Assignment for Lecture 5

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Question 1

Proof Because the first row of A and $A^{(1)}$ are identical, therefore

$$\left|A_{11}^{(1)}\right| \geq \sum_{k=2}^{n} \left|A_{1k}^{(1)}\right| \tag{1}$$

can be established.

For $u = 2, 3, \cdots, n$, we have

$$A_{uk}^{(1)} = A_{uk} - \frac{A_{u1}A_{1k}}{A_{11}} \tag{2}$$

with $A_{u1}^{(1)} = 0$. Because

$$\begin{aligned}
\left|A_{uu}^{(1)}\right| &= \left|A_{uu} - \frac{A_{u1}A_{1u}}{A_{11}}\right| \geq |A_{uu}| - \left|\frac{A_{u1}A_{1u}}{A_{11}}\right| \\
&\geq \sum_{\substack{k=1 \\ k\neq u}}^{n} |A_{uk}| + \sum_{\substack{k=2 \\ k\neq u}}^{n} \left|\frac{A_{u1}A_{1k}}{A_{11}}\right| - |A_{u1}| \\
&= \sum_{\substack{k=2 \\ k\neq u}}^{n} \left(|A_{uk}| + \left|\frac{A_{u1}A_{1k}}{A_{11}}\right|\right) \geq \sum_{\substack{k=2 \\ k\neq u}}^{n} \left|A_{uk} - \frac{A_{u1}A_{1k}}{A_{11}}\right| \\
&= \sum_{\substack{k=1 \\ k\neq u}}^{n} \left|A_{uk}^{(1)}\right|.
\end{aligned} \tag{3}$$

From the argument above, we can conclude that $A^{(1)}$ is diagonally dominant.

□

Question 2

Answer The results are shown in Table 1.

n	$\|x^* - x\|_2$	$\|x^* - x\|_\infty$
2	2.22045e-16	2.22045e-16
12	1.53738e-13	1.13687e-13
24	6.29651e-10	4.65633e-10
48	1.05638e-02	7.81202e-03
84	7.25938e+08	5.36838e+08

Table 1 Errors between x^* and x for different n using Gaussian elimination

Source codes are given in `Problem2.ipynb`.

It can be seen that the error goes large for increasing n.

Question 3

Proof From properties of Gaussian elimination with full pivoting, the i-th row of U is identical to that of

$$L_i P_i \cdots L_1 P_1 A Q_1 \cdots Q_i. \tag{4}$$

Therefore, followed by properties of Gaussian transform, it is also identical to

$$P_i \cdots L_1 P_1 A Q_1 \cdots Q_i =: \widetilde{U}^{(i)}. \tag{5}$$

Because P_i and Q_i swap the element of maximum absolute value to the (i, i) entry, therefore

$$\left|\widetilde{U}^{(i)}_{ii}\right| \geq \left|\widetilde{U}^{(i)}_{ij}\right| \tag{6}$$

for $j > i$ and consequently

$$|U_{ii}| \geq \left|U_{ij}\right|. \tag{7}$$

□

Question 4

Proof From the definition of Gaussian elimination, there exists lower triangular $k \times k$ matrix L_{11} and $(n - k) \times k$ matrix such that

$$\begin{bmatrix} L_{11} & \\ L_{21} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A^{(k)}_{11} & A^{(k)}_{12} \\ & A^{(k)}_{22} \end{bmatrix}. \tag{8}$$

Consequently,

$$L_{21}A_{11} + A_{21} = 0, \tag{9}$$

$$L_{21}A_{12} + A_{22} = A_{22}^{(k)}, \tag{10}$$

which means (note that A_{11} is invertible because Gaussian elimination can be conducted)

$$L_{21} = -A_{21}A_{11}^{-1}, A_{22}^{(k)} = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{11}$$

as desired.

□

Question 5

Proof Suppose there exists upper triangular matrices U_1, U_2 and lower triangular matrices with diagonal 1 L_1, L_2 such that

$$A = L_1U_1 = L_2U_2. \tag{12}$$

Because principle submatrices of A are non-vanishing, diagonal entries of U_1 and U_2 are all non-zero. Therefore

$$L_2^{-1}L_1 = U_2U_1^{-1}, \tag{13}$$

where $L_2^{-1}L_1$ are lower triangular matrices with diagonal 1 and $U_2U_1^{-1}$ are upper triangular matrices. Comparing matrix shape of two sides, we have

$$L_2^{-1}L_1 = U_2U_1^{-1} = I, \tag{14}$$

which means exactly $L_1 = L_2$, $U_1 = U_2$, the uniqueness.

□