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# Non-linear impulsive dynamical systems. Part I: Stability and dissipativity

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In this paper we develop Lyapunov and invariant set stability theorems for non-linear impulsive dynamical systems. Furthermore, we generalize dissipativity theory to non-linear dynamical systems with impulsive effects. Specifically, the classical concepts of system storage functions and supply rates are extended to impulsive dynamical systems providing a generalized hybrid system energy interpretation in terms of stored energy, dissipated energy over the continuous-time system dynamics and dissipated energy over the resetting instants. Furthermore, extended Kalman–Yakubovich–Popov conditions in terms of the impulsive system dynamics characterizing dissipativeness via system storage functions are derived. Finally, the framework is specialized to passive and non-expansive impulsive systems to provide a generalization of the classical notions of passivity and non-expansivity for non-linear impulsive systems. These results are used in the second part of this paper to develop extensions of the small gain and positivity theorems for feedback impulsive systems as well as to develop optimal hybrid feedback controllers.

## 1. Introduction

Modern complex engineering systems as well as biological and physiological systems typically possess a multi-echelon hierarchical hybrid architecture characterized by continuous-time dynamics at the lower levels of hierarchy and discrete-time dynamics at the higher levels of the hierarchy. Hence, it is not surprising that hybrid systems have been the subject of intensive research over the past recent years (see Branicky *et al.* 1998, Ye *et al.* 1998 b, Haddad and Chellaboina 2001 and references therein). Such systems include dynamical switching systems (Branicky 1998, Leonessa *et al.* 2000), non-smooth impact and constrained mechanical systems (Back *et al.* 1993, Brogliato 1996, Brogliato *et al.* 1997), biological systems (Lakshmikantham *et al.* 1989), demographic systems (Liu 1994), sampled-data systems (Hagiwara and Araki 1988), discrete-event systems (Passino *et al.* 1994), intelligent vehicle/highway systems (Lygeros *et al.* 1998) and flight control systems (Tomlin *et al.* 1998), to cite but a few examples. The mathematical descriptions of many of these systems can be characterized by impulsive differential equations (Simeonov and Bainov 1985, 1987, Liu 1988, Lakshmikantham *et al.* 1989, 1994, Bainov and Simeonov 1989, 1995, Kulev and Bainov 1989, Lakshmikantham and Liu 1989, Hu *et al.* 1989, Samoilenko and Perestyuk 1995). Impulsive dynamical systems can be viewed as a subclass of hybrid systems and consist of three elements: namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset. As in classical dynamical systems theory, it seems natural that dissipativity theory should play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections and optimality for hybrid dynamical systems.

The key foundation in developing dissipativity theory for general non-linear dynamical systems was presented by Willems (1972 a, b) in his seminal two-part paper on dissipative dynamical systems. In particular, Willems (1972 a) introduced a definition of dissipativity for general dynamical systems in terms of an inequality involving a generalized system power input, or, supply rate and a generalized energy function, or, storage function. Since Lyapunov functions can be viewed as generalizations of energy functions for non-linear dynamical systems, the notion of dissipativity, with appropriate storage functions and supply rates, can be used to construct Lyapunov functions for non-linear feedback systems by appropriately combining storage functions for each subsystem. Even though the original work on dissipative dynamical systems was formulated in the state space setting describing the system dynamics in terms of continuous flows on appropriate manifolds, an input–output formulation for dissipative dynamical systems extending the notions of passivity (Zames 1966), non-expansivity (Zames 1966) and conicity (Zames 1966, Safonov 1980) was presented in Moylan (1974) and Hill and Moylan 1976, 1980. More recently, the notion of dissipativity theory was generalized in Chellaboina and Haddad (2000) to formalize the concepts of the non-linear analogue of strict positive realness and strict bounded realness. In particular,

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using exponentially weighted system storage functions with appropriate exponentially weighted supply rates, the concept of exponential dissipativity was introduced in Chellaboina and Haddad (2000).

Dissipativity theory along with its connections to Lyapunov stability theory has been extensively developed for dynamical systems possessing continuous flows. However, in light of the increasingly complex nature of the dynamical systems discussed above, discontinuous system flows arise naturally. Alternatively, within the context of feedback control, active energy flow resetting control for interconnected subsystems also gives rise to discontinuous closed-loop system flows. Specifically, if a dissipative or lossless plant is at a high energy level, and a dissipative feedback controller at a low energy level is attached to it, then energy will generally tend to flow from the plant into the controller, decreasing the plant energy and increasing the controller energy (Kishimoto *et al.* 1995). Of course, emulated energy and not physical energy, is accumulated by the controller. Conversely, if the attached controller is at a high energy level and a plant is at a low energy level, then energy can flow from the controller to the plant, since a controller can generate real, physical energy to effect the required energy flow. Hence, if and when the controller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. In this case, the overall closed-loop system consisting of the plant and the controller possesses discontinuous flows characterized by impulsive differential equations (Lakshmikantham *et al.* 1989). Within the context of vibration control using resetting virtual absorbers, these ideas were first presented in Bupp *et al.* (2000).

Motivated by complex hybrid dynamical systems possessing discontinuous flows, in this paper we develop stability, dissipativity and exponential dissipativity concepts for non-linear impulsive dynamical systems. Specifically, we develop an invariance principle for impulsive dynamical systems wherein system trajectories converge to a largest invariant set contained in a hybrid level surface composed of a union involving vanishing Lyapunov derivatives and differences of the continuous-time trajectories and resetting instants, respectively. Furthermore, we extend the notions of classical dissipativity theory using generalized storage functions and supply rates for impulsive dynamical systems. The overall approach provides an interpretation of a generalized hybrid energy balance for an impulsive dynamical system in terms of the stored or, accumulated generalized energy, dissipated energy over the continuous-time dynamics and dissipated energy at the resetting instants. Furthermore, as in the case of dynamical systems possessing continuous flows (Willems 1972 a), we show that the set of all possible storage functions of an impulsive dynamical system forms a convex set and is bounded from below by the system's available stored generalized energy which can be recovered from the system and bounded from above by the system's required generalized energy supply needed to transfer the system from an initial state of minimum generalized energy to a given state. In addition, for two kinds of non-linear impulsive dynamical systems; namely, time-dependent and state-dependent impulsive systems, we develop extended Kalman–Yakubovich–Popov algebraic conditions in terms of the system dynamics for characterizing dissipativeness via system storage functions for impulsive dynamical systems.

Although the results of this paper are confined to analysis, stability and optimality results of feedback non-linear impulsive systems are discussed in the second part of this paper (Haddad *et al.* 2001). The main contribution of this two-part paper is to develop a unified framework for the analysis and control synthesis of non-linear impulsive systems. However, since impulsive dynamical systems involve a hybrid formulation of continuous-time and discrete-time dynamics, these papers also provide a tutorial for stability, dissipativity, feedback interconnections and optimality of continuous-time and discrete-time dynamical systems which can be viewed as a specialization of impulsive systems.

The contents of the paper are as follows. In § 2 we establish definitions, notation and review some basic results on impulsive dynamical systems. In § 3 we present Lyapunov, asymptotic and exponential stability results for impulsive dynamical systems. Furthermore, *new* invariant set theorems are derived wherein system trajectories converge to a largest invariant set contained in a hybrid Lyapunov level surface composed of a union involving vanishing Lyapunov derivatives and differences of the hybrid system dynamics. Then, in § 4, we extend the notion of dissipative dynamical systems to develop the concept of dissipativity for impulsive dynamical systems. In § 5 we develop extended Kalman–Yakubovich–Popov algebraic conditions in terms of the hybrid system dynamics for characterizing dissipativeness via system storage functions for impulsive systems. Furthermore, a generalized hybrid energy balance interpretation involving the system's stored or, accumulated energy, dissipated energy over the continuous-time dynamics and dissipated energy at the resetting instants is given. Specialization of these results to passive and non-expansive impulsive systems is also provided. In § 6 we specialize the results of § 5 to linear impulsive systems to obtain extended hybrid Kalman–Yakubovich–Popov equations for positive real and bounded real impulsive systems. Finally, we draw conclusions in § 7.

## 2. Non-linear impulsive dynamical systems

In this section we establish definitions, notation and review some basic results on impulsive dynamical systems (Simeonov and Bainov 1985, 1987, Liu 1988, Lakshmikanthan *et al.* 1989, 1994, Bainov and Simeonov 1989, 1995, Kulev and Bainov 1989, Lakshmikantham and Liu 1989, Hu *et al.* 1989, Samoilenko and Perestyuk 1995). Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $(\ )^{\mathrm{T}}$ denote transpose, $\mathcal{N}$ denote the set of non-negative integers, $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denote the set of $n \times n$ non-negative (resp., positive) definite matrices and let $I_n$ or $I$ denote the $n \times n$ identity matrix. Furthermore, let $\partial\mathcal{S}$, $\mathring{\mathcal{S}}$ and $\bar{\mathcal{S}}$ denote the boundary, the interior and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centred at $\alpha$ with radius $\varepsilon$, $V'(x)$ for the Fréchet derivative of $V$ at $x$ and $M \geq 0$ (resp., $M > 0$) to denote the fact that the Hermitian matrix $M$ is non-negative (resp., positive) definite. Finally, let $\mathrm{C}^0$ denote the set of continuous functions and $\mathrm{C}^r$ denote the set of functions with $r$ continuous derivatives.

As discussed in the introduction, an impulsive dynamical system consists of three elements:

(1) a continuous-time dynamical equation, which governs the motion of the system between resetting events;
(2) a difference equation, which governs the way the states are instantaneously changed when a resetting event occurs; and
(3) criterion for determining when the states of the system are to be reset.

For the characterization of an impulsive dynamical system $\tilde{\mathcal{U}} \triangleq \tilde{\mathcal{U}}_{\mathrm{c}} \times \tilde{\mathcal{U}}_{\mathrm{d}}$ is an input space and consists of bounded continuous $\mathcal{U}$-valued functions on the semi-infinite interval $[0,\infty)$. The set $\mathcal{U} \triangleq \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$, where $\mathcal{U}_{\mathrm{c}} \subseteq \mathbb{R}^{m_{\mathrm{c}}}$ and $\mathcal{U}_{\mathrm{d}} \subseteq \mathbb{R}^{m_{\mathrm{d}}}$, contains the set of input values; that is, for every $u = (u_{\mathrm{c}}, u_{\mathrm{d}}) \in \tilde{\mathcal{U}}$ and $t \in [0,\infty)$, $u(t) \in \mathcal{U}$, $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}}$ and $u_{\mathrm{d}}(t) \in \mathcal{U}_{\mathrm{d}}$. Furthermore, $\tilde{\mathcal{Y}} \triangleq \tilde{\mathcal{Y}}_{\mathrm{c}} \times \tilde{\mathcal{Y}}_{\mathrm{d}}$ is an output space and consists of bounded continuous $\mathcal{Y}$-valued functions on the semi-infinite interval $[0,\infty)$. The set $\mathcal{Y} \triangleq \mathcal{Y}_{\mathrm{c}} \times \mathcal{Y}_{\mathrm{d}}$, where $\mathcal{Y}_{\mathrm{c}} \subseteq \mathbb{R}^{l_{\mathrm{c}}}$ and $\mathcal{Y}_{\mathrm{d}} \subseteq \mathbb{R}^{l_{\mathrm{d}}}$, contains the set of output values; that is, for every $y = (y_{\mathrm{c}}, y_{\mathrm{d}}) \in \tilde{\mathcal{Y}}$ and $t \in [0,\infty)$, $y(t) \in \mathcal{Y}$, $y_{\mathrm{c}}(t) \in \mathcal{Y}_{\mathrm{c}}$ and $y_{\mathrm{d}}(t) \in \mathcal{Y}_{\mathrm{d}}$. Thus, an impulsive dynamical system has the form

$$\left.\begin{aligned} \dot{x}(t) &= f_{\mathrm{c}}(x(t)) + G_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t) \\ x(0) &= x_0, \quad (t, x(t), u_{\mathrm{c}}(t)) \notin \mathcal{S} \end{aligned}\right\} \tag{1}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)) + G_{\mathrm{d}}(x(t))u_{\mathrm{d}}(t), \quad (t, x(t), u_{\mathrm{c}}(t)) \in \mathcal{S} \tag{2}$$

$$y_{\mathrm{c}}(t) = h_{\mathrm{c}}(x(t)) + J_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t), \quad (t, x(t), u_{\mathrm{c}}(t)) \notin \mathcal{S} \tag{3}$$

$$y_{\mathrm{d}}(t) = h_{\mathrm{d}}(x(t)) + J_{\mathrm{d}}(x(t))u_{\mathrm{d}}(t), \quad (t, x(t), u_{\mathrm{c}}(t)) \in \mathcal{S} \tag{4}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\Delta x(t) \triangleq x(t^+) - x(t)$, $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}} \subseteq \mathbb{R}^{m_{\mathrm{c}}}$, $u_{\mathrm{d}}(t_k) \in \mathcal{U}_{\mathrm{d}} \subseteq \mathbb{R}^{m_{\mathrm{d}}}$, $t_k$ denotes the $k$th instant of time at which $(t, x(t), u_{\mathrm{c}}(t))$ intersects $\mathcal{S}$ for a particular trajectory $x(t)$ and input $u_{\mathrm{c}}(t)$, $y_{\mathrm{c}}(t) \in \mathcal{Y}_{\mathrm{c}} \subseteq \mathbb{R}^{l_{\mathrm{c}}}$, $y_{\mathrm{d}}(t_k) \in \mathcal{Y}_{\mathrm{d}} \subseteq \mathbb{R}^{l_{\mathrm{d}}}$, $f_{\mathrm{c}}\colon \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0) = 0$, $G_{\mathrm{c}}\colon \mathcal{D} \to \mathbb{R}^{n \times m_{\mathrm{c}}}$, $f_{\mathrm{d}}\colon \mathcal{D} \to \mathbb{R}^n$ is continuous, $G_{\mathrm{d}}\colon \mathcal{D} \to \mathbb{R}^{n \times m_{\mathrm{d}}}$, $h_{\mathrm{c}}\colon \mathcal{D} \to \mathbb{R}^{l_{\mathrm{c}}}$ and satisfies $h_{\mathrm{c}}(0) = 0$, $J_{\mathrm{c}}\colon \mathcal{D} \to \mathbb{R}^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $h_{\mathrm{d}}\colon \mathcal{D} \to \mathbb{R}^{l_{\mathrm{d}}}$, $J_{\mathrm{d}}\colon \mathcal{D} \to \mathbb{R}^{l_{\mathrm{d}} \times m_{\mathrm{d}}}$ and $\mathcal{S} \subset [0,\infty) \times \mathcal{D} \times \mathcal{U}_{\mathrm{c}}$ is the *resetting set*. Here, we assume that $u_{\mathrm{c}}(\cdot)$ and $u_{\mathrm{d}}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions such that $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \triangleq \{k\colon 0 \leq t_k < t\}$, where the constraint set $\mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ is given with $(0,0) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$. We refer to the differential equation (1) as the *continuous-time dynamics*, and we refer to the difference equation (2) as the *resetting law*.

For convenience, we use the notation $s(t; \tau, x_0, u)$ to denote the solution $x(t)$ of (1), (2) at time $t > \tau$ with initial condition $x(\tau) = x_0$, where $u = (u_{\mathrm{c}}, u_{\mathrm{d}})\colon \mathbb{R} \times \mathcal{T} \to \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ and $\mathcal{T} \triangleq \{t_1, t_2, \ldots\}$. Furthermore, we call the times $t_k$ the *resetting times*. Thus the trajectory of the system (1) and (2) from the initial condition $x(0) = x_0$ is given by $\psi(t; 0, x_0, u)$ for $0 < t \leq t_1$, where $\psi(t; 0, x_0, u)$ denotes the solution to the continuous-time dynamics (1). If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $(t_1, x_1, u_1) \in \mathcal{S}$, where $u_1 \triangleq u_{\mathrm{c}}(t_1)$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_{\mathrm{d}}(x_1) + G_{\mathrm{d}}(x_1)u_{\mathrm{d}}$, where $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$ is a given input, according to the resetting law (2). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t; t_1, x_1^+, u)$ and so on. Note that the solution $x(t)$ of (1) and (2) is left-continuous; that is, it is continuous everywhere except at the resetting times $t_k$, and

$$x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon) \tag{5}$$

$$\begin{aligned} x_k^+ &\triangleq x(t_k) + f_{\mathrm{d}}(x(t_k)) + G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k) \\ &= \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon), \quad u_{\mathrm{d}}(t_k) \in \mathcal{U}_{\mathrm{d}} \end{aligned} \tag{6}$$

for $k = 1, 2, \ldots$.

We make the following additional assumptions:

**A1.** *If* $(t, x(t), u_{\mathrm{c}}(t)) \in \overline{\mathcal{S}} \backslash \mathcal{S}$, *then there exists* $\varepsilon > 0$ *such that, for all* $0 < \delta < \varepsilon$

$$s(t + \delta; t, x(t), u_{\mathrm{c}}(t + \delta)) \notin \mathcal{S}$$

**A2.** *If* $(t_k, x(t_k), u_c(t_k)) \in \partial\mathcal{S} \cap \mathcal{S}$, *then there exists* $\varepsilon > 0$ *such that, for all* $0 \le \delta < \varepsilon$ *and* $u_d(t_k) \in \mathcal{U}_d$

$$s(t_k + \delta; t_k, x(t_k) + f_d(x(t_k)) + G_d(x(t_k))u_d(t_k), u_c(t_k + \delta)) \notin \mathcal{S}$$

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{S}$ at a point that does not belong to $\mathcal{S}$, then the trajectory must be directed away from $\mathcal{S}$; that is, a trajectory cannot enter $\mathcal{S}$ through a point that belongs to the closure of $\mathcal{S}$ but not to $\mathcal{S}$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$. Finally, we note that if $(0, x_0, u_{c0}) \in \mathcal{S}$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) + G_d(x_0)u_d(0)$ which serves as the initial condition for the continuous dynamics (1).

**Remark 1:** It follows from A2 that resetting removes the pair $(t_k, x_k, u_c(t_k))$ from the resetting set $\mathcal{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (1) and not the resetting law (2), becomes the active element of the impulsive dynamical system. Furthermore, it follows from A1 and A2 that no trajectory can intersect the interior of $\mathcal{S}$. Specifically, it follows from A1 that a trajectory can only reach $\mathcal{S}$ through a point belonging to both $\mathcal{S}$ and its boundary. And, from A2, it follows that if a trajectory reaches a point in $\mathcal{S}$ that is on the boundary of $\mathcal{S}$, then the trajectory is instantaneously removed from $\mathcal{S}$. Since a continuous trajectory starting outside of $\mathcal{S}$ and intersecting the interior of $\mathcal{S}$ must first intersect the boundary of $\mathcal{S}$, it follows that no trajectory can reach the interior of $\mathcal{S}$.

To show that the resetting times $t_k$ are well defined and distinct, assume that for a given input $u \in \tilde{\mathcal{U}}$, $T = \inf\{t: \psi(t; 0, x_0, u) \in \mathcal{S}\} < \infty$. Now, *ad absurdum*, suppose $t_1$ is not well defined; that is, $\min\{t: \psi(t; 0, x_0, u) \in \mathcal{S}\}$ does not exist. Since $\psi(\cdot; 0, x_0, u)$ is continuous, it follows that $\psi(T; 0, x_0, u) \in \partial\mathcal{S}$ and since, by assumption, $\min\{t: \psi(t; 0, x_0, u) \in \mathcal{S}\}$ does not exist it follows that $\psi(T; 0, x_0, u) \in \overline{\mathcal{S}} \backslash \mathcal{S}$. Note that $\psi(t; 0, x_0, u) = s(t, 0, x_0, u)$, for every $t$ such that $\psi(\tau; 0, x, u) \notin \mathcal{S}$ for all $0 \le \tau \le t$. Now, it follows from A1 that there exists $\varepsilon > 0$ such that $s(T + \delta;\ 0, x_0, u) = \psi(T + \delta; 0, x_0, u)$, $\delta \in (0, \varepsilon)$, which implies that $\inf\{t: \psi(t; 0, x_0, u) \in \mathcal{S}\} > T$ which is a contradiction. Hence $\psi(T; 0, x_0, u) \in \partial\mathcal{S} \cap \mathcal{S}$ and $\inf\{t: \psi(t; 0,\ x_0, u) \in \mathcal{S}\} = \min\{t: \psi(t; 0, x_0, u) \in \mathcal{D}\}$ which implies that the first resetting time $t_1$ is well defined for all initial conditions $x_0 \in \mathcal{D}$. Next, it follows from A2 that $t_2$ is also well defined and $t_2 \ne t_1$. Repeating the above arguments it follows that the resetting times $t_k$ are well defined and distinct.

Since the resetting times are well defined and distinct and since the solution to (1) exists and is unique, it follows that the solution of the impulsive dynamical system (1), (2) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit Zenoness, beating, as well as confluence wherein solutions exhibit infinitely many resettings in a finite-time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a given point in time. In this paper we allow for the possibility of confluence and Zeno solutions. However, A2 precludes the possibility of beating. Furthermore, since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details see Lakshmikantham *et al.* (1989) and Bainov and Simeonov (1989, 1995).

In Simeonov and Bainov (1985, 1987), Liu (1988), Lakshmikantham *et al.* (1989, 1994), Bainov and Simeonov (1989), Kulev and Bainov (1989), Lakshmikantham and Liu (1989) and Hu *et al.* (1989), the resetting set $\mathcal{S}$ is defined in terms of a countable number of functions $\tau_k: \mathcal{D} \to (0, \infty)$ and is given by

$$\mathcal{S} = \bigcup_k \{(\tau_k(x), x, u_c(\tau_k(x))) :\ x \in \mathcal{D}\} \tag{7}$$

The analysis of impulsive dynamical systems with a resetting set of the form (7) can be quite involved. Furthermore, since impulsive dynamical systems of the form (1)–(4) involve impulses at variable times they are time-varying systems. Here we will consider impulsive dynamical systems involving two distinct forms of the resetting set $\mathcal{S}$. In the first case, the resetting set is defined by a prescribed sequence of times which are independent of the state $x$. These equations are thus called *time-dependent impulsive dynamical systems*. In the second case, the resetting set is defined by a region in the state space that is independent of time. These equations are called *state-dependent impulsive dynamical systems*.

### 2.1. *Time-dependent impulsive dynamical systems*

Time-dependent impulsive dynamical systems can be written as (1)–(4) with $\mathcal{S}$ defined as

$$\mathcal{S} \triangleq \mathcal{T} \times \mathcal{D} \times \mathcal{U}_c \tag{8}$$

where

$$\mathcal{T} \triangleq \{t_1, t_2, \ldots\} \tag{9}$$

and $0 \le t_1 < t_2 < \cdots$ are prescribed resetting times. Now (1)–(2) can be rewritten in the form of the *time-dependent impulsive dynamical system*

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t),\ x(0) = x_0,\ t \neq t_k \quad (10)$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad t = t_k \quad (11)$$

$$y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t), \quad t \neq t_k \quad (12)$$

$$y_d(t) = h_d(x(t)) + J_d(x(t))u_d(t), \quad t = t_k \quad (13)$$

Since $0 \notin \mathcal{T}$ and $t_k < t_{k+1}$, it follows that the Assumptions A1 and A2 are satisfied. Since time-dependent impulsive dynamical systems involve impulses at a fixed sequence of times, they are time-varying systems.

