Adjunction definitions

Cristvo Gomes Ferreira

November 23, 2019

Abstract

We are going to prove that three definitions of adjunction are equivalent.

1 Definitions

1.1 Prelude

For this lecture, we assume the following:

- $\mathcal{C}$, $\mathcal{D}$ or actually any calligraphed letter denotes a category.
- $Hom(x, y)$ is the set of arrows between objects x and y.
- I is the identity functor ($Ic = c$, $If = f$).
- Basic category theory identities are trivial. [1]

1.2 Adjunction

Let F and G such that:

$$\mathcal{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

We say that F is left adjoint[2] to G, written $F \dashv G$, when some rules apply.

There are several equivalent definitions of the rules in this relation, we will prove the equivalence between two of them.

Definition 1 (Isomorphism collection)**.**
$F \dashv G$ if there is a collection of functions $\phi_{x,y} \colon Hom(Fx, y) \to Hom(x, Gy)$ that forms a natural isomorphism, i.e.:

- $\phi_{x,y}$ has an inverse $\phi^{-1}_{x,y}$;
- The following diagrams commute:

[1] Namely, if F is a functor, $F(g \circ h) = Fg \circ Fh$, and $Fid_a = id_{Fa}$

[2] or F has a right adjoint G, or G is right adjoint to F, or other trivially equivalent expression.

$$
\begin{tikzcd}
Hom(Fa,b) \arrow[r, "\phi_{a,b}"] \arrow[d, "Hom(Ff,b)"] & Hom(a,Gb) \arrow[d, "Hom(f,Gb)"] \\
Hom(Fa',b) \arrow[r, "\phi_{a',b}"] & Hom(a',Gb)
\end{tikzcd}
\qquad
\begin{tikzcd}
Hom(Fa,b) \arrow[r, "\phi_{a,b}"] \arrow[d, "Hom(Fa,f)"] & Hom(a,Gb) \arrow[d, "Hom(a,Gf)"] \\
Hom(Fa,b') \arrow[r, "\phi_{a,b'}"] & Hom(a,Gb')
\end{tikzcd}
$$

Definition 2 (Universal morphism)**.**
$F \dashv G$ if there is a natural transformation $\epsilon\colon FG \stackrel{\cdot}{\longrightarrow} I$ such that each $epsilon_c\colon FGc \to c$ is universal from Fc to c, i.e., for all morphisms f there is a unique g such that:

$$
\begin{tikzcd}
d \arrow[d, dotted, "\exists! g"] & Fd \arrow[d, dotted, "Fg"] \arrow[dr, "f"] & \\
Gc & FGc \arrow[r, "\epsilon_c"] & c
\end{tikzcd}
$$

Definition 3 (Natural units)**.**
$F \dashv G$ if there are two natural transformations $\eta\colon I \stackrel{\cdot}{\longrightarrow} GF$ and $\epsilon\colon FG \stackrel{\cdot}{\longrightarrow} I$ such that $G\epsilon \circ \eta_G = I$, $\epsilon_F \circ F\eta = I$.

2 Main section

We are going to prove that all these definitions are equivalent, by proving $definition\ 1 \Rightarrow definition\ 2 \Rightarrow definition\ 3 \Rightarrow definition\ 1$

2.1 Proofs

2.1.1 $1 \Rightarrow 2$

We have a collection of isomorphisms $\phi_{x,y}$, such that the naturality squares in 1 commute.

We want to define a natural transformation $\epsilon\colon FG \stackrel{\cdot}{\longrightarrow} I$, such that for each arrow $f\colon Fd \to c$ there exists only one $g\colon d \to Gc$ such that $\epsilon_c \circ Fg = f$.

$$
\begin{tikzcd}
d \arrow[d, dotted, "\exists! g"] & Fd \arrow[d, dotted, "Fg"] \arrow[dr, "f"] & \\
Gc & FGc \arrow[r, "\epsilon_c"] & c
\end{tikzcd}
$$

We start by taking the arrow $f\colon Fd \to c$

$$
\begin{tikzcd}
d & Fd \arrow[dr, "f"] & \\
Gc & FGc & c
\end{tikzcd}
$$

A motivation for defining ϵ, is recognizing that $\epsilon_c \in Hom(FGc, c)$ and $\phi_{Gc,c}$ defines an isomorphism between $Hom(Gc, Gc)$ and $Hom(FGc, c)$. So $\phi_{Gc,c}(\epsilon_c) \in Hom(Gc, Gc)$.

