$$

(13): Let $f = (f_1, f_2)$ be an arrow and $a = (a_1, a_2)$ be an object in $\mathcal{C}_1 \times \mathcal{C}_2$ such that $\mathrm{cod}(f) = a$. We need to show that $\mathrm{id}(a) \circ f = f$ which goes as follows: Using (4) of definition (6) and the condition $\mathrm{cod}(f) = a$ we obtain the equation $(\,\mathrm{cod}(f_1)\,,\,\mathrm{cod}(f_2)\,) = (a_1, a_2)$. Hence, we have:

$$\begin{array}{rcl}
\mathrm{id}(a) \circ f & = & \mathrm{id}\,(a_1, a_2) \circ (f_1, f_2) \\
\text{(5) of def. (6)} \rightarrow & = & (\,\mathrm{id}(a_1)\,,\,\mathrm{id}(a_2)\,) \circ (f_1, f_2) \\
\text{(6) of def. (6)} \rightarrow & = & (\,\mathrm{id}(a_1) \circ f_1\,,\,\mathrm{id}(a_2) \circ f_2\,) \\
\mathcal{C}_1 \text{ category, } \mathrm{cod}(f_1) = a_1 \rightarrow & = & (f_1,\; \mathrm{id}(a_2) \circ f_2\,) \\
\mathcal{C}_2 \text{ category, } \mathrm{cod}(f_2) = a_2 \rightarrow & = & (f_1, f_2) \\
 & = & f
\end{array}$$

This completes our proof of properties (7) – (13). $\diamond$

1.7 Hom-sets of a Category

Definition 7 *Let $\mathcal{C}$ be a category and $a, b \in \mathcal{C}$. We call* hom-set of $\mathcal{C}$ associated with the ordered pair (a, b) *the collection denoted $\mathcal{C}(a, b)$ and defined as:*

$$\mathcal{C}(a, b) = \{\ f \in \mathrm{Arr}\ \mathcal{C}\ |\ f : a \rightarrow b\ \}$$

In other words the collection $\mathcal{C}(a, b)$ is the collection of all arrows f in $\mathcal{C}$ such that $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Note that despite being called a 'hom-set', the collection $\mathcal{C}(a, b)$ is generally not a set but an arbitary collection.

Proposition 11 *Let $\mathcal{C}$ be a category and $a, b \in \mathcal{C}$. Then $\mathcal{C}^{op}(a, b) = \mathcal{C}(b, a)$.*

Proof
Denoting $\mathrm{dom}' = \mathrm{cod}$ and $\mathrm{cod}' = \mathrm{dom}$ we have:

$$\begin{array}{rcl}
\mathcal{C}^{op}(a, b) & = & \{\ f \in \mathrm{Arr}\ \mathcal{C}^{op}\ |\ f : a \rightarrow b\ @\ \mathcal{C}^{op}\ \} \\
\text{def. (5)} \rightarrow & = & \{\ f \in \mathrm{Arr}\ \mathcal{C}\ |\ f : a \rightarrow b\ @\ \mathcal{C}^{op}\ \}
\end{array}$$

$$
\begin{array}{rcl}
\text{def. (5)} \rightarrow & = & \{ \ f \in \text{Arr } \mathcal{C} \mid \text{dom}'(f) = a \ , \ \text{cod}'(f) = b \ \} \\
& = & \{ \ f \in \text{Arr } \mathcal{C} \mid \text{cod}(f) = a \ , \ \text{dom}(f) = b \ \} \\
& = & \{ \ f \in \text{Arr } \mathcal{C} \mid f : b \rightarrow a \ @ \ \mathcal{C} \ \} \\
& = & \mathcal{C}(b, a)
\end{array}
$$

◇

Proposition 12 *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories and $a, b \in \mathcal{C}_1 \times \mathcal{C}_2$. Then:*

$$
\mathcal{C}_1 \times \mathcal{C}_2 \, (a, b) = \mathcal{C}_1(a_1, b_1) \times \mathcal{C}_2(a_2, b_2)
$$

where it is understood that $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

Proof
Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be objects in the product category $\mathcal{C}_1 \times \mathcal{C}_2$:

$$
\begin{array}{rcl}
\mathcal{C}_1 \times \mathcal{C}_2 \, (a, b) & = & \mathcal{C}_1 \times \mathcal{C}_2 \, [\, (a_1, a_2) \, , \, (b_1, b_2) \,] \\
\text{def. (7)} \rightarrow & = & \{ \ f \in \text{Arr } (\mathcal{C}_1 \times \mathcal{C}_2) \ \mid \ f : (a_1, a_2) \rightarrow (b_1, b_2) \ \} \\
\text{def. (6)} \rightarrow & = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C}_1 \ , \ f_2 \in \text{Arr } \mathcal{C}_2 \\
& & , \ (f_1, f_2) : (a_1, a_2) \rightarrow (b_1, b_2) \ \} \\
& = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C}_1 \ , \ f_2 \in \text{Arr } \mathcal{C}_2 \\
& & , \ \text{dom}(f_1, f_2) = (a_1, a_2) \\
& & , \ \text{cod}(f_1, f_2) = (b_1, b_2) \ \} \\
\text{(3) and (4) of def. (6)} \rightarrow & = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C}_1 \ , \ f_2 \in \text{Arr } \mathcal{C}_2 \\
& & , \ (\, \text{dom}(f_1) \, , \, \text{dom}(f_2) \,) = (a_1, a_2) \\
& & , \ (\, \text{cod}(f_1) \, , \, \text{cod}(f_2) \,) = (b_1, b_2) \ \} \\
& = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C}_1 \ , \ f_2 \in \text{Arr } \mathcal{C}_2 \\
& & , \ f_1 : a_1 \rightarrow b_1 \ , \ f_2 : a_2 \rightarrow b_2 \ \} \\
\text{product of collections} \rightarrow & = & \{ \ f_1 \in \text{Arr } \mathcal{C}_1 \mid f_1 : a_1 \rightarrow b_1 \ \} \\
& \times & \{ \ f_2 \in \text{Arr } \mathcal{C}_2 \mid f_2 : a_2 \rightarrow b_2 \ \} \\
\text{def. (7)} \rightarrow & = & \mathcal{C}_1(a_1, b_1) \times \mathcal{C}_2(a_2, b_2)
\end{array}
$$

◇

Proposition 13 *For all sets a, b, the hom-set $\mathbf{Set}(a, b)$ is given by:*

$$
\mathbf{Set}(a, b) = \{ \ (a, b, f) \mid f \ \textit{is a function} \ f : a \rightarrow b \ \}
$$

Proof

$$
\begin{array}{rcl}
\mathbf{Set}(a, b) & = & \{ \ f \in \text{Arr } \mathbf{Set} \mid f : a \rightarrow b \ \} \ \leftarrow \ \text{def. (7)} \\
\text{notation (5)} \ \rightarrow & = & \{ \ f \in \text{Arr } \mathbf{Set} \mid \text{dom}(f) = a \ , \ \text{cod}(f) = b \ \}
\end{array}
$$

$$
\begin{aligned}
\text{(2) of def. (3)} \;\rightarrow \quad &= \{\ (x,y,f) \mid x,y \text{ sets, } f \text{ is a function } f: x \rightarrow y \\
&\quad ,\ \mathrm{dom}(x,y,f) = a\ ,\ \mathrm{cod}(x,y,f) = b\ \} \\
\text{(3) and (4) of def. (3)} \;\rightarrow \quad &= \{\ (x,y,f) \mid x,y \text{ sets, } f \text{ is a function } f: x \rightarrow y \\
&\quad ,\ x = a\ ,\ y = b\ \} \\
&= \{\ (a,b,f) \mid f \text{ is a function } f: a \rightarrow b\ \}
\end{aligned}
$$

$\diamond$

1.8 Locally Small Category

Definition 8 *A category $\mathcal{C}$ is said to be* locally small *if and only if the hom-set $\mathcal{C}(a,b)$ associated with every ordered pair of objects (a,b) is actually a set.*

Proposition 14 *A small category is locally small.*

Proof
Let $\mathcal{C}$ be a small category and $a, b \in \mathcal{C}$. We need to show that $\mathcal{C}(a,b)$ is a set. Using axiom (5), it is sufficient to find a set A such that every member of $\mathcal{C}(a,b)$ is an element of A. Take $A = \mathrm{Arr}\,\mathcal{C}$. Then from definition (1) A is indeed a set, and every member of $\mathcal{C}(a,b)$ is an element of A by virtue of definition (7). $\diamond$

Proposition 15 *A category $\mathcal{C}$ is locally small if and only if $\mathcal{C}^{op}$ is locally small.*

Proof
The category $\mathcal{C}$ being locally small is equivalent to $\mathcal{C}(a,b)$ being a set for all $a, b \in \mathrm{Ob}\,\mathcal{C}$. Since $\mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{C}^{op}$ and $\mathcal{C}^{op}(a,b) = \mathcal{C}(b,a)$ from proposition (11), this is in turn equivalent to $\mathcal{C}^{op}(a,b)$ being a set for all $a, b \in \mathrm{Ob}\,\mathcal{C}^{op}$. Hence, it is equivalent to $\mathcal{C}^{op}$ being locally small. $\diamond$

Proposition 16 *The product $\mathcal{C}_1 \times \mathcal{C}_2$ of locally small categories is locally small.*

Proof
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two locally small categories. We need to show that the canonical product $\mathcal{C}_1 \times \mathcal{C}_2$ is itself locally small. In other words, given $a, b \in \mathcal{C}_1 \times \mathcal{C}_2$ we need to show that the collection $\mathcal{C}_1 \times \mathcal{C}_2(a,b)$ is actually a set. However from proposition (12) we have $\mathcal{C}_1 \times \mathcal{C}_2\,(a,b) = \mathcal{C}_1(a_1,b_1) \times \mathcal{C}_2(a_2,b_2)$ where $a = (a_1,a_2)$ and $b = (b_1,b_2)$. So the proposition follows from the fact that both $\mathcal{C}_1(a_1,b_1)$ and $\mathcal{C}_2(a_2,b_2)$ are sets, $\mathcal{C}_1$ and $\mathcal{C}_2$ being locally small. $\diamond$

Proposition 17 *The category* **Set** *is locally small.*

Proof
Given two sets a, b, we need to that the hom-set $\mathbf{Set}(a,b)$ is actually a set. Using axiom (5), it is sufficient to show that this collection is a sub-collection of another collection which is known to be a set. Now any (untyped) function $f : a \rightarrow b$ is a set of ordered pairs (x,y) where $x \in a$ and $y \in b$ (and with

additional properties). In particular, a function $f : a \to b$ is a subset of the cartesian product $a \times b$, i.e. $f \subseteq a \times b$. Equivalently, f is an element of the power set $\mathcal{P}(a \times b)$, i.e. $f \in \mathcal{P}(a \times b)$. Consequently, we have:

$$\begin{array}{rcl} \mathbf{Set}(a,b) & = & \{\ (a,b,f) \mid f \text{ is a function } f : a \to b\ \}\ \leftarrow\ \text{prop. (13)} \\ & \subseteq & \{\ (a,b,f) \mid f \in \mathcal{P}(a \times b)\ \} \\ & = & \{\ (x,y,f) \mid x \in \{a\}\ ,\ y \in \{b\}\ ,\ f \in \mathcal{P}(a \times b)\ \} \\ & = & \{a\} \times \{b\} \times \mathcal{P}(a \times b) \end{array}$$

◇

1.9 Isomorphism

Definition 9 *Let $\mathcal{C}$ be a category and $f : a \to b$ for some $a, b \in \mathcal{C}$. We say that $g : b \to a$ is a* left-inverse of f, *if and only if $g \circ f = \text{id}(a)$.*

Definition 10 *Let $\mathcal{C}$ be a category and $f : a \to b$ for some $a, b \in \mathcal{C}$. We say that $g : b \to a$ is a* right-inverse of f, *if and only if $f \circ g = \text{id}(b)$.*

Remark: Comparing definitions (9) and (10), given a category $\mathcal{C}$ and given $f, g \in \text{Arr}\ \mathcal{C}$, g is a left-inverse of f if and only if f is a right-inverse of g.

Definition 11 *Let $\mathcal{C}$ be a category and $f : a \to b$ for some $a, b \in \mathcal{C}$. We say that $g : b \to a$ is an* inverse of f, *if and only if it is a left and right-inverse of f.*

Remark : Given a category $\mathcal{C}$ and $f, g \in \text{Arr}\ \mathcal{C}$, g is an inverse of f if and only if f is an inverse of g.

Definition 12 *Let $\mathcal{C}$ be a category and $f \in \text{Arr}\ \mathcal{C}$. We say that f is an* isomorphism *if and only if f has an inverse.*

Proposition 18 *Let $\mathcal{C}$ be a category and $f \in \text{Arr}\ \mathcal{C}$. If f has a right-inverse and a left-inverse, then these are equal.*

Proof
Let $a = \text{dom}(f)$ and $b = \text{cod}(f)$. Suppose $g : b \to a$ is a left-inverse of f and $h : b \to a$ is a right-inverse of f. We need to show that $g = h$:

$$\begin{array}{rcl} g & = & g \circ \text{id}(b)\ \leftarrow\ \text{(12) of def (2)} \\ h \text{ is right inverse } \to & = & g \circ (f \circ h) \\ \circ \text{ assoc } \to & = & (g \circ f) \circ h \\ g \text{ is left inverse } \to & = & \text{id}(a) \circ h \\ \text{(12) of def (2) } \to & = & h \end{array}$$

◇

Proposition 19 *Let $\mathcal{C}$ be a category. If $f \in \mathrm{Arr}\ \mathcal{C}$ has an inverse, it is unique.*

Proof
Suppose $g, h \in \mathrm{Arr}\ \mathcal{C}$ are both inverses of f. Then in particular g is a left-inverse of f and h is a right-inverse of f. From proposition (18), we have $g = h$. $\diamond$

Notation 11 *Given a category $\mathcal{C}$, an inverse of $f \in \mathrm{Arr}\ \mathcal{C}$ is denoted f^{-1}.*

Proposition 20 *Let $\mathcal{C}$ be a category and $f \in \mathrm{Arr}\ \mathcal{C}$. Then f is an isomorphism if and only if f has a left-inverse and a right-inverse.*

Proof
Suppose f is an isomorphism. From definition (12), f has an inverse. Hence from definition (11), there is some $g \in \mathrm{Arr}\ \mathcal{C}$ which is both a left-inverse and a right-inverse of f. So in particular, f has a left-inverse and a right-inverse. Conversely, suppose f has a left-inverse g and a right-inverse h. from proposition (18) we have $g = h$. So g is in fact both a left-inverse and a right-inverse of f. From definition (11), g is in fact an inverse of f and from definition (12) we conclude that f is an isomorphism. $\diamond$

Proposition 21 *Let $\mathcal{C}$ be a category, $f : a \to b$ and $g : b \to c$ where $a, b, c \in \mathcal{C}$. If f and g are isomorphisms then $g \circ f$ is an isomophism and:*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof
We assume that f and g are isomophisms with inverses denoted f^{-1} and g^{-1} respectively. We need to show that $g \circ f$ is an isomorphism, i.e. that it has an inverse and that the inverse is actually $f^{-1} \circ g^{-1}$. So we need to show that $f^{-1} \circ g^{-1}$ is both a left-inverse and a right-inverse of $g \circ f$. Note that since $f : a \to b$ and $g : b \to c$, we have $f^{-1} : b \to a$ and $g^{-1} : c \to b$ and consequently $f^{-1} \circ g^{-1} : c \to a$. The proof for left-inverse goes as follows:

$$\begin{array}{rcl}
(f^{-1} \circ g^{-1}) \circ (g \circ f) & = & f^{-1} \circ (g^{-1} \circ g) \circ f \ \leftarrow\ \circ \text{ assoc} \\
g^{-1} \text{ left-inverse of } g \ \to & = & f^{-1} \circ \mathrm{id}\,(b) \circ f \\
(12) \text{ or } (13) \text{ of def. } (2) \ \to & = & f^{-1} \circ f \\
f^{-1} \text{ left-inverse of } f \ \to & = & \mathrm{id}\,(a)
\end{array}$$

The proof of right-inverse goes as follows:

$$\begin{array}{rcl}
(g \circ f) \circ (f^{-1} \circ g^{-1}) & = & g \circ (f \circ f^{-1}) \circ g^{-1} \ \leftarrow\ \circ \text{ assoc} \\
f^{-1} \text{ right-inverse of } f \ \to & = & g \circ \mathrm{id}\,(b) \circ g^{-1} \\
(12) \text{ or } (13) \text{ of def. } (2) \ \to & = & g \circ g^{-1} \\
g^{-1} \text{ right-inverse of } g \ \to & = & \mathrm{id}\,(c)
\end{array}$$

$\diamond$

Proposition 22 *Let $\mathcal{C}$ be a category and $a \in \mathcal{C}$. Then* $\mathrm{id}\,(a)$ *is an isomorphim.*

Proof
From $\mathrm{id}\,(a) \circ \mathrm{id}\,(a) = \mathrm{id}\,(a)$, $\mathrm{id}(a)$ is both a left and right inverse of itself. $\diamond$

Proposition 23 *Let $\mathcal{C}$ be a category. If $f \in \mathrm{Arr}\,\mathcal{C}$ is an isomorphism, so is f^{-1}.*

Proof
If f is an isomorphism with imverse f^{-1}, f is an inverse of f^{-1}. $\diamond$

Definition 13 *Let $\mathcal{C}$ be a category and $a, b \in \mathcal{C}$. We say that the object a is* isomorphic *to the object b if and only if there exists an isomorphism $f : a \to b$.*

Notation 12 *We write $a \simeq b$ to say that a is isomorphic to b.*

Proposition 24 *Let $\mathcal{C}$ be a category. The relation $\simeq$ is an equivalence on* $\mathrm{Ob}\,\mathcal{C}$.

Proof
We need to show that $\simeq$ is reflexive, symmetric and transitive. Given $a \in \mathcal{C}$, from propostion (22) the arrow $\mathrm{id}\,(a) : a \to a$ is an isomorpshim, so $a \simeq a$ and $\simeq$ is reflexive. Given $a, b \in \mathcal{C}$ with $a \simeq b$, there exists an isomorphism $f : a \to b$. From proposition (23) the arrow $f^{-1} : b \to a$ is also an isomorphism. So $b \simeq a$ and $\simeq$ is symmetric. Suppose $a, b, c \in \mathcal{C}$ with $a \simeq b$ and $b \simeq c$. There exist isomorphisms $f : a \to b$ and $g : b \to c$. From proposition (21) the arrow $g \circ f$ is an isomorphism from a to c. So $a \simeq c$ and $\simeq$ is transitive. $\diamond$

Chapter 2

Functor

2.1 Functor

Definition 14 *We call* functor *from categories* $\mathcal{C}$ *to* $\mathcal{D}$ *any tuple* (F_0, F_1) *with:*

$$
\begin{array}{ll}
(1) & F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{D} \text{ is a map} \\
(2) & F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D} \text{ is a map} \\
(3) & F_1(f) : F_0(a) \to F_0(b) \\
(4) & F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\,F_0(a)\,) \\
(5) & F_1(g \circ f) = F_1(g) \circ F_1(f)
\end{array}
$$

where $(3) - (5)$ *hold for all* $a, b, c \in \mathcal{C}$, $f : a \to b$ *and* $g : b \to c$.

Notation 13 *We shall use* $F : \mathcal{C} \to \mathcal{D}$ *as a notational shortcut for the statement that* F is a functor from the category $\mathcal{C}$ to the category $\mathcal{D}$.

Notation 14 *If* $F = (F_0, F_1)$ *if a functor from* $\mathcal{C}$ *to* $\mathcal{D}$*, we shall also commonly denote* F_0 *and* F_1 *simply by* F.

So if F is a functor $F : \mathcal{C} \to \mathcal{D}$ we effectively have a map $F : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{D}$ acting on objects, and a map $F : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$ acting on arrows. These two maps satisfy the consistency condition (3) of definition (14) i.e. that if f is an arrow $f : a \to b$ in $\mathcal{C}$, then $F(f)$ must be an arrow $F(f) : F(a) \to F(b)$ in $\mathcal{D}$. Furthermore, the functor F must preserve the identity operators on $\mathcal{C}$ and $\mathcal{D}$ which is condition (4) of definition (14): for all objects $a \in \mathcal{C}$, we must have $F(\,\mathrm{id}(a)\,) = \mathrm{id}(\,F(a)\,)$. Note that since $\mathrm{id}(a) : a \to a$, by consistency we have $F(\,\mathrm{id}(a)\,) : F(a) \to F(a)$, and since $\mathrm{id}(\,F(a)\,) : F(a) \to F(a)$ the equality makes sense. Another way to express the preservation of identity operators by F is simply $F \circ \mathrm{id} = \mathrm{id} \circ F$ or $F_1 \circ \mathrm{id} = \mathrm{id} \circ F_0$ to be more explicit. However, we should remember that the notation '$\circ$' in these equality does not refer to the composition operator $\circ$ of either $\mathcal{C}$ or $\mathcal{D}$, nor does it in general refer to the usual function composition since id, F_0 and F_1 are maps between collections and not functions between sets. Now going back to our functor F, it must also preserve the composition operators on $\mathcal{C}$ and $\mathcal{D}$, which is condition (5) of definition (14): For all objects $a, b, c \in \mathcal{C}$ and arrows $f : a \to b$ and $g : b \to c$, we must have $F(g \circ f) = F(g) \circ F(f)$. Note that given these assumptions, the composition arrow $g \circ f$ is well-defined, and by consistency we have $F(f) : F(a) \to F(b)$ and $F(g) : F(b) \to F(c)$, so $F(g) \circ F(f)$ is also well-defined. Furthermore, since $g \circ f : a \to c$ by consistency we have $F(g \circ f) : F(a) \to F(c)$ and since $F(g) \circ F(f) : F(a) \to F(c)$, the equality $F(g \circ f) = F(g) \circ F(f)$ makes sense.

2.2 Identity Functor

Definition 15 *Let $\mathcal{C}$ be a category. We call* identity functor on $\mathcal{C}$ *the functor $F : \mathcal{C} \to \mathcal{C}$ defined by $F = (F_0, F_1)$ with:*

$$\begin{array}{ll} (1) & F_0(a) = a \\ (2) & F_1(f) = f \end{array}$$

where (1) *holds for all $a \in \mathrm{Ob}\ \mathcal{C}$ and* (2) *holds for all $f \in \mathrm{Arr}\ \mathcal{C}$.*

Notation 15 *The identity functor on a category $\mathcal{C}$ is denoted $I_{\mathcal{C}}$.*

We shall now check our definition is indeed that of a functor.

Proposition 25 *The identity functor on a category $\mathcal{C}$ is a functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.*

Proof
Let $\mathcal{C}$ be a category and $I_{\mathcal{C}} = (F_0, F_1)$ be the identity functor on $\mathcal{C}$. We need to check that properties $(1) - (5)$ of definition (14) are satisfied:

(1): F_0 is indeed a map $F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{C}$ from (1) of definition (15).
(2): F_1 is indeed a map $F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{C}$ from (2) of definition (15).
(3): We have $F_1(f) : F_0(a) \to F_0(b)$ whenever $f : a \to b$, since:

$$F_1(f) = f,\ F_0(a) = a,\ F_0(b) = b$$

(4): We have $F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\,F_0(a)\,)$ for all $a \in \mathcal{C}$, since:

$$F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(a),\ F_0(a) = a$$

(5): $F_1(g \circ f) = F_1(g) \circ F_1(f)$ whenever $f : a \to b$ and $g : b \to c$, since:

$$F_1(g \circ f) = g \circ f,\ F_1(g) = g,\ F_1(f) = f$$

$\diamond$

2.3 Constant Functor

Definition 16 *Let $\mathcal{C}$ be a category and $c \in \mathcal{C}$. We call* constant functor on $\mathcal{C}$ at c *the functor $F : \mathcal{C} \to \mathcal{C}$ defined by $F = (F_0, F_1)$ with:*

$$\begin{array}{ll} (1) & F_0(a) = c \\ (2) & F_1(f) = \mathrm{id}(c) \end{array}$$

where (1) *holds for all $a \in \mathrm{Ob}\ \mathcal{C}$ and* (2) *holds for all $f \in \mathrm{Arr}\ \mathcal{C}$.*

Notation 16 *The constant functor on a category $\mathcal{C}$ at $c \in \mathcal{C}$ is denoted K_c.*

We shall now check our definition is indeed that of a functor.

Proposition 26 *Given a category $\mathcal{C}$ and $c \in \mathcal{C}$, K_c is a functor $K_c : \mathcal{C} \to \mathcal{C}$.*

Proof
Let $\mathcal{C}$ be a category, $c \in \mathcal{C}$ and $K_c = (F_0, F_1)$ be the constant functor on $\mathcal{C}$ at c. We need to check that properties $(1) - (5)$ of definition (14) are satisfied:
(1): F_0 is indeed a map $F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{C}$ from (1) of definition (16).
(2): F_1 is indeed a map $F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{C}$ from (2) of definition (16).
(3): We have $F_1(f) : F_0(a) \to F_0(b)$ whenever $f : a \to b$, since:

$$F_1(f) = \mathrm{id}(c),\ F_0(a) = c,\ F_0(b) = c$$

(4): We have $F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\,F_0(a)\,)$ for all $a \in \mathcal{C}$, since:

$$F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(c),\ F_0(a) = c$$

(5): $F_1(g \circ f) = F_1(g) \circ F_1(f)$ whenever $f : a \to b$ and $g : b \to c$, since:

$$F_1(g \circ f) = \mathrm{id}(c),\ F_1(g) = \mathrm{id}(c),\ F_1(f) = \mathrm{id}(c)$$

and $\mathrm{id}(c) = \mathrm{id}(c) \circ \mathrm{id}(c)$ from (12) (or (13)) of definition (2). $\diamond$

2.4 Functor and Opposite Category

Proposition 27 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then F is also a functor $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$.*

Proof
We need to check that properties $(1) - (5)$ of definition (14) are satisfied:
(1): F_0 is indeed a map $F_0 : \mathrm{Ob}\ \mathcal{C}^{op} \to \mathrm{Ob}\ \mathcal{D}^{op}$ since F is a functor and F_0 is therefore a map $F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{D}$. Furthermore, we have $\mathrm{Ob}\ \mathcal{C}^{op} = \mathrm{Ob}\ \mathcal{C}$ and $\mathrm{Ob}\ \mathcal{D}^{op} = \mathrm{Ob}\ \mathcal{D}$ by virtue of definition (5).
(2): F_1 is indeed a map $F_1 : \mathrm{Arr}\ \mathcal{C}^{op} \to \mathrm{Arr}\ \mathcal{D}^{op}$ since $\mathrm{Arr}\ \mathcal{C}^{op} = \mathrm{Arr}\ \mathcal{C}$ and $\mathrm{Arr}\ \mathcal{D}^{op} = \mathrm{Arr}\ \mathcal{D}$ from definition (5), and F_1 is a map $F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$.

(3): We have $F_1(f) : F_0(a) \to F_0(b)$ @ $\mathcal{D}^{op}$ whenever $f : a \to b$ @ $\mathcal{C}^{op}$. Indeed, the assumption $f : a \to b$ @ $\mathcal{C}^{op}$ is equivalent to $f : b \to a$ @ $\mathcal{C}$. F being a functor, this implies $F_1(f) : F_0(b) \to F_0(a)$ @ $\mathcal{D}$ which is equivalent to $F_1(f) : F_0(a) \to F_0(b)$ @ $\mathcal{D}^{op}$.

(4): We have $F_1(\,\mathrm{id}(a)$ @ $\mathcal{C}^{op}) = \mathrm{id}(\,F_0(a)\,)$ @ $\mathcal{D}^{op}$ for all $a \in \mathcal{C}^{op}$. This follows from the equality $F_1(\,\mathrm{id}(a)$ @ $\mathcal{C}) = \mathrm{id}(\,F_0(a)\,)$ @ $\mathcal{D}$ and the fact that the notions of objects and identity coincide on a category and its opposite.

