Imp: **Curl of a vector:**

Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $(\nabla \times \bar{f})$ or Curl $\bar{f}$ and is denoted by

$$\boxed{\text{Curl } \bar{f} = \hat{i} \times \frac{\partial \bar{f}}{\partial x} + \hat{j} \times \frac{\partial \bar{f}}{\partial y} + \hat{k} \times \frac{\partial \bar{f}}{\partial z}}$$

(or) $$\nabla \times \bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note: If $\bar{f}$ is a constant vector then

Curl $\bar{f} = 0$.

Irrotational vector:

If Curl $\bar{f} = 0$ then $\bar{f}$ is called Irrotational vector.

Scanned with CamScanner

Examples:

Eg1: If $\bar{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ find Curl $\bar{f}$ at the point $(1,-1,1)$

Sol; Let $\bar{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$$\text{Curl } \bar{f} = \nabla \times \bar{f} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \times \bar{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right\}$$

$$\nabla \times \bar{f} = \hat{i}\{-3z^2 - 2x^2y\} - \hat{j}\{0-0\} + \hat{k}\{4xyz - 2xy\}$$

$$(\nabla \times \bar{f})_{\substack{(1,-1,1)\\ x\ y\ z}} = \hat{i}\{-3+2\} - \hat{j}\{0\} + \hat{k}\{-4+2\}$$

$$= -1\hat{i} - 2\hat{k}$$

$$\boxed{\text{Curl } \bar{f}_{\text{at }(1,-1,1)} = -\hat{i} - 2\hat{k}}$$

Scanned with CamScanner

Eg 2): Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$.

2

Sol: Here we have

$$\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$$

where $\phi = x^3+y^3+z^3-3xyz$

First find grad ϕ:

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)+$$

$$\hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\nabla\phi = \hat{i}\{3x^2-3yz\}+\hat{j}\{3y^2-3xz\}+\hat{k}\{3z^2-3xy\} = \bar{f}$$

Now To find curl $\bar{f}$:

$$\nabla\times\bar{f} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\bar{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right\} -$$

$$\hat{j}\left\{\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right\} +$$

$$\hat{k}\left\{\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right\}$$

$$= \hat{i}\{0-\cancel{3x}-0+\cancel{3x}\} - \hat{j}\{0-\cancel{3y}-0+\cancel{3y}\} + \hat{k}\{-\cancel{3z}+\cancel{3z}\}$$

$= 0.$

Hence Curl $\bar{F} = 0$

$\therefore$ $\bar{F}$ is irrotational vector.

Eg 3): If $\bar{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ then show that $\bar{F} \cdot \text{Curl}\,\bar{F} = 0$.

Sol: $\bar{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$

$$\text{curl}\,\bar{F} = \nabla \times \bar{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right\} + \hat{k}\left\{0 - \frac{\partial}{\partial y}(x+y+1)\right\}$$

$$= \hat{i}\{-1\} - \hat{j}\{-1-0\} + \hat{k}\{-1\}$$

$$\therefore\ \text{curl}\,\bar{F} = -\hat{i} + \hat{j} - \hat{k}$$

<u>now $\bar{F}\cdot\text{curl}\,\bar{F}$:</u>

$$= \left\{(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}\right\}\cdot\left\{-\hat{i} + \hat{j} - \hat{k}\right\}$$

$$= -(x+y+1) + 1 + x + y$$

$$= \cancel{-x} \cancel{-y} \cancel{-1} \cancel{+1} \cancel{+x} \cancel{+y}$$

$$= 0$$

Hence Proved.

Eg 4): Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n\bar{r}$ is Irrotational.

(Or)

Show that $\text{Curl}(r^n\bar{r}) = 0$.

Proof:

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \quad r^2 = x^2+y^2+z^2 \quad \text{——} (1)$$

P. diff. w.r.t 'x', 'y' & 'z' Respectively.

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \text{——} (a), \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \text{——} (b) \quad \& \quad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}} \text{——} (c).$$

To find Curl $(r^n\bar{r})$:

$$\nabla \times (r^n\bar{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (x\hat{i} + y\hat{j} + z\hat{k})r^n$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n . x & r^n . y & r^n . z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(r^n \cdot z) - \frac{\partial}{\partial z}(r^n \cdot y)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(r^n \cdot z) - \frac{\partial}{\partial z} r^n \cdot x\right\} + \hat{k}\left\{\frac{\partial}{\partial x} r^n \cdot y - \frac{\partial}{\partial y} r^n \cdot x\right\}$$

$$= \hat{i}\left\{z \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial y} - y \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial z}\right\} - \hat{j}\left\{z \cdot n \cdot r^{n-1} \frac{\partial r}{\partial x} - x \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial z}\right\}$$
$$+ \hat{k}\left\{y \cdot n \cdot r^{n-1} \frac{\partial r}{\partial x} - x \cdot n \cdot r^{n-1} \frac{\partial r}{\partial y}\right\}$$

