Assignment for Lecture 6

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Question 1

Proof Consider

$$\widehat{x} = x / \|x\|_p, \tag{1}$$
$$\widehat{y} = y / \|y\|_q, \tag{2}$$

which satisfies $\|\widehat{x}\|_p = 1$, $\|\widehat{y}\|_q = 1$. Because of the Young's inequality, we have

$$\left|\widehat{x}^{\mathrm{T}}\widehat{y}\right| \le \sum_{i=1}^{n} |\widehat{x}_i \widehat{y}_i| \le \frac{1}{p}\sum_{i=1}^{n} |\widehat{x}_i|^p + \frac{1}{q}\sum_{i=1}^{n} |\widehat{y}_i|^q = \frac{1}{p}\|\widehat{x}\|_p + \frac{1}{q}\|\widehat{y}\|_q = \frac{1}{p} + \frac{1}{q} = 1, \tag{3}$$

which consequently gives

$$\left|x^{\mathrm{T}} y\right| = \left|\widehat{x}^{\mathrm{T}}\widehat{y}\right| \|x\|_p \|y\|_q \le \|x\|_p \|y\|_q . \tag{4}$$

□

Question 2

Proof (1) Positive definiteness: because $\|Ax\| \ge 0$ for all x, therefore $\|A\| = \max_{\|x\|=1} \|Ax\| \ge 0$. If $\|A\| = 0$, then for all $x \ne 0$, $\|Ax\| = \|A(x/\|x\|)\| \|x\| = 0$ and $Ax = 0$, which means $A = 0$.

(2) Absolute homogeneity:

$$\|aA\| = \max_{\|x\|=1} \|aAx\| = |a| \max_{\|x\|=1} \|Ax\| = |a| \|A\| . \tag{5}$$

(3) Triangle inequality:

$$\|A\|+\|B\| = \max_{\|x\|=1} \|Ax\|+\max_{\|x\|=1} \|Bx\| \geq \max_{\|x\|=1} (\|Ax\| + \|Bx\|) \geq \max_{\|x\|=1} \|(A+B)\,x\| = \|A+B\| . \tag{6}$$

Combining these condition, $\|\cdot\| : A \mapsto \|A\|$ is indeed a norm.

□

Question 3

Proof Choose some x such that $\|x\| \neq 0$. Because A is invertible,

$$\kappa(A)\,\|x\| = \|A\|\,\|A^{-1}\|\,\|x\| \geq \|A\|\,\|A^{-1}x\| \geq \|AA^{-1}x\| \geq \|x\|, \tag{7}$$

which means

$$\kappa(A) \geq 1. \tag{8}$$

□

Question 4

Answer Analytical solution of the equation is

$$x^* = \frac{1}{(3+\epsilon)(1+\epsilon)} \begin{bmatrix} -4-\epsilon \\ -1-\epsilon \\ 5+2\epsilon \end{bmatrix}. \tag{9}$$

The result is shown in Table 1.

Table 1 Number of Gauss-Seidel iterations for different ϵ

ϵ	iterations
1.0e+00	11
1.0e-01	77
1.0e-02	730
1.0e-03	7262
1.0e-04	72581
1.0e-05	725774
1.0e-06	7257699
1.0e-07	72577097

Source codes are given in Python in `Problem4.ipynb`.

Question 5

Proof It suffices to prove that $\rho(H) < 1$. If λ is an eigenvalue of H and v is the corresponding eigenvector, then (note that H is real)

$$0 < v^* B v = v^* P v - v^* H^* P H v = \left(1 - |\lambda|^2\right) v^* P v. \tag{10}$$

Because $v^* P v^* > 0$ also holds, therefore $1 - |\lambda|^2 > 0$, which implies that $|\lambda| < 1$ and $\rho(H) < 1$.

□

Question 6

Proof Proof by contradiction. That A is singular leads to the existence of non-zero v such that $Av = 0$.

Consider the case that A is diagonally dominant. Suppose $|v_i| = \max_{j=1}^{n} \left|v_j\right|$ (which is grater than zero according to the hypothesis), and therefore

$$0 = \left|(Av)_i\right| = \left|\sum_{j=1}^{n} A_{ij} v_j\right| \geq |A_{ii}|\,|v_i| - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|A_{ij}\right| \left|v_j\right| \geq \left(|A_{ii}| - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|A_{ij}\right|\right) |v_i| > 0, \tag{11}$$

which leads to contradiction.

Consider the case that A is irreducibly diagonally dominant. Let $S = \left\{i : |v_i| = \max_{j=1}^{n} \left|v_j\right|\right\}$ and $T = \{1, 2, \cdots, n\} \setminus S$. Because A is irreducible, therefore there exists $s \in S$, $t \in T$ such that $A_{st} \neq 0$. Consequently,

$$0 = \left|(Av)_s\right| = \left|\sum_{j=1}^{n} A_{sj} v_j\right| \geq |A_{ss}|\,|v_s| - \sum_{\substack{j=1 \\ j \neq s}}^{n} \left|A_{sj}\right| \left|v_j\right| > \left(|A_{ss}| - \sum_{\substack{j=1 \\ j \neq s}}^{n} \left|A_{sj}\right|\right) |v_s| \geq 0, \tag{12}$$

which also leads to contradiction.

□