(5): We have $F_1(g \circ f$ @ $\mathcal{C}^{op}) = F_1(g) \circ F_1(f)$ @ $\mathcal{D}^{op}$ when $f : a \to b$ @ $\mathcal{C}^{op}$ and $g : b \to c$ @ $\mathcal{C}^{op}$. Indeed, the assumptions $f : a \to b$ @ $\mathcal{C}^{op}$ and $g : b \to c$ @ $\mathcal{C}^{op}$ are equivalent to $f : b \to a$ @ $\mathcal{C}$ and $g : c \to b$ @ $\mathcal{C}$. Hence we have:

$$\begin{array}{rcl} F_1(g \circ f \text{ @ } \mathcal{C}^{op}) & = & F_1(f \circ g \text{ @ } \mathcal{C}) \;\leftarrow\; \text{def. (5)} \\ \text{(5) of def. (14)} \;\rightarrow & = & F_1(f) \circ F_1(g) \text{ @ } \mathcal{D} \\ \text{def. (5)} \;\rightarrow & = & F_1(g) \circ F_1(f) \text{ @ } \mathcal{D}^{op} \end{array}$$

$\diamond$

Proposition (27) illustrates the fact that the knowledge of a functor F by itself does not tell us what its domain and codomain are. In fact, we have not even defined what the *domain* or *codomain* of a functor should be. By virtue of notation (13), writing $F : \mathcal{C} \to \mathcal{D}$ is simply a notational shortcut for the statement that F is a functor between $\mathcal{C}$ and $\mathcal{D}$. This is very similar to writing $f : a \to b$ for an untyped function f. Now we cannot simply define the domain and codomain of a functor F to be the categories $\mathcal{C}$ and $\mathcal{D}$ respectively, whenever $F : \mathcal{C} \to \mathcal{D}$. This is because the categories $\mathcal{C}$ and $\mathcal{D}$ are not unique, as whenever the statement $F : \mathcal{C} \to \mathcal{D}$ holds, the statement $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$ also holds. To resolve this issue, similarly to (4), we define:

Definition 17 *Given a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$, $\mathcal{D}$, we say that F is the* untyped functor *while the triple $(\mathcal{C}, \mathcal{D}, F)$ is called the* typed *functor.*

2.5 Composition of Functors

Definition 18 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors where $\mathcal{C}$,$\mathcal{D}$,$\mathcal{E}$ are categories. We call* composition of G and F *the functor $H : \mathcal{C} \to \mathcal{E}$ defined by:*

$$\begin{array}{ll} (1) & H_0(a) = G_0(F_0(a)) \\ (2) & H_1(f) = G_1(F_1(f)) \end{array}$$

where (1) *holds for all $a \in \mathrm{Ob}\ \mathcal{C}$ and* (2) *holds for all $f \in \mathrm{Arr}\ \mathcal{C}$.*

Notation 17 *The composition of two functors G and F is denoted GF or $G \circ F$.*

Remark: In view of notation (8), (1) and (2) of definition (18) could equally have been written $H_0 = G_0 \circ F_0$ and $H_1 = G_1 \circ F_1$. Note that the overloaded symbol '$\circ$' has many possible meanings: it is the usual symbol for set-theoretic function composition, it also refers to composition of maps between collections as per notation (8), it is the generic symbol for the composition operator in an arbitrary category as per notation (6), and finally it is also used to denote composition of functors as per notation (17).

Proposition 28 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors where $\mathcal{C}$,$\mathcal{D}$,$\mathcal{E}$ are categories. Then $G \circ F$ is indeed a functor $G \circ F : \mathcal{C} \to \mathcal{E}$.*

Proof
We need to check that properties (1) – (5) of definition (14) are satisfied:

(1): $(GF)_0 = G_0 \circ F_0$ is indeed a map $(GF)_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{E}$, since:

$$F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{D},\ G_0 : \mathrm{Ob}\ \mathcal{D} \to \mathrm{Ob}\ \mathcal{E}$$

(2): $(GF)_1 = G_1 \circ F_1$ is indeed a map $(GF)_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{E}$, since:

$$F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D},\ G_1 : \mathrm{Arr}\ \mathcal{D} \to \mathrm{Arr}\ \mathcal{E}$$

(3): We have $(GF)_1(f) : (GF)_0(a) \to (GF)_0(b)$ whenever $f : a \to b$, since:

$$\begin{array}{rcl} f : a \to b & \Rightarrow & F_1(f) : F_0(a) \to F_0(b) \ \leftarrow \text{(3) of def. (14)} \\ \text{(3) of def. (14)} \ \to & \Rightarrow & G_1(F_1(f)) : G_0(F_0(a)) \to G_0(F_0(b)) \\ \text{(1) and (2) of def. (18)} \ \to & \Leftrightarrow & (GF)_1(f) : (GF)_0(a) \to (GF)_0(b) \end{array}$$

(4): We have $(GF)_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\,(GF)_0(a)\,)$ for all $a \in \mathcal{C}$, since:

$$\begin{array}{rcl} (GF)_1(\,\mathrm{id}(a)\,) & = & G_1(F_1(\,\mathrm{id}(a)\,)) \ \leftarrow \ \text{(2) of def. (18)} \\ \text{(4) of def. (14)} \ \to & = & G_1(\,\mathrm{id}(\,F_0(a)\,)\,) \\ \text{(4) of def. (14)} \ \to & = & \mathrm{id}(\,G_0(\,F_0(a)\,)\,) \\ \text{(1) of def. (18)} \ \to & = & \mathrm{id}(\,(GF)_0(a)\,) \end{array}$$

(5): $(GF)_1(g \circ f) = (GF)_1(g) \circ (GF)_1(f)$ whenever $f : a \to b$ and $g : b \to c$:

$$\begin{array}{rcl} (GF)_1(g \circ f) & = & G_1(F_1(\,g \circ f\,)) \ \leftarrow \ \text{(2) of def. (18)} \\ \text{(5) of def. (14)} \ \to & = & G_1(\,F_1(g) \circ F_1(f)\,) \\ \text{(5) of def. (14)} \ \to & = & G_1(F_1(g)) \circ G_1(F_1(f)) \\ \text{(2) of def. (18)} \ \to & = & (GF)_1(g) \circ (GF)_1(f) \end{array}$$

$\diamond$

2.6 Equality between Functors

One question which we have not yet considered is that of equality between functors. Given two categories $\mathcal{C}$ and $\mathcal{D}$, and two functors $F, G : \mathcal{C} \to \mathcal{D}$, it will sometimes be convenient to argue that these two functors are equal. For one thing, we would like to state that functor composition is associative, a statement which is not meaningful without a notion of equality between functors. Such equality is not obvious and may depend on the specifics of the logical context in which we are working. Equality between sets is given but anything else requires careful consideration. Using the principles of section (1.3), we are able to state:

Proposition 29 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between categories $\mathcal{C}$,$\mathcal{D}$ with:*

$$\begin{array}{ll} (1) & \forall a \in \text{Ob } \mathcal{C} \ \ , \ F(a) = G(a) \\ (2) & \forall f \in \text{Arr } \mathcal{C} \ , \ F(f) = G(f) \end{array}$$

Then $F = G$, i.e. the two functors F and G are equal.

Proof
Let $F = (F_0, F_1)$ and $G = (G_0, G_1)$ be two functors between $\mathcal{C}$ and $\mathcal{D}$ such that $F_0(a) = G_0(a)$ for all $a \in \text{Ob}\,\mathcal{C}$ and $F_1(f) = G_1(f)$ for all $f \in \text{Arr}\,\mathcal{C}$. We need to show that $F = G$, which is $(F_0, F_1) = (G_0, G_1)$. In order to show this equality, using axiom (4) it is sufficient to prove that $F_0 = G_0$ and $F_1 = G_1$. However, by virtue of definition (14), both F_0 and G_0 are maps $F_0, G_0 : \text{Ob } \mathcal{C} \to \text{Ob } \mathcal{D}$, and both F_1, G_1 are maps $F_1, G_1 : \text{Arr } \mathcal{C} \to \text{Arr } \mathcal{D}$. Hence, using axiom (3), in order to show that $F_0 = G_0$, it is sufficient to prove that $F_0(a) = G_0(a)$ for all $a \in \text{Ob } \mathcal{C}$, and this is true by assumption. Likewise, in order to show that $F_1 = G_1$ it is sufficient to prove that $F_1(f) = G_1(f)$ for all $f \in \text{Arr } \mathcal{C}$ which also true by assumption. $\diamond$

Proposition 30 *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ and $H : \mathcal{E} \to \mathcal{F}$ be functors between categories $\mathcal{C}$,$\mathcal{D}$,$\mathcal{E}$ and $\mathcal{F}$. Then we have the equality:*

$$(H \circ G) \circ F = H \circ (G \circ F)$$

i.e. functor composition is associative.

Proof
Using proposition (29) we simply need to show that the two functors $(H \circ G) \circ F$ and $H \circ (G \circ F)$ act equally on both objects and arrows of the category $\mathcal{C}$. For $a \in \text{Ob } \mathcal{C}$, this goes as follows:

$$\begin{array}{rcl} ((H \circ G) \circ F)(a) & = & (H \circ G)(\,F(a)\,) \ \leftarrow \ \text{def. (18)} \\ \text{def. (18)} \ \to & = & H(\,G(\,F(a)\,)\,) \\ \text{def. (18)} \ \to & = & H(\,(G \circ F)(a)\,) \\ \text{def. (18)} \ \to & = & (H \circ (G \circ F))(a) \end{array}$$

and likewise for $f \in \text{Arr } \mathcal{C}$:

$$\begin{array}{rcl} ((H \circ G) \circ F)(f) & = & (H \circ G)(\,F(f)\,) \ \leftarrow \ \text{def. (18)} \\ \text{def. (18)} \ \to & = & H(\,G(\,F(f)\,)\,) \\ \text{def. (18)} \ \to & = & H(\,(G \circ F)(f)\,) \\ \text{def. (18)} \ \to & = & (H \circ (G \circ F))(f) \end{array}$$

$\diamond$

Proposition 31 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}$, $\mathcal{D}$. Then:*

$$I_{\mathcal{D}} \circ F = F$$

where $I_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is the identity functor on $\mathcal{D}$ as per definition (15).

Proof
Using proposition (29) we simply need to show that the two functors $I_{\mathcal{D}} \circ F$ and F act equally on both objects and arrows of the category $\mathcal{C}$. For $a \in \mathrm{Ob}\ \mathcal{C}$:

$$\begin{array}{rcll} (I_{\mathcal{D}} \circ F)(a) & = & I_{\mathcal{D}}(\,F(a)\,) & \leftarrow\ \text{(1) of def. (18)} \\ \text{(1) of def. (15)}\ \rightarrow & = & F(a) & \end{array}$$

and likewise for $f \in \mathrm{Arr}\ \mathcal{C}$:

$$\begin{array}{rcll} (I_{\mathcal{D}} \circ F)(f) & = & I_{\mathcal{D}}(\,F(f)\,) & \leftarrow\ \text{(2) of def. (18)} \\ \text{(2) of def. (15)}\ \rightarrow & = & F(f) & \end{array}$$

$\diamond$

Proposition 32 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}$, $\mathcal{D}$. Then:*

$$F \circ I_{\mathcal{C}} = F$$

where $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor on $\mathcal{C}$ as per definition (15).

Proof
Using proposition (29) we simply need to show that the two functors $F \circ I_{\mathcal{C}}$ and F act equally on both objects and arrows of the category $\mathcal{C}$. For $a \in \mathrm{Ob}\ \mathcal{C}$:

$$\begin{array}{rcll} (F \circ I_{\mathcal{C}})(a) & = & F(\,I_{\mathcal{C}}(a)\,) & \leftarrow\ \text{(1) of def. (18)} \\ \text{(1) of def. (15)}\ \rightarrow & = & F(a) & \end{array}$$

and likewise for $f \in \mathrm{Arr}\ \mathcal{C}$:

$$\begin{array}{rcll} (F \circ I_{\mathcal{C}})(f) & = & F(\,I_{\mathcal{C}}(f)\,) & \leftarrow\ \text{(2) of def. (18)} \\ \text{(2) of def. (15)}\ \rightarrow & = & F(f) & \end{array}$$

$\diamond$

Proposition 33 *Let $\mathcal{C}$ be a category. Then $I_{\mathcal{C}} = I_{\mathcal{C}^{op}}$, where $I_{\mathcal{C}}$ and $I_{\mathcal{C}^{op}}$ are the identity functors on $\mathcal{C}$ and $\mathcal{C}^{op}$ respectively as per definition (15).*

Proof
Since $I_{\mathcal{C}}$ is a functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, by virtue of proposition (27) it is also a functor $I_{\mathcal{C}} : \mathcal{C}^{op} \to \mathcal{C}^{op}$. Hence from proposition (29), in order to show that $I_{\mathcal{C}} = I_{\mathcal{C}^{op}}$ it is sufficient to prove that $I_{\mathcal{C}}(a) = I_{\mathcal{C}^{op}}(a)$ for all $a \in \mathrm{Ob}\ \mathcal{C} = \mathrm{Ob}\ \mathcal{C}^{op}$, as well as $I_{\mathcal{C}}(f) = I_{\mathcal{C}^{op}}(f)$ for all $f \in \mathrm{Arr}\ \mathcal{C} = \mathrm{Arr}\ \mathcal{C}^{op}$. This follows immediately from definition (15), since both functors act as the identity on objects and arrows. $\diamond$

2.7 Hom-functor of a Locally Small Category

Definition 19 *Let $\mathcal{C}$ be a locally small category. We call* hom-functor associated with $\mathcal{C}$ *the functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ defined by $F = (F_0, F_1)$ with:*

$$\begin{array}{ll} (1) & F_0(a_1, a_2) = \mathcal{C}(a_1, a_2) \\ (2) & F_1(f_1, f_2)(h) = f_2 \circ h \circ f_1 \end{array}$$

where (2) holds for $a_1, a_2, b_1, b_2 \in \mathcal{C}$, $f_1 : b_1 \to a_1$, $h : a_1 \to a_2$ and $f_2 : a_2 \to b_2$.

Notation 18 *Given a locally small category $\mathcal{C}$, the hom-functor $F = (F_0, F_1)$ associated with $\mathcal{C}$ is denoted $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$.*

Remark: Using the notation $\mathcal{C}$ to denote both the locally small category $\mathcal{C}$ and its associated hom-functor may appear confusing, but the notation actally makes sense since the equation $F_0(a_1, a_2) = \mathcal{C}(a_1, a_2)$ simply becomes the tautology $\mathcal{C}(a_1, a_2) = \mathcal{C}(a_1, a_2)$. In other words, using $\mathcal{C}$ to denote the hom-functor makes it very easy to remember that when applied to the object (a_1, a_2) of the category $\mathcal{C}^{op} \times \mathcal{C}$, we simply obtain the hom-set $\mathcal{C}(a_1, a_2)$ of the locally small category $\mathcal{C}$.

Given a locally small category $\mathcal{C}$, definition (19) defines a tuple $F = (F_0, F_1)$ where F_0 appears to be a map defined on Ob $\mathcal{C} \times$ Ob $\mathcal{C}$ with values in **Set**, and F_1 appears to be a map defined on Arr $\mathcal{C} \times$ Arr $\mathcal{C}$ with values in some functional space (since it takes an h as argument). Looking at this, it is far from obvious that definition (19) defines a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$. Hence we state:

Proposition 34 *Let $\mathcal{C}$ be a locally small category. Then the hom-functor F associated with $\mathcal{C}$ is indeed a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$.*

Proof
We need to check that properties $(1) - (5)$ of definition (14) are satisfied:

(1): We need to show that F_0 is a map $F_0 : \text{Ob } (\mathcal{C}^{op} \times \mathcal{C}) \to \text{Ob } \mathbf{Set}$. Having defined $F_0(a_1, a_2) = \mathcal{C}(a_1, a_2)$ and the category $\mathcal{C}$ being locally small, we see that $F_0(a_1, a_2)$ is a set for all $a_1, a_2 \in \mathcal{C}$. So F_0 is defined as a map $F_0 : (\text{Ob } \mathcal{C}) \times (\text{Ob } \mathcal{C}) \to \text{Ob } \mathbf{Set}$, and it remains to check that the collections $(\text{Ob } \mathcal{C}) \times (\text{Ob } \mathcal{C})$ and $\text{Ob } (\mathcal{C}^{op} \times \mathcal{C})$ actually coincide, which goes as follows:

$$\begin{array}{rcl} (\text{Ob } \mathcal{C}) \times (\text{Ob } \mathcal{C}) & = & \{ \ (a_1, a_2) \mid a_1 \in \text{Ob } \mathcal{C} \ , \ a_2 \in \text{Ob } \mathcal{C} \} \\ \text{def. (5)} \to & = & \{ \ (a_1, a_2) \mid a_1 \in \text{Ob } \mathcal{C}^{op} \ , \ a_2 \in \text{Ob } \mathcal{C} \} \\ \text{(1) of def. (6)} \to & = & \text{Ob } (\mathcal{C}^{op} \times \mathcal{C}) \end{array}$$

(2): We need to show that F_1 is a map $F_1 : \text{Arr } (\mathcal{C}^{op} \times \mathcal{C}) \to \text{Arr } \mathbf{Set}$. Having defined $F_1(f_1, f_2)$ for any $f_1 : b_1 \to a_1$ and $f_2 : a_2 \to b_2$ where a_1, a_2, b_1, b_2 are arbitrary objects in $\mathcal{C}$, we see that F_1 is a map defined on $(\text{Arr } \mathcal{C}) \times (\text{Arr } \mathcal{C})$. So we need to check that the collections $(\text{Arr } \mathcal{C}) \times (\text{Arr } \mathcal{C})$ and $\text{Arr } (\mathcal{C}^{op} \times \mathcal{C})$ actually coincide, which goes as follows:

$$\begin{array}{rcl} (\text{Arr } \mathcal{C}) \times (\text{Arr } \mathcal{C}) & = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C} \ , \ f_2 \in \text{Arr } \mathcal{C} \} \\ \text{def. (5)} \to & = & \{ \ (f_1, f_2) \mid f_1 \in \text{Arr } \mathcal{C}^{op} \ , \ f_2 \in \text{Arr } \mathcal{C} \} \\ \text{(2) of def. (6)} \to & = & \text{Arr } (\mathcal{C}^{op} \times \mathcal{C}) \end{array}$$

However, given arrows f_1, f_2 in $\mathcal{C}$, we still need to check that $F_1(f_1, f_2)$ is a member of the collection Arr **Set**. In other words, we need to check that $F_1(f_1, f_2)$ is a typed function. Introducing the notations a_1, a_2, b_1, b_2 such that $f_1 : b_1 \to a_1$ and $f_2 : a_2 \to b_2$, our definition states that $F_1(f_1, f_2)(h)$ is defined for any $h : a_1 \to a_2$. In other words, $F_1(f_1, f_2)(h)$ is defined for any h which belongs to the hom-set $\mathcal{C}(a_1, a_2)$. Having assumed that $\mathcal{C}$ is a locally small category, the collection $\mathcal{C}(a_1, a_2)$ is in fact a set, and $F_1(f_1, f_2)$ is therefore an untyped function with domain $\mathcal{C}(a_1, a_2)$, defined by $F_1(f_1, f_2)(h) = f_2 \circ h \circ f_1$. Note that since $f_1 : b_1 \to a_1$ and $f_2 : a_2 \to b_2$, the composition $f_2 \circ h \circ f_1$ is a well-defined arrow in $\mathcal{C}$ whenever $h \in \mathcal{C}(a_1, a_2)$. This arrow has domain b_1 and codomain b_2 and we see that $F_1(f_1, f_2)(h)$ is in fact an element of the hom-set $\mathcal{C}(b_1, b_2)$. So $F_1(f_1, f_2)$ is an untyped function $F_1(f_1, f_2) : \mathcal{C}(a_1, a_2) \to \mathcal{C}(b_1, b_2)$. Now looking at definition (3), an arrow of the category **Set** is a typed function (a, b, f) where a and b are sets and f is an untyped function $f : a \to b$. Hence, strictly speaking our definition of $F_1(f_1, f_2)$ is not a member of Arr **Set** but simply an untyped function $F_1(f_1, f_2) : \mathcal{C}(a_1, a_2) \to \mathcal{C}(b_1, b_2)$. However, the typed function $(\mathcal{C}(a_1, a_2), \mathcal{C}(b_1, b_2), F_1(f_1, f_2))$ is an arrow of the category **Set** and we have agreed in notation (9) to simply refer to this arrow as $F_1(f_1, f_2)$, as the domain $\mathcal{C}(a_1, a_2)$ and intended codomain $\mathcal{C}(b_1, b_2)$ can easily be inferred from the formula $F_1(f_1, f_2)(h) = f_2 \circ h \circ f_1$ for all $h : a_1 \to a_2$. This completes our proof that F_1 is a map $F_1 : \text{Arr}\ (\mathcal{C}^{op} \times \mathcal{C}) \to \text{Arr}\ \mathbf{Set}$.

(3): We need to check that $F_1(f) : F_0(a) \to F_0(b)$ whenever $a, b \in \mathcal{C}^{op} \times \mathcal{C}$ and $f : a \to b$. So let $a = (a_1, a_2) \in \mathcal{C}^{op} \times \mathcal{C}$, $b = (b_1, b_2) \in \mathcal{C}^{op} \times \mathcal{C}$ and let f be an arrow $f = (f_1, f_2) : (a_1, a_2) \to (b_1, b_2)\ @\ \mathcal{C}^{op} \times \mathcal{C}$. From the equalities $F_0(a) = \mathcal{C}(a_1, a_2)$ and $F_0(b) = \mathcal{C}(b_1, b_2)$, it is clear that we simply need to check $F_1(f_1, f_2) : \mathcal{C}(a_1, a_2) \to \mathcal{C}(b_1, b_2)$. However, we have already established this fact in part (2) of this proof, provided we show that $f_1 : b_1 \to a_1\ @\ \mathcal{C}$ together with $f_2 : a_2 \to b_2\ @\ \mathcal{C}$. Hence we need $(f_1, f_2) \in \mathcal{C}(b_1, a_1) \times \mathcal{C}(a_2, b_2)$, knowing that $(f_1, f_2) \in (\mathcal{C}^{op} \times \mathcal{C})[\,(a_1, a_2)\,,\,(b_1, b_2)\,]$ by assumption. It is therefore sufficient to prove that the two collections $\mathcal{C}(b_1, a_1) \times \mathcal{C}(a_2, b_2)$ and $\mathcal{C}^{op} \times \mathcal{C}[\,(a_1, a_2)\,,\,(b_1, b_2)\,]$ coincide, which goes as follows:

$$\begin{array}{rcl} \mathcal{C}^{op} \times \mathcal{C}[\,(a_1, a_2)\,,\,(b_1, b_2)\,] & = & \mathcal{C}^{op}(a_1, b_1) \times \mathcal{C}(a_2, b_2) \ \leftarrow \ \text{prop. (12)} \\ \text{prop. (11)} \ \to & = & \mathcal{C}(b_1, a_1) \times \mathcal{C}(a_2, b_2) \end{array}$$

(4): We need to check that $F_1(\,\text{id}(a)\,) = \text{id}(\,F_0(a)\,)$ whenever $a \in \mathcal{C}^{op} \times \mathcal{C}$. So let $a = (a_1, a_2) \in \mathcal{C}^{op} \times \mathcal{C}$. Since $F_0(a) = \mathcal{C}(a_1, a_2)$, we need to check that $F_1(\,\text{id}(a)\,) = \text{id}(\,\mathcal{C}(a_1, a_2)\,)$. This is an equality between two arrows of the category **Set**, with identical domain and codomain, namely the set $\mathcal{C}(a_1, a_2)$. Given $h \in \mathcal{C}(a_1, a_2)$, from proposition (6) it is sufficient to check that $F_1(\,\text{id}(a)\,)(h) = h$:

$$\begin{array}{rcl} F_1(\,\text{id}(a)\,)(h) & = & F_1(\,\text{id}\,(a_1, a_2)\,)(h) \\ \text{(5) of def. (6)} \ \to & = & F_1(\,\text{id}(a_1)\ @\ \mathcal{C}^{op}\,,\,\text{id}(a_2)\,)(h) \\ \text{def. (5)} \ \to & = & F_1(\,\text{id}(a_1)\ @\ \mathcal{C}\,,\,\text{id}(a_2)\,)(h) \\ \text{(2) of def. (19)} \ \to & = & \text{id}(a_2) \circ h \circ \text{id}(a_1) \\ \text{(12) of def. (2)} \ \to & = & \text{id}(a_2) \circ h \end{array}$$

$$\text{(13) of def. (2)} \ \rightarrow \quad = \quad h$$

(5): We need to check that $F_1(g \circ f) = F_1(g) \circ F_1(f)$ whenever $f : a \rightarrow b$, $g : b \rightarrow c$ and $a, b, c \in \mathcal{C}^{op} \times \mathcal{C}$. So let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ be objects in $\mathcal{C}^{op} \times \mathcal{C}$, and $f = (f_1, f_1) : a \rightarrow b$ and $g = (g_1, g_2) : b \rightarrow c$. We have $g \circ f : a \rightarrow c$ and consequently $F_1(g \circ f) : F_0(a) \rightarrow F_0(c)$. Hence the arrows $F_1(g \circ f)$ and $F_1(g) \circ F_1(f)$ are two arrows in **Set**, with domain $F_0(a) = \mathcal{C}(a_1, a_2)$ and codomain $F_0(c) = \mathcal{C}(c_1, c_2)$. From proposition (6), in order to prove the equality $F_1(g \circ f) = F_1(g) \circ F_1(f)$ it is therefore sufficient to show that the underlying functions coincide for all $h \in \mathcal{C}(a_1, a_2)$ which goes as follows:

$$\begin{array}{rcl}
F_1(g \circ f)(h) & = & F_1(\,(g_1, g_2) \,\circ\, (f_1, f_2)\,)(h) \\
\text{(6) of def. (6)} \ \rightarrow & = & F_1(\,g_1 \circ f_1 \ @\ \mathcal{C}^{op},\ g_2 \circ f_2\,)\,(h) \\
\text{crucially, def. (5)} \ \rightarrow & = & F_1(\,f_1 \circ g_1\,,\ g_2 \circ f_2\,)\,(h) \\
\text{(2) of def. (19)} \ \rightarrow & = & (g_2 \circ f_2) \,\circ\, h \,\circ (f_1 \circ g_1) \\
\text{associativity of} \circ \text{in } \mathcal{C} \ \rightarrow & = & g_2 \,\circ\, (f_2 \circ h \circ f_1) \,\circ g_1 \\
\text{(2) of def. (19)} \ \rightarrow & = & g_2 \,\circ\, F_1(f_1, f_2)(h) \,\circ g_1 \\
\text{(2) of def. (19)} \ \rightarrow & = & F_1(g_1, g_2)(\,F_1(f_1, f_2)(h)\,) \\
 & = & F_1(g)(\,F_1(f)(h)\,) \\
 & = & (F_1(g) \circ F_1(f))(h)
\end{array}$$

$\diamond$

2.8 Canonical Product of Functors

Definition 20 *We call* canonical product *of two functors $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ and $F_2 : \mathcal{C}_2 \rightarrow \mathcal{D}_2$ the functor $G : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}_1 \times \mathcal{D}_2$ defined by $G = (G_0, G_1)$ with:*

$$\begin{array}{ll}
(1) & G_0(a_1, a_2) = (\,F_1(a_1)\,,\ F_2(a_2)\,) \\
(2) & G_1(f_1, f_2) = (\,F_1(f_1)\,,\ F_2(f_2)\,)
\end{array}$$

where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2$ are arbitrary categories, (1) *holds for all $a_1 \in \mathrm{Ob}\ \mathcal{C}_1$ and $a_2 \in \mathrm{Ob}\ \mathcal{C}_2$, and* (2) *holds for all $f_1 \in \mathrm{Arr}\ \mathcal{C}_1$ and $f_2 \in \mathrm{Arr}\ \mathcal{C}_2$.*

Remark: In accordance with notation (14), we are using the same notations F_1 and F_2 in definition (20) to describe the actions of these functors both on objects and on arrows. The alternative would be to use the notations $(F_1)_0$, $(F_1)_1$, $(F_2)_0$ and $(F_2)_1$ which would arguably be harder to read.

