Assignment for Lecture 8

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Question 1

Proof Let $e_n = x_n - x^\star$. Because

$$\begin{aligned} e_{n+1} &= e_n - \frac{kf(x_n)}{f'(x_n)} = e_n - \frac{k\frac{1}{k!}e_n^k f^{(k)}\left(x^\star\right) + k\frac{1}{(k+1)!}e_n^{k+1}f^{(k+1)}(\xi_1)}{\frac{1}{(k-1)!}e_n^{k-1}f^{(k)}(x^\star) + \frac{1}{k!}e_n^k f^{(k+1)}(\xi_2)} \\ &= e_n^2 \frac{\frac{1}{k+1}f^{(k+1)}(\xi_1)}{f^{(k)}(x^\star) + \frac{1}{k}e_n f^{(k+1)}(\xi_2)}, \end{aligned} \tag{1}$$

where $\xi_1, \xi_2 \in \left(x^\star, x_n\right)$ (or $\left(x_n, x^\star\right)$). Because $f^{(k)}\left(x^\star\right) \neq 0$, therefore when x lies in a neighborhood of $x^\star$, the coefficient

$$C = \frac{\frac{1}{k+1}f^{(k+1)}(\xi_1)}{f^{(k)}(x^\star) + \frac{1}{k}e_n f^{(k+1)}(\xi_2)} \tag{2}$$

is bounded, and hence second order convergence is derived.

□

Question 2

Proof Let $e_n = x_n - x^\star$.

Suppose $f'\left(x^\star\right) \neq 0$. Note that

$$\begin{aligned} g(x_n) &= \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)} = \frac{f(x_n)f'(x_n) + \frac{1}{2}f^2(x_n)f''(\xi_1)}{f(x_n)} \\ &= f'(x_n) + \frac{1}{2}f(x_n)f''(\xi_1). \end{aligned} \tag{3}$$

Because

$$e_{n+1} = e_n - \frac{f(x_n)}{g(x_n)} = \frac{e_n g(x_n) - f(x_n)}{g(x_n)} = \frac{e_n f'(x_n) + \frac{1}{2} e_n f(x_n) f''(\xi_1) - f(x_n)}{g(x_n)}. \tag{4}$$

supposing

$$0 = f\left(x^\star\right) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_2) \tag{5}$$

$$f(x_n) = f\left(x^\star\right) + e_n f'(\xi_3) = e_n f'(\xi_3), \tag{6}$$

yields

$$e_{n+1} = e_n^2 \frac{\frac{1}{2} f''(\xi_2) + \frac{1}{2} f'(\xi_3) f''(\xi_1)}{g(x_n)}. \tag{7}$$

Note that $f'\left(x^\star\right) \neq 0$ implies $|g(x_n)| > A > 0$ if x_n lies in a neighborhood of $x^\star$, therefore the coefficient of e_n^2 is bounded and second order convergence is shown.

Second order convergence cannot be proved if $f'\left(x^\star\right) = 0$. A counter example is $f(x) = x^2$ and

$$x_{n+1} = x_n - \frac{x_n^4}{\left(x_n + x_n^2\right)^2 - x_n^2} = x_n \frac{1 - x_n}{2 + x_n}. \tag{8}$$

$\square$