* WARSHALL'S ALGORITHM *

→ If relation R for a set is not transitive, we need to apply closure to make it transitive. This closure is known as transitive closure.

aRb bRC aRc
ex{(1,2) (2,3) (1,3)}

We use warshall's method to find the transitive closure.

eg:- let $A = \{a, b, c, d\}$ and $R = \{(a,a), (b,a) (b,c) (c,d)\}$

find the transitive closure.

Steps :-

Step 1: we first write the matrix M_R of the relation R and denote it by W_0.

Step 2 :- we write a blank matrix of order 4, denote it by w_1 & transfer all 1's from w_0.

⋮

Continue until you get same matrix.

Step 1

$$W_0 = M_R = \begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

column row
↓ ↓
1/1 p_i q_j
2 2 → relⁿ (2,2)

step 2

$$W_1 = \begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

p_i q_j → (1,1)(1,2)(1,3)
1 1 (2,1)(2,2)(2,3)
2 2
3

step 3

$$W_2 = \begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 4

$$W_3 = \begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$$

$p_i \downarrow$ 1, 2 $\quad q_j \downarrow$ 4 $\quad$ (1,4) (2,4)

Step 5

$$W_4 = \begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & & \end{array}$$

$p_i \downarrow$ 1, 2, 3 $\quad q_j \downarrow$ ϕ (no relations found)

$$\boxed{R = \{(a,a)\ (a,b)\ (a,c)\ (a,d)\ (b,a)\ (b,b)\ (b,c)\ (b,d)\ (c,d)\}}$$

example :- Transitive closure (Warshall's Algorithm)

Relation → Closure

let $A = \{1, 2, 3, 4\}$ $\quad n(A) = 4 \quad R = \{(1,2)\ (2,1)\ (2,3)\ (3,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array} = W_0$$

Step 1 → 1^{st} column → 2
1^{st} row → 2 $\quad R = \{(2,2)\}$

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Step 2 $\rightarrow$ 2^{nd} col $\rightarrow$ 1, 2
2^{nd} row $\rightarrow$ 1, 2, 3

(1,1) (1,2) (1,3) (2,1) (2,2) (2,3)

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 3 3^{rd} col $\rightarrow$ 1, 2
3^{rd} row $\rightarrow$ 4

{(1,4) (2,4)}

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 4 4^{th} col $\rightarrow$ 1, 2, 3
4^{th} row $\rightarrow$ 0

{no rel^n found}

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In ~~step 4~~ Here, In ~~W_4~~ Above matrix 1 is absent in 4th row hence we can not find a relation for W_4

hence.

$$M_R^{\infty} = W_3 = W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Fansitive Closure}$$