Assignment for Lecture 10

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Question 1

Proof Consider the $(J-1)\times(J-1)$-matrix

$$A=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}, \tag{1}$$

and

$$v_j=\begin{bmatrix} \sin\frac{j\pi}{J} & \sin\frac{2j\pi}{J} & \cdots & \sin\frac{(J-1)j\pi}{J} \end{bmatrix}^{\mathrm{T}} \tag{2}$$

for $j=1,2,\cdots,J-1$. Note that

$$\sin\frac{0j\pi}{J}=\sin\frac{Jj\pi}{J}=0, \tag{3}$$

and

$$\begin{aligned} & 2\sin\frac{kj\pi}{J}-\sin\frac{(k+1)j\pi}{J}-\sin\frac{(k-1)j\pi}{J} \\ =& 2\sin\frac{kj\pi}{J}-2\sin\frac{kj\pi}{J}\cos\frac{j\pi}{J} \\ =& 4\sin\frac{kj\pi}{J}\sin^2\frac{j\pi}{2J}. \end{aligned} \tag{4}$$

Therefore, we have

$$Av_j = 4\sin^2\frac{j\pi}{2J}v_j. \tag{5}$$

Because $v_j \neq 0$ and $4\sin^2\frac{j\pi}{2J}$ are distinct, therefore all eigenvalues of the $(J-1)\times(J-1)$-matrix A are $4\sin^2\frac{j\pi}{2J}$ for $j = 1, 2, \cdots, J-1$.

As a result, spectrum of

$$1 - \lambda A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & 1-2\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & 1-2\lambda \end{bmatrix} \tag{6}$$

is exactly $1 - 4\lambda\sin^2\frac{j\pi}{2J}$ with $j = 1, 2, \cdots, J-1$.

□

Question 2

Proof It suffices to prove the matrix

$$M = (1 + \lambda A)^{-1} \tag{7}$$

satisfies $\rho(M) < 1$, where A is defined as (1). This directly follows from that eigenvalues of M are $-1 < 0 < 1/\left(1 + 4\lambda\sin^2\frac{j\pi}{2J}\right) < 1$ for $j = 1, 2, \cdots, J-1$.

□

Question 3

Proof Suppose Courant condition

$$|a(x,t)|\frac{k}{h} \leq 1 \tag{8}$$

is satisfied.

Denote the error at (j,n) by e_j^n. Linearity and Courant condition yields

$$\begin{cases} e_j^{n+1} = \left(1 - a_j^n \frac{k}{h}\right) e_j^n + a_j^n \frac{k}{h} e_{j-1}^n, & 0 \le a_j^n \frac{k}{h} \le 1; \\ e_j^{n+1} = \left(1 + a_j^n \frac{k}{h}\right) e_j^n - a_j^n \frac{k}{h} e_{j+1}^n, & -1 \le a_j^n \frac{k}{h} < 0. \end{cases} \tag{9}$$

Suppose $\left|e_j^n\right| \le M$ for all j. Because

$$\left|e_j^{n+1}\right| \le \left(1 - a_j^n \frac{k}{h}\right) \left|e_j^n\right| + a_j^n \frac{k}{h} \left|e_{j-1}^n\right| \le \left(1 - a_j^n \frac{k}{h}\right) M + a_j^n \frac{k}{h} M = M \tag{10}$$

for $0 \le a_j^n \frac{k}{h} \le 1$, and

$$\left|e_j^{n+1}\right| \le \left(1 + a_j^n \frac{k}{h}\right) \left|e_j^n\right| - a_j^n \frac{k}{h} \left|e_{j-1}^n\right| \le \left(1 + a_j^n \frac{k}{h}\right) M - a_j^n \frac{k}{h} M = M \tag{11}$$

for $-1 \le a_j^n \frac{k}{h} < 0$, we have

$$\left|e_j^{n+1}\right| \le M. \tag{12}$$

Applying mathematical induction, this algorithm is numerical stable.

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