Report for Project 1

Zhihan Li

1600010653

June 19, 2018

Question 1

Proof We have $u \in C_0^2([0,L])$ and $f \in C^0([0,L])$. Because for $v \in H_0^1([0,L])$, it follows that

$$\int_0^L f v = \int_0^L u' v' = u' v|_0^L - \int_0^L u'' v = -\int_0^L u'' v \tag{1}$$

and therefore

$$\int_0^L (-u'' - f) v = 0. \tag{2}$$

If $-u'' - f \not\equiv 0$, there exists $x_0 \in [0,L]$ such that $c := (-u'' - f)(x_0) \neq 0$. Assume $c > 0$ without loss of generality. Because $-u'' - f$ is continuous, therefore there exists $[a,b] \ni x_0$, such that $(-u'' - f)(x) > \frac{1}{2}c \ \ (x \in [a,b])$. Because

$$v(x) = \begin{cases} (x-a) / \left(\frac{b-a}{2}\right), & x \in \left[a, \frac{a+b}{2}\right]; \\ (b-x) / \left(\frac{b-a}{2}\right), & x \in \left[\frac{a+b}{2}, b\right]; \\ 0, & x \in [0,L] \text{ otherwise} \end{cases} \tag{3}$$

lies in $H_0^1([0,L])$, therefore from (2)

$$0 = \int_0^L (-u'' - f) v \geq \frac{1}{2}(b-a)\frac{1}{2}c > 0. \tag{4}$$

This leads to contradiction, and we conclude $-u'' - f \equiv 0$. Because $u \in C_0^2([0,L])$, this further implies that u is a solution of the original function.

$\square$

Question 2

Proof Denote

$$\langle g, h\rangle = \int_0^L gh. \tag{5}$$

The linear system is exactly

$$\left\langle u_h', \phi_i' \right\rangle = \langle f, \phi_i \rangle = \left\langle u', \phi_i' \right\rangle, \tag{6}$$

for $i = 1, 2, \cdots, n-1$, where the last equality follows from (2). Because $v_h \in V_h = \operatorname{span}\{\phi_1, \phi_2, \cdots, \phi_{n-1}\}$, there exists v_i such that

$$v_h = \sum_{i=1}^{n-1} v_i \phi_i. \tag{7}$$

This directly leads to

$$\left\langle u' - u_h', v_h' \right\rangle = \left\langle u', v_h' \right\rangle - \left\langle u_h', v_h' \right\rangle = \sum_{i=1}^{n-1} v_i \left(\left\langle u', \phi_i' \right\rangle - \left\langle u_h', \phi_i' \right\rangle \right) = 0. \tag{8}$$

□

Question 3

Proof Because

$$A = \left[a_{ij}\right]_{i,j=1}^{n-1} = \begin{bmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & & & & \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & & & \\ & -\frac{1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \frac{1}{h_{n-2}} + \frac{1}{h_{n-1}} & -\frac{1}{h_{n-1}} \\ & & & & -\frac{1}{h_{n-1}} & \frac{1}{h_{n-1}} + \frac{1}{h_n} \end{bmatrix}, \tag{9}$$

therefore $A^{\mathrm{T}} = A$ and A is symmetric.

Let

$$e_i = \begin{bmatrix} \delta_{i1} - \delta_{i2} & \delta_{i2} - \delta_{i3} & \cdots & \delta_{i(n-1)} - \delta_{in} \end{bmatrix}^{\mathrm{T}} \tag{10}$$

be unit column vectors for $i = 1, 2, \cdots, n$, where δ is the Kronecker delta. Note that A can be decomposed into semi-positive definite matrices, that is,

$$A = \frac{1}{h_1} e_1 e_1^{\mathrm{T}} + \frac{1}{h_2} e_2 e_2^{\mathrm{T}} + \cdots + \frac{1}{h_n} e_n e_n^{\mathrm{T}}. \tag{11}$$

As a result, $A \geq 0$. It is sufficient to prove A is non-singular for $A \succ 0$. Otherwise, there exists $p \neq 0$ such that $Ap = 0$. This means

$$0 = p^{\mathrm{T}} A p = \frac{1}{h_1} \left(e_1^{\mathrm{T}} p \right)^2 + \frac{1}{h_2} \left(e_2^{\mathrm{T}} p \right)^2 + \cdots + \frac{1}{h_n} \left(e_n^{\mathrm{T}} p \right)^2 = 0, \tag{12}$$

and consequently $e_i^{\mathrm{T}} p = 0$. This can be expanded into $p_1 = p_2 - p_1 = p_3 - p_2 = \cdots = p_n - p_{n-1} = -p_n = 0$ and we obtain $p = 0$ as a result, contradicting the hypothesis. Therefore, we conclude that A is non-singular and $A \succ 0$.

