PARTICLE SWARM OPTIMISATION

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Nature-Inspired Learning Algorithms (7CCSMBIM)

Outline

Learning Objectives

- To get an idea of swarm intelligence.
- To get the concept of particle swarm optimisation and know how it works.
- To apply the particle swarm optimisation to optimisation problems.

Introduction

Swarm Intelligence

- Swarm intelligence is an artificial intelligence technique based on the study of collective behaviour in decentralised, self-organised systems.
- Swarm intelligence systems are typical made up of a population of simple agents interacting locally with one another and with their environment.
- Although there is no centralised control structure, local interactions between such agents often lead to the emergence of global behaviour.
- Examples of systems in nature, including social insect colonies, bird flocking, fish schooling and animal herding.

Self-organisation

- A set of dynamical mechanisms whereby structures appear at the global level of a system from interactions of its lower-level components.

Self-organisation

- A set of dynamical mechanisms whereby structures appear at the global level of a system from interactions of its lower-level components.
- **Four basic ingredients:**
  - **Positive feedback (amplification)**: To show the right of direction to the food source (optimal solution); to reinforce those portions of good solutions that contribute to the quality of these solutions.
  - **Negative feedback:** to introduce a time scale into the algorithm through pheromone evaporation, to prevent premature convergence (stagnation), for counter-balance and stabilisation.
  - **Amplification of fluctuation:** Randomness or errors, e.g., lost ant foragers can find new food sources. An element moves more randomly to search for a solution and then amplified by a positive feedback loop.
  - **Multiple interactions:** Direct or indirect communication (e.g., modification of the environment).

Why do Animals Swarm?

- Defence against predators.
  - Enhance the detection of predators.
  - Minimise the chance of being captured.
- Enhance success in foraging.
- Better chances to find a mate.
- Decrease of energy consumption.

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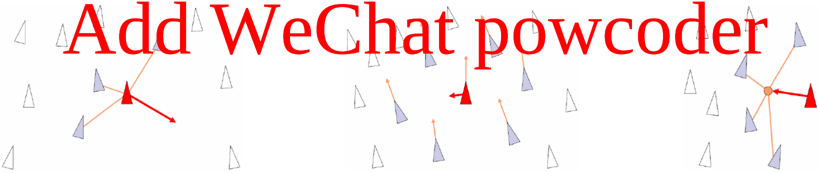
Coordinated Collective Behaviour

- Reynolds (1987) proposed a behaviour model to interpret bird flocking, fish schooling and animal herds.
- Biologically and physically sound assumption:
  1. Individual has only local knowledge.
  2. Has certain cognitive capabilities.
  3. Bound by laws of physics.

- Comply with only three simple rules.

Coordinated Collective Behaviour Reynolds (1987) Behaviour Model

1. **Separation**: each agent tries to move away from its neighbours if they are too close.
2. **Alignment**: each agent steers towards the averaging heading of its neighbours.
3. **Cohesion**: each agent tries to go towards the average position of its neighbours.



(a) Separation (b) Alignment (c) Cohesion

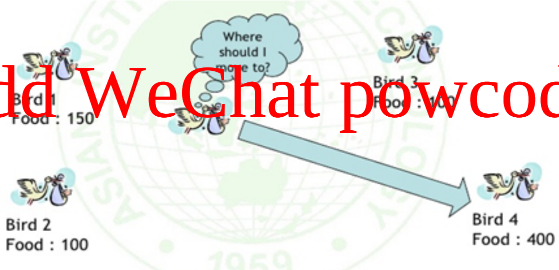
Building a Metaheuristic

- They expand the swarm behavioural model to n-dimensional psychosocial space without constraints of physical laws.
- A swarm consists of a set of particles, where each particle represents a potential solution for an n-dimensional optimisation problem.
- Particles are flown through the hyperspace, where position of each particle is changed according to its own experience and that of its neighbours.

Basic Particle Swarm Optimisation

- **Particle Swarm Optimisation (PSO):** A numerical population-based optimisation techniques discovers optimal regions of a high dimensional search space through collective behaviour of individuals.

- **Particle Swarm Optimisation (PSO):** A numerical population-based optimisation techniques discovers optimal regions of a high dimensional search space through collective behaviour of individuals.
- **Ingredients:**
  - *A particle*: an individual (a potential solution)
  - *A swarm*: a population.
  - *Update rules*.



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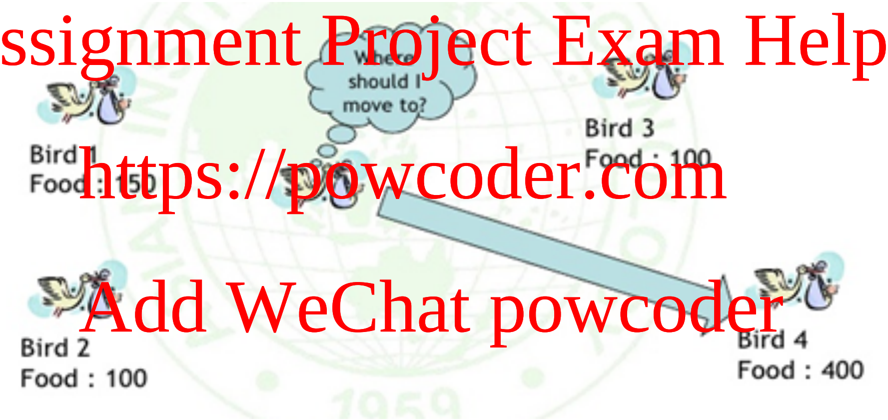

