Sequences and Series

- Sequences and Series
- Convergence of Infinite Series
- Tests of Convergence
- P-Series Test
- Comparison Tests
- Ratio test
- Raabe's test
- Cauchy's Root test
- Integral test
- Leibnitz's test
- Absolute Convergence
- Conditional convergence

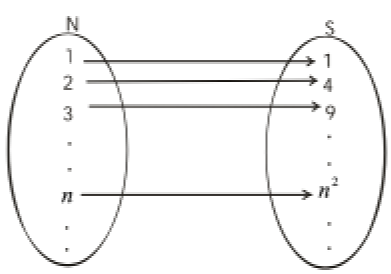
Sequence

A function f:N → S, where S is any nonempty set is called a Sequence i.e., for each n∈ N, ∃ a unique element f(n) ∈ S. The sequence is written as f(1), f(2), f(3), ...f(n)...., and is denoted by {f(n)}, or <f(n)>, or (f(n)). If f(n) =n

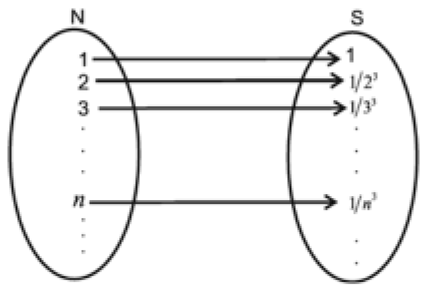
a, the sequence is

written as $a_1,a_2,a_3.........a_n$ denoted by {an} or (a_n). Here f(n) or a_n are the n^{th} terms of the Sequence.

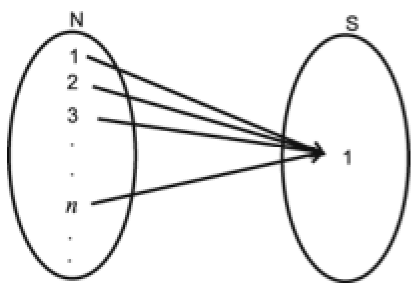
Ex. 1. $1, 4, 9, 16, ... n^2, ...(\text{or}) < n^2 >$



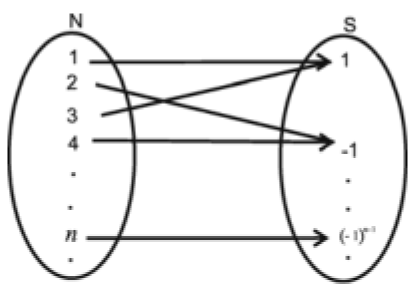
Ex. 2. $\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots\ldots \frac{1}{n^3} \ldots.(or) \left(\frac{1}{n^3}\right)$



Ex. 3. 1, 1, 1......1..... or <1>



Ex 4: $1, -1, 1, -1, \ldots\ldots\ldots$ or $\left\langle (-1)^{n-1} \right\rangle$



Note : 1. If S $\subseteq$ R then the sequence is called a *real sequence.*
2. The range of a sequence is almost a countable set.

Kinds of Sequences

1. **Finite Sequence:** A sequence $< a_n >$ in which $a_n = 0 \ \forall n > m \in N$ is said to be a finite Sequence. i.e., A finite Sequence has a finite number of terms.
2. **Infinite Sequence:** A sequence, which is not finite, is an infinite sequence.

Bounds of a Sequence and Bounded Sequence

1. If $\exists$ a number 'M' $\ni a_n \le$ M, $\forall n \in$ N, the Sequence $< a_n >$ is said to be bounded above or bounded on the right.

   Ex. $1, \frac{1}{2}, \frac{1}{3}$, ... here $a_n \le 1 \ \forall n \in \mathrm{N}$

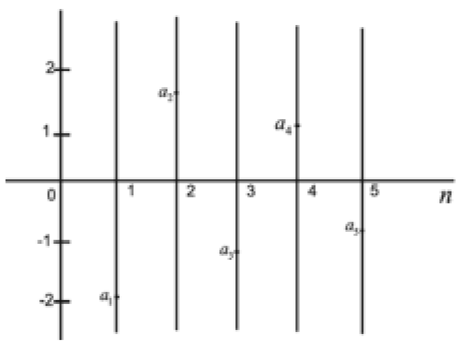
2. If $\exists$ a number 'm' $\ni a_n \ge m, \forall n \in \mathrm{N}$, the sequence $< a_n >$ is said to be bounded below or bounded on the left.

   Ex. 1 , 2 , 3 ,.....here $a_n \ge 1 \ \forall n \in \mathrm{N}$

3. A sequence which is bounded above and below is said to be bounded.

   Ex. Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

n	1	2	3	4	......
a_n	-2	3/2	-4/3	5/4	......



From the above figure (see also table) it can be seen that m = –2 and M = $\frac{3}{2}$.

$\therefore$ The sequence is bounded.

Limits of a Sequence

A Sequence $< a_n >$ is said to tend to limit 'l' when, given any + ve number '$\in$', however small, we can always find an integer 'm' such that $|a_n - l| < \in, \forall n \ge m$, and we write $\underset{n\to\infty}{Lt}\, a_n = l$ or $\langle a_n \to l \rangle$

Ex. If $a_n = \dfrac{n^2+1}{2n^2+3}$ then $< a_n > \to \frac{1}{2}$.

Convergent, Divergent and Oscillatory Sequences

1. **Convergent Sequence**:A sequence which tends to a finite limit, say 'l' is called a Convergent Sequence. We say that the sequence converges to 'l'
2. **Divergent Sequence**:A sequence which tends to ±∞ is said to be Divergent (or is said to diverge).
3. **Oscillatory Sequence**: A sequence which neither converges nor diverges ,is called an Oscillatory Sequence.

Ex. 1. Consider the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots$ here $a_n = 1 + \frac{1}{n}$

The sequence $< a_n >$ is convergent and has the limit 1

$a_n - 1 = 1 + \frac{1}{n} - 1 = \frac{1}{n}$ and $\frac{1}{n} < \in$ whenever $n > \frac{1}{\in}$

Suppose we choose $\in = .001$, we have $\frac{1}{n} < .001$ when n > 1000.

Ex. 2. If $a_n = 3 + (-1)^n \frac{1}{'n'} < a_n >$ converges to 3.

Ex. 3. If $a_n = n^2 + (-1)^n .n, < a_n >$ diverges.

Ex. 4. If $a_n = \frac{1}{n} + 2(-1)^n, < a_n >$ oscillates between -2 and 2.

