# Assignment for Lecture 4

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**Question 1**

**Proof** Prove by contradiction. Otherwise, there exists $c = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}^{\mathrm{T}}$, such that $Ac = 0$, and therefore $c^{\dagger} A c = 0$.

Note that

$$\begin{aligned} c^{\dagger} A c &= \sum_{i=1}^{m} \sum_{j=1}^{m} \overline{c_j} \left\langle g_j, g_i \right\rangle c_i \\ &= \left\langle \sum_{j=1}^{m} \overline{c_j} g_j, \sum_{i=1}^{m} \overline{c_i} g_i \right\rangle, \end{aligned} \tag{1}$$

and consequently from the positively definiteness

$$\sum_{i=1}^{m} \overline{c_i} g_i = 0. \tag{2}$$

However this contradicts the given linear independence, which shows $A$ is non-singular.

□

**Question 2**

**Answer** Let the best approximation be $g = \alpha x + \beta x^3 + \gamma x^5$. From

$$f - g \perp \operatorname{span} \left\{ x, x^3, x^5 \right\}, \tag{3}$$

it can be derived that

$$\langle f-g, x\rangle = \left\langle f-g, x^3\right\rangle = \left\langle f-g, x^5\right\rangle = 0. \tag{4}$$

Direct computation leads to the system

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{5} & \frac{2}{7} \\ \frac{2}{5} & \frac{2}{7} & \frac{2}{9} \\ \frac{2}{7} & \frac{2}{9} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -2\cos(1)+2\sin(1) \\ -6\sin(1)+10\cos(1) \\ -202\cos(1)+130\sin(1) \end{bmatrix}, \tag{5}$$

whose solution is

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\frac{139965}{8}\cos(1)+11235\sin(1) \\ -\frac{104265}{2}\sin(1)+\frac{324765}{4}\cos(1) \\ -\frac{582813}{8}\cos(1)+\frac{93555}{2}\sin(1) \end{bmatrix}. \tag{6}$$

Therefore, the final solution is

$$\begin{aligned} g = {} & x^5\left(-\frac{582813}{8}\cos(1)+\frac{93555}{2}\sin(1)\right) \\ & + x^3\left(-\frac{104265}{2}\sin(1)+\frac{324765}{4}\cos(1)\right) \\ & + x\left(-\frac{139965}{8}\cos(1)+11235\sin(1)\right). \end{aligned} \tag{7}$$

**Question 3**

**Proof** Because

$$\|u_i\| = \frac{\left\|v_i - \sum_{j=1}^{i-1}\left\langle v_i, u_j\right\rangle u_j\right\|}{\left\|v_i - \sum_{j=1}^{i-1}\left\langle v_i, u_j\right\rangle u_j\right\|} = 1 \tag{8}$$

for $1 \le i \le n$, it remains to prove $u_i$ are pairwise perpendicular.

Perform mathematical induction to prove that for a given $i$ with $1 \le i \le n$, $u_j \perp u_k$ is always true for $1 \le j < k \le i$. Suppose the case $i = m-1$ is done and then consider the case $i = m$, where $1 < m \le n$. It is sufficient to prove $u_m \perp u_k$ for $1 \le k < m$, which directly follows from

$$\langle u_m, u_k\rangle = \langle v_m, u_k\rangle - \sum_{j=1}^{m-1}\left\langle v_m, u_j\right\rangle\left\langle u_j, u_k\right\rangle = \langle v_m, u_k\rangle - \langle v_m, u_k\rangle\|u_k\| = 0. \tag{9}$$

Therefore, the induction is finished and we obtain $u_j \perp u_k$ for $1 \le j < k \le n$.

As shown above, $u_i$ are pairwise orthogonal and normalized, which means they form a orthonormal basis.

□

**Question 4**

**Proof** The leading coefficient of $L_n$ is

$$\frac{(2n)!}{2^n n! n!} = \frac{(2n-1)!!}{n!}, \tag{10}$$

and therefore

$$g = \frac{n!}{(2n-1)!!} L_n \tag{11}$$

is monic. Note that $L_0, L_1, \cdots, L_{n-1}$ form a basis of span $\left\{1, x, \cdots, x^{n-1}\right\}$. Therefore, for any monic polynomial $f$ of degree $n$, there exists $c_i$ $(0 \le i \le n-1)$ such that