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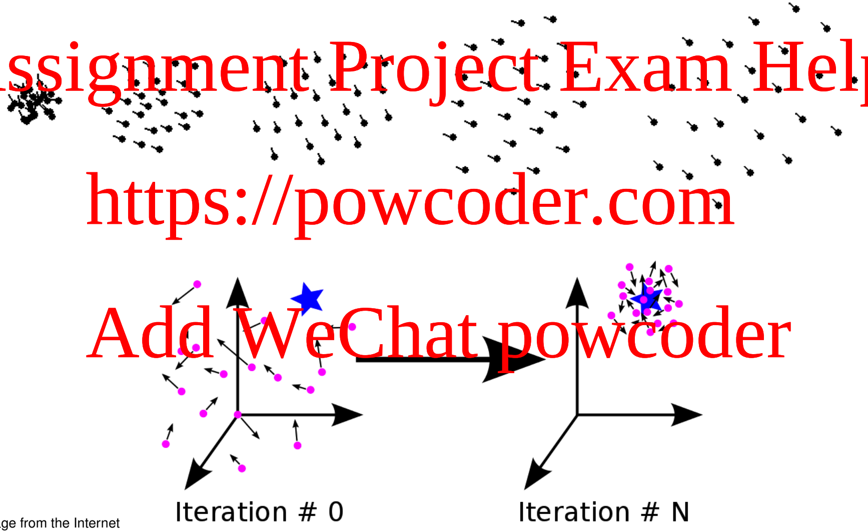

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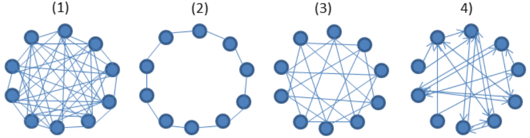
Notation

- $\mathbf{x}_i(t) = [x_{i1}, \cdots, x_{in_x}]$: the i^{th} particle (individual).
- n_x: number of elements in each particle.
- n_s: size of swarm (number of particles in the swarm).
- n_t: maximum number of iterations.
- $\hat{\mathbf{y}}(t) = [\hat{y}_1(t), \cdots, \hat{y}_{n_x}(t)]$: the global best position since the first generation.
- $\hat{\mathbf{y}}_i(t) = [\hat{y}_{i1}(t), \cdots, \hat{y}_{in_x}(t)]$: the local best position since the first generation.
- $\mathbf{y}_i(t) = [y_{i1}(t), \cdots, y_{in_x}(t)]$: the personal best position since the first generation.
- $\mathscr{N}_i$: the set of neighbourhoods of particle i.
- $\mathbf{x}_{\min} = [x_{1_{\min}}, \cdots, x_{n_{x\min}}]$: a vector of constants denoting the lower bound of $\mathbf{x}_i(t)$.
- $\mathbf{x}_{\max} = [x_{1_{\max}}, \cdots, x_{n_{x\max}}]$: a vector of constants denoting the upper bound of $\mathbf{x}_i(t)$.
- $\mathbf{v}_i = [v_{i1}, \cdots, v_{in_x}]$: a velocity vector.
- $r_{1j}(t)$, $r_{2j}(t) \in [0, 1]$: a random number .

- Two basic PSO algorithms:
  - The main difference is the size of their neighbourhoods.
- **Global best PSO** (*gbest* PSO):
  - Social network: star topology.
  - Neighbours of a particle: the whole swarm.
  - Particles are updated based on the social information from all particles in the swarm.
  - Social information: the best position (solution) found by the swarm.
- **Local best PSO** (*lbest* PSO):
  - Social network: ring topology.
  - Neighbours of a particle: a small number of particles in the swarm
  - Particles are updated based on the social information exchanged within the neighbourhood of the particle.
  - Social information: the local best position (solution) within the neighbourhood.



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Graphical representation of (1) fully connected, (2) ring, (3) von Neumann and (4) random topology

Global Best PSO

- Size of Swarm: n_s particles.
- The (**global**) best position found in the swarm:

$$\hat{\mathbf{y}}(t) = [\hat{y}_1(t), \cdots, \hat{y}_{n_x}(t)]$$

- The personal best position since the first generation:

$$\mathbf{y}_i(t) = [y_{i1}(t), \cdots, y_{in_x}(t)]$$

- **Velocity update rule:**

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)\big(y_{ij}(t) - x_{ij}(t)\big) + c_2 r_{2j}(t)\big(\hat{y}_j(t) - x_{ij}(t)\big) \quad (1)$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are acceleration constants; $r_{1j}(t)$ and $r_{2j}(t) \in [0, 1]$ are random numbers.

- **Particle update rule:**

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1),\ i = 1, \cdots, n_s$$

where $\mathbf{x}_i(t) = [x_{i1}, \cdots, x_{in_x}]$ and $\mathbf{v}_i = [v_{i1}, \cdots, v_{in_x}]$.

