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Non-linear impulsive dynamical systems. Part II: Stability of feedback interconnections and optimality

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In a companion paper (Nonlinear Impulsive Dynamical Systems. Part I: Stability and Dissipativity) Lyapunov and invariant set stability theorems and dissipativity theory were developed for non-linear impulsive dynamical systems. In this paper we build on these results to develop general stability criteria for feedback interconnections of non-linear impulsive systems. In addition, a unified framework for hybrid feedback optimal and inverse optimal control involving a hybrid non-linear-non-quadratic performance functional is developed. It is shown that the hybrid cost functional can be evaluated in closed-form as long as the cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the non-linear closed-loop impulsive system. Furthermore, the Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton–Jacobi–Bellman equation.

1. Introduction

In a companion paper (Haddad *et al*. 2001) stability and dissipativity theory for non-linear impulsive dynamical systems were developed. Using the concepts of dissipativity and exponential dissipativity for impulsive systems, in this paper we develop feedback interconnection stability results for non-linear impulsive dynamical systems. The feedback system can be impulsive, non-linear, and either dynamic or static. General stability criteria are given for Lyapunov, asymptotic and exponential stability of feedback impulsive systems. In the case of quadratic supply rates involving net system power and input–output energy, these results generalize the positivity and small gain theorems to the case of non-linear impulsive dynamical systems. In particular, we show that if the non-linear impulsive dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ are dissipative (respectively, exponentially dissipative) with respect to quadratic supply rates corresponding to net system power, or, weighted input and output energy, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov (respectively, asymptotically) stable.

Next, using the stability results developed in the first part of the paper, we consider a hybrid feedback optimal control problem over an infinite horizon involving a hybrid non-linear-non-quadratic performance functional. The performance functional involves a continuous-time cost for addressing performance of the continuous-time system dynamics and a discrete-time cost for addressing performance at the resetting instants. Furthermore, the hybrid cost functional can be evaluated in closed-form as long as the non-linear-non-quadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the non-linear closed-loop impulsive system. This Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton–Jacobi–Bellman equation and thus guaranteeing both optimality and stability of the feedback controlled impulsive system. The overall framework provides the foundation for extending linear-quadratic feedback control methods to non-linear impulsive dynamical systems. We note that the optimal control framework for impulsive dynamical systems developed herein is quite different from the quasivariational inequality methods for impulsive and hybrid control developed in the literature (e.g. Barles 1985 a, b, Bardi and Dolcetta 1997, Branicky *et al*. 1998). Specifically, quasivariational methods do not guarantee asymptotic stability via Lyapunov functions and do not necessarily yield feedback controllers. In contrast, the proposed approach provides hybrid *feedback* controllers guaranteeing closed-loop stability via an underlying Lyapunov function.

The contents of the paper are as follows. Feedback interconnection stability results for non-linear impulsive dynamical systems are given in § 2. In § 3 we address an optimal control problem with respect to a hybrid non-linear-non-quadratic performance functional for impulsive dynamical systems. To avoid the complexity in solving the hybrid Hamilton–Jacobi–Bellman equation, in § 4 we specialize the results of § 3 to address an inverse optimal control problem for non-linear affine (in the control) impulsive systems. In § 5 we specialize the results of § 4 to linear impulsive systems controlled by non-linear controllers that minimize polynomial and multilinear cost functionals. In § 6, we apply the results developed in this paper and Part I of this paper to the control of thermoacoustic instabilities in combustion processes. The overall framework demonstrates that hybrid controllers provide an extremely effective way for dissipating energy in combustion systems. Finally, we draw conclusions in § 7.

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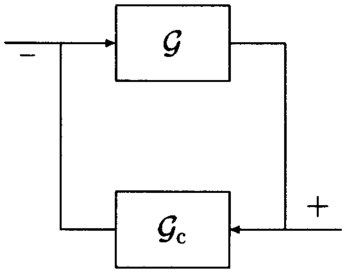


Figure 1. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

2. Feedback interconnections of dissipative impulsive dynamical systems

In this section we consider stability of feedback interconnections of dissipative impulsive dynamical systems. Specifically, using the notion of dissipative and exponentially dissipative impulsive dynamical systems (Haddad *et al.* 2001), with appropriate storage functions and supply rates, we construct Lyapunov functions for interconnected impulsive dynamical systems by appropriately combining storage functions for each subsystem. Here, we restrict our attention to input/state-dependent impulsive dynamical systems (Haddad *et al.* 2001). Analogous results, with the exception of results requiring the impulsive invariance principle (Haddad *et al.* 2001), hold for time-dependent impulsive dynamical systems. In this paper we use the notation and assumptions established in Part I of this paper (Haddad *et al.* 2001). We begin by considering the non-linear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \quad x(0) = x_0, \quad (x(t), u_c(t)) \notin \mathcal{Z} \tag{1}$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in \mathcal{Z} \tag{2}$$

$$y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t), \quad (x(t), u_c(t)) \notin \mathcal{Z} \tag{3}$$

$$y_d(t) = h_d(x(t)) + J_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in \mathcal{Z} \tag{4}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}, \Delta x(t) \triangleq x(t^+) - x(t)$, $u_c(t) \in \mathcal{U}_c \subseteq \mathbb{R}^{m_c}$, $u_d(t_k) \in \mathcal{U}_d \subseteq \mathbb{R}^{m_d}$, t_k denotes the k^{th} instant of time at which $(x(t), u_c(t))$ intersects $\mathcal{Z}$ for a particular trajectory $x(t)$ and input $u_c(t)$, $y_c(t) \in \mathbb{R}^{l_c}$, $y_d(t_k) \in \mathbb{R}^{l_d}$, $f_c: \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_c(0) = 0$, $G_c: \mathcal{D} \to \mathbb{R}^{n \times m_c}$, $f_d: \mathcal{D} \to \mathbb{R}^n$ is continuous, $G_d: \mathcal{D} \to \mathbb{R}^{n \times m_d}$, $h_c: \mathcal{D} \to \mathbb{R}^{l_c}$ and satisfies $h_c(0) = 0$, $J_c: \mathcal{D} \to \mathbb{R}^{l_c \times m_c}$, $h_d: \mathcal{D} \to \mathbb{R}^{l_d}$, $J_d: \mathcal{D} \to \mathbb{R}^{l_d \times m_d}$ and $\mathcal{Z} \triangleq \mathcal{Z}_x \times \mathcal{Z}_{u_c}$, where $\mathcal{Z}_x \subset \mathcal{D}$ and $\mathcal{Z}_{u_c} \subset \mathcal{U}_c$, is the resetting set. Furthermore, consider the impulsive non-linear feedback system $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = f_{cc}(x_c(t)) + G_{cc}(u_{cc}(t), x_c(t))u_{cc}(t), \quad x_c(0) = x_{c0}, \quad (x_c(t), u_{cc}(t)) \notin \mathcal{Z}_c \tag{5}$$

$$\Delta x_c(t) = f_{dc}(x_c(t)) + G_{dc}(u_{dc}(t), x_c(t))u_{dc}(t), \quad (x_c(t), u_{cc}(t)) \in \mathcal{Z}_c \tag{6}$$

$$y_{cc}(t) = h_{cc}(x_c(t)) + J_{cc}(u_{cc}(t), x_c(t))u_{cc}(t), \quad (x_c(t), u_{cc}(t)) \notin \mathcal{Z}_c \tag{7}$$

$$y_{dc}(t) = h_{dc}(x_c(t)) + J_{dc}(u_{dc}(t), x_c(t))u_{dc}(t), \quad (x_c(t), u_{cc}(t)) \in \mathcal{Z}_c \tag{8}$$

where $t \geq 0$, $\Delta x_c(t) \triangleq x_c(t^+) - x_c(t)$, $x_c(t) \in \mathbb{R}^{n_c}$, $u_{cc}(t) \in \mathcal{U}_{cc} \subseteq \mathbb{R}^{m_{cc}}$, $u_{dc}(t_k) \in \mathcal{U}_{dc} \subseteq \mathbb{R}^{m_{dc}}$, $y_{cc}(t) \in \mathbb{R}^{l_{cc}}$, $y_{dc}(t_k) \in \mathbb{R}^{l_{dc}}$, $f_{cc}: \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is Lipschitz continuous and satisfies $f_{cc}(0) = 0$, $G_{cc}: \mathbb{R}^{m_{cc}} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times m_{cc}}$, $f_{dc}: \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is continuous, $G_{dc}: \mathcal{Z}_c \to \mathbb{R}^{n_c \times m_{dc}}$, $J_{cc}: \mathbb{R}^{m_{cc}} \times \mathbb{R}^{n_c} \to \mathbb{R}^{l_{cc} \times m_{cc}}$, $h_{cc}: \mathbb{R}^{n_c} \to \mathbb{R}^{l_{cc}}$ and satisfies $h_{cc}(0) = 0$, $J_{dc}: \mathcal{Z}_c \to \mathbb{R}^{l_{dc} \times m_{dc}}$, $h_{dc}: \mathbb{R}^{n_c} \to \mathbb{R}^{l_{dc}}$, $m_{cc} = l_c$, $m_{dc} = l_d$, $l_{cc} = m_c$, $l_{dc} = m_d$ and $\mathcal{Z}_c \triangleq \mathcal{Z}_{cx_c} \times \mathcal{Z}_{cu_{cc}}$, where $\mathcal{Z}_{cx_c} \subset \mathbb{R}^{n_c}$ and $\mathcal{Z}_{cu_{cc}} \subset \mathcal{U}_{cc}$, is such that Assumptions A1 and A2 of Haddad *et al.* (2001) hold. Note that with the feedback interconnection given by figure 1, $(u_{cc}, u_{dc}) = (y_c, y_d)$ and $(y_{cc}, y_{dc}) = (-u_c, -u_d)$. Furthermore, even though the input–output pairs of the feedback interconnection shown on figure 1 consist of two-vector inputs/two-vector outputs, at any given instant of time a single-vector input/single-vector output is active. Here, we assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed; that is

$$\det[I_{m_c} + J_{cc}(y_c, x_c)J_c(x)] \neq 0$$

and

$$\det[I_{m_d} + J_{dc}(y_d, x_c)J_d(x)] \neq 0$$

for all y_c, y_d, x and x_c. The following results give sufficient conditions for Lyapunov, asymptotic and exponential stability of the negative feedback interconnection given by figure 1. In contrast to Haddad *et al.* (2001), in this paper we represent the resetting time $\tau_k(x_0)$ for a state-dependent impulsive dynamical system by t_k. This minor abuse of notation considerably simplifies the presentation. Furthermore, for the results of this section we define the closed-loop resetting set

$$\tilde{\mathcal{Z}}_{\tilde{x}} \triangleq \mathcal{Z}_x \times \mathcal{Z}_{cx_c} \cup \{(x, x_c): (\mathcal{F}_{cc}(x), \mathcal{F}_c(x_c)) \in \mathcal{Z}_{cu_{cc}} \times \mathcal{Z}_{u_c}\}$$

where $\mathcal{F}_{cc}(\cdot)$ and $\mathcal{F}_c(\cdot)$ are functions of x and x_c arising from the algebraic loops due to u_{cc} and u_c, respectively. Note that since the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed, it follows that $\tilde{\mathcal{Z}}_{\tilde{x}}$ is well defined and depends on the closed-loop states $\tilde{x} \triangleq [x^{\mathrm{T}} \, x_c^{\mathrm{T}}]^{\mathrm{T}}$. In the special case where $J_c(x) \equiv 0$ and $J_{cc}(u_{cc}, x_c) \equiv 0$ it follows that $\tilde{\mathcal{Z}}_{\tilde{x}} = \mathcal{Z}_x \times \mathcal{Z}_{cx_c} \cup \{(x, x_c): (h_c(x), h_{cc}(x_c)) \in \mathcal{Z}_{cu_{cc}} \times \mathcal{Z}_{u_c}\}$. Furthermore, note that in the case where

$\mathcal{Z} = \emptyset$; that is, the plant is a continuous-time dynamical system without any resetting, it follows that $\tilde{\mathcal{Z}}_{\tilde{x}} = \mathcal{Z}_{\mathrm{c}x_{\mathrm{c}}} \cup \{(x, x_{\mathrm{c}}): h_{\mathrm{c}}(x) \in \mathcal{Z}_{\mathrm{c}u_{\mathrm{cc}}}\}$ and hence knowledge of x_{c} and y_{c} is sufficient to determine whether or not the closed-loop state vector is in the set $\tilde{\mathcal{Z}}_{\tilde{x}}$. For the statement of the results of this section let $\mathcal{T}^{\mathrm{c}}_{x_0,u_{\mathrm{c}}}$ denote the set of resetting times of $\mathcal{G}$, let $\mathcal{T}_{x_0,u_{\mathrm{c}}}$ denote the complement of $\mathcal{T}^{\mathrm{c}}_{x_0,u_{\mathrm{c}}}$; that is, $[0,\infty)\backslash\mathcal{T}^{\mathrm{c}}_{x_0,u_{\mathrm{c}}}$, let $\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}0},u_{\mathrm{cc}}}$ denote the set of resetting times of $\mathcal{G}_{\mathrm{c}}$ and let $\mathcal{T}_{x_{\mathrm{c}0},u_{\mathrm{cc}}}$ denote the complement of $\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}0},u_{\mathrm{cc}}}$; that is, $[0,\infty)\backslash\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}0},u_{\mathrm{cc}}}$.

Theorem 1: *Consider the closed-loop system consisting of the non-linear impulsive dynamical systems $\mathcal{G}$ given by (1)–(4) and $\mathcal{G}_{\mathrm{c}}$ given by (5)–(8) with input–output pairs $(u_{\mathrm{c}}, u_{\mathrm{d}}; y_{\mathrm{c}}, y_{\mathrm{d}})$ and $(u_{\mathrm{cc}}, u_{\mathrm{dc}}; y_{\mathrm{cc}}, y_{\mathrm{dc}})$, respectively and with $(u_{\mathrm{cc}}, u_{\mathrm{dc}}) = (y_{\mathrm{c}}, y_{\mathrm{d}})$ and $(y_{\mathrm{cc}}, y_{\mathrm{dc}}) = (-u_{\mathrm{c}}, -u_{\mathrm{d}})$. Assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are zero-state observable and dissipative with respect to the supply rates $(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and $(r_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}), r_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}))$ and with continuously differentiable positive definite, radially unbounded storage functions $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$, respectively, such that $V_{\mathrm{s}}(0) = 0$, $V_{\mathrm{sc}}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) + \sigma r_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}) \le 0$ and $r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) + \sigma r_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}) \le 0$. Then the following statements hold:*

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is Lyapunov stable.*

(ii) *If $\mathcal{G}$ is strongly zero-state observable, $\mathcal{G}_{\mathrm{c}}$ is exponentially dissipative with respect to the supply rate $(r_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}), r_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}))$ and $\operatorname{rank}[G_{\mathrm{cc}}(u_{\mathrm{cc}}, 0)] = m_{\mathrm{cc}}$, $u_{\mathrm{cc}} \in \mathcal{U}_{\mathrm{cc}}$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable.*

(iii) *If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are exponentially dissipative with respect to supply rates $(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and $(r_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}), r_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}))$, respectively and $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$ are such that there exist constants α, α_{c}, β, $\beta_{\mathrm{c}} > 0$ such that*

$$\alpha\|x\|^2 \le V_{\mathrm{s}}(x) \le \beta\|x\|^2, \quad x \in \mathbb{R}^n \tag{9}$$

$$\alpha_{\mathrm{c}}\|x_{\mathrm{c}}\|^2 \le V_{\mathrm{sc}}(x_{\mathrm{c}}) \le \beta_{\mathrm{c}}\|x_{\mathrm{c}}\|^2, \quad x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}} \tag{10}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally exponentially stable.