Notation 19 *The canonical product of functors F_1 and F_2 is denoted $F_1 \times F_2$.*

Proposition 35 *Let $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ and $F_2 : \mathcal{C}_2 \rightarrow \mathcal{D}_2$ be two functors. Then the product of F_1 and F_2 is indeed a functor $F_1 \times F_2 : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}_1 \times \mathcal{D}_2$.*

Proof
Let $G = (G_0, G_1)$ denote the product functor $F_1 \times F_2$. We need to check that properties (1) – (5) of definition (14) are satisfied, which goes as follows:

(1): G_0 is indeed a map $G_0 : \mathrm{Ob}\ (\mathcal{C}_1 \times \mathcal{C}_2) \to \mathrm{Ob}\ (\mathcal{D}_1 \times \mathcal{D}_2)$: firstly, G_0 is defined on the collection of all (a_1, a_2) where $a_1 \in \mathrm{Ob}\ \mathcal{C}_1$ and $a_2 \in \mathrm{Ob}\ \mathcal{C}_2$. According to definition (6), this is precisely the collection $\mathrm{Ob}\ (\mathcal{C}_1 \times \mathcal{C}_2)$. Furthermore $G_0(a_1, a_2)$ is defined as $(\, F_1(a_1)\, ,\, F_2(a_2)\,)$ and since $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ while $F_2 : \mathcal{C}_2 \to \mathcal{D}_2$, we have $F_1(a_1) \in \mathrm{Ob}\ \mathcal{D}_1$ together with $F_2(a_2) \in \mathrm{Ob}\ \mathcal{D}_2$. Hence we see that $G_0(a_1, a_2)$ is indeed a member of the collection $\mathrm{Ob}\ (\mathcal{D}_1 \times \mathcal{D}_2)$.

(2): G_1 is indeed a map $G_1 : \mathrm{Arr}\ (\mathcal{C}_1 \times \mathcal{C}_2) \to \mathrm{Arr}\ (\mathcal{D}_1 \times \mathcal{D}_2)$: G_1 is defined on the collection of all (f_1, f_2) where $f_1 \in \mathrm{Arr}\ \mathcal{C}_1$ and $f_2 \in \mathrm{Arr}\ \mathcal{C}_2$. According to definition (6), this is precisely the collection $\mathrm{Arr}\ (\mathcal{C}_1 \times \mathcal{C}_2)$. Furthermore $G_1(f_1, f_2)$ is defined as $(\, F_1(f_1)\, ,\, F_2(f_2)\,)$ and since $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ while we have $F_2 : \mathcal{C}_2 \to \mathcal{D}_2$, we see that $F_1(f_1) \in \mathrm{Arr}\ \mathcal{D}_1$ and $F_2(f_2) \in \mathrm{Arr}\ \mathcal{D}_2$. Hence we conclude that $G_1(f_1, f_2)$ is indeed a member of the collection $\mathrm{Arr}\ (\mathcal{D}_1 \times \mathcal{D}_2)$.

(3): We need to show that $G_1(f) : G_0(a) \to G_0(b)$ whenever $f : a \to b$: let $f \in \mathrm{Arr}\ (\mathcal{C}_1 \times \mathcal{C}_2)$ such that $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Then $f = (f_1, f_2)$ for some $f_1 \in \mathrm{Arr}\ \mathcal{C}_1$ and $f_2 \in \mathrm{Arr}\ \mathcal{C}_2$. Furthermore, since $a, b \in \mathrm{Ob}\ (\mathcal{C}_1 \times \mathcal{C}_2)$, we have $a = (a_1, a_2)$ and $b = (b_1, b_2)$ for some $a_1, b_1 \in \mathrm{Ob}\ \mathcal{C}_1$ and $a_2, b_2 \in \mathrm{Ob}\ \mathcal{C}_2$:

$$\begin{array}{rcl}
(a_1, a_2) & = & a \\
& = & \mathrm{dom}\,(f) \\
& = & \mathrm{dom}\,(f_1, f_2) \\
\text{(3) of def. (6)} \ \to & = & (\,\mathrm{dom}\,(f_1)\,,\,\mathrm{dom}\,(f_2)\,)
\end{array}$$

Hence we have $a_1 = \mathrm{dom}(f_1)$ and $a_2 = \mathrm{dom}(f_2)$ and similarly:

$$\begin{array}{rcl}
(b_1, b_2) & = & b \\
& = & \mathrm{cod}\,(f) \\
& = & \mathrm{cod}\,(f_1, f_2) \\
\text{(4) of def. (6)} \ \to & = & (\,\mathrm{cod}\,(f_1)\,,\,\mathrm{cod}\,(f_2)\,)
\end{array}$$

from which we conclude that $b_1 = \mathrm{cod}(f_1)$ and $b_2 = \mathrm{cod}(f_2)$. Hence we see that $f_1 : a_1 \to b_1 \ @\ \mathcal{C}_1$ and $f_2 : a_2 \to b_2 \ @\ \mathcal{C}_2$. Since F_1 and F_2 are functors, using (3) of definition (14) we obtain $F_1(f_1) : F_1(a_1) \to F_1(b_1) \ @\ \mathcal{D}_1$ and likewise $F_2(f_2) : F_2(a_2) \to F_2(b_2) \ @\ \mathcal{D}_2$. In order to show that $G_1(f) : G_0(a) \to G_0(b)$, since we already know that $G_1(f) \in \mathrm{Arr}(\mathcal{D}_1 \times \mathcal{D}_2)$, it remains to show that $\mathrm{dom}\,(G_1(f)) = G_0(a)$ and $\mathrm{cod}\,(G_1(f)) = G_0(b)$, which goes as follows:

$$\begin{array}{rcl}
\mathrm{dom}\,(G_1(f)) & = & \mathrm{dom}\,(\,G_1(f_1, f_2)\,) \\
\text{(2) of def. (20)} \ \to & = & \mathrm{dom}\,(\,F_1(f_1)\,,\,F_2(f_2)) \\
\text{(3) of def. (6)} \ \to & = & (\,\mathrm{dom}\,(\,F_1(f_1)\,)\,,\,\mathrm{dom}\,(\,F_2(f_2)\,)\,)
\end{array}$$

$$
\begin{array}{rcl}
F_1(f_1) : F_1(a_1) \to F_1(b_1) \ \to & = & (\, F_1(a_1)\, ,\, \mathrm{dom}\, (\, F_2(f_2)\,)\,) \\
F_2(f_2) : F_2(a_2) \to F_2(b_2) \ \to & = & (\, F_1(a_1)\, ,\, F_2(a_2)\,) \\
(1) \text{ of def. } (20) \ \to & = & G_0(a_1, a_2) \\
& = & G_0(a)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{cod}\, (G_1(f)) & = & \mathrm{cod}\, (\, G_1(f_1, f_2)\,) \\
(2) \text{ of def. } (20) \ \to & = & \mathrm{cod}\, (\, F_1(f_1)\, ,\, F_2(f_2)) \\
(4) \text{ of def. } (6) \ \to & = & (\, \mathrm{cod}\, (\, F_1(f_1)\,)\, ,\, \mathrm{cod}\, (\, F_2(f_2)\,)\,) \\
F_1(f_1) : F_1(a_1) \to F_1(b_1) \ \to & = & (\, F_1(b_1)\, ,\, \mathrm{cod}\, (\, F_2(f_2)\,)\,) \\
F_2(f_2) : F_2(a_2) \to F_2(b_2) \ \to & = & (\, F_1(b_1)\, ,\, F_2(b_2)\,) \\
(1) \text{ of def. } (20) \ \to & = & G_0(b_1, b_2) \\
& = & G_0(b)
\end{array}
$$

(4): We have $G_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\, G_0(a)\,)$ for all $a = (a_1, a_2) \in \mathcal{C}_1 \times \mathcal{C}_2$:

$$
\begin{array}{rcl}
G_1(\,\mathrm{id}(a)\,) & = & G_1(\,\mathrm{id}(\, a_1, a_2\,)\,) \\
(5) \text{ of def. } (6) \ \to & = & G_1(\,\mathrm{id}(a_1)\, ,\, \mathrm{id}(a_2)\,) \\
(2) \text{ of def. } (20) \ \to & = & (\, F_1(\,\mathrm{id}(a_1)\,)\, ,\, F_2(\,\mathrm{id}(a_2)\,)\,) \\
(4) \text{ of def. } (14) \ \to & = & (\,\mathrm{id}(\, F_1(a_1)\,)\, ,\, \mathrm{id}(\, F_2(a_2)\,)\,) \\
(5) \text{ of def. } (6) \ \to & = & \mathrm{id}\, (\, F_1(a_1)\, ,\, F_2(a_2)\,) \\
(1) \text{ of def. } (20) \ \to & = & \mathrm{id}\, (\, G_0(a_1, a_2)\,) \\
& = & \mathrm{id}\, (G_0(a))
\end{array}
$$

(5): We need to show that $G_1(g \circ f) = G_1(g) \circ G_1(f)$ for all $f : a \to b \ @\ \mathcal{C}_1 \times \mathcal{C}_2$ and $g : b \to c \ @\ \mathcal{C}_1 \times \mathcal{C}_2$. So let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ with $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$. Following the same details as in (3), we have $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$ and similarly $g_1 : b_1 \to c_1$ and $g_2 : b_2 \to c_2$:

$$
\begin{array}{rcl}
G_1(g \circ f) & = & G_1(\,(g_1, g_2)\, \circ\, (f_1, f_2)\,) \\
(6) \text{ of def. } (6) \ \to & = & G_1(\, g_1 \circ f_1\, ,\, g_2 \circ f_2\,) \\
(2) \text{ of def. } (20) \ \to & = & (\, F_1(\, g_1 \circ f_1\,)\, ,\, F_2(\, g_2 \circ f_2\,)\,) \\
(5) \text{ of def. } (14) \ \to & = & (\, F_1(g_1)\, \circ\, F_1(f_1)\, ,\, F_2(g_2)\, \circ\, F_2(f_2)\,) \\
(6) \text{ of def. } (6) \ \to & = & (\, F_1(g_1)\, ,\, F_2(g_2)\,)\, \circ\, (\, F_1(f_1)\, ,\, F_2(f_2)\,) \\
(2) \text{ of def. } (20) \ \to & = & G_1(g_1, g_2)\, \circ\, G_1(f_1, f_2) \\
& = & G_1(g)\, \circ\, G_1(f)
\end{array}
$$

$\diamond$

2.9 Cartesian Functor

Definition 21 *The* cartesian functor $F : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ *is defined by:*

$$
(1) \qquad F_0(a_1, a_2) = a_1 \times a_2
$$

$$(2) \qquad F_1(f_1, f_2)(x_1, x_2) = (\, f_1(x_1)\, ,\, f_2(x_2)\,)$$

where (1) *holds for all sets* a_1, a_2*, and* (2) *holds for all sets* a_1, a_2, b_1, b_2 *as well as* $f_1 : a_1 \to b_1$*,* $f_2 : a_2 \to b_2$ *together with* $x_1 \in a_1$ *and* $x_2 \in a_2$*.*

Remark: Recall that given two sets a_1 and a_2, the cartesian product $a_1 \times a_2$ of a_1 and a_2 is the set of all ordered pairs (x_1, x_2) where $x_1 \in a_1$ and $x_2 \in a_2$:

$$a_1 \times a_2 = \{(x_1, x_2) \mid x_1 \in a_1,\ x_2 \in a_2\}$$

Notation 20 *The cartesian functor is denoted* $(\times)$ *as an infix operator.*

Remark: So if $F : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is the cartesian functor, we shall typicallly write $a_1 \times a_2$ and $f_1 \times f_2$ instead of $F_0(a_1, a_2)$ and $F_1(f_1, f_2)$ respectively.
Remark: If f_1 and f_2 are arrows of the category **Set**, strictly speaking from definition (3) f_1 and f_2 are typed functions (a, b, f) where $f : a \to b$ is untyped. In particular f_1 and f_2 are sets and the cartesian product $f_1 \times f_2$ is meaningful, so we have a notational conflict with $f_1 \times f_2$, the cartesian functor evaluated at (f_1, f_2). Similarly to the composition $\circ$, the symbol $\times$ is highly overloaded and may also be used to denote the canonical product of two categories as in definition (6), or the canonical product of two functors as in notation (19).

Proposition 36 *The cartesian functor* $(\times)$ *is a functor* $(\times) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$*.*

Proof
Let $F = (F_0, F_1)$ denote the cartesian functor $(\times)$. We need to check that properties $(1) - (5)$ of definition (14) are satisfied:

(1): F_0 is indeed a map $F_0 : \mathrm{Ob}\ (\mathbf{Set} \times \mathbf{Set}) \to \mathrm{Ob}\ \mathbf{Set}$, since F_0 is defined on the collection of all (a_1, a_2) where a_1 and a_2 are sets and this collection is indeed the collection of all objects of $\mathbf{Set} \times \mathbf{Set}$. Furthemore $F_0(a_1, a_2) = a_1 \times a_2 \in \mathbf{Set}$.

(2): F_1 is indeed a map $F_1 : \mathrm{Arr}\ (\mathbf{Set} \times \mathbf{Set}) \to \mathrm{Arr}\ \mathbf{Set}$: $F_1(f_1, f_2)$ is defined on the collection of all (f_1, f_2) where $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$ for arbitrary a_1, a_2, b_1, b_2. So it is defined on the collection of all (f_1, f_2) where f_1, f_2 are arrows in **Set**, and this collection is indeed $\mathrm{Arr}\ (\mathbf{Set} \times \mathbf{Set})$. Furthemore, if we have $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$, then $F_1(f_1, f_2)(x_1, x_2)$ is defined for all (x_1, x_2) where $x_1 \in a_1$ and $x_2 \in a_2$. So $F_1(f_1, f_2)$ is a function defined on the cartesian product $a_1 \times a_2$. Since $F_1(f_1, f_2)(x_1, x_2)$ is defined as $(\, f_1(x_1)\, ,\, f_2(x_2)\,)$ we see that $F_1(f_1, f_2)$ is in fact a function $F_1(f_1, f_2) : a_1 \times a_2 \to b_1 \times b_2$. However, definition (21) does not spell out the fact that $b_1 \times b_2$ is the intended codomain of $F_1(f_1, f_2)$, but this is pretty clear from the context. Furthermore, definition (21) does not explicitely define a typed function $(a_1 \times a_2, b_1 \times b_2, F_1(f_1, f_2))$ of **Set**, but only an untyped function $F_1(f_1, f_2)$. This is also fair enough given the context and in line with notation (9). We conlude that $F_1(f_1, f_2) \in \mathrm{Arr}\ \mathbf{Set}$.

(3): We need to show that $F_1(f) : F_0(a) \to F_0(b)$ whenever $f : a \to b$: let $f \in \mathrm{Arr}\ (\mathbf{Set} \times \mathbf{Set})$ such that $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Then $a = (a_1, a_2)$ for some sets a_1, a_2, $b = (b_1, b_2)$ for some sets b_1, b_2 and $f = (f_1, f_2)$ for some arrows $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$. We just established in (2) the fact that $F_1(f_1, f_2) : a_1 \times a_2 \to b_1 \times b_2$, which is $F_1(f) : F_0(a) \to F_0(b)$ as requested.

(4): We need to show that $F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(\,F_0(a)\,)$ for all $a \in \mathbf{Set} \times \mathbf{Set}$. So let $a = (a_1, a_2)$. We need to show that $F_1(\,\mathrm{id}(a)\,) = \mathrm{id}(a_1 \times a_2)$. We already know that $F_1(\,\mathrm{id}(a)\,) : a_1 \times a_2 \to a_1 \times a_2$. So we simply need to check that the underlying function is the usual identity on the set $a_1 \times a_2$. Let $(x_1, x_2) \in a_1 \times a_2$:

$$\begin{array}{rcl}
F_1(\,\mathrm{id}(a)\,)(x_1, x_2) & = & F_1(\,\mathrm{id}(a_1, a_2)\,)(x_1, x_2) \\
\text{(5) of def. (6)} \;\to & = & F_1(\,\mathrm{id}(a_1)\,,\,\mathrm{id}(a_2)\,)(x_1, x_2) \\
\text{(2) of def. (21)} \;\to & = & (\,\mathrm{id}(a_1)(x_1)\,,\,\mathrm{id}(a_2)(x_2)\,) \\
\text{(5) of def. (3)} \;\to & = & (\,(a_1, a_1, i(a_1))(x_1)\,,\,(a_2, a_2, i(a_2))(x_2)\,) \\
\text{notation (10)} \;\to & = & (\,i(a_1)(x_1)\,,\,i(a_2)(x_2)\,) \\
i(a)(x) = x \;\to & = & (x_1, x_2)
\end{array}$$

(5): We need to show that $F_1(g \circ f) = F_1(g) \circ F_1(f)$ for $f : a \to b$ @ $\mathbf{Set} \times \mathbf{Set}$ and $g : b \to c$ @ $\mathbf{Set} \times \mathbf{Set}$. So let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ with $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$. Then we have $f_1 : a_1 \to b_1$ and $f_2 : a_2 \to b_2$ and similarly $g_1 : b_1 \to c_1$ and $g_2 : b_2 \to c_2$. We know that $F_1(f) : a_1 \times a_2 \to b_1 \times b_2$ and $F_1(g) : b_1 \times b_2 \to c_1 \times c_2$. Hence using proposition (6), in order to show that $F_1(g \circ f) = F_1(g) \circ F_1(f)$ it is sufficient to prove that the underlying functions coincide on the set $a_1 \times a_2$. Given $(x_1, x_2) \in a_1 \times a_2$, we have:

$$\begin{array}{rcl}
F_1(g \circ f)(x_1, x_2) & = & F_1(\,(g_1, g_2) \,\circ\, (f_1, f_2)\,)(x_1, x_2) \\
\text{(6) of def. (6)} \;\to & = & F_1(\,g_1 \circ f_1\,,\,g_2 \circ f_2\,)(x_1, x_2) \\
\text{(2) of def. (21)} \;\to & = & (\,(g_1 \circ f_1)(x_1)\,,\,(g_2 \circ f_2)(x_2)\,) \\
\circ \text{ in } \mathbf{Set} \;\to & = & (\,g_1(f_1(x_1))\,,\,g_2(f_2(x_2))\,) \\
\text{(2) of def. (21)} \;\to & = & F_1(g_1, g_2)(\,f_1(x_1)\,,\,f_2(x_2)\,) \\
\text{(2) of def. (21)} \;\to & = & F_1(g_1, g_2)(F_1(f_1, f_2)(x_1, x_2)) \\
\circ \text{ in } \mathbf{Set} \;\to & = & (\,F_1(g_1, g_2) \,\circ\, F_1(f_1, f_2)\,)(x_1, x_2) \\
 & = & (\,F_1(g) \,\circ\, F_1(f)\,)(x_1, x_2)
\end{array}$$

$\diamond$

2.10 Category of Small Categories

We are now familiar with the notion of *category* as defined in (2) as well as that of functor as defined in (14). Thanks to definition (18), we know how to *compose functors*, and we also have a notion of *identity functor* as defined in (15). So it is very tempting at this stage to wonder whether the *collection of all categories* could be turned into a category itself, in which the objects are categories and the arrows are functors. However, we know from set theory that assuming the existence of the *set of all sets* leads to a contradiction. Those familiar with proof assistants such that Coq, Agda and Lean will also be used to the idea that *the type of all types* does not exist, as we have *universes* and *type levels* instead.

So we shall not attempt to define *the category of all catgeories* here. Instead, we shall focus on a collection which is a lot smaller by considering only those categories which are *small*, as per definition (1).

Definition 22 *We call* **cat** *the category* $\mathbf{cat} = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ *where*

$$
\begin{array}{ll}
(1) & \mathrm{Ob} = \{\ \mathcal{C} \mid \mathcal{C} \textit{ is a small category }\} \\
(2) & \mathrm{Arr} = \{\ (\mathcal{C}, \mathcal{D}, F) \mid \mathcal{C}, \mathcal{D} \in \mathrm{Ob} \textit{ and } F : \mathcal{C} \to \mathcal{D}\ \} \\
(3) & \mathrm{dom}\,(\mathcal{C}, \mathcal{D}, F) = \mathcal{C} \\
(4) & \mathrm{cod}\,(\mathcal{C}, \mathcal{D}, F) = \mathcal{D} \\
(5) & \mathrm{id}(\mathcal{C}) = (\mathcal{C}, \mathcal{C}, I_{\mathcal{C}}) \\
(6) & (\mathcal{D}, \mathcal{E}, G) \circ (\mathcal{C}, \mathcal{D}, F) = (\mathcal{C}, \mathcal{E}, G \circ F)
\end{array}
$$

where $(3) - (6)$ *hold for all small categories* $\mathcal{C}, \mathcal{D}, \mathcal{E}$*, and functors* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$*,* $I_{\mathcal{C}}$ *is the identity functor of definition (15) and* $G \circ F$ *is the composition of* G *and* F *as per definition (18).*

Remark: Similarly to definition (3) where the arrows of the category **Set** are not functions but *typed functions*, the arrrows of the category **cat** are not functors but *typed functors* as per defintion (17).

The definition of **cat** is formally very similar to that of the category **Set**. However, we still need to do our due diligence and check it forms a category. Before we do so, we shall emphasize a couple of results:

Proposition 37 *Let* $\mathcal{C}$ *be a small category. Then* $\mathcal{C}$ *is a set.*

Proof
According to definition (1), a small category is a tuple $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ in which all the entries are sets, functions between sets (which are also sets) or a partial function between sets (which is also a set). Hence all the entries of this tuple are in fact sets which makes the tuple itself a set. $\diamond$

Proposition 38 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *between small categories* $\mathcal{C}$,$\mathcal{D}$ *is a set.*

Proof
Using definition (14), a functor F is a tuple $F = (F_0, F_1)$ where F_0 and F_1 are maps $F_0 : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Ob}\ \mathcal{D}$ and $F_1 : \mathrm{Arr}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$. When both $\mathcal{C}$ and $\mathcal{D}$ are small, the collections $\mathrm{Ob}\ \mathcal{C}$, $\mathrm{Ob}\ \mathcal{D}$, $\mathrm{Arr}\ \mathcal{C}$ and $\mathrm{Arr}\ \mathcal{D}$ are sets and F_0, F_1 are therefore maps between sets, i.e. functions which are also sets. So $F = (F_0, F_1)$ is a tuple where both entries are sets, which makes F itself a set. $\diamond$

Proposition 39 *The category* **cat** *of definition (22) is a category.*

Proof
Given $(\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ of definition (22), we need to check that this data satisfies condition $(1) - (13)$ of definition (2).

(1): The collection $\mathrm{Ob} = \{\, \mathcal{C} \mid \mathcal{C}$ is a small category $\}$ should be a collection with equality. However, from proposition (37) every small category is a set and Ob is therefore a collection of sets for which set equality is defined.

(2): The collection $\mathrm{Arr} = \{\, (\mathcal{C}, \mathcal{D}, F) \mid \mathcal{C}, \mathcal{D} \in \mathrm{Ob}$ and $F : \mathcal{C} \to \mathcal{D} \,\}$ should be a collection with equality. It is sufficient for us to establish that all members of this collection are sets. We have already seen from proposition (37) that a small category is a set, so each triple $(\mathcal{C}, \mathcal{D}, F)$ is also a set provided we can show that any functor $F : \mathcal{C} \to \mathcal{D}$ between two small categories is a set, which is exactly proposition (38).

(3): dom should be a map $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$. The equation $\mathrm{dom}(\mathcal{C}, \mathcal{D}, F) = \mathcal{C}$ holds for all small categories $\mathcal{C}$,$\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$. So dom is indeed defined on the collection Arr as requested. Furthermore $\mathrm{dom}\,(f) \in \mathrm{Ob}$ for all f.

(4): cod should be a map $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$ which is the case as per (3).

(5): id should be a map $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$. The equation $\mathrm{id}(\mathcal{C}) = (\mathcal{C}, \mathcal{C}, I_{\mathcal{C}})$ holds for all small category $\mathcal{C}$. So id is indeed defined on Ob as requested. So it remains to show that $(\mathcal{C}, \mathcal{C}, I_{\mathcal{C}}) \in \mathrm{Arr}$ for all $\mathcal{C}$, which the case since the identity functor $I_{\mathcal{C}}$ is a functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ as per proposition (25).

(6): $\circ$ should be a partial map $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$. From definition (22), $g \circ f$ is defined whenever f and g are of the form $f = (\mathcal{C}, \mathcal{D}, F)$ and $g = (\mathcal{D}, \mathcal{E}, G)$ where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$. So $g \circ f$ is defined on a sub-collection of $\mathrm{Arr} \times \mathrm{Arr}$ as requested. So it remains to show that $g \circ f \in \mathrm{Arr}$ when defined. However, $g \circ f$ is defined as $(\mathcal{C}, \mathcal{E}, G \circ F)$ where $G \circ F$ is the functor composition, so it remains to show that $G \circ F : \mathcal{C} \to \mathcal{E}$ which follows from proposition (28).

(7): $g \circ f$ should be defined exactly when $\mathrm{cod}(f) = \mathrm{dom}(g)$. From definition (22), $g \circ f$ is defined exactly when f is of the form $f = (\mathcal{C}, \mathcal{D}, F)$ and g is of the form $(\mathcal{D}, \mathcal{E}, G)$. Since $\mathrm{cod}(f) = \mathcal{D}$ and $\mathrm{dom}(g) = \mathcal{D}$, we see that $g \circ f$ is defined for all arrows f, g for which $\mathrm{cod}(f) = \mathrm{dom}(g)$ as requested.

(8): We should have $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ when $g \circ f$ is defined. So let $f = (\mathcal{C}, \mathcal{D}, F)$ and $g = (\mathcal{D}, \mathcal{E}, G)$. Then we have $g \circ f = (\mathcal{C}, \mathcal{E}, G \circ F)$ and consequently $\mathrm{dom}(g \circ f) = \mathcal{C}$ which is $\mathrm{dom}(f)$ as requested.

(9): We should have $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$ when $g \circ f$ is defined. So let $f = (\mathcal{C}, \mathcal{D}, F)$ and $g = (\mathcal{D}, \mathcal{E}, G)$. Then we have $g \circ f = (\mathcal{C}, \mathcal{E}, G \circ F)$ and consequently $\mathrm{cod}(g \circ f) = \mathcal{E}$ which is $\mathrm{cod}(g)$ as requested.