□

Question 4

Answer The variational problem is to find $u \in C_0^2\left([0, L]\right)$, such that for $v \in H_0^1\left([0, L]\right)$,

$$\int_0^1 u' v' = \int_0^1 v. \tag{13}$$

Here the matrix A is

$$A = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & & & \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & & \\ & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & \\ & & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ & & & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}. \tag{14}$$

The solution u_h of different n is given in Figure 1.

Because the analytical solution is $u\left(x\right) = x\left(1 - x\right)/2$, therefore errors can be calculated by

$$\sqrt{\int_0^1 \left(u' - u_h' \right)^2}. \tag{15}$$

The table of errors with respect to different n is shown in Table 1.

It can be observed that the error deceases as n increases.

For higher precision, the quadrature here is composite Simpson formula instead of midpoint formula. For integrand f on interval $[a, b]$, we divide $[a, b]$ into k pieces of identical length,

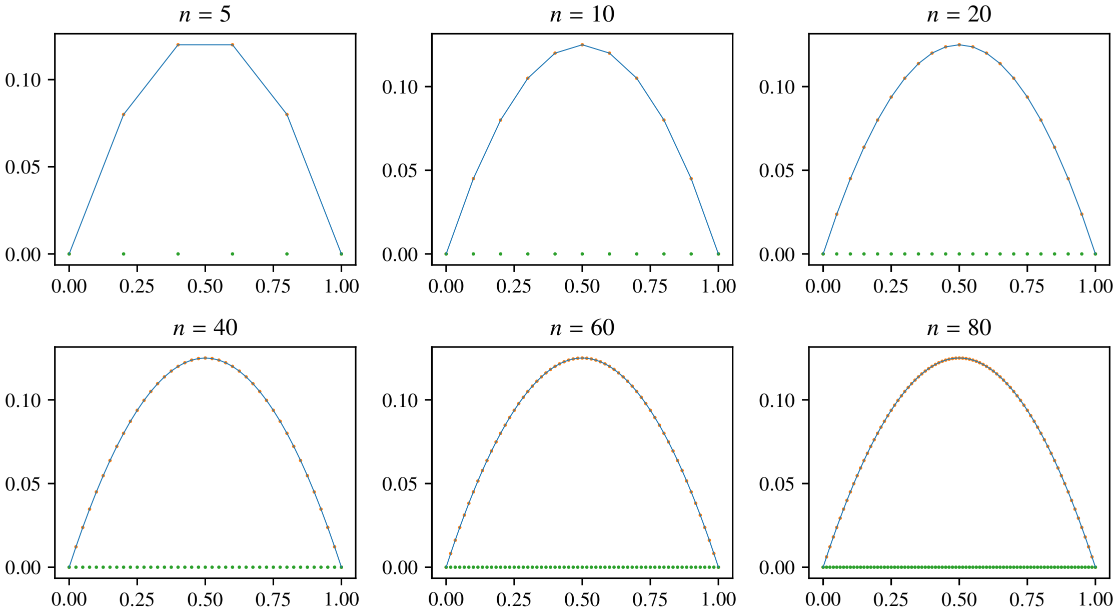

Figure 1 Solution u_h for different n

Table 1 Errors between u_h and u with respect to different n

n	Error
2	1.44338e-01
4	7.21688e-02
8	3.60844e-02
16	1.80422e-02
32	9.02110e-03
64	4.51055e-03

where k is the smallest integer such that

$$\frac{1}{k}(b-a) < \epsilon. \tag{16}$$

In computational practice, ϵ is evaluated 10^{-3}.

Question 5

Answer The solutions with respect to different ϵ given $\alpha = 0.5$ is shown in Figure 2. It can be seen that the number of nodes increases when ϵ decreases, and most nodes are located near 0, where $|f|$ is relatively large.

