Introduction

At higher secondary level, we have studied the definition of a matrix, operations on the matrices, types of matrices inverse of a matrix etc.

In this chapter, we are studying adjoint method of finding the inverse of a square matrix and also the rank of a matrix.

Definition:

A system of $m \times n$ numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called an matrix.

The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} & a_{2n} \\ .... & ..... & .... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{n \times n}$$

Note:

i) Matrices are generally denoted by capital letters.

ii) The elements are generally denoted by corresponding small letters.

Types of Matrices:

Rectangular matrix :-

Any mxn Matrix where $m \neq n$ is called

rectangular matrix. For e.g

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$$

Column Matrix :

It is a matrix in which there is only one column.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

Row Matrix :

It is a matrix in which there is only one row.

Example:$[1\ 2\ 3\ 4]$

Square Matrix :

It is a matrix in which number of rows equals the number of columns.
i.e its order is n x n.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}_{2\times2}$$

Diagonal Matrix:

It is a square matrix in which all non-diagonal elements are zero.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

Scalar Matrix:

It is a square diagonal matrix in which all diagonal elements

are equal. e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

Unit Matrix:

It is a scalar matrix with diagonal elements

as unity. e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

Upper Triangular Matrix:

It is a square matrix in which all the elements below the principle diagonal are zero.
Example

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

Lower Triangular Matrix:

It is a square matrix in which all the elements above the principle diagonal are zero.
Example:

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}_{3\times 3}$$

Transpose of Matrix:

It is a matrix obtained by interchanging rows into columns or columns into rows.
Example:

$$\mathrm{A} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \end{bmatrix}_{2\times 3}$$

$$A^T = Transpose\ of\ \mathrm{A} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \\ 5 & 9 \end{bmatrix}_{3\times 2}$$

Symmetric Matrix:

If for a square matrix A, $A = \mathrm{A}^T$ then A is symmetric

$$\mathrm{A} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 1 \\ 5 & 1 & 9 \end{bmatrix}$$

Skew Symmeric Matrix:

if for a square matrix A,$A = \mathrm{A}^T$ then it is skew -symmetric matrix.

$$A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix}$$

Note : For a skew Symmetric matrix, diagonal elements are zero

Determinant of a Matrix:

Let A be a square matrix then
$|A|$ = *determinant of A i.e det A= A*

If (i) then $|A| \neq 0$ matrix A is called as non-singular and
If (i) then $|A| = 0$, matrix A is singular.

Note : for non-singular matrix A^{-1} exists.

Minor of an element :

Consider a square matrix A of order n
Let A $= \left[a_{ij}\right]_{n\times n}$
The matrix is also can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & - & - & - & a_{nn} \end{bmatrix}$$

Minor of an element a_{ij} is a determinant of order (nd) by deleting the elements of the matrix A, which are in 6th row and 5th column of A.

E.g. Consider,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{11} = Minor of an element a_{11}

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

II y

$$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix}$$

Example:
Let

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}, M_{22} = \begin{bmatrix} 2 & 8 \\ 0 & 6 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

(b) Cofactor of an element :-

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is a square matrix of order n and a_{ij} denotes cofactor of the element a_{ij}

$C_{ij} = (-1)^{i+j}. M_{ij}$ Where M_{ij} is minor of a_{ij} .

$$\begin{bmatrix} a_1\ b_1\ c_1 \\ a_2\ b_2\ c_2 \\ a_3\ b_3\ c_3 \end{bmatrix}$$

A_1 = The cofactor of A_1 $=(-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

B_1 = The cofactor of b_1 $=(-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

C_1 = The cofactor of b_1 $=(-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Example:
consider

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

$c_{11} = (-1)^{1+1} M_{11}$ $\quad c_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix}$

$= (-1)^{1+1}. \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \quad = (-1)^3 \times (0-3)$

$= (-1)^{1+1}. \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \quad = (-1)^3 \times (0-3)$

$= (1) \times (12-7) \quad = (-1) \times (-3)$

$= (1) \times (12-7) \quad = (-1) \times (-3)$

$= 5 \quad = 3$

(C) Cofactor Matrix :-

A matrix $C = [C_{ij}]$ where C_{ij} denotes cofactor of the element a_{ij} Of a matrix A of order nxn, is called a cofactor matrix.
In above matrix A, cofactor matrix is

$$C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} A^1 & B^1 & C^1 \\ A_2 & B_2 & C_2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Similarly for a matrix A=$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ the cofactor matrix is $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(d) Adjoint of Matrix :-

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A.

Thus in the notations used,

Adjoint of A = C^T

$$\Rightarrow Adj\,A = \begin{bmatrix} A^1 & B^1 & C^1 \\ A_2 & B_2 & C_2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Adjoint of a matrix A is denoted as Adj.AThus if,

$$\text{Adjoint of } A = C^T$$

$$\Rightarrow Adj\;A = \begin{bmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Adjoint of a matrix A is denoted as Adj.A

Thus if,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{ than Adj. } A = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note :

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \text{ than Adj . } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(d) Inverse of a square Matrix:-

Two non-singular square matrices of order n A and B are said to be inverse of each other if,

AB=BA=I, where I is an identity matrix of order n.