(10): We should have $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $g \circ f$ and $h \circ g$ are well defined. So let $f = (\mathcal{C}, \mathcal{D}, F)$, $g = (\mathcal{D}, \mathcal{E}, G)$ and $h = (\mathcal{E}, \mathcal{F}, H)$. We have:

$$
\begin{array}{rcl}
(h \circ g) \circ f & = & (\,(\mathcal{E}, \mathcal{F}, H) \circ (\mathcal{D}, \mathcal{E}, G)\,) \circ (\mathcal{C}, \mathcal{D}, F) \\
\text{(6) of def. (22)} \;\rightarrow & = & (\mathcal{D}, \mathcal{F}, H \circ G) \circ (\mathcal{C}, \mathcal{D}, F) \\
\text{(6) of def. (22)} \;\rightarrow & = & (\mathcal{C}, \mathcal{F}, (H \circ G) \circ F) \\
\text{functor} \circ \text{assoc, prop (30)} \;\rightarrow & = & (\mathcal{C}, \mathcal{F}, H \circ (G \circ F)) \\
\text{(6) of def. (22)} \;\rightarrow & = & (\mathcal{E}, \mathcal{F}, H) \circ (\mathcal{C}, \mathcal{E}, G \circ F) \\
\text{(6) of def. (22)} \;\rightarrow & = & (\mathcal{E}, \mathcal{F}, H) \circ (\,(\mathcal{D}, \mathcal{E}, G) \circ (\mathcal{C}, \mathcal{D}, F)\,) \\
 & = & h \circ (g \circ f)
\end{array}
$$

(11): We should have $\mathrm{dom}\,(\,\mathrm{id}(\mathcal{C})\,) = \mathcal{C} = \mathrm{cod}\,(\,\mathrm{id}(\mathcal{C})\,)$ whenever $\mathcal{C} \in \mathrm{Ob}$:

$$\begin{array}{rcll} \mathrm{dom}\,(\,\mathrm{id}(\mathcal{C})\,) & = & \mathrm{dom}\,(\,\mathcal{C}, \mathcal{C}, I_{\mathcal{C}}\,) & \leftarrow\ \text{(5) of def. (22)} \\ \text{(3) of def. (22)}\ \rightarrow & = & \mathcal{C} & \\ \text{(4) of def. (22)}\ \rightarrow & = & \mathrm{cod}\,(\,\mathcal{C}, \mathcal{C}, I_{\mathcal{C}}\,) & \\ \text{(5) of def. (22)}\ \rightarrow & = & \mathrm{cod}\,(\,\mathrm{id}(\mathcal{C})\,) & \end{array}$$

(12): We should have $f \circ \mathrm{id}(\mathcal{C}) = f$ whenever $\mathrm{dom}(f) = \mathcal{C}$ so let $f = (\mathcal{C}, \mathcal{D}, F)$:

$$\begin{array}{rcl} f \circ \mathrm{id}(\mathcal{C}) & = & (\mathcal{C}, \mathcal{D}, F) \circ \mathrm{id}(\mathcal{C}) \\ \text{(5) of def. (22)}\ \rightarrow & = & (\mathcal{C}, \mathcal{D}, F)\ \circ\ (\mathcal{C}, \mathcal{C}, I_{\mathcal{C}}) \\ \text{(6) of def. (22)}\ \rightarrow & = & (\mathcal{C}, \mathcal{D}, F \circ I_{\mathcal{C}}) \\ \text{prop. (32)}\ \rightarrow & = & (\mathcal{C}, \mathcal{D}, F) \\ & = & f \end{array}$$

(13): We should have $\mathrm{id}(\mathcal{D}) \circ f = f$ whenever $\mathrm{cod}(f) = \mathcal{D}$ so let $f = (\mathcal{C}, \mathcal{D}, F)$:

$$\begin{array}{rcl} \mathrm{id}(\mathcal{D}) \circ f & = & \mathrm{id}(\mathcal{D}) \circ (\mathcal{C}, \mathcal{D}, F) \\ \text{(5) of def. (22)}\ \rightarrow & = & (\mathcal{D}, \mathcal{D}, I_{\mathcal{D}}) \circ (\mathcal{C}, \mathcal{D}, F) \\ \text{(6) of def. (22)}\ \rightarrow & = & (\mathcal{C}, \mathcal{D}, I_{\mathcal{D}} \circ F) \\ \text{prop. (31)}\ \rightarrow & = & (\mathcal{C}, \mathcal{D}, F) \\ & = & f \end{array}$$

$\diamond$

Proposition 40 *The category* **cat** *is not small.*

Proof
Suppose **cat** is a small category. Then **cat** $= (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ where Ob is a set by virtue of definition (1). Using axiom (4) we obtain Ob **cat** $=$ Ob and it follows that Ob **cat** is itself a set. In order to obtain a contradiction, using axiom (6), it is sufficient to construct an injective map $f : \mathcal{U} \rightarrow$ Ob **cat** defined on the collection of all sets $\mathcal{U}$. Given a set x consider the small category $\mathcal{C}_x = (\{x\}, \{x\}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ with a single object x and a single arrow x, where $\mathrm{dom}(x) = x = \mathrm{cod}(x)$ and $\mathrm{id}(x) = x$ and $x \circ x = x$. Being a small category, from definition (22) we see that $\mathcal{C}_x$ is a member of the collection Ob **cat**. We then define a map $f : \mathcal{U} \rightarrow$ Ob **cat** by setting $f(x) = \mathcal{C}_x$. This map is injective as the equality $f(x) = f(y)$ clearly implies $x = y$. $\diamond$

Proposition 41 *The category* **cat** *is locally small.*

Proof
From definition (8) we need to show that given two small categories $\mathcal{C}$ and $\mathcal{D}$, the hom-set **cat**$(\mathcal{C}, \mathcal{D})$ is actually a set. In order to do so, we shall use axiom (5) and argue that the collection **cat**$(\mathcal{C}, \mathcal{D})$ is a sub-collection of some set. An member

of this collection is a triple $(\mathcal{C}, \mathcal{D}, F)$ where F is a functor $F : \mathcal{C} \to \mathcal{D}$. Now every functor $F : \mathcal{C} \to \mathcal{D}$ is an ordered pair $F = (F_0, F_1)$ where F_0 and F_1 are maps $F_0 : \text{Ob } \mathcal{C} \to \text{Ob } \mathcal{D}$ and $F_1 : \text{Arr } \mathcal{C} \to \text{Arr } \mathcal{D}$. Since $\mathcal{C}$ and $\mathcal{D}$ are small categories, all the collections Ob $\mathcal{C}$, Ob $\mathcal{D}$, Arr $\mathcal{C}$ and Arr $\mathcal{D}$ are sets, and F_0 and F_1 are in fact (untyped) functions between sets and in particular F_0 is a subset of the set $\text{Ob } \mathcal{C} \times \text{Ob } \mathcal{D}$ and F_1 is a subset of the set $\text{Arr } \mathcal{C} \times \text{Arr } \mathcal{D}$. In other words, F_0 is an element of the power set $\mathcal{P}(\text{Ob } \mathcal{C} \times \text{Ob } \mathcal{D})$ and F_1 is an element of the power set $\mathcal{P}(\text{Arr } \mathcal{C} \times \text{Arr } \mathcal{D})$, which means that $F = (F_0, F_1)$ is an element of the cartesian product:

$$A = \mathcal{P}(\text{Ob } \mathcal{C} \times \text{Ob } \mathcal{D}) \times \mathcal{P}(\text{Arr } \mathcal{C} \times \text{Arr } \mathcal{D})$$

Furthermore, recall from proposition (37) that the small categories $\mathcal{C}$ and $\mathcal{D}$ are sets and $\{\mathcal{C}\}$ and $\{\mathcal{D}\}$ are therefore well-defined singleton sets. Having established that F is an element of the set A above, we see that the triple $(\mathcal{C}, \mathcal{D}, F)$ is an element of the set $\{\mathcal{C}\} \times \{\mathcal{D}\} \times A$. So given small categories $\mathcal{C}$ and $\mathcal{D}$ we have found a set containing every member of the collection $\mathbf{cat}(\mathcal{C}, \mathcal{D})$.
$\diamond$

Chapter 3

Natural Transformation

3.1 Natural Transformation

Definition 23 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We call* natural transformation *from the typed functor $(\mathcal{C}, \mathcal{D}, F)$ to the typed functor $(\mathcal{C}, \mathcal{D}, G)$ any map $\alpha : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$ with the following properties:*

$$\begin{array}{ll} (1) & \alpha(a) : F(a) \to G(a) \\ (2) & G(f) \circ \alpha(a) = \alpha(b) \circ F(f) \end{array}$$

where (1) *holds for all $a \in \mathrm{Ob}\ \mathcal{C}$ and* (2) *holds for all $a, b \in \mathrm{Ob}\ \mathcal{C}$ and $f : a \to b$.*

Remark: It is very common to casually say that α *is a natural transformation between F and G*, i.e. to refer only to the untyped functors F and G rather than spell out the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{C}, \mathcal{D}, G)$. This is fine as long as the typed functors under consideration are clear from the context. It should be remembered however that being a natural transformation is a statement about typed functors, not untyped functors, as the knowledge of which categories $\mathcal{C}$ and $\mathcal{D}$ are involved crucially matters and these categories are not determined from F and G alone, which are also functors from $\mathcal{C}^{op} \to \mathcal{D}^{op}$ (proposition (27)).

Notation 21 *We shall use $\alpha : (\mathcal{C}, \mathcal{D}, F) \Rightarrow (\mathcal{C}, \mathcal{D}, G)$ as a notational shortcut for the statement that α* is a natural transformation between the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{C}, \mathcal{D}, G)$. *We shall write $\alpha : F \Rightarrow G$ when the context is clear.*

Remark: A mental picture of a natural transformation $\alpha : F \Rightarrow G$ where F and G are two functors between categories $\mathcal{C}$ and $\mathcal{D}$ is as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

Remark: Given $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$, given $a, b \in \mathrm{Ob}\,\mathcal{C}$ and $f : a \to b$, since F and G are functors we have $F(f) : F(a) \to F(b)$, $G(f) : G(a) \to G(b)$ and from (1) of definition (23), $\alpha(a) : F(a) \to G(a)$ and $\alpha(b) : F(b) \to G(b)$. It follows that both arrows $G(f) \circ \alpha(a)$ and $\alpha(b) \circ F(f)$ are well defined arrows in $\mathcal{D}$ (from $F(a)$ to $G(b)$), and the equality (2) of definition (23) is always meaningful.
Remark: Equality (2) of definition (23) is commonly visualized as:

$$\begin{tikzcd} a \arrow[d, "f"'] & F(a) \arrow[r, "\alpha(a)"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ b & F(b) \arrow[r, "\alpha(b)"'] & G(b) \end{tikzcd}$$

This diagram is called the *naturality square* of the natural transformation α relative to $f : a \to b$. Equality (2) is informally expressed by saying that *the naturality square commutes*, i.e. that both arrows obtained by composition along the two paths from $F(a)$ to $G(b)$ are equal.

Definition 24 *Given $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$, given $a \in \mathcal{C}$ we call* component at a of the natural transformation α, *the arrow $\alpha(a) : F(a) \to G(a)$.*

Remark: The component $\alpha(a)$ of α at $a \in \mathcal{C}$ is an arrow in the category $\mathcal{D}$.

Notation 22 *The component $\alpha(a)$ of α at a is commonly denoted α_a.*

3.2 Identity Natural Transformation

Definition 25 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}, \mathcal{D}$ are categories. We call* identity natural transformation on F, *the natural transformation $\iota_F : F \Rightarrow F$:*

$$(1) \qquad \iota_F(a) = \mathrm{id}(\,F(a)\,)$$

where (1) *holds for all $a \in \mathcal{C}$.*

Remark: A mental picture of the identity natural transformation $\iota_F : F \Rightarrow F$ where F is a functor between categories $\mathcal{C}$ and $\mathcal{D}$ is as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\iota_F"] & \mathcal{D} \end{tikzcd}$$

Proposition 42 *The identity natural transformation $\iota_F : F \Rightarrow F$ on a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are categories is indeed a natural transformation.*

Proof
We need to check that ι_F is a map defined on $\mathrm{Ob}\,\mathcal{C}$ which satisfies (1) and (2)

of definition (23). The fact that it is map defined on Ob $\mathcal{C}$ is clear, since $\iota_F(a)$ is defined for all $a \in \mathcal{C}$, that is for all $a \in$ Ob $\mathcal{C}$.

(1): We have $\iota_F(a) : F(a) \to F(a)$ for all $a \in \mathcal{C}$, since $\iota_F(a) = \mathrm{id}(\,F(a)\,)$

(2): We need to check that $F(f) \circ \iota_F(a) = \iota_F(b) \circ F(f)$ for all $f : a \to b$ where $a, b \in \mathcal{C}$. So we need to show that $F(f) \circ \mathrm{id}(\,F(a)\,) = \mathrm{id}(\,F(b)\,) \circ F(f)$, which follows from (12) and (13) of definition (2) and the fact that $F(f) : F(a) \to F(b)$.
$\diamond$

3.3 Natural Transformation, Opposite Category

Given categories $\mathcal{C}$ and $\mathcal{D}$, and a functor $F : \mathcal{C} \to \mathcal{D}$, we saw in proposition (27) that F is also a functor $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$. Now consider two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F \Rightarrow G$. By virtue of notation (21), α is a natural transformation between the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{C}, \mathcal{D}, G)$, which are clearly understood from the context. But the notation is nonetheless ambiguous as it could also indicate that α is a natural transformation between $(\mathcal{C}^{op}, \mathcal{D}^{op}, F)$ and $(\mathcal{C}^{op}, \mathcal{D}^{op}, G)$ among others. The two statements are not identical and in fact as it turns out, we have the equivalence:

Proposition 43 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between categories $\mathcal{C}$, $\mathcal{D}$. Then:*

$$\alpha \,:\, (\mathcal{C}, \mathcal{D}, F) \,\Rightarrow\, (\mathcal{C}, \mathcal{D}, G)$$

is equivalent to:

$$\alpha \,:\, (\mathcal{C}^{op}, \mathcal{D}^{op}, G) \,\Rightarrow\, (\mathcal{C}^{op}, \mathcal{D}^{op}, F)$$

In other words, being a natural transformation $\alpha : F \Rightarrow G$ (w.r. to $\mathcal{C}$ and $\mathcal{D}$) is equivalent to being a natural transformation $\alpha : G \Rightarrow F$ (w.r. to $\mathcal{C}^{op}$ and $\mathcal{D}^{op}$):

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}^{op} \end{tikzcd} \tag{3.1}$$

Proof
from proposition (9) we have $(\mathcal{C}^{op})^{op} = \mathcal{C}$ and $(\mathcal{D}^{op})^{op} = \mathcal{D}$ and it is therefore sufficient to prove the implication $\Rightarrow$. So we assume that α is a natural transformation $\alpha : (\mathcal{C}, \mathcal{D}, F) \Rightarrow (\mathcal{C}, \mathcal{D}, G)$, and we need to show that it is also a natural transformation $\alpha : (\mathcal{C}^{op}, \mathcal{D}^{op}, G) \Rightarrow (\mathcal{C}^{op}, \mathcal{D}^{op}, F)$. Looking at definition (23), we first need to establish that α is a map α : Ob $\mathcal{C}^{op} \to$ Arr $\mathcal{D}^{op}$. However, it is certainly a map α : Ob $\mathcal{C} \to$ Arr $\mathcal{D}$ by assumption and from definition (5) we have Ob $\mathcal{C}$ = Ob $\mathcal{C}^{op}$ and Arr $\mathcal{D}$ = Arr $\mathcal{D}^{op}$. Next we need to show property (1) of definition (23), namely that $\alpha(a) : G(a) \to F(a)$ @ $\mathcal{D}^{op}$ for all $a \in$ Ob $\mathcal{C}^{op}$. This follows immediately from the fact that $\alpha(a) : F(a) \to G(a)$ @ $\mathcal{D}$ for all $a \in$ Ob $\mathcal{C}$. So it remains to show property (2) of definition (23). So we assume that $a, b \in$ Ob $\mathcal{C}^{op}$ = Ob $\mathcal{C}$ and $f : a \to b$ @ $\mathcal{C}^{op}$. We need to show that

$F(f) \circ \alpha(a)$ @ $\mathcal{D}^{op} = \alpha(b) \circ G(f)$ @ $\mathcal{D}^{op}$, i.e. the following square commutes:

$$
@\ \mathcal{C}^{op} \quad
\begin{array}{c} a \\ {\scriptstyle f}\downarrow \\ b \end{array}
\qquad
\begin{array}{ccc}
G(a) & \xrightarrow{\alpha(a)} & F(a) \\
{\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\
G(b) & \xrightarrow[\alpha(b)]{} & F(b)
\end{array}
\quad @\ \mathcal{D}^{op}
$$

A formal proof goes as follows:

$$
\begin{array}{rcll}
F(f) \circ \alpha(a) \ @\ \mathcal{D}^{op} & = & \alpha(a) \circ F(f) \ @\ \mathcal{D} & \leftarrow \text{ def. } (5) \\
\text{def. } (23),\ \alpha : F \Rightarrow G,\ f : b \to a \ @\ \mathcal{C} \ \rightarrow & = & G(f) \circ \alpha(b) \ @\ \mathcal{D} & \\
\text{def. } (5) \ \rightarrow & = & \alpha(b) \circ G(f) \ @\ \mathcal{D}^{op} &
\end{array}
$$

$\diamond$

Remark: proving that the above naturality square in relation to $\mathcal{C}^{op}$ and $\mathcal{D}^{op}$ commutes amounts to proving the same square commutes, in relation to $\mathcal{C}$ and $\mathcal{D}$ after all the arrows have been reversed, and this follows from $\alpha : F \Rightarrow G$.

Proposition (43) illustrates the fact that the knowledge of a natural transformation α by itself does not tell us what its intended domain and codomain are. This situation is similar to that encountered with untyped functions and untyped functors. In line with definitions (4) and (17), we define:

Definition 26 *Given a natural transformation $\alpha : (\mathcal{C}, \mathcal{D}, F) \Rightarrow (\mathcal{C}, \mathcal{D}, G)$ where $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{C}, \mathcal{D}, G)$ are typed functors, α is called the* untyped natural transformation *while (F, G, α) is called the* typed *natural transformation.*

Remark: It may appear surprising that a typed natural transformation is defined as a triple (F, G, α) which does not contain the data allowing us to recover the categories $\mathcal{C}$ and $\mathcal{D}$. We are adopting this simpler definition as opposed to the expected $((\mathcal{C}, \mathcal{D}, F), (\mathcal{C}, \mathcal{D}, G), \alpha)$ because it allows us to define the functor category of definition (28) while keeping notations light.

3.4 Composition of Natural Transformations

Definition 27 *Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations where F, G and H are functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C}$, $\mathcal{D}$ are categories. We call* composition of β and α *the natural transformation $\beta \circ \alpha : F \Rightarrow H$ with:*

$$(1) \qquad (\beta \circ \alpha)(a) = \beta(a) \circ \alpha(a)$$

where (1) *holds for all $a \in \mathcal{C}$.*

Remark: If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations then for all $a \in \mathcal{C}$ we have $\alpha(a) : F(a) \to G(a)$ and $\beta(a) : G(a) \to H(a)$, and $\beta(a) \circ \alpha(a)$ is therefore a well-defined arrow in $\mathcal{D}$.

Proposition 44 *Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations where F, G and H are functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C}$, $\mathcal{D}$ are categories. Then $\beta \circ \alpha$ is indeed a natural transformation $\beta \circ \alpha : F \Rightarrow H$.*

Proof
We need to check that $\beta \circ \alpha$ is a map defined on Ob $\mathcal{C}$ which satisfies (1) and (2) of definition (23). As noted above, since $\alpha(a) : F(a) \to G(a)$ and $\beta(a) : G(a) \to H(a)$, the expression $\beta(a) \circ \alpha(a)$ is a well-defined arrow in $\mathcal{D}$ and $(\beta \circ \alpha)(a)$ is thus well-defined for all $a \in \mathcal{C}$. So $\beta \circ \alpha$ is a map defined on Ob $\mathcal{C}$.

(1): We need to check that $(\beta \circ \alpha)(a) : F(a) \to H(a)$ for all $a \in \mathcal{C}$, which is clear since $(\beta \circ \alpha)(a) = \beta(a) \circ \alpha(a)$, $\alpha(a) : F(a) \to G(a)$ and $\beta(a) : G(a) \to H(a)$.

(2): We need $H(f) \circ (\beta \circ \alpha)(a) = (\beta \circ \alpha)(b) \circ F(f)$ for all $a, b \in \mathcal{C}$ and $f : a \to b$, which goes as follows:

$$\begin{array}{rcll}
H(f) \circ (\beta \circ \alpha)(a) & = & H(f) \circ (\beta(a) \circ \alpha(a)) & \leftarrow \text{ (1) of def. (27)} \\
\circ \text{ associative in } \mathcal{D} \;\to & = & (H(f) \circ \beta(a)) \circ \alpha(a) & \\
\text{(2) of def. (23)}, \beta : G \Rightarrow H \;\to & = & (\beta(b) \circ G(f)) \circ \alpha(a) & \\
\circ \text{ associative in } \mathcal{D} \;\to & = & \beta(b) \circ (G(f) \circ \alpha(a)) & \\
\text{(2) of def. (23)}, \alpha : F \Rightarrow G \;\to & = & \beta(b) \circ (\alpha(b) \circ F(f)) & \\
\circ \text{ associative in } \mathcal{D} \;\to & = & (\beta(b) \circ \alpha(b)) \circ F(f) & \\
\text{(1) of def. (27)} \;\to & = & (\beta \circ \alpha)(b) \circ F(f) &
\end{array}$$

$\diamond$

Remark: Showing that $\beta \circ \alpha$ is a natural transformation is essentially about proving the equality $H(f) \circ \beta(a) \circ \alpha(a) = \beta(b) \circ \alpha(b) \circ F(f)$, where we no longer care about brackets as composition is associative in $\mathcal{D}$. Informally, this equality amounts to saying that the big rectangle below commutes:

$$\begin{array}{ccccccc}
a & \quad & F(a) & \xrightarrow{\alpha(a)} & G(a) & \xrightarrow{\beta(a)} & H(a) \\
{\scriptstyle f}\downarrow & & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\
b & & F(b) & \xrightarrow[\alpha(b)]{} & G(b) & \xrightarrow[\beta(b)]{} & H(b)
\end{array}$$

We formally proved the commutativity of this rectangle using the commutativity of the individual squares. Forgetting about associativity details:

$$H(f) \circ \beta(a) \circ \alpha(a) = \beta(b) \circ G(f) \circ \alpha(a) = \beta(b) \circ \alpha(b) \circ F(f)$$

We obtain a proof which is a lot simpler.

3.5 Equality between Natural Transformations

Now that we have a notion of composition between natural transformations, we would like to argue that this operation is associative. However, the statement is not meaningful unless we are very clear as to what it means for two natural transformations to be equal. Now if $F, G : \mathcal{C} \to \mathcal{D}$ are two functors between categories $\mathcal{C}$ and $\mathcal{D}$, and if $\alpha, \beta : F \Rightarrow G$ are natural transformations between the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{C}, \mathcal{D}, G)$, then $\alpha, \beta : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Arr}\,\mathcal{D}$, i.e. both α and β are maps from the collection $\mathrm{Ob}\,\mathcal{C}$ to the collection $\mathrm{Arr}\,\mathcal{D}$. Furthermore since $\mathcal{D}$ is a category, from (2) of definition (2) there is a clear notion of equality defined on $\mathrm{Arr}\,\mathcal{D}$. Hence given $a \in \mathrm{Ob}\,\mathcal{C}$, it is perfectly meaningful to ask whether $\alpha(a) = \beta(a)$. In fact, using the extentionality axiom (3) we have:

Proposition 45 *Let $\alpha, \beta : F \Rightarrow G$ be two natural transformations between functors $F, G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are categories, such that:*

$$\forall a \in \mathrm{Ob}\,\mathcal{C}\ ,\ \alpha(a) = \beta(a)$$

Then $\alpha = \beta$, i.e. the two natural transformations α and β are equal.

Proof
This is an immediate consequence of the extensionality axiom (3). $\diamond$

Proposition 46 *Let $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$, $\gamma : H \Rightarrow J$ be natural transformations between functors $F, G, H, J : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$, $\mathcal{D}$ are categories. Then:*

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

i.e. composition of natural transfromations is associative.

Proof
Using proposition (45) it is sufficient to show that for all $a \in \mathcal{C}$, the component at a of both natural transformations $(\gamma \circ \beta) \circ \alpha$ and $\gamma \circ (\beta \circ \alpha)$ are equal:

$$\begin{array}{rcl}
(\,(\gamma \circ \beta)\,\circ \alpha\,)(a) & = & (\gamma \circ \beta)(a) \circ \alpha(a)\ \leftarrow\ (1) \text{ of def. } (27) \\
(1) \text{ of def. } (27)\ \rightarrow & = & (\,\gamma(a)\,\circ\,\beta(a)\,)\,\circ\,\alpha(a) \\
\circ \text{ assoc in } \mathcal{D}\ \rightarrow & = & \gamma(a) \circ (\,\beta(a)\,\circ\,\alpha(a)\,) \\
(1) \text{ of def. } (27)\ \rightarrow & = & \gamma(a) \circ (\beta \circ \alpha)(a) \\
(1) \text{ of def. } (27)\ \rightarrow & = & (\,\gamma\,\circ\,(\beta \circ \alpha)\,)(a)
\end{array}$$

$\diamond$

Proposition 47 *Let $\alpha : F \Rightarrow G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$, $\mathcal{D}$ are categories. Then, we have:*

$$\iota_G \circ \alpha = \alpha$$

where $\iota_G : G \Rightarrow G$ is the identity natural transformation as per definition (25).

Proof
Using proposition (45) it is sufficient to show that for all $a \in \mathcal{C}$, the component at a of both natural transformations $\iota_G \circ \alpha$ and α are equal:

$$\begin{array}{rcl}
(\iota_G \circ \alpha)(a) & = & \iota_G(a) \circ \alpha(a)\ \leftarrow\ (1) \text{ of def. } (27) \\
(1) \text{ of def. } (25)\ \rightarrow & = & \mathrm{id}(\,G(a)\,) \circ \alpha(a) \\
(13) \text{ of def. } (2)\ \rightarrow & = & \alpha(a)
\end{array}$$

◇

Proposition 48 *Let $\alpha : F \Rightarrow G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$, $\mathcal{D}$ are categories. Then, we have:*

$$\alpha \circ \iota_F = \alpha$$

where $\iota_F : F \Rightarrow F$ is the identity natural transformation as per definition (25).

Proof
Using proposition (45) it is sufficient to show that for all $a \in \mathcal{C}$, the component at a of both natural transformations $\alpha \circ \iota_F$ and α are equal:

$$\begin{array}{rcl} (\alpha \circ \iota_F)(a) & = & \alpha(a) \circ \iota_F(a) \ \leftarrow \ \text{(1) of def. (27)} \\ \text{(1) of def. (25)} \ \rightarrow & = & \alpha(a) \circ \mathrm{id}(\, F(a)\,) \\ \text{(12) of def. (2)} \ \rightarrow & = & \alpha(a) \end{array}$$

◇

3.6 Functor Category

In this section, given two categories $\mathcal{C}$ and $\mathcal{D}$, we define a new category denoted $[\mathcal{C}, \mathcal{D}]$ and called the *functor category between $\mathcal{C}$ and $\mathcal{D}$*. Heuristically, the functor category between $\mathcal{C}$ and $\mathcal{D}$ is the category in which the objects are the functors between $\mathcal{C}$ and $\mathcal{D}$, and the arrows are the natural transformations between them.

Definition 28 *We call* functor category between the categories $\mathcal{C}$ and $\mathcal{D}$*, the category denoted $[\mathcal{C}, \mathcal{D}]$ and defined by $[\mathcal{C}, \mathcal{D}] = (\mathrm{Ob}, \mathrm{Arr}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ where:*

$$\begin{array}{ll} (1) & \mathrm{Ob} = \{\ F \mid F : \mathcal{C} \to \mathcal{D}\ \} \\ (2) & \mathrm{Arr} = \{\ (F, G, \alpha) \mid F\ ,\ G : \mathcal{C} \to \mathcal{D} \text{ and } \alpha : F \Rightarrow G\ \} \\ (3) & \mathrm{dom}\,(F, G, \alpha) = F \\ (4) & \mathrm{cod}\,(F, G, \alpha) = G \\ (5) & \mathrm{id}(F) = (F, F, \iota_F) \\ (6) & (G, H, \beta) \circ (F, G, \alpha) = (F, H, \beta \circ \alpha) \end{array}$$

where (3)−(6) *hold for all functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, $\iota_F : F \Rightarrow F$ is the identify natural transformation of definition (25) and $\beta \circ \alpha$ is the composition of β and α of definition (27).*

Remark: The objects of the category $[\mathcal{C}, \mathcal{D}]$ are untyped functors, not typed functors. This makes the notations lighter without changing the essence of the category being defined: there is an obvious one-to-one mapping between the collections $\{\ F \mid F : \mathcal{C} \to \mathcal{D}\ \}$ and $\{\ (\mathcal{C}, \mathcal{D}, F) \mid F : \mathcal{C} \to \mathcal{D}\ \}$, given $\mathcal{C}$ and $\mathcal{D}$. There is also a one-to-one mapping between a collection containing triples of the form (F, G, α) and a collection containing triples of the form $((\mathcal{C}, \mathcal{D}, F), (\mathcal{C}, \mathcal{D}, G), \alpha)$. This explains why we decided to define a *typed natural transformation* simply as (F, G, α) rather than $((\mathcal{C}, \mathcal{D}, F), (\mathcal{C}, \mathcal{D}, G), \alpha)$ in definition (26).