Proof: Let $\tilde{\mathcal{T}}^{\mathrm{c}} \triangleq \mathcal{T}^{\mathrm{c}}_{x_0,u_{\mathrm{c}}} \cup \mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}0},u_{\mathrm{cc}}}$ and $t_k \in \tilde{\mathcal{T}}^{\mathrm{c}}$, $k \in \mathcal{N}$. First, note that it follows from Assumptions A1 and A2 of Haddad *et al.* (2001) that the resetting times $t_k(= \tau_k(\tilde{x}_0))$ for the feedback system are well defined and distinct for every closed-loop trajectory. (i) Consider the Lyapunov function candidate $V(x, x_{\mathrm{c}}) = V_{\mathrm{s}}(x) + \sigma V_{\mathrm{sc}}(x_{\mathrm{c}})$. Now, the corresponding Lyapunov derivative of $V(x, x_{\mathrm{c}})$ along the state trajectories $(x(t), x_{\mathrm{c}}(t))$, $t \in (t_k, t_{k+1}]$, is given by

$$\begin{aligned}\dot{V}(x(t), x_{\mathrm{c}}(t)) &= \dot{V}_{\mathrm{s}}(x(t)) + \sigma\dot{V}_{\mathrm{sc}}(x_{\mathrm{c}}(t))\\ &\le r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) + \sigma r_{\mathrm{cc}}(u_{\mathrm{cc}}(t), y_{\mathrm{cc}}(t))\\ &\le 0, \quad (x(t), x_{\mathrm{c}}(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}}\end{aligned} \tag{11}$$

and the Lyapunov difference of $V(x, x_{\mathrm{c}})$ at the resetting times t_k, $k \in \mathcal{N}$, is given by

$$\begin{aligned}\Delta V(x(t_k), x_{\mathrm{c}}(t_k)) &= \Delta V_{\mathrm{s}}(x(t_k)) + \sigma\Delta V_{\mathrm{sc}}(x_{\mathrm{c}}(t_k)\\ &\le r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\\ &\quad + \sigma r_{\mathrm{dc}}(u_{\mathrm{dc}}(t_k), y_{\mathrm{dc}}(t_k))\\ &\le 0, \quad (x(t_k), x_{\mathrm{c}}(t_k)) \in \tilde{\mathcal{Z}}_{\tilde{x}}\end{aligned} \tag{12}$$

Now, Lyapunov stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ follows as a direct consequence of Theorem 2 of Haddad *et al.* (2001). (ii) Next, if $\mathcal{G}_{\mathrm{c}}$ is exponentially dissipative it follows that for some scalar $\varepsilon_{\mathrm{cc}} > 0$

$$\begin{aligned}\dot{V}(x(t), x_{\mathrm{c}}(t)) &= \dot{V}_{\mathrm{s}}(x(t)) + \sigma\dot{V}_{\mathrm{sc}}(x_{\mathrm{c}}(t))\\ &\le -\sigma\varepsilon_{\mathrm{cc}}V_{\mathrm{sc}}(x_{\mathrm{c}}(t)) + r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\\ &\quad + \sigma r_{\mathrm{cc}}(u_{\mathrm{cc}}(t), y_{\mathrm{cc}}(t))\\ &\le -\sigma\varepsilon_{\mathrm{cc}}V_{\mathrm{sc}}(x_{\mathrm{c}}(t)), \quad (x(t), x_{\mathrm{c}}(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\\ &\qquad t_k < t \le t_{k+1}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\Delta V(x(t_k), x_{\mathrm{c}}(t_k)) &= \Delta V_{\mathrm{s}}(x(t_k)) + \sigma\Delta V_{\mathrm{sc}}(x_{\mathrm{c}}(t_k))\\ &\le r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\\ &\quad + \sigma r_{\mathrm{dc}}(u_{\mathrm{dc}}(t_k), y_{\mathrm{dc}}(t_k))\\ &\le 0, \quad (x(t_k), x_{\mathrm{c}}(t_k)) \in \tilde{\mathcal{Z}}_{\tilde{x}},\\ &\qquad k \in \mathcal{N}\end{aligned} \tag{14}$$

Now, let

$$\begin{aligned}\mathcal{R} \triangleq {}& \{(x, x_{\mathrm{c}}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}}: (x, x_{\mathrm{c}}) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\ \dot{V}(x, x_{\mathrm{c}}) = 0\}\\ &\cup\{(x, x_{\mathrm{c}}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}}: (x, x_{\mathrm{c}}) \in \tilde{\mathcal{Z}}_{\tilde{x}},\ \Delta V(x, x_{\mathrm{c}}) = 0\}\end{aligned}$$

where $\dot{V}(x, x_{\mathrm{c}})$ and $\Delta V(x, x_{\mathrm{c}})$ denote the total derivative and difference of $V(x, x_{\mathrm{c}})$ of the closed-loop system for all $(x, x_{\mathrm{c}}) \notin \tilde{\mathcal{Z}}_{\tilde{x}}$ and $(x, x_{\mathrm{c}}) \in \tilde{\mathcal{Z}}_{\tilde{x}}$, respectively. Since $V_{\mathrm{sc}}(x_{\mathrm{c}})$ is positive definite, note that $\dot{V}(x, x_{\mathrm{c}}) = 0$ for all $(x, x_{\mathrm{c}}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}}\backslash\tilde{\mathcal{Z}}_{\tilde{x}}$ only if $x_{\mathrm{c}} = 0$. Now, since $\operatorname{rank}[G_{\mathrm{cc}}(u_{\mathrm{cc}}, 0)] = m_{\mathrm{cc}}$, $u_{\mathrm{cc}} \in \mathcal{U}_{\mathrm{cc}}$, it follows that on every invariant set $\mathcal{M}$ contained in $\mathcal{R}$, $u_{\mathrm{cc}}(t) = y_{\mathrm{c}}(t) \equiv 0$ and hence $y_{\mathrm{cc}}(t) \equiv -u_{\mathrm{c}}(t) \equiv 0$ so that $\dot{x}(t) = f_{\mathrm{c}}(x(t))$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that

$$\begin{aligned}\mathcal{R} = {}& \{(0,0)\} \cup \{(x, x_{\mathrm{c}}) \in \mathbb{R}^n\\ &\times \mathbb{R}^{n_{\mathrm{c}}}: (x, x_{\mathrm{c}}) \in \tilde{\mathcal{Z}}_{\tilde{x}}, \Delta V(x, x_{\mathrm{c}}) = 0\}\end{aligned}$$

contains no solution other than the trivial solution $(x(t), x_c(t)) \equiv (0, 0)$. Hence, it follows from Theorem 4 of Haddad *et al.* (2001) that $(x(t), x_c(t)) \to \mathcal{M} = \{(0,0)\}$ as $t \to \infty$. Now, global asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded. (iii) Finally, if $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative and (9) and (10) hold, it follows that

$$\begin{aligned}\dot V(x(t), x_c(t)) &= \dot V_s(x(t)) + \sigma \dot V_{sc}(x_c(t))\\ &\le -\varepsilon_c V_s(x(t)) - \sigma\varepsilon_{cc} V_{sc}(x_c(t))\\ &\quad + r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t))\\ &\le -\min\{\varepsilon_c, \varepsilon_{cc}\}\, V(x(t), x_c(t)),\\ &\quad (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde x}, \qquad t_k < t \le t_{k+1} \end{aligned} \tag{15}$$

and $\Delta V(x(t_k), x_c(t_k))$, $(x(t_k), x_c(t_k)) \in \tilde{\mathcal{Z}}_{\tilde x}$, $k \in \mathcal{N}$, satisfies (14). Now, Theorem 2 of Haddad *et al.* (2001) implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally exponentially stable. □

The next result presents Lyapunov, asymptotic and exponential stability of dissipative feedback systems with quadratic supply rates.

Theorem 2: *Let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c \times m_c}$, $R_c \in \mathbb{S}^{m_c}$, $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d \times m_d}$, $R_d \in \mathbb{S}^{m_d}$, $Q_{cc} \in \mathbb{S}^{l_{cc}}$, $S_{cc} \in \mathbb{R}^{l_{cc} \times m_{cc}}$, $R_{cc} \in \mathbb{S}^{m_{cc}}$, $Q_{dc} \in \mathbb{S}^{l_{dc}}$, $S_{dc} \in \mathbb{R}^{l_{dc} \times m_{dc}}$ and $R_{dc} \in \mathbb{S}^{m_{dc}}$. Consider the closed-loop system consisting of the nonlinear impulsive dynamical systems $\mathcal{G}$ given by (1)–(4) and $\mathcal{G}_c$ given by (5)–(8) and assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable. Furthermore, assume $\mathcal{G}$ is dissipative with respect to the quadratic supply rate*

$$\begin{aligned}(r_c(u_c, y_c), r_d(u_d, y_d)) = (&y_c^T Q_c y_c + 2 y_c^T S_c u_c\\ &+ u_c^T R_c u_c,\ y_d^T Q_d y_d\\ &+ 2 y_d^T S_d u_d + u_d^T R_d u_d)\end{aligned}$$

and has a continuously differentiable, radially unbounded storage function $V_s(\cdot)$ and $\mathcal{G}_c$ is dissipative with respect to the quadratic supply rate

$$\begin{aligned}(r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) = (&y_{cc}^T Q_{cc} y_{cc} + 2 y_{cc}^T S_{cc} u_{cc}\\ &+ u_{cc}^T R_{cc} u_{cc},\ y_{dc}^T Q_{dc} y_{dc}\\ &+ 2 y_{dc}^T S_{dc} u_{dc} + u_{dc}^T R_{dc} u_{dc})\end{aligned}$$

and has a continuously differentiable, radially unbounded storage function $V_{sc}(\cdot)$. Finally, assume there exists a scalar $\sigma > 0$ such that

$$\left.\begin{aligned}\hat Q_c &\triangleq \begin{bmatrix} Q_c + \sigma R_{cc} & -S_c + \sigma S_{cc}^T \\ -S_c^T + \sigma S_{cc} & R_c + \sigma Q_{cc}\end{bmatrix} \le 0\\ \hat Q_d &\triangleq \begin{bmatrix} Q_d + \sigma R_{dc} & -S_d + \sigma S_{dc}^T \\ -S_d^T + \sigma S_{dc} & R_d + \sigma Q_{dc}\end{bmatrix} \le 0\end{aligned}\right\} \tag{16}$$

Then the following statements hold:

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable.*

(ii) *If $\mathcal{G}$ is strongly zero-state observable, $\mathcal{G}_c$ is exponentially dissipative with respect to the supply rate $(r_{cc}(u_{cc}, y_{cc}),\ r_{dc}(u_{dc},\ y_{dc}))$ and* $\operatorname{rank}[G_{cc}(u_{cc}, 0)] = m_{cc}$, *$u_{cc} \in \mathcal{U}_{cc}$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable.*

(iii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative with respect to supply rates $(r_c(u_c, y_c),\ r_d(u_d, y_d))$ and $(r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc}))$, respectively and there exist constants α, α_c, β, $\beta_c > 0$ such that* (9) *and* (10) *hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally exponentially stable.*

(iv) *If $\hat Q_c < 0$ and $\hat Q_d < 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable.*

Proof: (i)–(iii) are a direct consequence of Theorem 2 by noting

$$r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) = \begin{bmatrix} y_c \\ y_{cc}\end{bmatrix}^T \hat Q_c \begin{bmatrix} y_c \\ y_{cc}\end{bmatrix} \tag{17}$$

$$r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) = \begin{bmatrix} y_d \\ y_{dc}\end{bmatrix}^T \hat Q_d \begin{bmatrix} y_d \\ y_{dc}\end{bmatrix} \tag{18}$$

and hence $r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) \le 0$ and $r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) \le 0$. To show (iv) consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x)$. Noting that $u_{cc} = y_c$ and $y_{cc} = -u_c$, it follows that the corresponding Lyapunov derivative satisfies

$$\begin{aligned}\dot V(x(t), x_c(t)) &= \dot V_s(x(t)) + \sigma \dot V_{sc}(x_c(t))\\ &\le r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t))\\ &= y_c^T(t) Q_c y_c(t) + 2 y_c^T(t) S_c u_c(t)\\ &\quad + u_c^T(t) R_c u_c(t)\\ &\quad + \sigma[y_{cc}^T(t) Q_{cc} y_{cc}(t) + 2 y_{cc}^T(t) S_{cc} u_{cc}(t)\\ &\quad + u_{cc}^T(t) R_{cc} u_{cc}(t)]\\ &= \begin{bmatrix} y_c(t) \\ y_{cc}(t)\end{bmatrix}^T \hat Q_c \begin{bmatrix} y_c(t) \\ y_{cc}(t)\end{bmatrix}\\ &\le \quad 0, \quad (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde x}, \quad t_k < t \le t_{k+1}\end{aligned} \tag{19}$$

and, similarly, the Lyapunov difference satisfies

$$\Delta V(x(t_k), x_{\rm c}(t_k)) = \begin{bmatrix} y_{\rm d}(t_k) \\ y_{\rm dc}(t_k) \end{bmatrix}^{\rm T} \hat{Q}_{\rm d} \begin{bmatrix} y_{\rm d}(t_k) \\ y_{\rm dc}(t_k) \end{bmatrix} \le 0, \quad (x(t_k), x_{\rm c}(t_k)) \in \tilde{\mathcal{Z}}_{\tilde{x}}, \quad k \in \mathcal{N} \tag{20}$$

which implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is Lyapunov stable. Next, let

$$\mathcal{R} \triangleq \{(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} : (x, x_{\rm c}) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\ \dot{V}(x, x_{\rm c}) = 0\} \cup \{(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} : (x, x_{\rm c}) \in \tilde{\mathcal{Z}}_{\tilde{x}},\ \Delta V(x, x_{\rm c}) = 0\}$$

where $\dot{V}(x, x_{\rm c})$ and $\Delta V(x, x_{\rm c})$ denote the total derivative and difference of $V(x, x_{\rm c})$ of the closed-loop system for all $(x, x_{\rm c}) \notin \tilde{\mathcal{Z}}_{\tilde{x}}$ and $(x, x_{\rm c}) \in \tilde{\mathcal{Z}}_{\tilde{x}}$, respectively. Now, note that $\dot{V}(x, x_{\rm c}) = 0$ for all $(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} \backslash \tilde{\mathcal{Z}}_{\tilde{x}}$ if and only if $(y_{\rm c}, y_{\rm cc}) = (0, 0)$ and $\Delta V(x, x_{\rm c}) = 0$ for all $(x, x_{\rm c}) \in \tilde{\mathcal{Z}}_{\tilde{x}}$ if and only if $(y_{\rm d}, y_{\rm dc}) = (0, 0)$. Since $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are zero-state observable it follows that

$$\mathcal{R} \triangleq \{(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} : (x, x_{\rm c}) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\ (h_{\rm c}(x), h_{\rm cc}(x_{\rm c})) = (0, 0)\} \cup \{(x, x_{\rm c}) \in \tilde{\mathcal{Z}}_{\tilde{x}},\ (h_{\rm d}(x), h_{\rm dc}(x_{\rm c})) = (0, 0)\}$$

which contains no solution other than the trivial solution $(x(t), x_{\rm c}(t)) \equiv (0, 0)$. Hence, it follows from Theorem 4 of Haddad *et al.* (2001) that $(x(t), x_{\rm c}(t)) \to \mathcal{M} = \{(0, 0)\}$ as $t \to \infty$. Finally, global asymptotic stability follows from the fact that $V_{\rm s}(\cdot)$ and $V_{\rm sc}(\cdot)$ are, by assumption, radially unbounded and hence $V(x, x_{\rm c}) \to \infty$ as $\|(x, x_{\rm c})\| \to \infty$. $\square$

