Divergence of a vector:

Let $\bar{f}$ be any continuously differentiable vector point function, Then

$$\operatorname{div} \bar{f} = \left(\hat{i} \frac{\partial \bar{f}}{\partial x} + \hat{j} \frac{\partial \bar{f}}{\partial y} + \hat{k} \frac{\partial \bar{f}}{\partial z} \right)$$

$$\boxed{\operatorname{div} \bar{f} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial y} \right) \bar{f}}$$

where $\bar{f}$ vector point function

Note: $\operatorname{div} \bar{f} = \nabla . \bar{f}$

Hence it is scalar point function.

Solenoidal vector:

A vector point function $\bar{f}$ is said to be solenoidal vector if $\operatorname{div} \bar{f} = 0$ (or) $\nabla . \bar{f} = 0$.

Note: 1. Grad $\phi = \nabla . \phi \rightarrow$ is vector point fun.

2. $\operatorname{div} \bar{f} = \nabla . \bar{f} \rightarrow$ is scalar point fun.

3. Curl $\bar{f} = \nabla \times \bar{f} \rightarrow$ is vector point fun.

Examples:

1. If $\bar{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ find div $\bar{f}$ at $(1,-1,1)$

Sol: $\text{div}\,\bar{f} = \nabla\cdot\bar{f}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2\frac{\partial}{\partial x}(x) + 2x^2z\frac{\partial}{\partial y}(y) + (-3y)\frac{\partial}{\partial z}(z^2)$$

$$= y^2 + 2x^2z - 3y(2z)$$

$$\nabla\cdot\bar{f} = y^2 + 2x^2z - 6yz$$

$$(\nabla\cdot\bar{f})_{\text{at }(1,-1,1)} = (-1)^2 + 2(1)^2(1) - 6(-1)(1)$$

$(1,-1,1)$ = (x, y, z)

$$= 1 + 2 + 6 = 9.$$

2. Find div $\bar{f}$ when $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol: Here $\bar{f} = \text{grad}(\underbrace{x^3+y^3+z^3-3xyz}_{\phi})$

$$= \nabla\phi$$

where $\phi = x^3+y^3+z^3-3xyz$

first find $\nabla\phi$ then find $\nabla\bar{f}$

find $\nabla\phi$:

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$+ \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = \hat{i}(3x^2-3yz.1) + \hat{j}(3y^2-3xz) + \hat{k}(3z^2-3xy)$$

now find $\nabla\bar{f}$

$$\nabla\bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{\hat{i}(3x^2-3yz) + \hat{j}(3y^2-3xz) + \hat{k}(3z^2-3xy)\right\}$$

multiply corresponding unit vectors

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(3z^2)$$

$$= 3(2x) + 3(2y) + 3(2z)$$

$$\nabla\bar{f} = 6x + 6y + 6z = 6(x+y+z).$$

3. If $\bar{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+pz)\hat{k}$ is solenoidal, find p.

Sol: we know that,

def. of solenoidal,

$\bar{f}$ is said to be solenoidal if $\nabla \cdot \bar{f} = 0$.

∴ The given $\bar{f}$ is solenoidal then

$$\nabla \bar{f} = 0$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(x+3y)\hat{i} + (y-2z)\hat{j} + (x+pz)\hat{k}\right\} = 0.$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+pz) = 0$$

$$\Rightarrow (1+0) + (1-0) + (0+p) = 0$$

$$\Rightarrow 2+p = 0$$

$$\boxed{p = -2}$$

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4. Find div $\bar{f}$ where $\bar{f} = r^n \bar{r}$. Find 'n' if it is solenoidal?

Sol: Here we know $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\therefore \boxed{r^2 = x^2+y^2+z^2} \quad \text{--- (1)}$$

P. diff. w.r.t 'x', 'y' & 'z' Respectively to Eq. (1)

$r^2 = x^2+y^2+z^2$

$\frac{\partial}{\partial x} r^2 = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \quad \text{--- (a)}$$

$r^2 = x^2+y^2+z^2$

$\cancel{2} r \frac{\partial r}{\partial y} = \cancel{2} y$

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \quad \text{--- (b)}$$

$r^2 = x^2+y^2+z^2$

$\cancel{2} r \frac{\partial r}{\partial z} = \cancel{2} z$

$$\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}} \quad \text{--- (c)}$$