Put Eq. (a), (b) & (c) in the above

$$= \hat{i}\left\{z \cdot n \cdot \cancel{r^{n-1}} \cdot \frac{y}{r} - y \cdot n \cdot \cancel{r^{n-1}} \cdot \frac{z}{r}\right\} - \hat{j}\left\{z \cdot n \cdot \cancel{r^{n-1}} \frac{x}{r} - x \cdot n \cdot \cancel{r^{n-1}} \frac{z}{r}\right\} + \hat{k}\left\{y \cdot n \cdot \cancel{r^{n-1}} \frac{x}{r} -\right.$$
$$\left. x \cdot n \cdot \cancel{r^{n-1}} \frac{y}{r}\right\}$$

$= 0$

$\therefore$ Hence $\bar{F} \cdot \operatorname{curl} \bar{F} = 0$. **Proved.**

Eg 5): Prove that curl $\bar{r} = \bar{0}$.

Proof: Here we know that

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{curl } \bar{r} = \nabla \times \bar{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right] - \hat{j}\left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right] + \hat{k}\left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]$$

$$= \hat{i}[0] - \hat{j}[0] + \hat{k}[0]$$

$$= 0. \quad \Longrightarrow \quad \underline{\text{curl } \bar{r} = 0}$$

Hence $\bar{r}$ is Irrotational vector.

Scanned with CamScanner

Scalar potential:

If $\bar{F}$ is an irrotational vector then

$$\nabla \times \bar{F} = 0 \quad \text{——— (1)}$$

But, by the identity

$$\nabla \times \nabla \phi = 0 \quad \text{——— (2)}$$

from (1) & (2) we get

$$\boxed{\bar{F} = \nabla \phi}$$

Then $\bar{F}$ is called conservative and ϕ is called the scalar potential.

Examples:

1. Show that the vector

$$\bar{f} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} \text{ is}$$

irrotational then find its scalar potential.

Sol: $\bar{f} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

To find $\nabla \times \bar{f}$

$$\nabla \times \bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \bar{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2-zx) - \frac{\partial}{\partial y}(x^2-yz)\right]$$

$$= \hat{i}[-x+x] - \hat{j}[-y+y] + \hat{k}[-z+z]$$

$= 0.$

Hence $\nabla \times \bar{f} = 0 \Rightarrow \bar{f}$ is Irrotational.

Then $\exists\ \phi$ such that $\boxed{\bar{f} = \nabla\phi}$

$$(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

comparing the co-efficients of $\hat{i}, \hat{j}$ & $\hat{k}$

$$\frac{\partial\phi}{\partial x} = x^2 - yz$$

Apply S.O.B.S. with respect to 'x'

$$\int \frac{\partial\phi}{\partial x} = \int(x^2-yz)\,dx + c$$

$$\phi = \int x^2\,dx - \int yz\,dx + c$$

$$\boxed{\phi = \frac{x^3}{3} - xyz + c} \quad \text{——— (1)}$$

$$\frac{\partial\phi}{\partial y} = y^2 - 2x$$

— Apply ∫ o.b.s. w.r.t 'y'

$$\int \frac{\partial\phi}{\partial y} = \int (y^2 - 2x)\,dy + c$$

$$\boxed{\phi = \frac{y^3}{3} - xyz + c} \quad \text{—— (2)}$$

Ⅲ $$\frac{\partial\phi}{\partial z} = -z^2 - xy$$

— Apply ∫ o.b.s. w.r.t 'z'

$$\int \frac{\partial\phi}{\partial z} = \int (-z^2 - xy)\,dz + c$$

$$\boxed{\phi = \frac{-z^3}{3} - xyz + c} \quad \text{—— (3)}$$

from (1), (2) & (3) the scalar potential

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

$$\boxed{\phi = \frac{1}{3}(x^3 + y^3 + z^3) + c}$$

2. find the scalar potential function of an irrotational vector $\bar{f} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol: Here $\bar{f} = x\hat{i} + y\hat{j} + z\hat{k}$

Now to find $\nabla \times \bar{f}$:

$$\nabla \times \bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right] - \hat{j}\left[\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right] + \hat{k}\left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]$$

$$= 0$$

Hence $\bar{f}$ is irrotational then $\exists\, \phi$ such that $\boxed{\bar{f} = \nabla\phi}$

$$(x\hat{i} + y\hat{j} + z\hat{k}) = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

comparing the corresponding co-efficients

$$\frac{\partial \phi}{\partial x} = x$$

— Apply S.O.B.S

$$\int \frac{\partial \phi}{\partial x} = \int x + c$$

$$\boxed{\phi = \frac{x^2}{2} + c}$$

& $$\frac{\partial \phi}{\partial y} = y$$

- Apply $\int$ O.B.S

$$\int \frac{\partial \phi}{\partial y} = \int y \, dy + c$$

$$\boxed{\phi = \frac{y^2}{2} + c}$$

& also

$$\frac{\partial \phi}{\partial z} = z$$

- Apply $\int$ O.B.S

$$\int \frac{\partial \phi}{\partial z} = \int z \, dz + c$$

$$\boxed{\phi = \frac{z^2}{2} + c}$$

Hence the scalar potential

$$\boxed{\phi = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c}$$