MEAN VALUE THEOREMS

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OBJECTIVES

After going through this chapter you will be able to: State and prove three mean value theorems (MVT):

Rolle‟s MVT,

Lagrange‟s MVT

and Cauchy‟s MVT

INTRODUCTION:

The Mean Value Theorem is one of the most important theoretical tools in Calculus. Let us consider the following real life event to understand the concept of this theorem: If a train travels 120 km in one hour, then its average speed during is 120 km/hr. The car definitely either has to go at a constant speed of 120 km/hr during that whole journey, or, if it goes slower (at a speed less than 120 km/hr) at a moment, it has to go faster (at a speed more than 120 km/hr) at another moment, in order to end up with an average speed of 120 km/hr. Thus, the Mean Value Theorem tells us that at some point during the journey, the train must have been traveling at exactly 120 km/hr. This theorem form one of the most important results in Calculus.

Geometrically we can say that MVT states that given a continuous and differentiable curve in an interval [a, b], there exists a point c ∈ [a, b] such that the tangent at c is parallel to the secant joining (a, f(a)) and (b, f(b)).

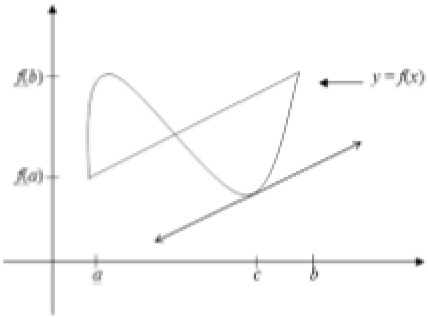
Rolle's Theorem:

If f is a real valued function such that

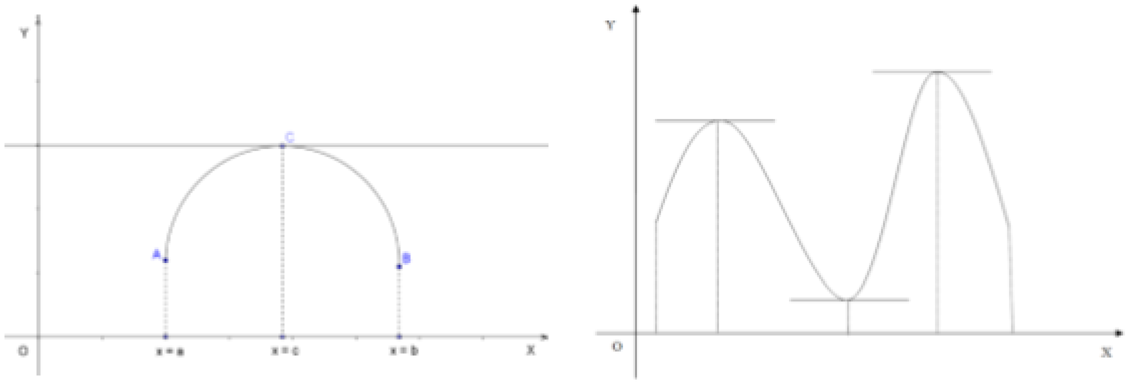
(i) f is continuous on [a, b],

(ii) f is differentiable in (a, b) and

(iii) f(a) = f(b) then there exists a point c (a, b) such that $f^1(c)=0$



Geometrical Interpretation of Rolle's theorem:



Algebraic Interpretation of Rolle's Theorem: We have seen that the third condition of the hypothesis of Rolle's theorem is that f(a) = f(b). If for a function f, both f(a) and f(b) are zero that is a and b are the roots of the equation f(x) = 0, then by the theorem there is a point c of (a, b), where fc ()⍰ = 0, which means that c is a root of the equation fx ()⍰ = 0. Thus Rolle's theorem implies that between two roots a and b of f(x) = 0 there always exists at least one root c of fx ()⍰ = 0 where a < c < b. This is the algebraic interpretation of the theorem.