Proposition 49 *The functor category* $[\mathcal{C},\mathcal{D}]$ *for categories* $\mathcal{C}$,$\mathcal{D}$ *is a category.*

Proof

Given $(\mathrm{Ob},\mathrm{Arr},\mathrm{dom},\mathrm{cod},\mathrm{id},\circ)$ of definition (28), we need to check that this data satisfies condition (1) – (13) of definition (2).

(1): The collection $\mathrm{Ob} = \{\ F \mid F : \mathcal{C} \to \mathcal{D}\ \}$ should be a collection with equality. This is the case by virtue of proposition (29), where two functors are equal if and only if they coincide on all objects and all arrows.

(2): The collection $\mathrm{Arr} = \{\ (F,G,\alpha) \mid F\ ,\ G : \mathcal{C} \to \mathcal{D} \text{ and } \alpha : F \Rightarrow G\ \}$ should be a collection with equality. This is the case by virtue of proposition (45), where two natural transformations $\alpha,\beta : F \Rightarrow G$ are equal if and only if they coincide on all objects of $\mathcal{C}$. Having equality for functors and natural transformations, we conclude from axiom (4) that we also have equality for triples (F,G,α).

(3): dom should be a map $\mathrm{dom} : \mathrm{Arr} \to \mathrm{Ob}$. The equation $\mathrm{dom}(F,G,\alpha) = F$ holds for all functors $F,G : \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha : F \Rightarrow G$. So dom is indeed defined on the collection Arr as requested. Also $\mathrm{dom}\,(f) \in \mathrm{Ob}$.

(4): cod should be a map $\mathrm{cod} : \mathrm{Arr} \to \mathrm{Ob}$ which is the case as per (3).

(5): id should be a map $\mathrm{id} : \mathrm{Ob} \to \mathrm{Arr}$. The equation $\mathrm{id}(F) = (F,F,\iota_F)$ holds for all functors $F : \mathcal{C} \to \mathcal{D}$. So id is indeed defined on Ob as requested. So it remains to show that $(F,F,\iota_F) \in \mathrm{Arr}$ for all F, which the case since the identity natural transformation ι_F of definition (25) is a natural transformation $\iota_F : F \Rightarrow F$ as per proposition (42).

(6): $\circ$ should be a partial map $\circ : \mathrm{Arr} \times \mathrm{Arr} \to \mathrm{Arr}$. From definition (28), $g \circ f$ is defined whenever f and g are of the form $f = (F,G,\alpha)$ and $g = (G,H,\beta)$ where $F,G,H : \mathcal{C} \to \mathcal{D}$, $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$. So $g \circ f$ is defined on a sub-collection of $\mathrm{Arr} \times \mathrm{Arr}$ as requested. So it remains to show that $g \circ f \in \mathrm{Arr}$ when defined. However, $g \circ f$ is defined as $(F,H,\beta \circ \alpha)$ where $\beta \circ \alpha$ is the composition of natural transformation, so it remains to show that $\beta \circ \alpha : F \Rightarrow H$ which follows from proposition (44).

(7): $g \circ f$ should be defined exactly when $\mathrm{cod}(f) = \mathrm{dom}(g)$. From definition (28), $g \circ f$ is defined exactly when f is of the form $f = (F,G,\alpha)$ and g is of the form (G,H,β). Since $\mathrm{cod}(f) = G$ and $\mathrm{dom}(g) = G$, we see that $g \circ f$ is defined for all arrows f,g for which $\mathrm{cod}(f) = \mathrm{dom}(g)$ as requested.

(8): We should have $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ when $g \circ f$ is defined. So let $f = (F,G,\alpha)$ and $g = (G,H,\beta)$. Then we have $g \circ f = (F,H,\beta \circ \alpha)$ and consequently $\mathrm{dom}(g \circ f) = F$ which is $\mathrm{dom}(f)$ as requested.

(9): We should have $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$ when $g \circ f$ is defined. So let $f = (F,G,\alpha)$ and $g = (G,H,\beta)$. Then we have $g \circ f = (F,H,\beta \circ \alpha)$ and consequently $\mathrm{cod}(g \circ f) = H$ which is $\mathrm{cod}(g)$ as requested.

(10): We should have $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $g \circ f$ and $h \circ g$ are well defined. So let $f = (F,G,\alpha)$, $g = (G,H,\beta)$ and $h = (H,J,\gamma)$. We have:

$$\begin{array}{rcl}
(h \circ g) \circ f & = & (\,(H,J,\gamma)\,\circ\,(G,H,\beta)\,)\,\circ\,(F,G,\alpha) \\
\text{(6) of def. (28)} \ \to & = & (G,J,\gamma \circ \beta)\,\circ\,(F,G,\alpha) \\
\text{(6) of def. (28)} \ \to & = & (F,J,(\gamma \circ \beta) \circ \alpha) \\
\text{nats} \circ \text{assoc, prop (46)} \ \to & = & (F,J,\gamma \circ (\beta \circ \alpha))
\end{array}$$

$$
\begin{array}{rcl}
(6) \text{ of def. } (28) \ \rightarrow & = & (H, J, \gamma) \circ (F, H, \beta \circ \alpha) \\
(6) \text{ of def. } (28) \ \rightarrow & = & (H, J, \gamma) \circ (\, (G, H, \beta) \circ (F, G, \alpha) \,) \\
& = & h \circ (g \circ f)
\end{array}
$$

(11): We should have $\mathrm{dom}\,(\,\mathrm{id}(F)\,) = F = \mathrm{cod}\,(\,\mathrm{id}(F)\,)$ whenever $F \in \mathrm{Ob}$:

$$
\begin{array}{rcl}
\mathrm{dom}\,(\,\mathrm{id}(F)\,) & = & \mathrm{dom}\,(\,F, F, \iota_F\,) \ \leftarrow \ (5) \text{ of def. } (28) \\
(3) \text{ of def. } (28) \ \rightarrow & = & F \\
(4) \text{ of def. } (28) \ \rightarrow & = & \mathrm{cod}\,(\,F, F, \iota_F\,) \\
(5) \text{ of def. } (28) \ \rightarrow & = & \mathrm{cod}\,(\,\mathrm{id}(F)\,)
\end{array}
$$

(12): We should have $f \circ \mathrm{id}(F) = f$ when $\mathrm{dom}(f) = F$ so let $f = (F, G, \alpha)$:

$$
\begin{array}{rcl}
f \circ \mathrm{id}(F) & = & (F, G, \alpha) \circ \mathrm{id}(F) \\
(5) \text{ of def. } (28) \ \rightarrow & = & (F, G, \alpha) \circ (F, F, \iota_F) \\
(6) \text{ of def. } (28) \ \rightarrow & = & (F, G, \alpha \circ \iota_F) \\
\text{prop. } (48) \ \rightarrow & = & (F, G, \alpha) \\
& = & f
\end{array}
$$

(13): We should have $\mathrm{id}(G) \circ f = f$ when $\mathrm{cod}(f) = G$ so let $f = (F, G, \alpha)$:

$$
\begin{array}{rcl}
\mathrm{id}(G) \circ f & = & \mathrm{id}(G) \circ (F, G, \alpha) \\
(5) \text{ of def. } (28) \ \rightarrow & = & (G, G, \iota_G) \circ (F, G, \alpha) \\
(6) \text{ of def. } (28) \ \rightarrow & = & (F, G, \iota_G \circ \alpha) \\
\text{prop. } (47) \ \rightarrow & = & (F, G, \alpha) \\
& = & f
\end{array}
$$

$\diamond$

3.7 Natural Isomorphism

Definition 29 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We say that $\alpha : F \Rightarrow G$ is a* natural isomorphism *if and only if the arrow (F, G, α) of the functor category $[\mathcal{C}, \mathcal{D}]$ of definition (28) is an isomorphism.*

Proposition 50 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then $\alpha : F \Rightarrow G$ is a natural isomorphism if and only if there exists a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$.*

Proof
We first show the *only if* part. So we assume that $\alpha : F \Rightarrow G$ is a natural isomorphism. From definition (29), it follows that $f = (F, G, \alpha)$ is an isomorphism of the functor category $[\mathcal{C}, \mathcal{D}]$. Thus, from definition (12), the arrow f has an inverse. Hence there exists some $g = (F', G', \beta)$ arrow in $[\mathcal{C}, \mathcal{D}]$ which is both

a left and right-inverse of f. Since $f : F \to G$ we must have $g : G \to F$ and consequently $g = (G, F, \beta)$ with $\beta : G \Rightarrow F$. So we have found a natural transformation $\beta : G \Rightarrow F$ and it remains to show that $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$. Since g is a left-inverse of f, we have:

$$\begin{array}{rcll} (F, F, \iota_F) & = & \mathrm{id}(F) & \leftarrow \text{ (5) of def. (28)} \\ g \text{ left-inverse of } f \;\to & = & g \circ f & \\ & = & (G, F, \beta) \circ (F, G, \alpha) & \\ \text{(6) of def. (28)} \;\to & = & (F, F, \beta \circ \alpha) & \end{array}$$

and it follows that $\beta \circ \alpha = \iota_F$ as requested. Since g is a right-inverse of f:

$$\begin{array}{rcll} (G, G, \iota_G) & = & \mathrm{id}(G) & \leftarrow \text{ (5) of def. (28)} \\ g \text{ right-inverse of } f \;\to & = & f \circ g & \\ & = & (F, G, \alpha) \circ (G, F, \beta) & \\ \text{(6) of def. (28)} \;\to & = & (G, G, \alpha \circ \beta) & \end{array}$$

and it follows that $\alpha \circ \beta = \iota_G$ which completes the proof of the *only if* part.

We now show the *if* part. So we assume the existence of a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$. We need to show that α is a natural isomorphim, i.e. that $f = (F, G, \alpha)$ has an inverse in the functor category $[\mathcal{C}, \mathcal{D}]$. Consider the arrow $g = (G, F, \beta)$. From definition (28), this is an arrow of $[\mathcal{C}, \mathcal{D}]$ with domain G and codomain F. We can complete the proof by showing that g is an inverse of f, i.e. that $g \circ f = \mathrm{id}(F)$ and $f \circ g = \mathrm{id}(G)$. This follows easily reversing the above derivations with $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$. $\diamond$

Proposition 51 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then $\alpha : F \Rightarrow G$ is a natural isomorphism if and only if for all $a \in \mathcal{C}$, the component $\alpha(a) : F(a) \to G(a)$ is an isomorphism.*

Proof
We first show the *only if* part. So we assume that α is a natural isomorphism and we need to show that $\alpha(a) : F(a) \to G(a)$ is an isomorphism for all $a \in \mathcal{C}$. Using proposition (50), there exists a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$. It follows that $\beta(a)$ is the inverse of $\alpha(a)$, since:

$$\begin{array}{rcll} \beta(a) \circ \alpha(a) & = & (\beta \circ \alpha)(a) & \leftarrow \text{ (1) of def. (27)} \\ & = & (\iota_F)(a) & \\ \text{(1) of def. (25)} \;\to & = & \mathrm{id}(\,F(a)\,) & \end{array}$$

and:

$$\begin{array}{rcll} \alpha(a) \circ \beta(a) & = & (\alpha \circ \beta)(a) & \leftarrow \text{ (1) of def. (27)} \\ & = & (\iota_G)(a) & \\ \text{(1) of def. (25)} \;\to & = & \mathrm{id}(\,G(a)\,) & \end{array}$$

Since $\alpha(a) : F(a) \to G(a)$ has an inverse, it is an isomophism as requested.

We now show the *if* part. So we assume that $\alpha : F \Rightarrow G$ is a natural transformation such that $\alpha(a) : F(a) \to G(a)$ is an isomorphism for all $a \in \mathcal{C}$. We need to show that α is in fact a natural isomorphism. From proposition (50) it is sufficient to prove the existence of a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$. By assumption, given $a \in \mathcal{C}$, there exists $\beta(a) : G(a) \to F(a)$ which is an inverse of $\alpha(a)$. In fact using proposition (19) this inverse is unique and collecting all $\beta(a)$'s for $a \in \text{Ob } \mathcal{C}$ we obtain a map $\beta : \text{Ob } \mathcal{C} \to \text{Arr } \mathcal{D}$. We do not know at this stage whether β is a natural transformation $\beta : G \Rightarrow F$ but let us assume that it is. Then we have:

$$
\begin{array}{rcl}
(\beta \circ \alpha)(a) & = & \beta(a) \circ \alpha(a) \;\leftarrow\; \text{(1) of def. (27)} \\
\beta(a) \text{ left-inverse of } \alpha(a) \;\rightarrow & = & \text{id}(\, F(a)\,) \\
\text{(1) of def. (25)} \;\rightarrow & = & (\iota_F)(a)
\end{array}
$$

and:

$$
\begin{array}{rcl}
(\alpha \circ \beta)(a) & = & \alpha(a) \circ \beta(a) \;\leftarrow\; \text{(1) of def. (27)} \\
\beta(a) \text{ right-inverse of } \alpha(a) \;\rightarrow & = & \text{id}(\, G(a)\,) \\
\text{(1) of def. (25)} \;\rightarrow & = & (\iota_G)(a)
\end{array}
$$

These equalities being true for all $a \in \text{Ob } \mathcal{C}$, from proposition (45) we obtain the equalities of natural transformations $\beta \circ \alpha = \iota_F$ and $\alpha \circ \beta = \iota_G$ as requested.

It remains to show that $\beta : \text{Ob } \mathcal{C} \to \text{Arr } \mathcal{D}$ is indeed a natural transformation $\beta : G \Rightarrow F$, i.e. that properies (1) and (2) of definition (23) are satisfied. We already know that $\beta(a) : G(a) \to F(a)$, being the inverse of $\alpha(a) : F(a) \to G(a)$. So (1) is done, and it remains to show property (2). So let $f : a \to b$ where $a, b \in \mathcal{C}$. We need to show the equality $F(f) \circ \beta(a) = \beta(b) \circ G(f)$:

$$
\begin{tikzcd}
a \arrow[d, "f"'] & G(a) \arrow[r, "\beta(a)"] \arrow[d, "G(f)"'] & F(a) \arrow[r, "\alpha(a)"] \arrow[d, "F(f)"] & G(a) \arrow[d, "G(f)"] \\
b & G(b) \arrow[r, "\beta(b)"'] & F(b) \arrow[r, "\alpha(b)"'] & G(b)
\end{tikzcd}
$$

The naturality squares are presented for convenience, and the proof is:

$$
\begin{array}{rcl}
F(f) \circ \beta(a) & = & \text{id}(\, F(b)\,) \circ (\, F(f) \circ \beta(a)\,) \\
\beta(b) \text{ inverse of } \alpha(b) \;\rightarrow & = & (\, \beta(b) \circ \alpha(b)\,) \circ (\, F(f) \circ \beta(a)\,) \\
\circ \text{ assoc} \;\rightarrow & = & \beta(b) \circ (\, \alpha(b) \circ F(f)\,) \circ \beta(a) \\
\alpha \text{ natural} \;\rightarrow & = & \beta(b) \circ (\, G(f) \circ \alpha(a)\,) \circ \beta(a) \\
\circ \text{ assoc} \;\rightarrow & = & (\, \beta(b) \circ G(f)\,) \circ (\, \alpha(a) \circ \beta(a)\,) \\
\beta(a) \text{ inverse of } \alpha(a) \;\rightarrow & = & (\, \beta(b) \circ G(f)\,) \circ \text{id}(\, G(a)\,) \\
 & = & \beta(b) \circ G(f)
\end{array}
$$

$\diamond$

3.8 Right-Multiplication by Functor

Let $T : \mathcal{B} \to \mathcal{C}$ and $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\alpha : F \Rightarrow G$ be a natural transformation. Then the situation is as follows:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

In particular the functors $F \circ T : \mathcal{B} \to \mathcal{D}$ and $G \circ T : \mathcal{B} \to \mathcal{D}$ are well-defined by virtue of definition (18). Furthermore, since α is a natural transformation $\alpha : F \Rightarrow G$, using definition (23), for all $a \in \mathcal{C}$ we have $\alpha(a) : F(a) \to G(a)$. It follows that for all $a \in \mathcal{B}$ we have $\alpha(T(a)) : F(T(a)) \to G(T(a))$. So the question arises as to whether the expression $\alpha(T(a))$ for all $a \in \mathcal{B}$ defines a natural transformation between $F \circ T$ and $G \circ T$. As we shall see the answer is 'yes' so we shall define a new natural transformation denoted αT:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} & \to & \mathcal{B} \arrow[r, bend left, "F\circ T"] \arrow[r, bend right, "G\circ T"'] \arrow[r, phantom, "\Downarrow\alpha T"] & \mathcal{D} \end{tikzcd}$$

Definition 30 *Let $T : \mathcal{B} \to \mathcal{C}$ and $F, G : \mathcal{C} \to \mathcal{D}$ be functors where $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\alpha : F \Rightarrow G$ be a natural transformation. We denote $\alpha T : F \circ T \Rightarrow G \circ T$ the natural transformation defined by, given $a \in \mathrm{Ob}\ \mathcal{B}$:*

$$(1) \qquad (\alpha T)(a) = \alpha(T(a))$$

Proposition 52 *αT of (30) is a natural transformation $\alpha T : F \circ T \Rightarrow G \circ T$.*

Proof
We need to show that αT is a map $\alpha T : \mathrm{Ob}\ \mathcal{B} \to \mathrm{Arr}\ \mathcal{D}$ which satisfies properties (1) and (2) of definition (23). If we explicit the functor T as the ordered pair $T = (T_0, T_1)$ (definition (14)), it is implicit in definition (30) that given $a \in \mathrm{Ob}\ \mathcal{B}$, $(\alpha T)(a)$ is defined as $\alpha(T_0(a))$. Since $\alpha : F \Rightarrow G$, in particular α is a map $\alpha : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$ while T_0 is a map $T_0 : \mathrm{Ob}\ \mathcal{B} \to \mathrm{Ob}\ \mathcal{C}$. It follows that $\alpha(T_0(a))$ is a well defined arrow in $\mathcal{D}$ and we see that $\alpha T : \mathrm{Ob}\ \mathcal{B} \to \mathrm{Arr}\ \mathcal{D}$.

(1): We need to show that $(\alpha T)(a) : (F \circ T)(a) \to (G \circ T)(a)$ for all $a \in \mathcal{B}$. This is the same as showing that $\alpha(T(a)) : F(T(a)) \to G(T(a))$, which is the case since $\alpha(a) : F(a) \to G(a)$ for all $a \in \mathcal{C}$ (α being a natural transformation).

(2): We need to show that the naturality square commutes, namely that given $a, b \in \mathcal{B}$ and $f : a \to b$, we have $(G \circ T)(f) \circ (\alpha T)(a) = (\alpha T)(b) \circ (F \circ T)(f)$.

$$\begin{tikzcd} a \arrow[d, "f"'] & (F\circ T)(a) \arrow[r, "(\alpha T)(a)"] \arrow[d, "(F\circ T)(f)"'] & (G\circ T)(a) \arrow[d, "(G\circ T)(f)"] \\ b & (F\circ T)(b) \arrow[r, "(\alpha T)(b)"'] & (G\circ T)(b) \end{tikzcd}$$

However, this equality is the same as $G(\,T(f)\,)\circ\alpha(\,T(a)\,)=\alpha(\,T(b)\,)\circ F(\,T(f)\,)$, and is simply the equality expressing the commutativity of the naturality square of α, associated with $T(a), T(b) \in \mathcal{C}$ and $T(f) : T(a) \to T(b)$.

$$\begin{array}{ccc} T(a) & & F(\,T(a)\,) \xrightarrow{\alpha(\,T(a)\,)} G(\,T(a)\,) \\ {\scriptstyle T(f)}\big\downarrow & & {\scriptstyle F(\,T(f)\,)}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle G(\,T(f)\,)} \\ T(b) & & F(\,T(b)\,) \xrightarrow[\alpha(\,T(b)\,)]{} G(\,T(b)\,). \end{array}$$

$\diamond$

3.9 Left-Multiplication by Functor

Let $F, G : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{E}$ be functors where $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories, and let $\alpha : F \Rightarrow G$ be a natural transformation. Then the situation is as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "T"] & \mathcal{E} \end{tikzcd}$$

In particular the functors $T \circ F : \mathcal{C} \to \mathcal{E}$ and $T \circ G : \mathcal{C} \to \mathcal{E}$ are well-defined by virtue of definition (18). Furthermore, since α is a natural transformation $\alpha : F \Rightarrow G$, using definition (23), for all $a \in \mathcal{C}$ we have $\alpha(a) : F(a) \to G(a)$. T being a functor, it follows that $T(\,\alpha(a)\,) : T(\,F(a)\,) \to T(\,G(a)\,)$. So the question arises as to whether the expression $T(\,\alpha(a)\,)$ for all $a \in \mathcal{C}$ defines a natural transformation between $T \circ F$ and $T \circ G$. As we shall see the answer is 'yes' so we shall define a new natural transformation denoted $T\alpha$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "T"] & \mathcal{E} & \rightarrow & \mathcal{C} \arrow[r, bend left, "T\circ F"] \arrow[r, bend right, "T\circ G"'] \arrow[r, phantom, "\Downarrow T\alpha"] & \mathcal{E} \end{tikzcd}$$

Definition 31 *Let $F, G : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{E}$ be functors where $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories, and let $\alpha : F \Rightarrow G$ be a natural transformation. We denote $T\alpha : T \circ F \Rightarrow T \circ G$ the natural transformation defined by, given $a \in \mathrm{Ob}\,\mathcal{C}$:*

$$(1) \qquad (T\alpha)(a) = T(\,\alpha(a)\,)$$

Proposition 53 *$T\alpha$ of (31) is a natural transformation $T\alpha : T \circ F \Rightarrow T \circ G$.*

Proof
We need to show that $T\alpha$ is a map $T\alpha : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Arr}\,\mathcal{E}$ which satisfies properties (1) and (2) of definition (23). If we explicit the functor T as the ordered pair $T = (T_0, T_1)$ (definition (14)), it is implicit in definition (31) that given

$a \in \mathrm{Ob}\ \mathcal{C}$, $(T\alpha)(a)$ is defined as $T_1(\,\alpha(a)\,)$. Since $\alpha : F \Rightarrow G$, in particular α is a map $\alpha : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{D}$ while T_1 is a map $T_1 : \mathrm{Arr}\ \mathcal{D} \to \mathrm{Arr}\ \mathcal{E}$. It follows that $T_1(\,\alpha(a)\,)$ is a well defined arrow in $\mathcal{E}$ and we see that $T\alpha : \mathrm{Ob}\ \mathcal{C} \to \mathrm{Arr}\ \mathcal{E}$.

(1): We need to show that $(T\alpha)(a) : (T \circ F)(a) \to (T \circ G)(a)$ for all $a \in \mathcal{C}$. This is the same as showing that $T(\,\alpha(a)\,) : T(\,F(a)\,) \to T(\,G(a)\,)$, which is the case since $\alpha(a) : F(a) \to G(a)$ for all $a \in \mathcal{C}$ and T is a functor.

(2): We need to show that the naturality square commutes, namely that given $a, b \in \mathcal{C}$ and $f : a \to b$, we have $(T \circ G)(f) \circ (T\alpha)(a) = (T\alpha)(b) \circ (T \circ F)(f)$.

$$\begin{array}{ccc} a & \quad & \begin{tikzcd} (T \circ F)(a) \arrow[r, "(T\alpha)(a)"] \arrow[d, "(T\circ F)(f)"'] & (T \circ G)(a) \arrow[d, "(T\circ G)(f)"] \\ (T \circ F)(b) \arrow[r, "(T\alpha)(b)"'] & (T \circ G)(b) \end{tikzcd} \\ \Big\downarrow f & & \\ b & & \end{array}$$

However, this equality is the same as $T(\,G(f)\,) \circ T(\,\alpha(a)\,) = T(\,\alpha(b)\,) \circ T(\,F(f)\,)$, and is simply a lifting by the functor T of the equality expressing the naturality of α, associated with $a, b \in \mathcal{C}$ and $f : a \to b$.

$$\begin{array}{ccc} a & \quad & \begin{tikzcd} T(\,F(a)\,) \arrow[r, "T(\,\alpha(a)\,)"] \arrow[d, "T(\,F(f)\,)"'] & T(\,G(a)\,) \arrow[d, "T(\,G(f)\,)"] \\ T(\,F(b)\,) \arrow[r, "T(\,\alpha(b)\,)"'] & T(\,G(b)\,). \end{tikzcd} \\ \Big\downarrow f & & \\ b & & \end{array}$$

However, we crucially need T to be a functor for this square to commute:

$$\begin{array}{rcll} T(\,G(f)\,) \circ T(\,\alpha(a)\,) & = & T(\,G(f) \circ \alpha(a)\,) & \leftarrow \text{(5) of def. (14)} \\ \text{(2) of def. (23)} \rightarrow & = & T(\,\alpha(b) \circ F(f)\,) & \\ \text{(5) of def. (14)} \rightarrow & = & T(\,\alpha(b)\,) \circ T(\,F(f)\,) & \end{array}$$

$\diamond$

Chapter 4

Adjunction

In this chapter, we are interested in pairs of functors as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

4.1 Unit of a Pair of Functors

Definition 32 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We call* unit *of the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{D}, \mathcal{C}, G)$, a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ such that for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \to G(d)$, there exists a unique $g : F(c) \to d$ such that $f = G(g) \circ \eta_c$.*

Remark: The equality $f = G(g) \circ \eta_c$ can be visualized with the diagram:

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "f"'] & G(\,F(c)\,) \arrow[d, "G(\,g\,)"] \\ & G(\,d\,) \end{tikzcd}$$

Remark: Recall from definition (23) that η being a natural transformation is a statement about typed functors, not just functors. The same is true of a unit. Being a unit of F and G viewed as functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is not the same as being a unit of F and G viewed as functor $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$ and $G : \mathcal{D}^{op} \to \mathcal{C}^{op}$. It is important to remember which categories are being considered, hence typed functors. The order also matters. Being a unit of F and G is not the same as being a unit of G and F.

Notation 23 *Whenever the categories $\mathcal{C}$ and $\mathcal{D}$ are clearly undertood from the context, we shall simply say that η* is a unit of *the ordered pair (F, G).*

Proposition 54 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\eta : I_\mathcal{C} \Rightarrow G \circ F$ be a unit of (F, G). Then, for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g_1, g_2 : F(c) \to d$, we have the implication:*

$$G(g_1) \circ \eta_c = G(g_2) \circ \eta_c \;\Rightarrow\; g_1 = g_2$$

Proof
This is an immediate consequence of the uniqueness property of definition (32). Consider the arrow $f = G(g_1) \circ \eta_c$. Since $g_1 : F(c) \to d$ and G is a functor, we have $G(g_1) : G(F(c)) \to G(d)$ and since $\eta_c : c \to G(F(c))$, f is a well-defined arrow $f : c \to G(d)$ in the category $\mathcal{C}$. Having assumed that η is a unit of (F, G), from definition (32) there exists a unique arrow $g : F(c) \to d$ in the category $\mathcal{D}$ such that $f = G(g) \circ \eta_c$. However the equality $f = G(g) \circ \eta_c$ is satified by the arrow $g = g_1$ since we defined f as $f = G(g_1) \circ \eta_c$. Now if we assume that $G(g_1) \circ \eta_c = G(g_2) \circ \eta_c$, then the equality $f = G(g) \circ \eta_c$ is also satisfied by g_2. By uniquenes, it follows that $g_1 = g_2$. $\diamond$

The following proposition applies to any $\eta : I_\mathcal{C} \Rightarrow G \circ F$ which is a natural transformation, so in particular it applies to any unit of (F, G).

