Example of ODEs

$$\ddot{x}_i = F_i^x(t, x_1, y_1, z_1, \cdots x_n, y_n, z_n)$$

$$\ddot{y}_i = F_i^y(t, x_1, y_1, z_1, \cdots x_n, y_n, z_n) \qquad i = 1, \cdots n$$

$$\ddot{z}_i = F_i^z(t, x_1, y_1, z_1, \cdots x_n, y_n, z_n)$$

Newton's equ: Moleculer dynamics and planetary motion

For lecture 19, slide 3

Derivation using Taylor series

Taylor series of $y(t_{n+1})$ about t_n:

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h\,y'(t_n) + O(h^2)$$

$$\Rightarrow \quad y_{n+1} = y_n + h f(t_n, y_n) + O(h^2)$$

Derivation by approximating y'

Use forward difference approximating y'

$$\Rightarrow \quad y'\big|_{t_n} = \frac{y(t_{n+1}) - y(t_n)}{h} + O(h)$$

$$\Rightarrow \quad \frac{y_{n+1} - y_n}{h} = f(t_n, y_n) + O(h)$$

$$\Rightarrow \quad y_{n+1} = y_n + h f(t_n, y_n) + O(h^2)$$

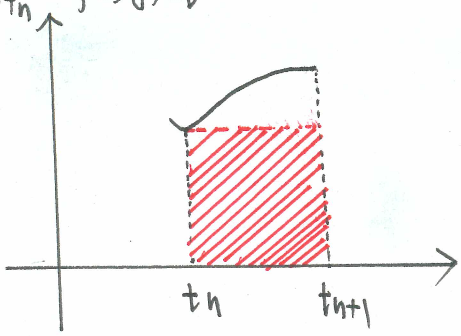
Derivation using quadrature

$$y' = f(t, y)$$

$$\Rightarrow \quad \int_{t_n}^{t_{n+1}} y'\,d\tau = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau$$

$$\Big\downarrow \text{FTC}$$

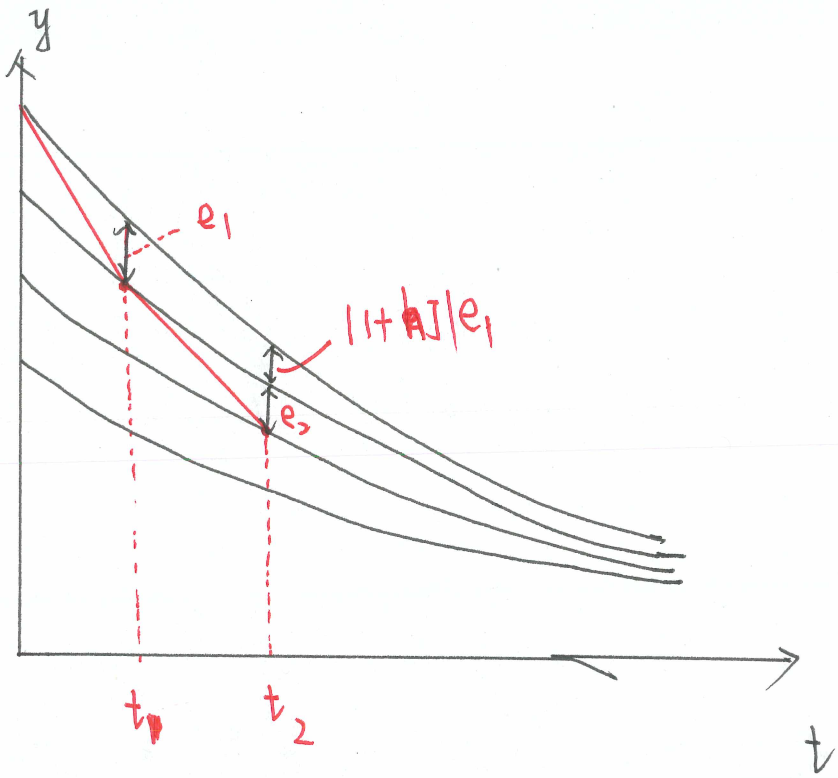
$$\Rightarrow \quad y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(\tau, y)\,d\tau \longrightarrow \text{area below } f$$



approximate the area by left-rectangle rule

$$\Rightarrow \quad y_{n+1} = y_n + h f(t_n, y_n) + O(h^2)$$

For lecture 19, slides 15 – 17



contractive sol: $J<0$

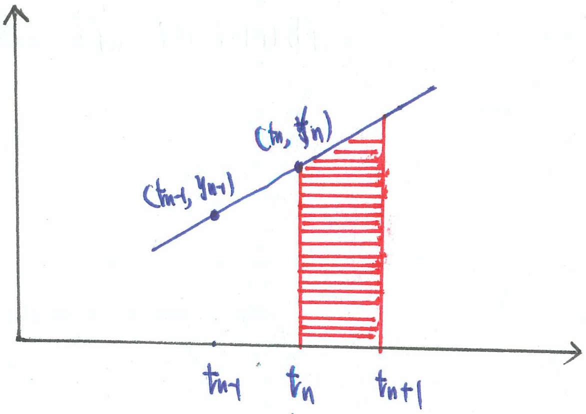
For lecture 20, slide 5

Derive of Adams – Bashforth methods (AB 2)