Global Best PSO

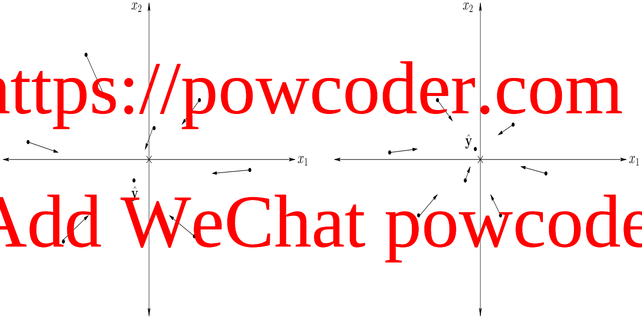
- Update of personal best position:

$$\mathbf{y}_i(t+1) = \begin{cases} \mathbf{y}_i(t) & \text{if } f(\mathbf{x}_i(t+1)) \geq f(\mathbf{y}_i(t)) \\ \mathbf{x}_i(t+1) & \text{otherwise} \end{cases}, i = 1, \cdots, n_s$$

- The global best position:

$$\hat{\mathbf{y}}(t+1) \in \{\mathbf{y}_1(t+1), \cdots, \mathbf{y}_{n_s}(t+1)\} | f(\hat{\mathbf{y}}(t+1)) = \min\{f(\mathbf{y}_1(t+1)), \cdots, f(\mathbf{y}_{n_s}(t+1))\}$$

- What is the swarm size?
- How many elements does each particle have?
- How many elements does the velocity vector have?
- What is the initial velocity, $\mathbf{v}(0)$?



(a) At time $t = 0$

(b) At time $t = 1$

Figure 1: Multi-particle *gbest* PSO illustration. '$\times$' indicates the optimum.

***gbest* PSO algorithm**

Create and initialise an n_x-dimensional swarm, $\mathbf{y}_i(0)$ and $\hat{\mathbf{y}}(0)$;
Set $t = 0$, $\mathbf{v}_i(0) = \mathbf{0}$; Choose values for c_1 and c_2;

while STOP-CRIT **do**
    for *each particle* $i = 1, \cdots, n_s$ **do**
        Update the velocity, $v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)\big(y_{ij}(t) - x_{ij}(t)\big) + c_2 r_{2j}(t)\big(\hat{y}_j(t) - x_{ij}(t)\big)$;
        Update the position of particles, $\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1)$;
    end
    for *each particle* $i = 1, \cdots, n_s$ **do**
        if $f(\mathbf{x}_i(t+1)) < f(\mathbf{y}_i(t))$ **then**
            $\mathbf{y}_i(t+1) = \mathbf{x}_i(t+1)$;
        else
            $\mathbf{y}_i(t+1) = \mathbf{y}_i(t)$;
        end
        if $f(\mathbf{y}_i(t+1)) < f(\hat{\mathbf{y}}(t))$ **then**
            $\hat{\mathbf{y}}(t+1) = \mathbf{y}_i(t+1)$;
        end
    end
    $t \leftarrow t + 1$;
end

Table 1: Pseudocode of *gbest* PSO algorithm.

Global Best PSO

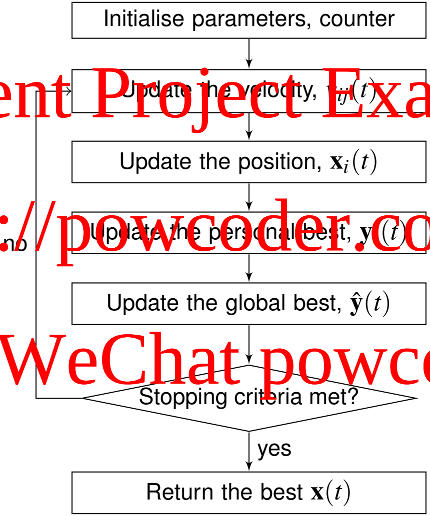
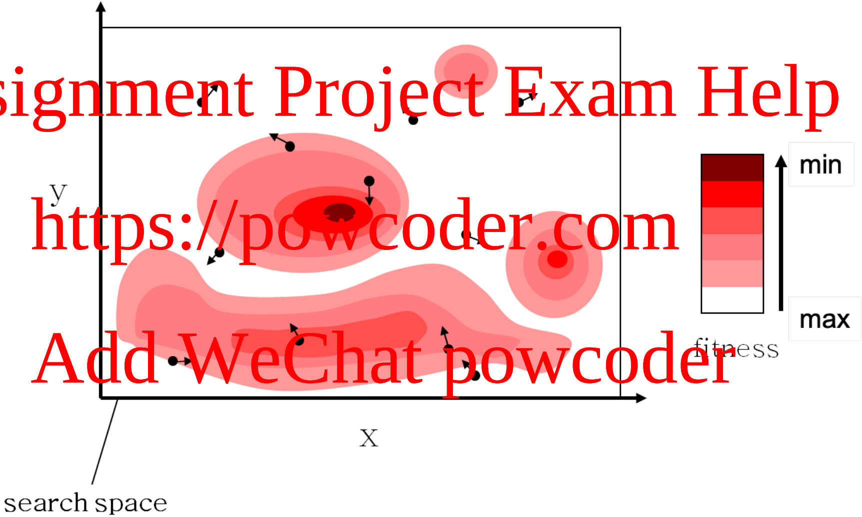
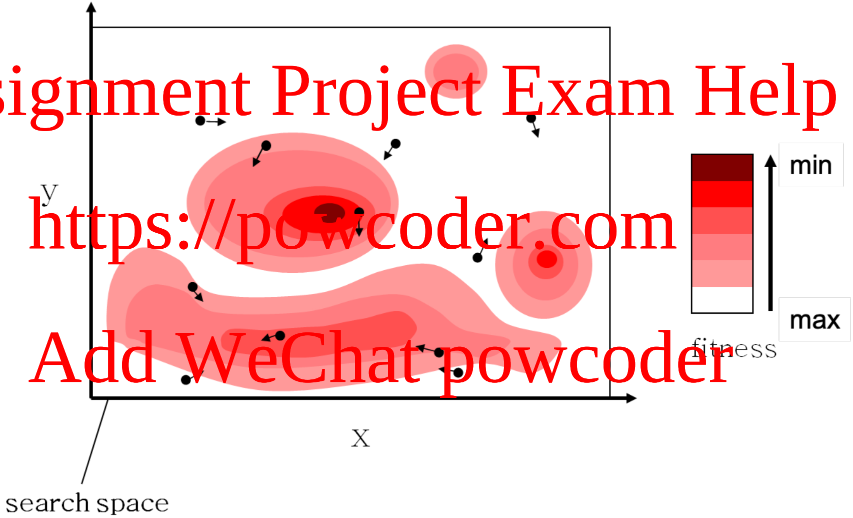


