# Week 5: aim to cover



- root-finding: bisection, fixed point iteration (Lecture 9)



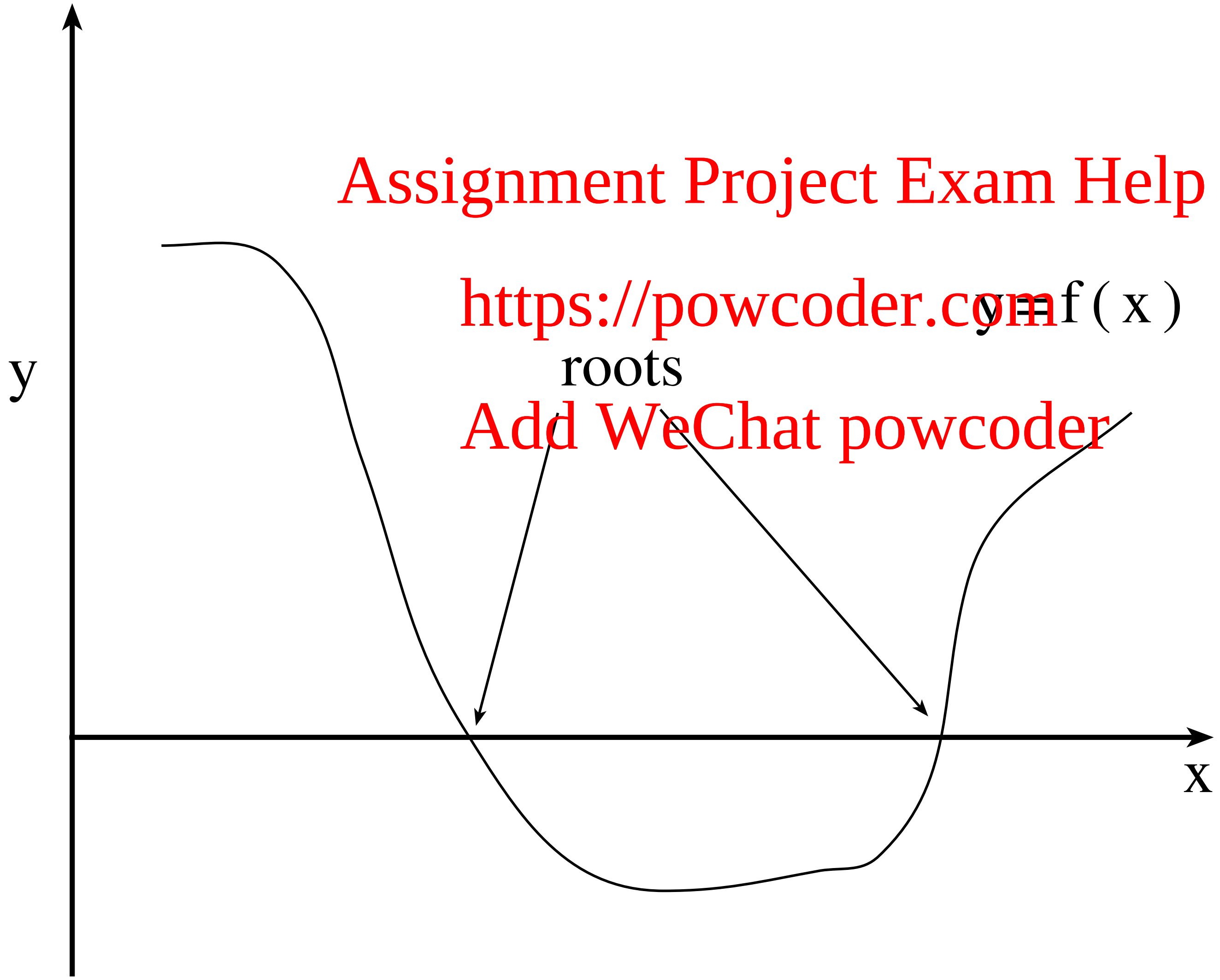
- error propagation, bisection, fixed point iteration ( Lab 5)



- Newton's method, secant method, `fzero` (Lecture 10)

# Root-finding methods

Find $x$ such that $f(x) = 0$



# Iterative processes

‘find zeroes of $f$’, ‘find roots of $f$’, ‘root-finding’

We look for **iterative procedures**



guess $x_0 \to x_1 \to x_2 \cdots$



So we must construct a rule so the sequence of iterates **converges to the root $x^*$**. Since we have to **stop** the iteration, we get a truncation error (this dominates roundoff error until v. close to root)



> Most problems of continuous mathematics cannot be solved by finite algorithms.

Trefethen’s Maxims

# Fixed point iteration

Rearrange $f(x) = 0$ to $x = g(x)$ (not uniquely)



**Definition**

A point $x^*$ that satisfies $x^* = g(x^*)$ is a fixed point of the function $g$



then try the iteration



$$x_{n+1} = g(x_n)$$

does it converge to the fixed point?

Let's try it ...