The canonical arrow to take from $Hom(Gc, Gc)$ is id_{Gc}, so we'll define $\epsilon_c = \phi^{-1}_{Gc,c}(id_{Gc})$, and see how that goes (the proof that ϵ is a natural transformation is lemma 2).

$$\begin{tikzcd}
d & Fd \arrow[dr, "f"] & \\
Gc & FGc \arrow[r, "\phi^{-1}_{Gc,c}(id_{Gc})"'] & c
\end{tikzcd}$$

The motivation for defining the unique g for each f is actually quite intuitive: g is just $\phi_{d,c}(f)$.

$$\begin{tikzcd}
d \arrow[d, "\phi_{d,c}(f)"'] & Fd \arrow[d, "F(\phi_{d,c}(f))"'] \arrow[dr, "f"] & \\
Gc & FGc \arrow[r, "\phi^{-1}_{Gc,c}(id_{Gc})"'] & c
\end{tikzcd}$$

We actually have to prove that this diagram commutes, but note that, by lemma 1, $\phi^{-1}_{Gc,c}(id_{Gc}) \circ F(\phi_{d,c}(f)) = \phi^{-1}_{d,c}(\phi_{d,c}(f)) = f$, so it does!

At last, we have to prove the uniqueness of our g, but note that, if $\phi^{-1}_{d,c}(g') = \phi^{-1}_{Gc,c}(id_{Gc}) \circ Fg' = f = \phi^{-1}_{d,c}(\phi_{d,c}(f))$ then $g' = \phi_{d,c}(f)$ by injectiveness of $\phi^{-1}_{d,c}$.

And with that we're done. □

2.1.2 $2 \Rightarrow 3$

We have a natural transformation $\epsilon \colon FG \xrightarrow{\bullet} I$, such that for each arrow $f \colon Fd \to c$ there exists only one $g \colon d \to Gc$ such that $\epsilon_c \circ Fg = f$.

$$\begin{tikzcd}
d \arrow[d, dotted, "\exists! g"] & Fd \arrow[d, dotted, "Fg"] \arrow[dr, "f"] & \\
Gc & FGc \arrow[r, "\epsilon_c"] & c
\end{tikzcd}$$

We want to get two natural transformations $\epsilon' \colon FG \xrightarrow{\bullet} I$, $\eta \colon I \xrightarrow{\bullet} GF$, such that $G\epsilon' \circ \eta_G = I$, $\epsilon'_F \circ F\eta = I$.

Let us define $\epsilon' = \epsilon$ (and stop using the $'$ notation) and η_d the unique g for $f = id_{Fd}$. We now have to prove that:

1. $\epsilon_F \circ F\eta = I$
2. $G\epsilon \circ \eta_G = I$
3. η is a natural transformation

Item 1 is immediate, by definition of η.

■

We prove item 2 by noting that

$$
\begin{aligned}
\epsilon_a \circ F(G\epsilon_a \circ \eta_{Ga}) &= \\
\epsilon_a \circ FG\epsilon_a \circ F\eta_{Ga} &= && \text{(by naturality of } \epsilon) \\
\epsilon_a \circ \epsilon_{FGa} \circ F\eta_{Ga} &= && \text{(by item 1)} \\
\epsilon_a \circ id_{FGa} &= \\
\epsilon_a &= \\
\epsilon_a \circ id_{FGa} &= \\
\epsilon_a \circ Fid_{Ga}
\end{aligned}
$$

hence, by lemma 3 on $G\epsilon_a \circ \eta_{Ga}$ and id_{Ga}, we get $G\epsilon_a \circ \eta_{Ga} = id_{Ga}$.

■

To prove item 3 we need to prove that

$$
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & GFa \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle GFf} \\
b & \xrightarrow{\eta_b} & GFb
\end{array}
$$

commutes, i.e., $GFf \circ \eta_a = \eta_b \circ f$. We prove this by resorting to Lemma 3 again, but this time on

$$
\begin{aligned}
\epsilon_{Fb} \circ F(GFf \circ \eta_a) &= \\
\epsilon_{Fb} \circ FGFf \circ F\eta_a &= && \text{(by naturality of } \epsilon) \\
Ff \circ \epsilon_{Fa} \circ F\eta_a &= && \text{(by item 1)} \\
Ff \circ id_{Fa} &= \\
Ff &= \\
id_{Fb} \circ Ff &= && \text{(by item 1)} \\
\epsilon_{Fb} \circ F\eta_b \circ Ff &= \\
\epsilon_{Fb} \circ F(\eta_b \circ f)
\end{aligned}
$$

This proves that $GFf \circ \eta_a = \eta_b \circ f$ which is exactly the equation defined by the naturality square.