Figure 2: Flowchart of *gbest* PSO algorithm.

Global Best PSO

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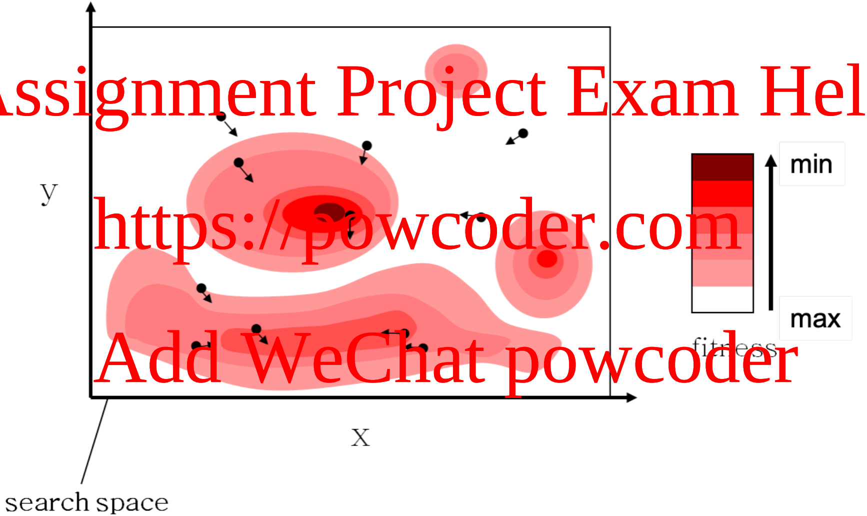
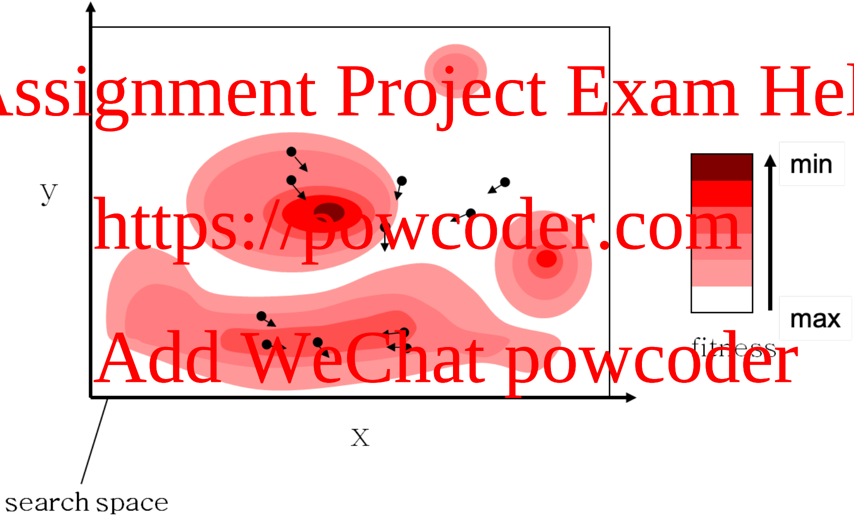
Global Best PSO