**Remark 2:** Standard continuous-time and discrete-time dynamical systems as well as sampled-data systems can be treated as special cases of impulsive dynamical systems. In particular, setting $f_d(x) = 0$, $G_d(x) = 0$, $h_d(x) = 0$ and $J_d(x) = 0$, it follows that (10)–(13) has an identical state trajectory as the non-linear continuous-time system

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t)$$
$$x(0) = x_0, \quad t \ge 0 \quad (14)$$

$$y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t) \quad (15)$$

Alternatively, setting $f_c(x) = 0$, $G_c(x) = 0$, $h_c(x) = 0$, $J_c(x) = 0$, $t_k = kT$ and $T = 1$ and assuming $f_d(0) = 0$, it follows that (10)–(13) has an identical state trajectory as the non-linear discrete-time system

$$x(k+1) = f_d(x(k)) + G_d(x(k))u_d(k)$$
$$x(0) = x_0, \quad k \in \mathcal{N} \quad (16)$$

$$y_d(k) = h_d(x(k)) + J_d(x(k))u_d(k) \quad (17)$$

Finally, to show that (10)–(13) can be used to represent sampled-data systems, consider the continuous-time non-linear system (14) and (15) with piecewise constant input $u_c(t) = u_d(t_k)$, $t \in (t_k, t_{k+1}]$ and sampled measurements $y_d(t_k) = h_d(x(t_k)) + J_d(x(t_k))u_d(t_k)$. Defining $\hat{x} = [x^T \ u_c^T]^T$, it follows that the sampled-data system can be represented as

$$\dot{\hat{x}} = \hat{f}(\hat{x}(t)), \quad t \neq t_k \quad (18)$$

$$\Delta\hat{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}\hat{x}(t) + \begin{bmatrix} 0 \\ I \end{bmatrix}u_d(t), \quad t = t_k \quad (19)$$

$$y(t) = \hat{h}(\hat{x}(t)), \quad t \neq t_k \quad (20)$$

$$y_d(t) = \hat{h}_d(\hat{x}(t)) + \hat{J}_d(\hat{x}(t))u_d(t), \quad t = t_k \quad (21)$$

where

$$\hat{f}(\hat{x}) = \begin{bmatrix} f_c(x) + G_c(x)u_c \\ 0 \end{bmatrix}$$

$$\hat{h}(\hat{x}) = h_c(x) + J_c(x)u_c$$

$$\hat{h}_d(\hat{x}) = h_d(x), \hat{J}_d(\hat{x}) = J_d(x)$$

and new input variable $u_d(t_k)$.

**Remark 3:** The time-dependent impulsive dynamical system (10)–(13) includes as a special case the impulsive control problem addressed in Yang (1999) wherein at least one of the state variables of the continuous-time plant can be changed instantaneously to any value given by an impulsive control at a set of control instants $\mathcal{T}$.

### 2.2. *State-dependent impulsive dynamical systems*

State-dependent impulsive dynamical systems can be written as (1)–(4) with $\mathcal{S}$ defined as

$$\mathcal{S} \triangleq [0, \infty) \times \mathcal{Z} \quad (22)$$

where $\mathcal{Z} \triangleq \mathcal{Z}_x \times \mathcal{U}_c$ and $\mathcal{Z}_x \subset \mathcal{D}$. Therefore, (1)–(4) can be rewritten in the form of the *state-dependent impulsive dynamical system*

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t),$$
$$x(0) = x_0, \quad (x(t), u_c(t)) \notin \mathcal{Z} \quad (23)$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t),$$
$$(x(t), u_c(t)) \in \mathcal{Z} \quad (24)$$

$$y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t),$$
$$(x(t), u_c(t)) \notin \mathcal{Z} \quad (25)$$

$$y_d(t) = h_d(x(t)) + J_d(x(t))u_d(t),$$
$$(x(t), u_c(t)) \in \mathcal{Z} \quad (26)$$

We assume that if $(x, u_c) \in \mathcal{Z}$, then $(x + f_d(x) + G_d(x)u_d, u_c) \notin \mathcal{Z}$, $u_d \in \mathcal{U}_d$. In addition, we assume that if at time $t$ the trajectory $(x(t), u_c(t)) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, then there exists $\varepsilon > 0$ such that for $0 < \delta < \varepsilon$, $(x(t+\delta), u_c(t+\delta)) \notin \mathcal{Z}$. These assumptions represent the specialization of A1 and A2 for the particular resetting set (22). It follows from these assumptions that for a particular initial condition, the resetting times $\tau_k(x_0, u_c)$ are distinct and well defined. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, state-dependent impulsive dynamical systems are time-invariant systems. Finally, in the case where $\mathcal{S} \triangleq [0, \infty) \times \mathcal{D} \times \mathcal{Z}_{u_c}$, where $\mathcal{Z}_{u_c} \subset \mathcal{U}_c$, we refer to (23)–(26) as an *input-dependent impulsive dynamical system*, while in the case where $\mathcal{S} \triangleq [0, \infty) \times \mathcal{Z}_x \times \mathcal{Z}_{u_c}$ we refer to (23)–(26) as an *input/state-dependent impulsive dynamical system*. Both these cases represent a generalization to the impulsive control problem considered in Yang (1999).

**Remark 4:** For the state-dependent impulsive dynamical system given by (23)–(26) let $x^* \in \mathbb{R}^n$ satisfy $f_\mathrm{d}(x^*) = 0$. Then $x^* \notin \mathcal{Z}_x$. To see this, suppose $x^* \in \mathcal{Z}_x$. Then $x^* + f_\mathrm{d}(x^*) = x^* \in \mathcal{Z}_x$, which contradicts the assumption that if $x \in \mathcal{Z}_x$ then $x + f_\mathrm{d}(x) + G_\mathrm{d}(x)u_\mathrm{d} \notin \mathcal{Z}_x, u_\mathrm{d} \in \mathcal{U}_\mathrm{d}$, since $0 \in \mathcal{U}_\mathrm{d}$. Specifically, we note that $0 \notin \mathcal{Z}_x$.

## 3. Stability theory of impulsive dynamical systems

In this section we present Lyapunov, asymptotic and exponential stability theorems for non-linear time-dependent and state-dependent impulsive dynamical systems. Furthermore, for state-dependent impulsive dynamical systems we present *new* invariant set stability theorems that generalize the Barbashin–Krasovskii–LaSalle invariance principle (Barbashin and Krasovskii 1952, Krasovskii 1959, LaSalle 1960) to impulsive systems. Even though versions of the Lyapunov stability results in this section have appeared in the literature (Bainov and Simeonov 1989, 1995, Samoilenko and Perestyuk 1995), the invariant set stability theorems are *new* to this paper. Note that for addressing the stability of the zero solution of an impulsive dynamical system the usual stability definitions are valid.

**Theorem 1:** *Suppose there exists a continuously differentiable function $V: \mathcal{D} \to [0, \infty)$ satisfying $V(0) = 0$, $V(x) > 0$, $x \neq 0$ and*

$$V'(x) f_\mathrm{c}(x) \leq 0, \quad x \in \mathcal{D} \tag{27}$$

$$V(x + f_\mathrm{d}(x)) \leq V(x), \quad x \in \mathcal{D} \tag{28}$$

*Then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_\mathrm{c}(t), u_\mathrm{d}(t_k)) \equiv (0, 0))$ system (10),(11) is Lyapunov stable. Furthermore, if the inequality (27) is strict for all $x \neq 0$, then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_\mathrm{c}(t), u_\mathrm{d}(t_k)) \equiv (0, 0))$ system (10), (11) is asymptotically stable. Alternatively, if there exist scalars $\alpha$, $\beta$, $\varepsilon > 0$ and $p \geq 1$ such that*

$$\alpha \|x\|^p \leq V(x) \leq \beta \|x\|^p, \quad x \in \mathcal{D} \tag{29}$$

$$V'(x) f_\mathrm{c}(x) \leq -\varepsilon V(x), \quad x \in \mathcal{D} \tag{30}$$

*and (28) holds, then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_\mathrm{c}(t), u_\mathrm{d}(t)) \equiv (0, 0))$ system (10), (11) is exponentially stable. Finally, if $\mathcal{D} = \mathbb{R}^n$ and*

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty \tag{31}$$

*then the above results are global.*

**Proof:** Prior to the first resetting time, we can determine the value of $V(x(t))$ as

$$V(x(t)) = V(x(0)) + \int_0^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau, \quad t \in [0, t_1] \tag{32}$$

Between consecutive resetting times $t_k$ and $t_{k+1}$, we can determine the value of $V(x(t))$ as its initial value plus the integral of its rate of change along the trajectory $x(t)$; that is,

$$V(x(t)) = V(x(t_k) + f_\mathrm{d}(x(t_k))) + \int_{t_k}^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau, \quad t \in (t_k, t_{k+1}] \tag{33}$$

for $k = 1, 2, \ldots$. Adding and subtracting $V(x(t_k))$ to and from the right hand side of (33) yields

$$V(x(t)) = V(x(t_k)) + [V(x(t_k) + f_\mathrm{d}(x(t_k))) - V(x(t_k))] + \int_{t_k}^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau, \quad t \in (t_k, t_{k+1}] \tag{34}$$

and in particular at time $t_{k+1}$

$$V(x(t_{k+1})) = V(x(t_k)) + [V(x(t_k) + f_\mathrm{d}(x(t_k))) - V(x(t_k))] + \int_{t_k}^{t_{k+1}} V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau \tag{35}$$

By recursively substituting (35) into (34) and ultimately into (32), we obtain

$$V(x(t)) = V(x(0)) + \int_0^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau + \sum_{i=1}^{k} [V(x(t_i) + f_\mathrm{d}(x(t_i))) - V(x(t_i))], \quad t \in (t_k, t_{k+1}] \tag{36}$$

If we allow $t_0 \triangleq 0$ and $\sum_{i=1}^{0} \triangleq 0$, then (36) is valid for $k \in \mathcal{N}$. From (36) and (28) we obtain

$$V(x(t)) \leq V(x(0)) + \int_0^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau, \quad t \geq 0 \tag{37}$$

Furthermore, it follows from (27) that

$$V(x(t)) \leq V(x(0)), \quad t \geq 0 \tag{38}$$

so that Lyapunov stability follows from standard arguments.

Next, it follows from (28) and (36) that

$$V(x(t)) - V(x(s)) \leq \int_s^t V'(x(\tau)) f(x(\tau)) \, \mathrm{d}\tau, \quad t > s \tag{39}$$

and, assuming strict inequality in (27), we obtain

$$V(x(t)) < V(x(s)), \quad t > s \tag{40}$$

provided $x(s) \neq 0$. Asymptotic and exponential stability and, with (31), global asymptotic and exponential stability, then follow from standard arguments. □

**Remark 5:** If in Theorem 1 the inequality (28) is strict for all $x \neq 0$ as opposed to the inequality (27) and an infinite number of resetting times are used; that is, the set $\mathcal{T} = \{t_1, t_2, \ldots\}$ is infinitely countable, then the zero solution $x(t) \equiv 0$ of the undisturbed system (10), (11) is also asymptotically stable. A similar remark holds for Theorem 2 below.

**Remark 6:** In the proof of Theorem 1, we note that assuming strict inequality in (27), the inequality (40) is obtained *provided* $x(s) \neq 0$. This proviso is necessary since it may be possible to reset the states to the origin, in which case $x(s) = 0$ for a finite value of $s$. In this case, for $t > s$, we have $V(x(t)) = V(x(s)) = V(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.

**Remark 7:** Theorem 1 presents sufficient conditions for time-dependent impulsive dynamical systems in terms of Lyapunov functions that do not depend explicitly on time. Since time-dependent impulsive dynamical systems are time-varying, Lyapunov functions that explicitly depend on time can also be considered. However, in this case the conditions on the Lyapunov functions required to guarantee stability are significantly harder to verify. For further details see Bainov and Simeonov (1989), Samoilenko and Perestyuk (1995) and Ye *et al.* (1998 a).

Next, we state a stability theorem for non-linear state-dependent impulsive dynamical systems.

**Theorem 2:** *Suppose there exists a continuously differentiable function $V: \mathcal{D} \to [0, \infty)$ satisfying $V(0) = 0$, $V(x) > 0$, $x \neq 0$ and*

$$V'(x) f_{\mathrm{c}}(x) \leq 0, \quad x \notin \mathcal{Z}_x \tag{41}$$

$$V(x + f_{\mathrm{d}}(x)) \leq V(x), \quad x \in \mathcal{Z}_x \tag{42}$$

*Then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0, 0))$ system (23), (24) is Lyapunov stable. Furthermore, if the inequality (41) is strict for all $x \neq 0$, then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0, 0))$ system (23), (24) is asymptotically stable. Alternatively, if there exist scalars $\alpha$, $\beta$, $\varepsilon > 0$ and $p \geq 1$ such that (29) holds*

$$V'(x) f_{\mathrm{c}}(x) \leq -\varepsilon V(x), \quad x \notin \mathcal{Z}_x \tag{47}$$

*and (42) holds, then the zero solution $x(t) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t)) \equiv (0, 0))$ system (23), (23) is exponentially stable. Finally, if $\mathcal{D} = \mathbb{R}^n$ and (31) is satisfied, then the above results are global.*

**Proof:** For $\mathcal{S} = [0, \infty) \times \mathcal{Z}_x$ it follows from Assumptions A1 and A2 that the resetting times $\tau_k(x_0)$ are well defined and distinct for every trajectory of (23), (24) with $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0, 0)$. Now, the proof follows as in the proof of Theorem 1 with $t_k$ replaced by $\tau_k(x_0)$. □

**Remark 8:** To examine the stability of linear state-dependent impulsive systems set $f_{\mathrm{c}}(x) = A_{\mathrm{c}} x$ and $f_{\mathrm{d}}(x) = (A_{\mathrm{d}} - I_n) x$ in Theorem 2. Considering the quadratic Lyapunov function candidate $V(x) = x^{\mathrm{T}} P x$, where $P > 0$, it follows from Theorem 2 that the conditions

$$x^{\mathrm{T}} (A_{\mathrm{c}}^{\mathrm{T}} P + P A_{\mathrm{c}}) x < 0, \quad x \notin \mathcal{Z}_x \tag{44}$$

$$x^{\mathrm{T}} (A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} - P) x \leq 0, \quad x \in \mathcal{Z}_x \tag{48}$$

establish asymptotic stability for linear state-dependent impulsive systems. These conditions are implied by $P > 0$, $A_{\mathrm{c}}^{\mathrm{T}} P + P A_{\mathrm{c}} < 0$ and $A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} - P \leq 0$ which can be solved using a linear matrix inequality (LMI) feasibility problem (Boyd *et al.* 1994).

Next, we generalize the Barbashin–Krasovskii–LaSalle invariance principle (Barbashin and Krasovskii 1952, Krasovskii 1959, LaSalle 1960) to state-dependent impulsive dynamical systems. Recall that a state-dependent impulsive dynamical system is time-invariant and hence $s(t + \tau; \tau, x_0, 0) = s(t; 0, x_0, 0)$ for all $x_0 \in \mathcal{D}$, $t, \tau \in [0, \infty)$. For simplicity of exposition, in the remainder of this section we denote the trajectory $s(t; 0, x_0, 0)$ by $s(t, x_0)$ and let the map $s_t: \mathcal{D} \to \mathcal{D}$ be defined by $s_t(x) \triangleq s(t, x_0)$, $x_0 \in \mathcal{D}$, for a given $t \geq 0$. The following definitions and key theorem are needed for this result.

**Definition 1:** Consider the non-linear impulsive dynamical system $\mathcal{G}$ given by (23), (24) with $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0, 0)$. The *trajectory* $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq 0$, of $\mathcal{G}$ denotes the solution to (23), (24) corresponding to the initial condition $x(0) = x_0$ evaluated at time $t$. The trajectory $x(t)$, $t \geq 0$, of $\mathcal{G}$ is *bounded* if there exists $\gamma > 0$ such that $\|x(t)\| < \gamma$, $t \geq 0$.

**Definition 2:** Consider the non-linear impulsive dynamical system $\mathcal{G}$ given by (23), (24) with $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0, 0)$. A set $\mathcal{M} \subseteq \mathcal{D}$ is a *positively invariant set* for the dynamical system $\mathcal{G}$ if $s_t(\mathcal{M}) \subseteq \mathcal{M}$, for all $t \geq 0$, where $s_t(\mathcal{M}) \triangleq \{s_t(x) : x \in \mathcal{M}\}$. A set $\mathcal{M} \subseteq \mathcal{D}$ is an *invariant set* for the dynamical system $\mathcal{G}$ if $s_t(\mathcal{M}) = \mathcal{M}$ for all $t \geq 0$.

**Definition 3:** $p \in \bar{\mathcal{D}} \subset \mathbb{R}^n$ is a *positive limit point* of the trajectory $x(t)$, $t \geq 0$, if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of non-negative real numbers, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$, $t \geq 0$, is the *positive limit set* $\omega(x_0)$ of $x(t)$, $t \geq 0$.

The following key assumption is needed for the statement of the next result.

**Assumption 1:** *Consider the impulsive dynamical system $\mathcal{G}$ given by* (23), (24) *with* $(u_c(t), u_d(t_k)) \equiv (0, 0)$ *and let* $s(t, x_0)$, $t \geq 0$, *denote the solution to* (23), (24) *with initial condition* $x_0$. *Then, for every* $x_0 \in \mathcal{D}$, *there exists* $\mathcal{T}_{x_0} \subseteq [0, \infty)$ *such that* $[0, \infty) \backslash \mathcal{T}_{x_0}$ *is countable and for every* $\varepsilon > 0$ *and* $t \in \mathcal{T}_{x_0}$, *there exists* $\delta(\varepsilon, x_0, t) > 0$ *such that if* $\|x_0 - y\| < \delta(\varepsilon, x_0, t)$, $y \in \mathcal{D}$, *then* $\|s(t, x_0) - s(t, y)\| < \varepsilon$.

Assumption 1 is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with discontinuous flows. Specifically, by letting $\mathcal{T}_{x_0} = \overline{\mathcal{T}}_{x_0} = [0, \infty)$, where $\overline{\mathcal{T}}_{x_0}$ denotes the closure of the set $\mathcal{T}_{x_0}$, Assumption 1 specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $x_0 \in \mathcal{D}$ (Vidyasagar 1993). If, in addition, $x_0 = 0$, $s(t, 0) = 0$, $t \geq 0$ and $\delta(\varepsilon, 0, t)$ can be chosen independent of $t$, then continuous dependence implies the classical Lyapunov stability of the zero trajectory $s(t, 0) = 0$, $t \geq 0$. Hence, Lyapunov stability of motion can be interpreted as continuous dependence of solutions uniformly in $t$ for all $t \geq 0$. Conversely, continuous dependence of solutions can be interpreted as Lyapunov stability of motion for every fixed time $t$ (Vidyasagar 1993). Analogously, Lyapunov stability of impulsive dynamical systems as defined in Lakshmikantham *et al.* (1989) can be interpreted as *quasi-continuous dependence of solutions* (i.e. Assumption 1) uniformly in $t$ for all $t \in \mathcal{T}_{x_0}$.

For the next result note that $p$ is a positive limit point of the trajectory $s(t, x_0)$, $t \geq 0$, if and only if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty} \subset \mathcal{T}_{x_0}$, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x_0) \to p$ as $n \to \infty$. To see this, let $p \in \omega(x_0)$ and let $\mathcal{T}_{x_0}$ be a dense subset of the semi-infinite interval $[0, \infty)$. In this case, it follows that there exists an unbounded sequence $\{t_n\}_{n=0}^{\infty}$, such that $\lim_{n\to\infty} s(t_n, x_0) = p$. Hence, for every $\varepsilon > 0$, there exists $n > 0$ such that $\|s(t_n, x_0) - p\| < \varepsilon/2$. Furthermore, since $s(\cdot, x_0)$ is left-continuous and $\mathcal{T}_{x_0}$ is a dense subset of $[0, \infty)$, there exists $\hat{t}_n \in \mathcal{T}_{x_0}$, $\hat{t}_n \leq t_n$, such that $\|s(\hat{t}_n, x_0) - s(t_n, x_0)\| < \varepsilon/2$ and hence $\|s(\hat{t}_n, x_0) - p\| \leq \|s(t_n, x_0) - p\| + \|s(\hat{t}_n, x_0) - s(t_n, x_0)\| < \varepsilon$. Using this procedure, with $\varepsilon = 1, 1/2, 1/3, \ldots$, we can construct an unbounded sequence $\{\hat{t}_k\}_{k=1}^{\infty} \subset \mathcal{T}_{x_0}$, such that $\lim_{k\to\infty} s(\hat{t}_k, x_0) = p$. Hence, $p \in \omega(x_0)$ if and only if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty} \subset \mathcal{T}_{x_0}$, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x_0) \to p$ as $n \to \infty$.

Next, we state and prove a fundamental result on positive limit sets for impulsive dynamical systems. The result generalizes the classical results on positive limit sets to systems with left-continuous flows. For the remainder of the paper the notation $s(t, x_0) \to \mathcal{M} \subseteq \mathcal{D}$ as $t \to \infty$ denotes the fact that $\lim_{t\to\infty} s(t, x_0)$ evolves in $\mathcal{M}$; that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\operatorname{dist}(s(t, x_0), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$.

**Theorem 3:** *Consider the impulsive dynamical system $\mathcal{G}$ given by* (23), (24) *with* $(u_c(t), u_d(t_k)) \equiv (0, 0)$, *assume Assumption 1 holds and suppose the trajectory* $x(t)$ *of* $\mathcal{G}$ *is bounded for all* $t \geq 0$. *Then the positive limit set* $\omega(x_0)$ *of* $x(t)$, $t \geq 0$, *is a non-empty, compact invariant set. Furthermore,* $x(t) \to \omega(x_0)$ *as* $t \to \infty$.

**Proof:** Let $s(t, x_0)$, $t \geq 0$, denote the solution to $\mathcal{G}$ with initial condition $x_0 \in \mathcal{D}$. Since $s(t, x_0)$ is bounded for all $t \geq 0$, it follows from the Bolzano–Weierstrass theorem (Royden 1988) that every sequence in the positive orbit $\gamma^+(x_0) \triangleq \{s(t, x_0) : t \in [0, \infty)\}$ has at least one accumulation point $y \in \mathcal{D}$ as $t \to \infty$ and hence $\omega(x_0)$ is non-empty. Furthermore, since $s(t, x_0)$, $t \geq 0$, is bounded it follows that $\omega(x_0)$ is bounded. To show that $\omega(x_0)$ is closed let $\{y_i\}_{i=0}^{\infty}$ be a sequence contained in $\omega(x_0)$ such that $\lim_{i\to\infty} y_i = y$. Now, since $y_i \to y$ as $i \to \infty$ it follows that for every $\varepsilon > 0$ there exists $i$ such that $\|y - y_i\| < \varepsilon/2$. Next, since $y_i \in \omega(x_0)$ it follows that for every $T > 0$ there exists $t \geq T$ such that $\|s(t, x_0) - y_i\| < \varepsilon/2$. Hence, it follows that for every $\varepsilon > 0$ and $T > 0$ there exists $t \geq T$ such that $\|s(t, x_0) - y\| \leq \|s(t, x_0) - y_i\| + \|y - y_i\| < \varepsilon$ which implies that $y \in \omega(x_0)$ and hence $\omega(x_0)$ is closed. Thus, since $\omega(x_0)$ is closed and bounded, $\omega(x_0)$ is compact.