$$f - g = \sum_{i=0}^{n-1} c_i L_i. \tag{12}$$

Therefore,

$$\begin{aligned} \|f\|^2 &= \left\| g + \sum_{i=0}^{n-1} c_i L_i \right\|^2 \\ &= \|g\|^2 + \sum_{i=0}^{n-1} c_i^2 \|L_i\|^2 \end{aligned} \tag{13}$$

where the last equality follows from the orthogonality of $L_i$. Because $\|L_i\|^2 > 0$, we have $\|f\| \ge \|g\|$, and the equality is reached iff $c_i = 0$ $(0 \le i \le n-1)$, which is equivalent to $f = g$. Consequently,

$$g = \underset{f \text{ monic}, \deg f = n}{\arg\min} \|f\|, \tag{14}$$

$$\|g\| = \min_{f \text{ monic}, \deg f = n} \|f\|. \tag{15}$$

□

**Answer**

**Question 5** Suppose the best uniform approximation is $g(x) = ax + b$, and therefore there exists a Chebyshev alternance of 3 points for the $R(x) = \sin \frac{\pi x}{2} - ax - b$. Becuase $R$ is concave over $(0, 1)$, therefore two of the points are $0, 1$ respectively. Because $R(0) = R(1)$, it follows that $a = 1$ and consequently the third point is

$$\xi = \frac{2}{\pi} \arccos \frac{2}{\pi}. \tag{16}$$

From $R(0) = -R(\xi)$, we derive that

$$b = \frac{1}{2} \sin \arccos \frac{2}{\pi} - \frac{1}{\pi} \arccos \frac{2}{\pi} = \frac{\sqrt{\pi^2 - 4}}{2\pi} - \frac{1}{\pi} \arccos \frac{2}{\pi} \tag{17}$$

and

$$g(x) = x + \frac{\sqrt{\pi^2 - 4}}{2\pi} - \frac{1}{\pi} \arccos \frac{2}{\pi}. \tag{18}$$

**Answer**

**Question 6** Let $T_n$ be the Chebyshev polynomial of degree $n$, $r_n = \frac{1}{2^{n-1}} T_n$ and $f_n = x^n - r_n$. Because $r_n$ is monic, therefore $f_n \in \mathcal{P}_{n-1}$.

Note that $r_n = x^n - f_n$ have a Chebyshev alternance of $n + 1$ points: let

$$x_k = \cos \frac{k\pi}{2n}, \ (0 \le k \le n), \tag{19}$$

and then

(1) $x_k$ are $n + 1$ distinct points arranged from right to left on the axis;

(2)

$$r_n(x_k) = \frac{1}{2^{n-1}} T_n(x_k) = \frac{1}{2^{n-1}} (-1)^k ; \tag{20}$$

(3) $|r_n(x)| \le \frac{1}{2^{n-1}}$;

(4) the equality in (3) is only reached at $x_k$.

Because $f_n \in \mathcal{P}_{n-1}$, therefore $f_n$ is the best uniform approximation of $x^n$.

In conclusion, $f_n = x^n - \frac{1}{2^{n-1}} T_n$ is the best uniform approximation of $x^n$, and the Chebyshev alternance consists of $x_k = \cos \frac{k\pi}{2n}$ $(0 \le k \le n)$.

**Question 7**

**Proof** Consider

$$u_i = \begin{bmatrix} \phi_i(x_1) & \phi_i(x_2) & \phi_i(x_3) & \cdots & \phi_i(x_{i+1}) \end{bmatrix}^{\mathrm{T}} \tag{21}$$

and the matrix

$$M = \begin{bmatrix} u_i & u_1 & u_2 & u_3 & \cdots & u_i \end{bmatrix}. \tag{22}$$

Because $M$ is linear dependent in terms of columns, we have

$$\begin{aligned} 0 &= \det M \qquad (23) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \phi_i \left(x_j\right) \left(x_j\right) D_j \qquad (24) \\ &= -\sum_{j=1}^{n} \phi_i \left(x_j\right) \sigma_j. \qquad (25) \end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} \phi_i \left(x_j\right) \sigma_j = 0 \tag{26}$$

follows as desired.