The following result generalizes the classical positivity and small gain theorems to the case of impulsive dynamical systems. For this result note that if a non-linear dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}),\ r_{\rm d}(u_{\rm d}, y_{\rm d})) = (2u_{\rm c}^{\rm T} y_{\rm c}, 2u_{\rm d}^{\rm T} y_{\rm d})$, then, with $(\kappa_{\rm c}(y_{\rm c}), \kappa_{\rm d}(y_{\rm d})) = (-k_{\rm c} y_{\rm c}, -k_{\rm d} y_{\rm d})$, where $k_{\rm c}, k_{\rm d} > 0$, it follows that $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (-k_{\rm c} y_{\rm c}^{\rm T} y_{\rm c}, -k_{\rm d} y_{\rm d}^{\rm T} y_{\rm d}) < (0, 0)$, $(y_{\rm c}, y_{\rm d}) \neq (0, 0)$. Alternatively, if a non-linear dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}),\ r_{\rm d}(u_{\rm d}, y_{\rm d})) = (\gamma_{\rm c}^2 u_{\rm c}^{\rm T} u_{\rm c} - y_{\rm c}^{\rm T} y_{\rm c}, \gamma_{\rm d}^2 u_{\rm d}^{\rm T} u_{\rm d} - y_{\rm d}^{\rm T} y_{\rm d})$, where $\gamma_{\rm c}, \gamma_{\rm d} > 0$, then, with $(\kappa_{\rm c}(y_{\rm c}), \kappa_{\rm d}(y_{\rm d})) = (0, 0)$, it follows that $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (-y_{\rm c}^{\rm T} y_{\rm c}, -y_{\rm d}^{\rm T} y_{\rm d}) < (0, 0)$, $(y_{\rm c}, y_{\rm d}) \neq (0, 0)$. Hence, if $\mathcal{G}$ is zero-state observable it follows from Theorem 7 of Haddad *et al.* (2001) that all storage functions of $\mathcal{G}$ are positive definite.

Corollary 1: *Consider the closed-loop system consisting of the non-linear impulsive dynamical systems $\mathcal{G}$ given by* (1)–(4) *and $\mathcal{G}_{\rm c}$ given by* (5)–(8). *Assume $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are zero-state observable. Then the following statements hold:*

(i) *If $\mathcal{G}$ is passive and strongly zero-state observable, $\mathcal{G}_{\rm c}$ is exponentially passive and* $\operatorname{rank}[G_{\rm cc}(u_{\rm cc}, 0)] = m_{\rm cc}$, $u_{\rm cc} \in \mathcal{U}_{\rm cc}$, *then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are exponentially passive with storage functions $V_{\rm s}(\cdot)$ and $V_{\rm sc}(\cdot)$, respectively, such that* (9) *and* (10) *hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is exponentially stable.*

(iii) *If $\mathcal{G}$ is non-expansive with gains $\gamma_{\rm c}, \gamma_{\rm d} > 0$ and strongly zero-state observable, $\mathcal{G}_{\rm c}$ is exponentially non-expansive with gains $\gamma_{\rm cc} > 0$, $\gamma_{\rm dc} > 0$,* $\operatorname{rank}[G_{\rm cc}(u_{\rm cc}, 0)] = m_{\rm cc}$, $u_{\rm cc} \in \mathcal{U}_{\rm cc}$, $\gamma_{\rm c}\gamma_{\rm cc} \le 1$ *and* $\gamma_{\rm d}\gamma_{\rm dc} \le 1$, *then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.*

(iv) *If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are exponentially non-expansive with storage functions $V_{\rm s}(\cdot)$ and $V_{\rm sc}(\cdot)$, respectively, such that* (9) *and* (10) *hold and gains $\gamma_{\rm c}, \gamma_{\rm d} > 0$ and $\gamma_{\rm cc}, \gamma_{\rm dc} > 0$, respectively, such that $\gamma_{\rm c}\gamma_{\rm cc} \le 1$ and $\gamma_{\rm d}\gamma_{\rm dc} \le 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is exponentially stable.*

Proof: The proof is a direct consequence of Theorem 2. Specifically, (i) and (ii) follow from Theorem 2 with $Q_{\rm c} = 0$, $Q_{\rm d} = 0$, $Q_{\rm cc} = 0$, $Q_{\rm dc} = 0$, $S_{\rm c} = I_{m_{\rm c}}$, $S_{\rm d} = I_{m_{\rm d}}$, $S_{\rm cc} = I_{m_{\rm cc}}$, $S_{\rm dc} = I_{m_{\rm dc}}$, $R_{\rm c} = 0$, $R_{\rm d} = 0$, $R_{\rm cc} = 0$ and $R_{\rm dc} = 0$; while (iii) and (iv) follow from Theorem 2 with $Q_{\rm c} = -I_{l_{\rm c}}$, $Q_{\rm d} = -I_{l_{\rm d}}$, $Q_{\rm cc} = -I_{l_{\rm cc}}$, $Q_{\rm dc} = -I_{l_{\rm dc}}$, $S_{\rm c} = 0$, $S_{\rm d} = 0$, $S_{\rm cc} = 0$, $S_{\rm dc} = 0$ and $R_{\rm c} = \gamma_{\rm c}^2 I_{m_{\rm c}}$, $R_{\rm d} = \gamma_{\rm d}^2 I_{m_{\rm d}}$, $R_{\rm cc} = \gamma_{\rm cc}^2 I_{m_{\rm cc}}$ and $R_{\rm dc} = \gamma_{\rm dc}^2 I_{m_{\rm dc}}$. $\square$

Remark 1: Global asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is also guaranteed if the non-linear impulsive system $\mathcal{G}$ is input strict passive (see Remark 17 of Haddad *et al.* (2001)) (resp., output strict passive) and the non-linear impulsive compensator $\mathcal{G}_{\rm c}$ is input strict passive (resp., output strict passive). Hence, the results of this section present a generalization of the results given in Hill and Moylan (1977) and Chellaboina and Haddad (2000) to non-linear impulsive dynamical systems.

3. Optimal control for impulsive dynamical systems

In this section we consider an optimal control problem for non-linear impulsive dynamical systems involving a notion of optimality with respect to a hybrid non-linear-non-quadratic performance functional. Specifically, we consider the following impulsive optimal control problem.

Impulsive optimal control problem: Consider the non-linear impulsive controlled system given by

$$\left.\begin{aligned} \dot{x}(t) &= F_{\rm c}(x(t), u_{\rm c}(t), t), \quad x(t_0) = x_0, \\ x(t_{\rm f}) &= x_{\rm f}, \quad u_{\rm c}(t) \in \mathcal{U}_{\rm c}, \quad (t, x(t)) \notin \mathcal{S}_x \end{aligned}\right\} \tag{21}$$

$$\begin{aligned} \Delta x(t) &= F_{\rm d}(x(t), u_{\rm d}(t), t), \\ & u_{\rm d}(t) \in \mathcal{U}_{\rm d}, \quad (t, x(t)) \in \mathcal{S}_x \end{aligned} \tag{22}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the state vector, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d} \subseteq \mathbb{R}^{m_{\rm c}} \times \mathbb{R}^{m_{\rm d}}$, $t \in [t_0, t_{\rm f}]$, $k \in \mathcal{N}_{[t_0, t_{\rm f})}$, is the hybrid control input, $x(t_0) = x_0$ is given, $x(t_{\rm f}) = x_{\rm f}$ is fixed, $F_{\rm c}\colon \mathcal{D} \times \mathcal{U}_{\rm c} \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_{\rm c}(0, 0, 0) = 0$, $F_{\rm d}\colon \mathcal{S}_x \times \mathcal{U}_{\rm d} \to \mathbb{R}^n$ is continuous and $\mathcal{S}_x \subset [0, \infty) \times \mathcal{D}$. Then determine the control inputs $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$, $t \in [t_0, t_{\rm f}]$, $k \in \mathcal{N}_{[t_0, t_{\rm f})}$, such that the hybrid performance functional

$$J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot), t_0) = \int_{t_0}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t_0, t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \tag{23}$$

is minimized, where $L_{\rm c}\colon \mathcal{D} \times \mathcal{U}_{\rm c} \times \mathbb{R} \to \mathbb{R}$ and $L_{\rm d}\colon \mathcal{S}_x \times \mathcal{U}_{\rm d} \to \mathbb{R}$ are given.

Next, we present a hybrid version of Bellman's principle of optimality which provides necessary and sufficient conditions, with a given hybrid control $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$, $t \geq t_0$, $k \in \mathcal{N}_{[t, t_0)}$, for minimizing the performance functional (23).

Lemma 1: *Let $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$, $t \in [t_0, t_{\rm f}]$, $k \in \mathcal{N}_{[t_0, t_{\rm f})}$, be an optimal hybrid control that generates the trajectory $x(t)$, $t \in [t_0, t_{\rm f}]$, with $x(t_0) = x_0$. Then the trajectory $x(\cdot)$ from (t_0, x_0) to $(t_{\rm f}, x_{\rm f})$ is optimal if and only if for all $t', t \in [t_0, t_{\rm f}]$, the portion of the trajectory $x(\cdot)$ going from $(t', x(t'))$ to $(t'', x(t''))$ optimizes the same cost functional over $[t', t'']$, where $x(t') = x_1$ is a point on the optimal trajectory generated by $(u_{\rm c}(t), u_{\rm d}(t_k))$, $t \in [t_0, t')$, $k \in \mathcal{N}_{[t_0, t')}$.*

Proof: Let $u_{\rm c}(t)$, $t \in [t_0, t_{\rm f}]$, $u_{\rm d}(t_k)$, $k \in \mathcal{N}_{[t_0, t_{\rm f})}$, solve the impulsive optimal control problem and let $x(t)$, $t \in [t_0, t_{\rm f}]$, be the solution to (21) and (22). Next, *ad absurdum*, suppose there exist $t' \geq t_0$, $t'' \leq t_{\rm f}$ and $\hat{u}_{\rm c}(t)$, $t \in [t', t'']$, $\hat{u}_{\rm d}(t_k)$, $k \in \hat{\mathcal{N}}_{[t', t'')}$, such that

$$\begin{aligned} &\int_{t'}^{t''} L_{\rm c}(\hat{x}(t), \hat{u}_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \hat{\mathcal{N}}_{[t', t'')}} L_{\rm d}(\hat{x}(t_k), \hat{u}_{\rm d}(t_k), t_k) \\ &\quad < \int_{t'}^{t''} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t', t'')}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \end{aligned} \tag{24}$$

where $\hat{x}(t)$ is a solution of (21) and (22) for all $t \in [t', t'']$ with $u_{\rm c}(t) = \hat{u}_{\rm c}(t)$, $u_{\rm d}(t_k) = \hat{u}_{\rm d}(t_k)$, $\hat{x}(t') = x(t')$ and $\hat{x}(t'') = x(t'')$. Now, define

$$u_{\rm co}(t) \triangleq \begin{cases} u_{\rm c}(t), & [t_0, t') \\ \hat{u}_{\rm c}(t), & [t', t''] \\ u_{\rm c}(t), & (t'', t_{\rm f}] \end{cases}$$

$$u_{\rm do}(t_k) \triangleq \begin{cases} u_{\rm d}(t_k), & k \in \mathcal{N}_{[t_0, t')} \\ \hat{u}_{\rm d}(t_k), & k \in \hat{\mathcal{N}}_{[t', t'')} \\ u_{\rm d}(t_k), & k \in \mathcal{N}_{[t'', t_{\rm f})} \end{cases}$$

Then

$$\begin{aligned} J(x_0, u_{\rm co}(\cdot), u_{\rm do}(\cdot), t_0) &= \int_{t_0}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t_0, t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &= \int_{t_0}^{t'} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t_0, t')}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &\quad + \int_{t'}^{t''} L_{\rm c}(\hat{x}(t), \hat{u}_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \hat{\mathcal{N}}_{[t', t'')}} L_{\rm d}(\hat{x}(t_k), \hat{u}_{\rm d}(t_k), t_k) \\ &\quad + \int_{t''}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t'', t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &< \int_{t_0}^{t'} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t_0, t')}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &\quad + \int_{t'}^{t''} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t', t'')}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &\quad + \int_{t''}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\,{\rm d}t + \sum_{k \in \mathcal{N}_{[t'', t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \\ &= J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot), t_0) \end{aligned}$$

which is a contradiction.

Conversely, if $(u_{\rm c}(t), u_{\rm d}(t_k))$ minimizes $J(\cdot,\cdot,\cdot)$ over $[t', t'']$ and $k \in \mathcal{N}_{[t',t'')}$ for all $t' \geq t_0$ and $t'' \leq t_{\rm f}$, then it minimizes $J(\cdot,\cdot,\cdot)$ over $[t_0, t_{\rm f}]$. $\square$

Next, let $(u_{\rm c}^*(t), u_{\rm d}^*(t_k))$, $t \in [t_0, t_{\rm f}]$, $k \in \mathcal{N}_{[t_0,t_{\rm f})}$, solve the impulsive optimal control problem and define the optimal cost $J^*(x_0, t_0) \triangleq J(x_0, u_{\rm c}^*(\cdot), u_{\rm d}^*(\cdot), t_0)$. Furthermore, define, for $p: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $q: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, the Hamiltonians

$$H_{\rm c}(x, u_{\rm c}, p, t) \triangleq L_{\rm c}(x, u_{\rm c}, t) + p^{\rm T}(x, t) F_{\rm c}(x, u_{\rm c}, t)$$

and

$$\begin{aligned} H_{\rm d}(x, u_{\rm d}, q(x, t_k), t_k) \triangleq\ & L_{\rm d}(x, u_{\rm d}, t_k) \\ & + q(x + F_{\rm d}(x, u_{\rm d}, t_k), t_k) \\ & - q(x, t_k) \end{aligned}$$

Theorem 3: *Let $J^*(x, t)$ denote the minimal cost for the impulsive optimal control problem with $x_0 = x$ and $t_0 = t$ and assume that $J^*(\cdot,\cdot)$ is continuously differentiable in x. Then*

$$0 = \frac{\partial J^*(x(t), t)}{\partial t} + \min_{u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c}} H_{\rm c}(x(t), u_{\rm c}(t), p(x(t), t), t), \quad (t, x(t)) \notin \mathcal{S}_x \tag{25}$$

$$0 = \min_{u_{\rm d}(\cdot) \in \tilde{\mathcal{U}}_{\rm d}} H_{\rm d}(x(t), u_{\rm d}(t), q(x(t), t), t), \quad (t, x(t)) \in \mathcal{S}_x \tag{26}$$

where $p(x(t), t) \triangleq \left(\partial J^*(x(t), t)/\partial x\right)^{\rm T}$ and $q(x(t), t) = J^*(x(t), t)$. Furthermore, if $(u_{\rm c}^*(\cdot), u_{\rm d}^*(\cdot))$ solves the impulsive optimal control problem, then

$$0 = \frac{\partial J^*(x(t), t)}{\partial t} + H_{\rm c}(x(t), u_{\rm c}^*(t), p(x(t), t), t), \quad (t, x(t)) \notin \mathcal{S}_x \tag{27}$$

$$0 = H_{\rm d}(x(t), u_{\rm d}^*(t), q(x(t), t), t), \quad (t, x(t)) \in \mathcal{S}_x \tag{28}$$