Example 1: Verify Rolle‟s Theorem for the following

(1) x^2 in $[-1,1]$ (2) x^2 in $[1,3]$

Solution: (1) Let $f(x)=x^2$

, $x \in [-1,1]$

As ⍰ ⍰ xf is a polynomial in x, it is continuous and differentiable everywhere on its domain. Also

F(-1)=f(1)=1

The conditions of the Rolle‟s theorem are satisfied.

We may have to find some c £[1,1] such that $f^1(c)=0$

Now $f(x)=x^2$ $\quad\therefore f'(x)=2x.$ $\quad\therefore f'(c)=2c.$

$\therefore f'(c)=0 \Rightarrow 2c=0$ $\quad\therefore c=0$ and lies in $[-1,1]$

$\therefore$ Rolle's Theorem is verified.

2) Let $f(x)=x^2, \quad x\in[1,3]$

$f(x)$ is polynomial in x. $\therefore$ $f(x)$ is continuous and differentiable everywhere on its domain. i.e. (i) f is continuous on [1, 3] and (ii) f is differentiable in (1, 3).

But we have $f(1)=1$ and $f(3)=9$ which are not equal.

$\therefore$ The values of f at the end points are not equal i.e. $f(1)\neq f(3)$

$\therefore$ The function x^2 in (1, 3) do not satisfy all the conditions of Rolle's Theorem.

Example2: verify rolle's theorem for f(x)= in [-3,0] x(x+3)$e^{-x/2}$ in [-3,0]

Solution: given f(x)= in [-3,0] x(x+3)$e^{-x/2}$ in [3,0]

i). f(x) is continuous in [-3,0] since it is is a product of continuous functions.

(ii) $f'(x)=(2x+3)e^{-x/2}+(x^2+3x)\left(-\frac{1}{2}\right)e^{-x/2}=e^{-x/2}\left[2x+3-\frac{x^2}{2}-\frac{3x}{2}\right]$

$$=e^{-x/2}\left[-\frac{x^2}{2}+\frac{x}{2}+3\right] \text{ exists in } (-3, 0)$$

ii). $F^1(-3)=f(0)=0$

All conditions of Rolle‟s Theorem are satisfied

$\therefore$ There exists $c\in(-3,0)$ such

that $f'(c)=0 \quad \Rightarrow e^{-c/2}\left[-\frac{c^2}{2}+\frac{c}{2}+3\right]=0$

$\Rightarrow -c^2+c+6=0 \quad \Rightarrow c^2-c-6=0$

$\therefore c=3, -2$

$\because 3\notin -3,0 \qquad \therefore c\neq 3, \quad \Rightarrow c=-2\in -3,0$

Hence Rolle‟s theorem is verified and c = - 2 is the required value.

Example 3: verify rolle's theorem for f(x)=$log\left[\frac{x^2+ab}{x(a+b)}\right]$ in [a,b] in a,b >0

Solution: $f(x)$ is continuous in (a, b) and $f(x) = \log(x^2+ab) - \log x - \log(a+b)$

$\therefore f'(x) = \dfrac{2x}{x^2+ab} - \dfrac{1}{x} = \dfrac{x^2-ab}{x(x^2+ab)}$ exists, since it is not indeterminate or infinite.

Also $f(a) = f(b) = 0$ $\therefore$ All conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists $c \in (a,b)$ such that $f'(c) = 0$

$\therefore \dfrac{c^2-ab}{c(c^2+ab)} = 0$ (i.e.) $c^2 - ab = 0$ $\therefore c = \sqrt{ab}$, which lies in (a, b).

Example 4: verify rolle's theorem for f(x)=$e^{-x}(sinx - cosx)$ in $[\frac{\pi}{4},\frac{5\pi}{4}]$

Solution: Since e^{-x}, sinx, cosx are continuous and differentiable functions, the given functions is also continuous in $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ and differentiable in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

Also, $f(\pi/4) = e^{-\pi/4}(\sin\pi/4 - \cos\pi/4) = 0$

$$f(5\pi/4) = e^{-5\pi/4}(\sin 5\pi/4 - \cos 5\pi/4) = 0$$

$$\therefore f(\pi/4) = f(5\pi/4) = 0$$

Hence, Rolle's Theorem is applicable.