□

2.1.3 $3 \Rightarrow 1$

We have two natural transformations $\epsilon\colon FG \xrightarrow{\cdot} I$, $\eta\colon I \xrightarrow{\cdot} GF$, such that $G\epsilon \circ \eta_G = I$, $\epsilon_F \circ F\eta = I$.

We want to get a collection of isomorphisms $\phi_{x,y}\colon Hom(Fx,y) \xrightarrow{\simeq} Hom(x,Gy)$ that is natural in x and in y.

We do this by defining $\phi_{x,y}$, $\phi^{-1}_{x,y}$ as follows.

$$
\begin{array}{ccc}
Hom(Fx,y) & \underset{\phi^{-1}_{x,y}}{\overset{\phi_{x,y}}{\rightleftarrows}} & Hom(x,Gy) \\
f & \longmapsto & Gf \circ \eta_x \\
\epsilon_y \circ Fg & \longleftarrow\!\shortmid & g
\end{array}
$$

We now have to prove:

1. $\phi_{x,y}$ is an isomorphism
2. The first naturality square in definition 1 commutes.
3. The second naturality square in definition 1 commutes.

We prove item 1 by showing that our definitions of $\phi_{x,y}$ and $\phi^{-1}_{x,y}$ are actually inverses. This happens because:

$$\begin{aligned}
\phi_{x,y}(\phi^{-1}_{x,y}(g)) &= \\
G(\phi^{-1}_{x,y}(g)) \circ \eta_x &= \\
G(\epsilon_y \circ Fg) \circ \eta_x &= \\
G\epsilon_y \circ GFg \circ \eta_x &= \quad \text{(by naturality of } \eta) \\
G\epsilon_y \circ \eta_{Gy} \circ g &= \\
id_{Gy} \circ g &= \\
g &
\end{aligned}$$

and

$$\begin{aligned}
\phi^{-1}_{x,y}(\phi_{x,y}(f)) &= \\
\epsilon_y \circ F(\phi_{x,y}(f)) &= \\
\epsilon_y \circ F(Gf \circ \eta_x) &= \\
\epsilon_y \circ FGf \circ F\eta_x &= \quad \text{(by naturality of } \epsilon) \\
f \circ \epsilon_{Fx} \circ F\eta_x &= \\
f \circ id_{Fx} &= \\
f &
\end{aligned}$$

■

To prove item 2, let us take a look at the first naturality square for ϕ:

$$\begin{tikzcd}
Hom(Fa,b) \arrow[r, "\phi_{a,b}"] \arrow[d, "Hom(Ff,b)"] & Hom(a,Gb) \arrow[d, "Hom(f,Gb)"] \\
Hom(Fa',b) \arrow[r, "\phi_{a',b}"] & Hom(a',Gb)
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & \phi_{a,b}(h) \arrow[d, mapsto] \\
h \circ Ff \arrow[r, mapsto] & \phi_{a',b}(h \circ Ff) = \phi_{a,b}(h) \circ f
\end{tikzcd}$$

Replacing with our definition for ϕ, we get that we have to prove:

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & Gh \circ \eta_a \arrow[d, mapsto] \\
h \circ Ff \arrow[r, mapsto] & G(h \circ Ff) \circ \eta_{a'} = Gh \circ \eta_a \circ f
\end{tikzcd}$$

But this is true, for:

$$
\begin{aligned}
G(h \circ Ff) \circ \eta_{a'} &= \\
Gh \circ GFf \circ \eta_{a'} &= \quad \text{(by naturality of } \eta\text{)} \\
Gh \circ \eta_a \circ f &
\end{aligned}
$$

■

To prove item 3, let us take a look at the second naturality square for ϕ:

$$
\begin{tikzcd}
Hom(Fa,b) \arrow[r, "\phi_{a,b}"] \arrow[d, "Hom(Fa,f)"] & Hom(a,Gb) \arrow[d, "Hom(a,Gf)"] \\
Hom(Fa,b') \arrow[r, "\phi_{a,b'}"] & Hom(a,Gb')
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & \phi_{a,b}(h) \arrow[d, mapsto] \\
f \circ h \arrow[r, mapsto] & \phi_{a,b'}(f \circ h) = Gf \circ \phi_{a,b}(h)
\end{tikzcd}
$$

Replacing with our definition for ϕ, we get that we have to prove:

$$
\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & Gh \circ \eta_a \arrow[d, mapsto] \\
f \circ h \arrow[r, mapsto] & G(f \circ h) \circ \eta_a = Gf \circ Gh \circ \eta_a
\end{tikzcd}
$$

This is trivial. □

2.2 Conclusions

So... Why did we go through all of this? What do we have now that we didn't before?

The answer is two-fold: We now have more understanding of what an adjunction is, and we have more tools to work with adjunctions in the future.