Proposition 55 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\eta : I_\mathcal{C} \Rightarrow G \circ F$ be a natural transformation. Then we have:*

$$(G \circ F)(f) \circ \eta_a = \eta_b \circ f$$

for all objects $a, b \in \mathcal{C}$ and arrow $f : a \to b$.

Proof
This is an immediate consequence of definition (23) and the fact that η is a natural transformation $\eta : I_\mathcal{C} \Rightarrow G \circ F$, so the following square commutes:

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "f"'] & G(F(a)) \arrow[d, "(G\circ F)(f)"] \\
b \arrow[r, "\eta_b"'] & G(F(b))
\end{tikzcd}$$

$\diamond$

Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$, $\mathcal{D}$ are categories, if $\eta : I_\mathcal{C} \Rightarrow G \circ F$ is a unit of (F, G), then a question arises as to whether this unit is unique. As the following proposition shows, the answer is 'no' in general as every natural isomorphism $\alpha : F \Rightarrow F$ potentially gives rise to a new unit.

Proposition 56 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\eta : I_\mathcal{C} \Rightarrow G \circ F$ be a unit of (F, G). Then, a natural transformation $\eta' : I_\mathcal{C} \Rightarrow G \circ F$ is a unit of (F, G) if and only if there exists a natural isomorphism $\alpha : F \Rightarrow F$ such that:*

$$\eta' = (G\alpha) \circ \eta$$

Remark: If $\alpha : F \Rightarrow F$ and $G : \mathcal{D} \to \mathcal{C}$, then $G\alpha$ is the natural transformation $G\alpha : G \circ F \Rightarrow G \circ F$ as per definition (31), and if $\eta : I_\mathcal{C} \Rightarrow G \circ F$ then the composition $(G\alpha) \circ \eta$ is a well-defined natural transformation from the identity functor $I_\mathcal{C}$ to the functor $G \circ F$ as per definition (27).
Proof
First we show the *if* part. So we assume that $\eta' = (G\alpha) \circ \eta$ for some natural isomorphism $\alpha : F \Rightarrow F$. We need to show that η' is a unit of (F, G). As already indicated η' is a well-defined natural transformation $\eta' : I_\mathcal{C} \Rightarrow G \circ F$. Hence we simply need to show that it satisfies the universal property of definition (32). So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \to G(d)$. We need to show the existence of a unique $g : F(c) \to d$ such that $f = G(g) \circ \eta'_c$. However, by assumption we have:

$$\begin{array}{rrcl} & \eta'_c & = & ((G\alpha) \circ \eta)_c \\ \text{(1) of def. (27)} & \rightarrow & = & (G\alpha)_c \circ \eta_c \\ \text{(1) of def. (31)} & \rightarrow & = & G(\ \alpha_c\ ) \circ \eta_c \end{array}$$

So we need to prove the existence of a unique arrow $g : F(c) \to d$ such that $f = G(g) \circ G(\alpha_c) \circ \eta_c$, which is $f = G(g \circ \alpha_c) \circ \eta_c$ since G is a functor:

Existence: having assumed η is a unit of (F, G), since $f : c \to G(d)$ using definition (32) there exists a (unique) $h : F(c) \to d$ with $f = G(h) \circ \eta_c$. However, by assumption $\alpha : F \Rightarrow F$ is a natural isomorphism. From proposition (51), every component of α is an isomorphism and in particular $\alpha_c : F(c) \to F(c)$ has an inverse $\alpha_c^{-1} : F(c) \to F(c)$. Defining $g = h \circ \alpha_c^{-1}$ we obtain an arrow $g : F(c) \to d$ such that $f = G(g \circ \alpha_c) \circ \eta_c$ as requested.

Uniqueness: Suppose $g_1, g_2 : F(c) \to d$ are two arrows in $\mathcal{D}$ such that $G(g_1 \circ \alpha_c) \circ \eta_c = f = G(g_2 \circ \alpha_c) \circ \eta_c$. Having assumed η is a unit of (F, G) using proposition (54) we obtain $g_1 \circ \alpha_c = g_2 \circ \alpha_c$ and composing both sides to the right by α_c^{-1} we conclude that $g_1 = g_2$ as requested.

We now prove the *only if* part. So we assume that $\eta' : I_\mathcal{C} \Rightarrow G \circ F$ is a unit of (F, G) and we need to show the existence of a natural isomorphism $\alpha : F \Rightarrow F$ such that $\eta' = (G\alpha) \circ \eta$. So let $c \in \mathcal{C}$. First we need to define $\alpha_c : F(c) \to F(c)$. Define $d = F(c) \in \mathcal{D}$. Then we have $\eta'_c : c \to G(d)$. Since η is a unit of (F, G) there exists a unique arrow $g : F(c) \to d$ such that $\eta'_c = G(g) \circ \eta_c$. Define α_c to be precisely this unique arrow g. Then we have $\alpha_c : F(c) \to F(c)$ and:

$$\eta'_c = G(\alpha_c) \circ \eta_c \tag{4.1}$$

Collecting all these arrows $\alpha_c : F(c) \to F(c)$ for $c \in \mathcal{C}$, we obtain a map $\alpha : \text{Ob } \mathcal{C} \to \text{Arr } \mathcal{D}$. Next we need to show that α is a natural transformation, so it remains to show that condition (2) of definition (23) is satisfied. So let $a, b \in \mathcal{C}$ and $f : a \to b$. We need to show the equality $F(f) \circ \alpha_a = \alpha_b \circ F(f)$:

$$\begin{tikzcd} a \arrow[d, "f"'] & F(a) \arrow[r, "\alpha_a"] \arrow[d, "F(f)"'] & F(a) \arrow[d, "F(f)"] \\ b & F(b) \arrow[r, "\alpha_b"'] & F(b) \end{tikzcd}$$

Define $c = a \in \mathcal{C}$ and $d = F(b) \in \mathcal{D}$. Then both arrows $F(f) \circ \alpha_a$ and $\alpha_b \circ F(f)$ have domain $F(c)$ and codomain d in $\mathcal{D}$. Appplying proposition (54) since η is a unit of (F, G), in order to show that these are equal it is sufficient to prove:

$$G(\,F(f) \circ \alpha_a\,) \circ \eta_c = G(\,\alpha_b \circ F(f)\,) \circ \eta_c$$

The proof goes as follows:

$$\begin{array}{rcl}
G(\,F(f) \circ \alpha_a\,) \circ \eta_c & = & G(\,F(f) \circ \alpha_a\,) \circ \eta_a \\
G \text{ functor } \rightarrow & = & G(F(f)) \circ G(\alpha_a) \circ \eta_a \\
\text{eqn. (4.1) } \rightarrow & = & G(F(f)) \circ \eta'_a \\
\text{prop. (55), } \eta' \text{ unit } \rightarrow & = & \eta'_b \circ f \\
\text{eqn. (4.1) } \rightarrow & = & G(\alpha_b) \circ \eta_b \circ f \\
\text{prop. (55), } \eta \text{ unit } \rightarrow & = & G(\alpha_b) \circ G(F(f)) \circ \eta_a \\
G \text{ functor } \rightarrow & = & G(\,\alpha_b \circ F(f)\,) \circ \eta_a \\
 & = & G(\,\alpha_b \circ F(f)\,) \circ \eta_c
\end{array}$$

So we have now proved that α is a natural transformation $\alpha : F \Rightarrow F$. However, we aim to show that it is actually a natural isomorphism. Using proposition (51), it is sufficient to prove that each component $\alpha_c : F(c) \rightarrow F(c)$ for $c \in \mathcal{C}$ is an isomorphisn in $\mathcal{D}$. So let $c \in \mathcal{C}$. Define $d = F(c) \in \mathcal{D}$. Then we have $\eta_c : c \rightarrow G(d)$. Having assumed that η' is a unit of (F, G) there exists a unique arrow $g : F(c) \rightarrow d$ such that $\eta_c = G(g) \circ \eta'_c$. Define β_c to be precisely this unique arrow g. Then we have $\beta_c : F(c) \rightarrow F(c)$ and:

$$\eta_c = G(\beta_c) \circ \eta'_c \tag{4.2}$$

We claim that β_c is an inverse of α_c. So we need to prove that $\beta_c \circ \alpha_c = \mathrm{id}(\,F(c)\,)$ and $\alpha_c \circ \beta_c = \mathrm{id}(\,F(c)\,)$. In order to show that $\beta_c \circ \alpha_c = \mathrm{id}(\,F(c)\,)$, since both arrows are from $F(c)$ to $d = F(c)$ and η is a unit of (F, G), applying proposition (54) it is sufficient to prove that:

$$G(\,\beta_c \circ \alpha_c\,) \circ \eta_c = G(\,\mathrm{id}(F(c))\,) \circ \eta_c$$

The proof goes as follows:

$$\begin{array}{rcl}
G(\,\beta_c \circ \alpha_c\,) \circ \eta_c & = & G(\beta_c) \circ G(\alpha_c) \circ \eta_c \;\leftarrow\; G \text{ functor} \\
\text{eqn. (4.1) } \rightarrow & = & G(\beta_c) \circ \eta'_c \\
\text{eqn. (4.2) } \rightarrow & = & \eta_c \\
\text{identity } \rightarrow & = & \mathrm{id}(\,G(F(c))\,) \circ \eta_c \\
G \text{ functor } \rightarrow & = & G(\,\mathrm{id}(F(c))\,) \circ \eta_c
\end{array}$$

In order to show that $\alpha_c \circ \beta_c = \mathrm{id}(\,F(c)\,)$, since both arrows are from $F(c)$ to $d = F(c)$ and η' is assumed to be a unit of (F, G), applying proposition (54) it is sufficient to prove that:

$$G(\,\alpha_c \circ \beta_c\,) \circ \eta'_c = G(\,\mathrm{id}(F(c))\,) \circ \eta'_c$$

The proof goes as follows:

$$\begin{array}{rcll} G(\,\alpha_c \circ \beta_c\,) \circ \eta'_c & = & G(\alpha_c) \circ G(\beta_c) \circ \eta'_c & \leftarrow\ G \text{ functor} \\ \text{eqn. (4.2)} \ \rightarrow & = & G(\alpha_c) \circ \eta_c & \\ \text{eqn. (4.1)} \ \rightarrow & = & \eta'_c & \\ \text{identity} \ \rightarrow & = & \mathrm{id}(\,G(F(c))\,) \circ \eta'_c & \\ G \text{ functor} \ \rightarrow & = & G(\,\mathrm{id}(F(c))\,) \circ \eta'_c & \end{array}$$

So we now know that $\alpha : F \Rightarrow F$ is a natural isomorphism and it remains to show that $\eta' = (G\alpha) \circ \eta$. Using proposition (45), this equality between natural transformations is obtained if we can show that all components are equal:

$$\begin{array}{rcll} \eta'_c & = & G(\alpha_c) \circ \eta_c & \leftarrow\ \text{eqn. (4.1)} \\ \text{(1) of def. (31)} \ \rightarrow & = & (G\alpha)_c \circ \eta_c & \\ \text{(1) of def. (27)} \ \rightarrow & = & (\,G\alpha \circ \eta\,)_c & \end{array}$$

$\diamond$

4.2 Counit of a Pair of Functors

Definition 33 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We call* counit *of the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{D}, \mathcal{C}, G)$, a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ such that for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g : F(c) \to d$, there exists a unique $f : c \to G(d)$ such that $g = \epsilon_d \circ F(f)$.*

Remark: The equality $g = \epsilon_d \circ F(f)$ can be visualized with the diagram:

$$\begin{tikzcd} F(\,G(d)\,) \arrow[r, "\epsilon_d"] & d \\ F(\,c\,) \arrow[u, "F(\,f\,)"] \arrow[ur, "g"'] & \end{tikzcd}$$

Remark: Just like for *units*, being a *counit* is a statement relating typed functors (not just functors), and the order between F and G matters.

Notation 24 *Whenever the categories $\mathcal{C}$ and $\mathcal{D}$ are clearly undertood from the context, we shall simply say that ϵ* is a counit of *the ordered pair (F, G).*

Proposition 57 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be a counit of (F, G). Then, for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f_1, f_2 : c \to G(d)$, we have the implication:*

$$\epsilon_d \circ F(f_1) = \epsilon_d \circ F(f_2) \ \Rightarrow \ f_1 = f_2$$

Proof
This is an immediate consequence of the uniqueness property of definition (33). Consider the arrow $g = \epsilon_d \circ F(f_1)$. Since $f_1 : c \to G(d)$ and F is a functor, we have $F(f_1) : F(c) \to F(G(d))$ and since $\epsilon_d : F(G(d)) \to d$, g is a well-defined arrow $g : F(c) \to d$ in the category $\mathcal{D}$. Having assumed that ϵ is a counit of (F, G), from definition (33) there exists a unique arrow $f : c \to G(d)$ in the category $\mathcal{C}$ such that $g = \epsilon_d \circ F(f)$. However the equality $g = \epsilon_d \circ F(f)$ is satified by the arrow $f = f_1$ since we defined g as $g = \epsilon_d \circ F(f_1)$. Now if we assume that $\epsilon_d \circ F(f_1) = \epsilon_d \circ F(f_2)$, then the equality $g = \epsilon_d \circ F(f)$ is also satisfied by f_2. By uniquenes, it follows that $f_1 = f_2$. $\diamond$

The following proposition applies to any $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ which is a natural transformation, so in particular it applies to any counit of (F, G).

Proposition 58 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be a natural transformation. Then we have:*

$$\epsilon_b \circ (F \circ G)(g) = g \circ \epsilon_a$$

for all objects $a, b \in \mathcal{C}$ and arrow $g : a \to b$.

Proof
This is an immediate consequence of definition (23) and the fact that ϵ is a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$, so the following square commutes:

$$\begin{tikzcd} F(G(a)) \arrow[r, "\epsilon_a"] \arrow[d, "(F\circ G)(g)"'] & a \arrow[d, "g"] \\ F(G(b)) \arrow[r, "\epsilon_b"'] & b \end{tikzcd}$$

$\diamond$

Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$, $\mathcal{D}$ are categories, if $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ is a counit of (F, G), then a question arises as to whether this counit is unique. As the following proposition shows, the answer is 'no' in general as every natural isomorphism $\alpha : G \Rightarrow G$ potentially gives rise to a new counit.

Proposition 59 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories, and let $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be a counit of (F, G). Then, a natural transformation $\epsilon' : F \circ G \Rightarrow I_{\mathcal{D}}$ is a counit of (F, G) if and only if there exists a natural isomorphism $\alpha : G \Rightarrow G$ such that:*

$$\epsilon' = \epsilon \circ (F\alpha)$$

Remark: If $\alpha : G \Rightarrow G$ and $F : \mathcal{C} \to \mathcal{D}$, then $F\alpha$ is the natural transformation $F\alpha : F \circ G \Rightarrow F \circ G$ as per definition (31), and if $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ then the composition $\epsilon \circ (F\alpha)$ is a well-defined natural transformation from the functor $F \circ G$ to the functor $I_{\mathcal{D}}$ as per definition (27).

Proof

First we show the *if* part. So we assume that $\epsilon' = \epsilon \circ (F\alpha)$ for some natural isomorphism $\alpha : G \Rightarrow G$. We need to show that ϵ' is a counit of (F, G). As already seen ϵ' is a well-defined natural transformation $\epsilon' : F \circ G \Rightarrow I_{\mathcal{D}}$. Hence we simply need to show that it satisfies the universal property of definition (33). So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g : F(c) \to d$. We need to show the existence of a unique $f : c \to G(d)$ such that $g = \epsilon'_d \circ F(f)$. However, by assumption we have:

$$
\begin{array}{rcl}
 & \epsilon'_d & = & (\epsilon \circ (F\alpha))_d \\
\text{(1) of def. (27)} \ \to & & = & \epsilon_d \circ (F\alpha)_d \\
\text{(1) of def. (31)} \ \to & & = & \epsilon_d \circ F(\,\alpha_d\,)
\end{array}
$$

So we need to prove the existence of a unique arrow $f : c \to G(d)$ such that $g = \epsilon_d \circ F(\alpha_d) \circ F(f)$, which is $g = \epsilon_d \circ F(\alpha_d \circ f)$ since F is a functor:

Existence: having assumed ϵ is a counit of (F, G), since $g : F(c) \to d$ using definition (33) there exists a (unique) $h : c \to G(d)$ with $g = \epsilon_d \circ F(h)$. However, by assumption $\alpha : G \Rightarrow G$ is a natural isomorphism. From proposition (51), every component of α is an isomorphism and in particular $\alpha_d : G(d) \to G(d)$ has an inverse $\alpha_d^{-1} : G(d) \to G(d)$. Defining $f = \alpha_d^{-1} \circ h$ we obtain an arrow $f : c \to G(d)$ such that $g = \epsilon_d \circ F(\alpha_d \circ f)$ as requested.

Uniqueness: Suppose $f_1, f_2 : c \to G(d)$ are two arrows in $\mathcal{C}$ with the equality $\epsilon_d \circ F(\alpha_d \circ f_1) = g = \epsilon_d \circ F(\alpha_d \circ f_2)$. Having assumed ϵ is a counit of (F, G) using proposition (57) we obtain $\alpha_d \circ f_1 = \alpha_d \circ f_2$ and composing both sides to the left by α_d^{-1} we conclude that $f_1 = f_2$ as requested.

We now prove the *only if* part. So we assume that $\epsilon' : F \circ G \Rightarrow I_{\mathcal{D}}$ is a counit of (F, G) and we need to show the existence of a natural isomorphism $\alpha : G \Rightarrow G$ such that $\epsilon' = \epsilon \circ (F\alpha)$. So let $d \in \mathcal{D}$. First we need to define $\alpha_d : G(d) \to G(d)$. Define $c = G(d) \in \mathcal{C}$. Then we have $\epsilon'_d : F(c) \to d$. Since ϵ is a counit of (F, G) there exists a unique arrow $f : c \to G(d)$ such that $\epsilon'_d = \epsilon_d \circ F(f)$. Define α_d to be precisely this unique arrow f. Then we have $\alpha_d : G(d) \to G(d)$ and:

$$\epsilon'_d = \epsilon_d \circ F(\alpha_d) \tag{4.3}$$

Collecting all these arrows $\alpha_d : G(d) \to G(d)$ for $d \in \mathcal{D}$, we obtain a map $\alpha : \mathrm{Ob}\ \mathcal{D} \to \mathrm{Arr}\ \mathcal{C}$. Next we need to show that α is a natural transformation, so it remains to show that condition (2) of definition (23) is satisfied. So let $a, b \in \mathcal{D}$ and $g : a \to b$. We need to show the equality $G(g) \circ \alpha_a = \alpha_b \circ G(g)$:

$$
\begin{tikzcd}
a \arrow[d, "g"'] & G(a) \arrow[r, "\alpha_a"] \arrow[d, "G(g)"'] & G(a) \arrow[d, "G(g)"] \\
b & G(b) \arrow[r, "\alpha_b"'] & G(b)
\end{tikzcd}
$$

Define $c = G(a) \in \mathcal{C}$ and $d = b \in \mathcal{D}$. Then both arrows $G(g) \circ \alpha_a$ and $\alpha_b \circ G(g)$ have domain c and codomain $G(d)$ in $\mathcal{C}$. Appplying proposition (57) since ϵ is a counit of (F, G), in order to show that these are equal it is sufficient to prove:

$$\epsilon_d \circ F(\, G(g) \circ \alpha_a\,) = \epsilon_d \circ F(\, \alpha_b \circ G(g)\,)$$

The proof goes as follows:

$$\begin{array}{rcl}
\epsilon_d \circ F(\, G(g) \circ \alpha_a\,) & = & \epsilon_b \circ F(\, G(g) \circ \alpha_a\,) \\
F \text{ functor } \rightarrow & = & \epsilon_b \circ F(G(g)) \circ F(\alpha_a) \\
\text{prop. (58), } \epsilon \text{ counit } \rightarrow & = & g \circ \epsilon_a \circ F(\alpha_a) \\
\text{eqn. (4.3) } \rightarrow & = & g \circ \epsilon'_a \\
\text{prop. (58), } \epsilon\text{' counit } \rightarrow & = & \epsilon'_b \circ F(G(g)) \\
\text{eqn. (4.3) } \rightarrow & = & \epsilon_b \circ F(\alpha_b) \circ F(G(g)) \\
F \text{ functor } \rightarrow & = & \epsilon_b \circ F(\, \alpha_b \circ G(g)\,) \\
 & = & \epsilon_d \circ F(\, \alpha_b \circ G(g)\,)
\end{array}$$

So we have now proved that α is a natural transformation $\alpha : G \Rightarrow G$. However, we aim to show that it is actually a natural isomorphism. Using proposition (51), it is sufficient to prove that each component $\alpha_d : G(d) \rightarrow G(d)$ for $d \in \mathcal{D}$ is an isomorphisn in $\mathcal{C}$. So let $d \in \mathcal{D}$. Define $c = G(d) \in \mathcal{C}$. Then we have $\epsilon_d : F(c) \rightarrow d$. Having assumed that ϵ' is a counit of (F, G) there exists a unique arrow $f : c \rightarrow G(d)$ such that $\epsilon_d = \epsilon'_d \circ F(f)$. Define β_d to be precisely this unique arrow f. Then we have $\beta_d : G(d) \rightarrow G(d)$ and:

$$\epsilon_d = \epsilon'_d \circ F(\beta_d) \tag{4.4}$$

We claim that β_d is an inverse of α_d. So we need to prove that $\beta_d \circ \alpha_d = \mathrm{id}(\, G(d)\,)$ and $\alpha_d \circ \beta_d = \mathrm{id}(\, G(d)\,)$. In order to show that $\beta_d \circ \alpha_d = \mathrm{id}(\, G(d)\,)$, since both arrows are from $c = G(d)$ to $G(d)$ and ϵ' is assumed to be a counit of (F, G), applying proposition (57) it is sufficient to prove that:

$$\epsilon'_d \circ F(\, \beta_d \circ \alpha_d\,) = \epsilon'_d \circ F(\, \mathrm{id}(G(d))\,)$$

The proof goes as follows:

$$\begin{array}{rcl}
\epsilon'_d \circ F(\, \beta_d \circ \alpha_d\,) & = & \epsilon'_d \circ F(\beta_d) \circ F(\alpha_d) \;\leftarrow\; F \text{ functor} \\
\text{eqn. (4.4) } \rightarrow & = & \epsilon_d \circ F(\alpha_d) \\
\text{eqn. (4.3) } \rightarrow & = & \epsilon'_d \\
\text{identity } \rightarrow & = & \epsilon'_d \circ \mathrm{id}(\, F(G(d))\,) \\
F \text{ functor } \rightarrow & = & \epsilon'_d \circ F(\, \mathrm{id}(G(d))\,)
\end{array}$$

In order to show that $\alpha_d \circ \beta_d = \mathrm{id}(\, G(d)\,)$, since both arrows are from $c = G(d)$ to $G(d)$ and ϵ is a counit of (F, G), applying proposition (57) it is sufficient to prove that:

$$\epsilon_d \circ F(\,\alpha_d \circ \beta_d\,) = \epsilon_d \circ F(\,\mathrm{id}(G(d))\,)$$

The proof goes as follows:

$$\begin{array}{rcll}
\epsilon_d \circ F(\,\alpha_d \circ \beta_d\,) & = & \epsilon_d \circ F(\alpha_d) \circ F(\beta_d) & \leftarrow\ F \text{ functor} \\
\text{eqn. (4.3)}\ \rightarrow & = & \epsilon'_d \circ F(\beta_d) & \\
\text{eqn. (4.4)}\ \rightarrow & = & \epsilon_d & \\
\text{identity}\ \rightarrow & = & \epsilon_d \circ \mathrm{id}(\,F(G(d))\,) & \\
F \text{ functor}\ \rightarrow & = & \epsilon_d \circ F(\,\mathrm{id}(G(d))\,) &
\end{array}$$

So we now know that $\alpha : G \Rightarrow G$ is a natural isomorphism and it remains to show that $\epsilon' = \epsilon \circ (F\alpha)$. Using proposition (45), this equality between natural transformations is obtained if we can show that all components are equal:

$$\begin{array}{rcll}
\epsilon'_d & = & \epsilon_d \circ F(\alpha_d) & \leftarrow\ \text{eqn. (4.3)} \\
\text{(1) of def. (31)}\ \rightarrow & = & \epsilon_d \circ (F\alpha)_d & \\
\text{(1) of def. (27)}\ \rightarrow & = & (\,\epsilon \circ F\alpha\,)_d &
\end{array}$$

$\diamond$

4.3 Unit, Counit and Opposite Category

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Suppose η is a unit of (F, G). Then in particular, η is a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$. Now if we are precise and wish to spell out the relevant typed functors, η is a natural transformation between $(\mathcal{C}, \mathcal{C}, I_{\mathcal{C}})$ and $(\mathcal{C}, \mathcal{C}, G \circ F)$. However, using proposition (43), it follows that η is also a natural transformation between the typed functors $(\mathcal{C}^{op}, \mathcal{C}^{op}, G \circ F)$ and $(\mathcal{C}^{op}, \mathcal{C}^{op}, I_{\mathcal{C}})$. Furthermore, from proposition (33) we have $I_{\mathcal{C}} = I_{\mathcal{C}^{op}}$ so η is in fact a natural transformation between the typed functors $(\mathcal{C}^{op}, \mathcal{C}^{op}, G \circ F)$ and $(\mathcal{C}^{op}, \mathcal{C}^{op}, I_{\mathcal{C}^{op}})$. In short, we have $\eta : G \circ F \Rightarrow I_{\mathcal{C}^{op}}$ where $G : \mathcal{D}^{op} \rightarrow \mathcal{C}^{op}$ and $F : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$, as per proposition (27). This is exactly the signature we would expect from a counit of (G, F) in relation to the categories $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$. As the following proposition shows, whenever η is a unit of (F, G) in relation to categories $\mathcal{C}$ and $\mathcal{D}$, it is indeed also a counit of (G, F) in relation to the categories $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}^{op} \arrow[r, bend left, "G"] & \mathcal{C}^{op} \arrow[l, bend left, "F"]
\end{tikzcd}$$

Proposition 60 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then every unit of (F, G) is a counit of (G, F) w.r. to $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$.*

Proof
Let η be a unit of (F, G), i.e a unit between the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{D}, \mathcal{C}, G)$ as per definition (32). We need to show that η is a counit of (G, F) in relation to the categories $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$, i.e. a counit between the typed functors $(\mathcal{D}^{op}, \mathcal{C}^{op}, G)$ and $(\mathcal{C}^{op}, \mathcal{D}^{op}, F)$ as per definition (33). First we need to show that η is a natural transformation $\eta : G \circ F \Rightarrow I_{\mathcal{C}^{op}}$, or more precisely a natural transformation between the typed functors $(\mathcal{C}^{op}, \mathcal{C}^{op}, G \circ F)$ and $(\mathcal{C}^{op}, \mathcal{C}^{op}, I_{\mathcal{C}^{op}})$. From proposition (43), this amounts to showing that η is a natural transformation between $(\mathcal{C}, \mathcal{C}, I_{\mathcal{C}^{op}})$ and $(\mathcal{C}, \mathcal{C}, G \circ F)$. Since $I_{\mathcal{C}^{op}} = I_{\mathcal{C}}$ as per proposition (33), we need to show that η is a natural transformation between the typed functors $(\mathcal{C}, \mathcal{C}, I_{\mathcal{C}})$ and $(\mathcal{C}, \mathcal{C}, G \circ F)$ which is $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$. This follows immediately from definition (32) and our assumption that η is a unit of (F, G). So it remains to show that η satisfies the universal property of definition (33), where F and G have been swapped and $\mathcal{C}$,$\mathcal{D}$ replaced by $\mathcal{D}^{op}$,$\mathcal{C}^{op}$ respectively. So let $d \in \mathcal{D}^{op}$, $c \in \mathcal{C}^{op}$ and $f : G(d) \to c$ @ $\mathcal{C}^{op}$. We need to show the existence of a unique $g : d \to F(c)$ @ $\mathcal{D}^{op}$ such $f = \eta_c \circ G(g)$ @ $\mathcal{C}^{op}$. In other words, using definition (5), given $d \in \mathcal{D}$, $c \in \mathcal{C}$ and $f : c \to G(d)$, we need to show the existence of a unique $g : F(c) \to d$ such that $f = G(g) \circ \eta_c$. Once again, this follows from definition (32) and our assumption that η is a unit of (F, G). $\diamond$