$y' = f(t, y(t))$

$\Rightarrow \int_{t_n}^{t_{n+1}} y' \, dt = \int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt$

$\Rightarrow y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt$



Idea: Use the area below the line passing through (t_{n-1}, f_{n-1}) and (t_n, f_n) to approximate $\int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt$

The equ of the line passing (t_{n-1}, f_{n-1}) and (t_n, f_n) is given by

$$y = \frac{f_n - f_{n-1}}{t_n - t_{n-1}} (t - t_n) + f_n = \frac{f_n - f_{n-1}}{h} (t - t_n) + f_n$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt \approx \int_{t_n}^{t_{n+1}} \frac{f_n - f_{n-1}}{h} (t - t_n) + f_n \, dt$$

$$= \frac{f_n - f_{n-1}}{2h} (t - t_n)^2 \Big|_{t=t_n}^{t_{n+1}} + h f_n$$

$$= \frac{f_n - f_{n-1}}{2h} h^2 + h f_n$$

$$= \frac{3h}{2} f_n - \frac{h}{2} f_{n-1}$$

$$\Rightarrow y_{n+1} = y_n + h \left(\frac{3}{2} f_n - \frac{1}{2} f_{n-1} \right)$$

For lecture 22, slide 2.

RAS for Backward Euler

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$$

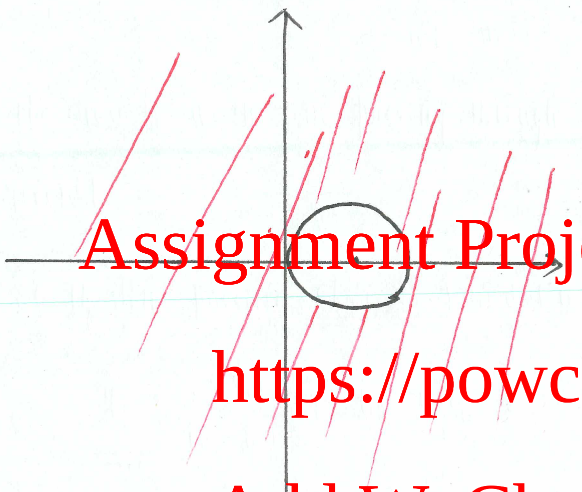
$f = \lambda y$

$$\Rightarrow \quad y_{n+1} = y_n + \lambda h y_{n+1}$$

$$\Rightarrow \quad (1 - h\lambda) y_{n+1} = y_n$$

$$\Rightarrow \quad \frac{y_{n+1}}{y_n} = \frac{1}{1 - h\lambda}$$

so $\left|\frac{y_{n+1}}{y_n}\right| < 1 \Rightarrow |1 - h\lambda| > 1$



$\Rightarrow$ Backward Euler is A-stable.

Note: $\left|\frac{y_{n+1}}{y_n}\right| \to 0$ as $\lambda \to -\infty$ which is called L-stable

For lecture 22, slide 6.

RAS for Trapezoid method.

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

$$f = \lambda y$$

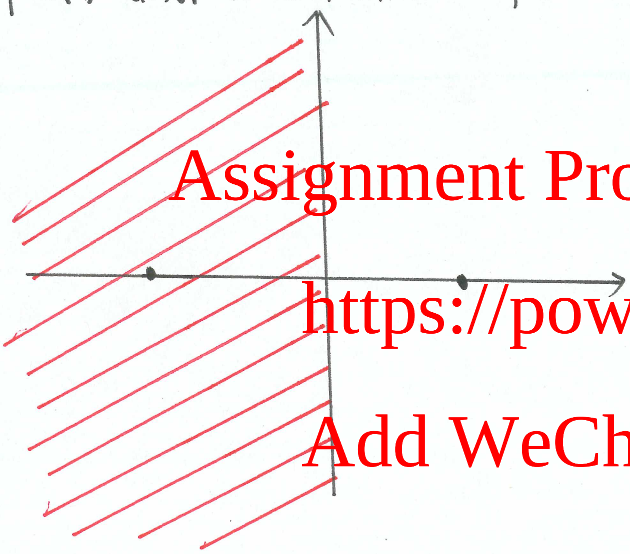
$$\Rightarrow \quad y_{n+1} = y_n + \tfrac{1}{2}h\lambda y_n + \tfrac{1}{2}h\lambda y_{n+1}$$

$$\Rightarrow \quad y_{n+1}(1 - \tfrac{1}{2}h\lambda) = y_n(1 + \tfrac{1}{2}h\lambda)$$

$$\Rightarrow \quad \left|\frac{y_{n+1}}{y_n}\right| = \left|\frac{2+h\lambda}{2-h\lambda}\right|$$

$$\left|\frac{y_{n+1}}{y_n}\right| < 1 \quad \Rightarrow \quad |2+h\lambda| < |2-h\lambda|$$

$\Rightarrow \{ h\lambda :$ closer to -2 than to $2 \}$ $\Rightarrow \mathrm{Re}(h\lambda) < 0$



$\Rightarrow$ A-stable

But $\left|\frac{y_{n+1}}{y_n}\right| \to -1$ as $\lambda \to \infty$

So not L-stable

For lecture 22, slide 7.