Infinite Series

If $< u_n >$ is a sequence, then the expression $u_1 + u_2 + u_3 + \ldots\ldots.. + u_n + \ldots..$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$ or simply Σu_n

The sum of the first n terms of the series is denoted by s_n

i.e., $s_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n; s_1, s_2, s_3, \ldots. s_n$ are called *partial sums.*

Convergent, Divergent and Oscillatory Series

Let Σu_n be an infinite series. As $n \to \infty$, there are three possibilities.

(a) Convergent series: As $n \to \infty, s_n \to$ a finite limit, say 's' in which case the series is said to be convergent and 's' is called its sum to infinity.

Thus $\underset{n\to\infty}{Lt}\, s_n = s$ (or) simply $Lt s_n = s$

This is also written as $u_1 + u_2 + u_3 + \ldots.. + u_n + \ldots to\infty = s.$ (or) $\sum_{n=1}^{\infty} u_n = s$ (or) simply $\Sigma u_n = s.$

(b) Divergent series: If $s_n \to \infty$ or $-\infty$, the series said to be divergent.

(c) Oscillatory Series: If s_n does not tend to a unique limit either finite or infinite it is said to be an *Oscillatory Series.*

Note: Divergent or Oscillatory series are sometimes called non convergent series.

Geometric Series

The series, $1 + x + x^2 + \ldots.. x^{n-1} + \ldots$ is

(i) Convergent when $|x| < 1$, and its sum is $\frac{1}{1-x}$

(ii) Divergent when $x \geq 1$.

(iii) Oscillates finitely when x = -1 and oscillates infinitely when $x <$ -1.

Proof: The given series is a geometric series with common ratio 'x'

$\therefore\ s_n = \frac{1-x^n}{1-x}$ when $x \neq 1$ [By actual division – verify]

(i) When $|x| < 1$:

$$\underset{n\to\infty}{Lt}\, s_n = \underset{n\to\infty}{Lt}\left(\frac{1}{1-x}\right) - \underset{n\to\infty}{Lt}\left(\frac{x^n}{1-x}\right) = \frac{1}{1-x} \qquad \left[\text{since } x^n \to 0 \text{ as } n \to \infty\right]$$

$\therefore$ The series converges to $\dfrac{1}{1-x}$

(ii) When $x \ge 1 : s_n = \dfrac{x^n - 1}{x-1}$ and $s_n \to \infty$ as $n \to \infty$

$\therefore$ The series is divergent.

(iii) When $x = -1$: when n is even, $s_n \to 0$ and when n is odd, $s_n \to 1$

$\therefore$ The series oscillates finitely.

(iv) When $x < -1, s_n \to \infty$ or $-\infty$ according as n is odd or even.

$\therefore$ The series oscillates infinitely.

Some Elementary Properties of Infinite Series

1. The convergence or divergence of an infinites series is unaltered by an addition or deletion of a finite number of terms from it.
2. If some or all the terms of a convergent series of positive terms change their signs, the series will still be convergent.
3. Let Σu_n converge to 's'
   Let 'k' be a non – zero fixed number. Then $\Sigma k u_n$ converges to ks.
   Also, if Σu_n diverges or oscillates, so does $\Sigma k u_n$
4. Let Σu_n converge to 'l' and Σv_n converge to 'm'. Then
   (i) $\Sigma(u_n + v_n)$ converges to ($l + m$) and **(ii)** $\Sigma(u_n + v_n)$ converges to ($l - m$)

Series of Positive Terms

Consider the series in which all terms beginning from a particular term are +ve.
Let the first term from which all terms are +ve be u_1
Let Σu_n be such a convergent series of +ve terms. Then, we observe that the convergence is unaltered by any rearrangement of the terms of the series.

Theorem

If Σu_n is convergent, then $\underset{n\to\infty}{Lt}\, u_n = 0$.

Proof : $s_n = u_1 + u_2 + ... + u_n$

$s_{n-1} = u_1 + u_2 + ... + u_{n-1}$, so that, $u_n = s_n - s_{n-1}$

Suppose $\Sigma u_n = l$ then $\underset{n\to\infty}{Lt}\, s_n = l$ and $\underset{n\to\infty}{Lt}\, s_{n-1} = l$

$$\therefore \underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\left(s_n - s_{n-1}\right) \;;\; \underset{n\to\infty}{Lt}\, s_n - \underset{n\to\infty}{Lt}\, s_{n-1} = l - l = 0$$

Note: The converse of the above theorem need not be always true. This can be observed from the following examples.

(i) Consider the series, $1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{n}+....$; $u_n=\frac{1}{n}, \underset{n\to\infty}{Lt}\, u_n=0$

But from p-series test (1.3.1) it is clear that $\Sigma\frac{1}{n}$ is divergent.

(ii) Consider the series, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.....+\frac{1}{n^2}+......$

$u_n=\frac{1}{n^2}, \underset{n\to\infty}{Lt}\, u_n=0$, by p series test, clearly $\Sigma\frac{1}{n^2}$ converges,

Note : If $\underset{n\to\infty}{Lt}\, u_n\neq 0$ the series is divergent;

Ex. $u_n=\frac{2^n-1}{2^n}$, here $\underset{n\to\infty}{Lt}\, u_n=1$ $\therefore$ Σu_n is divergent.

Tests for the Convergence of an Infinite Series

In order to study the nature of any given infinite series of +ve terms regarding convergence or otherwise, a few tests are given below.

P-Series Test

The infinite series, $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+......$, is

(i) Convergent when $p>1$, and **(ii)** Divergent when $p\leq 1$. (JNTU 2002, 2003)

Proof :

***Case* (i)** Let $p>1$; $p>1, 3^p>2^p$; $\Rightarrow \frac{1}{3^p}<\frac{1}{2^p}$

$$\therefore \quad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}$$

Similarly, $\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}$

$\frac{1}{8^p}+\frac{1}{9^p}+....+\frac{1}{16^p}<\frac{8}{8^p}$, and so on.

Adding we get

$$\Sigma\frac{1}{n^p}<1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+....$$

i.e., $$\Sigma\frac{1}{n^p}<1+\frac{1}{2^{(p-1)}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+......$$

The RHS of the above inequality is an infinite geometric series with common ratio $\frac{1}{2^{p-1}} < 1 \left(\text{since } p > 1\right)$ The sum of this geometric series is finite.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also finite.

$\therefore$ The given series is convergent.