As proposition (60) shows a unit of two functors can be seen as a counit, if we switch the two functors and the two catgeories around, as well as replace each category by its opposite. Likewise we have:

Proposition 61 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then every counit of (F, G) is a unit of (G, F) w.r. to $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$.*

Proof
Let ϵ be a counit of (F, G), i.e a counit between the typed functors $(\mathcal{C}, \mathcal{D}, F)$ and $(\mathcal{D}, \mathcal{C}, G)$ as per definition (33). We need to show that ϵ is a unit of (G, F) in relation to the categories $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$, i.e. a unit between the typed functors $(\mathcal{D}^{op}, \mathcal{C}^{op}, G)$ and $(\mathcal{C}^{op}, \mathcal{D}^{op}, F)$ as per definition (32). First we need to show that ϵ is a natural transformation $\epsilon : I_{\mathcal{D}^{op}} \Rightarrow F \circ G$, or more precisely a natural transformation between the typed functors $(\mathcal{D}^{op}, \mathcal{D}^{op}, I_{\mathcal{D}^{op}})$ and $(\mathcal{D}^{op}, \mathcal{D}^{op}, F \circ G)$. From proposition (43), this amounts to showing that ϵ is a natural transformation between $(\mathcal{D}, \mathcal{D}, F \circ G)$ and $(\mathcal{D}, \mathcal{D}, I_{\mathcal{D}^{op}})$. Since $I_{\mathcal{D}^{op}} = I_{\mathcal{D}}$ as per proposition (33), we need to show that ϵ is a natural transformation between the typed functors $(\mathcal{D}, \mathcal{D}, F \circ G)$ and $(\mathcal{D}, \mathcal{D}, I_{\mathcal{D}})$ which is $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$. This follows immediately from definition (33) and our assumption that ϵ is a counit of (F, G). So it remains to show that ϵ satisfies the universal property of definition (32), where F and G have been swapped and $\mathcal{C}$,$\mathcal{D}$ replaced by $\mathcal{D}^{op}$,$\mathcal{C}^{op}$ respectively. So let $d \in \mathcal{D}^{op}$, $c \in \mathcal{C}^{op}$ and $g : d \to F(c)$ @ $\mathcal{D}^{op}$. We need to show the existence of a unique $f : G(d) \to c$ @ $\mathcal{C}^{op}$ such $g = F(f) \circ \epsilon_d$ @ $\mathcal{D}^{op}$. In other words, using definition (5), given $d \in \mathcal{D}$, $c \in \mathcal{C}$ and $g : F(c) \to d$, we need to show the

existence of a unique $f : c \to G(d)$ such that $g = \epsilon_d \circ F(f)$. Once again, this follows from definition (33) and our assumption that ϵ is a counit of (F, G). $\diamond$

4.4 Related Units and Counits

Definition 34 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Let $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be a unit and counit of (F, G) respectively. We say that η and ϵ are* related *if and only if one of these holds:*

$$\begin{array}{ll} (1) & G\epsilon \circ \eta G = \iota_G \\ (2) & \epsilon F \circ F\eta = \iota_F \end{array}$$

Remark: recall that ι_F and ι_G are identity natural transformations on F and G respectively, as per definition (25). Hence he have $\iota_F : F \Rightarrow F$ as well as $\iota_G : G \Rightarrow G$. Furthermore, since $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ and $G : \mathcal{D} \to \mathcal{C}$, ηG is the natural transformation $\eta G : I_{\mathcal{C}} \circ G \Rightarrow G \circ F \circ G$ as per definition (30), which is the same as $\eta G : G \Rightarrow G \circ F \circ G$. Since $F : \mathcal{C} \to \mathcal{D}$, we also obtain $F\eta : F \Rightarrow F \circ G \circ F$ from definition (31). Likewise, since $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$, we have $\epsilon F : F \circ G \circ F \Rightarrow F$ from definition (30) and $G\epsilon : G \circ F \circ G \Rightarrow G$ from definition (31). Hence we see that $\epsilon F \circ F\eta : F \Rightarrow F$ is a well-defined natural transformation, as per definition (27) and likewise $G\epsilon \circ \eta G : G \Rightarrow G$ is well-defined. So both equations (1) and (2) of definition (34) make perfect sense.

In order for a unit η and counit ϵ to be related, according to definition (34) only one of equations (1) and (2) needs to be satisfied. However, as the following proposition shows, each of these equalities imply the other, so that related units and counits will always in fact satisfy both equations.

Proposition 62 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Let η and ϵ be a unit and counit of (F, G) respectively. Then if η and ϵ are related, both equations* (1) *and* (2) *of definition (34) hold.*

Proof

We assume that η and ϵ are related unit and counit of (F, G). Then one of equation (1) and (2) holds, and we need to show that both equations (1) and (2) hold. It is therefore sufficient to prove that given a unit η and a counit ϵ, we have (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). First we show that (1) $\Rightarrow$ (2). So we assume that (1) holds, and we need to show (2), which is an equality between two natural transformations. Using proposition (45), if sufficient to show that for all $c \in \mathcal{C}$:

$$(\epsilon F \circ F\eta)_c = (\iota_F)_c$$

However, from definition (25) we have $(\iota_F)_c = \mathrm{id}(\,F(c)\,)$ and furthermore:

$$\begin{array}{rcll} (\epsilon F \circ F\eta)_c & = & (\epsilon F)_c \circ (F\eta)_c & \leftarrow \text{ def. } (27) \\ \text{def. } (30) \;\rightarrow & = & \epsilon_{F(c)} \circ (F\eta)_c & \\ \text{def. } (31) \;\rightarrow & = & \epsilon_{F(c)} \circ F(\eta_c) & \end{array}$$

So we need to show that $\epsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}(\,F(c)\,)$ for all $c \in \mathcal{C}$. Note that a similar derivation using assumption (1) allows us to obtain for all $d \in \mathcal{D}$:

$$G(\epsilon_d) \circ \eta_{G(d)} = \mathrm{id}(\,G(d)\,) \tag{4.5}$$

Now given $c \in \mathcal{C}$, let us pick $d = F(c)$. Then both $\epsilon_{F(c)} \circ F(\eta_c)$ and $\mathrm{id}(\,F(c)\,)$ are arrows from $F(c)$ to d in $\mathcal{D}$. In order to show that these two arrows coincide, having assumed η is a unit of (F,G), from proposition (54) it is sufficient to prove that $G(\,\epsilon_{F(c)} \circ F(\eta_c)\,) \circ \eta_c = G(\,\mathrm{id}(\,F(c)\,)\,) \circ \eta_c$ which goes as follows:

$$\begin{array}{rcl}
G(\,\epsilon_{F(c)} \circ F(\eta_c)\,) \circ \eta_c & = & G(\,\epsilon_d \circ F(\eta_c)\,) \circ \eta_c \\
G \text{ functor } \rightarrow & = & G(\epsilon_d) \circ (G \circ F)(\eta_c) \circ \eta_c \\
\text{prop (55)}, \ \eta_c : c \rightarrow G(d) \ \rightarrow & = & G(\epsilon_d) \circ \eta_{G(d)} \circ \eta_c \\
\text{eqn. (4.5) } \rightarrow & = & \mathrm{id}(\,G(d)\,) \circ \eta_c \\
G \text{ functor } \rightarrow & = & G(\,\mathrm{id}(d)\,) \circ \eta_c \\
 & = & G(\,\mathrm{id}(\,F(c)\,)\,) \circ \eta_c
\end{array}$$

So we now show that (2) $\Rightarrow$ (1). In this case for all $c \in \mathcal{C}$ we have:

$$\epsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}(\,F(c)\,) \tag{4.6}$$

and we need to prove that equation (4.5) holds for all $d \in \mathcal{D}$. However, given $d \in \mathcal{D}$ and setting $c = G(d)$, both arrows of equation (4.5) are arrows from c to $G(d)$ in $\mathcal{C}$. In order to show that these two arrows coincide, having assumed ϵ is a counit of (F,G), from proposition (57) it is sufficient to prove the equality $\epsilon_d \circ F(\,G(\epsilon_d) \circ \eta_{G(d)}\,) = \epsilon_d \circ F(\,\mathrm{id}(\,G(d)\,)\,)$ which goes as follows:

$$\begin{array}{rcl}
\epsilon_d \circ F(\,G(\epsilon_d) \circ \eta_{G(d)}\,) & = & \epsilon_d \circ F(\,G(\epsilon_d) \circ \eta_c\,) \\
F \text{ functor } \rightarrow & = & \epsilon_d \circ (F \circ G)(\epsilon_d) \circ F(\eta_c) \\
\text{prop (58)}, \ \epsilon_d : F(c) \rightarrow d \ \rightarrow & = & \epsilon_d \circ \epsilon_{F(c)} \circ F(\eta_c) \\
\text{eqn. (4.6) } \rightarrow & = & \epsilon_d \circ \mathrm{id}(\,F(c)\,) \\
F \text{ functor } \rightarrow & = & \epsilon_d \circ F(\,\mathrm{id}(c)\,) \\
 & = & \epsilon_d \circ F(\,\mathrm{id}(\,G(d)\,)\,)
\end{array}$$

$\diamond$

Proposition 63 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then every unit of (F,G) has a unique related counit.*

Proof

Let $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ be a unit of (F,G). We need to show the existence of a unique counit $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ which is related to η as per definition (34). First we shall define a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$, then prove that it is a counit which is related to η, and finally prove that it is unique. So let $d \in \mathcal{D}$.

We need to define an arrow $\epsilon_d : (F \circ G)(d) \to d$ in the category $\mathcal{D}$. Define $c = G(d)$. Then we need to define an arrow $\epsilon_d : F(c) \to d$. However, η is a unit of (F, G) and $\mathrm{id}(c)$ is an arrow $\mathrm{id}(c) : c \to G(d)$. Applying definition (32) we can define ϵ_d to be the unique arrow $\epsilon_d : F(c) \to d$ such that $\mathrm{id}(c) = G(\epsilon_d) \circ \eta_c$. Rewriting $c = G(d)$ we obtain:

$$G(\epsilon_d) \circ \eta_{G(d)} = \mathrm{id}(\, G(d)\,) \tag{4.7}$$

Having defined $\epsilon_d : (F \circ G)(d) \to d$ for all $d \in \mathcal{D}$ we have a map $\epsilon : \mathrm{Ob}\,\mathcal{D} \to \mathrm{Arr}\,\mathcal{D}$ which satisfies (1) of definition (23) in relation to the functors $F \circ G$ and $I_{\mathcal{D}}$. In order to show that ϵ is a natural transformation, it remains to prove property (2), namely that the naturality square commutes. So let $a, b \in \mathcal{D}$ and $g : a \to b$:

$$\begin{array}{ccc} F(G(a)) & \xrightarrow{\epsilon_a} & a \\ {\scriptstyle (F\circ G)(g)}\downarrow & & \downarrow{\scriptstyle g} \\ F(G(b)) & \xrightarrow[\epsilon_b]{} & b \end{array}$$

We need to show that $g \circ \epsilon_a = \epsilon_b \circ (F \circ G)(g)$. However, defining $c = G(a)$ and $d = b$, both arrows are from $F(c)$ to d. Having assumed that η is a unit of (F, G), in order to show that these arrows are equal, from proposition (54) it is sufficient to prove that $G(\, g \circ \epsilon_a\,) \circ \eta_c = G(\, \epsilon_b \circ (F \circ G)(g)\,) \circ \eta_c$. Both of these arrows are equal to $G(g)$ as can be seen from:

$$\begin{array}{rcl} G(\, g \circ \epsilon_a\,) \circ \eta_c & = & G(g) \circ G(\epsilon_a) \circ \eta_c \;\leftarrow\; G \text{ functor} \\ & = & G(g) \circ G(\epsilon_a) \circ \eta_{G(a)} \\ \text{eqn. 4.7 } \to & = & G(g) \circ \mathrm{id}(\, G(a)\,) \\ & = & G(g) \end{array}$$

and:

$$\begin{array}{rcl} G(\, \epsilon_b \circ (F \circ G)(g)\,) \circ \eta_c & = & G(\epsilon_b) \circ (G \circ F \circ G)(g) \circ \eta_c \;\leftarrow\; G \text{ functor} \\ & = & G(\epsilon_b) \circ (G \circ F)(G(g)) \circ \eta_{G(a)} \\ \text{prop. (55), } G(g) : G(a) \to G(b) \;\to & = & G(\epsilon_b) \circ \eta_{G(b)} \circ G(g) \\ \text{eqn. 4.7 } \to & = & \mathrm{id}(\, G(b)\,) \circ G(g) \\ & = & G(g) \end{array}$$

So we have now proved that ϵ is a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$, and we need to show that it is a counit of (F, G) as per definition (33). So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g : F(c) \to d$. We need to show the existence of a unique $f : c \to G(d)$ such that $g = \epsilon_d \circ F(f)$. We first show the existence: let $f = G(g) \circ \eta_c$. Since η is a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$, in particular we have $\eta_c : c \to (G \circ F)(c)$. Since G is a functor and $g : F(c) \to d$, we have $G(g) : (G \circ F)(c) \to G(d)$. It follows that f is a well-defined arrow $f : c \to G(d)$. We need to show that $g = \epsilon_d \circ F(f)$. Having assumed that η is a unit of (F, G), using proposition (54) it is sufficient to prove that $G(g) \circ \eta_c = G(\,\epsilon_d \circ F(f)\,) \circ \eta_c$, which is $f = G(\,\epsilon_d \circ F(f)\,) \circ \eta_c$. The proof goes as follows:

$$\begin{array}{rcll} G(\,\epsilon_d \circ F(f)\,) \circ \eta_c & = & G(\epsilon_d) \circ (G \circ F)(f) \circ \eta_c & \leftarrow\ G \text{ functor} \\ \text{prop. (55)},\ f : c \to G(d)\ \to & = & G(\epsilon_d) \circ \eta_{G(d)} \circ f & \\ \text{eqn. 4.7}\ \to & = & \mathrm{id}(\,G(d)\,) \circ f & \\ & = & f & \end{array}$$

We now prove the uniqueness. So we assume that $f_1, f_2 : c \to G(d)$ are such that $g = \epsilon_d \circ F(f_1) = \epsilon_d \circ F(f_2)$ and we need to show that $f_1 = f_2$:

$$\begin{array}{rcl} f_1 & = & \mathrm{id}(\,G(d)\,) \circ f_1 \\ \text{eqn. 4.7}\ \to & = & G(\epsilon_d) \circ \eta_{G(d)} \circ f_1 \\ \text{prop. (55)},\ f_1 : c \to G(d)\ \to & = & G(\epsilon_d) \circ (G \circ F)(f_1) \circ \eta_c \\ G \text{ functor}\ \to & = & G(\,\epsilon_d \circ F(f_1)\,) \circ \eta_c \\ \text{assumption}\ \to & = & G(\,\epsilon_d \circ F(f_2)\,) \circ \eta_c \\ \text{same derivation}\ \to & = & f_2 \end{array}$$

So we have now proved that ϵ is a counit of (F, G) as per definition (33) and it remains to show that it is related to η. From definition (34), it is sufficient to show that $G\epsilon \circ \eta G = \iota_G$ or equivalently that $G(\epsilon_d) \circ \eta_{G(d)} = \mathrm{id}(\,G(d)\,)$ for all $d \in \mathcal{D}$. This follows directly from equation (4.7).

So we have now proved the existence of a counit ϵ which is related to η. It remains to show that such counit is unique. So we assume that $\epsilon' : F \circ G \Rightarrow I_{\mathcal{D}}$ is another counit which is related to η. We need to show that $\epsilon = \epsilon'$ or equivalently that $\epsilon_d = \epsilon'_d$ for all $d \in \mathcal{D}$. Given $d \in \mathcal{D}$ define $c = G(d)$. Then both ϵ_d and ϵ'_d are arrows from $F(c)$ to d in $\mathcal{D}$. Using proposition (54), having assumed that η is a unit of (F, G), in order to show that $\epsilon_d = \epsilon'_d$ it is sufficient to prove that $G(\epsilon_d) \circ \eta_c = G(\epsilon'_d) \circ \eta_c$. Rewriting $c = G(d)$, we need to show that:

$$G(\epsilon_d) \circ \eta_{G(d)} = G(\epsilon'_d) \circ \eta_{G(d)}$$

However this follows from (1) of definition (34) and our assumption that both ϵ and ϵ' are related to η, which implies that both sides of this equation are equal to $\mathrm{id}(\,G(d)\,)$. Note that (1) of definition (34) is true from proposition (62). $\diamond$

Proposition 64 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then every counit of (F, G) has a unique related unit.*

Proof
Let $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be a counit of (F, G). We need to show the existence of a unique unit $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ which is related to ϵ as per definition (34). First we shall define a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$, then prove that it is a unit which is related to ϵ, and finally prove that it is unique. So let $c \in \mathcal{C}$. We need to define an arrow $\eta_c : c \to (G \circ F)(c)$ in the category $\mathcal{C}$. Define $d = F(c)$. Then we need to define an arrow $\eta_c : c \to G(d)$. However, ϵ is a counit of (F, G)

and $\mathrm{id}(d)$ is an arrow $\mathrm{id}(d) : F(c) \to d$. Applying definition (33) we can define η_c to be the unique arrow $\eta_c : c \to G(d)$ such that $\mathrm{id}(d) = \epsilon_d \circ F(\eta_c)$. Rewriting $d = F(c)$ we obtain:

$$\epsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}(\, F(c)\,) \tag{4.8}$$

Having defined $\eta_c : c \to (G \circ F)(c)$ for all $c \in \mathcal{C}$ we have $\eta : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Arr}\,\mathcal{C}$ which satisfies (1) of definition (23) in relation to the functors $I_{\mathcal{C}}$ and $G \circ F$. In order to show that η is a natural transformation, it remains to prove property (2), namely that the naturality square commutes. So let $a, b \in \mathcal{C}$ and $f : a \to b$:

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "f"'] & G(F(a)) \arrow[d, "(G\circ F)(f)"] \\
b \arrow[r, "\eta_b"'] & G(F(b))
\end{tikzcd}$$

We need to show that $(G \circ F)(f) \circ \eta_a = \eta_b \circ f$. However, defining $c = a$ and $d = F(b)$, both arrows are from c to $G(d)$. Having assumed that ϵ is a counit of (F, G), in order to show that these arrows are equal, from proposition (57) it is sufficient to prove that $\epsilon_d \circ F(\,(G \circ F)(f) \circ \eta_a\,) = \epsilon_d \circ F(\,\eta_b \circ f)$. Both of these arrows are equal to $F(f)$ as can be seen from:

$$\begin{array}{rcll}
\epsilon_d \circ F(\,(G \circ F)(f) \circ \eta_a\,) & = & \epsilon_d \circ (F \circ G \circ F)(f) \circ F(\eta_a) & \leftarrow\ F \text{ functor} \\
 & = & \epsilon_{F(b)} \circ (F \circ G)(F(f)) \circ F(\eta_a) & \\
\text{prop. (58), } F(f) : F(a) \to F(b)\ \to & = & F(f) \circ \epsilon_{F(a)} \circ F(\eta_a) & \\
\text{eqn. 4.8}\ \to & = & F(f) \circ \mathrm{id}(\,F(a)\,) & \\
 & = & F(f) &
\end{array}$$

and:

$$\begin{array}{rcll}
\epsilon_d \circ F(\,\eta_b \circ f) & = & \epsilon_d \circ F(\eta_b) \circ F(f) & \leftarrow\ F \text{ functor} \\
 & = & \epsilon_{F(b)} \circ F(\eta_b) \circ F(f) & \\
\text{eqn. 4.8}\ \to & = & \mathrm{id}(\,F(b)\,) \circ F(f) & \\
 & = & F(f) &
\end{array}$$

So we have now proved that η is a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$, and we need to show that it is a unit of (F, G) as per definition (32). So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \to G(d)$. We need to show the existence of a unique $g : F(c) \to d$ such that $f = G(g) \circ \eta_c$. We first show the existence: let $g = \epsilon_d \circ F(f)$. Since ϵ is a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$, in particular we have $\epsilon_d : (F \circ G)(d) \to d$. Since F is a functor and $f : c \to G(d)$, we have $F(f) : F(c) \to (F \circ G)(d)$. It follows that g is a well-defined arrow $g : F(c) \to d$. We need to show that $f = G(g) \circ \eta_c$. Having assumed that ϵ is a counit of (F, G), using proposition (57) it is sufficient to prove that $\epsilon_d \circ F(f) = \epsilon_d \circ F(\,G(g) \circ \eta_c\,)$, which is $g = \epsilon_d \circ F(\,G(g) \circ \eta_c\,)$. The proof goes as follows:

$$\epsilon_d \circ F(\,G(g) \circ \eta_c\,) \;=\; \epsilon_d \circ (F \circ G)(g) \circ F(\eta_c) \;\leftarrow\; F \text{ functor}$$

$$
\begin{array}{rcl}
\text{prop. (58), } g : F(c) \to d \ \to & = & g \circ \epsilon_{F(c)} \circ F(\eta_c) \\
\text{eqn. 4.8} \ \to & = & g \circ \mathrm{id}(\, F(c)\,) \\
& = & g
\end{array}
$$

We now prove the uniqueness. So we assume that $g_1, g_2 : F(c) \to d$ are such that $f = G(g_1) \circ \eta_c = G(g_2) \circ \eta_c$ and we need to show that $g_1 = g_2$:

$$
\begin{array}{rcl}
g_1 & = & g_1 \circ \mathrm{id}(\, F(c)\,) \\
\text{eqn. 4.8} \ \to & = & g_1 \circ \epsilon_{F(c)} \circ F(\eta_c) \\
\text{prop. (58), } g_1 : F(c) \to d \ \to & = & \epsilon_d \circ (F \circ G)(g_1) \circ F(\eta_c) \\
F \text{ functor} \ \to & = & \epsilon_d \circ F(\, G(g_1) \circ \eta_c\,) \\
\text{assumption} \ \to & = & \epsilon_d \circ F(\, G(g_2) \circ \eta_c\,) \\
\text{same derivation} \ \to & = & g_2
\end{array}
$$

So we have now proved that η is a unit of (F, G) as per definition (32) and it remains to show that it is related to ϵ. From definition (34), it is sufficient to show that $\epsilon F \circ F\eta = \iota_F$ or equivalently that $\epsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}(\, F(c)\,)$ for all $c \in \mathcal{C}$. This follows directly from equation (4.8).

So we have now proved the existence of a unit η which is related to ϵ. It remains to show that such unit is unique. So we assume that $\eta' : I_{\mathcal{C}} \Rightarrow G \circ F$ is another unit which is related to ϵ. We need to show that $\eta = \eta'$ or equivalently that $\eta_c = \eta'_c$ for all $c \in \mathcal{C}$. Given $c \in \mathcal{C}$ define $d = F(c)$. Then both η_c and η'_c are arrows from c to $G(d)$ in $\mathcal{C}$. Using proposition (57), having assumed that ϵ is a counit of (F, G), in order to show that $\eta_c = \eta'_c$ it is sufficient to prove that $\epsilon_d \circ F(\eta_c) = \epsilon_d \circ F(\eta'_c)$. Rewriting $d = F(c)$, we need to show that:

$$
\epsilon_{F(c)} \circ F(\eta_c) = \epsilon_{F(c)} \circ F(\eta'_c)
$$

However this follows from (2) of definition (34) and our assumption that both η and η' are related to ϵ, which implies that both sides of this equation are equal to $\mathrm{id}(\, F(c)\,)$. Note that (2) of definition (34) is true from proposition (62). $\diamond$

The following proposition is not an immediate consequence of definition (34). We do not assume that η and ϵ are a unit and counit of (F, G) to start with. We simply require them to be natural transformations. The remarkable thing is that when equations (1) and (2) below are met, then both η and ϵ satisfy the universal property which makes them a unit and counit of (F, G).

Proposition 65 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Let $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be natural transformations which satisfy both of the following equations:*

$$
\begin{array}{ll}
(1) & G\epsilon \circ \eta G = \iota_G \\
(2) & \epsilon F \circ F\eta = \iota_F
\end{array}
$$

Then η and ϵ are related unit and counit of (F, G) respectively.

Proof
It is sufficient to prove that η is a unit of (F, G) and ϵ is a counit. The fact that η and ϵ are related will then follow immediately from definition (34), given our assumption of equations (1) and (2). First we show that η is a unit of (F, G). By assumption, we already know that it is a natural transformation $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$. So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \to G(d)$. We need to prove the existence of a unique $g : F(c) \to d$ such that $f = G(g) \circ \eta_c$. We shall first prove the uniqueness. So suppose that $g : F(c) \to d$ is such that $f = G(g) \circ \eta_c$. Then:

$$
\begin{array}{rcl}
\epsilon_d \circ F(f) & = & \epsilon_d \circ F(\, G(g) \circ \eta_c \,) \\
F \text{ functor } \rightarrow & = & \epsilon_d \circ (F \circ G)(g) \circ F(\eta_c) \\
\text{prop. (58) } \rightarrow & = & g \circ \epsilon_{F(c)} \circ F(\eta_c) \\
 & = & g \circ (\epsilon F \circ F\eta)(c) \\
\text{equation (2) } \rightarrow & = & g \circ \iota_F(c) \\
\text{def. (25) } \rightarrow & = & g \circ \mathrm{id}(\, F(c) \,) \\
 & = & g
\end{array}
$$

So we see that g must be equal to $\epsilon_d \circ F(f)$ and is therefore unique. To prove the existence, suppose conversely that $g = \epsilon_d \circ F(f)$. We need to show that $f = G(g) \circ \eta_c$ which goes as follows:

$$
\begin{array}{rcl}
G(g) \circ \eta_c & = & G(\, \epsilon_d \circ F(f) \,) \circ \eta_c \\
G \text{ functor } \rightarrow & = & G(\epsilon_d) \circ (G \circ F)(f) \circ \eta_c \\
\text{prop. (55) } \rightarrow & = & G(\epsilon_d) \circ \eta_{G(d)} \circ f \\
 & = & (G\epsilon \circ \eta G)(d) \circ f \\
\text{equation (1) } \rightarrow & = & \iota_G(d) \circ f \\
\text{def. (25) } \rightarrow & = & \mathrm{id}(\, G(d) \,) \circ f \\
 & = & f
\end{array}
$$

So we have proved that η is a unit of (F, G) as requested, and it remains to show that ϵ is a counit of (F, G). By assumption, we already know that it is a natural transformation $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$. So let $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g : F(c) \to d$. We need to prove the existence of a unique $f : c \to G(d)$ such that $g = \epsilon_d \circ F(f)$. We shall first prove the uniqueness. So suppose that $f : c \to G(d)$ is such that $g = \epsilon_d \circ F(f)$. Following the same derivation as above, we see that f must be equal to $G(g) \circ \eta_c$ and is therefore unique. To prove the existence, suppose conversely that $f = G(g) \circ \eta_c$. We need to show that $g = \epsilon_d \circ F(f)$. Once again, this is obtained by following the same derivation as above. $\diamond$

4.5 Adjunction

Definition 35 *We call* adjunction *between two categories $\mathcal{C}$ and $\mathcal{D}$ any pair of functors (F, G) where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ which has a unit.*

Definition 36 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We say that F and G are* adjoint functors*, or that F is* left adjoint *to G, or that G is* right-adjoint *to F, if and only if (F, G) is an adjunction.*

Notation 25 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. We write $F \dashv G$ to express the fact that (F, G) is an adjunction.*

Proposition 66 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Then the following are equivalent:*

$$\begin{array}{rl} (i) & (F,G) \text{ is an adjunction between } \mathcal{C} \text{ and } \mathcal{D} \\ (ii) & F \dashv G \\ (iii) & F \text{ and } G \text{ are adjoint functors} \\ (iv) & F \text{ is left adjoint to } G \\ (v) & G \text{ is right adjoint to } F \\ (vi) & (F,G) \text{ has a unit} \\ (vii) & (F,G) \text{ has a counit} \end{array}$$

Proof
$(i) \Leftrightarrow (ii)$: is a restatement of notation (25).
$(i) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ is restatement of definition (36).
$(i) \Leftrightarrow (vi)$: is a restatement of definition (35).
$(vi) \Rightarrow (vii)$: Suppose (F, G) has a unit η. Using proposition (63), η has a unique related counit. In particular, (F, G) has a counit.
$(vii) \Rightarrow (vi)$: Suppose (F, G) has a counit ϵ. Using proposition (64), ϵ has a unique related unit. In particular, (F, G) has a unit. $\diamond$

The following propostion provides an easy criterium to determine whether a pair of functors (F, G) is an adjunction between two categories $\mathcal{C}$ and $\mathcal{D}$. This criterium only demands that two natural transformations with the appropriate properties be defined. It does not require proving any universal property.

