Eigen Values And Eigen Vectors

Definition:-

Let A be a given square matrix.
Then there exists a scalar and non-zero vector X such that

$$AX = \lambda X ....(1)$$

Our aim is to find and x for given matrix A using equation (1)
λ is called as eigen value, latent roots of a matrix value, characteristic value or root of a matrix A and x is called as eigen vector or characteristic vector etc.
X is a column matrix

Method of finding λand x :-

We have,

AX=**λ**X

AX-**λ** IX=0

(A-**λ**I)X=0 ------------(2)

Equation 2 is a set of homogenous equation and for non-zero x, we have

$|A - \lambda I|$=0 ---------(3)

This equation is called the characteristic equation of
First we solve equation (3) to find eigen values or roots. Then we solve equation (2) to find Eigen vectors.

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Equation 2 (A-**λ**I)X=0 becomes

$$\left\{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e. \begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (2)$$

and equation (3) i.e. $|\text{A-}\lambda\ \text{x}| = 0$ is

$$\begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix} = 0 \rightarrow (3)$$

Note :

1) Equation (2) is called as matrix equation of A in
2) Equation (3) is called as characteristic equation of A in
3) Usually given matrix A is of order 3X3 . Therefore it will have 3 eigen values and for every eigen value there will be corresponding eigen vector which is a column matrix of order 3X1. There are 3 such column matrices.
4) Eigen vectors are linearly independent.
5) Method of finding eigen values is same for any given matrix A. Method of finding eigen vectors is slightly different and we study 3 types of such problems.

Type (I) : When all eigen values are distinct and matrix A may be symmetric or non- symmetric.

Type (II) : When eigen values are repeated and A is non-symmetric

Type (III) : When eigen values are repeated and A is symmetric.

Solved examples :-

Type (I) : All roots are non- repeated.

Example 4: Find eigen values and given vectors for

$$A=\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

Solution: *Step* (1) : Charactristic equation of A in λ i s

$$|A-\lambda I|=0$$

$$i.e.\begin{vmatrix}2-\lambda & -2 & 3\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{vmatrix}=0$$

$$\therefore\ \lambda^3-(\textit{sum of}\ \text{diagonal elements of A})\ \lambda^2+$$
$$(\textit{sum of}\ \text{minors of diagonal elements of A})\ \lambda-|A|=0$$

$\therefore |A|$ = 2(-1-3)+2(-1-1)+3(3-1)

$|A|$= -6

Characteristic equation is given by

$$\therefore\ \lambda^3-2\lambda^2+(-4-5+4)\ \lambda-(-6)=0$$
$$\therefore\ \lambda^3-2\lambda^2-5\lambda+6=0$$

$$\begin{array}{c|cccc}1 & 1 & -2 & -5 & 6\\ & & 1 & -1 & -6\\ \hline & 1 & -1 & -6 & 0\end{array}$$

$(\lambda-1)(\lambda^2-\lambda-6)=0$

$(\lambda-1)(\lambda-3)(\lambda+2)=0$

$\lambda=1,-2,3$

The roots are non- repeated.

Step (ii) :- Now we find eigen vectors Matrix equations is given by

$$(A-\lambda I)X=0$$

$$i.e.\begin{bmatrix}2-\lambda & -2 & 3\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$Case\ (i):-$ When $\lambda=1$, matrix eq^n becomes

$$\begin{bmatrix}1 & -2 & 3\\ 1 & 0 & 1\\ 1 & 3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Solving first two rows by Cramer‟s rule. We have,

$$x_1-2x_2+3x_3=0$$
$$x_1+x_2+x_3=0$$

$$\therefore\ \frac{x_1}{-2}=\frac{-x_2}{-2}=\frac{x_3}{-2}$$

$$\therefore\ \frac{x_1}{-1}=\frac{x_2}{1}=\frac{x_3}{1}$$

$$\therefore\ x_1=\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

Case (ii) :- When the Matrix equation is given by

$$\begin{bmatrix}4 & -2 & 3\\ 1 & 3 & 1\\ 1 & 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore \quad \frac{x_1}{-11} = \frac{x_2}{1} = \frac{x_3}{14}$$

$$\therefore \quad \frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$$\therefore \quad x_2 = \begin{bmatrix} -11 \\ -1 \\ 14 \end{bmatrix}$$

Case (iii) : When 3 3 ▯ ▯▯ matrix equation is given by

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore \quad \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{4} \qquad \therefore \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Type (II) :- Repeated eigen values and A is non- symmetric.

Example 5: Find eigen values and eigen vectors for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution: Step (1) :- Characteristic equation of A in ƛ

$$[A - \lambda\, I] = 0$$

$$i.e. \quad \lambda^3 - 9\lambda^2 + (6+5+4)\lambda - 7 = 0$$

$$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

since sum of co-efficients 0

ƛ1 is a factor ▯ ▯▯ synthetic division

$$\begin{array}{cccccc} 1 & 1 & -9 & 15 & -7 \\ & & 1 & -8 & 7 \\ & 1 & -8 & 7 & 0 \end{array}$$

$\therefore \lambda^2 - 8\lambda + 7$
$= (\lambda - 7)(\lambda - 1)$

$\therefore \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$
$\therefore (\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$
$\lambda = 7, 1, 1$

Here two roots are repeated. First we find eigen vectors for non-repeated

Root.