***Case* (ii)** Let p =1; $\Sigma \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$

We have, $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \ldots\ldots\frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \ldots\ldots\frac{1}{16} = \frac{1}{2} \text{ and so on}$$

$$\therefore \quad \Sigma \frac{1}{n^p} = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \ldots\ldots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots$$

The sum of RHS series is ∞

$$\left(\text{since } s_n = 1 + \frac{n-1}{2} = \frac{n+1}{2} \text{ and } \underset{n\to\infty}{Lt}\, s_n = \infty\right)$$

$\therefore$ The sum of the given series is also ∞; $\therefore$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 1$) diverges.

Case (iii) Let p<1, $\Sigma \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots$

Since $p < 1, \frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}\ldots\ldots$ and so on

$$\therefore \quad \Sigma \frac{1}{n^p} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

From the Case (ii), it follows that the series on the RHS of above inequality is divergent.

$\therefore$ $\Sigma \frac{1}{n^p}$ is divergent, when $P < 1$

Note: This theorem is often helpful in discussing the nature of a given infinite series.

Comparison Tests

1. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be convergent.

Then Σu_n converges,

(a) If $u_n \leq v_n, \forall \; n \in N$

(b) or $\frac{u_n}{v_n} \leq k \forall n \in N$ where k is > 0 and finite.

(c) or $\frac{u_n}{v_n} \to$ a finite limit > 0

Proof : (a) Let $\Sigma v_n = l$ (finite)

Then, $u_1 + u_2 + ... + u_n + ... \leq v_1 + v_2 + ...v_n + ... \leq l > 0$

Since l is finite it follows that Σu_n is convergent

(c) $\frac{u_n}{v_n} \leq k \Rightarrow u_n \leq kv_n, \forall n \in N$, since Σv_n is convergent and k (>0) is finite, Σkv_n is convergent $\therefore$ Σu_n is convergent.

(d) Since $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite, we can find a +ve constant $k, \ni \frac{u_n}{v_n} < k \forall n \in N$

$\therefore$ from (2) , it follows that Σu_n is convergent

2. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be divergent. Then Σu_n diverges,

* 1. If $u_n \geq v_n, \forall n \in N$

or * 2. If $\frac{u_n}{v_n} \geq k, \forall n \in N$ where k is finite and $\neq 0$

or * 3. If $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite and non-zero.

Note :

(a) In (1) and (2), it is sufficient that the conditions with * hold ∀>∈nmN

Alternate form of comparison tests : The above two types of comparison tests 2.8.(1) and 2.8.(2) can be clubbed together and stated as follows :

If Σu_n and Σv_n are two series of + ve terms such that $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = k$, where k is non- zero and finite, then Σu_n and Σv_n both converge or both diverge.

(b) 1. The above form of comparison tests is mostly used in solving problems.

2. In order to apply the test in problems, we require a certain seriesΣv_n
whose
nature is already known i.e., we must know whether Σv_n
is convergent are
divergent. For this reason, we call Σv_n
as an ‘auxiliary series’.

3. In problems, the geometric series (1.2.2.) and the p-series (1.3.1) can be conveniently used as ‘auxiliary series’.

Solved Examples

EXAMPLE 1

Test the convergence of the following series:

(a) $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+.....$ **(b)** $\frac{4}{1}+\frac{5}{4}+\frac{6}{9}+\frac{7}{16}+.....$ **(c)** $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]$

SOLUTION

(a)Step 1: To find "u_n" the n^{th} term of the given series. The numerators 3, 4, 5,

6......of the terms, are in AP.

n^{th} term $t_n=3+(n-1).1=n+2$

Denominators are $1^3,2^3,3^3,4^3.....n^{th}$ term $=n^3$; $\therefore\ u_n=\frac{n+2}{n^3}$

Step 2: To choose the auxiliary series Σv_n. In u_n the highest degree of n in the numerator is 1 and that of denominator is 3.

$\therefore$ we take, $v_n=\frac{1}{n^{3-1}}=\frac{1}{n^2}$

Step 3: $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{n+2}{n^3}\times n^2=\underset{n\to\infty}{Lt}\frac{n+2}{n}=\underset{n\to\infty}{Lt}\left(1+\frac{2}{n}\right)=1,$ which is non- zero and finite.

Step 4: *Conclusion:* $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1$

$\therefore$ Σu_n and Σv_n both converge or diverge (by comparison test). But $\Sigma v_n=\Sigma\frac{1}{n^2}$ is convergent by p-series test ($p=2>1$); $\therefore$ Σu_n is convergent.

(b) $\frac{4}{1}+\frac{5}{4}+\frac{6}{9}+\frac{7}{16}+.....$

Step 1: 4 , 5, 6, 7, ...in AP , $t_n=4+(n-1)1=n+3$ $\therefore\ u_n=\frac{n+3}{n^2}$

Step 2: Let $\Sigma v_n=\frac{1}{n}$ be the auxiliary series

Step 3: $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\left(\frac{n+3}{n^2}\right)\times n=\underset{n\to\infty}{Lt}\left(1+\frac{3}{n}\right)=1$, which is non-zero and finite.

Step 4: $\therefore$ By comparison test, both Σu_n and Σv_n converge are diverge together.

But $\Sigma v_n=\Sigma\frac{1}{n}$ is divergent, by p-series test ($p=1$); $\therefore$ Σu_n is divergent.

(c) $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]=\left\{n^4\left(1+\frac{1}{n^4}\right)\right\}^{\frac{1}{4}}-n=n\left[\left(1+\frac{1}{n^4}\right)^{\frac{1}{4}}-1\right]$

$$=n\left[1+\frac{1}{4n^4}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}.\frac{1}{n^8}+.....-1\right]=n\left[\frac{1}{4n^4}-\frac{3}{32n^8}+.....\right]$$

$$=\frac{1}{4n^3}-\frac{3}{32n^7}+....=\frac{1}{n^3}\left[\frac{1}{4}-\frac{3}{32n^4}+.....\right]$$

Here it will be convenient if we take $v_n=\frac{1}{n^3}$

$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\left(\frac{1}{4}-\frac{1}{32n^4}+.....\right)=\frac{1}{4}$, which is non-zero and finite

$\therefore$ By comparison test, Σu_n and Σv_n both converge or both diverge. But by p-series test $\Sigma v_n=\frac{1}{n^3}$ is convergent. $(p=3>1)$; $\therefore$ Σu_n is convergent.

EXAMPLE 2

If $u_n=\frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{2n^3+3n+5}}$ show that $\sum u_n$ is divergent.