Now, $f'(x) = -e^{-x}(\sin x - \cos x) + e^{-x}(\cos x + \sin x) = 2e^{-x}\cos x$

$f'(c) = 2e^{-c}\cos c = 0$ $\therefore c = \pi/2$, which lies in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

Example 5: verify rolle's theorem for f(x)=sin^{2}x ,0$\leq x \leq \pi$

Solution: we have f(x)=sin^{2}x ,0$\leq x \leq \pi$

Since sinx continuous and differentiable in [0,π]

And sin^{2}x is also continuous and differentiable in [0,π]

Now f(0)=f(π)=0

∴ all the conditions of rolle's theorem are satisfied.

At least one point c∈(0,π) such

that $f'(c)=0$ Now, $f'(x) = 2\sin x\cos x = \sin 2x$.

$\therefore f'(c) = \sin 2c. \Rightarrow \quad f'(c) = 0 \Rightarrow \sin 2c = 0 \Rightarrow \quad 2c = 0, \pi, 2\pi, 3\pi, \ldots$

$\therefore c = 0, \frac{\pi}{2}, \pi, \ldots$

Since $c = \frac{\pi}{2}$ lies in $(0,\pi)$, it is the required value. Hence Rolle's theorem is verified.

Example 7: if f(x)=x(x+1)(x+2)(x+3) then show that f(x) has three real roots in [-3,0].

Solution: We apply Rolle's Theorem to $f(x)$ in three intervals $[-1,0]$, $[-2,-1]$, $[-3,-2]$

We observe that

(i) $f(x)$ is continuous in all the intervals since it is a polynomial in x.

(ii) $f(x)$ is differentiable in all the intervals $\therefore$ polynomial in x.

(iii) $f(-3) = f(-2) = f(-1) = f(0) = 0$.

Hence Rolle's Theorem is applicable in all each interval such that $f'(c) = 0$

$\therefore f(x)$ has three real roots.

LAGRANGE'S MEAN VALUE THEOREM

Theorem 6.1 : *If $y = f(x)$ is a real valued function defined on $[a,b]$, such that, (i) $f(x)$ is continuous on a closed interval $[a,b]$, (ii) $f(x)$ is differentiable in (a, b) then there exists at least one point $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Geometrical Interpretation of the Langrange's Mean Value Theorem:

Let $A(a, f(a))$ and $B(b, f(b))$ and , be two points on the curve $y = f(x)$ The slope m of the line AB is given by,

M

$= \frac{f(b)-f(a)}{b-a}$ Also, fc is the slope of the tangent at the point C $(c, f(c))$. . Lagrange‟s Mean Value Theorem says that there exists at least one point C $(c, f(c))$. , on the graph where the slope of the tangent line is same as the slope of line AB. (i.e.) C is a point on the graph where the tangent is parallel to the chord joining the extremities of the curve.

Some Important Deductions from the Mean Value Theorem

Example 11: Verify mean value theorem for f(x)=log x on [1,e]

Solution: The given function is $f(x) = \log x$ on $[1, e]$

We know that $f(x) = \log x$ is continuous on $[1, e]$ and differentiable on (1, e).

Thus all the conditions of Lagrange's mean value theorem are satisfied.

$\therefore\ \exists c \in (1, e)$ such that $\dfrac{f(e) - f(1)}{e - 1} = f'(c)$

$$\therefore \frac{\log e - \log 1}{e - 1} = f'(c)$$

Since $\log e = 1$, $\log 1 = 0$ and $f'(x) = \dfrac{1}{x}$ we get $\dfrac{1}{e-1} = \dfrac{1}{c}$

$\therefore c = e - 1$ which lies in the interval (1, 2) and hence in (1, e), since $2 < e < 3$.

Example 13: show that if x>0, x - $\frac{x^2}{2}$ < log (1+x) < x - $\frac{x^2}{2(1+x)}$ for x>0

Solution: Let us assume, $f \;\; x \;\; = \log \;\; 1 + x \;\; - x + \frac{x^2}{2}$

$\therefore f'\left(x\right) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}$.