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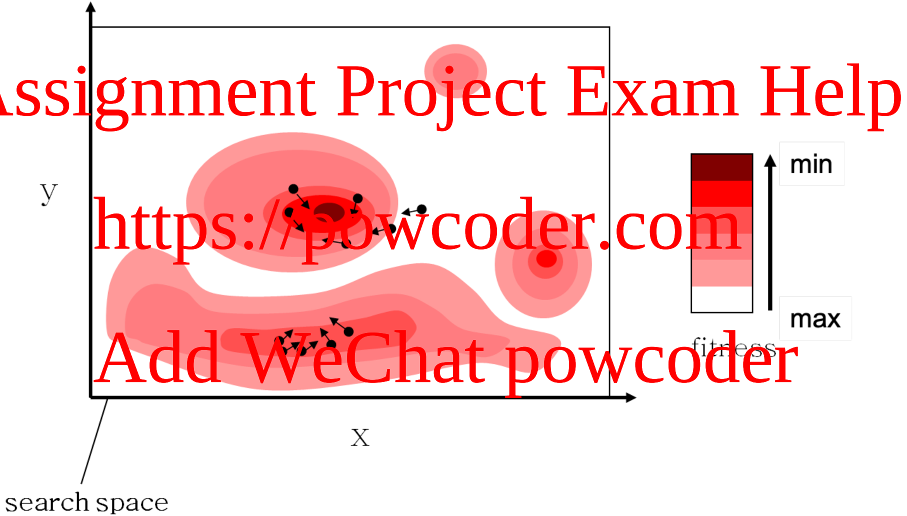
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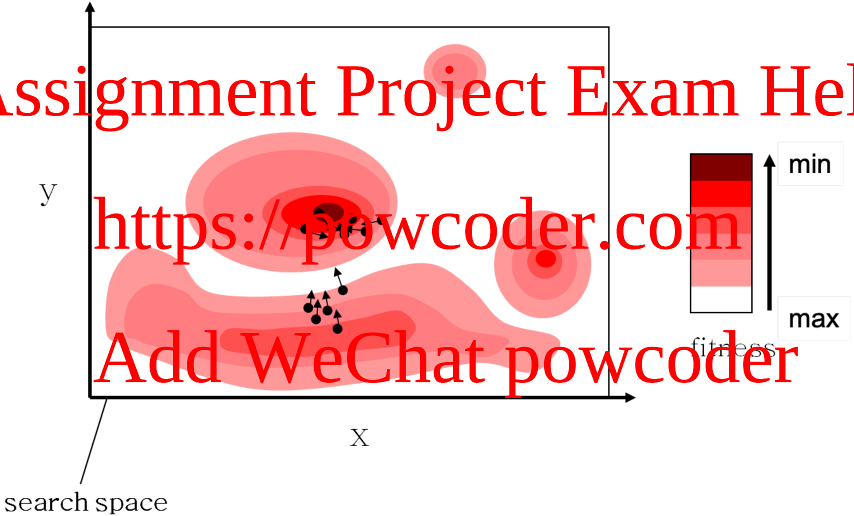
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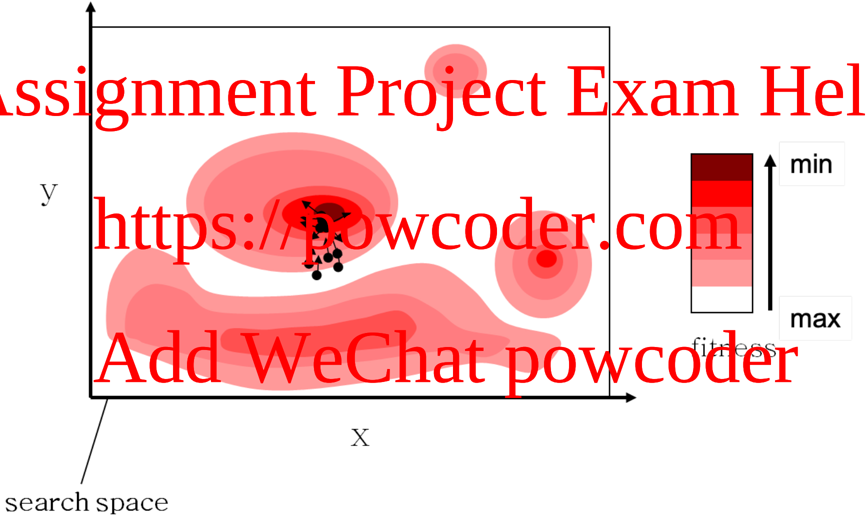
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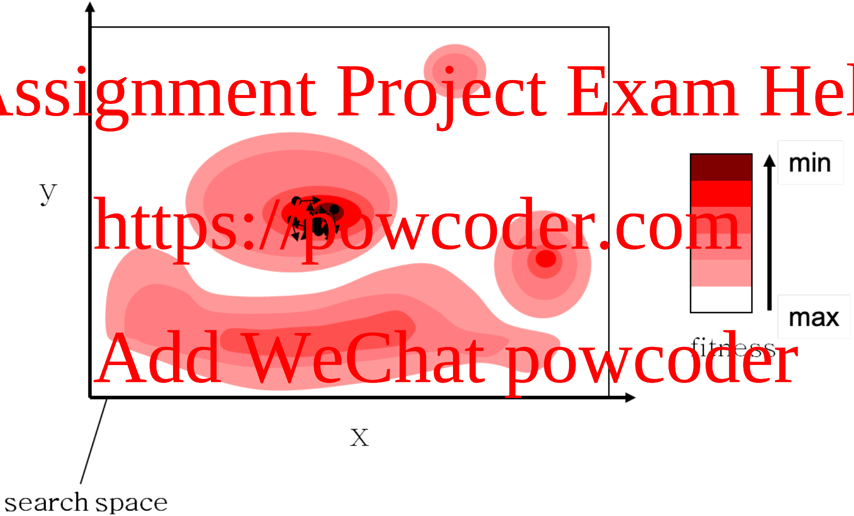
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Local Best PSO

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Local Best PSO

- Size of Swarm: n_s particles.
- The (**local**) best position found in the swarm (found by the neighbourhood of particle i):

$$\hat{\mathbf{y}}_i(t) = [\hat{y}_{i1}(t), \cdots, \hat{y}_{in_x}(t)]$$

- The personal best position since the first generation:

$$\mathbf{y}_i(t) = [y_{i1}(t), \cdots, y_{in_x}(t)]$$

- **Velocity update rule:**

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)\left(y_{ij}(t) - x_{ij}(t)\right) + c_2 r_{2j}(t)\left(\hat{y}_{ij}(t) - x_{ij}(t)\right) \tag{2}$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are acceleration constants; $r_{1j}(t)$ and $r_{2j}(t) \in [0, 1]$ are random numbers.

- **Particle update rule:**

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1),\ i = 1, \cdots, n_s$$

- The (**local**) best position found in the swarm (found by the neighbourhood of particle i):

$$\hat{\mathbf{y}}_i(t+1) \in \{\mathcal{N}_i | f(\hat{\mathbf{y}}_i(t+1)) = \min\{f(\mathbf{x}(t))\}, \forall \mathbf{x}(t) \in \mathcal{N}_i\}$$

$\mathcal{N}_i$: the set of neighbourhoods of particle i.