Proposition 67 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. Let $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$ be natural transformations which satisfy both of the following equations:*

$$\begin{array}{ll} (1) & G\epsilon \circ \eta G = \iota_G \\ (2) & \epsilon F \circ F\eta = \iota_F \end{array}$$

Then (F, G) is an adjunction between the categories $\mathcal{C}$ and $\mathcal{D}$, i.e. $F \dashv G$.

Proof
Suppose such natural transformations η and ϵ do exist. Using proposition (65) it follows that η and ϵ are related unit and counit of (F, G) respectively. In particular, (F, G) has a unit and it is therefore an adjunction between $\mathcal{C}$ and $\mathcal{D}$. $\diamond$

4.6 Adjunction and Locally Small Category

In this section, we consider the case when both categories $\mathcal{C}$ and $\mathcal{D}$ are locally small, as per definition (8). The difference this makes is the availability of the associated hom-functors $\mathcal{C} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ and $\mathcal{D} : \mathcal{D}^{op} \times \mathcal{D} \to \mathbf{Set}$ as per definition (19). So suppose we have two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We also have identity functors $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $I_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ as per definition (15). Using proposition (27), we know that F is also a functor $F : \mathcal{C}^{op} \to \mathcal{D}^{op}$, while from proposition (33) we see that $I_{\mathcal{C}}$ and $I_{\mathcal{C}^{op}}$ are the same functors. Using definition (20) we can define the product functors $F \times I_{\mathcal{D}} : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{D}^{op} \times \mathcal{D}$. and $I_{\mathcal{C}} \times G : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{C}^{op} \times \mathcal{C}$. Composing these functors with the hom-functors $\mathcal{D}$ and $\mathcal{C}$ respectively as per definition (18), we obtain two functors $\mathcal{D} \circ (F \times I_{\mathcal{D}}) : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ and $\mathcal{C} \circ (I_{\mathcal{C}} \times G) : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$. These functors are very important in what follows, so we shall give then a name:

Definition 37 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. We call* left-hand-side functor *associated with the pair (F, G) the functor $\mathcal{D} \circ (F \times I_{\mathcal{D}}) : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$.*

Definition 38 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. We call* right-hand-side functor *associated with the pair (F, G) the functor $\mathcal{C} \circ (I_{\mathcal{C}} \times G) : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$.*

Proposition 68 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let $F^* = \mathcal{D} \circ (F \times I_{\mathcal{D}})$ be the left-hand-side functor associated with (F, G). Then for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, we have:*

$$F^*(c, d) = \mathcal{D}(F(c), d)$$

Proof

$$\begin{array}{rcl}
F^*(c,d) & = & \mathcal{D} \circ (F \times I_{\mathcal{D}})(c,d) \\
\text{(1) of def. (18)} \rightarrow & = & \mathcal{D}(\,(F \times I_{\mathcal{D}})(c,d)\,) \\
\text{(1) of def. (20)} \rightarrow & = & \mathcal{D}(F(c), I_{\mathcal{D}}(d)) \\
\text{(1) of def. (15)} \rightarrow & = & \mathcal{D}(F(c), d)
\end{array}$$

$\diamond$

Remark: In light of proposition (68), it is common to refer to the left-hand-side functor casually as '$\mathcal{D}(F(c), d)$' where 'c' and 'd' are dummy variables. We should not forget however that a functor is more than a mere transformation on objects, and that the left-hand-side functor is really $\mathcal{D} \circ (F \times I_{\mathcal{D}})$.

Proposition 69 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let $G^* = \mathcal{C} \circ (I_{\mathcal{C}} \times G)$ be the right-hand-side functor associated with (F, G). Then for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, we have:*

$$G^*(c, d) = \mathcal{C}(c, G(d))$$

Proof

$$\begin{array}{rcl} G^*(c,d) & = & \mathcal{C} \circ (I_{\mathcal{C}} \times G)(c,d) \\ (1) \text{ of def. } (18) \ \rightarrow & = & \mathcal{C}(\,(I_{\mathcal{C}} \times G)(c,d)\,) \\ (1) \text{ of def. } (20) \ \rightarrow & = & \mathcal{C}(I_{\mathcal{C}}(c), G(d)) \\ (1) \text{ of def. } (15) \ \rightarrow & = & \mathcal{C}(c, G(d)) \end{array}$$

$\diamond$

Remark: In light of proposition (69), it is common to refer to the right-hand-side functor casually as '$\mathcal{C}(c, G(d))$' where 'c' and 'd' are dummy variables. Once again, a functor is more than a mere transformation on objects, and we should remember that the right-hand-side functor is really $\mathcal{C} \circ (I_{\mathcal{C}} \times G)$.

The left-hand side functor $F^* = \mathcal{D} \circ (F \times I_{\mathcal{D}})$ is a functor $F^* : \mathcal{C}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$. If $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, if $f : c' \rightarrow c$ @ $\mathcal{C}$ and $g : d \rightarrow d'$ @ $\mathcal{D}$ then we have $(f, g) : (c, d) \rightarrow (c', d')$ @ $\mathcal{C}^{op} \times \mathcal{D}$ and $F^*(f, g) : F^*(c, d) \rightarrow F^*(c', d')$ @ $\mathbf{Set}$, which is $F^*(f, g) : \mathcal{D}(F(c), d) \rightarrow \mathcal{D}(F(c'), d')$. So if $g' : F(c) \rightarrow d$ @ $\mathcal{D}$ then $F^*(f, g)(g')$ is well-defined in $\mathcal{D}(F(c'), d')$, i.e. $F^*(f, g)(g') : F(c') \rightarrow d'$ @ $\mathcal{D}$.

Proposition 70 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let $F^* = \mathcal{D} \circ (F \times I_{\mathcal{D}})$ be the left-hand-side functor associated with (F, G). Then for all $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, for all $f : c' \rightarrow c$, $g : d \rightarrow d'$ and $g' : F(c) \rightarrow d$, we have:*

$$F^*(f, g)(g') = g \circ g' \circ F(f)$$

Remark: Note that the expression $g \circ g' \circ F(f)$ is a well-defined arrow in $\mathcal{D}$ from $F(c')$ to d', since $F(f) : F(c') \rightarrow F(c)$, $g' : F(c) \rightarrow d$ and $g : d \rightarrow d'$.
Proof

$$\begin{array}{rcl} F^*(f,g)(g') & = & \mathcal{D} \circ (F \times I_{\mathcal{D}})(f,g)(g') \\ (2) \text{ of def. } (18) \ \rightarrow & = & \mathcal{D}(\,(F \times I_{\mathcal{D}})(f,g)\,)(g') \\ (2) \text{ of def. } (20) \ \rightarrow & = & \mathcal{D}(F(f), I_{\mathcal{D}}(g))(g') \\ (2) \text{ of def. } (15) \ \rightarrow & = & \mathcal{D}(F(f), g)(g') \\ (2) \text{ of def. } (19) \ \rightarrow & = & g \circ g' \circ F(f) \end{array}$$

$\diamond$

The right-hand side functor $G^* = \mathcal{C} \circ (I_{\mathcal{C}} \times G)$ is a functor $G^* : \mathcal{C}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$. If $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, if $f : c' \rightarrow c$ @ $\mathcal{C}$ and $g : d \rightarrow d'$ @ $\mathcal{D}$ then we have $(f, g) : (c, d) \rightarrow (c', d')$ @ $\mathcal{C}^{op} \times \mathcal{D}$ and $G^*(f, g) : G^*(c, d) \rightarrow G^*(c', d')$ @ $\mathbf{Set}$, which is $G^*(f, g) : \mathcal{C}(c, G(d)) \rightarrow \mathcal{C}(c', G(d'))$. So if $f' : c \rightarrow G(d)$ @ $\mathcal{C}$ then $G^*(f, g)(f')$ is well-defined in $\mathcal{C}(c', G(d'))$, i.e. $G^*(f, g)(f') : c' \rightarrow G(d')$ @ $\mathcal{C}$.

Proposition 71 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let $G^* = \mathcal{C} \circ (I_{\mathcal{C}} \times G)$ be the right-hand-side functor* *associated with (F, G). Then for all $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, for all $f : c' \to c$, $g : d \to d'$ and $f' : c \to G(d)$, we have:*

$$G^*(f, g)(f') = G(g) \circ f' \circ f$$

Remark: Note that the expression $G(g) \circ f' \circ f$ is a well-defined arrow in $\mathcal{C}$ from c' to $G(d')$, since $f : c' \to c$, $f' : c \to G(d)$ and $G(g) : G(d) \to G(d')$.
Proof

$$\begin{array}{rcl}
G^*(f, g)(f') & = & \mathcal{C} \circ (I_{\mathcal{C}} \times G)(f, g)(f') \\
\text{(2) of def. (18)} \;\to & = & \mathcal{C}(\,(I_{\mathcal{C}} \times G)(f, g)\,)(f') \\
\text{(2) of def. (20)} \;\to & = & \mathcal{C}(I_{\mathcal{C}}(f), G(g))(f') \\
\text{(2) of def. (15)} \;\to & = & \mathcal{C}(f, G(g))(f') \\
\text{(2) of def. (19)} \;\to & = & G(g) \circ f' \circ f
\end{array}$$

$\diamond$

Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let us assume that (F, G) is an adjunction, or equivalently that we have a unit $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$. Then given $c \in \mathcal{C}$ and $d \in \mathcal{D}$, we can consider the function $\alpha(c, d) : \mathcal{D}(F(c), d) \to \mathcal{C}(c, G(d))$ which maps any arrow $g : F(c) \to d$ @ $\mathcal{D}$ to the arrow $\alpha(c, d)(g) : c \to G(d)$ @ $\mathcal{C}$ defined by the equation $\alpha(c, d)(g) = G(g) \circ \eta_c$. This is a valid definition since we have $\eta_c : c \to G(F(c))$ and $G(g) : G(F(c)) \to G(d)$ and $G(g) \circ \eta_c$ is a well-defined element of $\mathcal{C}(c, G(g))$. So $\alpha(c, d)$ is a well-defined function from the hom-set $\mathcal{D}(F(c), d)$ to the hom-set $\mathcal{C}(c, G(d))$. In short, we have $\alpha(c, d) : F^*(c, d) \to G^*(c, d)$ where F^* and G^* are the left and right-hand side functors associated with (F, G).

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "F^*"] \arrow[r, bend right, "G^*"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{Set}
\end{tikzcd}$$

The obvious question to ask is whether the map $\alpha : \text{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \text{Arr}\ \mathbf{Set}$ we have just defined is in fact a natural transformation. The answer is 'yes':

Proposition 72 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let η be a unit of (F, G). Consider the map $\alpha : \text{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \text{Arr}\ \mathbf{Set}$ defined for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ by:*

$$\alpha(c, d)(g) = G(g) \circ \eta_c$$

for all $g : F(c) \to d$. Then α is a natural transformation $\alpha : F^* \Rightarrow G^*$, where F^* and G^* denote the left-hand-side and right-hand-side functors respectively.

Proof
The left-hand-side functor F^* as defined in (37) and the right-hand-side functor

G^* as defined in (38) are both functors $F^*, G^* : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$. Using definition (23), in order for α to qualify as a natural transformation $\alpha : F^* \Rightarrow G^*$ it needs to be a map $\alpha : \mathrm{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \mathrm{Arr}\ \mathbf{Set}$ (which is the case) satisfying (1) and (2) of definition (23). We have seen that $\alpha(c,d) : F^*(c,d) \to G^*(c,d)$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, that is for all $(c,d) \in \mathcal{C}^{op} \times \mathcal{D}$. Hence (1) of definition (23) is satisfied, and it remains to show (2). So let $c, c' \in \mathcal{C}$, $d, d' \in \mathcal{D}$ and $k : (c,d) \to (c',d')$ @ $\mathcal{C}^{op} \times \mathcal{D}$. Then $k = (f,g)$ for some $f : c' \to c$ @ $\mathcal{C}$ and $g : d \to d'$ @ $\mathcal{D}$, and we need to show that the following square commutes:

$$\begin{array}{ccccc} c & & d & & \mathcal{D}(F(c),d) \xrightarrow{\alpha(c,d)} \mathcal{C}(c,G(d)) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle g} & & {\scriptstyle F^*(f,g)}\downarrow \qquad\qquad \downarrow{\scriptstyle G^*(f,g)} \\ c' & & d' & & \mathcal{D}(F(c'),d') \xrightarrow[\alpha(c',d')]{} \mathcal{C}(c',G(d')) \end{array}$$

Hence we need to show that $G^*(f,g) \circ \alpha(c,d) = \alpha(c',d') \circ F^*(f,g)$. This is an equality between arrows in the category $\mathbf{Set}$. Using proposition (6), we can prove this equality simply by showing the two underlying untyped functions coincide on every $g' \in \mathcal{D}(F(c),d)$. So let $g' \in \mathcal{D}(F(c),d)$. We need to show that $(G^*(f,g) \circ \alpha(c,d))(g') = (\alpha(c',d') \circ F^*(f,g))(g')$ which goes as follows:

$$\begin{array}{rcl} (G^*(f,g) \circ \alpha(c,d))(g') & = & G^*(f,g)(\,\alpha(c,d)(g')\,) \\ & = & G^*(f,g)(\,G(g') \circ \eta_c\,) \\ \text{prop. (71)} \ \rightarrow & = & G(g) \circ G(g') \circ \eta_c \circ f \\ \text{prop. (55)} \ \rightarrow & = & G(g) \circ G(g') \circ (G \circ F)(f) \circ \eta_{c'} \\ G \text{ functor} \ \rightarrow & = & G(\,g \circ g' \circ F(f)\,) \circ \eta_{c'} \\ \text{prop. (70)} \ \rightarrow & = & G(\,F^*(f,g)(g')\,) \circ \eta_{c'} \\ & = & \alpha(c',d')(\,F^*(f,g)(g')) \\ & = & (\alpha(c',d') \circ F^*(f,g))(g') \end{array}$$

$\diamond$

The natural transformation $\alpha : F^* \Rightarrow G^*$ above deserves to be named:

Definition 39 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let η be a unit of (F,G). We call* natural transformation associated with η *the transformation $\alpha : F^* \Rightarrow G^*$, defined by:*

$$\alpha(c,d)(g) = G(g) \circ \eta_c$$

for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $g : F(c) \to d$, where F^* and G^* denote the left-hand-side and right-hand-side functors respectively, as per definitions (37) and (38).

Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories and let us assume that (F,G) is an adjunction, or equivalently that we have a counit $\epsilon : F \circ G \Rightarrow I_{\mathcal{D}}$. Then given $c \in \mathcal{C}$ and $d \in \mathcal{D}$, we can consider the function $\beta(c,d) : \mathcal{C}(c, G(d)) \to \mathcal{D}(F(c), d)$ which maps any arrow $f : c \to G(d)$ @ $\mathcal{C}$ to the arrow $\beta(c,d)(f) : F(c) \to d$ @ $\mathcal{D}$ defined by the equation $\beta(c,d)(f) = \epsilon_d \circ F(f)$. This is a valid definition since we see that $F(f) : F(c) \to F(G(d))$ and $\epsilon_d : F(G(d)) \to d$ and $\epsilon_d \circ F(f)$ is a well-defined element of $\mathcal{D}(F(c), d)$. So $\beta(c,d)$ is a well-defined function from the hom-set $\mathcal{C}(c, G(d))$ to the hom-set $\mathcal{D}(F(c), d)$. In short, $\beta(c,d) : G^*(c,d) \to F^*(c,d)$ where F^* and G^* are the left and right-hand side functors of (F, G).

$$\begin{tikzcd} \mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left=50, "G^*"{name=U}] \arrow[r, bend right=50, "F^*"'{name=L}] & \mathbf{Set} \arrow[Rightarrow, from=U, to=L, "\beta"] \end{tikzcd}$$

The obvious question to ask is whether the map $\beta : \mathrm{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \mathrm{Arr}\ \mathbf{Set}$ we have just defined is in fact a natural transformation. The answer is 'yes':

Proposition 73 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let ϵ be a counit of (F, G). Consider the map $\beta : \mathrm{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \mathrm{Arr}\ \mathbf{Set}$ defined for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ by:*

$$\beta(c,d)(f) = \epsilon_d \circ F(f)$$

for all $f : c \to G(d)$. Then β is a natural transformation $\beta : G^* \Rightarrow F^*$, where F^* and G^* denote the left-hand-side and right-hand-side functors respectively.

Proof
The left-hand-side functor F^* as defined in (37) and the right-hand-side functor G^* as defined in (38) are both functors $F^*, G^* : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$. Using definition (23), in order for β to qualify as a natural transformation $\beta : G^* \Rightarrow F^*$ it needs to be a map $\beta : \mathrm{Ob}\ (\mathcal{C}^{op} \times \mathcal{D}) \to \mathrm{Arr}\ \mathbf{Set}$ (which is the case) satisfying (1) and (2) of definition (23). We have seen that $\beta(c,d) : G^*(c,d) \to F^*(c,d)$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, that is for all $(c,d) \in \mathcal{C}^{op} \times \mathcal{D}$. Hence (1) of definition (23) is satisfied, and it remains to show (2). So let $c, c' \in \mathcal{C}$, $d, d' \in \mathcal{D}$ and $k : (c,d) \to (c',d')$ @ $\mathcal{C}^{op} \times \mathcal{D}$. Then $k = (f,g)$ for some $f : c' \to c$ @ $\mathcal{C}$ and $g : d \to d'$ @ $\mathcal{D}$, and we need to show that the following square commutes:

$$\begin{tikzcd} c & d & \mathcal{C}(c, G(d)) \arrow[r, "\beta(c,d)"] \arrow[d, "G^*(f,g)"'] & \mathcal{D}(F(c), d) \arrow[d, "F^*(f,g)"] \\ c' \arrow[u, "f"] & d' \arrow[from=u, "g"'] & \mathcal{C}(c', G(d')) \arrow[r, "\beta(c',d')"'] & \mathcal{D}(F(c'), d') \end{tikzcd}$$

Hence we need to show that $F^*(f,g) \circ \beta(c,d) = \beta(c',d') \circ G^*(f,g)$. This is an equality between arrows in the category $\mathbf{Set}$. Using proposition (6), we can prove this equality simply by showing the two underlying untyped functions coincide on every $f' \in \mathcal{C}(c, G(d))$. So let $f' \in \mathcal{C}(c, G(d))$. We need to show that

$(F^*(f,g)\circ\beta(c,d))(f') = (\beta(c',d')\circ G^*(f,g))(f')$ which goes as follows:

$$\begin{array}{rcl}
(F^*(f,g)\circ\beta(c,d))(f') & = & F^*(f,g)(\,\beta(c,d)(f')\,) \\
 & = & F^*(f,g)(\,\epsilon_d\circ F(f')\,) \\
\text{prop. (70)} \ \rightarrow & = & g\circ\epsilon_d\circ F(f')\circ F(f) \\
\text{prop. (58)} \ \rightarrow & = & \epsilon_{d'}\circ(F\circ G)(g)\circ F(f')\circ F(f) \\
F \text{ functor} \ \rightarrow & = & \epsilon_{d'}\circ F(\,G(g)\circ f'\circ f\,) \\
\text{prop. (71)} \ \rightarrow & = & \epsilon_{d'}\circ F(\,G^*(f,g)(f')\,) \\
 & = & \beta(c',d')(\,G^*(f,g)(f')\,) \\
 & = & (\beta(c',d')\circ G^*(f,g))(f')
\end{array}$$

$\diamond$

The natural transformation $\beta : G^* \Rightarrow F^*$ above deserves to be named:

Definition 40 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let ϵ be a counit of (F, G). We call* natural transformation associated with ϵ *the transformation $\beta : G^* \Rightarrow F^*$, defined by:*

$$\beta(c,d)(f) = \epsilon_d \circ F(f)$$

for all $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \to G(d)$, where F^* and G^* denote the left-hand-side and right-hand-side functors respectively, as per definitions (37) and (38).

Proposition 74 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let η be a unit of (F, G) and $\alpha : F^* \Rightarrow G^*$ be the natural transformation associated with η, as per (39). Then α is a natural isomorphism with inverse $\beta : G^* \Rightarrow F^*$ associated with the related counit ϵ.*

Remark: Recall from proposition (63) that every unit η of (F, G) has a unique related counit ϵ and $\beta : G^* \Rightarrow F^*$ is therefore well-defined by virtue of (40).
Proof
Using proposition (50), we need to show that $\beta \circ \alpha = \iota_{F^*}$ and $\alpha \circ \beta = \iota_{G^*}$, where F^* and G^* denote the left-hand-side and right-hand-side functors respectively, as per definitions (37) and (38). We shall start with $\beta \circ \alpha = \iota_{F^*}$. From proposition (45), in order to prove this equality it is sufficient to show that all components are equal. So given $c \in \mathcal{C}$ and $d \in \mathcal{D}$, i.e. given an arbitrary $(c, d) \in \mathcal{C}^{op} \times \mathcal{D}$, we need to show that $(\beta \circ \alpha)(c, d) = \iota_{F^*}(c, d)$. This is an equality between two arrows of the category **Set** with domain and codomain $F^*(c, d) = \mathcal{D}(F(c), d)$. Using proposition (6), it is sufficient to show that for all $g : F(c) \to d$ @ $\mathcal{D}$ we have $(\beta \circ \alpha)(c, d)(g) = \iota_{F^*}(c, d)(g)$:

$$\begin{array}{rcl}
(\beta\circ\alpha)(c,d)(g) & = & (\,\beta(c,d)\circ\alpha(c,d)\,)(g) \ \leftarrow \ \text{def. (27)} \\
\circ \text{ on } \textbf{Set} \ \rightarrow & = & \beta(c,d)(\,\alpha(c,d)(g)\,) \\
\text{def. (39)} \ \rightarrow & = & \beta(c,d)(\,G(g)\circ\eta_c\,)
\end{array}$$

$$
\begin{array}{rcl}
\text{def. (40)} \;\rightarrow & = & \epsilon_d \circ F(\,G(g) \circ \eta_c\,) \\
F \text{ functor} \;\rightarrow & = & \epsilon_d \circ (F \circ G)(g) \circ F(\eta_c) \\
\text{prop. (58)} \;\rightarrow & = & g \circ \epsilon_{F(c)} \circ F(\eta_c) \\
\text{def. (31) and (30)} \;\rightarrow & = & g \circ (\epsilon F \circ F\eta)(c) \\
\text{def. (62), } \eta \text{ and } \epsilon \text{ related} \;\rightarrow & = & g \circ \iota_F(c) \\
\text{def. (25)} \;\rightarrow & = & g \circ \mathrm{id}(\,F(c)\,) \\
 & = & g \\
g : F(c) \rightarrow d \text{ @ } \mathcal{D} \;\rightarrow & = & \mathrm{id}(\,\mathcal{D}(F(c), d)\,)(g) \\
\text{prop. (68)} \;\rightarrow & = & \mathrm{id}(\,F^*(c, d)\,)(g) \\
\text{def. (25)} \;\rightarrow & = & \iota_{F^*}(c, d)(g)
\end{array}
$$

So it remains to show that $\alpha \circ \beta = \iota_{G^*}$. Similarly, given $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $f : c \rightarrow G(d)$ @ $\mathcal{C}$, we need to show that $(\alpha \circ \beta)(c, d)(f) = \iota_{G^*}(c, d)(f)$:

$$
\begin{array}{rcl}
(\alpha \circ \beta)(c, d)(f) & = & (\,\alpha(c, d) \circ \beta(c, d)\,)(f) \;\leftarrow\; \text{def. (27)} \\
\circ \text{ on } \mathbf{Set} \;\rightarrow & = & \alpha(c, d)(\,\beta(c, d)(f)\,) \\
\text{def. (40)} \;\rightarrow & = & \alpha(c, d)(\,\epsilon_d \circ F(f)\,) \\
\text{def. (39)} \;\rightarrow & = & G(\,\epsilon_d \circ F(f)\,) \circ \eta_c \\
G \text{ functor} \;\rightarrow & = & G(\epsilon_d) \circ (G \circ F)(f) \circ \eta_c \\
\text{prop. (55)} \;\rightarrow & = & G(\epsilon_d) \circ \eta_{G(d)} \circ f \\
\text{def. (31) and (30)} \;\rightarrow & = & (G\epsilon \circ \eta G)(d) \circ f \\
\text{def. (62), } \eta \text{ and } \epsilon \text{ related} \;\rightarrow & = & \iota_G(d) \circ f \\
\text{def. (25)} \;\rightarrow & = & \mathrm{id}(\,G(d)\,) \circ f \\
 & = & f \\
f : c \rightarrow G(d) \text{ @ } \mathcal{C} \;\rightarrow & = & \mathrm{id}(\,\mathcal{C}(c, G(d))\,)(f) \\
\text{prop. (69)} \;\rightarrow & = & \mathrm{id}(\,G^*(c, d)\,)(f) \\
\text{def. (25)} \;\rightarrow & = & \iota_{G^*}(c, d)(f)
\end{array}
$$

$\diamond$

Proposition 75 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Let ϵ be a counit of (F, G) and $\beta : G^* \Rightarrow F^*$ be the natural transformation associated with ϵ, as per (40). Then β is a natural isomorphism with inverse $\alpha : F^* \Rightarrow G^*$ associated with the related unit η.*

Remark: Recall from proposition (64) that every counit ϵ of (F, G) has a unique related unit η and $\alpha : F^* \Rightarrow G^*$ is therefore well-defined by virtue of (39).
Proof
Saying that β is a natural isomorphism with inverse α is the same as saying that

α is a natural isomorphism with inverse β. So the result follows immediately from proposition (74), since η and ϵ are related unit and counit of (F, G) and $\alpha : F^* \Rightarrow G^*$ is associated with η while $\beta : G^* \Rightarrow F^*$ is associated with ϵ. $\diamond$

Proposition 76 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are locally small categories. Then we have the implication:*

$$F \dashv G \;\Rightarrow\; F^* \simeq G^*$$

where F^* and G^* denote the left-hand-side and right-hand-side functors respectively associated with (F, G), as per definitions (37) and (38). In other words, if (F, G) is an adjunction then F^* and G^* are naturally isomorphic.

Proof
We assume that $F \dashv G$, i.e. that (F, G) is an adjunction. In other words we assume that (F, G) has a unit $\eta : I_{\mathcal{C}} \Rightarrow G \circ F$ as per definition (32) and we need to show that $F^* \simeq G^*$ i.e. that the right-hand-side functor F^* is isomorphic to the left-hand-side functor G^* as per definition (13). Note that in order for the statement $F^* \simeq G^*$ to make sense, we need F^* and G^* to be objects of a common category. Since both F^* and G^* are functors between the categories $\mathcal{C}^{op} \times \mathcal{D}$ and **Set**, it is understood that the common category under consideration is the functor category $[\, \mathcal{C}^{op} \times \mathcal{D} \,, \mathbf{Set} \,]$ as per definition (28). From definition (13), in order to prove that $F^* \simeq G^*$ we need to show the existence of an isomophism $\alpha : F^* \Rightarrow G^*$. However, having assumed that (F, G) has a unit η, we can consider the associated natural transformation $\alpha : F^* \Rightarrow G^*$ as per definition (39) which is a natural isomorphism as follows from (74). $\diamond$

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