Step II :- Matrix equation of A in λ is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $\lambda = 7$

Matrix equation is

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Case (ii) :- Let λ=1

Matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By cramers rule we get

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$
$$i.e. \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But by definition we want non-zero x2

So we proceed as follows

Expanding by R_1

$X_1+x_2+x_3=0$

Assume any element to be zero say x1 and give any conventional value say 1 to x2 and find x3

Let

$$x_1 = 0, \quad \mathbf{x}_2 = 1$$
$$\therefore x_3 = -1$$
$$\therefore x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (iii) :- Let x=1

Again consider

$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 0$$
$$Let \quad x_2 = 0, \quad \mathbf{x}_1 = 1$$
$$\therefore x_3 = -1$$
$$\therefore x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Type (iii) :- A is symmetric and eigen values are repeated

Example 6: Find eigen values and eigen vectors for

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: Step :- Characteristic equations of A in ƛ

$$[A - \lambda \bar{I}] = 0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$$i.e.\ \lambda^3 - 12\lambda^2 + (8+14+14)\ \lambda\ \text{-}32 = 0$$

$$\therefore\ \lambda^3 - 12\lambda^2 + 36\ \lambda\ \text{-}32 = 0$$

$(\lambda\ \text{-}2)$ is a factor

Synthetic division

$$\begin{array}{cccccc} 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ & 1 & -10 & 16 & 0 \end{array}$$

$$\lambda^2 - 10\lambda + 6$$
$$= (\lambda - 8)(\lambda - 2)$$
$$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$
$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$$\therefore\ \lambda = 8,\ 2,\ 2$$

Step (ii) :- Matrix equation is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $\lambda = 8$

Matrix equation is given by

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6} \text{........} By \text{ cramer's rule}$$

$$\frac{x_1}{2} = \frac{-x_2}{-1} = \frac{x_3}{1}$$

$$\therefore x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) :- Let $\lambda = 2$

Matrix equation is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding R_1

$$4x_1 - 2x_2 + 2x_3 = 0$$

Let $x_1 = 0, \ x_2 = 1$

$$\therefore x_3 = 1$$

$$\therefore x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) :- Let $\lambda = 2$

A is symmetric

X_1, x_2, x_3 are orthogonal

$$\text{Let, } x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$\because\ x_1, x_3$ are orthogonal

$\therefore\ x_1^{1}, x_3 = 0$

$\therefore\ 2l\text{-}m + n = 0..........(1)$

x_2, x_3 are orthogonal

$\therefore\ x_2^{1}, x_3 = 0$

$\therefore\ ol + m + n = 0..........(2)$

solving (1) and (2) by cramer's rule

$$\frac{1}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\therefore\ \frac{1}{+1} = \frac{m}{1} = \frac{n}{-1}$$

$$\therefore\ x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

CAYLEY – HAMILTON THEORY

OBJECTIVE

After going through this chapter you will able to

- Find by using Cayley Hamilton Theorem. Application of Cayley- Hamilton Theorem.
- Find diagonal matrix on similar matrix.
- Characteristic Polynomial & Minimal Polynomial of matrix A. Derogatory & non-derogatory matrix.
- Complex matrix like Hermitian, Skew-Hermitian unitary matrix

INTRODUCTION

In previous chapter we learn about Eigen values & Eigen Vector. How here we are going to discuss Cayley Hamilton Theory & its application also we had study only Real matrix. We introduce here complex matrix with type of complex matrix also minimal polynomial CAYLEY – HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation. If the characteristic Equation for the nth order square matrix A is

$$|A-\ \lambda I|=(-1)^n\left[\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}........+a_n\right]\text{then}$$

$$(-1)^n\left(A^n+a_1A^{n-1}+a_2A^{n-2}........+a_nI\right)=0$$

Example 1:

Show that the given matrix A satisfies its characteristic equation

$$A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$$

Solution: The characteristic equation of the matrix A is $|A-\lambda I|$=0

$$\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$$

$$\therefore(2-\lambda)\left[(1-\lambda)(2-\lambda)-0\right]-1(0)+1\left(0-(1-\lambda)\right)=0$$

$$\therefore(2-\lambda)\left[2-3\lambda+\lambda^2\right]+1(-1+\lambda)=0$$

$$\therefore 4-6\lambda+2\lambda^2-2\lambda+3\lambda^2-\lambda^3-1+\lambda=0$$

$$\therefore-\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\therefore\lambda^3-5\lambda^2+7\lambda-3=0$$

By Cayley Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \text{.............................}(1)$$

Now, we have

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\therefore A^3 - 5A^2 + 7A - 3I =$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} \text{-5} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} \text{-5} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 20 & 20 \\ 0 & 8 & 0 \\ 20 & 20 & 28 \end{bmatrix} - \begin{bmatrix} 28 & 20 & 20 \\ 0 & 8 & 0 \\ 20 & 20 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$$\therefore A^3 - 5A^2 + 7A - 3I = 0$$

Thus the matrix A satisfies its characteristic equation.