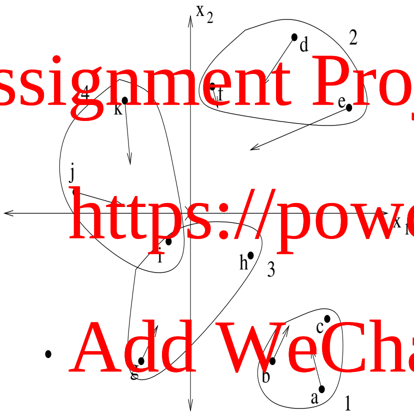
- The local best position is the neighbourhood best position.
- The personal best position since the first generation:

$$\mathbf{y}_i(t) = [y_{i1}(t), \cdots, y_{in_x}(t)]$$

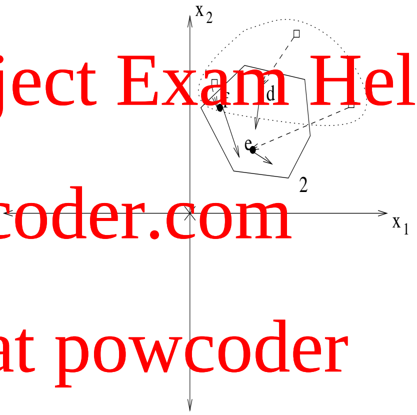
- **Particle update rule:**

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1), i = 1, \cdots, n_x$$

- Note: When the neighbourhoods of all particles are the entire swarm, the *lbest* PSO becomes the *gbest* PSO.



(a) Local Best Illustrated – Initial Swarm



(b) Local Best – Second Swarm

Figure 3: Illustration of *lbest* PSO. '$\times$' indicates the optimum. $v_{ij}(0) = 0$ except for particle f.

Local Best PSO

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***lbest* PSO algorithm**

Create and initialise an n_x-dimensional swarm, $\mathbf{y}_i(0)$ and $\hat{\mathbf{y}}_i(0)$;
Set $t = 0$, $\mathbf{v}_i(0) = \mathbf{0}$; Choose values for c_1 and c_2;

while STOP-CRIT **do**
  for *each particle* i = 1, $\cdots$, n_s **do**
    Update the velocity, $v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)(y_{ij}(t) - x_{ij}(t)) + c_2 r_{2j}(t)(\hat{y}_{ij}(t) - x_{ij}(t))$;
    Update the position of particles, $\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1)$;
  end
  for *each particle* i = 1, $\cdots$, n_s **do**
    if $f(\mathbf{x}_i(t+1)) < f(\mathbf{y}_i(t))$ **then**
      $\mathbf{y}_i(t+1) = \mathbf{x}_i(t+1)$;
    else
      $\mathbf{y}_i(t+1) = \mathbf{y}_i(t)$;
    end
  end
  for *each particle* i = 1, $\cdots$, n_s **do**
    $\hat{\mathbf{y}}_i(t+1) \in \{\mathscr{N}_i | f(\hat{\mathbf{y}}_i(\mathbf{x}(t))) = \min\{f(\mathbf{x}(t))\}, \forall \mathbf{x}(t) \in \mathscr{N}_i\}$
  end
  $t \leftarrow t+1$;
end

$n_{\mathscr{N}_i}$: number of neighbourhoods

Table 2: Pseudo Code of *lbest* PSO algorithm.

Local Best PSO

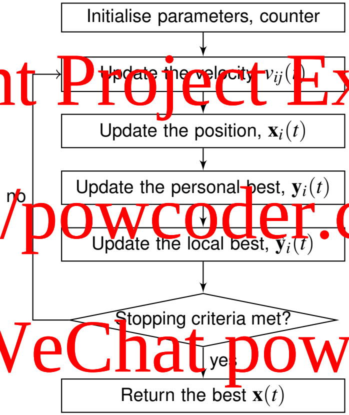

Figure 4: Flowchart of *lbest* PSO algorithm.

Selection of Neighbourhoods:

- **Particles indices:**
  - Computationally inexpensive.
  - Promotion of the spread of information irrespective of the position of the particles.
- **Spatial similarity:**
  - Computationally expensive.
  - Information of similar particles can be used for a local search.
- **Overlapping of neighbourhoods:** A particle can be a member of a number of neighbourhoods. The interconnection of neighbourhoods can promote information sharing such that all particles come to consensus faster (converge faster to a single solution).

***lbest* vs *gbest* PSO:**

	lbest PSO	*gbest* PSO
computational demand	higher	lower
diversity	larger	smaller
convergence speed	slower	faster
trapped in local minima	less likely	more likely

Velocity Components

- **Velocity Components:** previous velocity, cognition component and social component

- ***gbest* best PSO:**

$$v_{ij}(t+1) = \underbrace{v_{ij}(t)}_{\text{previous velocity}} + \underbrace{c_1 r_{1j}(t)\left(y_{ij}(t) - x_{ij}(t)\right)}_{\text{cognitive component}} + \underbrace{c_2 r_{2j}(t)\left(\hat{y}_j(t) - x_{ij}(t)\right)}_{\textit{social component}}$$