$\therefore f'\left(x\right) > 0$ for all $x > 0$ except at $x = 0$. and $f(0) = 0$.

$\therefore f\left(x\right)$ is an increasing function in $\left(0, \infty\right)$

$\therefore f\left(x\right)$ increasing from 0 and hence $f\left(x\right) > 0$.

$\log(1+x) < x - \frac{x^2}{2}$, for $x > 0$

… (i)

Consider,

$f\left(x\right) = x - \frac{x^2}{2\left(1+x\right)} - \log\left(1+x\right)$

$f'\left(x\right) = 1 - \frac{2x - x^2}{2\left(1+x\right)^2} - \frac{1}{1+x} = \frac{x^2}{2\left(1+x\right)^2}$

$\therefore f'\left(x\right) > 0$ for $x > 0$ except at $x = 0$ when it is zero.

$f\left(x\right)$ is an increasing function in $\left(0, \infty\right)$

$f\left(x\right)$ increasing from 0 and hence $f\left(x\right) > 0$.

$\therefore x - \frac{x^2}{2\left(1+x\right)^2} > \log\left(1+x\right)$ for $x > 0$.

… (ii)

From (i) and (ii), $x - \frac{x^2}{2} < \log\left(1+x\right) < x - \frac{x^2}{2\left(1+x\right)^2}$ for $x > 0$.

Show that $\left|\tan^{-1} x - \tan^{-1} y\right| < \left|x - y\right|$

Let $f(x) = \tan^{-1}(x)$

$\therefore$ By Lagrange's Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\therefore \frac{\tan^{-1}(x) - \tan^{-1}(y)}{x - y} = \frac{1}{1 + c^2} \quad \text{for } -\pi/2 < x < c < y < \pi/2$$

But, $\frac{1}{1 + c^2} < 1 \qquad (\because c^2 \text{ is positive})$

$$\therefore \left|\frac{\tan^{-1} x - \tan^{-1} y}{x - y}\right| < 1$$

$$\therefore \left|\tan^{-1} x - \tan^{-1} y\right| < \left|x - y\right|$$

example:

show that , $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$

hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

Let $f(x) = \tan^{-1}(x)$ in $[a, b]$

$$\therefore f'(x) = \frac{1}{1 + x^2}$$

$\therefore$ By Lagrange's M. V. T.

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{where } c \in (a, b)$$

$$\therefore \frac{1}{1 + c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b - a} \qquad (1)$$

Since $a < c < b, \quad a^2 < c^2 < b^2$

$$\therefore 1 + a^2 < 1 + c^2 < 1 + b^2$$

$$\therefore \frac{1}{1 + a^2} > \frac{1}{1 + c^2} > \frac{1}{1 + b^2} \qquad (2)$$

From (1) and (2)

$$\frac{1}{1 + b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1 + a^2}$$

$$\therefore \frac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b - a}{1 + a^2} \qquad (3)$$

For the second part;

Since $\tan^{-1} = \pi/4$ we put $a = 1$ and $b = \frac{4}{3}$ in (3)

$$\therefore \frac{4/3 - 1}{1 + (16/9)} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{4/3 - 1}{1+1}$$

$$\therefore \frac{3}{25} + \pi/4 < \tan^{-1} 4/3 < \frac{1}{6} + \pi/4 .$$

Example: prove that , $\frac{b-a}{b}$ < log(b/a) < $\frac{b-a}{a}$ for 0< a < b

Hence deduce that $\frac{1}{4}$ < log ($\frac{4}{3}$) < $\frac{1}{3}$

Solution: Let $f(x) = \log x$ in $[a, b]$

Since $f(x)$ is (i) continuous in $[a, b]$ and (ii) differentiable in (a, b)

by Lagrange's M. V. T. $\exists c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

But $f(x) = \log x$

$$\therefore f'(x) = \frac{1}{x} \qquad \therefore f'(c) = \frac{1}{c}$$

$$\therefore \frac{\log b - \log a}{b - a} = \frac{1}{c}$$

(1)

