Assignment for Lecture 7

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Question 1

Proof Suppose $f \in C^4[a,b]$.

Assume a and b to be -1 and 1 respectively, and it remains to prove the existence of $\xi \in [-1,1]$ such that

$$\int_{-1}^{1} f(x)\,\mathrm{d}x - \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right) = -\frac{1}{90} f^{(4)}(\xi). \tag{1}$$

Consider

$$\begin{aligned} P(x) = {} & \frac{1}{2} f(-1)\,x(x-1) - f(0)(x+1)(x-1) + \frac{1}{2} f(1)(x+1)x \\ & + \left(-\frac{1}{2} f(-1) - \frac{1}{2} f(1) - f^{(1)}(0)\right)(x+1)x(x-1). \end{aligned} \tag{2}$$

and $d(x) := f(x) - P(x)$. It can be verified that d satisfies $d(-1) = d(0) = d(1) = d'(0) = 0$. For $x \in (-1,0) \cup (0,1)$, there exists (proof postponed) $\xi_x \in [-1,1]$, such that

$$f(x) - P(x) = \frac{1}{24} x^2 (x-1)(x+1) f^{(4)}(\xi_x). \tag{3}$$

Therefore from continuity of $f^{(4)}$ there exists ξ_3, such that

$$\begin{aligned}
&\int_{-1}^{1} f(x)\,\mathrm{d}x - \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right) \\
=& \int_{-1}^{1} f(x)\,\mathrm{d}x - \int_{-1}^{1} P(x)\,\mathrm{d}x \\
=& \int_{-1}^{1} \frac{1}{24} x^2 (x-1)(x+1) f^{(4)}(\xi_x)\,\mathrm{d}x \\
=& \frac{1}{24}\int_{-1}^{1} x^2 (x-1)(x+1)\,\mathrm{d}x f^{(4)}(\xi) \\
=& -\frac{1}{90} f^{(4)}(\xi)
\end{aligned} \tag{4}$$

as desired.

Existence of ξ_x: Without loss of generality, assume $x \in (0,1)$. Consider

$$Q(y) = f(y) - P(y) - \frac{f(x) - P(x)}{x^2(x-1)(x+1)} y^2 (y-1)(y+1), \tag{5}$$

which satisfies $Q(-1) = Q(0) = Q(1) = Q'(0) = Q(x) = 0$. Applying Rolle's mean value theorem, there exists $\xi_x \in [-1,1]$ such that $Q^{(4)}(\xi_x) = 0$, which is equivalent to

$$f(x) - P(x) = \frac{1}{24} f^{(4)}(\xi) x^2 (x-1)(x+1). \tag{6}$$

□

Question 2

Proof Let the (equally spaced) interpolation nodes be $x_0 = a, x_1, \cdots, x_n = b$ respectively,

$$\phi_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}, \tag{7}$$

and

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\,\phi_k(x) \tag{8}$$

and therefore the Newton-Cotes formula can be written as

$$I = \int_a^b P_n(x)\,\mathrm{d}x = \sum_{k=0}^{n} f(x_k) \int_a^b \phi_k(x)\,\mathrm{d}x. \tag{9}$$

When $f(x) = x^n$, Lagrangian interpolation yields $f = P_n$, and therefore

$$I = \int_a^b P_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \tag{10}$$

exactly.

If n is even, then $n+1$ is odd. It suffices to prove that for the case $f(x) = \left(x - \frac{a+b}{2}\right)^{n+1}$,

$$I = \int_a^b P_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x = 0. \tag{11}$$

Note that $\phi_k(x) = \phi_{n-k}(a+b-x)$ follows from definition, and therefore

$$\begin{aligned} 2I &= \sum_{k=0}^{n} f(x_k)\left(\int_a^b \phi_k(x)\,\mathrm{d}x + \int_a^b \phi_{n-k}(a+b-k)\,\mathrm{d}x\right) \\ &= \sum_{k=0}^{n} \left(f(x_k) + f(x_{n-k})\right)\int_a^b \phi_k(x)\,\mathrm{d}x = 0 \end{aligned} \tag{12}$$

as desired.

□

Question 3

Answer The result is shown in table 1. Error corresponds to $|I - \pi|$ here.