SOLUTION

As n increases, u_n approximates to

$$\frac{\sqrt[3]{3n^2}}{\sqrt[4]{2n^3}}=\frac{3^{1/3}}{2^{1/4}}\times\frac{n^{2/3}}{n^{3/4}}=\frac{3^{1/3}}{2^{1/4}}.\frac{1}{n^{1/12}}$$

$\therefore$ If we take $v_n=\frac{1}{n^{1/12}}$, $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{3^{1/3}}{2^{1/4}}$ which is finite.

[(or) *Hint:* Take $v_n=\frac{1}{n^{l_1-l_2}}$, where l_1 and l_2 are indices of 'n' of the largest terms in denominator and nominator respectively of u_n. Here $v_n=\frac{1}{n^{\frac{3}{4}-\frac{2}{3}}}=\frac{1}{n^{\frac{1}{12}}}$]

By comparison test, Σv_n and Σu_n converge or diverge together. But $\Sigma v_n=\Sigma\frac{1}{n^{1/12}}$ is divergent by p – series test (since $p=\frac{1}{12}<1$)

$\therefore$ Σu_n is divergent.

Example:3

Test for convergence of the series $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\sqrt{\frac{4}{5}}+\ldots\ldots\ldots\ldots$

Solution:

Here, $u_n=\sqrt{\frac{n}{n+1}}$; Take $v_n=\frac{1}{n^{\frac{1}{2}-\frac{1}{2}}}=\frac{1}{n^0}=1$, $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\sqrt{\frac{1}{1+\frac{1}{n}}}=1$ (finite)

Σv_n is divergent by p – series test. ($p = 0 < 1$)

∴ By comparison test, Σu_n is divergent, (Students are advised to follow the procedure given in ex. 1.2.9(a) and (b) to find "u_n" of the given series.

Example:4

Show that $1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots\ldots.+\frac{1}{n!}+\ldots\ldots.$ **is convergent.**

Solution:

$$u_n=\frac{1}{\lfloor n} \quad \text{(neglecting } 1^{st} \text{ term)}$$

$$=\frac{1}{1.2.3\ldots\ldots n}<\frac{1}{1.2.2.2\ldots\ldots \overline{n-1 times}}=\frac{1}{(2^{n-1})}$$

$$\therefore \qquad \Sigma u_n<1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots$$

which is an infinite geometric series with common ratio $\frac{1}{2}<1$

$\therefore \qquad \Sigma\frac{1}{2^{n-1}}$ is convergent. (1.2.3(a)). Hence Σu_n is convergent.

Example:5

Test for the convergence of the series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots\ldots\ldots$

Solution:

$u_n=\frac{1}{n(n+1)(n+2)}$; Take $v_n=\frac{1}{n^3}$ $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{n^3}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=1$ (finite)

∴ By comparison test, Σu_n, and Σv_n converge or diverge together. But by p-series test, $\Sigma v_n=\Sigma\frac{1}{n^3}$ is convergent ($p=3>1$); ∴ Σu_n is convergent .

Example:6

If $u_n=\sqrt{n^4+1}-\sqrt{n^4-1}$**, show that** $\sum u_n$ **is convergent.**

Solution:

$$u_n = n^2\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} - n^2\left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}$$

$$= n^2\left[\left(1+\frac{1}{2n^4}-\frac{1}{8n^8}+\frac{1}{16n^{12}}-....\right)-\left(1-\frac{1}{2n^4}-\frac{1}{8n^8}-\frac{1}{16n^{12}}-.....\right)\right]$$

$$= n^2\left[\frac{1}{n^4}+\frac{1}{8n^{12}}+....\right] = \frac{1}{n^2}\left[1+\frac{1}{8n^{10}}+....\right]$$

Take $\quad v_n = \frac{1}{n^2}$, hence $\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = 1$

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \frac{1}{n^2}$ is convergent by p –series test ($p = 2 > 1$) $\therefore \quad \Sigma u_n$ is convergent.

EXAMPLE 7

Test the series $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+.....$ for convergence.

Solution:

$$u_n = \frac{1}{n+x}; \qquad \text{take} \quad v_n = \frac{1}{n}, \qquad \text{then} \qquad \frac{u_n}{v_n} = \frac{n}{n+x} = \frac{1}{1+\frac{x}{n}}$$

$$\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{x}{n}}\right) = 1; \Sigma v_n = \Sigma\frac{1}{n} \text{ is divergent by } p\text{-series test } (p = 1)$$

$\therefore$ By comparison test, Σu_n is divergent.

EXAMPLE 8

Show that $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ is divergent.

Solution:

$$u_n = \sin\left(\frac{1}{n}\right); \qquad \text{take} \quad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \underset{t\to 0}{Lt}\frac{\sin t}{t} \text{ (where } t = 1/n\text{)} = 1$$

$\therefore \quad \Sigma u_n, \Sigma v_n$ both converge or diverge . But $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent

(p -series test, $p = 1$); $\therefore \quad \Sigma u_n$ is divergent.

EXAMPLE 9

Test the series $\Sigma \sin^{-1}\left(\frac{1}{n}\right)$ for convergence.

SOLUTION

$$u_n = \sin^{-1}\frac{1}{n}; \qquad \text{Take} \qquad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin^{-1}\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}; = \underset{\theta\to 0}{Lt}\left(\frac{\theta}{\sin\theta}\right) = 1\left(Taking\,\sin^{-1}\frac{1}{n} = \theta\right)$$

But Σv_n is divergent. Hence Σu_n is divergent.

EXAMPLE 10

Show that the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^3} + ...$ is divergent.

Solution:

Neglecting the first term, the series is $\frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + ...$. Therefore

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n} = \frac{n^n}{n\left(1+\frac{1}{n}\right).n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n};$$

Take $v_n = \frac{1}{n}$

$$\therefore \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^n.1} = \frac{1}{e}$$

which is finite and $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent by p –series test ($p = 1$)

$\therefore \quad \Sigma u_n$ is divergent.

D' Alembert's Ratio Test

Let (i) $\sum u_n$ be a series of +ve terms and (ii) $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = k(\geq 0)$

Then the series $\sum u_n$ is (i) convergent if $k < 1$ and (ii) divergent if $k > 1$.

***Note*: 1** The ratio test fails when $k = 1$. As an example, consider the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Here $$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^p = \underset{n\to\infty}{Lt} \left(\frac{1}{1+1/n}\right)^p = 1$$

i.e., $k = 1$ for all values of p,

But the series is convergent if $p > 1$ and divergent if $p \le 1$, which shows that when $k = 1$, the series may converge or diverge and hence the test fails .