Next, to show positive invariance of $\omega(x_0)$ let $y \in \omega(x_0)$ so that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty} \subset \mathcal{T}_{x_0}$ such that $s(t_n, x_0) \to y$ as $n \to \infty$. Now, it follows from Assumption 1 that for every $\varepsilon > 0$ and $t \in \mathcal{T}_y$ there exists $\delta(\varepsilon, y, t) > 0$ such that $\|y - z\| < \delta(\varepsilon, y, t)$, $z \in \mathcal{D}$, implies $\|s(t, y) - s(t, z)\| < \varepsilon$ or, equivalently, for every sequence $\{y_i\}_{i=1}^{\infty}$ converging to $y$ and $t \in \mathcal{T}_y$, $\lim_{i\to\infty} s(t, y_i) = s(t, y)$. Now, since by assumption there exists a unique solution to $\mathcal{G}$, it follows that the semi-group property $s(\tau, s(t, x_0)) = s(t + \tau, x_0)$ holds. Furthermore, since $s(t_n, x_0) \to y$ as $n \to \infty$, it follows from the semi-group property that $s(t, y) = s(t, \lim_{n\to\infty} s(t_n, x_0)) = \lim_{n\to\infty} s(t + t_n, x_0) \in \omega(x_0)$ for all $t \in \mathcal{T}_y$. Hence, $s(t, y) \in \omega(x_0)$ for all $t \in \mathcal{T}_y$. Next, let $t \in [0, \infty) \backslash \mathcal{T}_y$ and note that, since $\mathcal{T}_y$ is dense in $[0, \infty)$, there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ such that $\tau_n \leq t$, $\tau_n \in \mathcal{T}_y$ and $\lim_{n\to\infty} \tau_n = t$. Now, since $s(\cdot, y)$ is left-continuous it follows that $\lim_{n\to\infty} s(\tau_n, y) = s(t, y)$. Finally, since $\omega(x_0)$ is closed and $s(\tau_n, y) \in \omega(x_0)$, $n = 1, 2, \ldots$, it follows that $s(t, y) = \lim_{n\to\infty} s(\tau_n, y) \in \omega(x_0)$. Hence, $s_t(\omega(x_0)) \subseteq \omega(x_0)$, $t \geq 0$, establishing positive invariance of $\omega(x_0)$.

Now, to show invariance of $\omega(x_0)$ let $y \in \omega(x_0)$ so that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$ such that $s(t_n, x_0) \to y$ as $n \to \infty$. Next, let $t \in \mathcal{T}_{x_0}$ and note that there exists $N$ such that $t_n > t$,

$n \geq N$. Hence, it follows from the semi-group property that $s(t, s(t_n - t, x_0)) = s(t_n, x_0) \to y$ as $n \to \infty$. Now, it follows from the Bolzano–Weierstass theorem (Royden 1988) that there exists a subsequence $z_{n_k}$ of the sequence $z_n = s(t_n - t, x_0)$, $n = N, N+1, \ldots$, such that $z_{n_k} \to z \in \mathcal{D}$ and, by definition, $z \in \omega(x_0)$. Next, it follows from Assumption 1 that $\lim_{k\to\infty} s(t, z_{n_k}) = s(t, \lim_{k\to\infty} z_{n_k})$ and hence $y = s(t, z)$ which implies that $\omega(x_0) \subseteq s_t(\omega(x_0))$, $t \in \mathcal{T}_{x_0}$. Next, let $t \in [0, \infty)\backslash\mathcal{T}_{x_0}$, let $\hat{t} \in \mathcal{T}_{x_0}$ be such that $\hat{t} > t$ and consider $y \in \omega(x_0)$. Now, there exists $\hat{z} \in \omega(x_0)$ such that $y = s(\hat{t}, \hat{z})$ and it follows from the positive invariance of $\omega(x_0)$ that $z = s(\hat{t} - t, \hat{z}) \in \omega(x_0)$. Furthermore, it follows from the semi-group property that $s(t, z) = s(t, s(\hat{t} - t, \hat{z})) = s(\hat{t}, \hat{z}) = y$ which implies that for all $t \in [0, \infty)\backslash\mathcal{T}_{x_0}$ and for every $y \in \omega(x_0)$ there exists $z \in \omega(x_0)$ such that $y = s(t, z)$. Hence, $\omega(x_0) \subseteq s_t(\omega(x_0))$, $t \geq 0$. Now, using positive invariance of $\omega(x_0)$ it follows that $s_t(\omega(x_0)) = \omega(x_0)$, $t \geq 0$, establishing invariance of the positive limit set $\omega(x_0)$.

Finally, to show $s(t, x_0) \to \omega(x_0)$ as $t \to \infty$, suppose, *ad absurdum*, $s(t, x_0) \not\to \omega(x_0)$ as $t \to \infty$. In this case, there exists an $\epsilon > 0$ and a sequence $\{t_n\}_{n=0}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that

$$\inf_{p \in \omega(x_0)} \|s(t_n, x_0) - p\| \geq \varepsilon, \quad n \geq 0$$

However, since $s(t, x_0)$, $t \geq 0$, is bounded, the bounded sequence $\{s(t_n, x_0)\}_{n=0}^{\infty}$ contains a convergent subsequence $\{s(t_n^*, x_0)\}_{n=0}^{\infty}$ such that $s(t_n^*, x_0) \to p^* \in \omega(x_0)$ as $n \to \infty$ which contradicts the original supposition. Hence, $s(t, x_0) \to \omega(x_0)$ as $t \to \infty$. □

**Remark 9:** Note that the compactness of the positive limit set $\omega(x_0)$ depends only on the boundedness of the trajectory $s(t, x_0)$, $t \geq 0$, whereas the left-continuity and Assumption 1 are key in proving invariance of the positive limit set $\omega(x_0)$. In classical dynamical systems where the trajectory $s(\cdot, \cdot)$ is assumed to be continuous in both its arguments, both the left-continuity and Assumption 1 are trivially satisfied. Finally, we note that unlike dynamical systems with continuous flows, the omega limit set of an impulsive dynamical system may not be connected.

Henceforth, we assume that $f_c(\cdot)$, $f_d(\cdot)$ and $\mathcal{Z}_x$ are such that Assumption 1 holds. Sufficient conditions that guarantee that the non-linear impulsive dynamical system $\mathcal{G}$ given by (23), (24), satisfies Assumption 1 are given in Chellaboina *et al*. (2000). Next, we present the main result of this section characterizing impulsive dynamical system limit sets in terms of $C^1$ functions. For this result define the notation $V^{-1}(\gamma) \triangleq \{x \in \mathcal{Q}: V(x) = \gamma\}$, where $\gamma \in \mathbb{R}$, $\mathcal{Q} \subseteq \mathcal{D}$ and $V: \mathcal{Q} \to \mathbb{R}$ is a continuously differentiable function and let $\mathcal{M}_\gamma$ denote the largest invariant set (with respect to $\mathcal{G}$) contained in $V^{-1}(\gamma)$.

**Theorem 4:** *Consider the impulsive dynamical system $\mathcal{G}$ given by (23), (24) with $(u_c(t), u_d(t_k)) \equiv (0, 0)$, assume $\mathcal{D}_c \subset \mathcal{D}$ is a compact positively invariant set with respect to (23), (24) and assume that there exists a continuously differentiable function $V: \mathcal{D}_c \to \mathbb{R}$ such that*

$$V'(x) f_c(x) \leq 0, \quad x \in \mathcal{D}_c, \quad x \notin \mathcal{Z}_x \tag{46}$$

$$V(x + f_d(x)) \leq V(x), \quad x \in \mathcal{D}_c, \quad x \in \mathcal{Z}_x \tag{47}$$

*Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_c: x \notin \mathcal{Z}_x, V'(x) f_c(x) = 0\} \cup \{x \in \mathcal{D}_c: x \in \mathcal{Z}_x, V(x + f_d(x)) = V(x)\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $x_0 \in \mathcal{D}_c$, then $x(t) \to \mathcal{M}$ as $t \to \infty$.*

**Proof:** Using identical arguments as in the proof of Theorem 1 it follows that for all $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$

$$\begin{aligned} V(x(t)) - V(x(0)) &= \int_0^t V'(x(\tau)) f_c(x(\tau))\, d\tau \\ &\quad + \sum_{i=1}^{k} [V(x(\tau_i(x_0)) + f_d(x(\tau_i(x_0)))) \\ &\quad - V(x(\tau_i(x_0)))] \end{aligned}$$

Hence, it follows from (46) and (47) that $V(x(t)) \leq V(x(0))$, $t \geq 0$. Using a similar argument it follows that $V(x(t)) \leq V(x(\tau))$, $t \geq \tau$, which implies that $V(x(t))$ is a non-increasing function of time. Since $V(\cdot)$ is continuous on a compact set $\mathcal{D}_c$ there exists $\beta \in \mathbb{R}$ such that $V(x) \geq \beta$, $x \in \mathcal{D}_c$. Furthermore, since $V(x(t))$, $t \geq 0$, is non-increasing, $\gamma_{x_0} \triangleq \lim_{t\to\infty} V(x(t))$, $x_0 \in \mathcal{D}_c$, exists. Now, for all $y \in \omega(x_0)$ there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$ such that $x(t_n) \to y$ as $n \to \infty$ and, since $V(\cdot)$ is continuous, it follows that

$$V(y) = V\Big(\lim_{n\to\infty} x(t_n)\Big) = \lim_{n\to\infty} V(x(t_n)) = \gamma_{x_0}$$

Hence, $y \in V^{-1}(\gamma_{x_0})$ for all $y \in \omega(x_0)$, or, equivalently, $\omega(x_0) \subseteq V^{-1}(\gamma_{x_0})$. Now, since $\mathcal{D}_c$ is compact and positively invariant, it follows that $x(t)$, $t \geq 0$, is bounded for all $x_0 \in \mathcal{D}_c$ and hence it follows from Theorem 3 that $\omega(x_0)$ is a non-empty, compact invariant set. Thus, $\omega(x_0)$ is a subset of the largest invariant set contained in $V^{-1}(\gamma_{x_0})$; that is, $\omega(x_0) \subseteq \mathcal{M}_{\gamma_{x_0}}$. Hence, for every $x_0 \in \mathcal{D}_c$ there exists $\gamma_{x_0} \in \mathbb{R}$ such that $\omega(x_0) \subseteq \mathcal{M}_{\gamma_{x_0}}$, where $\mathcal{M}_{\gamma_{x_0}}$ is the largest invariant set contained in $V^{-1}(\gamma_{x_0})$ which implies that $V(x) = \gamma_{x_0}$, $x \in \omega(x_0)$. Now, since $\mathcal{M}_{\gamma_{x_0}}$ is an invariant set it follows that for all $x(0) \in \mathcal{M}_{\gamma_{x_0}}$, $x(t) \in \mathcal{M}_{\gamma_{x_0}}$, $t \geq 0$ and thus $\dot{V}(x(t)) \triangleq dV(x(t))/dt = V'(x(t)) f_c(x(t)) = 0$, for all $x(t) \notin \mathcal{Z}_x$ and $V(x(t) + f_d(x(t))) = V(x(t))$, for all $x(t) \in \mathcal{Z}_x$. Thus, $\mathcal{M}_{\gamma_{x_0}}$ is contained in $\mathcal{M}$ which is the largest invariant set contained in $\mathcal{R}$. Hence, $x(t) \to \mathcal{M}$ as $t \to \infty$. □

Finally, using Theorem 4 we provide a generalization of Theorem 2 for local asymptotic stability of a non-linear state-dependent impulsive dynamical system.

**Corollary 1:** *Consider the impulsive dynamical system $\mathcal{G}$ given by (23), (24) with $(u_c(t), u_d(t_k)) \equiv (0,0)$, assume $\mathcal{D}_c \subset \mathcal{D}$ is a compact positively invariant set with respect to (23), (24) such that $0 \in \mathring{\mathcal{D}}$ and assume that there exists a continuously differentiable function $V: \mathcal{D}_c \to [0,\infty)$ such that $V(0)=0$, $V(x)>0$, $x \neq 0$ and (46), (47) are satisfied. Furthermore, assume that the set $\mathcal{R} \triangleq \{x \in \mathcal{D}_c: x \notin \mathcal{Z}_x, V'(x) f_c(x) = 0\} \cup \{x \in \mathcal{D}_c: x \in \mathcal{Z}_x, V(x + f_d(x)) = V(x)\}$ contains no invariant set other than the set $\{0\}$. Then the zero solution $x(t) \equiv 0$ to (23), (24) is asymptotically stable and $\mathcal{D}_c$ is a subset of the domain of attraction of (23), (24).*

**Proof:** Lyapunov stability of the zero solution $x(t) \equiv 0$ to (23), (24) follows from Theorem 2. Next, it follows from Theorem 4 that if $x_0 \in \mathcal{D}_c$, then $\omega(x_0) \subseteq \mathcal{M}$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$, which implies that $\mathcal{M} = \{0\}$. Hence, $x(t) \to \mathcal{M} = \{0\}$ as $t \to \infty$ establishing asymptotic stability of the zero solution $x(t) \equiv 0$ to (23), (24). □

**Remark 10:** Setting $\mathcal{D} = \mathbb{R}^n$ and requiring $V(x) \to \infty$ as $\|x\| \to \infty$ in Corollary 1, it follows that the zero solution $x(t) \equiv 0$ of the undisturbed system (23), (24) is globally asymptotically stable.

## 4. Dissipative impulsive dynamical systems: input–output and state properties

Many of the great landmarks of feedback control theory are associated with the theory of absolute stability and dissipativity. The Aizerman conjecture and the Luré problem, as well as the circle and Popov criteria, are extensively developed in the classical monographs of Aizerman and Gantmacher (1964), Lefschetz (1965) and Popov (1973). Since absolute stability theory concerns the stability of a dynamical system for classes of feedback non-linearities which, as noted in Zames (1966), Safonov (1980) and Haddad and Bernstein (1993), can readily be interpreted as an uncertainty model, it is not surprising that dissipativity theory forms the basis of modern-day robust stability analysis and synthesis (Haddad and Bernstein 1993, 1994, Hadded *et al.* 1994). Furthermore, since Lyapunov functions can be viewed as generalizations of energy functions for general non-linear dynamical systems, the notion of dissipativity, with appropriate storage functions and supply rates, can be used to construct Lyapunov functions for non-linear feedback systems by appropriately combining storage functions for each subsystem. In this section we extend the notion of dissipative dynamical systems to develop the concept of dissipativity for impulsive dynamical systems.

In this section we consider non-linear impulsive dynamical systems $\mathcal{G}$ of the form given by (1)–(4) with $t \in \mathbb{R}$, $(t, x(t), u_c(t)) \notin \mathcal{S}$ and $(t, x(t), u_c(t)) \in \mathcal{S}$ replaced by $\mathcal{X}(t, x(t), u_c(t)) \neq 0$ and $\mathcal{X}(t, x(t), u_c(t)) = 0$, respectively, where $\mathcal{X}: \mathbb{R} \times \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}$. Note that setting $\mathcal{X}(t, x(t), u_c(t)) = (t - t_1)(t - t_2) \cdots$, where $t_k \to \infty$ as $k \to \infty$, (1)–(4) reduce to (10)–(13), while setting $\mathcal{X}(t, x(t), u_c(t)) = \mathcal{X}(x(t))$, where $\mathcal{X}: \mathcal{D} \to \mathcal{D}$ is a support function characterizing the manifold $\mathcal{Z}$, (1)–(4) reduce to (23)–(26). Furthermore, we assume that the system functions $f_c(\cdot)$, $f_d(\cdot)$, $G_c(\cdot)$, $G_d(\cdot)$, $h_c(\cdot)$, $h_d(\cdot)$, $J_c(\cdot)$ and $J_d(\cdot)$ are smooth (at least continuously differentiable mappings). In addition, for the non-linear dynamical system (1) we assume that the required properties for the existence and uniqueness of solutions are satisfied such that (1) has a unique solution for all $t \in \mathbb{R}$ (Lakshmikantham *et al.* 1989, Bainov and Simeonov 1995). For the impulsive dynamical system $\mathcal{G}$ given by (1)–(4) a function $(r_c(u_c, y_c), r_d(u_d, y_d))$, where $r_c: \mathcal{U}_c \times \mathcal{Y}_c \to \mathbb{R}$ and $r_d: \mathcal{U}_d \times \mathcal{Y}_d \to \mathbb{R}$ are such that $r_c(0,0) = 0$ and $r_d(0,0) = 0$, is called a *supply rate* if $r_c(u_c, y_c)$ is locally integrable; that is, for all input–output pairs $u_c(t) \in \mathcal{U}_c$, $y_c(t) \in \mathcal{Y}_c$, $r_c(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |r_c(u_c(s), y_c(s))| \, \mathrm{d}s < \infty$, $t, \hat{t} \geq 0$. Note that since all input–output pairs $u_d(t_k) \in \mathcal{U}_d$, $y_d(t_k) \in \mathcal{Y}_d$, are defined for discrete instants, $r_d(\cdot,\cdot)$ satisfies $\sum_{k \in \mathcal{N}_{[t,\hat{t})}} |r_d(u_d(t_k), y_d(t_k))| < \infty$, where $k \in \mathcal{N}_{[t,\hat{t})} \triangleq \{k: t \leq t_k < \hat{t}\}$.

**Definition 4:** An impulsive dynamical system $\mathcal{G}$ of the form (1)–(4) is *dissipative with respect to the supply rate* $(r_c, r_d)$ if the *dissipation inequality*

$$0 \leq \int_{t_0}^{T} r_c(u_c(t), y_c(t)) \, \mathrm{d}t + \sum_{k \in \mathcal{N}_{[t_0,T)}} r_d(u_d(t_k), y_d(t_k)), \quad T \geq t_0 \tag{48}$$

is satisfied for all $T \geq t_0$, with $x(t_0) = 0$. An impulsive dynamical system $\mathcal{G}$ of the form (1)–(4) is *exponentially dissipative with respect to the supply rate* $(r_c, r_d)$ if there exists a constant $\varepsilon > 0$, such that the dissipation inequality (48) is satisfied, with $r_c(u_c(t), y_c(t))$ replaced by $\mathrm{e}^{\varepsilon t} r_c(u_c(t), y_c(t))$ and $r_d(u_d(t_k), y_d(t_k))$ replaced by $\mathrm{e}^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k))$, for all $T \geq t_0$ with $x(t_0) = 0$. An impulsive dynamical system is *lossless with respect to the supply rate* $(r_c, r_d)$ if the dissipation inequality (48) is satisfied as an equality for all $T \geq t_0$ with $x(t_0) = x(T) = 0$.

Next, define the *available storage* $V_a(t_0, x_0)$ of the impulsive dynamical system $\mathcal{G}$ by

$$V_a(t_0, x_0) \triangleq - \inf_{(u_c(\cdot), u_d(\cdot)),\, T \geq t_0} \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t)) \, \mathrm{d}t + \sum_{k \in \mathcal{N}_{[t_0,T)}} r_d(u_d(t_k), y_d(t_k)) \right] \tag{49}$$

where $x(t)$, $t \geq t_0$, is the solution to (1)–(4) with admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $x(t_0) = x_0$. Note that $V_a(t_0, x_0) \geq 0$ for all $(t, x) \in \mathbb{R} \times \mathcal{D}$ since $V_a(t_0, x_0)$ is the supremum over a set of numbers containing the zero element $(T = t_0)$. It follows from (49) that the available storage of a non-linear impulsive dynamical system $\mathcal{G}$ is the maximum amount of generalized stored energy which can be extracted from $\mathcal{G}$ at any time $T$. Furthermore, define the *available exponential storage* of the impulsive dynamical system $\mathcal{G}$ by

$$V_a(t_0, x_0) \triangleq -\inf_{(u_c(\cdot), u_d(\cdot)),\, T \geq t_0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right] \tag{50}$$

where $x(t)$, $t \geq t_0$, is the solution of (1)–(4) with admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $x(t_0) = x_0$.

**Remark 11:** Note that in the case of (time-invariant) state-dependent impulsive dynamical systems, the available storage is time-invariant; that is, $V_a(t_0, x_0) = V_a(x_0)$. Furthermore, the available exponential storage satisfies

$$\begin{aligned} V_a(t_0, x_0) &= -\inf_{(u_c(\cdot), u_d(\cdot)),\, T \geq t_0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right] \\ &= -e^{\varepsilon t_0} \inf_{(u_c(\cdot), u_d(\cdot)),\, T \geq 0} \left[ \int_{0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right] \\ &= e^{\varepsilon t_0} \hat{V}_a(x_0) \end{aligned} \tag{51}$$

where

$$\hat{V}_a(x_0) \triangleq -\inf_{(u_c(\cdot), u_d(\cdot)),\, T \geq 0} \left[ \int_{0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right] \tag{52}$$

Next, we show that the available storage (resp., available exponential storage) is finite if and only if $\mathcal{G}$ is dissipative (resp., exponentially dissipative). In order to state this result we require two more definitions.

**Definition 5:** Consider the impulsive dynamical system $\mathcal{G}$ given by (1)–(4). Assume $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$. A continuous non-negative-definite function $V_s\colon \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ satisfying

$$V_s(T, x(T)) \leq V_s(t_0, x(t_0)) + \int_{t_0}^{T} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \tag{53}$$

where $x(t)$, $t \geq t_0$, is a solution to (1)–(4) with $(u_c(t), u_d(t_k)) \in \mathcal{U}_c \times \mathcal{U}_d$ and $x(t_0) = x_0$, is called a *storage function for* $\mathcal{G}$. *A continuous non-negative-definite function* $V_s\colon \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ *satisfying*

$$e^{\varepsilon T} V_s(T, x(T)) \leq e^{\varepsilon t_0} V_s(t_0, x(t_0)) + \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t))\, dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \tag{54}$$

is called an *exponential storage function* for $\mathcal{G}$.

Note that $V_s(t, x(t))$ is left-continuous on $[t_0, \infty)$ and is continuous everywhere on $[t_0, \infty)$ except on an unbounded closed discrete set $\mathcal{T} = \{t_1, t_2, \ldots\}$, where $\mathcal{T}$ is the set of times when the jumps occur for $x(t)$, $t \geq 0$.