Table 1 Errors of different integration methods

$1/h$	Trapezoid	Simpson	Romberg 3	Romberg 4	Romberg 5	Gauss 2
1	1.41593e-01	8.25932e-03	5.24993e-04	6.86983e-06	1.16879e-08	5.94833e-03
2	4.15927e-02	2.40261e-05	1.44054e-06	1.51930e-08	5.98170e-11	1.72394e-05
3	1.85157e-02	8.72654e-07	4.40069e-08	2.38405e-11	9.85878e-14	6.20794e-07
4	1.04162e-02	1.51131e-07	7.55277e-09	2.35811e-13	8.88178e-16	1.07469e-07
5	6.66654e-03	3.96506e-08	1.98203e-09	3.10862e-14	0.00000e+00	2.81956e-08
8	2.60416e-03	2.36497e-09	1.18244e-10	8.88178e-16	4.44089e-16	1.68175e-09
10	1.66666e-03	6.20008e-10	3.09996e-11	4.44089e-16	0.00000e+00	4.40895e-10
20	4.16667e-04	9.68825e-12	4.84501e-13	0.00000e+00	4.44089e-16	6.88960e-12
30	1.85185e-04	8.49987e-13	4.21885e-14	4.44089e-16	0.00000e+00	6.04850e-13
40	1.04167e-04	1.51879e-13	7.99361e-15	0.00000e+00	0.00000e+00	1.07914e-13
50	6.66667e-05	3.95239e-14	2.22045e-15	0.00000e+00	0.00000e+00	2.84217e-14

Question 4

Proof Suppose zeros of ϕ_i are $x_1 < x_2 < \cdots < x_k$ with $k < i$, and without loss of generality assume $\phi_i, -\phi_i, \cdots, (-1)^k \phi_i$ are positive on $(x_k, b], (x_{k-1}, x_k), \cdots, (x_1, x_2), [a, x_1)$ respectively. (We may change the sign of ϕ_i by multiplying -1, and ignore repeated zeros away if signs do not change around ϕ_i) Consider

$$\psi(x) = (x - x_1)(x - x_2) \cdots (x - x_k). \tag{13}$$

Because $\deg g = k < i$ and consequently

$$\psi \in \operatorname{span} \langle \phi_0, \phi_1, \cdots, \phi_{i-1} \rangle, \tag{14}$$

therefore

$$\int_a^b \rho(x) \phi_i(x) \psi(x) \, \mathrm{d}x = 0. \tag{15}$$

However, this contradicts $\rho\phi_i\psi$ being continuous and non-negative, with finite points reaching equality.

As a result, $k \geq i$. Note that $k > i$ can be ruled out because of the degree constraint, and therefore we can conclude $k = i$.

□

Question 5

Proof Because L_n is n-th Legendre polynomial, therefore

$$f_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \tag{16}$$

satisfies $\deg f_i = 2n - 2 \leq 2n - 1$ and the quadrature is exact for f_i. Note that the quadrature gives

$$\begin{aligned} I &= \sum_{k=1}^{n} A_k l_i(x_k) = \sum_{k=1}^{n} A_k \delta_{ik} = A_i \\ &= \int_a^b f_i(x) \, \mathrm{d}x > 0 \end{aligned} \tag{17}$$

as desired.

□

Question 6

Proof It suffices to prove the case $a = 0$, $b = 1$.

Because

$$\int_a^b 1\,\mathrm{d}x = 1 = \frac{1}{6}(1+4+1), \tag{18}$$

$$\int_a^b x\,\mathrm{d}x = \frac{1}{2} = \frac{1}{6}\left(0+4\cdot\frac{1}{2}+1\right), \tag{19}$$

$$\int_a^b x^2\,\mathrm{d}x = \frac{1}{3} = \frac{1}{6}0+4\cdot\frac{1}{4}+1, \tag{20}$$

$$\int_a^b x^3\,\mathrm{d}x = \frac{1}{4} = \frac{1}{6}0+4\cdot\frac{1}{8}+1, \tag{21}$$

$$\int_a^b x^4\,\mathrm{d}x = \frac{1}{5} \neq \frac{5}{24} = \frac{1}{6}0+4\cdot\frac{1}{16}+1, \tag{22}$$

therefore the quadrature applies for polynomials of degree no greater than 3 exactly. That is, algebraic precision of the quadrature is 3.

□