Assignment for Lecture 3

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Question 1

Answer The graph for interpolations to $f_1(x) = \frac{1}{1+x^2}$ using equally spaced nodes is shown in Figure 1. The graph for interpolations of $f_2(x) = \mathrm{e}^{-x^2}$ using equally spaced nodes is shown in Figure 2. Note that polynomials of odd and even orders behave differently, and therefore plots are given separately.

Question 2

Answer The coefficient is given by

$$c_n = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}. \tag{1}$$

Question 3

Answer The algorithm is described in Algorithm 1.

Data: d_i where i ranges from 1 to n
Result: u
$u \leftarrow 0$;
for i *from* n *to* 1 **do**
  $u \leftarrow u + 1$;
  $u \leftarrow u d_i$;
end

Algorithm 1: Calculation of u

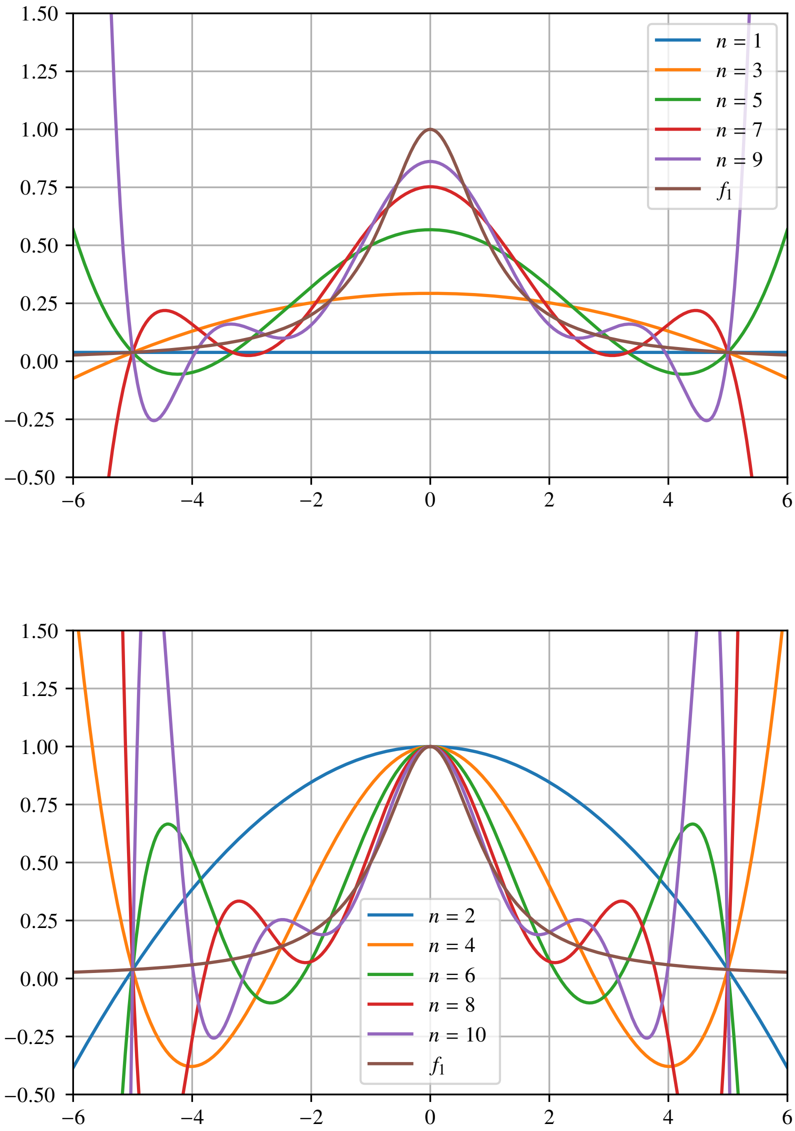

Figure 1 Interpolating polynomials of degree n to f_1 using equally spaced nodes

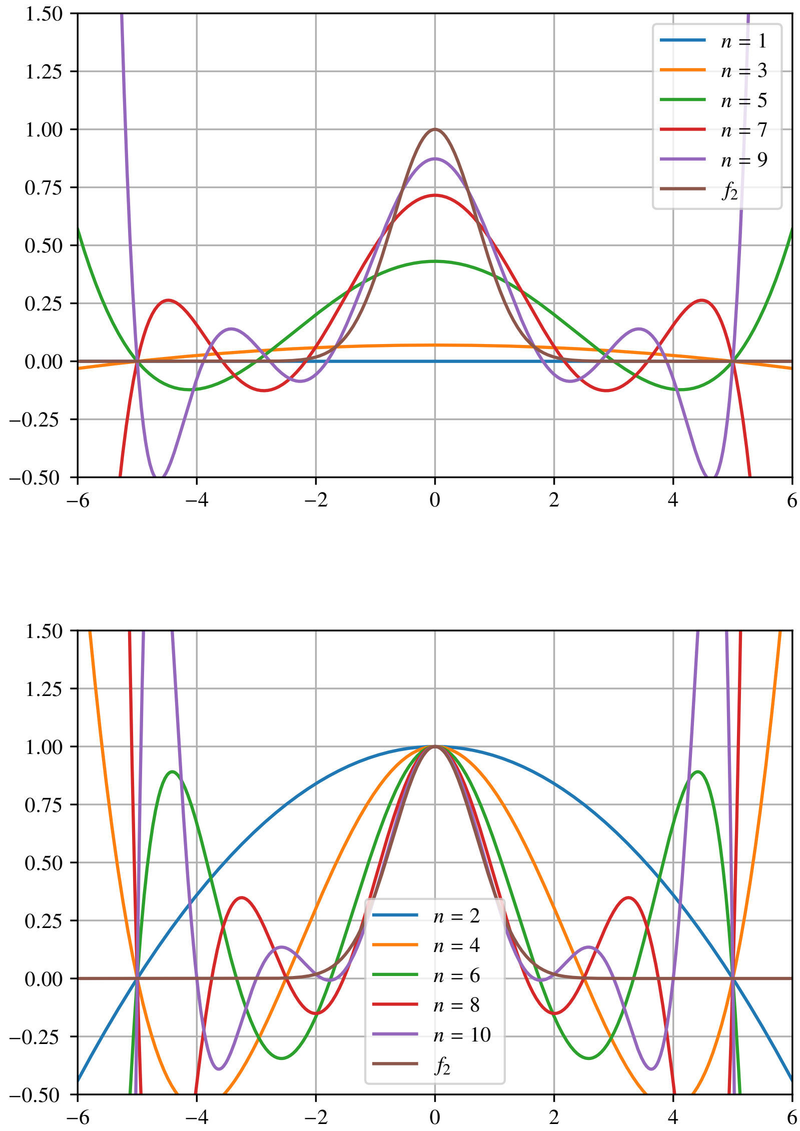

Figure 2 Interpolation polynomials of degree n to f_2 using equally spaced nodes

Question 4

Answer Claim that T_n is even if n is even, or odd if n is odd. Perform mathematical induction on n.

When $n = 0, 1$, $T_n(x) = 1, x$ and is even and odd respective, satisfying the claim.

Suppose the case $n \leq k$ is done where $k \geq 1$, and then consider the case $n = k + 1$. If k is itself even, then T_k is even and T_{k-1} is odd. Therefore, from

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{2}$$

is odd. If k is odd, then T_k is odd and T_{k-1} is even, and similarly T_{k+1} being even follows. Combing these two situation, the claim holds for $n = k + 1$.

By mathematical induction, the claim that T_n and n have the same oddness and evenness for $n \in \mathbb{N}$ is proven.

$\square$

Question 5

Answer Zeros of Chebyshev polynomial T_n (scaled to the interval $[-5, 5]$) are given by

$$t_i = 5\cos\frac{(2i+1)\pi}{2n}. \, (i = 0, 1, \cdots, n-1) \tag{3}$$

The graph for interpolations to $f_1(x) = \frac{1}{1+x^2}$ using zeros of Chebyshev polynomials is shown in Figure 3. The graph for interpolations of $f_2(x) = \mathrm{e}^{-x^2}$ using zeros of Chebyshev polynomials is shown in Figure 4. Note that polynomials of odd and even orders behave differently, and therefore plots are given separately.

