FMEN21
Continuum Mechanics
Project work: Granular flow

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1 Introduction

In this report, we will use continuum mechanics to describe a granular flow along an inclined plane. Even though the granular material is discontinuous in nature, consisting of individual particles, it is well established that the continuum mechanics method can be used to successfully model the flow.

Our report is based on the experimental work done by S. B. Savage (Adopted from Savage, S. B., Advances in Applied Mechanics, vol. 24, p. 289-366, 1984), where the results for a granular flow downward a leaning plane. A schematic of the setup is given in figure 1.

The flow is modelled by solving the balance equation of momentum, where constitutive relations for the stress tensor must be introduced. We will use two different constitutive relations of varying complexity and compare the results in order to see how the choice of constitutive relation affects the prediction of the flow.

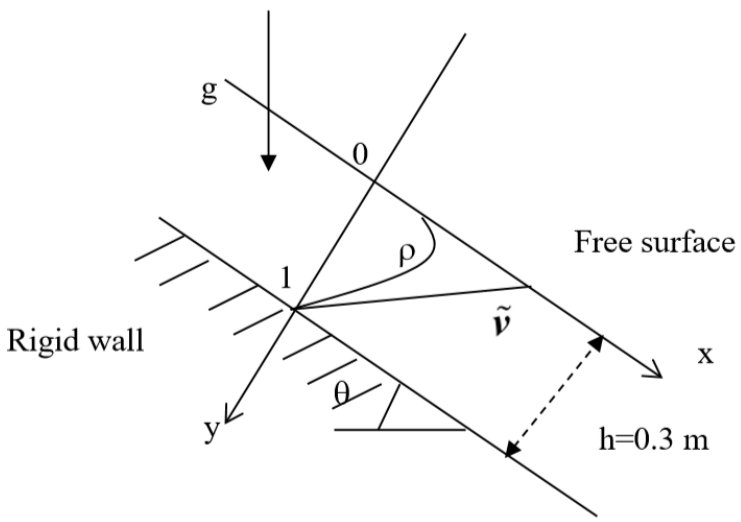

Figure 1: Schematic drawing of the setup of the granular flow.

2 Theoretical Background

From Newton's second law also called the Balance of Linear Momentum, here we have

$$\rho\ddot{\bar{x}} = div\underline{T} + \rho\bar{b} \tag{1}$$

with density ρ, stress tensor $\underline{R}$, body force vector $\bar{b}$, acceleration $\ddot{\bar{x}}$. In this project, the flow is acceleration free ($\ddot{\bar{x}} = 0$).

In the case of incompressible linear viscous flow, the constitutive relation for the stress tensor **T** can be written as

$$T = -pI + \lambda tr(D)I + 2\mu D \tag{2}$$

Here, λ, μ are material constants and p, $D = \frac{1}{2}(l + l^T)$ (l is velocity gradient), I are pressure, rate of deformation tensor, identity matrix. But $tr(D) = 0$ because of the incompressible fluid assumption. Therefore Eq. (2) reduces to

$$T = -pI + 2\mu D \tag{3}$$

and the rate of deformation tensor is given by

$$D = \begin{bmatrix} \frac{\partial\dot{x}}{\partial x} & \frac{1}{2}(\frac{\partial\dot{x}}{\partial y} + \frac{\partial\dot{y}}{\partial x}) & \frac{1}{2}(\frac{\partial\dot{x}}{\partial z} + \frac{\partial\dot{z}}{\partial x}) \\ \frac{1}{2}(\frac{\partial\dot{y}}{\partial x} + \frac{\partial\dot{x}}{\partial y}) & \frac{\partial\dot{y}}{\partial y} & \frac{1}{2}(\frac{\partial\dot{y}}{\partial z} + \frac{\partial\dot{z}}{\partial y}) \\ \frac{1}{2}(\frac{\partial\dot{z}}{\partial x} + \frac{\partial\dot{x}}{\partial z}) & \frac{1}{2}(\frac{\partial\dot{z}}{\partial y} + \frac{\partial\dot{y}}{\partial z}) & \frac{\partial\dot{z}}{\partial z} \end{bmatrix}$$