Proof: Let $(t, x(t)) \notin \mathcal{S}_x$. It follows from Lemma 1 that for small enough $\varepsilon > 0$ and $t' \in [t, t + \varepsilon]$

$$\begin{aligned} J^*(x(t), t) =& \min_{(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \in \tilde{\mathcal{U}}_{\rm c} \times \tilde{\mathcal{U}}_{\rm d}} \left[\int_t^{t_{\rm f}} L_{\rm c}(x(s), u_{\rm c}(s), s)\, {\rm d}s \right. \\ & \left. + \sum_{k \in \mathcal{N}_{[t,t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \right] \\ =& \min_{u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c}} \int_t^{t'} L_{\rm c}(x(s), u_{\rm c}(s), s)\, {\rm d}s \\ & + \min_{(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \in \tilde{\mathcal{U}}_{\rm c} \times \tilde{\mathcal{U}}_{\rm d}} \left[\int_{t'}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\, {\rm d}t \right. \\ & \left. + \sum_{k \in \mathcal{N}_{[t',t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \right] \\ =& \min_{u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c}} \left[\int_t^{t'} L_{\rm c}(x(s), u_{\rm c}(s), s)\, {\rm d}s + J^*(x(t'), t') \right] \end{aligned}$$

or, equivalently

$$\begin{aligned} 0 = \min_{u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c}} & \left[\frac{1}{t' - t} [J^*(x(t'), t') \right. \\ & \left. - J^*(x(t), t)] + \frac{1}{t' - t} \int_t^{t'} L_{\rm c}(x(s), u_{\rm c}(s), s)\, {\rm d}s \right] \end{aligned}$$

Letting $t' \to t$ yields

$$0 = \min_{u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c}} \left[\frac{{\rm d}J^*(x(t), t)}{{\rm d}t} + L_{\rm c}(x(t), u_{\rm c}(t), t) \right]$$

Now, (25) and (27) follow by noting that

$$\frac{{\rm d}J^*(x(t), t)}{{\rm d}t} = \frac{\partial J^*(x(t), t)}{\partial t} + \frac{\partial J^*(x(t), t)}{\partial x} F_{\rm c}(x(t), u_{\rm c}(t), t)$$

Next, let $(t, x(t)) \in \mathcal{S}_x$. It follows from Lemma 1 that

$$\begin{aligned} J^*(x(t), t) =& \min_{(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \in \tilde{\mathcal{U}}_{\rm c} \times \tilde{\mathcal{U}}_{\rm d}} \left[L_{\rm d}(x(t), u_{\rm d}(t), t) \right. \\ & + \int_{t^+}^{t_{\rm f}} L_{\rm c}(x(t), u_{\rm c}(t), t)\, {\rm d}t \\ & \left. + \sum_{k \in \mathcal{N}_{[t^+,t_{\rm f})}} L_{\rm d}(x(t_k), u_{\rm d}(t_k), t_k) \right] \\ =& \min_{u_{\rm d}(\cdot) \in \tilde{\mathcal{U}}_{\rm d}} L_{\rm d}(x(t), u_{\rm d}(t), t) + J^*(x(t^+), t^+) \\ =& \min_{u_{\rm d}(\cdot) \in \tilde{\mathcal{U}}_{\rm d}} L_{\rm d}(x(t), u_{\rm d}(t), t) + J^*(x(t) \\ & + F_{\rm d}(x(t), u_{\rm d}^*(t), t), t) \end{aligned}$$

which implies (26) and (28). $\square$

Next, we provide a converse result to Theorem 3.

Theorem 4: *Suppose there exists a continuously differentiable function $V: \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ and an optimal control $(u_{\rm c}^*(\cdot), u_{\rm d}^*(\cdot))$ such that $V(x(t_{\rm f}), t_{\rm f}) = 0$*

$$0 = \frac{\partial V(x, t)}{\partial t} + H_{\rm c}\left(x, u_{\rm c}^*(t), \frac{\partial V^{\rm T}(x, t)}{\partial x}, t\right), \quad (t, x) \notin \mathcal{S}_x \tag{29}$$

$$0 = H_{\rm d}(x, u_{\rm d}^*(t), V(x, t), t), \quad (t, x) \in \mathcal{S}_x, \tag{30}$$

$$H_{\rm c}\left(x, u_{\rm c}^*(t), \frac{\partial V^{\rm T}(x, t)}{\partial x}, t\right) \leq H_{\rm c}\left(x, u_{\rm c}(t), \frac{\partial V^{\rm T}(x, t)}{\partial x}, t\right), \quad (t, x) \notin \mathcal{S}_x,\ u_{\rm c}(\cdot) \in \tilde{\mathcal{U}}_{\rm c} \tag{31}$$

$$H_{\rm d}(x, u_{\rm d}^*(t), V(x, t), t) \leq H_{\rm d}(x, u_{\rm d}(t), V(x, t), t), \quad (t, x) \in \mathcal{S}_x,\ u_{\rm d}(\cdot) \in \tilde{\mathcal{U}}_{\rm d} \tag{32}$$

Then $(u_{\rm c}^*(\cdot), u_{\rm d}^*(\cdot))$ solves the impulsive optimal control problem; that is

$$J^*(x_0,t_0)=J(x_0,u_c^*(\cdot),u_d^*(\cdot),t_0)$$
$$\le J(x_0,u_c(\cdot),u_d(\cdot),t_0),$$
$$(u_c(\cdot),u_d(\cdot))\in\tilde{\mathcal{U}}_c\times\tilde{\mathcal{U}}_d \quad (33)$$

and

$$J^*(x_0,t_0)=V(x_0,t_0) \quad (34)$$

Proof: Let $x(t)$, $t\ge t_0$, satisfy (21) and (22) and, for all $(t,x(t))\notin\mathcal{S}_x$, define

$$\dot{V}(x(t),t)\triangleq\frac{\partial V(x(t),t)}{\partial t}+\frac{\partial V(x(t),t)}{\partial x}F_c(x(t),u_c(t),t)$$

Then, with $u_c(t)=u_c^*(t)$, it follows from (29) that

$$0=\dot{V}(x(t),t)+L_c(x(t),u_c^*(t),t),\quad (t,x(t))\notin\mathcal{S}_x \quad (35)$$

Furthermore, it follows from (30) that

$$0=V(x(t)+F_d(x(t),u_d^*(t),t))-V(x(t),t)+L_d(x(t),u_d^*(t),t),\quad (t,x(t))\in\mathcal{S}_x \quad (36)$$

Now, noting that $V(x(t_f),t_f)=0$, it follows from (35) and (36) that

$$\begin{aligned}J^*(x_0,t_0)&=J(x_0,u_c^*(\cdot),u_d^*(\cdot),t_0)\\&=\int_{t_0}^{t_f}L_c(x(t),u_c^*(t),t)\,dt\\&\quad+\sum_{k\in\mathcal{N}_{[t_0,t_f)}}L_d(x(t_k),u_d^*(t_k),t_k)\\&=V(x_0,t_0)\end{aligned}$$

Next, for all $(u_c(\cdot),u_d(\cdot))\in\tilde{\mathcal{U}}_c\times\tilde{\mathcal{U}}_d$ it follows from (29)–(32) that

$$\begin{aligned}J(x_0,u_c(\cdot),u_d(\cdot),t_0)&=\int_{t_0}^{t_f}L_c(x(t),u_c(t),t)\,dt\\&\quad+\sum_{k\in\mathcal{N}_{[t_0,t_f)}}L_d(x(t_k),u_d(t_k),t_k)\\&=\int_{t_0}^{t_f}\Bigg[-\dot{V}(x(t),t)+\frac{\partial V(x(t),t)}{\partial t}\\&\quad+H_c\left(x(t),u_c(t),\frac{\partial V^{\mathrm{T}}(x(t),t)}{\partial x},t\right)\Bigg]dt\\&\quad+\sum_{k\in\mathcal{N}_{[t_0,t_f)}}[V(x(t_k),t_k)-V(x(t_k)\\&\quad+F_d(x(t),u_d(t_k),t_k))\\&\quad+H_d(x(t_k),u_d(t_k),V(x(t_k),t_k),t_k)]\\&\ge\int_{t_0}^{t_f}-\dot{V}(x(t),t)\,dt\\&\quad+\sum_{k\in\mathcal{N}_{[t_0,t_f)}}[V(x(t_k),t_k)-V(x(t_k)\\&\quad+F_d(x(t),u_d(t_k),t_k))]\\&\quad+\int_{t_0}^{t_f}\Bigg[\frac{\partial V(x(t),t)}{\partial t}\\&\quad+H_c(x(t),u_c^*(t),\frac{\partial V^{\mathrm{T}}(x(t),t)}{\partial x},t)\Bigg]dt\\&\quad+\sum_{k\in\mathcal{N}_{[t_0,t_f)}}H_d(x(t_k),u_d^*(t_k),V(x(t_k),t_k),t_k)\\&=V(x_0,t_0)-V(x(t_f),t_f)\\&=J^*(x_0,t_0)\end{aligned}$$

which completes the proof. □

Next, we use Theorem 4 to characterize optimal hybrid *feedback* controllers for non-linear impulsive dynamical systems. In order to obtain time-invariant controllers, we restrict our attention to state-dependent impulsive dynamical systems with non–Zeno solutions and optimality notions over the infinite horizon with an infinite number of resetting times. Hence, the impulsive optimal control problem becomes

$$\dot{x}(t)=F_c(x(t),u_c(t)),\quad x(0)=x_0,\quad x(t)\notin\mathcal{Z}_x \quad (37)$$

$$\Delta x(t)=F_d(x(t),u_d(t)),\quad x(t)\in\mathcal{Z}_x \quad (38)$$

where $\mathcal{Z}_x\subset\mathcal{D}$ and $u_c(\cdot)$ and $u_d(\cdot)$ are restricted to the class of *admissible* hybrid controls consisting of measurable functions such that $(u_c(t),u_d(t_k))\in\mathcal{U}_c\times\mathcal{U}_d$ for all $t\ge 0$ and $k\in\mathcal{N}_{[0,\infty)}$, where the constraint set $\mathcal{U}_c\times\mathcal{U}_d$ is given with $(0,0)\in\mathcal{U}_c\times\mathcal{U}_d$. To address the optimal non-linear hybrid feedback control problem let $\phi_c\colon\mathcal{D}\to\mathcal{U}_c$ be such that $\phi_c(0)=0$ and let $\phi_d\colon\mathcal{Z}_x\to\mathcal{U}_d$. If $(u_c(t),u_d(t_k))=(\phi_c(x(t)),\phi_d(x(t_k)))$, where $x(t)$, $t\ge 0$, satisfies (21) and (22), then $(u_c(\cdot),u_d(\cdot))$ is a *hybrid feedback control*. Given the hybrid feedback control $(u_c(t),u_d(t_k))=(\phi_c(x(t)),\phi_d(x(t_k)))$, the closed-loop state-dependent impulsive dynamical system has the form

$$\dot{x}(t)=F_c(x(t),\phi_c(x(t))),\quad x(0)=x_0,\quad x(t)\notin\mathcal{Z}_x \quad (39)$$

$$\Delta x(t)=F_d(x(t),\phi_d(x(t))),\quad x(t)\in\mathcal{Z}_x \quad (40)$$

Now, we present the main theorem for characterizing hybrid feedback controllers that guarantee closed-loop stability and minimize a hybrid non-linear-non-quadratic performance functional over the infinite horizon. For the statement of this result, recall that with $\mathcal{S}_x\triangleq[0,\infty)\times\mathcal{Z}_x$ it follows from Assumptions A1 and A2 of Haddad *et al.* (2001) that the resetting times $t_k(=\tau_k(x_0))$ are well defined and distinct for every trajectory of (39) and (40). Furthermore, define the set of regulation hybrid controllers by

$$\mathcal{C}(x_0) \triangleq \{(u_c(\cdot), u_d(\cdot)) : (u_c(\cdot), u_d(\cdot)) \text{ is admissible and}$$

$$x(\cdot) \text{ given by (37) and (38) satisfies } x(t) \to 0 \text{ as } t \to \infty\}$$

Theorem 5: *Consider the non-linear controlled impulsive system* (37) *and* (38) *with hybrid performance functional*

$$J(x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty L_c(x(t), u(t))\,dt + \sum_{k\in\mathcal{N}_{[0,\infty)}} L_d(x(t_k), u_d(t_k)) \quad (41)$$

where $(u_c(\cdot), u_d(\cdot))$ *is an admissible hybrid control. Assume there exists a continuously differentiable function* $V: \mathcal{D} \to \mathbb{R}$ *and a hybrid control law* $\phi_c: \mathcal{D} \to \mathcal{U}_c$ *and* $\phi_d: \mathcal{Z}_x \to \mathcal{U}_d$ *such that* $V(0) = 0$, $V(x) > 0$, $x \neq 0$, $\phi_c(0) = 0$ *and*

$$V'(x)F_c(x, \phi_c(x)) < 0, \quad x \notin \mathcal{Z}_x, \quad x \neq 0 \quad (42)$$

$$V(x + F_d(x, \phi_d(x))) - V(x) \leq 0, \quad x \in \mathcal{Z}_x \quad (43)$$

$$H_c(x, \phi_c(x)) = 0, \quad x \notin \mathcal{Z}_x \quad (44)$$

$$H_d(x, \phi_d(x)) = 0, \quad x \in \mathcal{Z}_x \quad (45)$$

$$H_c(x, u_c) \geq 0, \quad x \notin \mathcal{Z}_x, \quad u_c \in \mathcal{U}_c \quad (46)$$

$$H_d(x, u_d) \geq 0, \quad x \in \mathcal{Z}_x, \quad u_d \in \mathcal{U}_d \quad (47)$$

where

$$H_c(x, u_c) \triangleq L_c(x, u_c) + V'(x)F_c(x, u_c) \quad (48)$$

$$H_d(x, u_d) \triangleq L_d(x, u_d) + V(x + F_d(x, u_d)) - V(x) \quad (49)$$

Then, with the hybrid feedback control $(u_c(\cdot), u_d(\cdot)) = (\phi_c(x(\cdot)), \phi_d(x(\cdot)))$, *there exists a neighbourhood of the origin* $\mathcal{D}_0 \subseteq \mathcal{D}$ *such that if* $x_0 \in \mathcal{D}_0$, *the zero solution* $x(t) \equiv 0$ *of the closed-loop system* (39) *and* (40) *is locally asymptotically stable. Furthermore*

$$J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = V(x_0), \quad x_0 \in \mathcal{D}_0 \quad (50)$$

In addition, if $x_0 \in \mathcal{D}_0$ *then the hybrid feedback control* $(u_c(\cdot), u_d(\cdot)) = (\phi_c(x(\cdot)), \phi_d(x(\cdot)))$ *minimizes* $J(x_0, u_c(\cdot), u_d(\cdot))$ *in the sense that*

$$J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot))\in\mathcal{C}(x_0)} J(x_0, u_c(\cdot), u_d(\cdot)) \quad (51)$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_c = \mathbb{R}^{m_c}$, $\mathcal{U}_d = \mathbb{R}^{m_d}$ *and* $V(x) \to \infty$ *as* $\|x\| \to \infty$, *then the zero solution* $x(t) \equiv 0$ *of the closed-loop system* (39) *and* (40) *is globally asymptotically stable.*

Proof: Local and global asymptotic stability is a direct consequence of (42) and (43) by applying Theorem 2 of Haddad *et al.* (2001) to the closed-loop system (39), (40). Conditions (50) and (51) are a direct consequence of Theorem 4 with $V(x,t) = V(x)$, $t_0 = 0$, $t_f \to \infty$ and using the fact that $\lim_{t\to\infty} V(x(t)) = 0$ and $\lim_{k\to\infty} V(x(t_k)) = 0$. □

Remark 2: Note that (44) and (45) are the steady state hybrid Hamilton–Jacobi–Bellman equations for the non-linear hybrid system (37) and (38) with the hybrid performance criterion $J(x_0, u_c(\cdot), u_d(\cdot))$ given by (41). Furthermore, Theorem 5 guarantees optimality with respect to the set of admissible stabilizing hybrid controllers $\mathcal{C}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{C}(x_0)$ is not required. In addition, the optimal stabilizing hybrid *feedback* control law $(u_c, u_d) = (\phi_c(x), \phi_d(x))$ is independent of the initial condition x_0. Finally, in order to assure asymptotic stability of the hybrid closed-loop system (39) and (40), Theorem 5 requires that $V(\cdot)$ satisfy (42) and (43) which implies that $V(\cdot)$ is a Lyapunov function for the hybrid closed-loop system (39) and (40).