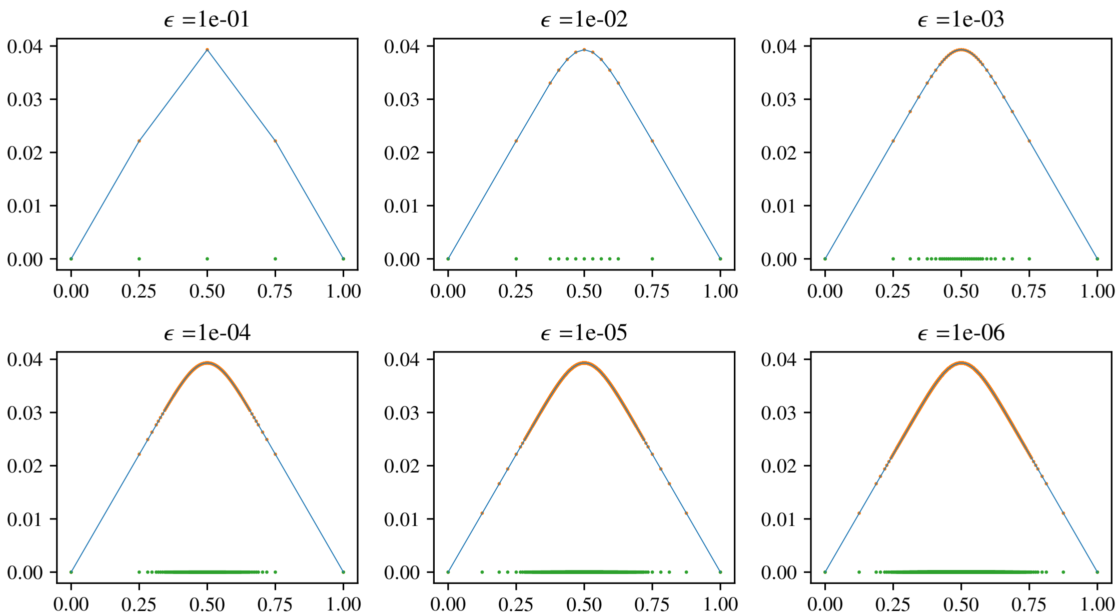

Figure 2 Solution u_h for different ϵ given $\alpha = 0.5$

The solutions with respect to different α given $\epsilon = 10^{-6}$ is shown in Figure 3.

To magnify the distinction between different α, we also record the number of iterations and nodes. This is shown in Table 2. It can be seen that the number of iterations goes large as α increases, but fewer nodes are added eventually.

For von Neumann boundary condition and Robin boundary condition, the space of test functions must be modified. We choose it to be $H^1([0,1])$ instead of $H_0^1([0,1])$. Therefore, the variational problem is to find $u \in C^2([0,1])$ such that for all $v \in H^1([0,1])$,