This constitutive relation is used in the exercise 2 as the case there is incompressible linear viscous flow. In exercise 3, we have stress tensor linearly depending on the rate of deformation tensor **D** and quadratically depending on the gradient of the density $\tilde{\nu} = \nabla\rho$. The constitutive relation for this stress tensor is given by

$$T = -pI + \lambda tr(D)I + 2\mu D + \mu_1\tilde{\nu}\otimes\tilde{\nu} + \mu_2(\tilde{\nu}\otimes D\tilde{\nu} + D\tilde{\nu}\otimes\tilde{\nu}) \tag{4}$$

Here, μ_1, μ_2, λ, μ are material constants. Again, $tr(D) = 0$ because of the incompressible fluid assumption. Therefore, Eq. (4) reduces to

$$T = -pI + 2\mu D + \mu_1\tilde{\nu}\otimes\tilde{\nu} + \mu_2(\tilde{\nu}\otimes D\tilde{\nu} + D\tilde{\nu}\otimes\tilde{\nu}) \tag{5}$$

Assumptions made in this project are:

- The inclination of the plane is $\theta = 30°$
- Velocity field and density depend on y coordinate

$$\dot{x} = \dot{x}(y), \rho = \rho(y) \tag{6}$$

- Pressure p varies only in y direction ($p = p(y)$)
- Only gravity force acts on the system and the body force vector is given by $\bar{b} = [g_x, g_y, 0]$ where $g = 9.8\ ms^{-1}$ is gravitational acceleration.
- Density is given by

$$\rho = \rho(y/h) = \rho_0(Ae^{\frac{y}{h}L} + Be^{-\frac{y}{h}L} + C\frac{y}{h} + D) \tag{7}$$

- A, B, C, D, L are constants, ρ_0 is the density of granular grains and $\rho_0 = 1$

The above equation can be written in terms of volume fraction as follows.

$$\nu(y) = \frac{\rho}{\rho_0} = (Ae^{\frac{y}{h}L} + Be^{-\frac{y}{h}L} + C\frac{y}{h} + D) \tag{8}$$

The boundary conditions used are: For $0 < y/h < 1$,

$$\begin{aligned}\nu(y/h = 0) &= \nu_{0y}\\ \nu(y/h = 0.3) &= \nu_{0.3y}\\ \nu(y/h = 1) &= \nu_{1y}\\ \nu'(y/h = 0.3) &= 0\end{aligned}$$

Here, ν_{0y} is the volume fraction at $y = 0$, $\nu_{0.3y}$ is the maximum volume fraction at $y/h = 0.3$, ν_{1y} is the volume fraction at $y/h = 1$ and $\nu'(y/h = 0.3)$ is the derivative with respect to y. All these values are available from experimental data given in Figure 1.