Next, we specialize Theorem 5 to linear impulsive systems. For the following result let $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m_c}$, $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m_d}$, $R_{1c} \in \mathbb{R}^{n\times n}$, $R_{2c} \in \mathbb{R}^{m_c\times m_c}$, $R_{1d} \in \mathbb{R}^{n\times n}$ and $R_{2d} \in \mathbb{R}^{m_d\times m_d}$ be given, where R_{1c}, R_{2c}, R_{1d} and R_{2d} are positive definite.

Corollary 2: *Consider the linear controlled impulsive system*

$$\dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \quad (52)$$

$$\Delta x(t) = (A_d - I_n)x(t) + B_d u_d(t), \quad x(t) \in \mathcal{Z}_x \quad (53)$$

with quadratic hybrid performance functional

$$J(x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty [x^T(t)R_{1c}x(t) + u_c^T(t)R_{2c}u_c(t)]\,dt + \sum_{k\in\mathcal{N}_{[0,\infty)}} [x^T(t_k)R_{1d}x(t_k) + u_d^T(t_k)R_{2d}u_d(t_k)] \quad (54)$$

where $(u_c(\cdot), u_d(\cdot))$ *is an admissible hybrid control. Furthermore, assume there exists a positive-definite matrix* $P \in \mathbb{R}^{n\times n}$ *such that*

$$0 = x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}} + R_{1\mathrm{c}} - PB_{\mathrm{c}}R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}P)x, \quad x \notin \mathcal{Z}_x \tag{55}$$

$$0 = x^{\mathrm{T}}(A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - P + R_{1\mathrm{d}} - A_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}}(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}})x, \quad x \in \mathcal{Z}_x \tag{56}$$

Then, the zero solution $x(t) \equiv 0$ to (52) and (53) is globally asymptotically stable with the hybrid feedback controller

$$u_{\mathrm{c}} = \phi_{\mathrm{c}}(x) = -R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}Px, \quad x \notin \mathcal{Z}_x \tag{57}$$

$$u_{\mathrm{d}} = \phi_{\mathrm{d}}(x) = -(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}}x, \quad x \in \mathcal{Z}_x \tag{58}$$

and

$$J(x_0, \phi_{\mathrm{c}}(\cdot), \phi_{\mathrm{d}}(\cdot)) = x_0^{\mathrm{T}}Px_0, \quad x_0 \in \mathbb{R}^n \tag{59}$$

Furthermore

$$J(x_0, \phi_{\mathrm{c}}(\cdot), \phi_{\mathrm{d}}(\cdot)) = \min_{(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) \in \mathcal{C}(x_0)} J(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))$$

where $\mathcal{C}(x_0)$ is the set of regulation hybrid controllers for (52) and (53) and $x_0 \in \mathbb{R}^n$.

Proof: The result is a direct consequence of Theorem 5 with $F_{\mathrm{c}}(x, u_{\mathrm{c}}) = A_{\mathrm{c}}x + B_{\mathrm{c}}u_{\mathrm{c}}$, $L_{\mathrm{c}}(x, u_{\mathrm{c}}) = x^{\mathrm{T}}R_{1\mathrm{c}}x + u_{\mathrm{c}}^{\mathrm{T}}R_{2\mathrm{c}}u_{\mathrm{c}}$, for $x \notin \mathcal{Z}_x$, $F_{\mathrm{d}}(x, u_{\mathrm{d}}) = (A_{\mathrm{d}} - I_n)x + B_{\mathrm{d}}u_{\mathrm{d}}$, $L_{\mathrm{d}}(x, u_{\mathrm{d}}) = x^{\mathrm{T}}R_{1\mathrm{d}}x + u_{\mathrm{d}}^{\mathrm{T}}R_{2\mathrm{d}}u_{\mathrm{d}}$, for $x \in \mathcal{Z}_x$, $V(x) = x^{\mathrm{T}}Px$, $\mathcal{D} = \mathbb{R}^n$ and $\mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}} = \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{d}}}$. Specifically, it follows from (55) that $H_{\mathrm{c}}(x, \phi_{\mathrm{c}}(x)) = 0$, $x \notin \mathcal{Z}_x$ and hence $V'(x)F_{\mathrm{c}}(x, \phi_{\mathrm{c}}(x)) < 0$ for all $x \neq 0$ and $x \notin \mathcal{Z}_x$. Similarly, it follows from (56) that $H_{\mathrm{d}}(x, \phi_{\mathrm{d}}(x)) = 0$, $x \in \mathcal{Z}_x$ and hence $V(x + F_{\mathrm{d}}(x, \phi_{\mathrm{d}}(x))) - V(x) < 0$ for all $x \neq 0$ and $x \in \mathcal{Z}_x$. Thus,

$$\begin{aligned} H_{\mathrm{c}}(x, u_{\mathrm{c}}) &= H_{\mathrm{c}}(x, u_{\mathrm{c}}) - H_{\mathrm{c}}(x, \phi_{\mathrm{c}}(x)) \\ &= [u_{\mathrm{c}} - \phi_{\mathrm{c}}(x)]^{\mathrm{T}}R_{2\mathrm{c}}[u_{\mathrm{c}} - \phi_{\mathrm{c}}(x)] \geq 0, \quad x \notin \mathcal{Z}_x \end{aligned}$$

and

$$\begin{aligned} H_{\mathrm{d}}(x, u_{\mathrm{d}}) &= H_{\mathrm{d}}(x, u_{\mathrm{d}}) - H_{\mathrm{d}}(x, \phi_{\mathrm{d}}(x)) \\ &= [u_{\mathrm{d}} - \phi_{\mathrm{d}}(x)]^{\mathrm{T}}(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})[u_{\mathrm{d}} - \phi_{\mathrm{d}}(x)] \\ &\geq 0, \quad x \in \mathcal{Z}_x \end{aligned}$$

so that all conditions of Theorem 5 are satisfied. Finally, since $V(\cdot)$ is radially unbounded, the zero solution $x(t) \equiv 0$ to (52) and (53) with $u_{\mathrm{c}}(t) = \phi_{\mathrm{c}}(x(t)) = -R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}Px(t)$, $x(t) \notin \mathcal{Z}_x$ and $u_{\mathrm{d}}(t) = \phi_{\mathrm{d}}(x(t)) = -(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}}x(t)$, $x(t) \in \mathcal{Z}_x$, is globally asymptotically stable. □

Remark 3: The optimal hybrid feedback control law $(\phi_{\mathrm{c}}(x), \phi_{\mathrm{d}}(x))$ in Corollary 2 is derived using the properties of $H_{\mathrm{c}}(x, u_{\mathrm{c}})$ and $H_{\mathrm{d}}(x, u_{\mathrm{d}})$ as defined in Theorem 5. Specifically, since

$$H_{\mathrm{c}}(x, u_{\mathrm{c}}) = x^{\mathrm{T}}R_{1\mathrm{c}}x + u_{\mathrm{c}}^{\mathrm{T}}R_{2\mathrm{c}}u_{\mathrm{c}} + x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}})x + 2x^{\mathrm{T}}PB_{\mathrm{c}}u_{\mathrm{c}}, \quad x \notin \mathcal{Z}_x$$

and

$$\begin{aligned} H_{\mathrm{d}}(x, u_{\mathrm{d}}) &= x^{\mathrm{T}}R_{1\mathrm{d}}x + u_{\mathrm{d}}^{\mathrm{T}}R_{2\mathrm{d}}u_{\mathrm{d}} \\ &\quad + (A_{\mathrm{d}}x + B_{\mathrm{d}}u_{\mathrm{d}})^{\mathrm{T}}P(A_{\mathrm{d}}x + B_{\mathrm{d}}u_{\mathrm{d}}) \\ &\quad - x^{\mathrm{T}}Px, \quad x \in \mathcal{Z}_x \end{aligned}$$

it follows that $\partial^2 H_{\mathrm{c}}/\partial u_{\mathrm{c}}^2 = R_{2\mathrm{c}} > 0$ and $\partial^2 H_{\mathrm{d}}/\partial u_{\mathrm{d}}^2 = R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}} > 0$. Now, $\partial H_{\mathrm{c}}/\partial u_{\mathrm{c}} = 2R_{2\mathrm{c}}u_{\mathrm{c}} + 2B_{\mathrm{c}}^{\mathrm{T}}Px = 0$, $x \notin \mathcal{Z}_x$ and $\partial H_{\mathrm{d}}/\partial u_{\mathrm{d}} = 2(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})u_{\mathrm{d}} + 2B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}}x = 0$, $x \in \mathcal{Z}_x$, give the unique global minimum of $H_{\mathrm{c}}(x, u_{\mathrm{c}})$, $x \notin \mathcal{Z}_x$ and $H_{\mathrm{d}}(x, u_{\mathrm{d}})$, $x \in \mathcal{Z}_x$, respectively. Hence, since $\phi_{\mathrm{c}}(x)$ minimizes $H_{\mathrm{c}}(x, u_{\mathrm{c}})$ on $x \notin \mathcal{Z}_x$ and $\phi_{\mathrm{d}}(x)$ minimizes $H_{\mathrm{d}}(x, u_{\mathrm{d}})$ on $x \in \mathcal{Z}_x$, it follows that $\phi_{\mathrm{c}}(x)$ satisfies $\partial H_{\mathrm{c}}/\partial u_{\mathrm{c}} = 0$ and $\phi_{\mathrm{d}}(x)$ satisfies $\partial H_{\mathrm{d}}/\partial u_{\mathrm{d}} = 0$, or, equivalently, $\phi_{\mathrm{c}}(x) = -R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}Px$, $x \notin \mathcal{Z}_x$ and $\phi_{\mathrm{d}}(x) = -(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}}x$, $x \in \mathcal{Z}_x$.

Remark 4: For given $R_{1\mathrm{c}}$, $R_{2\mathrm{c}}$, $R_{1\mathrm{d}}$ and $R_{2\mathrm{d}}$, (55) and (56) can be solved using constrained non-linear programming methods using the structure of $\mathcal{Z}_x$. For example, in the case where $\mathcal{Z}_x$ is characterized by the hyperplane $\mathcal{Z}_x = \{x \in \mathbb{R}^n : Hx = 0\}$, where $H \in \mathbb{R}^{m \times n}$, it follows that (56) holds when $x \in \mathcal{N}(\mathcal{H})$ and (55) holds when $x \in [\mathcal{N}(\mathcal{H})]^{\perp} = \mathcal{R}(\mathcal{H}^{\mathrm{T}})$, where $\mathcal{N}(\mathcal{H})$ denotes the null space of H and $\mathcal{R}(\mathcal{H}^{\mathrm{T}})$ denotes the range space of H^{T}. Now, reformulating $\mathcal{Z}_x$ as $\{x \in \mathbb{R}^n : Ex = 0\}$, where E is an elementary matrix composed of zeroes and ones such that the columns of E span the nullspace of H and using the fact that $P > 0$, (55) and (56) will hold for $P > 0$ with a specific internal matrix structure. This of course reduces the number of free elements in P satisfying (55) and (56). Alternatively, to avoid the complexity in solving (55) and (56), an inverse optimal control problem can be solved wherein $R_{1\mathrm{c}}$, $R_{2\mathrm{c}}$, $R_{1\mathrm{d}}$ and $R_{2\mathrm{d}}$ are arbitrary. In this case, (55) and (56) are implied by

$$0 = A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}} + R_{1\mathrm{c}} - PB_{\mathrm{c}}R_{2\mathrm{c}}^{-1}B_{\mathrm{c}}^{\mathrm{T}}P \tag{60}$$

$$0 = A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - P + R_{1\mathrm{d}} - A_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}}(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}})^{-1}B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} \tag{61}$$

Since $R_{1\mathrm{c}}$, $R_{2\mathrm{c}}$, $R_{1\mathrm{d}}$ and $R_{2\mathrm{d}}$ are arbitrary, (60) and (61) can be cast as an LMI feasibility problem involving

$$P > 0$$

$$\begin{bmatrix} A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}} & PB_{\mathrm{c}} \\ B_{\mathrm{c}}^{\mathrm{T}}P & -R_{2\mathrm{c}} \end{bmatrix} < 0$$

$$\begin{bmatrix} A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - P & A_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}} \\ B_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} & -(R_{2\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}}PB_{\mathrm{d}}) \end{bmatrix} < 0$$

4. Inverse optimal control for nonlinear affine impulsive systems

In this section we use the results of § 3 to obtain controllers that are predicated on an *inverse optimal hybrid control problem.* In particular, to avoid the complexity in solving the steady-state hybrid Hamilton–Jacobi–Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parametrize a family of stabilizing hybrid controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the non-linear impulsive system dynamics, the Lyapunov function of the closed-loop system and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton–Jacobi–Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints. In addition, as shown by Haddad *et al.* (2000), the inverse optimal hybrid controllers guarantee hybrid disc, sector and gain margins to multiplicative input uncertainty. Hence, the inverse optimal hybrid controllers obtained in this section additionally provide robustness guarantees to multiplicative input uncertainty and thus guarantee robustness to unmodelled actuator dynamics.

