INTRODUCTION

We have already studied the notion of limit,continuity and differentiation in relation with functions of a single variable. In this chapter we introduce the notify a function of several variables i.e., function of two or more variables

OBJECTIVES

After studying this unit you should be able to compute the approximate value of a function at a given point.

Limit of a function of two variables:

A function f(x,y) is said to tend to the limit L as (x,y) tends to (a,b) I.e., x→ a and y→ $b$if correspomding to any given positive number ϵ their exist a positive number δ such that $|f(x,y)-l|<\epsilon$ for all points (x,y) whenever $|x-a|\leq\delta$, $|y-b|\leq\delta$

Continuity of a function of two variables at a point:

We say that a function f(x,y) is continuous at a point (a,b), if corresponding to any given positive number ϵ,there exist a positive number δ such that $|f(x,y)-f(a,b)|<\epsilon$ whenever

$0<(x-a)^2+(y-b)^2<\delta^2$. in other words f(x,y) is said to be continuous at (a,b), of its domain of definition if $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

Partial differentiation

Higher order partial derivatives:

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y

And they can be differentiable reapeatedly to get higher order partial derivatives.

So $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial y^3}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial x y^2} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial y x^2} \text{ and so on}$$

Example:

1). Find first and second order partial derivatives $ax^2+2hxy+by^2$ and verify $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

Solution: let $f(x,y)=ax^2+2hxy+by^2$ then

$$\frac{\partial f}{\partial x}=2ax+2hy$$

$$\frac{\partial f}{\partial y}=2hx+2by$$

And $\frac{\partial^2 f}{\partial x^2}=2a$ (since y is a constant)

$$\frac{\partial^2 f}{\partial x^2}=2b \text{ (since x is a constant)}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(2hx+2by)=2h$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}(2ax+2hy)=2h$$

Hence $\boldsymbol{\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}}$

Example:

2). Find first and second order partial derivatives x^3+y^3-3axy and verify $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

Solution: let $f(x,y)=$ **x^3+y^3-3axy** then

$$\frac{\partial f}{\partial x}=3x^2-3ay$$

$$\frac{\partial f}{\partial y}=3y^2-3ax$$

And $\frac{\partial^2 f}{\partial x^2}=6x$ (since y is a constant)

$$\frac{\partial^2 f}{\partial x^2}=6y \text{ (since x is a constant)}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(3y^2-3ax)$$

$$=-3a$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}(3x^2-3ay)$$

$$=-3a$$

Hence $\boldsymbol{\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}}$

Homogeneous function:

A function f(x,y) is said to be homogeneous function of degree n if the degree of each term in f(x,y) is n,where n is a real number.

Example: if $f(x,y) = a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots\dots\dots+a_{n-1}xy^n-1+a_ny^n$ ………………(1)

Then f(x,y) is a homogeneous function of degree 'n' since the degree of each term is n.

The above definition of homogeneity applies to polynomial functions only.

→ Euler's theorem on homogeneous function:

Theorem statement: if z= f(x,y) is a homogeneous function of degree n, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$=nz ,$\forall$ x, y in the domain of the function.

Jacobian

Let are two functions , then the Jacobian of w.r.t is denoted by and is defined as

J($\frac{u,v}{x,y}$)=$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ or $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Properties:

- $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$
- If u,v are functions of r,s and r,s are functions of x,y then $\frac{\partial(u,v)}{\partial(x.y)} = \frac{\partial(u,v)}{\partial(r.s)} . \frac{\partial(r,s)}{\partial(x,y)}$

Functionally Dependence:

Two functions are said to be functional dependent on one another if the Jacobian of is zero. If they are functionally dependent on one another, then it is possible to find the relation between these two functions.

Maxima and minima

Let us consider a function y=f(x)

To find the maxima and minima the following procedure must be followed:

Step1: first find the first derivative snd equate to zero. i.e., $\frac{dy}{dx}$ =0

Step2: since y=f(x) is a polynomial → $\frac{dy}{dx}$=0 is a polynomial equation. By solving this equation we get roots.

Step3: find second derivative i.e., $\frac{d^2y}{d^2x}$

Step4: now substitute the obtained roots in $\frac{d^2y}{d^2x}$

Step5: depending on the nature of ., $\frac{d^2y}{d^2x}$ at that point we will solve further. The following cases will be there.

Case (i): if $\frac{d^2y}{dx^2}$<0 at a point say =a, then f has maximum at x=a and the maximum value is given by $[f(x)]_{x=a}$ = f(a)

Case (ii): if $\frac{d^2y}{dx^2}$ > 0 at a point say =a, then f has minimum at x=a and the minimum value is

Given by $[f(x)]_{x=a}$ = f(a)

Case (iii): if $\frac{d^2y}{dx^2}$ =0 at a point say =a, then f neither minimum nor maximum value i.e.,

stationary

Maxima and minima for the function of two variable

Let us consider a function z=f(x,y)

To find the maxima and minima for the given fuction. The following procedure must be followed:

Step 1: first find the first derivatives and equate to zero. i.e., $\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0$

Step 2: by solving $\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0$ we get the different values of x and y

Write these values as a set of ordered pairs i.e., (x,y)

Step 3: now find second partial derivatives

i.e., $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$

step4 : let us consider *l*= ., $\frac{\partial^2 z}{\partial x^2}$,m= $\frac{\partial^2 z}{\partial x \partial y}$ and n= $\frac{\partial^2 z}{\partial y^2}$

step 5: now , we have to see for what values of x and y.the given fuction is maximum/minimum/does not extreme values/fails to have maximum or minimum.