**Definition 6:** An impulsive dynamical system $\mathcal{G}$ given by (1)–(4) is *zero-state observable* if $(u_c(t), u_d(t_k)) \equiv (0, 0)$, $(y_c(t), y_d(t_k)) \equiv (0, 0)$ implies $x(t) \equiv 0$. An impulsive dynamical system $\mathcal{G}$ given by (1)–(4) is *strongly zero-state observable* if $(u_c(t), u_d(t_k)) \equiv (0, 0)$, $y_c(t) \equiv 0$ implies $x(t) \equiv 0$. An impulsive dynamical system $\mathcal{G}$ is *completely reachable* if for all $(t_0, x_i) \in \mathbb{R} \times \mathcal{D}$, there exist a finite time $t_i \leq t_0$, square integrable inputs $u_c(t)$ defined on $[t_i, t_0]$ and inputs $u_d(t_k)$ defined on $k \in \mathcal{N}_{[t_i, t_0)}$, such that the state $x(t)$, $t \geq t_i$, can be driven from $x(t_i) = 0$ to $x(t_0) = x_0$. Finally, an impulsive system $\mathcal{G}$ is *minimal* if it is zero-state observable and completely reachable.

**Remark 12:** Note that strong zero-state observability is a stronger condition than zero-state observability. In particular, strong zero-state observability implies zero-state observability but the converse is not necessarily true.

**Theorem 5:** *Consider the impulsive dynamical system* $\mathcal{G}$ *given by* (1)–(4) *and assume that* $\mathcal{G}$ *is completely reachable. Then* $\mathcal{G}$ *is dissipative* (*resp., exponentially dissipative*) *with respect to the supply rate* $(r_c, r_d)$ *if and only if the available system storage* $V_a(t_0, x_0)$ *given by* (49) (*resp., the available exponential system storage* $V_a(t_0, x_0)$ *given by* (50)) *is finite for all* $t_0 \in \mathbb{R}$ *and* $x_0 \in \mathcal{D}$. *Moreover, if* $V_a(t_0, x_0)$ *is finite for all* $t_0 \in \mathbb{R}$ *and* $x_0 \in \mathcal{D}$, *then* $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, *is a storage* *function (resp., exponential storage function) for $\mathcal{G}$. Finally, all storage functions (resp., exponential storage functions) $V_{\rm s}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, for $\mathcal{G}$ satisfy*

$$0\le V_{\rm a}(t,x)\le V_{\rm s}(t,x),\quad (t,x)\in\mathbb{R}\times\mathcal{D} \tag{55}$$

**Proof:** Suppose $V_{\rm a}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, is finite. Now, it follows from (49) (with $T=t_0$) that $V_{\rm a}(t,x)\ge 0$, $(t,x)\in\mathbb{R}\times\mathcal{D}$. Next, let $x(t)$, $t\ge t_0$, satisfy (1)–(4) with admissible inputs $(u_{\rm c}(t),u_{\rm d}(t_k))$, $t\ge t_0$, $k\in\mathcal{N}_{[t_0,t)}$ and $x(t_0)=x_0$. Since $-V_{\rm a}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, is given by the infimum over all admissible inputs $(u_{\rm c}(\cdot),u_{\rm d}(\cdot))$ and $T\ge t_0$ in (49), it follows that for all admissible inputs $(u_{\rm c}(\cdot),u_{\rm d}(\cdot))$ and $t\in[t_0,T]$

$$\begin{aligned}
-V_{\rm a}(t_0,x_0)&\le\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&=\int_{t_0}^{t} r_{\rm c}(u_{\rm c}(s),y_{\rm c}(s))\,{\rm d}s+\sum_{k\in\mathcal{N}_{[t_0,t)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&\quad+\int_{t}^{T} r_{\rm c}(u_{\rm c}(s),y_{\rm c}(s))\,{\rm d}s+\sum_{k\in\mathcal{N}_{[t,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))
\end{aligned}$$

which implies

$$\begin{aligned}
-V_{\rm a}(t_0,x_0)&-\int_{t_0}^{t} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t-\sum_{k\in\mathcal{N}_{[t_0,t)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&\le\int_{t}^{T} r_{\rm c}(u_{\rm c}(s),y_{\rm c}(s))\,{\rm d}s+\sum_{k\in\mathcal{N}_{[t,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))
\end{aligned}$$

Hence

$$\begin{aligned}
V_{\rm a}(t_0,x_0)&+\int_{t_0}^{t} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,t)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&\ge-\inf_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot)),\,T\ge t}\left[\int_{t}^{T} r_{\rm c}(u_{\rm c}(s),y_{\rm c}(s))\,{\rm d}s+\sum_{k\in\mathcal{N}_{[t,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\right]\\
&=V_{\rm a}(t,x(t))
\end{aligned}\tag{56}$$

which shows that $V_{\rm a}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, is a storage function for $\mathcal{G}$.

Conversely, suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\rm c},r_{\rm d})$ and let $t_0\in\mathbb{R}$ and $x_0\in\mathcal{D}$. Since $\mathcal{G}$ is completely reachable it follows that there exists $\hat{t}<t_0$, $u_{\rm c}(t)$, $\hat{t}\le t<t_0$ and $u_{\rm d}(t_k)$, $k\in\mathcal{N}_{[\hat{t},t_0)}$, such that $x(\hat{t})=0$ and $x(t_0)=x_0$. Hence, since $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\rm c},r_{\rm d})$ it follows that, for all $T\ge t_0$

$$\begin{aligned}
0&\le\int_{\hat{t}}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[\hat{t},T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&=\int_{\hat{t}}^{t_0} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[\hat{t},t_0)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&\quad+\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))
\end{aligned}$$

and hence there exists $W:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ such that

$$-\infty<W(t_0,x_0)\le\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)) \tag{57}$$

Now, it follows from (57) that, for all $(t,x)\in\mathbb{R}\times\mathcal{D}$

$$\begin{aligned}
V_{\rm a}(t,x)&=-\inf_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot)),\,T\ge t_0}\left[\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\right]\\
&\le-W(t,x)
\end{aligned}\tag{58}$$

and hence the available storage $V_{\rm a}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, is finite.

Next, if $V_{\rm s}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, is a storage function then it follows that, for all $T\ge t_0$ and $x_0\in\mathcal{D}$

$$\begin{aligned}
V_{\rm s}(t_0,x_0)&\ge V_{\rm s}(T,x(T))-\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t-\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\\
&\ge-\left[\int_{t_0}^{T} r_{\rm c}(u_{\rm c}(t),y_{\rm c}(t))\,{\rm d}t+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\right]
\end{aligned}$$

which implies

$$
\begin{aligned}
V_{\mathrm{s}}(t_0,x_0) &\geq -\inf_{(u_{\mathrm{c}}(\cdot),u_{\mathrm{d}}(\cdot)),\, T\geq t_0}\left[\int_{t_0}^{T} r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t))\,\mathrm{d}t\right.\\
&\qquad\left.+\sum_{k\in\mathcal{N}_{[t_0,T)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))\right]\\
&= V_{\mathrm{a}}(t_0,x_0)
\end{aligned}
$$

Finally, the proof for the exponentially dissipative case follows a similar construction and hence is omitted. □

The following corollary is immediate from Theorem 5.

**Corollary 2:** *Consider the impulsive dynamical system $\mathcal{G}$ given by* (1)–(4) *and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if there exists a continuous storage function (resp., exponential storage function) $V_{\mathrm{s}}(t,x)$, $(t,x)\in\mathbb{R}\times\mathcal{D}$, satisfying* (53) (*resp.,* (54)).

The next result gives necessary and sufficient conditions for dissipativity, exponential dissipativity and losslessness over an interval $t\in(t_k,t_{k+1}]$ involving the consecutive resetting times $t_k$ and $t_{k+1}$.

**Theorem 6:** *Assume $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if there exists a continuous non-negative-definite function $V_{\mathrm{s}}:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ such that, for all $k\in\mathcal{N}$*

$$
V_{\mathrm{s}}(\hat t,x(\hat t))-V_{\mathrm{s}}(t,x(t))\leq\int_t^{\hat t} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s,\qquad t_k<t\leq\hat t\leq t_{k+1} \tag{59}
$$

$$
\begin{aligned}
V_{\mathrm{s}}(t_k,x(t_k)+f_{\mathrm{d}}(x(t_k))+G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k))&\\
-V_{\mathrm{s}}(t_k,x(t_k))\leq r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))&
\end{aligned}\tag{60}
$$

*Furthermore, $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if there exists a continuous non-negative-definite function $V_{\mathrm{s}}:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ such that*

$$
\mathrm{e}^{\varepsilon\hat t}V_{\mathrm{s}}(\hat t,x(\hat t))-\mathrm{e}^{\varepsilon t}V_{\mathrm{s}}(t,x(t))\leq\int_t^{\hat t}\mathrm{e}^{\varepsilon s} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s,\qquad t_k<t\leq\hat t\leq t_{k+1} \tag{61}
$$

$$
\begin{aligned}
V_{\mathrm{s}}(t_k,x(t_k)+f_{\mathrm{d}}(x(t_k))+G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k))&\\
-V_{\mathrm{s}}(t_k,x(t_k))\leq r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))&
\end{aligned}\tag{62}
$$

*Finally, $\mathcal{G}$ is lossless with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if there exists a continuous non-negative-definite function $V_{\mathrm{s}}:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ such that* (59) *and* (60) *are satisfied as equalities.*

**Proof:** Let $k\in\mathcal{N}$ and suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$. Then, there exists a continuous non-negative-definite function $V_{\mathrm{s}}:\mathbb{R}\times\mathcal{D}\to\mathbb{R}$ such that (53) holds. Now, since for $t_k<t\leq\hat t\leq t_{k+1}$, $\mathcal{N}_{[t,\hat t)}=\varnothing$, (59) is immediate. Next, note that

$$
\begin{aligned}
&V_{\mathrm{s}}(t_k^+,x(t_k^+))-V_{\mathrm{s}}(t_k,x(t_k))\\
&\qquad\leq\int_{t_k}^{t_k^+} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s+r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))
\end{aligned}\tag{63}
$$

which, since $\mathcal{N}_{[t_k,t_k^+)}=\{k\}$, implies (60).

Conversely, suppose (59) and (60) hold, let $\hat t\geq t\geq 0$ and let $\mathcal{N}_{[t,\hat t)}=\{i,i+1,\ldots,j\}$. (Note that if $\mathcal{N}_{[t,\hat t)}=\varnothing$ the converse is a direct consequence of (53)). In this case, it follows from (59) and (60) that

$$
\begin{aligned}
&V_{\mathrm{s}}(\hat t,x(\hat t))-V_{\mathrm{s}}(t,x(t))\\
&\quad=V_{\mathrm{s}}(\hat t,x(\hat t))-V_{\mathrm{s}}(t_j^+,x(t_j^+))+V_{\mathrm{s}}(t_j^+,x(t_j^+))\\
&\qquad-V_{\mathrm{s}}(t_{j-1}^+,x(t_{j-1}^+))+V_{\mathrm{s}}(t_{j-1}^+,x(t_{j-1}^+))-\cdots\\
&\qquad-V_{\mathrm{s}}(t_i^+,x(t_i^+))+V_{\mathrm{s}}(t_i^+,x(t_i^+))-V_{\mathrm{s}}(t,x(t))\\
&\quad=V_{\mathrm{s}}(\hat t,x(\hat t))-V_{\mathrm{s}}(t_j^+,x(t_j^+))+V_{\mathrm{s}}(t_j,x(t_j)+f_{\mathrm{d}}(x(t_j))\\
&\qquad+G_{\mathrm{d}}(x(t_j))u_{\mathrm{d}}(t_j))-V_{\mathrm{s}}(t_j,x(t_j))+V_{\mathrm{s}}(t_j,x(t_j))\\
&\qquad-V_{\mathrm{s}}(t_{j-1}^+,x(t_{j-1}^+))+\cdots+V_{\mathrm{s}}(t_i,x(t_i)+f_{\mathrm{d}}(x(t_i))\\
&\qquad+G_{\mathrm{d}}(x(t_i))u_{\mathrm{d}}(t_i))-V_{\mathrm{s}}(t_i,x(t_i))+V_{\mathrm{s}}(t_i,x(t_i))\\
&\qquad-V_{\mathrm{s}}(t,x(t))\\
&\quad\leq\int_{t_j^+}^{\hat t} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s+r_{\mathrm{d}}(u_{\mathrm{d}}(t_j),y_{\mathrm{d}}(t_j))\\
&\qquad+\int_{t_{j-1}^+}^{t_j} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s+\cdots+r_{\mathrm{d}}(u_{\mathrm{d}}(t_i),y_{\mathrm{d}}(t_i))\\
&\qquad+\int_t^{t_i} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s\\
&\quad=\int_t^{\hat t} r_{\mathrm{c}}(u_{\mathrm{c}}(s),y_{\mathrm{c}}(s))\,\mathrm{d}s+\sum_{k\in\mathcal{N}_{[t,\hat t)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))
\end{aligned}
$$

which implies that $\mathcal{G}$ is dissipative with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$.

Finally, similar constructions show that $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if (61) and (62) are satisfied and $\mathcal{G}$ is lossless with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ if and only if (59) and (60) are satisfied as equalities. □

If in Theorem 6 $V_{\mathrm{s}}(\cdot,x(\cdot))$ is continuously differentiable a.e. on $[t_0,\infty)$ except on an unbounded closed discrete set $\mathcal{T}=\{t_1,t_2,\ldots\}$, where $\mathcal{T}$ is the set of times when jumps occur for $x(t)$, then an equivalent statement for dissipativeness of the impulsive dynamical system $\mathcal{G}$ with respect to the supply rate $(r_{\mathrm{c}},r_{\mathrm{d}})$ is

$$\dot{V}_{\mathrm{s}}(t,x(t)) \le r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t)), \quad t_k < t \le t_{k+1} \tag{64}$$

$$\Delta V_{\mathrm{s}}(t_k,x(t_k)) \le r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)), \quad k \in \mathcal{N} \tag{65}$$

where $\dot{V}_{\mathrm{s}}(\cdot,\cdot)$ denotes the total derivative of $V_{\mathrm{s}}(t,x(t))$ along the state trajectories $x(t)$, $t \in (t_k, t_{k+1}]$, of the impulsive dynamical system (1)–(4) and

$$\begin{aligned}\Delta V_{\mathrm{s}}(t_k,x(t_k)) &\triangleq V_{\mathrm{s}}(t_k^+,x(t_k^+)) - V_{\mathrm{s}}(t_k,x(t_k)) \\ &= V_{\mathrm{s}}(t_k, x(t_k)+f_{\mathrm{d}}(x(t_k)) \\ &\quad + G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)) - V_{\mathrm{s}}(t_k,x(t_k)), \\ &\qquad k \in \mathcal{N}\end{aligned}$$

denotes the difference of the storage function $V_{\mathrm{s}}(t,x)$ at the resetting times $t_k$, $k \in \mathcal{N}$, of the impulsive dynamical system (1)–(4). Furthermore, an equivalent statement for exponential dissipativeness of the impulsive dynamical system $\mathcal{G}$ with respect to the supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ is given by

$$\dot{V}_{\mathrm{s}}(t,x(t)) + \varepsilon V_{\mathrm{s}}(t,x(t)) \le r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t)), \quad t_k < t \le t_{k+1} \tag{66}$$

and (65).

The following theorem provides sufficient conditions for guaranteeing that all storage functions (resp., exponential storage functions) of a given dissipative (resp., exponentially dissipative) impulsive dynamical system are positive definite.

**Theorem 7:** *Consider the non-linear impulsive dynamical system $\mathcal{G}$ given by (1)–(4) and assume that $\mathcal{G}$ is completely reachable and zero-state observable. Furthermore, assume that $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ and there exist functions $\kappa_{\mathrm{c}}: \mathcal{Y}_{\mathrm{c}} \to \mathcal{U}_{\mathrm{c}}$ and $\kappa_{\mathrm{d}}: \mathcal{Y}_{\mathrm{d}} \to \mathcal{U}_{\mathrm{d}}$ such that $\kappa_{\mathrm{c}}(0)=0$, $\kappa_{\mathrm{d}}(0)=0$, $r_{\mathrm{c}}(\kappa_{\mathrm{c}}(y_{\mathrm{c}}),y_{\mathrm{c}})<0$, $y_{\mathrm{c}} \ne 0$ and $r_{\mathrm{d}}(\kappa_{\mathrm{d}}(y_{\mathrm{d}}),y_{\mathrm{d}})<0$, $y_{\mathrm{d}} \ne 0$. Then all the storage functions (resp., exponential storage functions) $V_{\mathrm{s}}(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, for $\mathcal{G}$ are positive definite, that is, $V_{\mathrm{s}}(\cdot,0)=0$ and $V_{\mathrm{s}}(t,x)>0$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, $x \ne 0$.*

**Proof:** It follows from Theorem 5 that the available storage $V_{\mathrm{a}}(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, is a storage function for $\mathcal{G}$. Next, suppose, *ad absurdum*, there exists $(t,x) \in \mathbb{R} \times \mathcal{D}$ such that $V_{\mathrm{a}}(t,x)=0$, $x \ne 0$, or, equivalently

$$\inf_{(u_{\mathrm{c}}(\cdot),u_{\mathrm{d}}(\cdot)),\, T \ge t_0} \left[ \int_{t_0}^{T} r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t))\,\mathrm{d}t + \sum_{k \in \mathcal{N}_{[t_0,T)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)) \right] = 0 \tag{67}$$

Furthermore, suppose there exists $[t_{\mathrm{s}}, t_{\mathrm{f}}) \subset \mathbb{R}$ such that $y_{\mathrm{c}}(t) \ne 0$, $t \in [t_{\mathrm{s}}, t_{\mathrm{f}})$, or $y_{\mathrm{d}}(t_k) \ne 0$, for some $k \in \mathcal{N}$. Now, since there exists $\kappa_{\mathrm{c}}: \mathcal{Y}_{\mathrm{c}} \to \mathcal{U}_{\mathrm{c}}$ and $\kappa_{\mathrm{d}}: \mathcal{Y}_{\mathrm{d}} \to \mathcal{U}_{\mathrm{d}}$ such that $r_{\mathrm{c}}(\kappa_{\mathrm{c}}(y_{\mathrm{c}}),y_{\mathrm{c}})<0$, $y_{\mathrm{c}} \ne 0$ and $r_{\mathrm{d}}(\kappa_{\mathrm{d}}(y_{\mathrm{d}}),y_{\mathrm{d}})<0$, $y_{\mathrm{d}} \ne 0$, the infimum in (67) occurs at a negative value which as a contradiction. Hence, $y_{\mathrm{c}}(t)=0$, a.e. $t \in \mathbb{R}$ and $y_{\mathrm{d}}(t_k)=0$ for all $k \in \mathcal{N}$. Next, since $\mathcal{G}$ is zero-state observable it follows that $x=0$ and hence $V_{\mathrm{a}}(t,x)=0$ if and only if $x=0$. The result now follows from (55). Finally, the proof for the exponentially dissipative case is similar and hence is omitted. □

Next, we introduce the concept of required supply of a non-linear impulsive dynamical system given by (1)–(4). Specifically, define the *required supply* $V_{\mathrm{r}}(t_0,x_0)$ of the non-linear impulsive dynamical system $\mathcal{G}$ by

$$V_{\mathrm{r}}(t_0,x_0) \triangleq \inf_{(u_{\mathrm{c}}(\cdot),u_{\mathrm{d}}(\cdot)),\, T \le t_0} \left[ \int_{T}^{t_0} r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t))\,\mathrm{d}t + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)) \right] \tag{68}$$

where $x(t)$, $t \ge T$, is the solution of (1)–(4) with $x(T)=0$ and $x(t_0)=x_0$. It follows from (68) that the required supply of a non-linear impulsive dynamical system is the minimum amount of generalized energy which can be delivered to the impulsive dynamical system in order to transfer it from an initial state $x(T)=0$ to a given state $x(t_0)=x_0$. Similarly, define the *required exponential supply* of the non-linear impulsive dynamical system $\mathcal{G}$ by

$$V_{\mathrm{r}}(t_0,x_0) \triangleq \inf_{(u_{\mathrm{c}}(\cdot),u_{\mathrm{d}}(\cdot)),\, T \le t_0} \left[ \int_{T}^{t_0} \mathrm{e}^{\varepsilon t} r_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t))\,\mathrm{d}t + \sum_{k \in \mathcal{N}_{[T,t_0)}} \mathrm{e}^{\varepsilon t_k} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)) \right] \tag{69}$$

where $x(t)$, $t \ge T$, is the solution of (1)–(4) with $x(T)=0$ and $x(t_0)=x_0$.

Next, using the notion of required supply, we show that all storage functions are bounded from above by the required supply and bounded from below by the available storage. Hence, as in the case of systems with continuous flows (Willems 1972 a), a dissipative impulsive dynamical system can only deliver to its surroundings a fraction of its stored generalized energy and can only store a fraction of the generalized work done to it.

**Theorem 8:** *Consider the non-linear impulsive dynamical system $\mathcal{G}$ given by (1)–(4) and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ if and only if $0 \le V_{\mathrm{r}}(t,x) < \infty$, $t \in \mathbb{R}$, $x \in \mathcal{D}$. Moreover, if $V_{\mathrm{r}}(t,x)$ is finite and non-negative for all $(t_0,x_0) \in \mathbb{R} \times \mathcal{D}$, then $V_{\mathrm{r}}(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, is a storage function (resp., exponential storage function) for $\mathcal{G}$. Finally, all storage functions (resp., exponential storage functions) $V_{\mathrm{s}}(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, for $\mathcal{G}$ satisfy*

$$0 \le V_a(t,x) \le V_s(t,x) \le V_r(t,x) < \infty, \quad (t,x) \in \mathbb{R} \times \mathcal{D} \tag{70}$$

**Proof:** Suppose $0 \le V_r(t,x) < \infty$, $(t,x) \in \mathbb{R} \times \mathcal{D}$. Next, let $x(t)$, $t \in \mathbb{R}$, satisfy (1)–(4) with admissible inputs $(u_c(t), u_d(t_k))$, $t \in \mathbb{R}$, $k \in \mathcal{N}_{[t_0,t)}$ and $x(t_0) = x_0$. Since $V_r(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, is given by the infimum over all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $T \le t_0$ in (68), it follows that for all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $T \le t \le t_0$

$$\begin{aligned} V_r(t_0,x_0) &\le \int_T^{t_0} r_c(u_c(t), y_c(t))\,dt + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \\ &= \int_T^{t} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[T,t)}} r_d(u_d(t_k), y_d(t_k)) \\ &\quad + \int_t^{t_0} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[t,t_0)}} r_d(u_d(t_k), y_d(t_k)) \end{aligned}$$

and hence

$$\begin{aligned} V_r(t_0,x_0) &\le \inf_{(u_c(\cdot),u_d(\cdot)),\,T\le t} \left[ \int_T^{t} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[T,t)}} r_d(u_d(t_k), y_d(t_k)) \right] \\ &\quad + \int_t^{t_0} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[t,t_0)}} r_d(u_d(t_k), y_d(t_k)) \\ &= V_r(t, x(t)) + \int_t^{t_0} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[t,t_0)}} r_d(u_d(t_k), y_d(t_k) \end{aligned} \tag{71}$$

which shows that $V_r(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$, is a storage function for $\mathcal{G}$ and hence $\mathcal{G}$ is dissipative.