Question 6

Answer The matrix is

$$A = \begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix}. \tag{4}$$

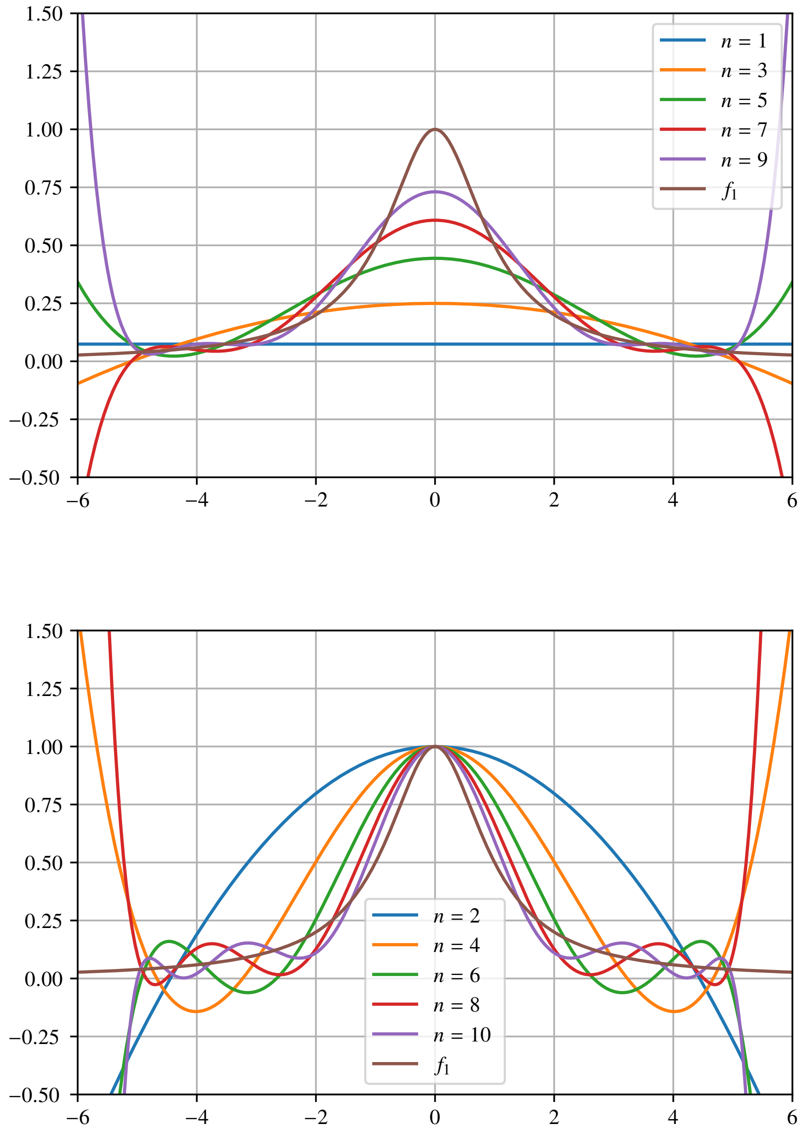

Figure 3 Interpolating polynomials of degree n to f_1 using zeros of T_n

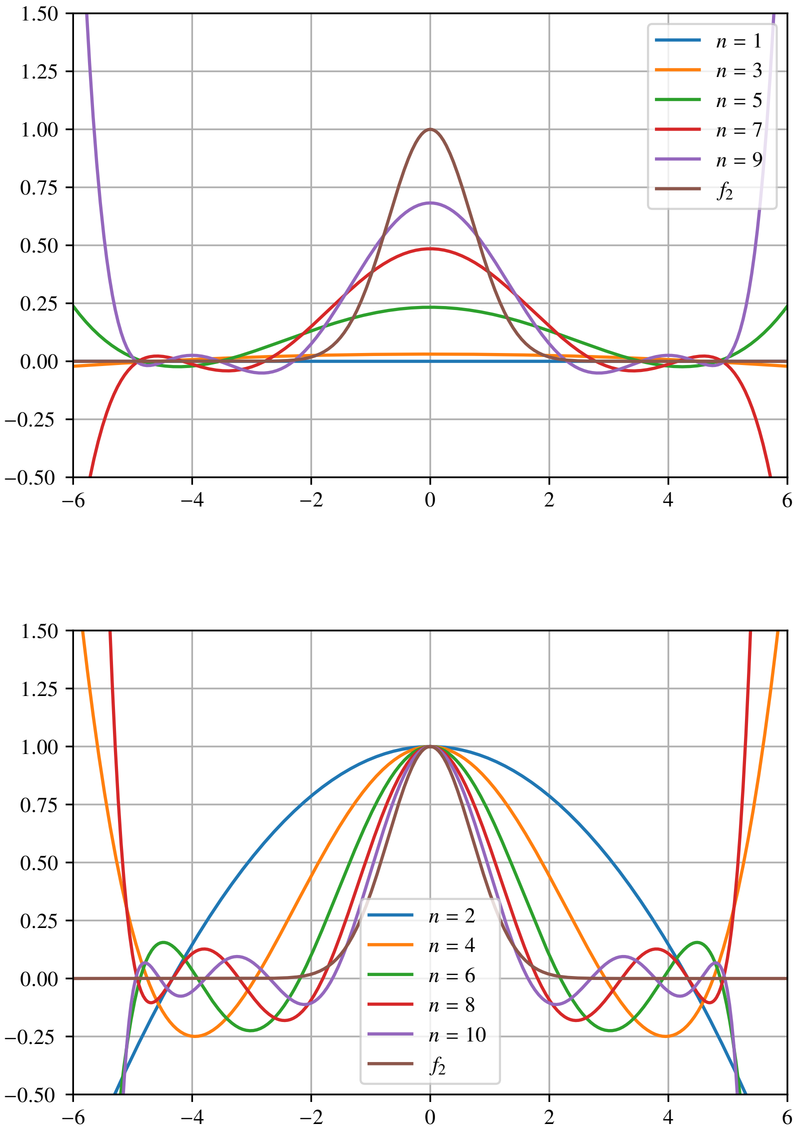

Figure 4　Interpolation polynomials of degree n to f_2 using zeros of T_n

Note that because $u_1 = 2(h_0 + h_1) > h_1$, $u_i = 2(h_i + h_{i-1}) > h_i + h_{i-1}$ $(i = 2, 3, \cdots, n-2)$, $u_{n-1} = 2(h_{n-2} + h_{n-1}) > h_{n-2}$, therefore A is strictly diagonally dominant. From the proof on the slide, one knows LU-decomposition can be perform on A. That is, $A = LU$ with lower triangular L and upper triangular U. From the slide, one knows the diagonal entries of U can be set to 1 and that of L are all non-zero. Therefore, L and U are non-singular and consequently A is invertible.

□

Question 7

Answer Note that there are $3n$ variables for a quadratic spline with $n+1$ nodes, while the number of constraints are $2(n-1) + (n-1) + 2 = 3n - 1$, where $2(n-1)$ stands for values at internal nodes, $(n-1)$ stands for the continuity of first-order derivative and 2 stands for values at end points. Therefore, there are (at least) one degree of freedom and one specific constraint should be forced for uniqueness. For example, such constraint may be that the first-order derivative at one end point is fixed, or that first-order derivatives at both end points coincides. Additionally, one may also consider the spline to be an optimization problem. For example, one may try to minimize

$$E = \int_{t_0}^{t_n} (S'(x))^2 \, \mathrm{d}x, \tag{5}$$

which stands for the elastic energy if the spline is thought to be a thin stick.

Question 8

Answer Assume u to be C^4. Therefore, we have

$$u_{i-1} = u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(x) + O\left(h^4\right), \tag{6}$$

$$u_{i+1} = u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(x) + O\left(h^4\right), \tag{7}$$

$$\tag{8}$$

and therefore

$$u_{i-1} - 2u_i + u_{i+1} = h^2 u''(x) + O\left(h^4\right), \tag{9}$$

which yields

$$\frac{\mathrm{d}^2(u)}{\mathrm{d}x^2} - \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = O\left(h^2\right). \tag{10}$$

□