Consider the state-dependent affine (in the control) impulsive dynamical system

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \tag{62}$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad x(t) \in \mathcal{Z}_x \tag{63}$$

Furthermore, we consider performance integrands $L_c(x, u_c)$ and $L_d(x, u_d)$ of the form

$$\left.\begin{aligned} L_c(x, u_c) &= L_{1c}(x) + u_c^T R_{2c}(x) u_c \\ L_d(x, u_d) &= L_{1d}(x) + u_d^T R_{2d}(x) u_d \end{aligned}\right\} \tag{64}$$

where $L_{1c}: \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{1c}(x) \geq 0$, $x \in \mathbb{R}^n$, $R_{2c}: \mathbb{R}^n \to \mathbb{P}^{m_c}$, $L_{1d}: \mathcal{Z}_x \to \mathbb{R}$ and satisfies $L_{1d}(x) \geq 0$, $x \in \mathbb{R}^n$ and $R_{2d}: \mathcal{Z}_x \to \mathbb{P}^{m_d}$ so that (23) becomes

$$\begin{aligned} J(x_0, u_c(\cdot), u_d(\cdot)) = &\int_0^\infty [L_{1c}(x(t)) + u_c^T(t) R_{2c}(x(t)) u_c(t)]\, dt \\ &+ \sum_{k \in \mathcal{N}_{[0,\infty)}} [L_{1d}(x(t_k)) \\ &+ u_d^T(t_k) R_{2d}(x(t_k)) u_d(t_k)] \end{aligned} \tag{65}$$

Corollary 3: *Consider the non-linear impulsive controlled system* (62) *and* (63) *with performance functional* (65). *Assume there exists a continuously differentiable function* $V: \mathbb{R}^n \to \mathbb{R}$ *and functions* $P_{12}: \mathcal{Z}_x \to \mathbb{R}^{1\times m_d}$ *and* $P_2: \mathcal{Z}_x \to \mathbb{N}^{m_d}$ *such that* $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$

$$V'(x)[f_c(x) - \tfrac{1}{2} G_c(x) R_{2c}^{-1}(x) G_c^T(x) V'^T(x)] < 0, \quad x \notin \mathcal{Z}_x, \quad x \neq 0 \tag{66}$$

$$V(x + f_d(x) - \tfrac{1}{2} G_d(x)(R_{2d}(x) + P_2(x))^{-1} P_{12}^T(x)) - V(x) \leq 0 \quad x \in \mathcal{Z}_x \tag{67}$$

$$V(x + f_d(x) + G_d(x)u_d) = V(x + f_d(x)) + P_{12}(x)u_d + u_d^T P_2(x) u_d \tag{68}$$

where u_d *is admissible and*

$$V(x) \to \infty \qquad \text{as } \|x\| \to \infty \tag{69}$$

Then the zero solution $x(t) \equiv 0$ *to the closed-loop system*

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))\phi_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \tag{70}$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))\phi_d(x(t)), \quad x(t) \in \mathcal{Z}_x \tag{71}$$

is globally asymptotically stable with the hybrid feedback control law

$$\phi_c(x) = -\tfrac{1}{2} R_{2c}^{-1}(x) G_c^T(x) V'^T(x), \quad x \notin \mathcal{Z}_x \tag{72}$$

$$\phi_d(x) = -\tfrac{1}{2}(R_{2d}(x) + P_2(x))^{-1} P_{12}^T(x), \quad x \in \mathcal{Z}_x \tag{73}$$

and performance functional (65)*, with*

$$L_{1c}(x) = \phi_c^T(x) R_{2c}(x) \phi_c(x) - V'(x) f_c(x) \tag{74}$$

$$L_{1d}(x) = \phi_d^T(x)(R_{2d}(x) + P_2(x))\phi_d(x) - V(x + f_d(x)) + V(x) \tag{75}$$

is minimized in the sense that

$$J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{C}(x_0)} J(x_0, u_c(\cdot), u_d(\cdot)), \quad x_0 \in \mathbb{R}^n \tag{76}$$

Finally

$$J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n \tag{77}$$

Proof: The result is a direct consequence of Theorem 5 with $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_c = \mathbb{R}^{m_c}$, $\mathcal{U}_d = \mathbb{R}^{m_d}$, $F_c(x, u_c) = f_c(x) + G_c(x)u_c$, $F_d(x, u_d) = f_d(x) + G_d(x)u_d$, $L_c(x, u_c) = L_{1c}(x) + u_c^T R_{2c}(x) u_c$ and $L_d(x, u_d) = L_{1d}(x) + u_d^T R_{2d}(x) u_d$. Specifically, with (64) the Hamiltonians have the form

$$H_c(x, u_c) = L_{1c}(x) + u_c^T R_{2c}(x) u_c + V'(x)(f_c(x) + G_c(x) u_c), \quad x \notin \mathcal{Z}_x, \quad u_c \in \mathbb{R}^{m_c} \tag{78}$$

$$H_d(x, u_d) = L_{1d}(x) + V(x + f_d(x)) + P_{12}(x) u_d + u_d^T (R_{2d}(x) + P_2(x)) u_d - V(x), \quad x \in \mathcal{Z}_x, u_d \in \mathbb{R}^{m_d} \tag{79}$$

Now, the hybrid feedback control law (72) and (73) is obtained by setting $\partial H_c / \partial u_c = 0$ and $\partial H_d / \partial u_d = 0$. With (72) and (73) it follows that (66) and (67) imply (42) and (43), respectively. Next, since $V(\cdot)$ is continuously differentiable and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$ and hence $\phi_c(0) = 0$. Next, with $L_{1c}(x)$ and $L_{1d}(x)$ given by (74) and (75), respectively and $\phi_c(x)$, $\phi_d(x)$ given by (72) and (73), (44) and (45) hold. Finally, since

$$H_c(x, u_c) = H_c(x, u_c) - H_c(x, \phi_c(x)) = [u_c - \phi_c(x)]^T R_{2c}(x) [u_c - \phi_c(x)], \quad x \notin \mathcal{Z}_x \tag{80}$$

$$H_d(x, u_d) = H_d(x, u_d) - H_d(x, \phi_d(x)) = [u_d - \phi_d(x)]^T (R_{2d}(x) + P_2(x)) [u_d - \phi_d(x)], \quad x \in \mathcal{Z}_x \tag{81}$$

where $R_{2c}(x) > 0$, $x \notin \mathcal{Z}_x$ and $R_{2d}(x) + P_2(x) > 0$, $x \in \mathcal{Z}_x$, conditions (46) and (47) hold. The result now follows as a direct consequence of Theorem 5. □

Remark 5: Note that (66) and (67) are equivalent to

$$\dot{V}(x) \triangleq V'(x)[f_c(x) + G_c(x)\phi_c(x)] < 0, \quad x \notin \mathcal{Z}_x, \quad x \neq 0 \tag{82}$$

$$\Delta V(x) \triangleq V(x + f_d(x) + G_d(x)\phi_d(x)) - V(x) \leq 0, \quad x \in \mathcal{Z}_x \tag{83}$$

with $\phi_c(x)$ and $\phi_d(x)$ given by (72) and (73), respectively. Furthermore, conditions (82) and (83) with $V(0) = 0$ and $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, assure that $V(x)$ is a Lyapunov function for the impulsive closed-loop system (70) and (71).

Remark 6: In the case where $R_{2c}(x)$, $x \notin \mathcal{Z}_x$, is a diagonal weighting function, the hybrid controller (72) and (73) guarantees hybrid disc, sector and gain margins to multiplicative input uncertainty of $((\frac{1}{2}, \infty), (1/1 + \theta_d, 1/1 - \theta_d))$, where $\theta_d = \sqrt{\underline{\gamma}_d / \overline{\gamma}_d}$, $\underline{\gamma}_d \triangleq \inf_{x \in \mathcal{Z}_x} \sigma_{\min}(R_{2d}(x))$ and $\overline{\gamma}_d \triangleq \sup_{x \in \mathcal{Z}_x} \sigma_{\max}(R_{2d}(x) + P_2(x))$. For details see Haddad *et al.* (2000).

5. Non-linear hybrid control with polynomial and multilinear performance functionals

In this section we specialize the results of §4 to linear impulsive systems controlled by inverse optimal non-linear hybrid controllers that minimize a derived polynomial and multilinear cost functional. These results generalize the non-linear feedback control results derived by Bass and Weber and Speyer (1976) to impulsive dynamical systems. For the results in this section we assume $u_d(t_k) \equiv 0$. Furthermore, let $R_{1c} \in \mathbb{P}^n$, $R_{1d} \in \mathbb{P}^n$, $R_{2c} \in \mathbb{P}^{m_c}$, $\hat{R}_q, \hat{\hat{R}}_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, be given, where r is a positive integer and define $S_c \triangleq B_c R_{2c}^{-1} B_c^T$.

Corollary 4: *Consider the linear controlled impulsive system*

$$\dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \tag{84}$$

$$\Delta x(t) = (A_d - I_n) x(t), \quad x(t) \in \mathcal{Z}_x \tag{85}$$

where u_c is admissible. Assume there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, such that

$$0 = x^T (A_c^T P + P A_c + R_{1c} - P B_c R_{2c}^{-1} B_c^T P) x, \quad x \notin \mathcal{Z}_x \tag{86}$$

$$0 = x^T [(A_c - S_c P)^T M_q + M_q (A_c - S_c P) + \hat{R}_q] x, \quad x \notin \mathcal{Z}_x, \quad q = 2, \ldots, r \tag{87}$$

$$0 = x^T (A_d^T P A_d - P + R_{1d}) x, \quad x \in \mathcal{Z}_x \tag{88}$$

$$0 = x^T (A_d^T M_q A_d - M_q + \hat{\hat{R}}_q) x, \quad x \in \mathcal{Z}_x, \quad q = 2, \ldots, r \tag{89}$$

Then the zero solution $x(t) \equiv 0$ of the closed-loop system

$$\dot{x}(t) = A_c x(t) + B_c \phi_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \tag{90}$$

$$\Delta x(t) = (A_d - I_n) x(t), \quad x(t) \in \mathcal{Z}_x \tag{91}$$

is globally asymptotically stable with the feedback control law

$$\phi_c(x) = -R_{2c}^{-1} B_c^T (P + \sum_{q=2}^{r} (x^T M_q x)^{q-1} M_q) x, \quad x \notin \mathcal{Z}_x \tag{92}$$

and the performance functional (65), with $R_{2c}(x) = R_{2c}$ and

$$L_{1c}(x) = x^{\mathrm{T}}\Bigg(R_{1c} + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} \times S_c\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]\Bigg)x \quad (93)$$

$$L_{1d}(x) = x^{\mathrm{T}}R_{1d}x + \sum_{q=2}^{r}\frac{1}{q}\left[(x^{\mathrm{T}}\hat{\hat{R}}_q x)\sum_{j=1}^{q}(x^{\mathrm{T}}M_q x)^{j-1} \times (x^{\mathrm{T}}A_d^{\mathrm{T}}M_q A_d x)^{q-j}\right] \quad (94)$$

is minimized in the sense that

$$J(x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot)\in\mathcal{C}(x_0)} J(x_0, u_c(\cdot)), \quad x_0 \in \mathbb{R}^n \quad (95)$$

Finally,

$$J(x_0, \phi_c(x(\cdot))) = x_0^{\mathrm{T}}Px_0 + \sum_{q=2}^{r}\frac{1}{q}(x_0^{\mathrm{T}}M_q x_0)^q, \quad x_0 \in \mathbb{R}^n \quad (96)$$

Proof: The result is a direct consequence of Corollary 3 with $f_c(x) = A_c x$, $f_d(x) = (A_d - I_n)x$, $G_c(x) = B_c$, $G_d(x) = 0$, $u_d = 0$, $R_{2c}(x) = R_{2c}$, $R_{2d}(x) = I_{m_d}$ and $V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{r}(1/q)(x^{\mathrm{T}}M_q x)^q$. Specifically, for $x \notin \mathcal{Z}_x$ it follows from (86), (87) and (92) that

$$V'(x)[f_c(x) - \tfrac{1}{2}G_c(x)R_{2c}^{-1}G_c^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)]$$
$$= -x^{\mathrm{T}}R_{1c}x - \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}\hat{R}_q x - \phi_c^{\mathrm{T}}(x)R_{2c}\phi_c(x)$$
$$-x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}}S_c\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]x$$

which implies (66). For $x \in \mathcal{Z}_x$ it follows from (88) and (89) that

$$\Delta V(x) = V(x + f_d(x)) - V(x)$$
$$= -x^{\mathrm{T}}R_{1d}x - \sum_{q=2}^{r}\frac{1}{q}\left[(x^{\mathrm{T}}\hat{\hat{R}}_q x) \times \sum_{j=1}^{q}(x^{\mathrm{T}}M_q x)^{j-1}(x^{\mathrm{T}}A_d^{\mathrm{T}}M_q A_d x)^{q-j}\right]$$

which implies (67) with $G_d(x) = 0$. Finally, with $u_d = 0$, (68) is automatically satisfied so that all the conditions of Corollary 3 are satisfied. □

Remark 7: As noted in Remark 4, viewing R_{1c}, R_{2c}, $\hat{R}_q$ and $\hat{\hat{R}}_q$, $q = 2, \ldots, r$, as arbitrary matrices, it follows that (86)–(89) are implied by a set of Bilinear matrix inequalities (BMIs). This considerably minimizes the numerical complexity for solving (86)–(89).

Finally, we specialize the results of §4 to linear impulsive systems controlled by inverse optimal hybrid controllers that minimize a derived multilinear functional. First, however, we give several definitions involving multilinear forms. A scalar function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *q-multilinear* if q is a positive integer and $\psi(x)$ is a linear combination of terms of the form $x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$, where i_j is a non-negative integer for $j = 1, \ldots, n$ and $i_1 + i_2 + \ldots + i_n = q$. Furthermore, a q-multilinear function $\psi(\cdot)$ is non-negative definite (resp., positive definite) if $\psi(x) \geq 0$ for all $x \in \mathbb{R}^n$ (resp., $\psi(x) > 0$ for all non-zero $x \in \mathbb{R}^n$). Note that if q is odd then $\psi(x)$ cannot be positive definite. If $\psi(\cdot)$ is a q-multilinear function then $\psi(\cdot)$ can be represented by means of Kronecker products; that is, $\psi(x)$ is given by $\psi(x) = \Psi x^{[q]}$, where $\Psi \in \mathbb{R}^{1\times n^q}$ and $x^{[q]} \triangleq x \otimes x \otimes \cdots \otimes x$ (q times), where $\otimes$ denotes Kronecker product. For the next result recall the definition of S_c, let $R_{1c} \in \mathbb{P}^n$, $R_{1d} \in \mathbb{P}^n$, $R_{2c} \in \mathbb{P}^{m_c}$, $\hat{R}_{2q}, \hat{\hat{R}}_{2q} \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, be given, where $\mathcal{N}^{(2q,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^{2q}} : \Psi x^{[2q]} \geq 0,\ x \in \mathbb{R}^n\}$ and define the repeated (q times) Kronecker sum as $\overset{q}{\oplus}A \triangleq A \oplus A \oplus \cdots \oplus A$.