Conversely, suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ and let $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D}$. Since $\mathcal{G}$ is completely reachable it follows that there exists $T < t_0$, $u_c(t)$, $T \le t < t_0$ and $u_d(t_k)$, $k \in \mathcal{N}_{[T,t_0)}$, such that $x(T) = 0$ and $x(t_0) = x_0$. Hence, since $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ it follows that, for all $T \le t_0$

$$0 \le \int_T^{t_0} r_c(u_c(t), y_c(t))\,dt + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \tag{72}$$

and hence

$$0 \le \inf_{(u_c(\cdot),u_d(\cdot)),\,T\le t_0} \left[ \int_T^{t_0} r_c(u_c(s), y_c(s))\,ds + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \right] \tag{73}$$

which implies that

$$0 \le V_r(t_0,x_0) < \infty, \quad (t_0,x_0) \in \mathbb{R} \times \mathcal{D} \tag{74}$$

Next, if $V_s(\cdot,\cdot)$ is a storage function for $\mathcal{G}$ then it follows from Theorem 5 that

$$0 \le V_a(t,x) \le V_s(t,x), \quad (t,x) \in \mathbb{R} \times \mathcal{D} \tag{75}$$

Furthermore, for all $T \in \mathbb{R}$ such that $x(T) = 0$ it follows that

$$V_s(t_0,x_0) \le V_s(T,0) + \int_T^{t_0} r_c(u_c(t), y_c(t))\,dt + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \tag{76}$$

and hence

$$\begin{aligned} V_s(t_0,x_0) &\le \inf_{(u_c(\cdot),u_d(\cdot)),\,T\le t_0} \left[ \int_T^{t_0} r_c(u_c(t), y_c(t))\,dt + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \right] \\ &= V_r(t_0,x_0) < \infty \end{aligned} \tag{77}$$

which implies (70). Finally, the proof for the exponentially dissipative case follows a similar construction and hence is omitted. □

Finally, as a direct consequence of Theorems 5 and 8, we show that the set of all possible storage functions of an impulsive dynamical system forms a convex set. An identical result holds for exponential storage functions.

**Proposition 1:** *Consider the non-linear impulsive dynamical system $\mathcal{G}$ given by (1)–(4) with available storage $V_a(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$ and required supply $V_r(t,x)$, $(t,x) \in \mathbb{R} \times \mathcal{D}$ and assume that $\mathcal{G}$ is completely reachable. Then*

$$V_s(t,x) \triangleq \alpha V_a(t,x) + (1-\alpha) V_r(t,x), \quad \alpha \in [0,1] \tag{78}$$

*is a storage function for $\mathcal{G}$.*

**Proof:** The result is a direct consequence of the dissipation inequality (53) by noting that if $V_a(t,x)$ and $V_r(t,x)$ satisfy (53), then $V_s(t,x)$ satisfies (53). □

## 5. Extended Kalman–Yakubovich–Popov conditions for impulsive dynamical systems

In this section we show that dissipativeness of an impulsive dynamical system can be characterized in terms of the system functions $f_c(\cdot)$, $G_c(\cdot)$, $h_c(\cdot)$, $J_c(\cdot)$, $f_d(\cdot)$, $G_d(\cdot)$, $h_d(\cdot)$ and $J_d(\cdot)$. Here we concentrate on the theory for dissipative time-dependent impulsive dynamical systems. Since in the case of dissipative state-dependent impulsive dynamical systems it follows from Assumptions A1 and A2 that, for $\mathcal{S} = [0, \infty) \times \mathcal{Z}$, the resetting times are well defined and distinct for every trajectory of (23), (24), the theory of dissipative state-dependent impulsive dynamical systems closely parallels that of dissipative time-dependent impulsive dynamical systems and hence many of the results are similar. In the case where the results for dissipative state-dependent impulsive dynamical systems deviate markedly from their time-dependent counterparts, we present a thorough treatment of these results. For the results in this section we consider the special case of dissipative impulsive systems with quadratic supply rates and set $\mathcal{U}_c = \mathbb{R}^{m_c}$ and $\mathcal{U}_d = \mathbb{R}^{m_d}$. Specifically, let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c \times m_c}$, $R_c \in \mathbb{S}^{m_c}$, $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d \times m_d}$ and $R_d \in \mathbb{S}^{m_d}$ be given and assume $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c$ and $r_d(u_d, y_d) = y_d^T Q_d y_d + 2y_d^T S_d u_d + u_d^T R_d u_d$. For simplicity of exposition, in the remainder of the paper we assume that for time-dependent impulsive dynamical systems the storage functions do not depend explicitly on time. This corresponds to the case in which $\mathcal{G}$ is time-varying but the energy storage mechanism does not reflect this. However, this is not to say that system energy dissipation does not have a time-varying character. Furthermore, we assume that there exist functions $\kappa_c: \mathbb{R}^{l_c} \to \mathbb{R}^{m_c}$ and $\kappa_d: \mathbb{R}^{l_d} \to \mathbb{R}^{m_d}$ such that $\kappa_c(0) = 0$, $\kappa_d(0) = 0$, $r_c(\kappa_c(y_c), y_c) < 0$, $y_c \neq 0$ and $r_d(\kappa_d(y_d), y_d) < 0$, $y_d \neq 0$, so that the storage function $V_s(x)$, $x \in \mathbb{R}^n$, is positive definite and we assume that $V_s(x)$, $x \in \mathbb{R}^n$, is continuously differentiable.

**Theorem 9:** *Let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c \times m_c}$, $R_c \in \mathbb{S}^{m_c}$, $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d \times m_d}$ and $R_d \in \mathbb{S}^{m_d}$. If there exist functions $V_s: \mathbb{R}^n \to \mathbb{R}$, $\ell_c: \mathbb{R}^n \to \mathbb{R}^{p_c}$, $\ell_d: \mathbb{R}^n \to \mathbb{R}^{p_d}$, $\mathcal{W}_c: \mathbb{R}^n \to \mathbb{R}^{p_c \times m_c}$, $\mathcal{W}_d: \mathbb{R}^n \to \mathbb{R}^{p_d \times m_d}$, $P_{1u_d}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_d}$ and $P_{2u_d}: \mathbb{R}^n \to \mathbb{N}^{m_d}$ such that $V_s(\cdot)$ is continuously differentiable, positive definite, $V_s(0) = 0$*

$$V_s(x + f_d(x) + G_d(x)u_d) = V_s(x + f_d(x)) + P_{1u_d}(x)u_d + u_d^T P_{2u_d}(x)u_d, \quad x \in \mathbb{R}^n, \quad u_d \in \mathbb{R}^{m_d} \tag{79}$$

*and, for all $x \in \mathbb{R}^n$*

$$0 = V_s'(x)f_c(x) - h_c^T(x)Q_c h_c(x) + \ell_c^T(x)\ell_c(x) \tag{80}$$

$$0 = \tfrac{1}{2}V_s'(x)G_c(x) - h_c^T(x) \times (Q_c J_c(x) + S_c) + \ell_c^T(x)\mathcal{W}_c(x) \tag{81}$$

$$0 = R_c + S_c^T J_c(x) + J_c^T(x)S_c + J_c^T(x)Q_c J_c(x) - \mathcal{W}_c^T(x)\mathcal{W}_c(x) \tag{82}$$

$$0 = V_s(x + f_d(x)) - V_s(x) - h_d^T(x)Q_d h_d(x) + \ell_d^T(x)\ell_d(x) \tag{83}$$

$$0 = \tfrac{1}{2}P_{1u_d}(x) - h_d^T(x)(Q_d J_d(x) + S_d) + \ell_d^T(x)\mathcal{W}_d(x) \tag{84}$$

$$0 = R_d + S_d^T J_d(x) + J_d^T(x)S_d + J_d^T(x)Q_d J_d(x) - P_{2u_d}(x) - \mathcal{W}_d^T(x)\mathcal{W}_d(x) \tag{85}$$

*then the non-linear impulsive system $\mathcal{G}$ given by (10)–(13) is dissipative with respect to the quadratic supply rate*

$$(r_c(u_c, y_c), r_d(u_d, y_d)) = (y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c,\ y_d^T Q_d y_d + 2y_d^T S_d u_d + u_d^T R_d u_d)$$

*If, alternatively*

$$\mathcal{N}_c(x) \triangleq R_c + S_c^T J_c(x) + J_c^T(x)S_c + J_c^T(x)Q_c J_c(x) > 0, \qquad x \in \mathbb{R}^n \tag{86}$$

*and there exist a continuously differentiable function $V_s: \mathbb{R}^n \to \mathbb{R}$ and matrix functions $P_{1u_d}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_d}$ and $P_{2u_d}: \mathbb{R}^n \to \mathbb{N}^{m_d}$ such that $V_s(\cdot)$ is positive definite, $V_s(0) = 0$, (79) holds, and for all $x \in \mathbb{R}^n$*

$$\mathcal{N}_d(x) \triangleq R_d + S_d^T J_d(x) + J_d^T(x)S_d + J_d^T(x)Q_d J_d(x) - P_{2u_d}(x) > 0 \tag{87}$$

$$0 \geq V_s'(x)f_c(x) - h_c^T(x)Q_c h_c(x) + [\tfrac{1}{2}V_s'(x)G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c)] \times \mathcal{N}_c^{-1}(x)[\tfrac{1}{2}V_s'(x)G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c)]^T \tag{88}$$

$$0 \geq V_s(x + f_d(x)) - V_s(x) - h_d^T(x)Q_d h_d(x) + [\tfrac{1}{2}P_{1u_d}(x) - h_d^T(x)(Q_d J_d(x) + S_d)] \times \mathcal{N}_d^{-1}(x)[\tfrac{1}{2}P_{1u_d}(x) - h_d^T(x)(Q_d J_d(x) + S_d)]^T \tag{89}$$

*then $\mathcal{G}$ is dissipative with respect to the quadratic supply rate*

$$(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (y_{\rm c}^{\rm T} Q_{\rm c} y_{\rm c} + 2y_{\rm c}^{\rm T} S_{\rm c} u_{\rm c} + u_{\rm c}^{\rm T} R_{\rm c} u_{\rm c},\ y_{\rm d}^{\rm T} Q_{\rm d} y_{\rm d} + 2y_{\rm d}^{\rm T} S_{\rm d} u_{\rm d} + u_{\rm d}^{\rm T} R_{\rm d} u_{\rm d})$$

**Proof:** For any admissible input $u_{\rm c}(\cdot)$, $t$, $\hat{t} \in \mathbb{R}$, $t_k < t \le \hat{t} \le t_{k+1}$ and $k \in \mathcal{N}$, it follows from (80)–(82) that

$$\begin{aligned}
V_{\rm s}(x(\hat{t})) - V_{\rm s}(x(t)) &= \int_t^{\hat{t}} \dot{V}_{\rm s}(x(s))\,{\rm d}s \\
&\le \int_t^{\hat{t}} \big[\dot{V}_{\rm s}(x(s)) + [\ell_{\rm c}(x(s)) + \mathcal{W}_{\rm c}(x(s))u_{\rm c}(s)]^{\rm T}[\ell_{\rm c}(x(s)) + \mathcal{W}_{\rm c}(x(s))u_{\rm c}(s)]\big]\,{\rm d}s \\
&= \int_t^{\hat{t}} [V_{\rm s}'(x(s))(f_{\rm c}(x(s)) + G_{\rm c}(x(s))u_{\rm c}(s)) + \ell_{\rm c}^{\rm T}(x(s))\ell_{\rm c}(x(s)) + 2\ell_{\rm c}^{\rm T}(x(s))\mathcal{W}_{\rm c}(x(s))u_{\rm c}(s) + u_{\rm c}^{\rm T}(s)\mathcal{W}_{\rm c}^{\rm T}(x(s))\mathcal{W}_{\rm c}(x(s))u_{\rm c}(s)]\,{\rm d}s \\
&= \int_t^{\hat{t}} [h_{\rm c}^{\rm T}(x(s))Q_{\rm c}h_{\rm c}(x(s)) + 2h_{\rm c}^{\rm T}(x(s))(S_{\rm c} + Q_{\rm c}J_{\rm c}(x(s)))u_{\rm c}(s) + u_{\rm c}^{\rm T}(s)(J_{\rm c}^{\rm T}(x(s))Q_{\rm c}J_{\rm c}(x(s)) + S_{\rm c}^{\rm T}J_{\rm c}(x(s)) + J_{\rm c}^{\rm T}(x(s))S_{\rm c} + R_{\rm c})u_{\rm c}(s)]\,{\rm d}s \\
&= \int_t^{\hat{t}} [y_{\rm c}^{\rm T}(s)Q_{\rm c}y_{\rm c}(s) + 2y_{\rm c}^{\rm T}(s)S_{\rm c}u_{\rm c}(s) + u_{\rm c}^{\rm T}(s)R_{\rm c}u_{\rm c}(s)]\,{\rm d}s \\
&= \int_t^{\hat{t}} r_{\rm c}(u_{\rm c}(s), y_{\rm c}(s))\,{\rm d}s
\end{aligned} \tag{90}$$

where $x(t)$, $t \in (t_k, t_{k+1}]$, satisfies (10) and $\dot{V}_{\rm s}(\cdot)$ denotes the total derivative of the storage function along the trajectories $x(t)$, $t \in (t_k, t_{k+1}]$, of (10). Next, for any admissible input $u_{\rm d}(t_k)$, $t_k \in \mathbb{R}$ and $k \in \mathcal{N}$, it follows that

$$\Delta V_{\rm s}(x(t_k)) = V_{\rm s}(x(t_k) + f_{\rm d}(x(t_k)) + G_{\rm d}(x(t_k))u_{\rm d}(t_k)) - V_{\rm s}(x(t_k)) \tag{91}$$

where $\Delta V_{\rm s}(\cdot)$ denotes the difference of the storage function at the resetting times $t_k$, $k \in \mathcal{N}$, of (11). Hence, it follows from (83)–(85), the structural storage function constraint (79), and (91), that for all $x \in \mathbb{R}^n$ and $u_{\rm d} \in \mathbb{R}^{m_{\rm d}}$

$$\begin{aligned}
\Delta V_{\rm s}(x) &= V_{\rm s}(x + f_{\rm d}(x) + G_{\rm d}(x)u_{\rm d}) - V_{\rm s}(x) \\
&= V_{\rm s}(x + f_{\rm d}(x)) - V_{\rm s}(x) + P_{1u_{\rm d}}(x)u_{\rm d} + u_{\rm d}^{\rm T}P_{2u_{\rm d}}(x)u_{\rm d} \\
&= h_{\rm d}^{\rm T}(x)Q_{\rm d}h_{\rm d}(x) - \ell_{\rm d}^{\rm T}(x)\ell_{\rm d}(x) + 2[h_{\rm d}^{\rm T}(x)(Q_{\rm d}J_{\rm d}(x) + S_{\rm d}) - \ell_{\rm d}^{\rm T}(x)\mathcal{W}_{\rm d}(x)]u_{\rm d} \\
&\quad + u_{\rm d}^{\rm T}[R_{\rm d} + S_{\rm d}^{\rm T}J_{\rm d}(x) + J_{\rm d}^{\rm T}(x)S_{\rm d} + J_{\rm d}^{\rm T}(x)Q_{\rm d}J_{\rm d}(x) - \mathcal{W}_{\rm d}^{\rm T}(x)\mathcal{W}_{\rm d}(x)]u_{\rm d} \\
&= r_{\rm d}(u_{\rm d}, y_{\rm d}) - [\ell_{\rm d}(x) + \mathcal{W}_{\rm d}(x)u_{\rm d}]^{\rm T} \times [\ell_{\rm d}(x) + \mathcal{W}_{\rm d}(x)u_{\rm d}] \\
&\le r_{\rm d}(u_{\rm d}, y_{\rm d})
\end{aligned} \tag{92}$$

Now, using (90) and (92) the result is immediate from Theorem 6.

To show (88), (89) imply that $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $(r_{\rm c}, r_{\rm d})$ note that (80)–(85) can be equivalently written as

$$\begin{bmatrix} \mathcal{A}_{\rm c}(x) & \mathcal{B}_{\rm c}(x) \\ \mathcal{B}_{\rm c}^{\rm T}(x) & \mathcal{C}_{\rm c}(x) \end{bmatrix} = -\begin{bmatrix} \ell_{\rm c}^{\rm T}(x) \\ \mathcal{W}_{\rm c}^{\rm T}(x) \end{bmatrix} [\, \ell_{\rm c}(x) \quad \mathcal{W}_{\rm c}(x) \,] \le 0, \qquad x \in \mathbb{R}^n \tag{93}$$

$$\begin{bmatrix} \mathcal{A}_{\rm d}(x) & \mathcal{B}_{\rm d}(x) \\ \mathcal{B}_{\rm d}^{\rm T}(x) & \mathcal{C}_{\rm d}(x) \end{bmatrix} = -\begin{bmatrix} \ell_{\rm d}^{\rm T}(x) \\ \mathcal{W}_{\rm d}^{\rm T}(x) \end{bmatrix} [\, \ell_{\rm d}(x) \quad \mathcal{W}_{\rm d}(x) \,] \le 0, \qquad x \in \mathbb{R}^n \tag{94}$$

where

$$\mathcal{A}_{\rm c}(x) \triangleq V_{\rm s}'(x)f_{\rm c}(x) - h_{\rm c}^{\rm T}(x)Q_{\rm c}h_{\rm c}(x)$$

$$\mathcal{B}_{\rm c}(x) \triangleq \tfrac{1}{2}V_{\rm s}'(x)G_{\rm c}(x) - h_{\rm c}^{\rm T}(x)(Q_{\rm c}J_{\rm c}(x) + S_{\rm c})$$

$$\mathcal{C}_{\rm c}(x) \triangleq -(R_{\rm c} + S_{\rm c}^{\rm T}J_{\rm c}(x) + J_{\rm c}^{\rm T}(x)S_{\rm c} + J_{\rm c}^{\rm T}(x)Q_{\rm c}J_{\rm c}(x))$$

$$\mathcal{A}_{\rm d}(x) \triangleq V_{\rm s}(x + f_{\rm d}(x)) - V_{\rm s}(x) - h_{\rm d}^{\rm T}(x)Q_{\rm d}h_{\rm d}(x)$$

$$\mathcal{B}_{\rm d}(x) \triangleq \tfrac{1}{2}P_{1u_{\rm d}}(x) - h_{\rm d}^{\rm T}(x)(Q_{\rm d}J_{\rm d}(x) + S_{\rm d})$$

and

$$\mathcal{C}_{\rm d}(x) \triangleq -(R_{\rm d} + S_{\rm d}^{\rm T}J_{\rm d}(x) + J_{\rm d}^{\rm T}(x)S_{\rm d} + J_{\rm d}^{\rm T}(x)Q_{\rm d}J_{\rm d}(x) - P_{2u_{\rm d}}(x))$$

Now, for all invertible $\mathcal{T}_{\rm c} \in \mathbb{R}^{(m_{\rm c}+1)\times(m_{\rm c}+1)}$ and $\mathcal{T}_{\rm d} \in \mathbb{R}^{(m_{\rm d}+1)\times(m_{\rm d}+1)}$ (93) and (94) hold if and only if $\mathcal{T}_{\rm c}^{\rm T}$ (93) $\mathcal{T}_{\rm c}$ and $\mathcal{T}_{\rm d}^{\rm T}$ (94) $\mathcal{T}_{\rm d}$ hold. Hence, the equivalence of (80)–(85) to (88) and (89) in the case when (86) and (87) hold follows from the (1, 1) block of $\mathcal{T}_{\rm c}^{\rm T}(93)\mathcal{T}_{\rm c}$, where

$$\mathcal{T}_{\mathrm{c}} \triangleq \begin{bmatrix} 1 & 0 \\ -\mathcal{C}_{\mathrm{c}}^{-1}(x)\mathcal{B}_{\mathrm{c}}^{\mathrm{T}}(x) & I_{m_{\mathrm{c}}} \end{bmatrix}$$

and $(1,1)$ block of $\mathcal{T}_{\mathrm{d}}^{\mathrm{T}}(94)\,\mathcal{T}_{\mathrm{d}}$, where

$$\mathcal{T}_{\mathrm{d}} \triangleq \begin{bmatrix} 1 & 0 \\ -\mathcal{C}_{\mathrm{d}}^{-1}(x)\mathcal{B}_{\mathrm{d}}^{\mathrm{T}}(x) & I_{m_{\mathrm{d}}} \end{bmatrix} \qquad \square$$

**Remark 13:** Note that Theorem 9 also holds for dissipative state-dependent impulsive dynamical systems. In this case, however, $x \in \mathbb{R}^n$ is replaced with $x \notin \mathcal{Z}_x$ for (80)–(82) and $x \in \mathcal{Z}_x$ for (79) and (83)–(85). Similar remarks hold for the remainder of the theorems in this section.

**Remark 14:** The structural constraint (79) on the system storage function is similar to the structural constraint invoked in standard discrete-time non-linear passivity theory (Byrnes *et al.* 1993, Byrnes and Lin 1994, Lin and Byrnes 1994, 1995, Chellaboina and Haddad 1998). This of course is not surprising since impulsive dynamical systems involve a hybrid formulation of continuous-time and discrete-time dynamics. In the case where $u_{\mathrm{d}} = 0$, or $\mathcal{G}$ is lossless with respect to a quadratic supply rate, or $\mathcal{G}$ is dissipative with respect to a quadratic supply rate of the form $(r_{\mathrm{c}}, 0)$, Condition (79) is necessary and sufficient (see Theorems 10 and 11 below) and hence is automatically satisfied. Similarly, in the case where $\mathcal{G}$ is linear and dissipative with respect to a quadratic supply rate, Condition (79) is also necessary and sufficient (see Theorem 14 below). In general, however, it is extremely difficult, if not impossible, to obtain (algebraic) sufficient conditions for dissipativity with respect to quadratic supply rates for impulsive dynamical systems without the structural constraint (79). Similar remarks hold for discrete-time non-linear systems (see Byrnes *et al.* 1993, Byrnes and Lin 1994, Lin and Byrnes 1994, 1995, Chellaboina and Haddad 1998 for further details).

**Remark 15:** Note that it follows from (66) that if the conditions in Theorem 9 are satisfied with (80) replaced by

$$0 = V_{\mathrm{s}}'(x)f_{\mathrm{c}}(x) + \varepsilon V_{\mathrm{s}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)Q_{\mathrm{c}}h_{\mathrm{c}}(x) + \ell_{\mathrm{c}}^{\mathrm{T}}(x)\ell_{\mathrm{c}}(x), \qquad x \in \mathbb{R}^n \quad (95)$$

where $\varepsilon > 0$, then the non-linear impulsive dynamical system $\mathcal{G}$ is exponentially dissipative. Similar remarks hold for Corollaries 3 and 4 below.