***Note:* 2** *Ratio test can also be stated as follows:*

If $\sum u_n$ is series of +ve terms and if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then $\sum u_n$ is convergent

If $k > 1$ and divergent if $k < 1$ (the test fails when $k = 1$).

Solved Examples

Test for convergence of Series

EXAMPLE 28

(a) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots\ldots\ldots\ldots\ldots\ldots$

SOLUTION

$$u_n = \frac{x^n}{n(n+1)};\ u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)};\quad \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^n} = \frac{1}{\left(1+\frac{2}{n}\right)}x.$$

Therefore $$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x$$

$\therefore$ By ratio test $\sum u_n$ is convergent When $|x| < 1$ and divergent when $|x| > 1$;

When $x = 1$, $u_n = \frac{1}{n^2(1+1/n)}$; Take $v_n = \frac{1}{n^2}$; $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test $\sum u_n$ is convergent.

Hence $\sum u_n$ is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

(b) $1+3x+5x^2+7x^3+.......$

SOLUTION

$$u_n=(2n-1)x^{n-1};\quad u_{n+1}=(2n+1)x^n;\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n-1}\right)x=x$$

$\therefore$ By ratio test $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$

When $x=1: u_n=2n-1;\ \underset{n\to\infty}{Lt}u_n=\infty;\ \therefore\ \sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|\geq 1$

(c) $\sum_{n=1}^{\infty}\frac{x^n}{n^2+1}$ ....

SOLUTION

$$u_n=\frac{x^n}{n^2+1};\qquad u_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}.$$

Hence $$\frac{u_{n+1}}{u_n}=\left(\frac{n^2+1}{n^2+2n+2}\right)x,\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+1/n^2\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\right](x)=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$ When $x=1: u_n=\frac{1}{n^2+1}$; Take $v_n=\frac{1}{n^2}$

$\therefore$ By comparison test, $\sum u_n$ is convergent when $|x|\leq 1$ and divergent when $|x|>1$

EXAMPLE 29

Test the series $\sum_{n\to\infty}^{\infty}\left(\frac{n^2-1}{n^2+1}\right)x^n, x>0$ for convergence.

Solution:

$$u_n=\left(\frac{n^2-1}{n^2+1}\right)x^n;u_{n+1}=\left[\frac{(n+1)^2-1}{(n+1)^2+1}\right]x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\left(\frac{n^2+2n}{n^2+2n+2}\right)\frac{\left(n^2+1\right)}{\left(n^2-1\right)}\right].x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{n^4\left(1+2/n\right)\left(1+1/n^2\right)}{n^4\left(1+2/n+2/n^2\right)\left(1-1/n^2\right)}\right]=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$ when $x=1$,

$$u_n=\frac{n^2-1}{n^2+1}\text{ Take }v_n=\frac{1}{n^0}$$

Applying p-series and comparison test, it can be seen that $\sum u_n$ is divergent when $x = 1$.

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent $x \geq 1$

EXAMPLE 30

Show that the series $1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+\ldots..$, is convergent for all values of p.

SOLUTION

$$u_n=\frac{n^p}{\lfloor n};\ u_{n+1}=\frac{\left(n+1\right)^p}{\lfloor n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{\left(n+1\right)^p}{\lfloor n+1}\times\frac{\lfloor n}{n^p}\right]=\underset{n\to\infty}{Lt}\left\{\frac{1}{\left(n+1\right)}\left(\frac{n+1}{n}\right)^p\right\}$$

$$=\underset{n\to\infty}{Lt}\frac{1}{\left(n+1\right)}\times\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^p=0<1;$$

$\sum u_n$ is convergent for all ‘ p ‘.

EXAMPLE 31

Test the convergence of the following series

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ \ldots\ldots\ldots\ldots$$

SOLUTION

$$u_n=\frac{1}{\left(2n-1\right)^p};\qquad u_{n+1}=\frac{1}{\left(2n+1\right)^p}$$

$$\frac{u_{n+1}}{u_n}=\frac{(2n-1)^p}{(2n+1)^p}=\frac{2^p.n^p(1-1/2n)^p}{2^p n^p(1+1/2n)^p}; \qquad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=1$$

$\therefore$ Ratio test fails.

Take $v_n=\frac{1}{n^p}$; $\frac{u_n}{v_n}=\frac{n^p}{(2n-1)^p}=\frac{1}{2^p\left(1-\frac{1}{2n}\right)^p}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{1}{2^p}$,

which is non – zero and finite

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge.

But by p – series test, $\sum v_n = \sum\frac{1}{n^p}$ converges when $p>1$ and diverges when $p\le 1$

$\therefore$ $\sum u_n$ is convergent if $p>1$ and divergent if $p\le 1$.

EXAMPLE 32

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(n+1)x^n}{n^3};x>0$

SOLUTION

$$u_n=\frac{(n+1)x^n}{n^3};u_{n+1}\frac{(n+2)x^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n}=\frac{n+2}{(n+1)^3}.x^{n+1}.\frac{n^3}{(n+1)x^n}=\left(\frac{n+2}{n+1}\right)\left(\frac{n}{n+1}\right)^3.x$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)\frac{1}{\left(1+\frac{1}{n}\right)^3}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x<1$ and diverges when $x>1$.

When $x=1$, $u_n=\frac{n+1}{n^3}$

Take $v_n=\frac{1}{n^2}$; By comparison test $\sum u_n$ is convergent (give proof)

$\therefore$ $\sum u_n$ is convergent if $x\le 1$ and divergent if $x>1$.

Raabe's Test

Let $\sum u_n$ be series of +ve terms and let $\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=k$

Then

(i) If $k>1$, $\sum u_n$ is convergent. **(ii)** If $k<1$, $\sum u_n$ is divergent. (The test fails if $k=1$)

Solved Examples

EXAMPLE 43

Test for convergence the series

$$x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$

SOLUTION

Neglecting the first tem ,the series can be taken as ,

$$\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+.....$$

1.3.5....are in A.P. n^{th} term $=1+(n-1)2=2n-1$

2.4.6...are in A.p. n^{th} term $=2+(n-1)2=2n$

3.5.7.....are in A.P n^{th} term $=3+(n-1)2=2n+1$

$\therefore$ u_n (n^{th} term of the series) $=\frac{1.3.5....(2n-1)}{2.4.6....(2n)}.\frac{x^{2n+1}}{2n+1}$

$$u_{n+1}=\frac{1.3.5....(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)}.\frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n}=\frac{1.3.5....(2n+1)}{2.4.6....(2n+2)}.\frac{x^{2n+3}}{(2n+3)}.\frac{2.4.6....2n}{1.3.5....(2n-1)}.\frac{(2n+1)}{x^{2n+1}}$$

$$=\frac{(2n+1)^2x^2}{(2n+2)(2n+3)}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)}x^2=x^2$$

$\therefore$ By ratio test, $\sum u_n$ converges if $|x|<1$ and diverges if $|x|>1$

If $|x|=1$ the test fails.