But $a < c < b$, $\quad \frac{1}{a} < \frac{1}{c} < \frac{1}{b}$ (2)

From (1) and (2) we get,

$$\frac{1}{b} < \frac{\log b - \log a}{b - a} < \frac{1}{a} \qquad \Rightarrow \frac{b - a}{b} < \log b - \log a < \frac{b - a}{a}$$

$$\therefore \frac{b - a}{b} < \log(b/a) < \frac{b - a}{a}$$

For the second part $a = 3$, $b = 4$.

$$\therefore \frac{1}{4} < \log 4/3 < \frac{1}{3}$$

Cauchy's Mean Value Theorem:

If functions f and g are (*i*) continuous in a closed interval [a, b], (*ii*) differentiable in the open interval (a, b) and (*iii*) $f'\ x \ \neq 0$ for any point of the open interval (a, b) then for some $c \in (a, b)$, $f'\ c\ [g\ b\ - g\ a\ ] = g'\ c\ [f\ b\ - f\ a\ ]$

$$\text{i.e. } \frac{g'\ c}{f'\ c} = \frac{g\ b\ - g\ a}{f\ b\ - f\ a} \qquad a < c < b.$$

Example 20: Verify Cauchy‟s MVT for the function x 2and x3in the interval [1, 2].

Solution:

Let $f(x) = x^2$ and let $g(x) = x^3$.

As $f(x)$ and $g(x)$ are polynomials (*i*) they are continuous on [1, 2], (*ii*) they are differentiable on (1, 2) and (*iii*) $g'(x) \neq 0$ for any value in (1, 2)

$\therefore$ Cauchy's mean value theorem can be applied. $\therefore$ If $c \in (1,2)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

$$\frac{2c^2}{3c^2} = \frac{2^2 - 1^2}{2^3 - 1^3} = \frac{4-1}{8-1} = 3/7 \qquad \Rightarrow \qquad \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow 9c = 14 \qquad \therefore c = \frac{14}{9} \in (1, 2)$$

$\therefore$ Cauchy mean value theorem is verified.

Example: using CMVT show that $\frac{sinb-sina}{cosa-cosb}$ = cotc,a < c <b., a>0 and b>0

Solution: Let $f(x) = \sin x$ and $g(x) = \cos x$.

Here, $f(x)$ and $g(x)$ are continuous on [*a*, *b*] and differentiable on (*a*, *b*) and for any c in (*a*, *b*), thus CMVT can be applied.

$\therefore c \in (a,b)$ such that, $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\therefore \frac{-\cos c}{\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a} \qquad \Rightarrow \qquad \cot c = \frac{\sin b - \sin a}{\cos a - \cos b}$$

Example: if in CMVT we write f(x)=e^x and g(x)=e^{-x} show that c is the arithmetic mean between a and b

Solution: Now $f(x)=e^x$ and $g(x)=e^{-x}$

If can be proved that function $f(x)$ and $g(x)$ are continuous on any closed interval $[a, b]$ and differentiable in (a, b). Also $g'(x) \neq 0$ and $x \in (a,b)$

Then CMVT can be applied. $\therefore \exists\ c \in (a,b)$ such that, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Now $\dfrac{f'(c)}{g'(c)} = \dfrac{e^c}{-e^{-c}} = -e^{2c}$ and $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{e^b - e^a}{e^{-b}-e^{-a}} = -e^{a+b}$ where $c \in (a,b)$

$\therefore -e^{2c} = -e^{a+b} \qquad \Rightarrow a+b=2c$

$\therefore c = \dfrac{a+b}{2} \in (a,b)$

Thus, c is the arithmetic mean between a and b.

Example: if 1<a<b , show that there exist c satisfying a<c<b such that log($\frac{b}{a}$)=$\frac{b^2-a^2}{2c^2}$

Solution: We have to prove that, $\dfrac{\log b - \log a}{b^2-a^2} = \dfrac{1}{2c^2}$

This suggests us to take $f(x) = \log x$ and $g(x) = x^2$ Now, $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for any c in (a, b).