# Behaviour of fixed point iteration

**what happens?**



1. sometimes it blows up!
2. if it converges , (absolute) error behaves like



$$e_{n+1} \approx k e_n$$



   **linear convergence**

3. $k$ is different for different $g$
4. the smaller $k$ is, the faster the convergence

# Explanation by Taylor series ...

# Some Theorems!

## Definition

If $g$ is defined on [a,b] and $x^* = g(x^*)$ for some $x^* \subset [a, b]$, $g$ has a **fixed point** $x^* \subset [a, b]$



## Theorem

*Sufficient conditions for a unique fixed point*

1. *Existence: If $g \in C[a, b]$ and $g(x) \in [a, b]$ (g maps [a,b] onto [a,b] or a subinterval) then g has a fixed point in [a,b]*
2. *Uniqueness: If, also, if $g'(x)$ exists on (a,b) and $\mid g'(x) \mid \leq k < 1$ on (a,b) then g has a unique fixed point in [a,b]*

# Useful theorems from analysis

**Theorem**

*Intermediate Value Theorem IVT*



*Given* $f \in C[a, b]$ *with* $f(a) > 0, f(b) < 0$

$\implies \exists c \in [a, b]$ *such that* $f(c) = 0$



1



**Theorem**

*Mean Value Theorem MVT*

*Given* $f \in C^1[a, b]$

$\implies \exists c \in [a, b]$ *such that* $\frac{f(b)-f(a)}{b-a} = f'(c)$

2

# Proofs!

## Proof.

Existence:

If $g(a) = a$ or $g(b) = b$ fixed point exists, so suppose not.

Then $g(a) > a$, $g(b) < b$. Define $h(x) = g(x) - x \implies \exists h \in C[a,b]$ with $h(a) > 0, h(b) < 0$

$\implies$ by IVT, $\exists c \in [a,b]$ such that $h(c) = 0$

$\implies g(c) = c \implies c$ is a fixed point of $g$. $\square$



1

## Proof.

Uniqueness: We have $| g'(x) | \leq k < 1$ and suppose $p, q$ are fixed points of $g$ with $p \neq q$. We prove a contradiction which implies that $p = q$

By MVT, $\exists c$ such that

$$g(p) - g(q) = g'(c)(p - q)$$

$\implies | p - q | = | g(p) - g(q) | = | g'(c) || p - q | \leq k | p - q | < | p - q |$

which can't be true, so we have proved that $p = q$

# Theorem: Convergence of fixed point iteration

$| g'(x) | \leq k < 1 \implies g$ is a **contraction mapping**

## Theorem

*Under conditions of previous theorem, for any $x_0 \in [a, b]$, the sequence $x_n = g(x_{n-1})$ converges to the unique fixed point $x^*$.*



## Proof.

By previous theorem, a unique fixed point exists. Since $g$ maps [a,b] into [a,b], the sequence of iterates $\{x_n\}$ is defined.

$$| x_n - x^* | = | g(x_{n-1}) - g(x^*) | = | g'(c) || x_{n-1} - x^* |$$

$$\leq k | x_{n-1} - x^* | \leq k^2 | x_{n-2} - x^* | \cdots \leq k^n | x_0 - x^* |$$

so $\lim_{n \to \infty} | x_n - x^* | = | x_0 - x^* | \lim_{n \to \infty} k^n = 0$ since $k < 1$
so $x_n \to x^*$ $\square$

so far so good

**BUT**



- not cheap to decide when conditions of theorem are met



- for a given $g$, this method may not find all roots



but this method is used for some difficult problems via the **Contraction Mapping Theorem**

# Pseudocode



```
input x0
compute x1
while (stopping criterion not satisfied) and (no. of iterations <
       compute next iteration
end
output results
```

**what stopping criterion to use?**

# Stopping criteria



Many possible:

1. tolerance on *residual* of function $| f(x_n) | < \text{fTol}$
2. tolerance on *absolute change* $| x_n - x_{n-1} | < \text{AbsTol}$
3. tolerance on *relative change* $| x_n - x_{n-1} | / | x_n | < \text{RelTol}$

- 1 is OK if $f$ is not shallow at the root
- 2 OK provided AbsTol not too small
- usually 3 is safe if RelTol $>$ a few $u$

# Mixed tolerance



Combine absolute and relative tolerances

1. mixed tolerance $| x_n - x_{n-1} | < \text{AbsTol} + \text{RelTol} \mid x_n \mid$
2. mixed residual $| f(x_n) | < \text{AbsfTol} + \text{RelfTol} \mid f(x_0) \mid$





1 switches between absolute tolerance if $x_n$ is small and relative tolerance for $| x_n | > 1$
2 uses $| f(x_0) |$ to 'set a scale' for relative residual

# Bisection



Fixed point iteration is v. simple but not guaranteed to converge!

Simplest **globally convergent** method is **bisection aka interval halving**

Start with an interval $[a_1, b_1]$ s.t. $f(a_1), f(b_1)$ of opposite sign

$\implies$ a root lies in $[a_1, b_1]$ by IVT

# Pseudocode

Find midpoint $p_1 = (a_1 + b_1)/2$
Repeat until convergence the step:



```
if f(p1) = 0
    x* = p1
else  ( choose subinterval s.t. root lies there)
   if f(p1) and f(a1) have same sign
       x* in [p1,b1]
   else
       x* in [a1,p1]
    end
```



This is just a floating point version of **binary search**.

# Properties of bisection method

- Since we keep $x^*$ within the interval at each step, we are guaranteed convergence.

- We have a precise error bound, since


$$| p_N - x^* | \leq \max(p_N - a_N, b_N - p_N) \leq | b_N - a_N | /2$$



- Interval halves at every step, so we get an extra **bit** of accuracy every step.
  $\implies$ For a given tolerance `tol`, required number of steps $N$ is known in advance:
  $2^{-N}(b-a) < \text{tol} \implies N > \log_2(\frac{b-a}{\text{tol}})$

## Example



error bound **halves** at every step (actual error jumps around a bit)

End of Lecture 9