$\therefore$ By ratio test, $\sum u_n$ converges if $|x| < 1$ and diverges if $|x| > 1$

If $|x| = 1$ the test fails.

Then $\quad x^2 = 1 \quad$ and $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 = \frac{6n+5}{(2n+1)^2}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{6n^2+5n}{4n^2+4n+1}\right)$$

$$= \underset{n\to\infty}{Lt}\frac{n^2\left(6+\dfrac{5}{n}\right)}{n^2\left(4+\dfrac{4}{n}+\dfrac{1}{n^2}\right)} = \frac{3}{2} > 1$$

By Raabe's test, $\sum u_n$ converges. Hence the given series is convergent when $|x| \le 1$ an divergent when $|x| > 1$.

EXAMPLE 44

Test for the convergence of the series

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + ...; x > 0$$

SOLUTION

Neglecting the first term,

$$u_n = \frac{3.6.9....3n}{7.10.13....3n+4}.x^n$$

$$u_{n+1} = \frac{3.6.9....3n(3n+3)}{7.10.13....(3n+4)(3n+7)}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}.x \ \ ; \ \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x < 1$ and divergent when $x > 1$.

When $x = 1$ The ratio test fails. Then

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}; \frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{4n}{3n+3}\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent .Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

EXAMPLE 45

Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}$

SOLUTION

$$u_n = \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}; \qquad u_{n+1} = \frac{1^2.5^2.9^2....(4n-3)^2(4n+1)^2}{4^2.8^2.12^2....(4n)^2(4n+4)^2}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{(4n+1)^2}{(4n+4)^2} = 1 \ \ \text{(verify)}$$

$\therefore$ The ratio test fails. Hence by Raabe's test, $\sum u_n$ is convergent. (give proof)

EXAMPLE 46

Find the nature of the series $\sum\frac{(\lfloor n)^2}{\lfloor 2n}x^n, (x > 0)$

SOLUTION

$$u_n = \frac{(\lfloor n)^2}{\lfloor 2n}.x^n \; ; \; u_{n+1} = \frac{(\lfloor n+1)^2}{\lfloor 2n+2}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)}x \; ;$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}.x = \frac{x}{4}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{x}{4} < 1$, i. e ; $x < 4$; and diverges when $x > 4$;

When $x = 4$, the test fails.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{4(n+1)^2}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{-2n-2}{4(n+1)^2} = \frac{-1}{2(n+1)} \; ; \quad \underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \frac{-1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent

Hence $\sum u_n$ is convergent when $x < 4$ and divergent when $x \geq 4$

EXAMPLE 47

Test for convergence of the series $\sum \frac{4.7....(3n+1)}{1.2.3....n}x^n$ (JNTU 1996)

Solution:

SOLUTION

$$u_n = \frac{4.7....(3n+1)}{1.2.3....n}x^n \; ; \; u_{n+1} = \frac{4.7....(3n+1)(3n+4)}{1.2.3....n(n+1).}x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(3n+4)}{(n+1)}.x\right] = 3x$$

$\therefore$ By ratio test $\sum u_n$ converges if $3x<1$ i.e., $x<\frac{1}{3}$ and diverges if $x>\frac{1}{3}$;

If $x=\frac{1}{3}$, the test fails

When $$x=\frac{1}{3},\ n\left[\frac{u_n}{u_{n+1}}-1\right]=n\left[\frac{(n+1)3}{3n+4}-1\right]=n\left[\frac{-1}{3n+4}\right]=-\frac{1}{\left(3+\frac{4}{n}\right)}$$

$$\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=-\frac{1}{3}<1$$

$\therefore$ By Raabe's test, $\sum u_n$ is divergent.

$\therefore$ $\sum u_n$ is convergent when $x<\frac{1}{3}$ and divergent when $x\geq\frac{1}{3}$

EXAMPLE 48

Test for convergence $2+\frac{3x}{2}+\frac{4x^2}{3}+\frac{5x^3}{4}+..........(x>0)$ (JNTU 2003)

SOLUTION

The n^{th} term $$u_n=\frac{(n+1)}{n}x^{n-1};\ u_{n+1}=\frac{(n+2)}{(n+1)}x^n;\qquad \frac{u_{n+1}}{u_n}=\frac{n(n+2)}{(n+1)^2}.x$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $x<1$ and divergent if $x>1$

If $x=1$, the test fails.

Then $$\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=\underset{n\to\infty}{Lt}\ n\left[\frac{(n+1)^2}{n(n+2)}-1\right]=\underset{n\to\infty}{Lt}\ n\left[\frac{1}{n(n+2)}\right]=0<1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent

$\therefore$ $\sum u_n$ is convergent when $x<1$ and divergent when $x\geq 1$

EXAMPLE 49

Find the nature of the series $\frac{3}{4}+\frac{3.6}{4.7}+\frac{3.6.9}{4.7.10}+......\infty$

SOLUTION

$$u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)};\ u_{n+1} = \frac{3.6.9.....3n(3n+3)}{4.7.10.....(3n+1)(3n+4)}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+4};\ \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{3n\left(1+{}^{3}\!/_{3n}\right)}{3n\left(1+{}^{4}\!/_{3n}\right)} = 1$$

Ratio test fails.

$$\therefore \qquad \underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right] = \underset{n\to\infty}{Lt}\left[n\left(\frac{3n+4}{3n+3}-1\right)\right]$$

$$= \underset{n\to\infty}{Lt}\frac{n}{3(n+1)} = \underset{n\to\infty}{Lt}\frac{n}{3n\left(1+{}^{1}\!/_{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent.

Cauchy's Root Test

Let $\sum u_n$ be a series of +ve terms and let $\underset{n\to\infty}{Lt}\ u_n^{1/n} = l$. Then $\sum u_n$ is convergent when $l < 1$ and divergent when $l > 1$

Note : When $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right) = 1$, the root test can't decide the nature of $\sum u_n$. The fact of this statement can be observed by the following two examples.