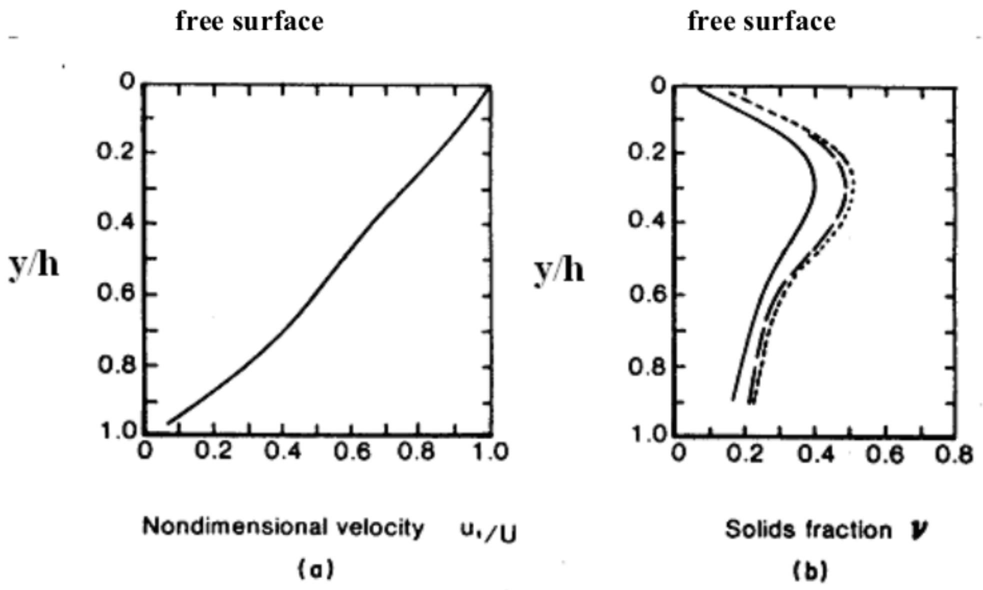

Figure 2: Experimental data for granular flow down the leaning plate

3 Exercise 1

The project considers two cases of granular flow for which the two different stress tensor constitutive relations have been introduced in the section Theoretical Background. The project is divided into 3 main tasks or exercises.

From the Balance of Linear Momentum equation (Eq. (1)) and stress tensor constitutive relation (Eq. (2)), we can see that we have 3 unknown ($\rho(y/h)$, $\dot{x} = \dot{x}(y/h)$, $p = p(y/h)$) and only 2 equations. Therefore, the density $\rho(y/h)$ is computed using experimental data (Figure 1). Exercise 1 deals with calculating the constants A, B, C, D, L by fitting the experimental data from Figure 1 to the Eq. (7) and using the BCs. With the help of the BCs, we can form a system 4 linear equations in A, B, C, D. Then this system of linear equations is solved in MATLAB using the function **solve()** for different values of L ($1 \leq L \leq 12$) and the curve for $\rho(y/h)$ is plotted. The values of A, B, C, D, L for which the curve best represents (with minimum error in the fit) the experimental data are taken as the optimal values. The optimal values of A, B, C, D, L obtained in this exercise are $A = 0.0063$, $B = -1.009$, $C = -1.298$, $D = 1.063$, $L = 4.1$ (shown in Figure 2).