- If at a point, say (a,b) : $ln-m^2$ >0 and l <0 then f has maximum at this point and the maximum value will be obtained by substituting (a,b) in the given function.
- If at a point ,say (a,b) *: $ln-m^2$ >0 and l>0* then f has minimum at this point and the minimum value will be obtained by substituting (a,b) in the given function.
- If at a point , say (a,b) : *ln-m2* <0 , then f has neither maximum nor mimnimum and such points are called as saddle point.
- If at a point say (a,b) : $ln-m^2$ =0 , then f fails to have maximum or minimum and case needs further investigation to decided maxima/minima i.e., no conclusion.

Problem:

1). Examine the function for extreme values $f=x^3+3xy^2-3x^2-3y^2+4$ (x>0,y>0)

Solution: given $f=x^3+3xy^2-3x^2-3y^2+4$ (x>0,y>0)

The first order partial derivatives of f are given by

$\frac{\partial f}{\partial x}=3x^2+3y^2-6x$

$\frac{\partial f}{\partial y}=6xy-6y$

Now, equating first order partial derivatives to zero, we get

$\frac{\partial f}{\partial x}=0 \rightarrow 3x^2+3y^2-6x=0$ ....(1)

$\frac{\partial f}{\partial y}=0 \rightarrow 6xy-6y=0$ ....(2)

Solving (1) and (2) we get

(2).......$6y(x-1)=0$

Y=0,x=1

Substituting y=0 in (1)..........> $3x^2-6x=0$

X=0,x=2

Substituting x=1 in (1) ....> $3y^2-3=0$

Y=1,-1

∴ all possible set of variables are (0,0) (2,0) (1,1) (1,-1)

Now , the second order partial derivatives are given by

$l=\frac{\partial^2 f}{\partial x^2}=6x-6$

$m=\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial x}(6xy-6y)=6y$

$n=\frac{\partial^2 f}{\partial y^2}=6x-6$

now, $ln-m^2=(6x-6)(6x-6)-(6y)^2=(6x-6)^2-36y^2$

at a point (0,0)............$ln-m^2=36>0$ and $-6<0$

∴ f has maximum at at (0,0) and the maximum value will be obtained by substituting (0,0) in the function.

Also, at a point (2,0) ..... $ln-m^2=36>0$ *and* $l=6>0$

∴ f has minimum at (2,0) and the minimum value is $[f(x,y)]_{(2,0)}=0$

Also, at a point (1,1) ….. *ln-m*^{2} <0

∴ f has neither minimum nor maximum at this point.

Again, at a piint (1,-1)……ln-m^2 <0

∴ f has neither minimum nor maximum at this point.

Lagrange's method of undetermined multipliers

The method is useful to find the extreme values(i.e., max and min) for the given function, whenever some condition is given involving the variables.

To find the max and min for the given function using lagrange's method, the following procedure must be followed:

Step1:

Let us consider given function to be f(x,y,z) subject to condition Ø(x,y,z)=0

Step2:

Let us define a lagrange's function F=f+$\lambda\varphi$, where λ is called the lagrange multiplier.

Step3:

find first order partial derivatives and equate to zero

i.e., $\frac{\partial f}{\partial x} = 0 \ldots\ldots \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \ldots \ldots. (1)$

., $\frac{\partial f}{\partial y} = 0 \ldots\ldots \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \ldots \ldots. (2)$

., $\frac{\partial f}{\partial z} = 0 \ldots\ldots \frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \ldots \ldots. (3)$

Let the given condition be Ø(x, y, z)=0 …………….(4)

Step4:

Solve (1),(2),(3) and (4) eliminate λ to get the values of x,y,z

Step5:

The values so obtained will give the stationary point of (x,y,z)

Step6:

The minimum/maximum value will be obtained by substituting the values of x,y,z in the given function.

Problem:

1). Find the minimum value of $x^2+y^2+z^2$ subject tonthe condition $xyz=a^3$

Solution: let us consider given function to be f= $x^2+y^2+z^2$ and φ = $xyz-a^3$

Let us define lagrangean function F=f+ƛφ, where ƛ is called the lagrange's multiplier.

F = (x2+y2+z2)+ƛ(xyz-a^3)

Now, $\frac{\partial F}{\partial x}$=0 ……………..> $2x + ƛyz = 0$ ………….> $\frac{ƛ}{2}$ = - $\frac{x}{yz}$ ……..(1)

$\frac{\partial F}{\partial y}$=0 ……………> 2y+ $ƛxz$ = 0 ………..> $\frac{ƛ}{2}$ = - $\frac{y}{xz}$ …………(2)

$\frac{\partial F}{\partial z}$=0 ………………> 2z+ $ƛxy$ =0 …………> $\frac{ƛ}{2}$ = - $\frac{z}{xy}$ …………….(3)

Solving (1),(2) and (3) …………> $\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$

Now, consider $\frac{x}{yz} = \frac{y}{xz}$ ………..> $x^2=y^2$ ……………….(4)

Again , consider $\frac{y}{xz} = \frac{z}{xy}$ ………….> $y^2=z^2$ ………………..(5)

Again solvimg, (4) and (5) ………> $x^2=y^2=z^2$

x=y=z

given Ø= xyz-a^3=0

at x=y=z…………….$x^3=a^3$

x=a

similarly , we get y=a , z=a

hence , the minimum value of the function is given by $f_{(a,a,a)}=a^2+a^2+a^2=3a^2$