We understand adjunctions better because we are able to see them through different angles now.

We no longer think an adjunction is just a relation between Hom-sets, we know that it also defines a family of universal morphisms (and we also know *in what way* it defines such a family).

And that also gives us more power: We can now use any of the lenses we have to see adjunctions to attack any problem related to them.

That will be essential, when we start attacking the relation between monads and adjunctions next class.

3 Appendix

3.1 Lemmas

Some lemmas were used above and are defined here to keep the flow of the proofs steady.

Lemma 1. *For any arrow* $g\colon x \to Gy$, $\phi^{-1}_{x,y}(g) = \phi^{-1}_{Gy,y}(id_{Gy}) \circ Fg$

The proof for this lemma can be easily obtained by chasing the first naturality diagram for ϕ^{-1} (which is just the first naturality diagram for ϕ reversed at the x-axis) at $a = Gy, a' = x, b = y$.

$$
\begin{tikzcd}
Hom(Gy,Gy) \arrow[r, "\phi^{-1}_{Gy,y}"] \arrow[d, "Hom(g,Gy)"] & Hom(FGy,y) \arrow[d, "Hom(Fg,y)"] \\
Hom(x,Gy) \arrow[r, "\phi^{-1}_{x,y}"'] & Hom(Fx,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
id_{Gy} \arrow[r, mapsto] \arrow[d, mapsto] & \phi^{-1}_{Gy,y}(id_{Gy}) \arrow[d, mapsto] \\
id_{Gy} \circ g = g \arrow[r, mapsto] & \phi^{-1}_{x,y}(g) = \phi^{-1}_{Gy,y}(id_{Gy}) \circ Fg
\end{tikzcd}
$$

□

Lemma 2. $\epsilon_c = \phi^{-1}_{Gc,c}(id_{Gc})\colon FG \stackrel{\cdot}{\longrightarrow} I$ *is a natural transformation*

In this case, the definition of natural transformation is that the following square commutes for all $c, c', f\colon c \to c'$:

$$
\begin{tikzcd}
FGc \arrow[r, "\epsilon_c"] \arrow[d, "FGf"] & c \arrow[d, "f"] \\
FGc' \arrow[r, "\epsilon_{c'}"] & c'
\end{tikzcd}
$$

Or, in algebraic terms, $\epsilon_{c'} \circ FGf = f \circ \epsilon_c$.

We get to that point by applying lemma 1 at $x = Gc, y = c', g = Gf$ and get $\epsilon_{c'} \circ FGf = \phi^{-1}_{Gc',c'}(id_{Gc'}) \circ FGf = \phi^{-1}_{Gc,c'}(Gf)$, and then chasing the second naturality diagram for ϕ^{-1} at $a = Gc, b = c, b' = c'$ to get:

$$
\begin{tikzcd}
Hom(Gc,Gc) \arrow[r, "\phi^{-1}_{Gc,c}"] \arrow[d, "Hom(Gc,Gf)"] & Hom(FGc,c) \arrow[d, "Hom(FGc,f)"] \\
Hom(Gc,Gc') \arrow[r, "\phi^{-1}_{Gc,c'}"'] & Hom(FGc,c')
\end{tikzcd}
\qquad
\begin{tikzcd}
id_{Gc} \arrow[r, mapsto] \arrow[d, mapsto] & \phi^{-1}_{Gc,c}(id_{Gc}) \arrow[d, mapsto] \\
Gf \arrow[r, mapsto] & \phi^{-1}_{Gc,c'}(Gf) = f \circ \phi^{-1}_{Gc,c}(id_{Gc})
\end{tikzcd}
$$

Hence, $\epsilon_{c'} \circ FGf = \phi^{-1}_{Gc',c'}(id_{Gc'}) \circ FGf = \phi^{-1}_{Gc,c'}(Gf) = f \circ \phi^{-1}_{Gc,c}(id_{Gc}) = f \circ \epsilon_c$. □

Lemma 3. *If two arrows $f_1, f_2 \colon a \to Gb$ are such that $\epsilon_b \circ Ff_1 = \epsilon_b \circ Ff_2$, then $f_1 = f_2$.*

We prove this lemma by observing that we can put both f_1 and f_2 in the universal diagram:

$$
\begin{tikzcd}
a \arrow[d, "f_1"', "f_2"] & Fa \arrow[d, "Ff_1"', "Ff_2"] \arrow[dr, "\epsilon_b \circ Ff_1 = \epsilon_b \circ Ff_2"] & \\
Gb & FGb \arrow[r, "\epsilon_b"] & b
\end{tikzcd}
$$

Due to the uniqueness of g for any f in the universal diagram, both arrows must be the same.