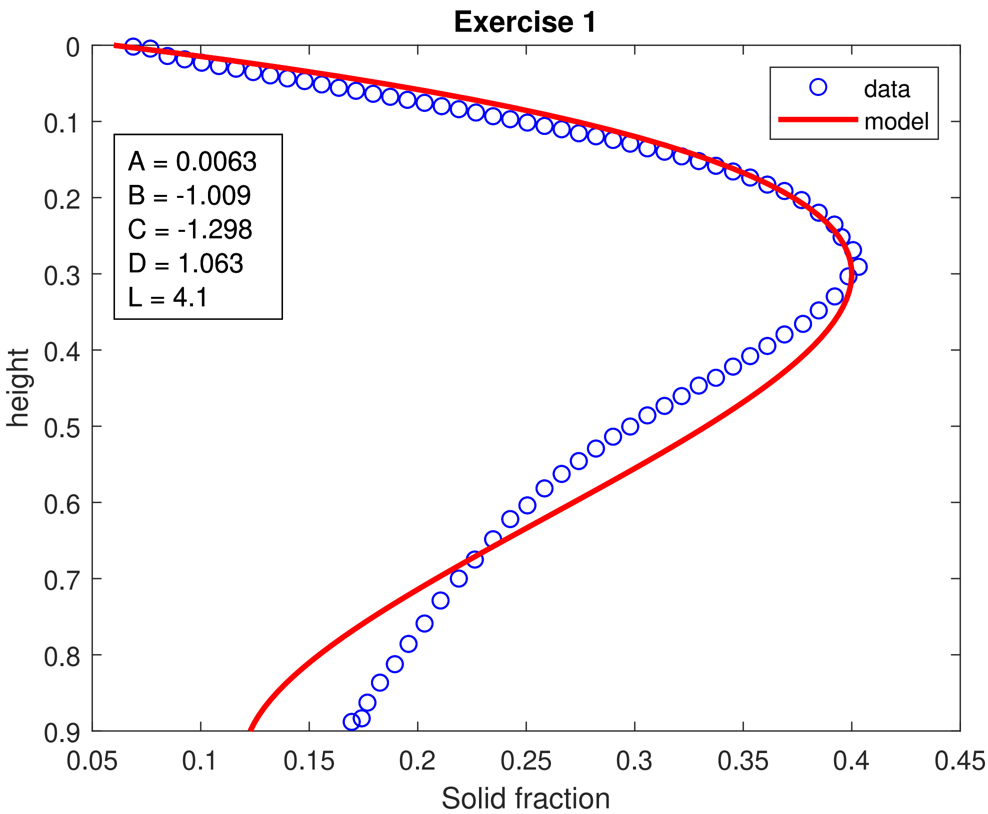

Figure 3: Exercise 1 - (Blue curve) Original experimental data, (red curve) data fit to Eq. 7

These optimal values are later used in the exercise 2 and exercise 3.

4 Exercise 2

While in exercise 1, we have computed $\rho(y/h)$ through the evaluation of the constants A, B, C, D, L, in exercise 2, we evaluate $\dot{x} = \dot{x}(y/h)$ and $p = p(y/h)$ using the density calculated in exercise 1. For this purpose the Eq. (1) which is a differential equation is separated into x and y components. Then the x component of the differential equation (i.e. Eq. (3)) is solved to obtain the velocity function $\dot{x}(y/h)$ with boundary conditions : at $y = 0.0$, shear stress $T_{xy} = 0.0$ ($\partial\dot{x}/\partial y = 0.0$) and at $y = 0.3$, velocity $\dot{x}(y) = 0.0$.

$$\mu\frac{\partial^2\dot{x}}{\partial y^2} + \rho * g * sin(\theta) = 0 \tag{9}$$

where $\theta = 30°$

The y component of the differential equation (i.e. Eq. (4)) is solved to obtain $p(y/h)$ with boundary conditions : at $y = 0.0$, atmospheric pressure $p = 101.3\ kPa$.