$$\int_0^1 u'v' + \kappa_0\, uv|_0 = \int_0^1 fv. \tag{17}$$

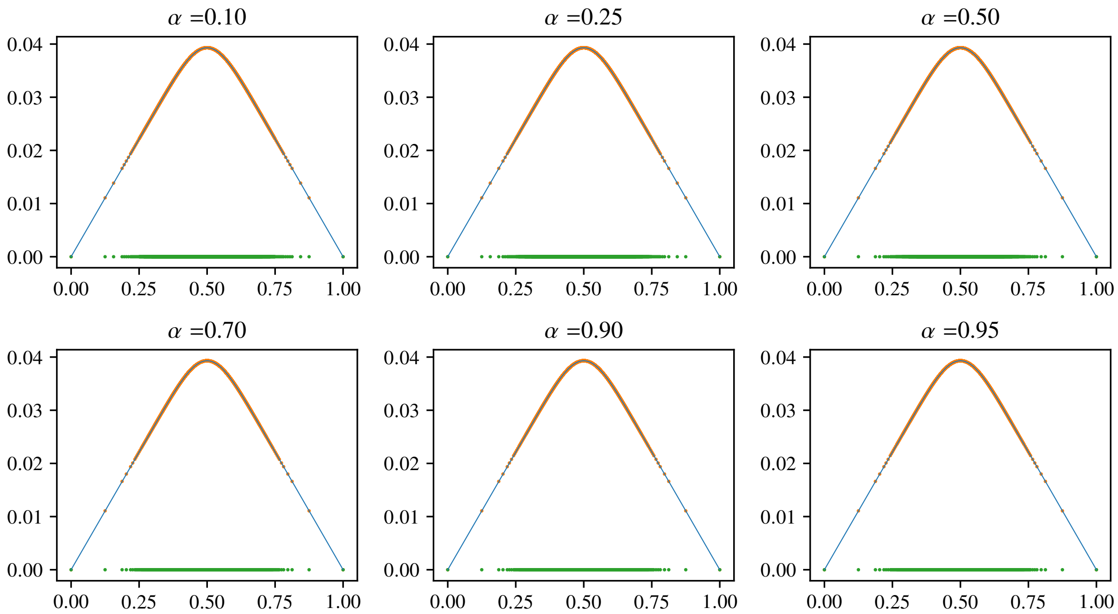

Figure 3 Solution u_h for different α given $\epsilon = 10^{-6}$

Table 2 Number of iterations nodes with respect to α given $\epsilon = 10^{-6}$

α	Iterations	Nodes
0.10	14	5756
0.25	15	7560
0.50	17	3704
0.70	29	3934
0.90	73	3522
0.95	123	3434

By assuming

$$u_h = c_0\phi_0 + c_1\phi_1 + \cdots + c_n\phi_n, \tag{18}$$

where

$$\phi_0(x) = \begin{cases} (x_1 - x)/h_1, & x \in I_1; \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

$$\phi_n(x) = \begin{cases} (x - x_{n-1})/h_n, & x \in I_n; \\ 0, & \text{otherwise}. \end{cases} \tag{20}$$

and testing the variational problem with ϕ_i for $i = 0, 1, \cdots, n$, we obtain the linear system $Ac = b$ with

$$A = \begin{bmatrix} \frac{1}{h_1} + \kappa_0 & -\frac{1}{h_1} & & & & & \\ -\frac{1}{h_1} & \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & & & & \\ & -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & & & \\ & & -\frac{1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \frac{1}{h_{n-2}} + \frac{1}{h_{n-1}} & -\frac{1}{h_{n-1}} & \\ & & & & & -\frac{1}{h_{n-1}} & \frac{1}{h_{n-1}} + \frac{1}{h_n} & -\frac{1}{h_n} \\ & & & & & & -\frac{1}{h_n} & \frac{1}{h_n} \end{bmatrix}, \tag{21}$$

and $b = \left[\int_0^1 f\phi_{i-1}\right]_{i=1}^{n+1}$. The solving process is similar to the previous one.

The final results of adaptive finite element method of the boundary value problem is shown in Figure 4. It can be seen that the grids get finer with ϵ going down.

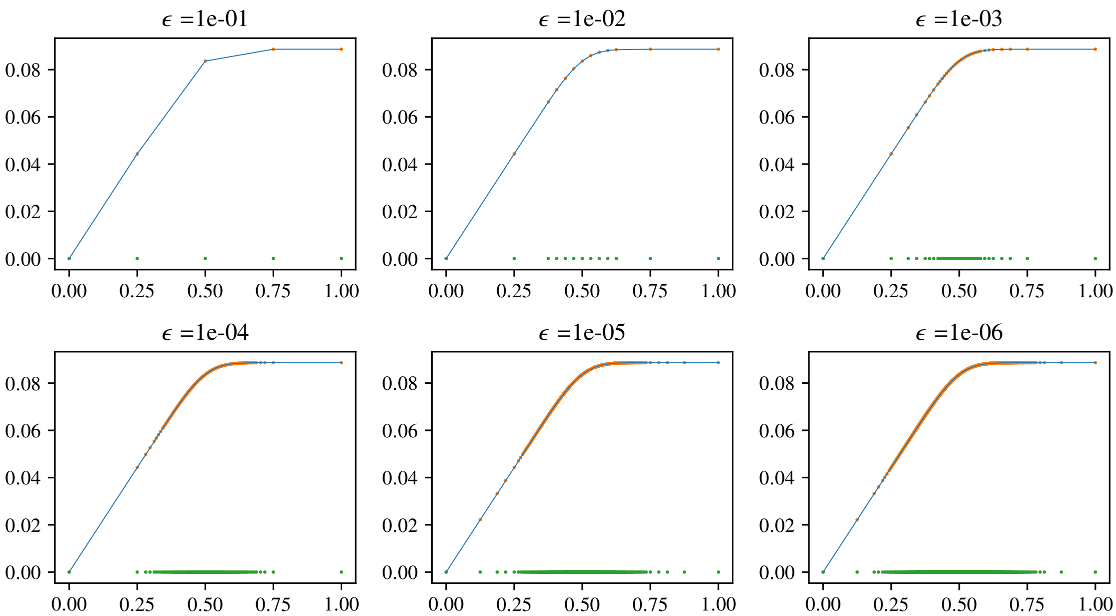

Figure 4 Solution u_h to the boundary value problem