$\therefore$ CMVT can be applied. $\qquad \therefore \exists\ c \in (a,b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \Rightarrow \frac{1/c}{2c} = \frac{\log b - \log a}{b^2-a^2}$$

$$\therefore \frac{1}{2c^2} = \frac{\log b - \log a}{b^2-a^2} \quad \Rightarrow \log b/a = \frac{b^2-a^2}{2c^2}$$

Beta and gamma function

Improper integrals

consider the integral $\int_a^b f(x)dx$ such an integral, for which

i). either the interval of integration is not finite. i.e., a=-∞ or b=∞ or both

ii). Or the function f(x) is unbounded at one or more points in [a,b] is called an improper integral.

Definition

Beta function:

The definite integral $\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$ is called the beta function and denoted by B(m,n) and read as "beta m,n". the above integral converges for m>0, n>0

Properties of beta function:

i). symmetry of beta function i.e., B(m,n)=B(n,m)

proof: by definition, we have

B(m,n)=$\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$

Put 1-x=y so that dx=-dy

$\therefore$ B(m,n)=$\int_1^0 (1-y)^{m-1}\,y^{n-1}\,(-dy)$ = $\int_0^1 y^{n-1}\,(1-y)^{m-1}\,dy$

=$\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$=B(n,m) $\left[\int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$

Hence B(m,n)=B(n,m)

ii). B(m,n)=2 $\int_0^{\frac{\pi}{2}} \boldsymbol{sin^{2m-1}\theta\ cos^{2n-1}\ \theta\ d\theta}$

proof: By definition ,we have

B(m,n)= $\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$

Put x= sin$^2\theta$ so that dx = sin2θ dθ

$\therefore$ B(m,n)= $\int_0^{\pi/2} (sin^2\,\theta)^{m-1}\,(1-sin^2\theta)^{n-1}\,sin2\theta\,d\theta$

B(m,n)= $\int\ \ sin^{2m-2}\theta\,cos^{2n-2}\theta\,2\,sin\theta\,cos\theta\,d\theta$

=2 $\int_0^{\pi/2} sin^{2m-1}\,cos^{2n-1}\,\theta\,d\theta$ ....(1)

Note : from (1) , we have

$\int_0^{\pi/2} \boldsymbol{sin^{2m-1}\,cos^{2n-1}\,\theta\,d\theta}$ $=\frac{1}{2}$**B(m,n)**

iii). B(m,n)= B(m+1,n) + B(m,n+1)

proof: B(m+1,n) + B(m,n+1)= $\int_0^1 x^m\,(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}\,(1-x)^n\,dx$

=$\int_0^1 [x^m\,(1-x)^{n-1} + \int_0^1 x^{m-1}\,(1-x)^n]\,dx$

= $\int_0^1 x^{m-1}\,(1-x)^{n-1}\,[x+(1-x)]dx$

$$= \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx = \text{B(m,n)}$$

Hence B(m,n)= B(m+1,n) + B(m,n+1)

Solved examples:

Example 1: show that $\int_0^{\pi/2} sin^m\theta\ cos^n\theta\ d\theta$

Solution: $\int_0^{\pi/2} sin^m\theta\ cos^n\theta\ d\theta = \int_0^{\pi/2} sin^{m-1}\theta\ cos^{n-1}\theta(sin\theta\ cos\theta)\ d\theta$

$$= \int_0^{\pi/2} (sin^2\ \theta)^{(m-1)/2}\ (cos^2\theta)^{(n-1)/2}\ sin\theta\ cos\theta\ d\theta$$

