Report of Assignment for Lecture 9

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Question 1

Proof We have

$$K_1 = f, \tag{1}$$

$$K_2 = f + ht\left(f_x + K_1 f_y\right) + h^2 t^2 \left(\frac{1}{2} f_{xx} + K_1 f_{xy} + \frac{1}{2} K_1^2 f_{yy}\right) + O\left(h^3\right), \tag{2}$$

$$K_2 = f + h\left(1-t\right)\left(f_x + K_1 f_y\right) + h^2 \left(1-t\right)^2 \left(\frac{1}{2} f_{xx} + K_1 f_{xy} + \frac{1}{2} K_1^2 f_{yy}\right) + O\left(h^3\right) \tag{3}$$

and

$$y' = f, \tag{4}$$

$$y'' = f_x + f f_y, \tag{5}$$

$$y''' = f_{xx} + 2 f f_{xy} + f f f_{yy} + f_x f_y + f_y f_y f \tag{6}$$

where f, f_x denote values at (x_n, y_n) and y, y' denote value at x_n. Combining these equations, the truncated error

$$\begin{aligned} T_n &= y\left(x_{n+1}\right) - y_{n+1} \\ &= y + hy' + \frac{1}{2} h^2 y'' + \frac{1}{6} h^3 y''' + O\left(h^4\right) - y - \frac{1}{2} h\left(K_2 + K_3\right) \\ &= h^3 \left(\frac{-1 + 6t - 6t^2}{12}\left(f_{xx} + 2 f f_{xy} + f f f_{yy}\right) + \frac{1}{6} f_x f_y + \frac{1}{6} f_y f_y f\right) + O\left(h^4\right) \end{aligned} \tag{7}$$

is $O\left(h^3\right)$, and this scheme is of second order. However, T_n being $O\left(h^4\right)$ cannot be established because terms like $h^3 f_x f_y$ are present.

In conclusion, the given scheme is of exact second order.

□

Question 2

Answer Consider the ordinary differential equation

$$y' = e^{x^2}, \tag{8}$$

$$y(0) = 0, \tag{9}$$

and it suffices to find values of y at $x = 0.5, 1.0, 1.5, 2.0$.

We deploy forward Euler method, backward Euler method and trapezoid method (which coincides the refined Euler method) here. Numerical results are shown in Table 1, and codes are given in Python in `Problem2.ipynb`. Note that n is the number of division and $h = x/n$.

Table 1 Numerical results of Euler methods

x	n	Forward	Backward	Trapezoid
0.5	5	0.53185	0.56026	0.54606
	10	0.53815	0.55236	0.54525
	20	0.54150	0.54860	0.54505
	50	0.54358	0.54642	0.54500
1.0	5	1.30883	1.65248	1.48065
	10	1.38126	1.55309	1.46717
	20	1.42083	1.50674	1.46378
	50	1.44565	1.48002	1.46283
1.5	5	2.99883	5.54515	4.27199
	10	3.47961	4.75277	4.11619
	20	3.75815	4.39473	4.07644
	50	3.93793	4.19256	4.06525
2.0	5	8.49060	29.92986	19.21023
	10	11.81040	22.53003	17.17021
	20	13.95405	19.31387	16.63396
	50	15.40977	17.55369	16.48173

It can be clearly seen that results of forward Euler method increases, and that of backward Euler methods decreases. Trapezoid method here also decreases, but in a slower manner.

Question 3

Proof We have

$$K_1 = f, \tag{10}$$

$$K_2 = f + h(\lambda_2 f_x + \mu_{21} K_1) + h^2 \left(\frac{1}{2}\lambda_2^2 f_{xx} + \lambda_2 \mu_{21} K_1 f_{xy} + \frac{1}{2}\mu_{21} K_1^2 f_{yy} \right) \tag{11}$$

and

$$y' = f, \tag{12}$$

$$y'' = f_x + f f_y, \tag{13}$$

$$y''' = f_{xx} + 2 f f_{xy} + f f f_{yy} + f_x f_y + f_y f_y f. \tag{14}$$

Combining these equations, the truncated error

$$\begin{aligned}
T_n &= y_{x_{n+1}} - y_{n+1} \\
&= y + hy' + \frac{1}{2}h^2 y'' + \frac{1}{6}h^3 y''' + O\left(h^4\right) - y - h(c_1 K_1 + c_2 K_2) \\
&= h(1 - c_1 - c_2) f + h^2 \left(\left(\frac{1}{2} - c_2 \lambda_2 \right) f_x + \left(\frac{1}{2} - c_2 \mu_{21} \right) f f_y \right) \\
&= h^3 \left(\left(\frac{1}{6} - \frac{1}{2} c_2 \lambda_2^2 \right) f_{xx} + \left(\frac{1}{3} - c_2 \lambda_2 \mu_{21} \right) f f_{xy} + \left(\frac{1}{6} - \frac{1}{2} c_2 \mu_{21}^2 \right) f f f_{yy} + \frac{1}{6} f_x f_y + \frac{1}{6} f f_y f_y \right) + O\left(h^4\right).
\end{aligned} \tag{15}$$

Note that terms like $h^3 f_x f_y$ are present and the coefficient is fixed (1/6), and therefore such scheme can never reach third order.

□