Using (80)–(85) it follows that, for $\hat{t} \geq t \geq 0$ and $k \in \mathcal{N}_{[t,\hat{t})}$

$$\begin{aligned} \int_t^{\hat{t}} r_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))\,\mathrm{d}s + \sum_{k\in\mathcal{N}_{[t,\hat{t})}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) &= V_{\mathrm{s}}(x(\hat{t})) - V_{\mathrm{s}}(x(t)) \\ &\quad + \int_t^{\hat{t}} [\ell_{\mathrm{c}}(x(s)) + \mathcal{W}_{\mathrm{c}}(x(s))u_{\mathrm{c}}(s)]^{\mathrm{T}} \\ &\quad \times [\ell_{\mathrm{c}}(x(s)) + \mathcal{W}_{\mathrm{c}}(x(s))u_{\mathrm{c}}(s)]\,\mathrm{d}s \\ &\quad + \sum_{k\in\mathcal{N}_{[t,\hat{t})}} [\ell_{\mathrm{d}}(x(t_k)) + \mathcal{W}_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)]^{\mathrm{T}} \\ &\quad \times [\ell_{\mathrm{d}}(x(t_k)) + \mathcal{W}_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)] \end{aligned} \quad (96)$$

which can be interpreted as a *generalized energy* balance equation where $V_{\mathrm{s}}(x(\hat{t})) - V_{\mathrm{s}}(x(t))$ is the stored or accumulated generalized energy of the impulsive dynamical system, the second path-dependent term on the right corresponds to the dissipated energy of the impulsive dynamical system over the continuous-time dynamics and the third discrete term on the right corresponds to the dissipated energy at the resetting instants. Equivalently, it follows from Theorem 6 that (96) can be rewritten as

$$\begin{aligned} \dot{V}_{\mathrm{s}}(x(t)) &= r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \\ &\quad - [\ell_{\mathrm{c}}(x(t)) + \mathcal{W}_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t)]^{\mathrm{T}} \\ &\quad \times [\ell_{\mathrm{c}}(x(t)) + \mathcal{W}_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t)], \qquad t_k < t \leq t_{k+1} \end{aligned} \quad (97)$$

$$\begin{aligned} \Delta V_{\mathrm{s}}(x(t_k)) &= r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \\ &\quad - [\ell_{\mathrm{d}}(x(t_k)) + \mathcal{W}_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)]^{\mathrm{T}} \\ &\quad \times [\ell_{\mathrm{d}}(x(t_k)) + \mathcal{W}_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)], \qquad k \in \mathcal{N} \end{aligned} \quad (98)$$

which yields a set of generalized energy conservation equations. Specifically, (97) shows that the rate of change in generalized energy, or generalized power, over the time interval $t \in (t_k, t_{k+1}]$ is equal to the generalized system power input minus the internal generalized system power dissipated; while (98) shows that the change of energy at the resetting times $t_k$, $k \in \mathcal{N}$, is equal to the external generalized system energy at the resetting times minus the generalized dissipated energy at the resetting times.

**Remark 16:** Note that if $\mathcal{G}$ with $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0,0)$ and a continuously differentiable, positive-definite, radially unbounded storage function is dissipative with respect to a quadratic supply rate where $Q_{\mathrm{c}} \leq 0$, $Q_{\mathrm{d}} \leq 0$, it follows that $\dot{V}_{\mathrm{s}}(x(t)) \leq y_{\mathrm{c}}^{\mathrm{T}}(t)Q_{\mathrm{c}}y_{\mathrm{c}}(t) \leq 0$, $t \geq 0$ and $\Delta V_{\mathrm{s}}(x(t_k)) \leq y_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}y_{\mathrm{d}}(t_k) \leq 0$, $k \in \mathcal{N}$. Hence, the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0,0))$ non-linear impulsive dynamical system (10)–(13) is Lyapunov stable. Alternatively, if $\mathcal{G}$ with $(u_c(t), u_d(t_k)) \equiv (0,0)$ and a continuously differentiable positive-definite, radially unbounded storage function is exponentially dissipative and $Q_c \le 0$, $Q_d \le 0$, it follows that $\dot{V}_s(x(t)) \le -\varepsilon V_s(x(t)) + y_c^T(t) Q_c y_c(t) \le -\varepsilon V_s(x(t))$, $t \ge 0$ and $\Delta V_s(x(t_k)) \le y_d^T(t_k) Q_d y_d(t_k) \le 0$, $k \in \mathcal{N}$. Hence, the undisturbed non-linear impulsive dynamical system (10)–(13) is asymptotically stable. If, in addition, there exist constants $\alpha, \beta > 0$ and $p \ge 1$ such that $\alpha \|x\|^p \le V_s(x) \le \beta \|x\|^p$, $x \in \mathbb{R}^n$, then the undisturbed non-linear impulsive dynamical system (10)–(13) is exponentially stable.

Next, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (10)–(13) is lossless with respect to a quadratic supply rate $(r_c, r_d)$.

**Theorem 10:** *Let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c \times m_c}$, $R_c \in \mathbb{S}^{m_c}$, $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d \times m_d}$ and $R_d \in \mathbb{S}^{m_d}$. Then the non-linear impulsive system $\mathcal{G}$ given by (10)–(13) is lossless with respect to the quadratic supply rate*

$$(r_c(u_c, y_c), r_d(u_d, y_d)) = (y_c^T Q_c y_c + 2 y_c^T S_c u_c + u_c^T R_c u_c,\; y_d^T Q_d y_d + 2 y_d^T S_d u_d + u_d^T R_d u_d)$$

*if and only if there exist functions $V_s: \mathbb{R}^n \to \mathbb{R}$, $P_{1u_d}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_d}$ and $P_{2u_d}: \mathbb{R}^n \to \mathbb{N}^{m_d}$ such that $V_s(\cdot)$ is continuously differentiable and positive definite, $V_s(0) = 0$ and, for all $x \in \mathbb{R}^n$, (79) holds and*

$$0 = V_s'(x) f_c(x) - h_c^T(x) Q_c h_c(x) \tag{99}$$

$$0 = \tfrac{1}{2} V_s'(x) G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c) \tag{100}$$

$$0 = R_c + S_c^T J_c(x) + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x) \tag{101}$$

$$0 = V_s(x + f_d(x)) - V_s(x) - h_d^T(x) Q_d h_d(x) \tag{102}$$

$$0 = \tfrac{1}{2} P_{1u_d}(x) - h_d^T(x)(Q_d J_d(x) + S_d) \tag{103}$$

$$0 = R_d + S_d^T J_d(x) + J_d^T(x) S_d + J_d^T(x) Q_d J_d(x) - P_{2u_d}(x) \tag{104}$$

*If, in addition, $V_s(\cdot)$ is two-times continuously differentiable, then*

$$P_{1u_d}(x) = V_s'(x + f_d(x)) G_d(x)$$
$$P_{2u_d}(x) = \tfrac{1}{2} G_d^T(x) V_s(x + f_d(x)) G_d(x)$$

**Proof:** Sufficiency follows as in the proof of Theorem 9. To show necessity, suppose that the non-linear impulsive dynamical system $\mathcal{G}$ is lossless with respect to the quadratic supply rate $(r_c, r_d)$. Then, it follows from Theorem 6 that for all $k \in \mathcal{N}$

$$V_s(x(\hat{t})) - V_s(x(t)) = \int_t^{\hat{t}} r_c(u_c(s), y_c(s))\, ds, \quad t_k < t \le \hat{t} \le t_{k+1} \tag{105}$$

and

$$V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k)) u_d(t_k)) = V_s(x(t_k)) + r_d(u_d(t_k), y_d(t_k)) \tag{106}$$

Now, dividing (105) by $\hat{t} - t^+$ and letting $\hat{t} \to t^+$, (105) is equivalent to

$$\dot{V}_s(x(t)) = V_s'(x(t))[f_c(x(t)) + G_c(x(t)) u_c(t)] = r_c(u_c(t), y_c(t)), \quad t_k < t \le t_{k+1} \tag{107}$$

Next, with $t = 0$, it follows from (107) that

$$V_s'(x_0)[f_c(x_0) + G_c(x_0) u_c(0)] = r_c(u_c(0), y_c(0)), \quad x_0 \in \mathbb{R}^n, \quad u_c(0) \in \mathbb{R}^{m_c} \tag{108}$$

Since $x_0 \in \mathbb{R}^n$ is arbitrary, it follows that

$$\begin{aligned} V_s'(x)[f_c(x) + G_c(x) u_c] &= y_c^T Q_c y_c + 2 y_c^T S_c u_c + u_c^T R_c u_c \\ &= h_c^T(x) Q_c h_c(x) + 2 h_c^T(x)(Q_c J_c(x) \\ &\quad + S_c) u_c + u_c^T (R_c + S_c^T J_c(x) \\ &\quad + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x)) u_c, \\ &\qquad x \in \mathbb{R}^n, \quad u_c \in \mathbb{R}^{m_c} \end{aligned}$$

Now, equating coefficients of equal powers yields (99)–(101). Next, it follows from (106) with $k = 1$ that

$$V_s(x(t_1) + f_d(x(t_1)) + G_d(x(t_1)) u_d(t_1)) = V_s(x(t_1)) + r_d(u_d(t_1), y_d(t_1)) \tag{109}$$

Now, since the continuous-time dynamics (10) are Lipschitz, it follows that for arbitrary $x \in \mathbb{R}^n$ there exists $x_0 \in \mathbb{R}^n$ such that $x(t_1) = x$. Hence, it follows from (109) that

$$\begin{aligned} V_s(x + f_d(x) + G_d(x) u_d) &= V_s(x) + y_d^T Q_d y_d + 2 y_d^T S_d u_d + u_d^T R_d u_d \\ &= V_s(x) + h_d^T(x) Q_d h_d(x) + 2 h_d^T(x)(Q_d J_d(x) \\ &\quad + S_d) u_d + u_d^T (R_d + S_d^T J_d(x) + J_d^T(x) S_d + J_d^T(x) \\ &\quad \times Q_d J_d(x)) u_d, \qquad x \in \mathbb{R}^n, \quad u_d \in \mathbb{R}^{m_d} \end{aligned} \tag{110}$$

Since the right-hand-side of (110) is quadratic in $u_d$ it follows that $V_s(x + f_d(x) + G_d(x) u_d)$ is quadratic in $u_d$ and hence there exists $P_{1u_d}: \mathbb{R}^n \to \mathbb{R}^{1 \times m_d}$ and $P_{2u_d}: \mathbb{R}^n \to \mathbb{N}^{m_d}$ such that

$$V_s(x + f_d(x) + G_d(x) u_d) = V_s(x + f_d(x)) + P_{1u_d}(x) u_d + u_d^T P_{2u_d}(x) u_d \tag{111}$$

Now, using (111) and equating coefficients of equal powers in (110) yields (102)–(104). Finally, if $V_s(\cdot)$ is two-times continuously differentiable, applying a Taylor series expansion on (111) about $u_{\rm d}=0$ yields

$$P_{1u_{\rm d}}(x)=\left.\frac{\partial V_{\rm s}(x+f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d})}{\partial u_{\rm d}}\right|_{u_{\rm d}=0}$$

$$=V_{\rm s}'(x+f_{\rm d}(x))G_{\rm d}(x) \qquad (112)$$

$$P_{2u_{\rm d}}(x)=\left.\frac{\partial^2 V_{\rm s}(x+f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d})}{\partial {u_{\rm d}}^2}\right|_{u_{\rm d}=0}$$

$$=\tfrac{1}{2}G_{\rm d}^{\rm T}(x)V_{\rm s}(x+f_{\rm d}(x))G_{\rm d}(x) \qquad (113)$$

□

Next, we provide two definitions of non-linear impulsive dynamical systems which are dissipative (resp., exponentially dissipative) with respect to supply rates of a specific form.

**Definition 7:** A system $\mathcal{G}$ of the form (1)–(4) with $m_{\rm c}=l_{\rm c}$ and $m_{\rm d}=l_{\rm d}$ is *passive* (resp., *exponentially passive)* if $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\rm c}(u_{\rm c},y_{\rm c}),r_{\rm d}(u_{\rm d},y_{\rm d}))=(2u_{\rm c}^{\rm T}y_{\rm c},2u_{\rm d}^{\rm T}y_{\rm d})$.

**Definition 8:** A system $\mathcal{G}$ of the form (1)–(4) is *non-expansive* (resp., *exponentially non-expansive*) if $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_{\rm c}(u_{\rm c},y_{\rm c}),r_{\rm d}(u_{\rm d},y_{\rm d}))=(\gamma_{\rm c}^2u_{\rm c}^{\rm T}u_{\rm c}-y_{\rm c}^{\rm T}y_{\rm c},\gamma_{\rm d}^2u_{\rm d}^{\rm T}u_{\rm d}-y_{\rm d}^{\rm T}y_{\rm d})$, where $\gamma_{\rm c},\gamma_{\rm d}>0$ are given.

**Remark 17:** Note that a mixed passive-non-expansive formulation of $\mathcal{G}$ can also be considered. Specifically, one can consider impulsive dynamical systems $\mathcal{G}$ which are dissipative with respect to supply rates of the form $(r_{\rm c}(u_{\rm c},y_{\rm c}),r_{\rm d}(u_{\rm d},y_{\rm d}))=(2u_{\rm c}^{\rm T}y_{\rm c},\gamma_{\rm d}^2u_{\rm d}^{\rm T}u_{\rm d}-y_{\rm d}^{\rm T}y_{\rm d})$, where $\gamma_{\rm d}>0$ and *vice versa*. Furthermore, supply rates for input strict passivity (Hill and Moylan 1977), output strict passivity (Hill and Moylan 1977) and input–output strict passivity (Hill and Moylan 1977) can also be considered. However, for simplicity of exposition we do not do so here.

The following results present the non-linear versions of the Kalman–Yakubovich–Popov positive real lemma and the bounded real lemma for non-linear impulsive systems $\mathcal{G}$ of the form (10)–(13).

**Corollary 3:** *Consider the non-linear impulsive system $\mathcal{G}$ given by (10)–(13). If there exist functions $V_{\rm s}:\mathbb{R}^n\to\mathbb{R}$, $\ell_{\rm c}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm c}}$, $\ell_{\rm d}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm d}}$, $\mathcal{W}_{\rm c}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm c}\times m_{\rm c}}$, $\mathcal{W}_{\rm d}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm d}\times m_{\rm d}}$, $P_{1u_{\rm d}}:\mathbb{R}^n\to\mathbb{R}^{1\times m_{\rm d}}$ and $P_{2u_{\rm d}}:\mathbb{R}^n\to\mathbb{N}^{m_{\rm d}}$ such that $V_{\rm s}(\cdot)$ is continuously differentiable and positive definite, $V_{\rm s}(0)=0$*

$$V_{\rm s}(x+f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d})=V_{\rm s}(x+f_{\rm d}(x))+P_{1u_{\rm d}}(x)u_{\rm d}+u_{\rm d}^{\rm T}P_{2u_{\rm d}}(x)u_{\rm d},\quad x\in\mathbb{R}^n,\ u_{\rm d}\in\mathbb{R}^{m_{\rm d}} \qquad (114)$$

*and, for all $x\in\mathbb{R}^n$*

$$0=V_{\rm s}'(x)f_{\rm c}(x)+\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) \qquad (115)$$

$$0=\tfrac{1}{2}V_{\rm s}'(x)G_{\rm c}(x)-h_{\rm c}^{\rm T}(x)+\ell_{\rm c}^{\rm T}(x)\mathcal{W}_{\rm c}(x) \qquad (116)$$

$$0=J_{\rm c}(x)+J_{\rm c}^{\rm T}(x)-\mathcal{W}_{\rm c}^{\rm T}(x)\mathcal{W}_{\rm c}(x) \qquad (117)$$

$$0=V_{\rm s}(x+f_{\rm d}(x))-V_{\rm s}(x)+\ell_{\rm d}^{\rm T}(x)\ell_{\rm d}(x) \qquad (118)$$

$$0=\tfrac{1}{2}P_{1u_{\rm d}}(x)-h_{\rm d}^{\rm T}(x)+\ell_{\rm d}^{\rm T}(x)\mathcal{W}_{\rm d}(x) \qquad (119)$$

$$0=J_{\rm d}(x)+J_{\rm d}^{\rm T}(x)-P_{2u_{\rm d}}(x)-\mathcal{W}_{\rm d}^{\rm T}(x)\mathcal{W}_{\rm d}(x) \qquad (120)$$

*then $\mathcal{G}$ is passive. If, alternatively, $J_{\rm c}(x)+J_{\rm c}^{\rm T}(x)>0$, $x\in\mathbb{R}^n$ and there exist a continuously differentiable function $V_{\rm s}:\mathbb{R}^n\to\mathbb{R}$ and matrix functions $P_{1u_{\rm d}}:\mathbb{R}^n\to\mathbb{R}^{1\times m_{\rm d}}$ and $P_{2u_{\rm d}}:\mathbb{R}^n\to\mathbb{N}^{m_{\rm d}}$ such that $V_{\rm s}(\cdot)$ is positive definite, $V_{\rm s}(0)=0$, (114) holds, and for all $x\in\mathbb{R}^n$*

$$0<J_{\rm d}(x)+J_{\rm d}^{\rm T}(x)-P_{2u_{\rm d}}(x) \qquad (121)$$

$$0\geq V_{\rm s}'(x)f_{\rm c}(x)+[\tfrac{1}{2}V_{\rm s}'(x)G_{\rm c}(x)-h_{\rm c}^{\rm T}(x)]\times[J_{\rm c}(x)+J_{\rm c}^{\rm T}(x)]^{-1}[\tfrac{1}{2}V_{\rm s}'(x)G_{\rm c}(x)-h_{\rm c}^{\rm T}(x)]^{\rm T} \qquad (122)$$

$$0\geq V_{\rm s}(x+f_{\rm d}(x))-V_{\rm s}(x)+[\tfrac{1}{2}P_{1u_{\rm d}}(x)-h_{\rm d}^{\rm T}(x)]\times[J_{\rm d}(x)+J_{\rm d}^{\rm T}(x)-P_{2u_{\rm d}}(x)]^{-1}[\tfrac{1}{2}P_{1u_{\rm d}}(x)-h_{\rm d}^{\rm T}(x)]^{\rm T} \qquad (123)$$

*then $\mathcal{G}$ is passive.*

**Proof:** The result is a direct consequence of Theorem 9 with $l_{\rm c}=m_{\rm c}$, $l_{\rm d}=m_{\rm d}$, $Q_{\rm c}=0$, $Q_{\rm d}=0$, $S_{\rm c}=I_{m_{\rm c}}$, $S_{\rm d}=I_{m_{\rm d}}$, $R_{\rm c}=0$ and $R_{\rm d}=0$. Specifically, with $\kappa_{\rm c}(y_{\rm c})=-y_{\rm c}$ and $\kappa_{\rm d}(y_{\rm d})=-y_{\rm d}$ it follows that $r_{\rm c}(\kappa_{\rm c}(y_{\rm c}),y_{\rm c})=-2y_{\rm c}^{\rm T}y_{\rm c}<0$, $y_{\rm c}\neq0$ and $r_{\rm d}(\kappa_{\rm d}(y_{\rm d}),y_{\rm d})=-2y_{\rm d}^{\rm T}y_{\rm d}<0$, $y_{\rm d}\neq0$, so that all of the assumptions of Theorem 9 are satisfied. □

**Corollary 4:** *Consider the non-linear impulsive system $\mathcal{G}$ given by (10)–(13). If there exist functions $V_{\rm s}:\mathbb{R}^n\to\mathbb{R}$, $\ell_{\rm c}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm c}}$, $\ell_{\rm d}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm d}}$, $\mathcal{W}_{\rm c}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm c}\times m_{\rm c}}$, $\mathcal{W}_{\rm d}:\mathbb{R}^n\to\mathbb{R}^{p_{\rm d}\times m_{\rm d}}$, $P_{1u_{\rm d}}:\mathbb{R}^n\to\mathbb{R}^{1\times m_{\rm d}}$ and $P_{2u_{\rm d}}:\mathbb{R}^n\to\mathbb{N}^{m_{\rm d}}$ such that $V_{\rm s}(\cdot)$ is continuously differentiable and positive definite, $V_{\rm s}(0)=0$*

$$V_{\rm s}(x+f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d})=V_{\rm s}(x+f_{\rm d}(x))+P_{1u_{\rm d}}(x)u_{\rm d}+u_{\rm d}^{\rm T}P_{2u_{\rm d}}(x)u_{\rm d},\quad x\in\mathbb{R}^n,\ u_{\rm d}\in\mathbb{R}^{m_{\rm d}} \qquad (124)$$

*and, for all $x\in\mathbb{R}^n$*

$$0 = V_s'(x)f_c(x) + h_c^T(x)h_c(x) + \ell_c^T(x)\ell_c(x) \quad (125)$$

$$0 = \tfrac{1}{2}V_s'(x)G_c(x) + h_c^T(x)J_c(x) + \ell_c^T(x)\mathcal{W}_c(x) \quad (126)$$

$$0 = \gamma_c^2 I_{m_c} - J_c^T(x)J_c(x) - \mathcal{W}_c^T(x)\mathcal{W}_c(x) \quad (127)$$

$$0 = V_s(x+f_d(x)) - V_s(x) + h_d^T(x)h_d(x) + \ell_d^T(x)\ell_d(x) \quad (128)$$

$$0 = \tfrac{1}{2}P_{1u_d}(x) + h_d^T(x)J_d(x) + \ell_d^T(x)\mathcal{W}_d(x) \quad (129)$$

$$0 = \gamma_d^2 I_{m_d} - J_d^T(x)J_d(x) - P_{2u_d}(x) - \mathcal{W}_d^T(x)\mathcal{W}_d(x) \quad (130)$$

*then $\mathcal{G}$ is non-expansive. If, alternatively, $\gamma_c^2 I_{m_c} - J_c^T(x)J_c(x) > 0$, $x \in \mathbb{R}^n$ and there exist a continuously differentiable function $V_s: \mathbb{R}^n \to \mathbb{R}$ and matrix functions $P_{1u_d}: \mathbb{R}^n \to \mathbb{R}^{1\times m_d}$ and $P_{2u_d}: \mathbb{R}^n \to \mathbb{N}^{m_d}$ such that $V_s(\cdot)$ is positive definite, $V_s(0) = 0$, (124) holds, and for all $x \in \mathbb{R}^n$*

$$0 < \gamma_d^2 I_{m_d} - J_d^T(x)J_d(x) - P_{2u_d}(x) \quad (131)$$

$$\begin{aligned} 0 \geq\ & V_s'(x)f_c(x) + h_c^T(x)h_c(x) \\ & + [\tfrac{1}{2}V_s'(x)G_c(x) + h_c^T(x)J_c(x)] \\ & \times [\gamma_c^2 I_{m_c} - J_c^T(x)J_c(x)]^{-1} \\ & \times [\tfrac{1}{2}V_s'(x)G_c(x) + h_c^T(x)J_c(x)]^T \end{aligned} \quad (132)$$

$$\begin{aligned} 0 \geq\ & V_s(x+f_d(x)) - V_s(x) + h_d^T(x)h_d(x) \\ & + [\tfrac{1}{2}P_{1u_d}(x) + h_d^T(x)J_d(x)] \\ & \times [\gamma_d^2 I_{m_d} - J_d^T(x)J_d(x) - P_{2u_d}(x)]^{-1} \\ & \times [\tfrac{1}{2}P_{1u_d}(x) + h_d^T(x)J_d(x)]^T \end{aligned} \quad (133)$$

*then $\mathcal{G}$ is non-expansive.*

**Proof:** The result is a direct consequence of Theorem 9 with $Q_c = -I_{l_c}$, $Q_d = -I_{l_d}$, $S_c = 0$, $S_d = 0$, $R_c = \gamma_c^2 I_{m_c}$ and $R_d = \gamma_d^2 I_{m_d}$. Specifically, with $\kappa_c(y_c) = -(1/2\gamma_c)y_c$ and $\kappa_d(y_d) = -(1/2\gamma_d)y_d$ it follows that $r_c(\kappa_c(y_c), y_c) = -\frac{3}{4}y_c^T y_c < 0$, $y_c \neq 0$ and $r_d(\kappa_d(y_d), y_d) = -\frac{3}{4}y_d^T y_d < 0$, $y_d \neq 0$, so that all of the assumptions of Theorem 9 are satisfied. □

Next, we provide necessary *and* sufficient conditions for dissipativity of a non-linear impulsive dynamical system $\mathcal{G}$ of the form (10)–(13) in the case where $r_d(u_d, y_d) \equiv 0$ and $G_d(x) \equiv 0$.