$$-\frac{\partial p}{\partial y} + \rho * g * cos(\theta) = 0 \tag{10}$$

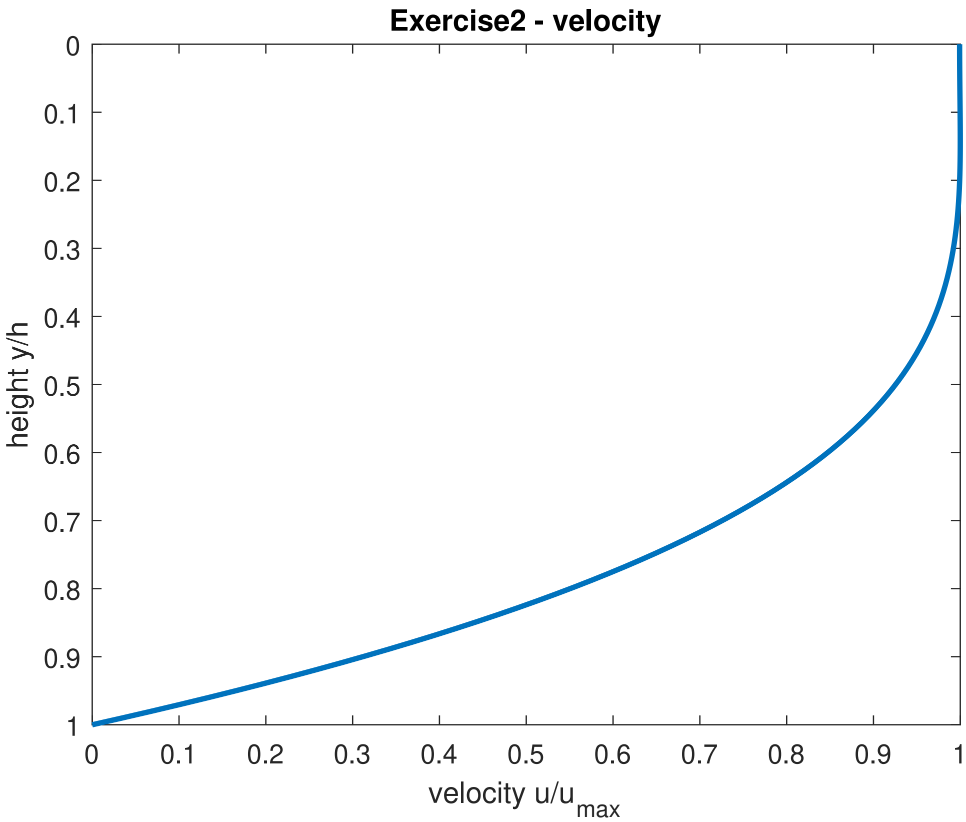

Figure 4: Exercise 2 - Velocity Profile

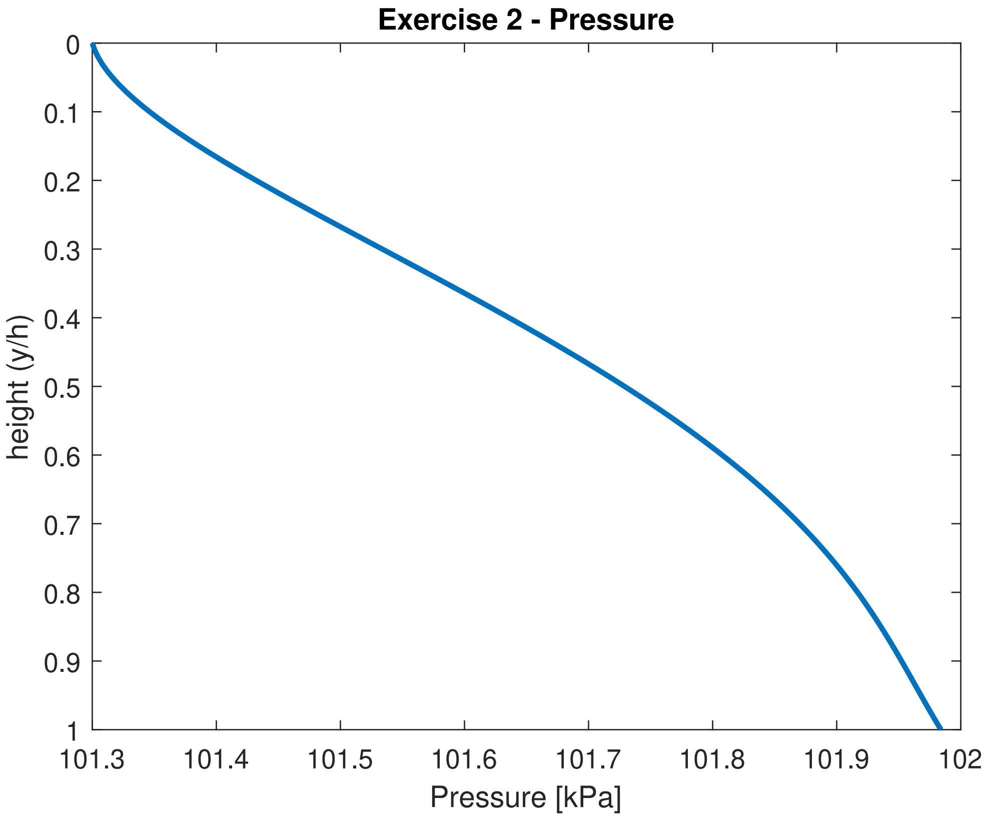

Figure 5: Exercise 2 - Pressure Profile

5 Exercise 3

In exercise 3, we evaluate $\dot{x} = \dot{x}(y/h)$ and $p = p(y/h)$ using the density calculated in exercise 1 but for the case where stress tensor $\mathbf{T}$ exhibits linear dependence on rate of deformation tensor $\mathbf{D}$ and quadratic dependence on density gradient $\tilde{\nu}$. This time, the constitutive relation for the stress tensor $\mathbf{T}$ given in Eq. 4 is used. The stress tensor for the flow found to be:

$$T = \begin{bmatrix} -P & \frac{\partial \dot{x}}{\partial y}[\mu + \frac{1}{2}\mu_2(\frac{\partial \rho}{\partial y})^2] & 0 \\ \frac{\partial \dot{x}}{\partial y}[\mu + \frac{1}{2}\mu_2(\frac{\partial \rho}{\partial y})^2] & -P + \mu_1(\frac{\partial \rho}{\partial y})^2 & 0 \\ 0 & 0 & -P \end{bmatrix}$$

We insert this stress tensor into the balance equation of momentum (Eq. (1)) and seperate the differential equation into its x and y components.

The x component (i.e. Eq. (11)) is solved to obtain the velocity function $\dot{x}(y/h)$ with boundary conditions : at $y = 0.0$, shear stress $T_{xy} = 0.0$ ($\partial \dot{x}/\partial y = 0.0$) and at $y = 0.3$, velocity $\dot{x}(y) = 0.0$. Also we use $\mu = -1$, $\mu_1 = 0.005$ and $\mu_2 = -1$.

$$\frac{\partial^2 \dot{x}}{\partial y^2}[\mu + \frac{1}{2}\mu_2(\frac{\partial \rho}{\partial y})^2] + \mu_2 \frac{\partial \dot{x}}{\partial y}\frac{\partial \rho}{\partial y}\frac{\partial^2 \rho}{\partial y^2} + \rho g \sin\theta = 0 \tag{11}$$

where $\theta = 30°$. When $\mu_2 = 0$ we will obtain the same result for the velocity as in exercise 2, since the equation reduces to the same form of the one in exercise 2.

The y component of the differential Eq. (1) (i.e. Eq. (12)) is solved to obtain $p(y/h)$ with boundary conditions at $y = 0.0$, atmospheric pressure $p = 101.3kPa$.

$$-\frac{\partial p}{\partial y} + 2\mu_1 \frac{\partial \rho}{\partial y}\frac{\partial^2 \rho}{\partial y^2} + \rho * g * cos(\theta) = 0 \tag{12}$$

Here, $\frac{\partial \rho}{\partial y} = \frac{AL}{h}e^{yL/h} - \frac{BL}{h}e^{-yL/h} + C\frac{y}{h}L$

By solving the differential equations Eq. (11) and Eq. (12), the following velocity (Figure 5) and pressure (Figure 6) profiles are obtained.