Inverse of A is denoted as A^{-1} and read as A inverse.

Thus

$AA^{-1}=A^{-1}A=I$

Inverse of a matrix can also be calculated by the Formula.

A^{-1} = Adj.A where denotes determinant of A.

Note:- From this relation it is clear that A^{-1} exist if and only if $|A \neq 0|$i.e A is non singular matrix.

definition:-

Orthogonal matrix.:-

If a square matrix it satisfies the relation AA^T then the matrix A is called an orthogonal matrix. &

$A^T=A^{-1}$

Example 3:

show that $\begin{bmatrix} \cos\theta & \cos\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix.

Solution:

To show that A is orthogonal i.e To show that $AA^T=I$

Solution:

$$A = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}\begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}^2\theta + \text{Sin}^2\theta & -\text{Cos}\theta\text{Sin}\theta + \text{Sin}\,\theta\text{Cos}\theta \\ -\text{Sin}\,\theta\text{Cos}\theta + \text{Cos}\theta\text{Sin}\theta & \text{Sin}^2\theta + \text{Cos}^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is an orthogonal matrix.

1.4 RANK OF A MATRIX

a) Minor of a matrix

Let A be any given matrix of order mxn. The determinant of any submatrix of a square order is called minor of the matrix A.
We observe that, if „r‟ denotes the order of a minor of a matrix of

order mxn then $1 \le r \le m$ if $m<n$ and $1 \le r \le n$ if $n<m$.

e.g. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 4 & 0 & 1 & 7 \\ 8 & 5 & 4 & -3 \end{bmatrix}$$

The determinants

$$\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 8 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 7 \\ 5 & 4 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 4 \\ 4 & 1 & 7 \\ 8 & 4 & -3 \end{bmatrix},$$

$$\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 0 & 7 \end{vmatrix}, |1|, |0|, |-3|,$$

Are some examples of minors of A.

b) Definition – Rank of a matrix:
A number „r" is called rank of a matrix of order mxn if there is almost one minor of the matrix which is of order r whose value is non-zero and all the minors of order greater than „r" will be zero.

e.g.(i) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Thus minor of order 3 is zero and atleast one minor of order 2 is non-zero

Consider e.g. Let

$$A_1 = \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4,\ A_2 = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \text{ etc.}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix} = 1(23) + 2(-2) = 19 \neq 0$$

$\therefore$ Rank of A= 3

Thus minor of order 3 is zero and atleast one minor of order 2 is non-zero Rank of A = 2.

Some results:

(i) Rank of null matrix is always zero.

(ii) Rank of any non-zero matrix is always greater than or equal to 1.

(iii) If A is any mxn non-zero matrix then Rank of A is always equal to rank of A.

(iv) Rank of transpose of matrix A is always equal to rank of A.

(v) Rank of product of two matrices cannot exceed the rank of both of the matrices.

(vi) Rank of a matrix remains unleasted by **elementary transformations**.

Elementary Transformations:

Following changes made in the elements of any matrix are called elementary transactions.

(i) Interchanging any two rows (or columns) .

(ii) Multiplying all the elements of any row (or column) by a non-zero real number.

(iii) Adding non-zero scalar multitudes of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

Definition:- Equivalent Matrix:

Two matrices A & B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order & the same rank. It can be denoted by

[it can be read as A equivalent to B]

1.5 CANONICAL FORM OR NORMAL FORM

If a matrix A of order mxn is reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ using a sequence of elementary transformations then it called canonical or normal form. Ir denotes identity matrix of order ‘r’

Note:-

If any given matrix of order mxn can be reduced to the canonical form which includes an identity matrix of order „r‟ then the matrix is of rank „r‟.

e.g. (1) Consider

Example 5: Determine rank of the matrix. A if

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 - 7R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_1 - R_2,\ R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$R_3 \times \dfrac{1}{28}$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 + 32\,R_3,\ \ R_2 - 33\,R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & o \end{bmatrix}$$

$\therefore$ Rank of A=3

Example 6: Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sim [I_2 \quad 0]$

$\therefore$ Rank of A= 2

Example 7: Determine the rank of matrix A if

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$$\sim\begin{bmatrix}1 & -1 & -2 & -4\\0 & 1 & -6 & -3\\0 & 4 & 9 & 10\\0 & 9 & 12 & 17\end{bmatrix}$$

$R_1+R_2,\ R_3-4R_2,\ R_4-9R_2$

$$\sim\begin{bmatrix}1 & 0 & -8 & -7\\0 & 1 & -6 & -3\\0 & 0 & 33 & 22\\0 & 0 & 66 & 44\end{bmatrix}$$