Corollary 5: *Consider the linear controlled impulsive system* (84) *and* (85). *Assume there exist* $P \in \mathbb{P}^n$ *and* $\hat{P}_q \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, *such that*

$$0 = x^{\mathrm{T}}(A_c^{\mathrm{T}}P + PA_c + R_{1c} - PB_cR_{2c}^{-1}B_c^{\mathrm{T}}P)x, \quad x \notin \mathcal{Z}_x \quad (97)$$

$$0 = x^{\mathrm{T}}(\hat{P}_q[\overset{2q}{\oplus}(A_c - S_cP)] + \hat{R}_{2q})x, \quad x \notin \mathcal{Z}_x, \quad q = 2, \ldots, r \quad (98)$$

$$0 = x^{\mathrm{T}}(A_d^{\mathrm{T}}PA_d - P + R_{1d})x, \quad x \in \mathcal{Z}_x \quad (99)$$

$$0 = x^{\mathrm{T}}(\hat{P}_q[A_d^{[2q]} - I_n^{[2q]}] + \hat{\hat{R}}_{2q})x, \quad x \in \mathcal{Z}_x, \quad q = 2, \ldots, r \quad (100)$$

Then the zero solution $x(t) \equiv 0$ *of the closed-loop system* (90) *and* (91) *is globally asymptotically stable with the feedback control law*

$$\phi_c(x) = -R_{2c}^{-1}B_c^{\mathrm{T}}(Px + \tfrac{1}{2}g'^{\mathrm{T}}(x)), \qquad x \notin \mathcal{Z}_x \quad (101)$$

where $g(x) \triangleq \sum_{q=2}^{r}\hat{P}_q x^{[2q]}$ *and the performance functional* (65), *with* $R_{2c}(x) = R_{2c}$ *and*

$$L_{1c}(x) = x^{\mathrm{T}} R_{1c} x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} + \tfrac{1}{4} g'g'(x) S_c g'^{\mathrm{T}}(x) \tag{102}$$

$$L_{1d}(x) = x^{\mathrm{T}} R_{1d} x + \sum_{q=2}^{r} \hat{\hat{R}}_{2q} x^{[2q]} \tag{103}$$

is minimized in the sense that

$$J(x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot)\in\mathcal{C}(x_0)} J(x_0, u_c(\cdot)), \quad x_0 \in \mathbb{R}^n \tag{104}$$

Finally

$$J(x_0, \phi_c(x(\cdot))) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}, \quad x_0 \in \mathbb{R}^n \tag{105}$$

Proof: The result is a direct consequence of Corollary 3 with $f_c(x) = A_c x$, $f_d(x) = (A_d - I_n)x$, $G_c(x) = B_c$, $G_d(x) = 0$, $u_d = 0$, $R_{2c}(x) = R_{2c}$, $R_{2d}(x) = I_{m_d}$ and $V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{r} \hat{P}_q x^{[2q]}$. Specifically, for $x \notin \mathcal{Z}_x$ it follows from (97), (98) and (101) that

$$\begin{aligned} V'(x)[f_c(x) - \tfrac{1}{2} G_c(x) R_{2c}^{-1}(x) G_c^{\mathrm{T}}(x) V'^{\mathrm{T}}(x)] &= -x^{\mathrm{T}} R_{1c} x - \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} \\ &\quad - \phi_c^{\mathrm{T}}(x) R_{2c} \phi_c(x) - \tfrac{1}{4} g'(x) S_c g'^{\mathrm{T}}(x) \end{aligned}$$

which implies (67). For $x \in \mathcal{Z}_x$ it follows from (68) and (100) that

$$\begin{aligned} \Delta V(x) &= V(x + f_d(x)) - V(x) \\ &= -x^{\mathrm{T}} R_{1d} x - \sum_{q=2}^{r} \hat{\hat{R}}_{2q} x^{[2q]} \end{aligned}$$

which implies (67) with $G_d(x) = 0$. Finally, with $u_d = 0$, (68) is automatically satisfied so that all the conditions of Corollary 3 are satisfied. $\square$

6. Hybrid controllers for combustion systems

High performance aeroengine afterburners and ramjets often experience combustion instabilities at some operating condition. Combustion in these high energy density engines is highly susceptible to flow disturbances, resulting in fluctuations to the instantaneous rate of heat release in the combustor. This unsteady combustion provides an acoustic source resulting in self-excited oscillations. In particular, unsteady combustion generates acoustic pressure and velocity oscillations which in turn perturb the combustion even further (Culick 1976, Candel 1992). These pressure oscillations, known as thermoacoustic instabilities, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates and even component melting. Hence, the need for active control to mitigate combustion induced pressure instabilities is severe.

In this section we apply the results developed in this paper and Part I of this paper (Haddad *et al.* 2001) to the control of thermoacoustic instabilities in combustion processes. We stress that the combustion model we use is *not* an impulsive dynamical system and hence can be stabilized by conventional non-linear control methods. The aim here however, is to show that hybrid control provides an extremely efficient mechanism for dissipating energy in the combustion process with far superior performance than any conventional control methodology. In particular, we show that the proposed hybrid controller provides finite-time stabilization. Since, as can be seen below, the combustion system dynamics are Lipschitz continuous, finite-time stabilization cannot be achieved by any other conventional controller.

To design hybrid controllers for combustion systems we concentrate on a two-mode, non-linear time-averaged combustion model with non-linearities present due to the second-order gas dynamics. This model is developed in Culick (1988) and is given by

$$\begin{aligned} \dot{x}_1(t) &= \alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) \\ &\quad + x_2(t)x_4(t)) + u_{s1}(t), \quad x_1(0) = x_{10}, \ t \geq 0 \end{aligned} \tag{106}$$

$$\begin{aligned} \dot{x}_2(t) &= -\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) \\ &\quad - x_1(t)x_4(t)) + u_{s2}(t), \quad x_2(0) = x_{20} \end{aligned} \tag{107}$$

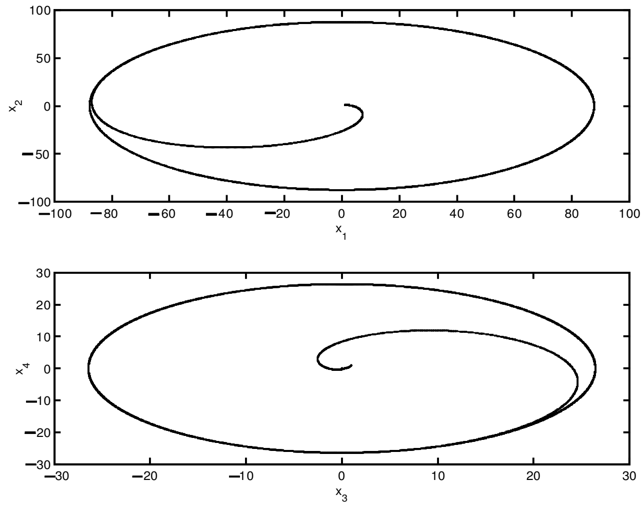
$$\begin{aligned} \dot{x}_3(t) &= \alpha_2 x_3(t) + \theta_2 x_4(t) + \beta(x_1^2(t) - x_2^2(t)) \\ &\quad + u_{s3}(t), \quad x_3(0) = x_{30} \end{aligned} \tag{108}$$

$$\begin{aligned} \dot{x}_4(t) &= -\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) \\ &\quad + u_{s4}(t), \quad x_4(0) = x_{40} \end{aligned} \tag{109}$$

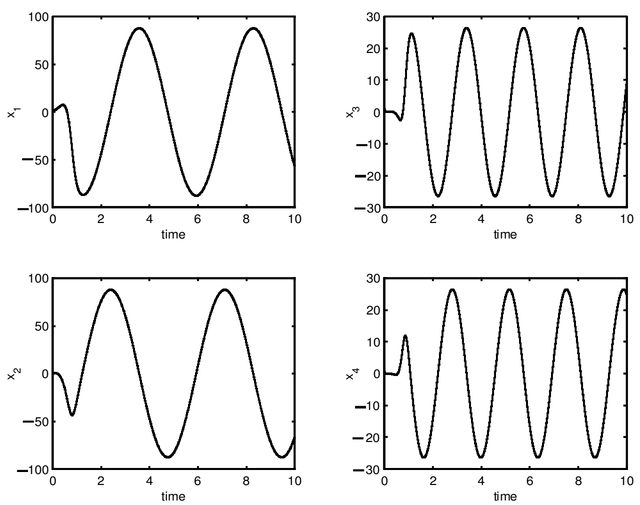
where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma+1)/8\gamma)\omega_1$, γ denotes the ratio of specific heats, ω_1 is frequency of the fundamental mode and u_{si}, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$ and $x_0 = [1\ 1\ 1\ 1]^{\mathrm{T}}$, the open-loop ($u_{si}(t) \equiv 0, i = 1, \ldots, 4$) dynamics (106)–(109) result in a limit cycle instability. Figure 2 shows the phase portrait, state response and *plant energy* $V_s(x) \triangleq x_1^2 + x_2^2 + x_3^2 + x_4^2$ versus time.

6.1. *Time-dependent hybrid controllers*

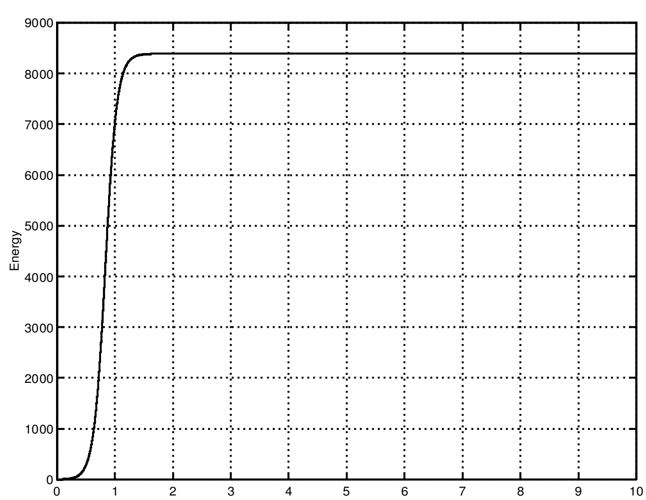
To design a stabilizing time-dependent hybrid controller for (106)–(109) we first design a control law $u_s = -K_s x + u_c$, where $K_s \triangleq \mathrm{diag}\,[k_{s1}, k_{s2}, k_{s3}, k_{s4}]$, $x \triangleq [x_1, x_2, x_3, x_4]^{\mathrm{T}}$, $u_s \triangleq [u_{s1}, u_{s2}, u_{s3}, u_{s4}]^{\mathrm{T}}$ and



(a)



(b)



(c)

Figure 2. (a) Phase portrait, (b) state response, and (c) plant energy versus time.

$u_c \triangleq [u_{c1}, u_{c2}, u_{c3}, u_{c4}]^T$. In this case, (106)–(109) are given by (1), (2) with $\mathcal{Z} = \varnothing$ and

$$\left.\begin{aligned} f_c(x) &= \begin{bmatrix} \alpha_1 x_1 + \theta_1 x_2 - \beta(x_1x_3 + x_2x_4) - k_{s1}x_1 \\ -\theta_1 x_1 + \alpha_1 x_2 + \beta(x_2x_3 - x_1x_4) - k_{s2}x_2 \\ \alpha_2 x_3 + \theta_2 x_4 + \beta(x_1^2 - x_2^2) - k_{s3}x_3 \\ -\theta_2 x_3 + \alpha_2 x_4 + 2\beta x_1x_2 - k_{s4}x_4 \end{bmatrix} \\ G_c(x) &= I_4 \end{aligned}\right\} \tag{110}$$

$$f_d(x) = 0, \quad G_d(x) = 0 \tag{111}$$

Now, with $y_c = x$, $k_{s1} = k_{s2} = \alpha_1$ and $k_{s3} = k_{s4} = 0$, it follows that (1), (3), with $f_c(x)$ and $G_c(x)$ given by (110) and $h_c(x) = x$ and $J_c(x) = 0$, is passive with input u_c, output y_c and plant energy function, or storage function, $V_s(x)$. Hence, $V_s'(x)f_c(x) \le 0$, $x \in \mathbb{R}^4$. Furthermore, (1), (3), with $f_c(x)$ and $G_c(x)$ given by (110) and $h_c(x) = x$ and $J_c(x) = 0$, is zero-state observable. Figure 3 shows the phase portrait, state response and plant energy of the controlled system (1), (3) with $u_s = -K_s x + u_c$ and $u_c \equiv 0$.

To improve the performance of the above controller, we use the flexibility in u_c to design a hybrid controller. Specifically, consider the hybrid controller emulating the plant structure given by (5)–(8) with $\mathcal{S}_c = \mathcal{T} \times \mathbb{R}^{n_c} \times \mathbb{R}^{m_{cc}}$

$$\left.\begin{aligned} f_{cc}(x_c) &= \begin{bmatrix} \alpha_1 x_{c1} + \theta_1 x_{c2} - \beta(x_{c1}x_{c3} + x_{c2}x_{c4}) - k_{c1}x_{c1} \\ -\theta_1 x_{c1} + \alpha_1 x_{c2} + \beta(x_{c2}x_{c3} - x_{c1}x_{c4}) - k_{c2}x_{c2} \\ \alpha_2 x_{c3} + \theta_2 x_{c4} + \beta(x_{c1}^2 - x_{c2}^2) - k_{c3}x_{c3} \\ -\theta_2 x_{c3} + \alpha_2 x_{c4} + 2\beta x_{c1}x_{c2} - k_{c4}x_{c4} \end{bmatrix} \\ G_{cc}(x_c) &= I_4 \end{aligned}\right\} \tag{112}$$

$$f_{dc}(x_c) = \begin{bmatrix} -x_{c1} \\ -x_{c2} \\ -x_{c3} \\ -x_{c4} \end{bmatrix}, \quad G_{dc}(x_c) = 0 \tag{113}$$

$$\left.\begin{aligned} h_{cc}(x_c) &= -[x_{c1}, x_{c2}, x_{c3}, x_{c4}]^T \\ J_{cc}(x_c) &= 0 \\ h_{dc}(x_c) &= 0 \\ J_{dc}(x_c) &= 0 \end{aligned}\right\} \tag{114}$$

where $k_{c1} > \alpha_1$, $k_{c2} > \alpha_1$, $k_{c3} > \alpha_2$ and $k_{c4} > \alpha_2$. It can be easily shown using Corollary 4 and Remark 15 of Haddad *et al.* (2001) that the hybrid controller (5)–(8) with dynamics given by (112)–(114), resetting set $\mathcal{S}_c = \mathcal{T} \times \mathbb{R}^{n_c} \times \mathbb{R}^{m_{cc}}$, input y_c and output $-u_c$ is exponentially passive with *controller energy*, or storage function,

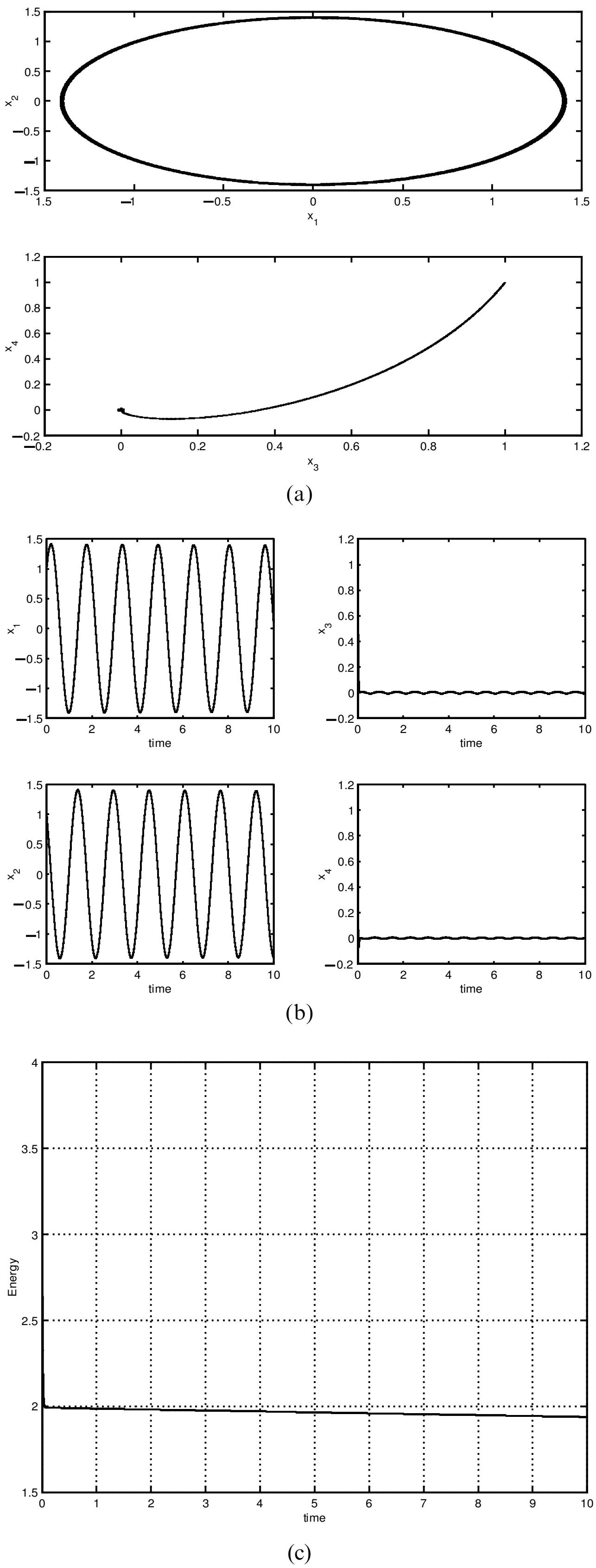


Figure 3. (a) Phase portrait, (b) state response, and (c) plant energy versus time.

