Report for Eigenvalue Problems

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For $n \in \mathbb{N}^2$, we consider the discretized Laplacian with Dirichlet boundary condition

$$\Delta = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}_{n\times n}. \tag{1}$$

Assume $\Delta = D - L - U$, where D is the diagonal part, and L, U are the negative lower triangular part and upper triangular part. The iteration matrix of Gauss-Seidel method is

$$M = (D-L)^{-1} U = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2^{-n+2} & 2^{-n+3} & 2^{-n+4} & \cdots & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 2^{-n+1} & 2^{-n+2} & 2^{-n+3} & \cdots & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ 0 & 2^{-n} & 2^{-n+1} & 2^{-n+2} & \cdots & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}. \tag{2}$$

It can be shown all eigenvalues of M are real. For non-zero eigenvalue λ of M, the matrix

$$\lambda D-\lambda L-U=\begin{bmatrix}2\lambda & -1 & 0 & \cdots & 0 & 0\\ -\lambda & 2\lambda & -1 & \cdots & 0 & 0\\ 0 & -\lambda & 2\lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2\lambda & -1\\ 0 & 0 & 0 & \cdots & -\lambda & 2\lambda\end{bmatrix} \tag{3}$$

is singular, and so as

$$P^{-1}\left(\lambda D-\lambda L-U\right)P=\sqrt{\lambda}\begin{bmatrix}2\sqrt{\lambda} & -1 & 0 & \cdots & 0 & 0\\ -1 & 2\sqrt{\lambda} & -1 & \cdots & 0 & 0\\ 0 & -1 & 2\sqrt{\lambda} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2\sqrt{\lambda} & -1\\ 0 & 0 & 0 & \cdots & -1 & 2\sqrt{\lambda}\end{bmatrix} \tag{4}$$

with

$$P=\begin{bmatrix}1 & & & & & \\ & \sqrt{\lambda} & & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda^{\frac{n-2}{2}} & \\ & & & & & \lambda^{\frac{n-1}{2}}\end{bmatrix}, \tag{5}$$

where $\sqrt{\lambda}$ is a selected fixed value. This means that $2-2\sqrt{\lambda}$ is an eigenvalue of Δ and therefore λ is real. To be exact, $\lambda=\left(1-2\sin\frac{j\pi}{n+1}\right)^2$, where $j=1,2,\cdots,n$. However, this can only provide $\left\lfloor\frac{n}{2}\right\rfloor$ non-zero eigenvalues.

We proceed to show that M has exactly $m:=\left\lfloor\frac{n}{2}\right\rfloor$ non-zero eigenvalues, together with a Jordan block of size $n-m=\left\lfloor\frac{n+1}{2}\right\rfloor$.

One proof aims to find the subspace $\ker M^n$. Consider

$$v_0=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & \cdots\end{bmatrix}^{\mathrm{T}}, \tag{6}$$

$$v_1=\begin{bmatrix}0 & 2 & -1 & 0 & 0 & \cdots\end{bmatrix}^{\mathrm{T}}, \tag{7}$$

$$v_2 = \begin{bmatrix} 0 & 0 & 4 & -4 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}, \tag{8}$$

$$\vdots \tag{9}$$

till v_{n-m-1} inclusively. Note that v_k are exactly coefficients of $(2-x)^k$. It can be verified that

$$v_0 = Mv_1, \tag{10}$$

$$v_1 = Mv_2, \tag{11}$$

$$v_2 = Mv_3, \tag{12}$$

$$\vdots, \tag{13}$$

$$v_{n-m-2} = Mv_{n-m-1}. \tag{14}$$

This indicates that $V = \operatorname{span}\{v_0, v_1, \cdots, v_{n-m-1}\}$ is a cyclic subspace of size $n-m$ of M. This means that M has a Jordan block of size $n-m$.

Another proof is given by Zeyu Jia. It is sufficient to check the characteristic polynomial of M. It is

$$\begin{aligned} f &= \det(xI - M) \\ &\sim \det(x(D-L) - U) \\ &= \det\left(\sqrt{x}\begin{bmatrix} 2\sqrt{x} & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2\sqrt{x} & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2\sqrt{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\sqrt{x} & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2\sqrt{x} \end{bmatrix}\right) \\ &= x^{\frac{n}{2}}\begin{vmatrix} 2\sqrt{x} & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2\sqrt{x} & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2\sqrt{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\sqrt{x} & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2\sqrt{x} \end{vmatrix}. \end{aligned} \tag{15}$$

This means $x^{\frac{n}{2}} \mid f$ and therefore $x^{n-m} \mid f$. Because $\dim\ker M = 1$, therefore a Jordan block of size $n - m$ is present.

Note that according to (4), there are two eigenvectors of the eigenvalue $\lambda = \left(1 - 2\sin\frac{j\pi}{n+1}\right)^2$, say $u_j = \left(\left(2\sin\frac{j\pi}{n+1}\right)^k \sin\frac{jk\pi}{n+1}\right)_{k=1}^n$ and $u_{n-j} = \left(\left(2\sin\frac{(n-j)\pi}{n+1}\right)^k \sin\frac{(n-j)k\pi}{n+1}\right)_{k=1}^n$. However, this two eigenvalues coinside, because $2\sin\frac{(n-j)\pi}{n+1} < 0$, $\left(2\sin\frac{j\pi}{n+1}\right)^k$ has alternating signs and this removes high frequency components.