1. Consider the series $\sum {}^{1}\!/_{n^3} :- \underset{n\to\infty}{Lt}u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^3}\right)^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n^{1/n}}\right)^3 = 1$

2. Consider the series $\sum {}^{1}\!/_{n}$, in which $\underset{n\to\infty}{Lt}u_n^{1/n} = \underset{n\to\infty}{Lt}\frac{1}{n^{1/n}} = 1$

In both the examples given above, $\underset{n\to\infty}{Lt}u_n^{1/n} = 1$. But series (1) is convergent (p-series test)

And series (2) is divergent. Hence when the *limit*=1, the test fails.

EXAMPLE 51

Test for convergence the infinite series whose nth terms are:

(i) $\dfrac{1}{n^{2n}}$ (ii) $\dfrac{1}{(\log n)^n}$ (iii) $\dfrac{1}{\left[1+\frac{1}{n}\right]^{n^2}}$

Solution:

(i) $u_n = \dfrac{1}{n^{2n}}, u_n^{1/n} = \dfrac{1}{n^2}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt} \dfrac{1}{n^2} = 0 < 1;$

By root test $\sum u_n$ is convergent.

(ii) $u_n = \dfrac{1}{(\log n)^n}; u_n^{1/n} = \dfrac{1}{\log n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt} \dfrac{1}{\log n} = 0 < 1;$

$\therefore$ By root test, $\sum u_n$ is convergent.

(iii) $u_n = \dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}; u_n^{1/n} = \dfrac{1}{\left(1+\frac{1}{n}\right)^n}$ $\underset{n\to\infty}{Lt} u_n^{1/n} = \underset{n\to\infty}{Lt} \dfrac{1}{\left(1+\frac{1}{n}\right)^n} = \dfrac{1}{e} < 1;$

$\therefore$ By root test $\sum u_n$ is convergent.

EXAMPLE 53

If $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}}$, show that $\sum u_n$ is convergent.

Solution:

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} ;= \underset{n\to\infty}{Lt} = \frac{n^n}{(n+1)^n} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^n$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 ; \therefore \sum u_n \text{ converges by root test .}$$

EXAMPLE 55

Test for the convergence of $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}.x^n$

SOLUTION :

$$u_n = \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x^n ; \ \underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}} .x = x$$

$\therefore$ By root test, $\sum u_n$ is convergent if $|x|<1$ and divergent if $|x|>1$.

When $|x|=1$: $u_n = \sqrt{\frac{n}{n+1}}$, taking $v_n = \frac{1}{n^0}$ and applying comparison test , it can be

seen that is divergent

$\sum u_n$ is convergent if $|x|<1$ and divergent if $|x|\geq 1$.

EXAMPLE 61

Test the convergence of the series $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\ldots\frac{(n+1)^n .x^n}{n^{n+1}}+\ldots\ldots, x>0$

Solution:

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^n .x^n}{n^{n+1}}\right]^{1/n} = \underset{n\to\infty}{Lt}\left[\left(\frac{n+1}{n}\right).\frac{1}{n^{1/n}}.x\right]$$

$$= \underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right).\frac{1}{n^{1/n}}.x\right] = 1.1.x = x\left[\text{since } \underset{n\to\infty}{Lt}\, n^{1/n}=1\right]$$

$\therefore$ By root test, $\sum u_n$ converges if $x<1$ and diverges when $x>1$.

When $x=1$, the test fails.

Then $\quad u_n = \left(1+\frac{1}{n}\right)^n .\frac{1}{n}$; Take $v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ By comparison test and *p*-series test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $x<1$ and divergent when $x\geq 1$.

Integral Test

+ve term series,

$$\phi(1)+\phi(2)+.....+\phi(n)+.....$$

where $\phi(n)$ decreases as n increases is convergent or divergent according as the integral $\int_1^\infty \phi(x)\,dx$ is finite or infinite.

Solved Examples

EXAMPLE 62

Test for convergence the series $\sum_{n=2}^{\infty} \frac{1}{n\log n}$

SOLUTION

$$\int_2^\infty \frac{1}{x\log x}dx = \underset{n\to\infty}{Lt}\left[\int_2^n \frac{1}{x\log x}dx\right] = \underset{n\to\infty}{Lt}\left[\log\log x\right]_2^n = \infty$$

$\therefore$ By integral test, the given series is divergent.

EXAMPLE 64

Test the series $\sum_1^\infty \frac{n}{e^{n^2}}$ for convergence.

SOLUTION

$$u_n = \frac{n}{e^{n^2}} = \phi(n)(say);$$

$\phi(n)$ is +ve and decreases as n increases. So let us apply the integral test.

$$\int_1^\infty \phi(x)\,dx = \int_1^\infty xe^{-x^2}\,dx = \frac{1}{2}\int_1^\infty e^{-t}dt\,\{t = x^2, dt = 2xdx\}$$

$$= -\frac{1}{2}e^{-t}\Big|_1^\infty = -\frac{1}{2}\left(0-\frac{1}{e}\right) = \frac{1}{2e}, \text{ which is finite.}$$

By integral test, $\sum u_n$ is convergent.

EXAMPLE 65

Apply integral test to test the convergence of the series $\sum_2^\infty \frac{1}{n^2}\sin\left(\frac{\pi}{n}\right)$

Solution:

Let $\phi(n)=\frac{1}{n^2}\sin\left(\frac{\pi}{n}\right)$; $\phi(n)$ decreases as n increases and is +ve .

$$\int_2^\infty \phi(x)\,dx=\int_2^\infty \frac{1}{x^2}\sin\left(\frac{\pi}{x}\right)dx\,; \qquad Let\ \pi/x=t$$

$$-\frac{1}{\pi}\int_{\pi/2}^{0}\sin t\,dt=\frac{1}{\pi}\cos t\Big|_{\pi/2}^{0}=\frac{1}{\pi}\ \text{finite,}\ -\pi/x^2\,dx=dt\,;\quad 1/x^2\,dx=-1/\pi\,dt$$

$\therefore$ By integral test, $\sum u_n$ converges $x=2\Rightarrow t=\pi/2$ $\quad x=\infty\Rightarrow t=0$

Alternating Series

A series, $u_1-u_2+u_3-u_4+-+.....+(-1)^{n-1}u_n+.....$, where u_n are all +ve, is an alternating series.

Leibneitz Test

If in an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}.u_n$, where u_n are all +ve,

(i) $u_n>u_{n+1},\forall n,$ and **(ii)** $\underset{n\to\infty}{Lt}\,u_n=0$, then the series is convergent.

Solved examples

EXAMPLE 68

Test for convergence $\sum\frac{(-1)^{n-1}}{2n-1}$

Solution:

The given series is an alternating series $\sum(-1)^{n-1}u_n$, where $u_n=\frac{1}{2n-1}$

We observe that (i) $u_n>0,\forall n$ (ii) $u_n>u_{n+1},\forall n$ (iii) $\underset{n\to\infty}{Lt}\,u_n=0$

$\therefore$ By Leibneitz's test, the given series is convergent.