- ***lbest* best PSO:**

$$v_{ij}(t+1) = \underbrace{v_{ij}(t)}_{\text{previous velocity}} + \underbrace{c_1 r_{1j}(t)\left(y_{ij}(t) - x_{ij}(t)\right)}_{\text{cognitive component}} + \underbrace{c_2 r_{2j}(t)\left(\hat{y}_{ij}(t) - x_{ij}(t)\right)}_{\textit{social component}}$$

- **Velocity Components:** previous velocity, cognition component and social component
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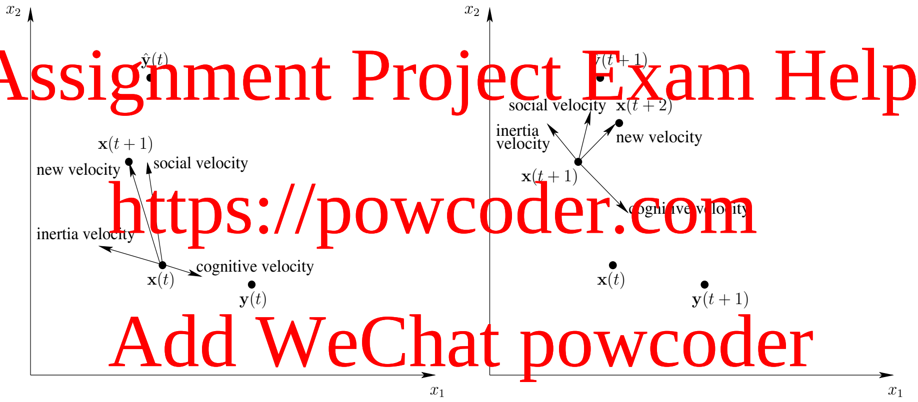
- ***lbest* best PSO:**

$$v_{ij}(t+1) = \underbrace{v_{ij}(t)}_{\text{previous velocity}} + \underbrace{c_1 r_{1j}(t)\big(y_{ij}(t) - x_{ij}(t)\big)}_{\text{cognitive component}} + \underbrace{c_2 r_{2j}(t)\big(\hat{y}_{ij}(t) - x_{ij}(t)\big)}_{\textit{social component}}$$

- Previous velocity: it is an inertia making the particle move in the same direction as in the t^{th} generation.
- Cognitive component: It is a personal influence which attempts to improve the individual by making the particle return to a previous good position.
- Social component: It is social influence which makes the particle follow the best neighbour's direction.
- Previous velocity is for exploration: Its searches new regions (new solutions) for potentially better solutions.
- Cognitive component and social component are for exploitation: It searches the previous regions (previous solutions) for better solutions.

Velocity Components



(a) Time Step t (b) Time Step $t+1$

Figure 5: Illustration of Velocity and Position Update for *gbest* PSO.

Particle Initialisation

- **Particle position:** $x_{ij}(0) = x_{\min,j} + r_j(x_{\max,j} - x_{\min,j})$, i = 1, $\cdots$, n_s; j = 1, $\cdots$, n_x.
  - $r_j \in [0, 1]$: a random number
  - $[x_{\min,j}, x_{\max,j}]$: boundary of the j-th element of particle, for all j
- **Initialised velocity:** $\mathbf{v}_i(0) = \mathbf{0}$, i = 1, $\cdots$, n_s.
- **Personal best position:** $\mathbf{y}_i(0) = \mathbf{x}_i(0)$, i = 1, $\cdots$, n_s.

Stopping Criteria

Stopping Criteria

- Maximum number of iterations has been reached.
- Acceptable solution has been found.
- No improvement has been observed over a period of iterations.
- The swarm radius is close to zero.
- The objective function slope is approximately zero.
  - Slope: $f'(t) = \frac{f(\hat{\mathbf{y}}(t)) - f(\hat{\mathbf{y}}(t-1))}{f(\hat{\mathbf{y}}(t))}$
  - algorithm terminates if $|f'(t)| \leq \varepsilon$ for a number of consecutive iterations.
  - $\varepsilon > 0$: a user-specified parameter

Swarm Radius:

1. **Maximum swarm radius:** The algorithm terminates if $R_{\max}(t) \leq \varepsilon$.
   - $\varepsilon > 0$: a user-specified parameter.
   - $R_{\max}(t) = \max_{m \in \{1,\cdots,n_s\}} ||\mathbf{x}_m(t) - \hat{\mathbf{y}}(t)||$.
   - $\hat{\mathbf{y}}(t)$ is the global best position of the *gbest* or *lbest* PSO.

2. **Particle clustering algorithm:** The algorithm terminates if $\frac{|C|}{n_s} \geq \delta$.
   - $0 < \delta \leq 1$: a user-specified parameter.
   - C: a cluster.
   - $|C|$: number of elements in cluster C.

Particle Clustering Algorithm

Initialise cluster $C = \{\hat{\mathbf{y}}(t)\}$;

for *about 5 times* **do**

  Calculate the centroid of cluster C: $\bar{\mathbf{x}}(t) = \frac{\sum_{i=1, \mathbf{x}_i(t) \in C}^{|C|} \mathbf{x}_i(t)}{|C|}$;

  for each particle i, $i = 1, \cdots, n_s$ **do**

    if $||\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)|| \leq \varepsilon$ **do**

      $C \leftarrow C \cup \{\mathbf{x}_i(t)\}$

    end

  end

end

$\varepsilon > 0$: a user-specified parameter

Table 3: Pseudo Code of Particle Clustering Algorithm.