Now find div $\bar{f}$:

$$\text{div } \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot r^n \bar{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k}) \cdot r^n$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r^n x\hat{i} + y r^n \hat{j} + z r^n \hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x r^n \hat{i} + y r^n \hat{j} + z r^n \hat{k}\right)$$

$$= \frac{\partial}{\partial x}(\underset{u}{x} . \underset{v}{r^n}) + \frac{\partial}{\partial y}(y . r^n) + \frac{\partial}{\partial z}(z . r^n)$$

$$= \left\{x . n . r^{n-1} \frac{\partial r}{\partial x} + r^n . 1\right\} + \left\{y . n . r^{n-1} \frac{\partial r}{\partial y} + r^n . 1\right\} + \left\{z . n . r^{n-1} \frac{\partial r}{\partial z} + r^n . 1\right\}$$

(∵ the Eqn of ⓐ, ⓑ, ⓒ we get)

$$= x . n . r^{n-1} \frac{x}{r} + y . n . r^{n-1} . \frac{y}{r} + z . n . r^{n-1} \frac{z}{r} + 3r^n$$

$$= n . r^{n-2} . x^2 + n . r^{n-2} . y^2 + n . r^{n-2} . z^2 + 3 . r^n$$

$$= n . r^{n-2}\left[x^2 + y^2 + z^2\right] + 3 . r^n$$

$$= n . r^{n-2} . r^2 + 3r^n$$

$$\boxed{\nabla . \bar{f} = n . r^n + 3r^n}$$

from the given $\bar{f}$ is solenoidal $\Rightarrow$

$$\nabla . \bar{f} = 0$$

$$n . r^n + 3r^n = 0$$

$$r^n[n + 3] = 0 \Rightarrow n + 3 = 0$$

$$\Rightarrow \boxed{n = -3}$$

5. Evaluate $\nabla\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$
V. Imp
& $|\bar{r}| = r$.

Sol:

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\Longrightarrow \boxed{r^2 = x^2+y^2+z^2} \quad \text{--- (1)}$$

P. diffn w.r.t 'x', 'y' & 'z' Respectively,

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \quad ; \quad \cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y \ ; \ \& \ \cancel{2}r\frac{\partial r}{\partial z} = \cancel{2}z$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \text{ --- (a)} \qquad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \text{ --- (b)} \qquad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}} \text{ --- (c)}$$

Now to find $\nabla\left(\frac{\bar{r}}{r^3}\right)$:

$$\nabla(\bar{r}.r^{-3}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k}).r^{-3}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xr^{-3}\hat{i} + yr^{-3}\hat{j} + zr^{-3}\hat{k})$$

$$= \frac{\partial}{\partial x}(xr^{-3}) + \frac{\partial}{\partial y}(yr^{-3}) + \frac{\partial}{\partial z}(zr^{-3})$$

$$= \left\{x.(-3)r^{-4}.\frac{\partial r}{\partial x} + r^{-3}.1\right\} + \left\{y.(-3).r^{-4}\frac{\partial r}{\partial y} + r^{-3}\right\}$$
$$+ \left\{z.(-3).r^{-4}\frac{\partial r}{\partial z} + r^{-3}\right\}$$

Put Eq" ⓐ, ⓑ & ⓒ in the above

$$= -3 \cdot x \cdot r^{-4} \cdot \frac{x}{r} + (-3) \cdot y \cdot r^{-4} \cdot \frac{y}{r} + (-3) \cdot z \cdot r^{-4} \frac{z}{r} + 3r^{-3}$$

$$= \frac{-3r^{-4}}{r}\left[x^2+y^2+z^2\right] + 3r^{-3}$$

$$= -3r^{-5} \cdot r^2 + 3 \cdot r^{-3}$$

$$= -3\cancel{r^{-3}} + 3\cancel{r^{-3}}$$

$$= 0.$$

$$\therefore \quad \nabla\left(\frac{\bar{r}}{r^3}\right) = 0$$

Hence $\frac{\bar{r}}{r^3}$ is solenoidal.

6. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then find div $\bar{r}$.

Sol: $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now To find div $\bar{r}$:

$$\text{div}\,\bar{r} = \nabla \cdot \bar{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3.$$

H.W:

1. Show that i) $3y^4z^2\hat{i} + z^3x^2\hat{j} - 3x^2y^2\hat{k}$
Imp
ii) $(x+3y)\hat{i} + (y-2z)\hat{j} + (x-2z)\hat{k}$
is solenoidal.

2. If $\bar{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$
V.Imp. is solenoidal then find a.