**Theorem 11:** *Let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c\times m_c}$ and $R_c \in \mathbb{S}^{m_c}$. Then the non-linear impulsive system $\mathcal{G}$ given by (10)–(13) with $G_d(x) \equiv 0$ is dissipative with respect to the supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c, 0)$ if and only if there exist functions $V_s: \mathbb{R}^n \to \mathbb{R}$, $\ell_c: \mathbb{R}^n \to \mathbb{R}^{p_c}$, $\ell_d: \mathbb{R}^n \to \mathbb{R}^{p_d}$ and $\mathcal{W}_c: \mathbb{R}^n \to \mathbb{R}^{p_c\times m_c}$ such that $V_s(\cdot)$ is continuously differentiable and positive definite, $V_s(0) = 0$ and for all $x \in \mathbb{R}^n$*

$$0 = V_s'(x)f_c(x) - h_c^T(x)Q_c h_c(x) + \ell_c^T(x)\ell_c(x) \quad (134)$$

$$\begin{aligned} 0 =\ & \tfrac{1}{2}V_s'(x)G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c) \\ & + \ell_c^T(x)\mathcal{W}_c(x) \end{aligned} \quad (135)$$

$$\begin{aligned} 0 =\ & R_c + S_c^T J_c(x) + J_c^T(x)S_c + J_c^T(x)Q_c J_c(x) \\ & - \mathcal{W}_c^T(x)\mathcal{W}_c(x) \end{aligned} \quad (136)$$

$$0 = V_s(x+f_d(x)) - V_s(x) + \ell_d^T(x)\ell_d(x) \quad (137)$$

**Proof:** Sufficiency follows from Theorem 9 with $Q_d = 0$, $S_d = 0$, $R_d = 0$, $G_d(x) = 0$, $P_{1u_d}(x) = 0$ and $P_{2u_d}(x) = 0$. Necessity follows from Theorem 6 using a similar construction as in the proof of Theorem 10. □

**Remark 18:** Note that in the case where $r_d(u_d, y_d) \equiv 0$ and $G_d(x) \equiv 0$, it follows from Theorem 11 that the non-linear impulsive system $\mathcal{G}$ given by (10)–(13) is passive (resp., non-expansive) if and only if there exist functions $V_s: \mathbb{R}^n \to \mathbb{R}$, $\ell_c: \mathbb{R}^n \to \mathbb{R}^{p_c}$, $\ell_d: \mathbb{R}^n \to \mathbb{R}^{p_d}$ and $\mathcal{W}_c: \mathbb{R}^n \to \mathbb{R}^{p_c\times m_c}$ such that $V_s(\cdot)$ is continuously differentiable and positive definite, $V_s(0) = 0$ and (115)–(117) and (137) (resp., (125)–(127) and (137)) are satisfied.

Finally, we present two key results on linearization of impulsive dynamical systems. For these results, we assume that there exist functions $\kappa_c: \mathbb{R}^{l_c} \to \mathbb{R}^{m_c}$ and $\kappa_d: \mathbb{R}^{l_d} \to \mathbb{R}^{m_d}$ such that $\kappa_c(0) = 0$, $\kappa_d(0) = 0$, $r_c(\kappa_c(y_c), y_c) < 0$, $y_c \neq 0$, $r_d(\kappa_d(y_d), y_d) < 0$, $y_d \neq 0$ and the available storage $V_a(x)$, $x \in \mathbb{R}^n$, is a three-times continuously differentiable function.

**Theorem 12:** *Let $Q_c \in \mathbb{S}^{l_c}$, $S_c \in \mathbb{R}^{l_c\times m_c}$, $R_c \in \mathbb{S}^{m_c}$, $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d\times m_d}$ and $R_d \in \mathbb{S}^{m_d}$ and suppose that the non-linear impulsive system $\mathcal{G}$ given by (10)–(13) is dissipative with respect to the quadratic supply rate*

$$\begin{aligned} (r_c(u_c, y_c), r_d(u_d, y_d)) = (& y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c, \\ & y_d^T Q_d y_d + 2y_d^T S_d u_d + u_d^T R_d u_d) \end{aligned}$$

*Then there exists matrices $P \in \mathbb{R}^{n\times n}$, $L_c \in \mathbb{R}^{p_c\times n}$, $W_c \in \mathbb{R}^{p_c\times m_c}$, $L_d \in \mathbb{R}^{p_d\times n}$ and $W_d \in \mathbb{R}^{p_d\times m_d}$, with $P$ non-negative definite, such that*

$$0 = A_c^T P + P A_c - C_c^T Q_c C_c + L_c^T L_c \quad (138)$$

$$0 = P B_c - C_c^T (Q_c D_c + S_c) + L_c^T W_c \quad (139)$$

$$0 = R_c + S_c^T D_c + D_c^T S_c + D_c^T Q_c D_c - W_c^T W_c \quad (140)$$

$$0 = A_d^T P A_d - P - C_d^T Q_d C_d + L_d^T L_d \quad (141)$$

$$0 = A_d^T P B_d - C_d^T (Q_d D_d + S_d) + L_d^T W_d \quad (142)$$

$$0 = R_d + S_d^T D_d + D_d^T S_d + D_d^T Q_d D_d - B_d^T P B_d - W_d^T W_d \quad (143)$$

*where*

$$\left.\begin{aligned} A_c &= \left.\frac{\partial f_c}{\partial x}\right|_{x=0}, \quad B_c = G_c(0) \\ C_c &= \left.\frac{\partial h_c}{\partial x}\right|_{x=0}, \quad D_c = J_c(0) \end{aligned}\right\} \quad (144)$$

$$\left.\begin{aligned} A_d &= \left.\frac{\partial f_d}{\partial x}\right|_{x=0} + I_n, \quad B_d = G_d(0) \\ C_d &= \left.\frac{\partial h_d}{\partial x}\right|_{x=0}, \quad D_d = J_d(0) \end{aligned}\right\} \quad (145)$$

*If, in addition,* $(A_c, C_c)$ *and* $(A_d, C_d)$ *are observable, then* $P > 0$.

**Proof:** First note that since $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ it follows from Theorem 6 that there exists a storage function $V_s: \mathbb{R}^n \to \mathbb{R}$ such that, for all $k \in \mathcal{N}$

$$V_s(x(\hat{t})) - V_s(x(t)) \le \int_t^{\hat{t}} r_c(u_c(s), y_c(s))\, ds, \quad t_k < t \le \hat{t} \le t_{k+1} \quad (146)$$

and

$$V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k)) u_d(t_k)) \le V_s(x(t_k)) + r_d(u_d(t_k), y_d(t_k)) \quad (147)$$

Now, dividing (146) by $\hat{t} - t^+$ and letting $\hat{t} \to t^+$, (146) is equivalent to

$$\begin{aligned} \dot{V}_s(x(t)) &= V_s'(x(t))[f_c(x(t)) + G_c(x(t)) u_c(t)] \\ &\le r_c(u_c(t), y_c(t)), \quad t_k < t \le t_{k+1} \end{aligned} \quad (148)$$

Next, with $t = 0$, it follows that

$$V_s'(x_0)[f_c(x_0) + G_c(x_0) u_c(0)] \le r_c(u_c(0), y_c(0)), \quad x_0 \in \mathbb{R}^n, \quad u_c(0) \in \mathbb{R}^{m_c} \quad (149)$$

Since $x_0 \in \mathbb{R}^n$ is arbitrary, it follows that

$$V_s'(x)[f_c(x) + G_c(x) u_c] \le r_c(u_c, h_c(x) + J_c(x) u_c), \quad x \in \mathbb{R}^n, \quad u_c \in \mathbb{R}^{m_c} \quad (150)$$

Furthermore, it follows from (147) with $k = 1$ that

$$V_s(x(t_1) + f_d(x(t_1)) + G_d(x(t_1)) u_d(t_1)) \le V_s(x(t_1)) + r_d(u_d(t_1), y_d(t_1)) \quad (151)$$

Now, since the continuous-time dynamics (10) are Lipschitz, it follows that for arbitrary $x \in \mathbb{R}^n$ there exists $x_0 \in \mathbb{R}^n$ such that $x(t_1) = x$. Hence, it follows from (151) that

$$V_s(x + f_d(x) + G_d(x) u_d) \le V_s(x) + r_d(u_d, h_d(x) + J_d(x) u_d), \quad x \in \mathbb{R}^n, \quad u_d \in \mathbb{R}^{m_d} \quad (152)$$

Next, it follows from (150) and (152) that there exists smooth functions $d_c: \mathbb{R}^n \times \mathbb{R}^{m_c} \to \mathbb{R}$ and $d_d: \mathbb{R}^n \times \mathbb{R}^{m_d} \to \mathbb{R}$ such that $d_c(x, u_c) \ge 0$, $d_c(0,0) = 0$, $d_d(x, u_d) \ge 0$, $d_d(0,0) = 0$ and

$$\begin{aligned} 0 = {} & V_s'(x)[f_c(x) + G_c(x) u_c] - r_c(u_c, h_c(x) \\ & + J_c(x) u_c) + d_c(x, u_c), \quad x \in \mathbb{R}^n, u_c \in \mathbb{R}^{m_c} \end{aligned} \quad (153)$$

$$\begin{aligned} 0 = {} & V_s(x + f_d(x) + G_d(x) u_d) - V_s(x) \\ & - r_d(u_d, h_d(x) + J_d(x) u_d) + d_d(x, u_d), \\ & x \in \mathbb{R}^n, u_d \in \mathbb{R}^{m_d} \end{aligned} \quad (154)$$

Now, expanding $V_s(\cdot)$, $d_c(\cdot,\cdot)$ and $d_d(\cdot,\cdot)$ via a Taylor series expansion about $x = 0$, $u_c = 0$, $u_d = 0$ and using the fact that $V_s(\cdot)$, $d_c(\cdot,\cdot)$ and $d_d(\cdot,\cdot)$ are non-negative definite and $V_s(0) = 0$, $d_c(0,0) = 0$ and $d_d(0,0) = 0$, it follows that there exist matrices $P \in \mathbb{R}^{n\times n}$, $L_c \in \mathbb{R}^{p_c \times n}$, $W_c \in \mathbb{R}^{p_c \times m_c}$, $L_d \in \mathbb{R}^{p_d \times n}$ and $W_d \in \mathbb{R}^{p_d \times m_d}$, with $P$ non-negative definite, such that

$$V_s(x) = x^T P x + V_r(x) \quad (155)$$

$$d_c(x, u_c) = (L_c x + W_c u_c)^T (L_c x + W_c u_c) + d_{cr}(x, u_c) \quad (156)$$

$$d_d(x, u_d) = (L_d x + W_d u_d)^T (L_d x + W_d u_d) + d_{dr}(x, u_d) \quad (157)$$

where $V_r: \mathbb{R}^n \to \mathbb{R}$, $d_{cr}: \mathbb{R}^n \times \mathbb{R}^{m_c} \to \mathbb{R}$ and $d_{dr}: \mathbb{R}^n \times \mathbb{R}^{m_d} \to \mathbb{R}$ contain the higher-order terms of $V_s(\cdot)$, $d_c(\cdot,\cdot)$ and $d_d(\cdot,\cdot)$, respectively. Next, let $f_c(x) = A_c x + f_{cr}(x)$, $h_c(x) = C_c x + h_{cr}(x)$, $f_d(x) = (A_d - I_n) x + f_{dr}(x)$ and $h_d(x) = C_d x + h_{dr}(x)$, where $f_{cr}(\cdot)$, $h_{cr}(\cdot)$, $f_{dr}(\cdot)$ and $h_{dr}(\cdot)$ contain the non-linear terms of $f_c(x)$, $h_c(x)$, $f_d(x)$ and $h_d(x)$, respectively and let $G_c(x) = B_c + G_{cr}(x)$, $J_c(x) = D_c + J_{cr}(x)$, $G_d(x) = B_d + G_{dr}(x)$, $J_d(x) = D_d + J_{dr}(x)$, where $G_{cr}(x)$, $J_{cr}(x)$, $G_{dr}(x)$ and $J_{dr}(x)$ contain the non–constant terms of $G_c(x)$, $J_c(x)$, $G_d(x)$ and

$J_{\rm d}(x)$, respectively. Using the above expressions, (153) and (154) can be written as

$$0 = 2x^{\rm T}P(A_{\rm c}x + B_{\rm c}u_{\rm c}) - (x^{\rm T}C_{\rm c}^{\rm T}Q_{\rm c}C_{\rm c}x + 2x^{\rm T}C_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c}u_{\rm c} + u_{\rm c}^{\rm T}D_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c}u_{\rm c} + 2x^{\rm T}C_{\rm c}^{\rm T}S_{\rm c}u_{\rm c} + 2u_{\rm c}^{\rm T}D_{\rm c}^{\rm T}S_{\rm c}u_{\rm c} + u_{\rm c}^{\rm T}R_{\rm c}u_{\rm c}) + (L_{\rm c}x + W_{\rm c}u_{\rm c})^{\rm T}(L_{\rm c}x + W_{\rm c}u_{\rm c}) + \delta_{\rm c}(x, u_{\rm c}) \tag{158}$$

$$0 = (A_{\rm d}x + B_{\rm d}u_{\rm d})^{\rm T}P(A_{\rm d}x + B_{\rm d}u_{\rm d}) - x^{\rm T}Px - (x^{\rm T}C_{\rm d}^{\rm T}Q_{\rm d}C_{\rm d}x + 2x^{\rm T}C_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d}u_{\rm d} + u_{\rm d}^{\rm T}D_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d}u_{\rm d} + 2x^{\rm T}C_{\rm d}^{\rm T}S_{\rm d}u_{\rm d} + 2u_{\rm d}^{\rm T}D_{\rm d}^{\rm T}S_{\rm d}u_{\rm d} + u_{\rm d}^{\rm T}R_{\rm d}u_{\rm d}) + (L_{\rm d}x + W_{\rm d}u_{\rm d})^{\rm T}(L_{\rm d}x + W_{\rm d}u_{\rm d}) + \delta_{\rm d}(x, u_{\rm d}) \tag{159}$$

where $\delta_{\rm c}(x, u_{\rm c})$ and $\delta_{\rm d}(x, u_{\rm d})$ are such that

$$\lim_{\|x\|^2+\|u_{\rm c}\|^2\to 0} \frac{|\,\delta_{\rm c}(x,u_{\rm c})\,|}{\|x\|^2 + \|u_{\rm c}\|^2} = 0$$

$$\lim_{\|x\|^2+\|u_{\rm d}\|^2\to 0} \frac{|\,\delta_{\rm d}(x,u_{\rm d})\,|}{\|x\|^2 + \|u_{\rm d}\|^2} = 0$$

Now, viewing (158) and (159) as the Taylor series expansion of (153) and (154), respectively, about $x = 0$, $u_{\rm c} = 0$ and $u_{\rm d} = 0$, it follows that

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} - C_{\rm c}^{\rm T}Q_{\rm c}C_{\rm c} + L_{\rm c}^{\rm T}L_{\rm c})x + 2x^{\rm T}(PB_{\rm c} - C_{\rm c}^{\rm T}S_{\rm c} - C_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c} + L_{\rm c}^{\rm T}W_{\rm c})u_{\rm c} + u_{\rm c}^{\rm T}(W_{\rm c}^{\rm T}W_{\rm c} - D_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c} - D_{\rm c}^{\rm T}S_{\rm c} - S_{\rm c}^{\rm T}D_{\rm c} - R_{\rm c})u_{\rm c}, \quad x \in \mathbb{R}^n, \quad u_{\rm c} \in \mathbb{R}^{m_{\rm c}} \tag{160}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PA_{\rm d} - P - C_{\rm d}^{\rm T}Q_{\rm d}C_{\rm d} + L_{\rm d}^{\rm T}L_{\rm d})x + 2x^{\rm T}(A_{\rm d}^{\rm T}PB_{\rm d} - C_{\rm d}^{\rm T}S_{\rm d} - C_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d} + L_{\rm d}^{\rm T}W_{\rm d})u_{\rm d} + u_{\rm d}^{\rm T}(W_{\rm d}^{\rm T}W_{\rm d} - D_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d} - D_{\rm d}^{\rm T}S_{\rm d} - S_{\rm d}^{\rm T}D_{\rm d} - R_{\rm d} + B_{\rm d}^{\rm T}PB_{\rm d})u_{\rm d}, \quad x \in \mathbb{R}^n, \quad u_{\rm d} \in \mathbb{R}^{m_{\rm d}} \tag{161}$$

Next, equating coefficients of equal powers in (160) and (161) yields (138)–(143).

Finally, to show that $P > 0$ in the case where $(A_{\rm c}, C_{\rm c})$ and $(A_{\rm d}, C_{\rm d})$ are observable, note that it follows from Theorem 9 and (138)–(143) that the linearized system $\mathcal{G}$ with storage function $V_{\rm s}(x) = x^{\rm T}Px$ is dissipative with respect to the quadratic supply rate $(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d}))$. Now, the positive definiteness of $P$ follows from Theorem 7. □

It is important to note that Theorem 12 does *not* hold for state-dependent impulsive dynamical systems. To see this, note that (138)–(143) follow from (160) and (161) if and only if $x \in \mathbb{R}^n$. For state-dependent impulsive dynamical systems $x \notin \mathcal{Z}_x$ in (160) and $x \in \mathcal{Z}_x$ in (161). For state-dependent impulsive dynamical systems we have the following linearization result.

**Theorem 13:** *Let $Q_{\rm c} \in \mathbb{S}^{l_{\rm c}}$, $S_{\rm c} \in \mathbb{R}^{l_{\rm c}\times m_{\rm c}}$, $R_{\rm c} \in \mathbb{S}^{m_{\rm c}}$, $Q_{\rm d} \in \mathbb{S}^{l_{\rm d}}$, $S_{\rm d} \in \mathbb{R}^{l_{\rm d}\times m_{\rm d}}$ and $R_{\rm d} \in \mathbb{S}^{m_{\rm d}}$ and suppose that the non-linear impulsive system $\mathcal{G}$ given by (23)–(26) is dissipative with respect to the quadratic supply rate*

$$(r_{\rm c}(u_{\rm c}, y_{\rm c}), r_{\rm d}(u_{\rm d}, y_{\rm d})) = (y_{\rm c}^{\rm T}Q_{\rm c}y_{\rm c} + 2y_{\rm c}^{\rm T}S_{\rm c}u_{\rm c} + u_{\rm c}^{\rm T}R_{\rm c}u_{\rm c},\ y_{\rm d}^{\rm T}Q_{\rm d}y_{\rm d} + 2y_{\rm d}^{\rm T}S_{\rm d}u_{\rm d} + u_{\rm d}^{\rm T}R_{\rm d}u_{\rm d})$$

*Then there exists matrices $P \in \mathbb{R}^{n\times n}$, $L_{\rm c} \in \mathbb{R}^{p_{\rm c}\times n}$, $W_{\rm c} \in \mathbb{R}^{p_{\rm c}\times m_{\rm c}}$, $L_{\rm d} \in \mathbb{R}^{p_{\rm d}\times n}$ and $W_{\rm d} \in \mathbb{R}^{p_{\rm d}\times m_{\rm d}}$, with $P$ non-negative definite, such that*

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} - C_{\rm c}^{\rm T}Q_{\rm c}C_{\rm c} + L_{\rm c}^{\rm T}L_{\rm c})x, \qquad x \notin \mathcal{Z}_x \tag{162}$$

$$0 = x^{\rm T}(PB_{\rm c} - C_{\rm c}^{\rm T}(Q_{\rm c}D_{\rm c} + S_{\rm c}) + L_{\rm c}^{\rm T}W_{\rm c}), \qquad x \notin \mathcal{Z}_x \tag{163}$$

$$0 = R_{\rm c} + S_{\rm c}^{\rm T}D_{\rm c} + D_{\rm c}^{\rm T}S_{\rm c} + D_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c} - W_{\rm c}^{\rm T}W_{\rm c} \tag{164}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PA_{\rm d} - P - C_{\rm d}^{\rm T}Q_{\rm d}C_{\rm d} + L_{\rm d}^{\rm T}L_{\rm d})x, \qquad x \in \mathcal{Z}_x \tag{165}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PB_{\rm d} - C_{\rm d}^{\rm T}(Q_{\rm d}D_{\rm d} + S_{\rm d}) + L_{\rm d}^{\rm T}W_{\rm d}), \qquad x \in \mathcal{Z}_x \tag{166}$$

$$0 = R_{\rm d} + S_{\rm d}^{\rm T}D_{\rm d} + D_{\rm d}^{\rm T}S_{\rm d} + D_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d} - B_{\rm d}^{\rm T}PB_{\rm d} - W_{\rm d}^{\rm T}W_{\rm d} \tag{167}$$

*where $A_{\rm c}$, $B_{\rm c}$, $C_{\rm c}$, $D_{\rm c}$, $A_{\rm d}$, $B_{\rm d}$, $C_{\rm d}$ and $D_{\rm d}$ are given by (144) and (145). If, in addition, $(A_{\rm c}, C_{\rm c})$ and $(A_{\rm d}, C_{\rm d})$ are observable, then $P > 0$.*

**Proof:** The proof is identical to the proof of Theorem 12 with (160) and (161) replaced by

$$0 = x^{\rm T}(A_{\rm c}^{\rm T}P + PA_{\rm c} - C_{\rm c}^{\rm T}Q_{\rm c}C_{\rm c} + L_{\rm c}^{\rm T}L_{\rm c})x + 2x^{\rm T}(PB_{\rm c} - C_{\rm c}^{\rm T}S_{\rm c} - C_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c} + L_{\rm c}^{\rm T}W_{\rm c})u_{\rm c} + u_{\rm c}^{\rm T}(W_{\rm c}^{\rm T}W_{\rm c} - D_{\rm c}^{\rm T}Q_{\rm c}D_{\rm c} - D_{\rm c}^{\rm T}S_{\rm c} - S_{\rm c}^{\rm T}D_{\rm c} - R_{\rm c})u_{\rm c}, \qquad x \notin \mathcal{Z}_x, \quad u_{\rm c} \in \mathbb{R}^{m_{\rm c}} \tag{168}$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T}PA_{\rm d} - P - C_{\rm d}^{\rm T}Q_{\rm d}C_{\rm d} + L_{\rm d}^{\rm T}L_{\rm d})x + 2x^{\rm T}(A_{\rm d}^{\rm T}PB_{\rm d} - C_{\rm d}^{\rm T}S_{\rm d} - C_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d} + L_{\rm d}^{\rm T}W_{\rm d})u_{\rm d} + u_{\rm d}^{\rm T}(W_{\rm d}^{\rm T}W_{\rm d} - D_{\rm d}^{\rm T}Q_{\rm d}D_{\rm d} - D_{\rm d}^{\rm T}S_{\rm d} - S_{\rm d}^{\rm T}D_{\rm d} - R_{\rm d} + B_{\rm d}^{\rm T}PB_{\rm d})u_{\rm d}, \quad x \in \mathcal{Z}_x,\ u_{\rm d} \in \mathbb{R}^{m_{\rm d}} \tag{169}$$