Put $\sin^2\theta$=x so that $\sin\theta cos\theta$ dθ =$\frac{dx}{2}$

$$\int_0^{\pi/2} sin^m\theta\ cos^n\theta\ d\theta = \frac{1}{2}\int_0^1 x^{(m-1)/2}\ (1-x)^{(n-1)/2}\ dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1}\ (1-x)^{\frac{n+1}{2}-1}\ dx$$

$$= \frac{1}{2}\,\text{B}\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Example 2: Express the following integrals in terms of beta function

i). $\int_0^1 \frac{x}{\sqrt{1-x^2}}$ dx **ii). $\int_0^1 \frac{1}{\sqrt{9-x^2}}$ dx**

solution: i) put x^2=y so that dx=$\frac{dy}{2x}$ =$\frac{1}{2}\ y^{\left(-\frac{1}{2}\right)}dy$

when x=0 , y=0 ; when x=1 ,y=1

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\ \text{dx} = \int_0^1 \frac{y^{\left(\frac{1}{2}\right)}}{\sqrt{1-y}}\ \frac{1}{2}\,y^{\left(-\frac{1}{2}\right)}\ dy$$

$$=(1/2)\ \text{y} \int_0^1 (1-y)^{\left(-\frac{1}{2}\right)}\ \text{dy}$$

$$=(1/2) \int_0^1 y^{1-1}\ (1-y)^{\frac{1}{2}-1}\ dy$$

$$=(1/2)\ \text{B}\left(1,\frac{1}{2}\right)$$

ii). Put x^2=9y so that dx=(3/2) $y^{\left(-\frac{1}{2}\right)}\ dy$

$$\int_0^1 \frac{1}{\sqrt{9-x^2}}\ \text{dx} = \int_0^3 (9-x^2)^{\left(-\frac{1}{2}\right)}\ dx$$

$$= \int_0^1 (9-9y)^{\left(-\frac{1}{2}\right)}\ \text{dy}$$

$$=(3/2) \int_0^1 y^{\left(-\frac{1}{2}\right)}\ \left(\frac{1}{3}\right)(1-y)^{\left(\frac{1}{2}\right)dy}\ \text{r}\Gamma$$

$=(1/2)\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{2}-1}\,dy$

$=(1/2)\ B(\frac{1}{2},\frac{1}{2})$

Gamma function

Definition: The definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$ is called the gamma function and is defined by Γ(n) and read as " gamma n"

The integral converges only for n>0

Thus $\Gamma(n)=\int_0^\infty e^{-x}\, x^{n-1}\, dx$, where n>0

Gamma function is also called eulerian integral of the second kind.

The integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$ does not conveges if n≤ 0.

Properties of gamma function

i).To show that $\Gamma(1)=1$

proof: by the definition of gamma function , we have

$$\Gamma(n)=\int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$\Gamma(1)=\int_0^\infty e^{-x}\, x^{1-1}\, dx$$

$$=\int_0^\infty e^{-x}\, dx$$

$$=(-e^{-x})_0{}^\infty$$

$$=-(0-1)$$

$$=1 \text{ for } \lim_{x\to\infty}\frac{1}{e^x}=0$$

ii). To show that Γ(n)=(n-1) Γ(n-1) ,where n>1

proof: $\Gamma(n)=\int_0^\infty e^{-x}\, x^{n-1}\, dx=\left[x^{n-1}\,\frac{e^{-x}}{(-1)}\right]_0^\infty-\int_0^\infty (n-1)x^{n-2}\,\frac{e^{-x}}{-1}\,dx$

$$=-\lim_{x\to\infty}\frac{x^{n-1}}{e^x}=0+(n-1)\int_0^\infty e^{-x}\, x^{n-2}\, dx$$

$$=(n-1)\int_0^\infty e^{-x}\, x^{n-2}\, dx \ (\text{ since } \lim_{x\to\infty}\frac{x^{n-1}}{e^x}=0 \text{ for } n>1$$

Γ(n)=(n-1) Γ(n-1)

iii). If n is a non-negative integer ,then Γ(n+1)=n!

proof: from property ii ,we have

Γ(n+1)= n Γ(n) = n(n-1) Γ(n-1) , by property ii again

=n(n-1)(n-2) Γ(n-2) ,by property ii again

$=n(n-1)(n-2)(n-3)\ \Gamma(n-3)$

$=n(n-1)(n-2)(n-3)\ldots\ldots.3.2.1.\ \Gamma(1)$ (since $\Gamma(1)=1$)

$=n!$

Thus $\Gamma(n+1)=n!$ $(n=0,1,2,3,\ldots\ldots)$