R_4-2R_3

$$\sim\begin{bmatrix}1 & 0 & -8 & -7\\0 & 1 & -6 & -3\\0 & 0 & 33 & 22\\0 & 0 & 0 & 0\end{bmatrix}$$

$R_3\times\ \dfrac{1}{11}$

$$\sim\begin{bmatrix}1 & 0 & -8 & -7\\0 & 1 & -6 & -2\\0 & 0 & 3 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

C_3- C_4

$$\sim\begin{bmatrix}1 & 0 & -1 & -7\\0 & 1 & -3 & -3\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

$R_1+R_3,\ R_2+3R_3$

$$\sim\begin{bmatrix}1 & 0 & 0 & -5\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

C_4- $(5C_1+3C_2+2C_2)$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}I_3 & 0\\ 0 & 0\end{bmatrix}$$

$\therefore$ Rank of A= 3

Gauss-elimination method

Example:1

Solve the given system of equations 3x+y+2z=3, 2x-3y-z=-3 and x+2y+z=4 using gauss-elimination method

Solution: the given system of equation can be written in the matrix form as AX=B

Where A=$\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}$, X=$\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and B=$\begin{bmatrix}3\\ -3\\ 4\end{bmatrix}$

The augmented matrix [A,B]= $\begin{bmatrix}3 & 1 & 2 & 3\\ 2 & -3 & -1 & -3\\ 1 & 2 & 1 & 4\end{bmatrix}$

Operatating R1 $\leftrightarrow$ R3

[A,B]= $\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & -3 & -1 & -3\\ 3 & 1 & 2 & 3\end{bmatrix}$

Operating R2 $\rightarrow$ R2 -2R1 and R3 $\rightarrow$ R3-3R3

[A,B]= $\begin{bmatrix}1 & 2 & 1 & 4\\ 0 & -7 & -3 & -11\\ 0 & -5 & -1 & -9\end{bmatrix}$

operating R3 → 7R3-5R2

$$[A,B] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & -11 & -8 \end{bmatrix}$$

Which is echelon matrix then the matrix form of the above system is

X+2y + z = 4 ……………(1)
-7y - 3z = -11 ……………(2)
8z = -8

By back substitution , we have

Z=-1 …………….(3)
Put equation (3) in equation (2) , we get
-7y = -11 – 3

Y=2
Now from (1) we have
X+4-1=4
X=1
∴ the solution is x=1 , y=2 and z=-1
Example:
x+y + z = 6
2x−y +z = 3
x+z = 4
2x+ y + z = 7
Form the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix}$$

Add −2 times the first row to the second, −1 times the first to the third, and −2 times the first to the fourth to get 0's in the first column below the diagonal.

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & -1 & -5 \end{pmatrix}$$

Multiply the second, third, and fourth equations by −1 and then switch the second and third equations to avoid fractions

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 9 \\ 0 & 1 & 1 & 5 \end{pmatrix}$$

Now add −3 times the second row to the third row, and add −1 times the second row to the fourth row

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Now add −1 times the third row to the fourth row.

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Re-interpret the matrix as a system of equations. The bottom equation says 0x+0y+0z = 0, i.e., 0 = 0. We can discard this equation since we already knew that, and use back substitution as before, to get z = 3, y = 2, x = 1.

Gauss-Seidel Method

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also,if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

What is the algorithm for the Gauss-Seidel method? Given a general set of nequations and nunknowns, we have

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2 \\
&. \qquad\qquad . \\
&. \qquad\qquad . \\
&. \qquad\qquad . \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n
\end{aligned}$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

Example

Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$
$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$
$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidelmethod.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1

$$x_1 = \frac{1 - 3(0) + 5(1)}{12}$$
$$= 0.50000$$
$$x_2 = \frac{28 - (0.50000) - 3(1)}{5}$$
$$= 4.9000$$
$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13}$$
$$= 3.0923$$

Iteration #2

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12}$$
$$= 0.14679$$
$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5}$$
$$= 3.7153$$
$$x_3 = \frac{76 - 3(0.14679) - 7(3.7153)}{13}$$
$$= 3.8118$$

The maximum absolute relative approximate error is 240.61%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows

Iteration	x_1	$\left\|\epsilon_a\right\|_1 \%$	x_2	$\left\|\epsilon_a\right\|_2 \%$	x_3	$\left\|\epsilon_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$
$$12x_1 + 3x_2 - 5x_3 = 1$$

find the solution using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess.

Solution

Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$
$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Iteration	x_1	$\|\in_a\|_1 \%$	x_2	$\|\in_a\|_2 \%$	x_3	$\|\in_a\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	–196.15	110.71	14.421	94.453	–462.30	110.96
3	1995.0	109.83	–116.02	112.43	4718.1	109.80
4	–20149	109.90	1204.6	109.63	–47636	109.90
5	2.0364×10^5	109.89	–12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^6	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$