Now, setting $u_{\rm c} = 0$ and $u_{\rm d} = 0$ in (168) and (169), respectively, yields (162) and (165). In this case, (168) and (169) become

$$0 = 2x^{\mathrm{T}} R_{\mathrm{c}xu} u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}uu} u_{\mathrm{c}}, \quad x \notin \mathcal{Z}_x, \quad u_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}} \tag{170}$$

$$0 = 2x^{\mathrm{T}} R_{\mathrm{d}xu} u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}} R_{\mathrm{d}uu} u_{\mathrm{d}}, \quad x \in \mathcal{Z}_x, \quad u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}} \tag{171}$$

where

$$\begin{aligned}
R_{\mathrm{c}xu} &\triangleq PB_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} D_{\mathrm{c}} + L_{\mathrm{c}}^{\mathrm{T}} W_{\mathrm{c}} \\
R_{\mathrm{c}uu} &\triangleq W_{\mathrm{c}}^{\mathrm{T}} W_{\mathrm{c}} - D_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} D_{\mathrm{c}} - D_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} - S_{\mathrm{c}}^{\mathrm{T}} D_{\mathrm{c}} - R_{\mathrm{c}} \\
R_{\mathrm{d}xu} &\triangleq A_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} D_{\mathrm{d}} + L_{\mathrm{d}}^{\mathrm{T}} W_{\mathrm{d}} \\
R_{\mathrm{d}uu} &\triangleq W_{\mathrm{d}}^{\mathrm{T}} W_{\mathrm{d}} - D_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} D_{\mathrm{d}} - D_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} \\
&\quad - S_{\mathrm{d}}^{\mathrm{T}} D_{\mathrm{d}} - R_{\mathrm{d}} + B_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}}
\end{aligned}$$

Next, let $x \notin \mathcal{Z}_x$ and $\hat{u}_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$ so that, with $u_{\mathrm{c}} = 2\hat{u}_{\mathrm{c}}$, (170) implies

$$0 = 4x^{\mathrm{T}} R_{\mathrm{c}xu} \hat{u}_{\mathrm{c}} + 4\hat{u}_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}uu} \hat{u}_{\mathrm{c}} \tag{172}$$

Now, forming $\frac{1}{2}$(172)–(171) yields $\hat{u}_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}uu} \hat{u}_{\mathrm{c}} = 0$, $\hat{u}_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$. Hence, $R_{\mathrm{c}uu} = 0$, or, equivalently, (164) holds. Furthermore, $\hat{u}_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}uu} \hat{u}_{\mathrm{c}} = 0$, $\hat{u}_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$, implies $2x^{\mathrm{T}} R_{\mathrm{c}xu} \hat{u}_{\mathrm{c}} = 0$, $\hat{u}_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$. Hence, $2x^{\mathrm{T}} R_{\mathrm{c}xu} = 0$, $x \notin \mathcal{Z}_x$, which implies (163). Using similar arguments (166) and (167) hold. Finally, the positive definiteness of $P$ in the case where $(A_{\mathrm{c}}, C_{\mathrm{c}})$ and $(A_{\mathrm{d}}, C_{\mathrm{d}})$ are observable follows as in the proof of Theorem 12 using Theorem 9, Remark 13 and Theorem 7. □

## 6. Specialization to linear impulsive dynamical systems

In this section we specialize the results of § 5 to the case of linear impulsive dynamical systems. Specifically, setting $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, $G_{\mathrm{c}}(x) = B_{\mathrm{c}}$, $h_{\mathrm{c}}(x) = C_{\mathrm{c}}x$, $J_{\mathrm{c}}(x) = D_{\mathrm{c}}$, $f_{\mathrm{d}}(x) = (A_{\mathrm{d}} - I_n)x$, $G_{\mathrm{d}}(x) = B_{\mathrm{d}}$, $h_{\mathrm{d}}(x) = C_{\mathrm{d}}x$ and $J_{\mathrm{d}}(x) = D_{\mathrm{d}}$, the non-linear time-dependent impulsive dynamical system given by (10)–(13) specializes to

$$\dot{x}(t) = A_{\mathrm{c}} x(t) + B_{\mathrm{c}} u_{\mathrm{c}}(t), \quad x(0) = x_0, \quad t \neq t_k \tag{173}$$

$$\Delta x(t) = (A_{\mathrm{d}} - I_n) x(t) + B_{\mathrm{d}} u_{\mathrm{d}}(t), \quad t = t_k \tag{174}$$

$$y_{\mathrm{c}}(t) = C_{\mathrm{c}} x(t) + D_{\mathrm{c}} u_{\mathrm{c}}(t), \quad t \neq t_k \tag{175}$$

$$y_{\mathrm{d}}(t) = C_{\mathrm{d}} x(t) + D_{\mathrm{d}} u_{\mathrm{d}}(t), \quad t = t_k \tag{176}$$

where $A_{\mathrm{c}} \in \mathbb{R}^{n\times n}$, $B_{\mathrm{c}} \in \mathbb{R}^{n\times m_{\mathrm{c}}}$, $C_{\mathrm{c}} \in \mathbb{R}^{l_{\mathrm{c}}\times n}$, $D_{\mathrm{c}} \in \mathbb{R}^{l_{\mathrm{c}}\times m_{\mathrm{c}}}$, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $B_{\mathrm{d}} \in \mathbb{R}^{n\times m_{\mathrm{d}}}$, $C_{\mathrm{d}} \in \mathbb{R}^{l_{\mathrm{d}}\times n}$ and $D_{\mathrm{d}} \in \mathbb{R}^{l_{\mathrm{d}}\times m_{\mathrm{d}}}$.

**Theorem 14:** *Let $Q_{\mathrm{c}} \in \mathbb{S}^{l_{\mathrm{c}}}$, $S_{\mathrm{c}} \in \mathbb{R}^{l_{\mathrm{c}}\times m_{\mathrm{c}}}$, $R_{\mathrm{c}} \in \mathbb{S}^{m_{\mathrm{c}}}$, $Q_{\mathrm{d}} \in \mathbb{S}^{l_{\mathrm{d}}}$, $S_{\mathrm{d}} \in \mathbb{R}^{l_{\mathrm{d}}\times m_{\mathrm{d}}}$, $R_{\mathrm{d}} \in \mathbb{S}^{m_{\mathrm{d}}}$, consider the linear impulsive dynamical system $\mathcal{G}$ given by* (173)–(176) *and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is dissipative with respect to the quadratic supply rate $(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})) = (y_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}} u_{\mathrm{c}},\ y_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} y_{\mathrm{d}} + 2y_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}} R_{\mathrm{d}} u_{\mathrm{d}})$.*

(ii) *There exist matrices $P \in \mathbb{R}^{n\times n}$, $L_{\mathrm{c}} \in \mathbb{R}^{p_{\mathrm{c}}\times n}$, $W_{\mathrm{c}} \in \mathbb{R}^{p_{\mathrm{c}}\times m_{\mathrm{c}}}$, $L_{\mathrm{d}} \in \mathbb{R}^{p_{\mathrm{d}}\times n}$ and $W_{\mathrm{d}} \in \mathbb{R}^{p_{\mathrm{d}}\times m_{\mathrm{d}}}$, with $P$ positive definite, such that* (138)–(143) *are satisfied.*

*If, alternatively, $R_{\mathrm{c}} + S_{\mathrm{c}}^{\mathrm{T}} D_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} D_{\mathrm{c}} > 0$, then $\mathcal{G}$ is dissipative with respect to the quadratic supply rate*

$$\begin{aligned}
(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})) = (&y_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}} u_{\mathrm{c}}, \\
&y_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} y_{\mathrm{d}} + 2y_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}} R_{\mathrm{d}} u_{\mathrm{d}})
\end{aligned}$$

*if and only if there exists an $n \times n$ positive-definite matrix $P$ such that*

$$0 < R_{\mathrm{d}} + S_{\mathrm{d}}^{\mathrm{T}} D_{\mathrm{d}} + D_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} + D_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} D_{\mathrm{d}} - B_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}} \tag{177}$$

$$\begin{aligned}
0 \geq\ & A_{\mathrm{c}}^{\mathrm{T}} P + PA_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} C_{\mathrm{c}} + [PB_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}(Q_{\mathrm{c}} D_{\mathrm{c}} + S_{\mathrm{c}})] \\
&\times [R_{\mathrm{c}} + S_{\mathrm{c}}^{\mathrm{T}} D_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} D_{\mathrm{c}}]^{-1} \\
&\times [PB_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}(Q_{\mathrm{c}} D_{\mathrm{c}} + S_{\mathrm{c}})]^{\mathrm{T}}
\end{aligned} \tag{178}$$

$$\begin{aligned}
0 \geq\ & A_{\mathrm{d}}^{\mathrm{T}} PA_{\mathrm{d}} - P - C_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} C_{\mathrm{d}} \\
&+ [A_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}}(Q_{\mathrm{d}} D_{\mathrm{d}} + S_{\mathrm{d}})] \\
&\times [R_{\mathrm{d}} + S_{\mathrm{d}}^{\mathrm{T}} D_{\mathrm{d}} + D_{\mathrm{d}}^{\mathrm{T}} S_{\mathrm{d}} + D_{\mathrm{d}}^{\mathrm{T}} Q_{\mathrm{d}} D_{\mathrm{d}} \\
&- B_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}}]^{-1} [A_{\mathrm{d}}^{\mathrm{T}} PB_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}}(Q_{\mathrm{d}} D_{\mathrm{d}} + S_{\mathrm{d}})]^{\mathrm{T}}
\end{aligned} \tag{179}$$

**Proof:** The fact that (ii) implies (i) follows from Theorem 9 with $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, $G_{\mathrm{c}}(x) = B_{\mathrm{c}}$, $h_{\mathrm{c}}(x) = C_{\mathrm{c}}x$, $J_{\mathrm{c}}(x) = D_{\mathrm{c}}$, $f_{\mathrm{d}}(x) = (A_{\mathrm{d}} - I_n)x$, $G_{\mathrm{d}}(x) = B_{\mathrm{d}}$, $h_{\mathrm{d}}(x) = C_{\mathrm{d}}x$, $J_{\mathrm{d}}(x) = D_{\mathrm{d}}$, $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$, $\ell_{\mathrm{c}}(x) = L_{\mathrm{c}}x$, $\ell_{\mathrm{d}}(x) = L_{\mathrm{d}}x$, $\mathcal{W}_{\mathrm{c}}(x) = W_{\mathrm{c}}$ and $\mathcal{W}_{\mathrm{d}}(x) = W_{\mathrm{d}}$. To show that (i) implies (ii), note that if the linear impulsive dynamical system given by (173)–(176) is dissipative, then it follows from Theorem 12 with $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, $G_{\mathrm{c}}(x) = B_{\mathrm{c}}$, $h_{\mathrm{c}}(x) = C_{\mathrm{c}}x$, $J_{\mathrm{c}}(x) = D_{\mathrm{c}}$, $f_{\mathrm{d}}(x) = (A_{\mathrm{d}} - I_n)x$, $G_{\mathrm{d}}(x) = B_{\mathrm{d}}$, $h_{\mathrm{d}}(x) = C_{\mathrm{d}}x$ and $J_{\mathrm{d}}(x) = D_{\mathrm{d}}$ that there exists matrices $P \in \mathbb{R}^{n\times n}$, $L_{\mathrm{c}} \in \mathbb{R}^{p_{\mathrm{c}}\times n}$, $W_{\mathrm{c}} \in \mathbb{R}^{p_{\mathrm{c}}\times m_{\mathrm{c}}}$, $L_{\mathrm{d}} \in \mathbb{R}^{p_{\mathrm{d}}\times n}$ and $W_{\mathrm{d}} \in \mathbb{R}^{p_{\mathrm{d}}\times m_{\mathrm{d}}}$, with $P$ positive definite, such that (138)–(143) are satisfied. Finally, (177)–(179) follow from (87)–(89) and Theorem 12 with the linearization given above. □

**Remark 19:** Note that the proof of Theorem 14 relies on Theorem 12 which *a priori* assumes that the storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is three-times continuously differentiable. Unlike linear, time-invariant dissipative dynamical systems with continuous flows (Willems 1972 b), there does not always exists a smooth (i.e. $\mathrm{C}^{\infty}$) storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, for linear dissipative impulsive dynamical systems.

**Remark 20:** Note that (138)–(143) are equivalent to

$$\begin{bmatrix} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{B}_c^T & \mathcal{D}_c \end{bmatrix} = \begin{bmatrix} L_c^T \\ W_c^T \end{bmatrix} [\, L_c \quad W_c \,] \geq 0 \qquad (180)$$

$$\begin{bmatrix} \mathcal{A}_d & \mathcal{B}_d \\ \mathcal{B}_d^T & \mathcal{D}_d \end{bmatrix} = \begin{bmatrix} L_d^T \\ W_d^T \end{bmatrix} [\, L_d \quad W_d \,] \geq 0 \qquad (181)$$

where

$$\mathcal{A}_c = -A_c^T P - P A_c + C_c^T Q_c C_c$$
$$\mathcal{B}_c = -P B_c + C_c^T (Q_c D_c + S_c)$$
$$\mathcal{D}_c = R_c + S_c^T D_c + D_c^T S_c + D_c^T Q_c D_c$$
$$\mathcal{A}_d = P - A_d^T P A_d + C_d^T Q_d C_d$$
$$\mathcal{B}_d = -A_d^T P B_d + C_d^T (Q_d D_d + S_d)$$

and

$$\mathcal{D}_d = R_d + S_d^T D_d + D_d^T S_d + D_d^T Q_d D_d - B_d^T P B_d$$

Hence dissipativity of linear impulsive dynamical systems with respect to quadratic supply rates can be characterized via linear matrix inequalities (LMIs) (Boyd *et al*. 1994). Similar remarks hold for the passivity and non-expansivity results given in Corollaries 5 and 6, respectively.

**Remark 21:** It follows from Theorem 13 that the equivalence between (138)–(143) and dissipativity of a linear *state-dependent* impulsive dynamical system does *not* hold. In particular, for linear state-dependent impulsive dynamical systems, (138)–(143) are only sufficient conditions for dissipativity. However, under the assumptions of Theorem 14 the equivalence between the more involved conditions (162)–(167) and dissipativity of a linear state-dependent impulsive dynamical system *does* hold. The proof of this fact is identical to the proof of Theorem 14 using Theorem 13 in place of Theorem 12. Similar remarks hold for the passivity and non-expansivity results given in Corollaries 5 and 6, respectively.

The following results present generalizations of the positive real lemma and the bounded real lemma for linear impulsive systems, respectively.

**Corollary 5:** *Consider the linear impulsive dynamical system $\mathcal{G}$ given by* (173)–(176) *with $m_c = l_c$ and $m_d = l_d$ and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is passive.*

(ii) *There exist matrices $P \in \mathbb{R}^{n\times n}$, $L_c \in \mathbb{R}^{p_c\times n}$, $W_c \in \mathbb{R}^{p_c\times m_c}$, $L_d \in \mathbb{R}^{p_d\times n}$ and $W_d \in \mathbb{R}^{p_d\times m_d}$, with $P$ positive definite, such that*

$$0 = A_c^T P + P A_c + L_c^T L_c \qquad (182)$$
$$0 = P B_c - C_c^T + L_c^T W_c \qquad (183)$$
$$0 = D_c + D_c^T - W_c^T W_c \qquad (184)$$
$$0 = A_d^T P A_d - P + L_d^T L_d \qquad (185)$$
$$0 = A_d^T P B_d - C_d^T + L_d^T W_d \qquad (186)$$
$$0 = D_d + D_d^T - B_d^T P B_d - W_d^T W_d \qquad (187)$$

*If, alternatively, $D_c + D_c^T > 0$, then $\mathcal{G}$ is passive if and only if there exists an $n \times n$ positive-definite matrix $P$ such that*

$$0 < D_d + D_d^T - B_d^T P B_d \qquad (188)$$
$$0 \geq A_c^T P + P A_c + (P B_c - C_c^T) \times (D_c + D_c^T)^{-1} (P B_c - C_c^T)^T \qquad (189)$$
$$0 \geq A_d^T P A_d - P + (A_d^T P B_d - C_d^T) \times (D_d + D_d^T - B_d^T P B_d)^{-1} (A_d^T P B_d - C_d^T)^T \qquad (190)$$

**Proof:** The result is a direct consequence of Theorem 14 with $m_c = l_c$, $m_d = l_d$, $Q_c = 0$, $S_c = I_{m_c}$, $R_c = 0$, $Q_d = 0$, $S_d = I_{m_d}$ and $R_d = 0$. □

**Remark 22:** Equations (182)–(184) are identical in form to the equations appearing in the continuous-time positive real lemma (Anderson 1967) used to characterize positive realness for continuous-time linear systems in the state-space; while (185)–(187) are identical in form to the equations appearing in the discrete-time positive real lemma (Hitz and Anderson 1969). This is not surprising since, as noted in Remark 14, impulsive dynamical systems involve a hybrid formulation of continuous-time and discrete-time dynamics. A key difference, however, is the fact that in the impulsive case a *single* positive-definite matrix $P$ is required to satisfy all six equations. Similar remarks hold for Corollary 6.

**Corollary 6:** *Consider the linear impulsive dynamical system $\mathcal{G}$ given by* (173)–(176) *and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is non-expansive.*

(ii) *There exist matrices $P \in \mathbb{R}^{n\times n}$, $L_c \in \mathbb{R}^{p_c\times n}$, $W_c \in \mathbb{R}^{p_c\times m_c}$, $L_d \in \mathbb{R}^{p_d\times n}$ and $W_d \in \mathbb{R}^{p_d\times m_d}$, with $P$ positive definite, such that*

$$0 = A_c^T P + P A_c + C_c^T C_c + L_c^T L_c \qquad (191)$$
$$0 = P B_c + C_c^T D_c + L_c^T W_c \qquad (192)$$
$$0 = \gamma_c^2 I_{m_c} - D_c^T D_c - W_c^T W_c \qquad (193)$$

$$0 = A_{\rm d}^{\rm T} P A_{\rm d} - P + C_{\rm d}^{\rm T} C_{\rm d} + L_{\rm d}^{\rm T} L_{\rm d} \tag{194}$$

$$0 = A_{\rm d}^{\rm T} P B_{\rm d} + C_{\rm d}^{\rm T} D_{\rm d} + L_{\rm d}^{\rm T} W_{\rm d} \tag{195}$$

$$0 = \gamma_{\rm d}^2 I_{\rm d} - D_{\rm d}^{\rm T} D_{\rm d} - B_{\rm d}^{\rm T} P B_{\rm d} - W_{\rm d}^{\rm T} W_{\rm d} \tag{196}$$

*If, alternatively, $\gamma_{\rm c}^2 I_{m_{\rm c}} - D_{\rm c}^{\rm T} D_{\rm c} > 0$, then $\mathcal{G}$ is non-expansive if and only if there exists an $n \times n$ positive-definite matrix $P$ such that*

$$0 < \gamma_{\rm d}^2 I_{m_{\rm d}} - D_{\rm d}^{\rm T} D_{\rm d} - B_{\rm d}^{\rm T} P B_{\rm d} \tag{197}$$

$$0 \geq A_{\rm c}^{\rm T} P + P A_{\rm c} + (P B_{\rm c} + C_{\rm c}^{\rm T} D_{\rm c})(\gamma_{\rm c}^2 I_{m_{\rm c}} - D_{\rm c}^{\rm T} D_{\rm c})^{-1} \times (P B_{\rm c} + C_{\rm c}^{\rm T} D_{\rm c})^{\rm T} + C_{\rm c}^{\rm T} C_{\rm c} \tag{198}$$

$$0 \geq A_{\rm d}^{\rm T} P A_{\rm d} - P + (A_{\rm d}^{\rm T} P B_{\rm d} + C_{\rm d}^{\rm T} D_{\rm d})(\gamma_{\rm d}^2 I_{m_{\rm d}} - D_{\rm d}^{\rm T} D_{\rm d} - B_{\rm d}^{\rm T} P B_{\rm d})^{-1} (A_{\rm d}^{\rm T} P B_{\rm d} + C_{\rm d}^{\rm T} D_{\rm d})^{\rm T} + C_{\rm d}^{\rm T} C_{\rm d} \tag{199}$$

**Proof:** The result is a direct consequence of Theorem 14 with $Q_{\rm c} = -I_{l_{\rm c}}$, $S_{\rm c} = 0$, $R_{\rm c} = \gamma_{\rm c}^2 I_{m_{\rm c}}$, $Q_{\rm d} = -I_{l_{\rm d}}$, $S_{\rm d} = 0$ and $R_{\rm d} = \gamma_{\rm d}^2 I_{m_{\rm d}}$. □

**Remark 23:** It follows from Remark 15 that if (182) and (191) are replaced, respectively, by

$$0 = A_{\rm c}^{\rm T} P + P A_{\rm c} + \varepsilon P + L_{\rm c}^{\rm T} L_{\rm c} \tag{200}$$

$$0 = A_{\rm c}^{\rm T} P + P A_{\rm c} + \varepsilon P + C_{\rm c}^{\rm T} C_{\rm c} + L_{\rm c}^{\rm T} L_{\rm c} \tag{201}$$

where $\varepsilon > 0$, then (200), (183)–(187) provide necessary and sufficient conditions for exponential passivity, while (201), (192)–(196) provide necessary and sufficient conditions for exponential non-expansivity. These conditions present generalizations of the strict positive real lemma and the strict bounded real lemma for linear impulsive systems, respectively.

## 7. Conclusion

In this paper we developed new invariant set stability theorems for non-linear impulsive dynamical systems. Furthermore, we extended the classical notions of dissipativity theory to non-linear dynamical systems with impulsive effects. Specifically, the concepts of storage functions and supply rates are extended to impulsive dynamical systems providing a generalized hybrid system energy interpretation in terms of stored energy, dissipated energy over the continuous-time dynamics and dissipated energy at the resetting instants. Furthermore, extended Kalman–Yakubovich–Popov algebraic conditions in terms of the impulsive system dynamics for characterizing dissipativeness via system storage functions are also derived. In the case of quadratic supply rates involving net system power and input–output energy, these results provide generalizations of the classical notions of passivity and non-expansivity. In addition, for linear impulsive systems, the proposed results provide a generalization of the positive real lemma and the bounded real lemma. In Part II of this paper (Haddad *et al.* 2001) we develop general stability criteria for feedback interconnections of non-linear impulsive systems as well as a unified framework for hybrid feedback optimal and inverse optimal control.

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