□

**Question 8**

**Answer** The Python code is placed in the file `Problem8.ipynb`. The algorithm succeeded in converging in the $k = 2$nd iteration, such that

$$\max_{j=1}^{n+1} \left|\epsilon_j^{(k)}\right| - \min_{j=1}^{n+1} \left|\epsilon_j^{(k)}\right| = 8.720947\text{e-}06 < 1\text{e-}4. \tag{27}$$

Plot of $f$ and $p.$ is shown in Figure 1, and plot of residue is shown in Figure 2. The final polynomial is

$$\begin{aligned} p_2(x) = &-9.600345 \cdot 10^{-11} x^5 + 0.603579 x^4 \\ &+ 1.136689 \cdot 10^{-10} x^3 + 0.414182 x^2 - 1.766548 \cdot 10^{-11} x - 0.008881, \end{aligned} \tag{28}$$

while the final control points are

$$-1.000000, -0.832979, -0.414353, -0.000000, 0.414353, 0.832979, 1.000000. \tag{29}$$

**Question 9**

**Proof** Consider the space to be continuos function on an interval $[a, b]$ and the inner

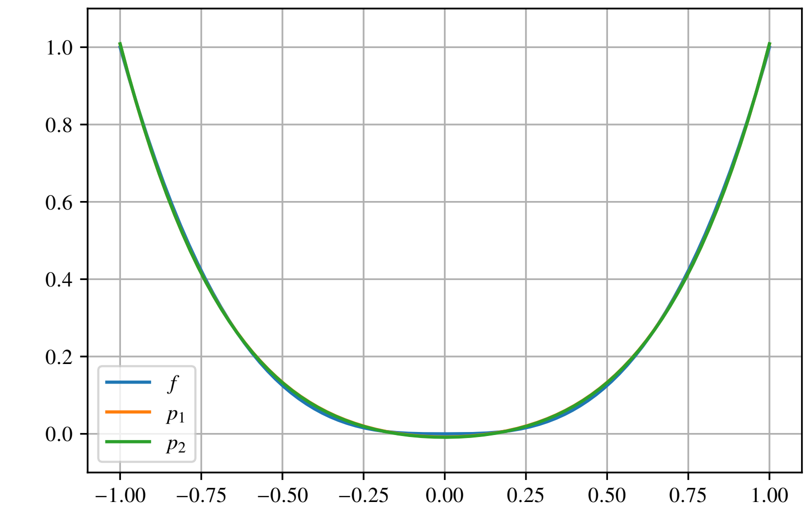


Figure 1 Graph of $f$ and $p$.

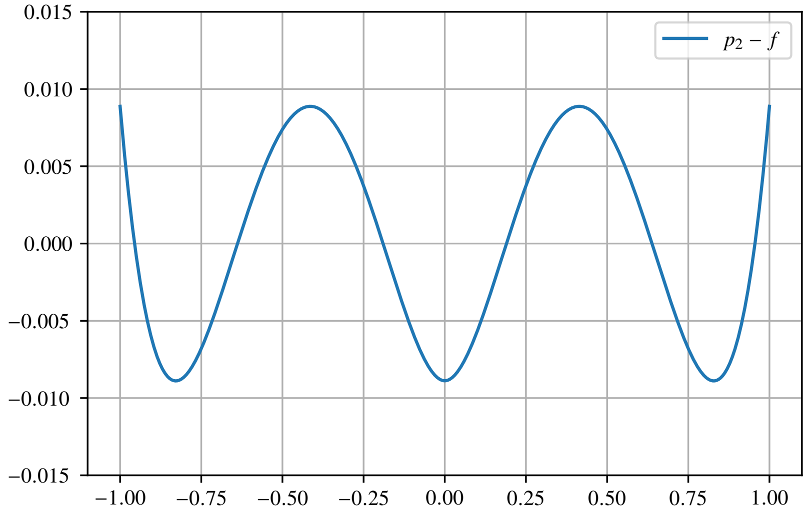


Figure 2 Graph of $p_2 - f$

product to be

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, \mathrm{d}x. \tag{30}$$

Suppose zeros of $p_i$ are $\xi_1 < \xi_2 < \cdots < \xi_k$ with $k < i$, and without loss of generality assume $p_i$ are positive, negative, positive, ... on $(\xi_k, b], (\xi_{k-1}, \xi_k), \cdots, (\xi_1, \xi_2), [a, \xi_1)$ respectively. Consider

$$g = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_k). \tag{31}$$

Because $p_i g \geq 0$, and $p_i g$ have some strictly positive points, therefore

$$\langle p_i, g \rangle = \int_a^b p_i(x)\, g(x)\, \mathrm{d}x > 0. \tag{32}$$

However, because $\deg g = k < i$, therefore $g \in \operatorname{span}\{p_0, p_1, \cdots, p_{i-1}\}$ and $\langle p_i, g \rangle = 0$, which leads to contradiction. Consequently, $p_i$ has at least $i$ zeros. Because $p_i$ is a polynomial of degree $i$, $p_i$ has exactly $i$ zeros in $[a, b]$.