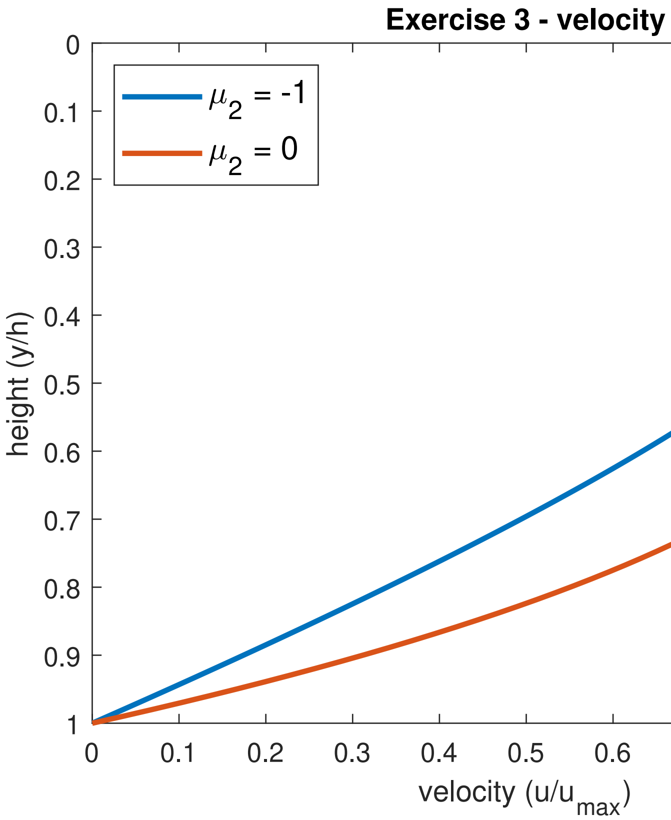

Figure 6: Velocity Profile for two different values of μ_2

The pressure $p(y)$ in the exercise 3 is obtained as

$$p(y) = p_1(y) + \mu_1 2 p_2(y) \tag{13}$$

where $p_1(y)$ is the pressure derived in exercise 2, $p2(y)$ is obtained by solving equation (12).

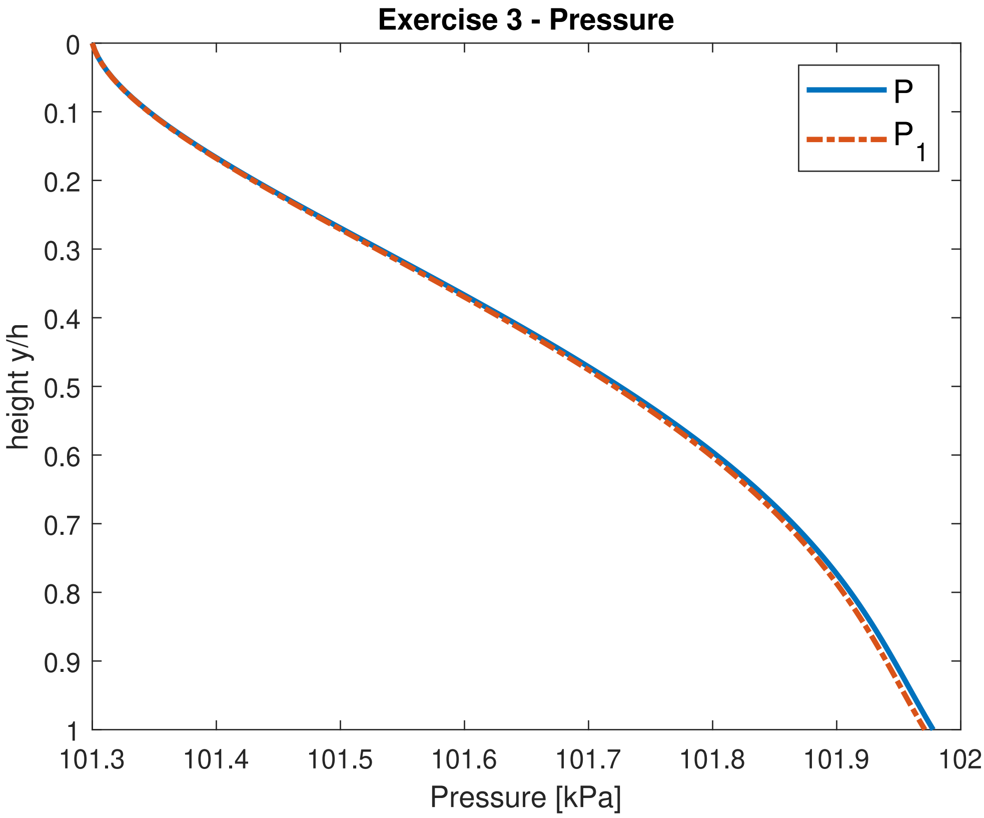

Figure 7: Pressure Profile. Comparing P(y) with P_1(y)

6 Conclusions

In this report we have modelled the flow of a granular material using continuum mechanics. The models are based on the results of the experimental work done S. B. Savage.

In order to solve the balance equation of motion we need to introduce a constitutive relation for stress tensor. We try out two different constitutive relations: one which depends linearly on the deformation tensor (exercise 2) and one which depends linearly on the deformation tensor and quadratic on the gradient of the density (exercise 3).

In figure 6 we show how the choice of constitutive relations affects the prediction of the velocity profile, and in figure 7 we do the for the pressure profile. The difference in the prediction shows the importance of establishing good constitutive relations for the models.