Social Network Structure

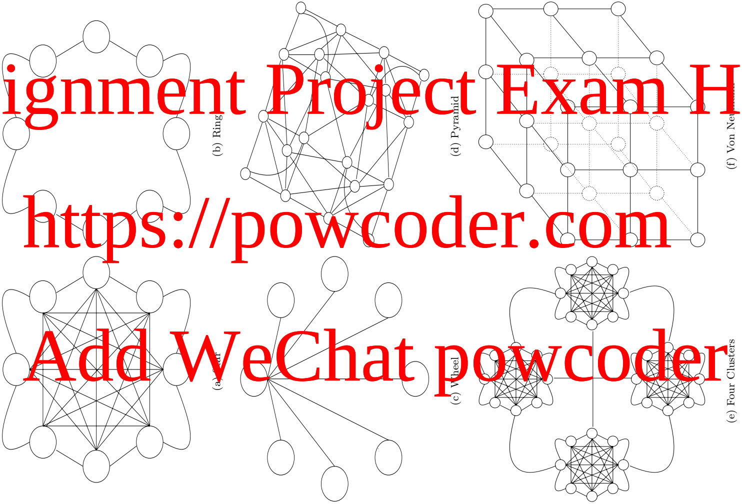


Figure 6: Examples of Social Network Structure.

Velocity Clamping

- **Exploration:** the ability to explore different regions to locate good optima.
- **Exploitation:** the ability to concentrate the search around a region to refine a solution.
- **Potential problem of *gbest* and *lbest* PSO:** Velocity explosion – velocities are updated quickly to large values.
  - It pushes particles to boundaries.

- **Velocity Clamping:** Particle velocity is adjusted before updating the particles' positions.

$$v'_{ij}(t+1) = \begin{cases} v_{ij}(t+1) & \text{if } v_{ij}(t+1) < V_{\max,j} \\ V_{\max,j} & \text{otherwise} \end{cases}$$

  - $V_{\max,j} > 0$: a user-specified maximum velocity of the $j^{th} element$ of the particle.
  - $V_{\max,j}$ controls 1) particle moving speed 2) ability of exploration and exploitation.
  - When $v_{ij}(t+1)$ is of negative, the above condition needs to be modified accordingly.

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- **Advantage:** Explosion of velocity is controlled.
- **Disadvantages:** 1)The searching direction of particle is changed. 2) When velocities reach maximum, particles will search on a hypercube defined by $[x_{ij}(t) - V_{\max,j}, x_{ij}(t) + V_{\max,j}]$ for all i.

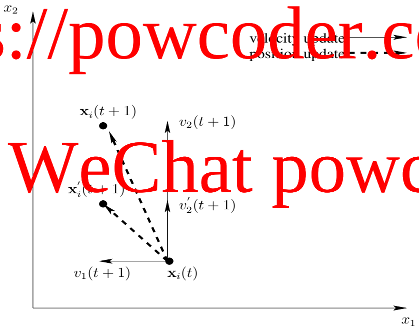

Figure 7: Effect of Velocity Clamping.

Dynamic Velocity Approaches:

- Change of the maximum velocity if the global best position does not improve over τ consecutive iterations:

$$V_{\max,j}(t+1) = \begin{cases} \gamma V_{\max,j}(t) & \text{if } f(\hat{\mathbf{y}}(t)) \geq f(\hat{\mathbf{y}}(t-t')) \ \forall\, t' = 1, \cdots, \tau \\ V_{\max,j}(t) & \text{otherwise} \end{cases}$$

  γ decreases linearly or exponentially from 1 to 0.01.

- Exponentially decay the maximum velocity:

$$V_{\max,j}(t+1) = \left(1 - \left(\frac{t}{n_t}\right)^{\alpha}\right) V_{\max,j}(t)$$

  $\alpha \geq 0$ is a user-specified parameter, n_t is the maximum number of iterations.

- Velocities are updated:

$$v_{ij}(t+1) = V_{\max,j}(t+1) \tanh\left(\frac{v'_{ij}(t+1)}{V_{\max,j}(t+1)}\right)$$

  where $v'_{ij}(t+1)$ is calculated from the *gbest* velocity update rule (1) or *lbest* velocity update rule (2).

Inertia Weight

Inertia Weight

- **Inertia Weight** - a mechanism 1) to control the exploration and exploitation abilities of the swarm, 2) to eliminate the need for velocity clamping.

- ***gbest* PSO:**

$$v_{ij}(t+1) = wv_{ij}(t) + c_1 r_{1j}(t)\left(y_{ij}(t) - x_{ij}(t)\right) + c_2 r_{2j}(t)\left(\hat{y}_j(t) - x_{ij}(t)\right)$$

- ***lbest* PSO:**

$$v_{ij}(t+1) = wv_{ij}(t) + c_1 r_{1j}(t)\left(y_{ij}(t) - x_{ij}(t)\right) + c_2 r_{2j}(t)\left(\hat{y}_{ij}(t) - x_{ij}(t)\right)$$

- $w > 0$
  - $w \geq 1$: Velocities increase over time.
  - $w < 1$: Velocities decrease over time.

- **Random adjustments:** $w \in [0, 1]$: a random number
- **Linear decreasing:**
  - $w(t) = (w(0) - w(n_t))\frac{n_t - t}{n_t} + w(n_t)$.
  - $w(0)$: initial inertia weight.
  - $w(n_t)$: final inertia weight.
  - $w(0) \geq w(n_t)$.
  - n_t : maximum number of iterations.
- **Nonlinear decreasing:**
  - $w(t+1) = \frac{(w(t)-0.4)(n_t - t)}{n_t + 0.4}$ with $w(0) = 0.9$.