$V_{\rm sc}(x_{\rm c}) \triangleq x_{\rm c1}^2 + x_{\rm c2}^2 + x_{\rm c3}^2 + x_{\rm c4}^2$. Hence, $V'_{\rm sc}(x_{\rm c}) f_{\rm cc}(x_{\rm c}) \le -\varepsilon V_{\rm sc}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^4$, where $\varepsilon = \min\{\alpha_1 - k_{\rm c1}, \alpha_1 - k_{\rm c2}, \alpha_2 - k_{\rm c3}, \alpha_2 - k_{\rm c4}\}$. Furthermore, note that rank $[G_{\rm cc}(0)] = 4$. Hence, stability of the closed-loop system (1), (3), (5)–(8), is guaranteed by Theorem 1. Finally, we note that the *total energy* of the closed-loop system (1), (3), (5)–(8) is given by

$$\begin{aligned} V(\tilde{x}) &\triangleq V_{\rm s}(x) + V_{\rm sc}(x_{\rm c}) \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_{\rm c1}^2 + x_{\rm c2}^2 + x_{\rm c3}^2 + x_{\rm c4}^2 \end{aligned} \quad (115)$$

where $\tilde{x} \triangleq [x^{\rm T}\, x_{\rm c}^{\rm T}]^{\rm T}$.

The effect of the resetting law (6) with $f_{\rm dc}(x_{\rm c})$ and $G_{\rm dc}(x_{\rm c})$ given by (113), is to cause all controller states to be instantaneously reset to zero; that is, the resetting law (6) implies $V_{\rm sc}(x_{\rm c} + \Delta x_{\rm c}) = 0$. The closed-loop resetting law is thus given by

$$\Delta\tilde{x} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; -x_{\rm c1} \;\; -x_{\rm c2} \;\; -x_{\rm c3} \;\; -x_{\rm c4}]^{\rm T} \quad (116)$$

Note that since

$$\tilde{x} + \Delta\tilde{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^{\rm T} \quad (117)$$

it follows that

$$V(\tilde{x} + \Delta\tilde{x}) = V_{\rm s}(x) \quad (118)$$

and

$$V(\tilde{x} + \Delta\tilde{x}) - V(\tilde{x}) = -V_{\rm sc}(x_{\rm c}) \le 0 \quad (119)$$

Now, from (119) it follows that the resetting law (6) causes the total energy to instantaneously decrease by an amount equal to the accumulated controller energy.

To illustrate the dynamic behaviour of the closed-loop system, let $\alpha_1 = 5$, $\alpha_2 = -55$, $k_{\rm s1} = \alpha_1$, $k_{\rm s2} = \alpha_1$, $k_{\rm s3} = 0$, $k_{\rm s4} = 0$, $k_{\rm c1} = \alpha_1 + 0.1$, $k_{\rm c2} = \alpha_1 + 0.1$, $k_{\rm c3} = 0$, $k_{\rm c4} = 0$ and $\mathcal{T} = \{2, 4, 6, \ldots\}$, so that the controller resets periodically with a period of 2 s. The response of the controlled system (1) and (3) with the resetting controller (5)–(8) and initial condition $x_0 = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^{\rm T}$ is shown on figure 4. Note that the control force versus time is discontinuous at the resetting times. A comparison of the plant energy, control energy and total energy is given in figure 5.

In this example the resetting times were chosen arbitrarily. However, with the same choice of controller parameters we can choose a resetting time to achieve finite-time stabilization. Specifically, the resetting time will correspond to the time at which all of the energy of the plant is drawn to the controller. This resetting time can be obtained from the energy history of the closed-loop system without resetting. In particular, the time instant when the plant and controller interchange energies such that plant energy is at zero, will correspond to the resetting time that achieves finite-time stabilization.

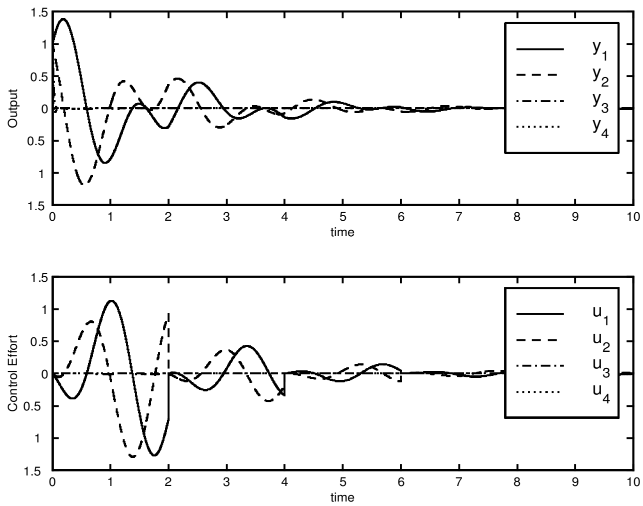


Figure 4. Time-dependent resetting controller: Output and control effort versus time.

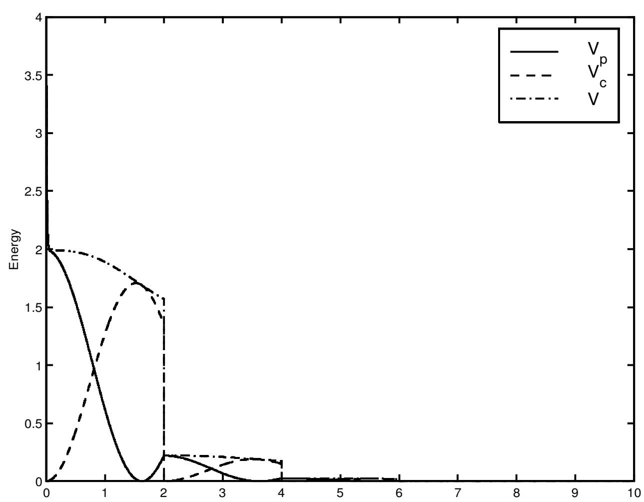


Figure 5. Time-dependent resetting controller: Plant, controller and total energy.

For this example, finite-time stability is achieved by choosing the resetting instant at $t = 1.6223$.

6.2. *Input/state-dependent hybrid controllers*

In this subsection we describe the mathematical setting and design of an input/state-dependent resetting controller. We consider the plant and resetting controller as described in §6.1 with $\mathcal{S}_c = [0, \infty) \times \mathcal{Z}_{cx_c} \times \mathcal{Z}_{cu_{cc}}$, where

$$\mathcal{Z}_{cx_c} \times \mathcal{Z}_{cu_{cc}} = \{(x_c, u_{cc}): f_{dc}(x_c) \neq 0 \quad \text{and} \quad V'_{sc}(x_c)[f_{cc}(x_c) + G_{cc}(x_c)u_{cc}] \leq 0\} \tag{120}$$

The resetting set (120) is thus defined to be the set of all controller state and input points that represent non–increasing control energy, except those that satisfy $f_{dc}(x_c) = 0$. As mentioned in Remark 4 of Haddad *et al.* (2001), the states x_c that satisfy $f_{dc}(x_c) = 0$ are states that do not change under the action of the resetting law and thus we need to exclude these states from the resetting set to ensure that the Assumption A1 of Haddad *et al.* (2001) is not violated. For the two-state, time-averaged combustion system given by (1) and (3) with $\mathcal{Z} = \varnothing$ and dynamics (110) and $h_c(x) = x$, the input/state-dependent resetting set (120) becomes

$$\begin{aligned}\mathcal{Z}_{cx_c} \times \mathcal{Z}_{u_{cc}} = \{(x_c, u_{cc}): f_{dc}(x_c) \neq 0 \text{ and } & \\ 2u_{cc1}(\alpha_1 x_{c1} + \theta_1 x_{c2} - \beta(x_{c1}x_{c3} + x_{c2}x_{c4}) & \\ - k_{c1}x_{c1} + u_{cc1}) + 2u_{cc2}(-\theta_1 x_{c1} + \alpha_1 x_{c2} & \\ + \beta(x_{c2}x_{c3} - x_{c1}x_{c4}) - k_{c2}x_{c2} + u_{cc2}) & \\ + 2u_{cc3}(\alpha_2 x_{c3} + \theta_2 x_{c4} + \beta(x_{c1}^2 - x_{c2}^2) & \\ - k_{c3}x_{c3} + u_{cc3}) + 2u_{cc4}(-\theta_2 x_{c3} + \alpha_2 x_{c4} & \\ + 2\beta x_{c1}x_{c2} - k_{c4}x_{c4} + u_{cc4})) \leq 0\} &\end{aligned} \tag{121}$$

where u_{cci}, $i = 1, \ldots, 4$, represents the ith component of u_{cc}. Now, it can be shown that Assumptions A1 and A2 of Haddad *et al.* (2001) are satisfied using straightforward calculations. Furthermore, since the resetting controller given in §6.1 is exponentially passive for $\mathcal{S}_c = [0, \infty) \times \mathbb{R}^{n_c} \times \mathbb{R}^{m_{cc}}$, it follows that the resetting controller is exponentially passive for $\mathcal{S}_c = [0, \infty) \times \mathcal{Z}_{cx_c} \times \mathcal{Z}_{cu_{cc}}$. Hence, asymptotic stability of the closed-loop system (1), (3) and (5)–(8) is guaranteed by Theorem 1. Finally, note that knowledge of x_c and y_c is sufficient to determine whether or not the closed-loop state vector $\tilde{x}$ is in the resetting set $\tilde{\mathcal{Z}}_{\tilde{x}}$, where

$$\tilde{\mathcal{Z}}_{\tilde{x}} = \mathcal{Z}_{cx_c} \cup \{(x, x_c): h_c(x) \in \tilde{\mathcal{U}}_{cc}\} = \{\tilde{x}: f_{dc}(x_c) \neq 0 \quad \text{and} \quad V'_{sc}(x_c)[f_{cc}(x_c) + G_{cc}(x_c)h_c(x)] \leq 0\} \tag{122}$$

By resetting the controller states, the plant energy can never increase. Hence, this approach allows the plant energy to 'flow' to the controller, where it increases the controller energy, but does not allow the controller energy to 'flow' back to the plant. This type of controller is referred to in Bupp *et al.* (2000) as a *one-way resetting controller*.

To illustrate the dynamics behaviour of the closed-loop system we again choose $\alpha_1 = 5$, $\alpha_2 = -55$, $k_{s1} = \alpha_1$, $k_{s2} = \alpha_1$, $k_{s3} = 0$, $k_{s4} = 0$, $k_{c1} = \alpha_1 + 0.1$, $k_{c2} = \alpha_1 + 0.1$, $k_{c3} = 0$ and $k_{c4} = 0$, with initial condition $x_0 = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^T$. The response of controlled system (1), (3), with dynamics (110) and $h_c(x) = x$ and the state-dependent resetting controller given by (5)–(8) with dynamics (112)–(114) and resetting set (121), is given in figure 6. The total energy, plant

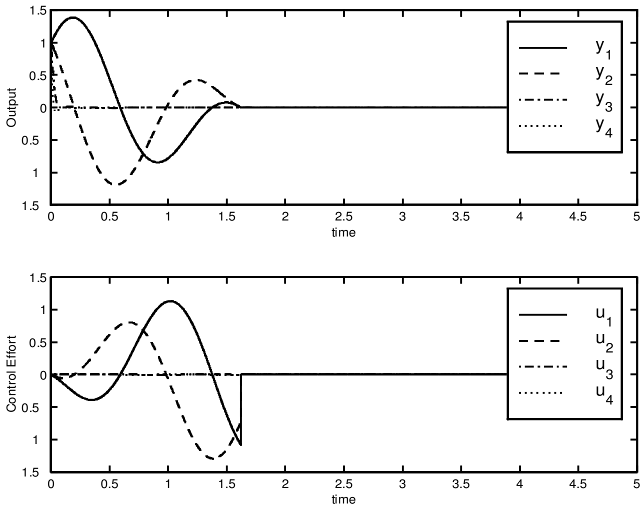


Figure 6. State-dependent resetting controller: Output and control effort versus time.

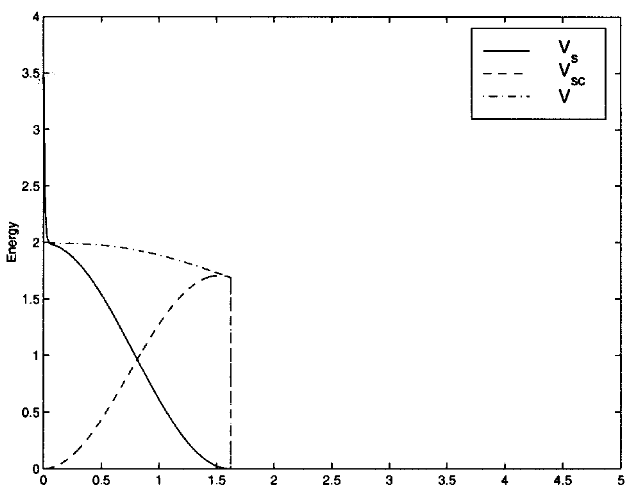


Figure 7. State-dependent resetting controller: Plant, controller and total energy.

energy and controller energy versus time are shown in figure 7. Note that the proposed input/state-dependent resetting controller achieves finite-time stabilization.

7. Conclusion

In this paper general stability criteria were given for Lyapunov, asymptotic and exponential stability of feedback interconnections of non-linear impulsive dynamical systems. A unified framework for hybrid feedback optimal control over an infinite horizon involving a hybrid non-linear-non-quadratic performance functional was also developed. The overall framework provides the foundation for generalizing optimal linear-quadratic control methods to non-linear impulsive dynamical systems. The proposed framework was applied to the control of thermoacoustic instabilities in combustion processes demonstrating the ability of hybrid controllers to significantly enhance the removal of energy in combustion systems.

The underlying intention of this two-part paper has been to develop a unified framework for analysis and control synthesis of non-linear impulsive dynamical systems. This exposition demonstrates that stability, dissipativity, stability of feedback interconnections and optimality of continuous-time and discrete-time systems are part of the more general system theory of impulsive dynamical systems.

Acknowledgements

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