Example 2 : Calculate A^7 by using Cayley Hamilton theorem.

Where $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation of A is given by

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\ (2-\lambda) - 6 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 - 6 = 0$$

$$\therefore\ \lambda^2 - 5\lambda = 0$$

By Cayley Hamilton theorem,

$$A^2 - 5A = 0$$

$$\text{i.e. } A^2 = 5A$$

Now to calculate

$$A^7 = A^5.A^2 = A^5.5A = 5A^6$$

$$= 5A^4.A^2 = 25A^5$$

$$= 25A^3.A^2 = 125A^4$$

$$= 125A^2.A^2 = 125(5A).(5A)$$

$$= 3125A^2 = 3125(5A)$$

$$= 15625A$$

$$A^7 = 15625\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$$

$$\therefore \text{ The value of } A^7 = \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$$

Example 3: By using Cayley Hamilton theorem find A^{-1}

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

The characteristic equation of A is given by

$$|A - \lambda I = 0|$$

$$\begin{bmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left[1-2\lambda+\lambda^2-4\right]+1\left[\lambda-1-2\right]+1\left[-2+\lambda-1\right]=0$$

$$\lambda^2-2\lambda-3+3\lambda+2\lambda^2-\lambda^3+\lambda-3-3+\lambda=0$$

$$-\lambda^3+3\lambda^2+3\lambda-9=0$$

$$\lambda^3-3\lambda^2-3\lambda+9=0$$

By Cayley Hamilton theorem

$$A^3-3A^2-3A+9I=0$$

Multiply by A^{-1}

$$\therefore A^3A^{-1} - 3A^2A^{-1} - 3AA^{-1} + 9IA^{-1} = 0A^{-1}$$

$$\therefore A^2 - 3A - 3I + 9A^{-1} = 0$$

$$A^{-1} = \frac{1}{9}\left[3A + 3I - A^2\right] \qquad \ldots\ldots.(1)$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$3A + 3I - A = 3\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$3A + 3I - A^2 = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left[3A + 3I - A^2\right]$$

$$= \frac{1}{9}\begin{bmatrix} 3 & -3 & 3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Diagonalization of a matrix

A matrix A is diagonalization if there exists an invertible matrix P suchthat $P^{-1}AP=D$ where D is a diagonal matrix.Also the matrix P is then said to be diagonalizable A or transform A to diagonal form.

Example:

diagonalize $A=\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{-4} & \mathbf{4} & \mathbf{3} \end{bmatrix}$ **and** A^4

solution: the characteristic equation of A is $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix}=0$

$(1-\lambda)(2-\lambda)(3-\lambda)=0$

The characteristic values of A are $\lambda = 1,2$ and 3

Characteristic vector corresponding to $\lambda = 1$

the eigen vector corresponding to $\lambda=1$ is given by $(A- 1.I)X=0$

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ -4 & 4 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Y+z=0 , y=z=0 and -4x+4y+2z=0

Take z=k, we have y=-k . then 4x=4y+2z ....> -4k+2k= -2k

Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} \\ -k \\ k \end{bmatrix}$

$= -k/2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$X1=\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$

Characteristic vector corresponding to ƛ = 2

the eigen vector corresponding to ƛ=1 is given by (A- 2.I)X=0

$$\begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

-x+y+z=0 , z=0 and -4x+4y+z=0

i.e., -x+y=0 , z=0

let x=k then y=k

now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$

$= k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

X2= $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to ƛ=2

Characteristic vector corresponding to ƛ = 3

the eigen vector corresponding to ƛ=1 is given by (A- 3.I)X=0

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we have -2x+y+z=0 , -y+z=0 and -4x+4y=0

Let y=k , then z=k and x=k

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{k} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

X3=$\begin{bmatrix}1\\1\\1\end{bmatrix}$ is the eigen vector corresponding to ƛ=3

Consider P=[x1 x2 x3] = $\begin{bmatrix}1 & 1 & 1\\2 & 1 & 1\\-2 & 0 & 1\end{bmatrix}$

We have $|P|$ = -1 and P^{-1} = $\begin{bmatrix}-1 & 1 & 0\\4 & -3 & -1\\-2 & 2 & 1\end{bmatrix}$

We have $P^{-1}AP$=$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$=D(1,2,3)

Also find A^4:

D^4 =$\begin{bmatrix}1^4 & 0 & 0\\0 & 2^4 & 0\\0 & 0 & 3^4\end{bmatrix}$=$\begin{bmatrix}1 & 0 & 0\\0 & 16 & 0\\0 & 0 & 81\end{bmatrix}$

Now A^4 = PD^4P^{-1}

A^4=$\begin{bmatrix}1 & 1 & 1\\2 & 1 & 1\\-2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 16 & 0\\0 & 0 & 81\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\4 & -3 & -1\\-2 & 2 & 1\end{bmatrix}$

$\begin{bmatrix}-99 & 115 & 65\\-100 & 116 & 65\\-160 & -160 & 81\end{bmatrix}$=$A^4$