EXAMPLE 69

Show that the series $S = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ...$ converges.

SOLUTION

The given series is $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{3^{n-1}} = \sum (-1)^{n-1} u_n$, where $u_n = \frac{1}{3^{n-1}}$ is an alternating series in which 1. $u_n > 0, \forall n$ 2. $u_n > u_{n+1}, \forall n$ and 3. $\underset{n \to \infty}{Lt}\, u_n = 0$;

Hence by Leibneitz's test, it is convergent.

EXAMPLE 70

Test for convergence of the series, $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - +....., 0 < x < 1$

SOLUTION

The given series is of the form $\sum \frac{(-1)^{n-1} . x^n}{1+x^n} = \sum (-1)^{n-1} u_n$,

where $u_n = \frac{x^n}{1+x^n}$ Since $0 < x < 1$, $u_n > 0, \forall n$;

Further, $u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$

$$= \frac{x^n - x^{n+1}}{(1+x^n)(1+x^{n+1})} = \frac{x^n (1-x)}{(1+x^n)(1+x^{n+1})}.$$

$0 < x < 1 \Rightarrow$ all terms in numerator and denominator of the above expression are +ve.

$\therefore \quad u_n > u_{n+1}, \forall n.$

Again, $x^n \to 0$ as $x^n \to \infty$ since $0 < x < 1$; $\therefore \underset{n \to \infty}{Lt}\, u_n = \frac{0}{1+0} = 0$

$\therefore$ By Leibneitz's test, the given series is convergent.

EXAMPLE 72

Test for the convergence of the following series,

$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - +....$

SOLUTION

Given series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{5n+1}=\sum(-1)^{n-1}u_n$ is an alternating series

$$u_n=\frac{n}{5n+1}>0\forall n\ ; \qquad \frac{n}{5n+1}-\frac{n+1}{5n+6}=\frac{-1}{(5n+1)(5n+6)}\Rightarrow u_n<u_{n+1},\forall n$$

Again, $\underset{n\to\infty}{Lt}\ u_n=\underset{n\to\infty}{Lt}\frac{n}{5n+1}=\frac{1}{5}\neq 0$

Thus conditions (ii) or (iii) of Leibnitz's test are not satisfied. The given series is not convergent. It is oscillatory.

Absolute convergence

A series $\sum u_n$ is said to be absolutely convergent if the series $\sum|u_n|$ is convergent

Ex. Consider the series

$$\sum u_n=1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+-+......$$

$$\sum|u_n|=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+......=\sum_{1}^{\infty}\frac{1}{n^3}$$

By $p-$ series test, $\sum|u_n|$ is convergent ($p=3>1$)

Hence $\sum u_n$ is absolutely convergent.

Note: 1. If $\sum u_n$ is a series of +ve terms, then $\sum u_n=\sum|u_n|$.

For such a series, there is no difference between convergence and absolute convergence. Thus a series of +ve terms is convergent as well as absolutely convergent.

2. An absolutely convergent series is convergent. But the converse need not be true.

Consider $\sum_{1}^{\infty}(-1)^{n-1}.\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......$

This series is convergent (1.7.3)

But $\sum\left|(-1)^{n-1}.\,{}^{1}\!/_{n}\right|=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......$ is divergent (p-series test).

Thus $\sum u_n$ is convergent need not imply that $\sum|u_n|$ is convergent (i.e., $\sum u_n$ is not absolutely convergent).

Conditional Convergence

If the series $\sum|u_n|$ is divergent and $\sum u_n$ is convergent, then $\sum u_n$ is said to be conditionally convergent.

Ex. Consider the Series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}.......$ $\sum u_n$ is convergent by Leibnitz's test. (Ex.1.7.3)

But $\sum|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}.....$ is divergent by $p-$ series test.

$\therefore$ $\sum u_n$ is conditionally convergent.

EXAMPLE 77

Find the interval of convergence of the series $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$

SOLUTION

$$u_n = \frac{x^n}{n^3}; u_{n+1} = \frac{x^{n+1}}{(n+1)^3}$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^3 .x = \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^3 .x = x$$

By ratio test, the given series converges when $|x|<1$, i.e., $x \in (-1,1)$

When $x=1, \sum u_n = \sum 1/n^3$, which, is convergent by p series test.

$\therefore$ $\sum u_n$ is convergent when $x=1$

Hence, the interval of convergence of the given series is (–1, 1)

EXAMPLE 80

Show that the series, $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^4}+........$ is absolutely convergent.

SOLUTION

$$\sum|u_n| = \sum_{n=1}^{\infty} 1/3^{n-1}$$, which is a geometric series with common ratio $\frac{1}{3}<1$

$\because$ It is convergent. Hence given series is absolutely convergent.

EXAMPLE 81

Test for convergence, absolute convergence and conditional convergence of the series,

$$1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\dots\dots\dots$$

SOLUTION

The given alternating series is of the form $\sum(-1)^{n-1}u_n$, where, $u_n=\frac{1}{4n-3}$.

Hence, $u_n>0\,\forall n\in N$; $u_{n+1}=\frac{1}{4(n+1)-3}=\frac{1}{4n+1}$

$$u_n-u_{n+1}=\frac{1}{4n-3}-\frac{1}{4n+1}$$

$$=\frac{4n+1-4n+3}{(4n-3)(4n+1)}=\frac{4}{(4n-3)(4n+1)}>0,\forall n\in N$$

i.e., $u_n>u_{n+1},\forall n\in N$ $\quad \underset{n\to\infty}{Lt}\,u_n=\underset{n\to\infty}{Lt}\frac{1}{4n-3}=0;$

All conditions of Leibnitz's test are satisfied.

Hence $\sum(-1)^{n-1}u_n$ is convergent.

$$|u_n|=\frac{1}{4n-3};\ \text{Take } v_n=\frac{1}{n}\ ;\ \underset{n\to\infty}{Lt}\frac{|u_n|}{v_n}=\underset{n\to\infty}{Lt}\frac{n}{n(4-3/n)}=\frac{1}{4}\neq 0 \text{ and finite.}$$

$\therefore$ By comparison test, $\sum|u_n|$ and $\sum v_n$ behave alike.

But by $p-$ series test, $\sum v_n$ is divergent (since $p=1$) .

$\sum|u_n|$ is divergent and $\therefore$ The given series is conditionally convergent.