A Minimax Lower Bound for EEG Source Localization

Praveen Venkatesh

Pulkit Grover

(vpraveen@cmu.edu)

Carnegie Mellon University

Electrical & Computer ENGINEERING

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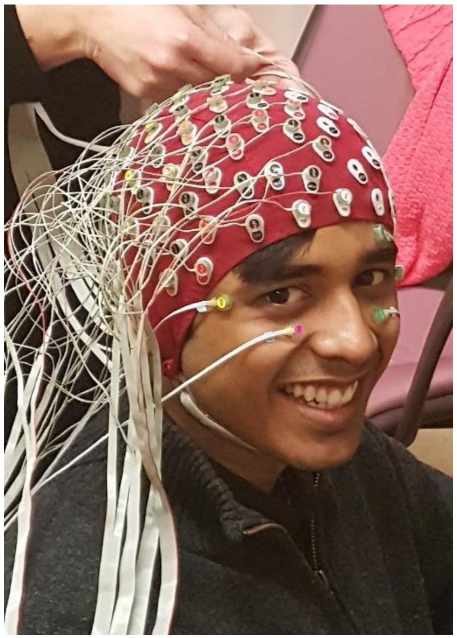
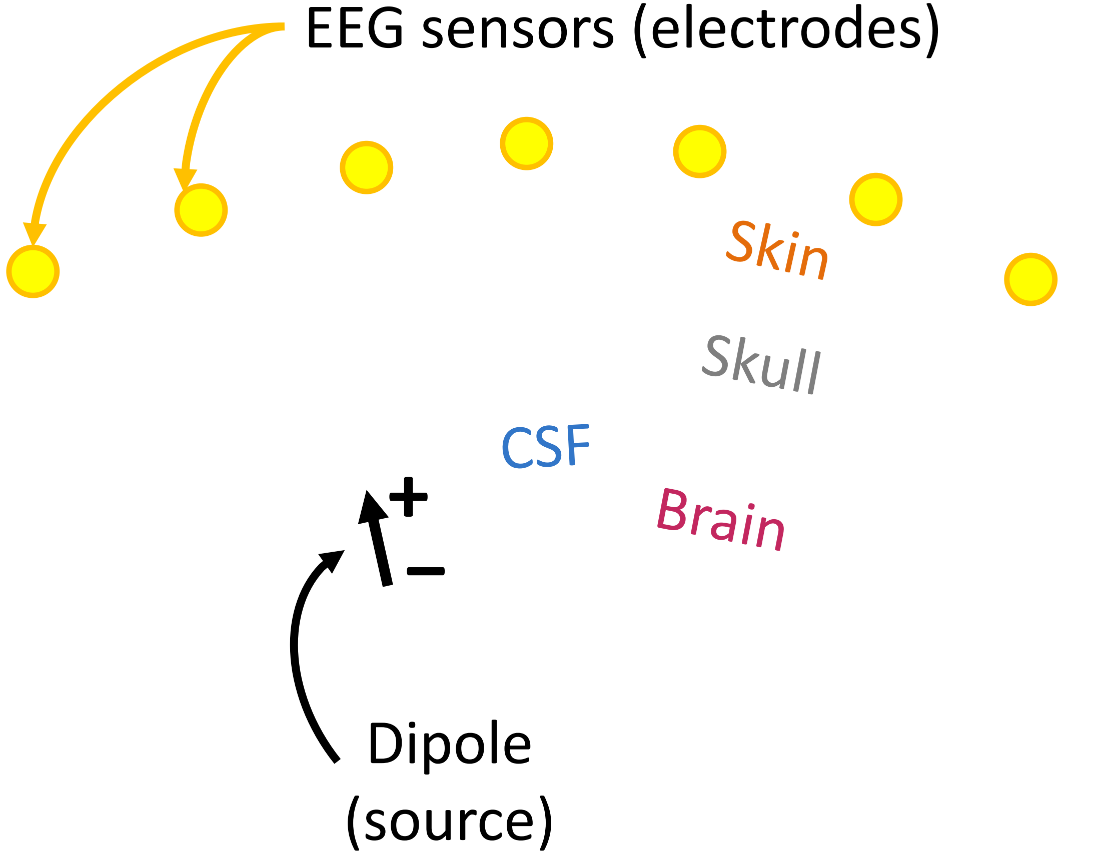
What is Electroencephalography (EEG)?

Measures brain activity (scalp potentials)





“Sources”: modeled as dipoles

Source Localization

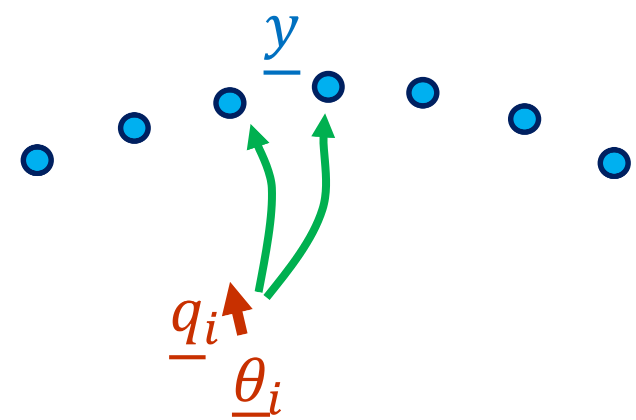
Estimate locations & dipole moments from sensor values

"Forward" problem:

Sensor values → $\underline{y} = \sum_{i=1}^{K} \boldsymbol{L}(\underline{\theta}_i) \cdot \underline{q}_i$

Location

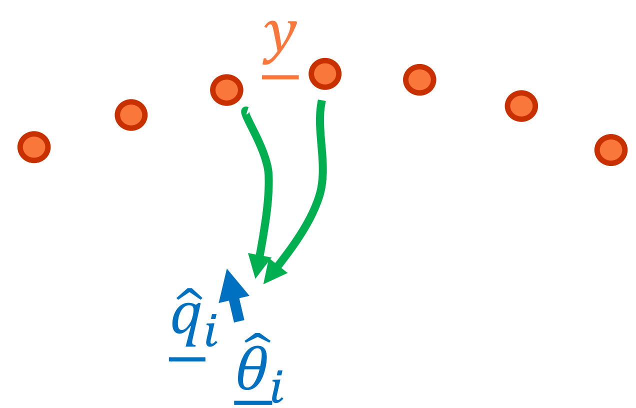
Dipole moment



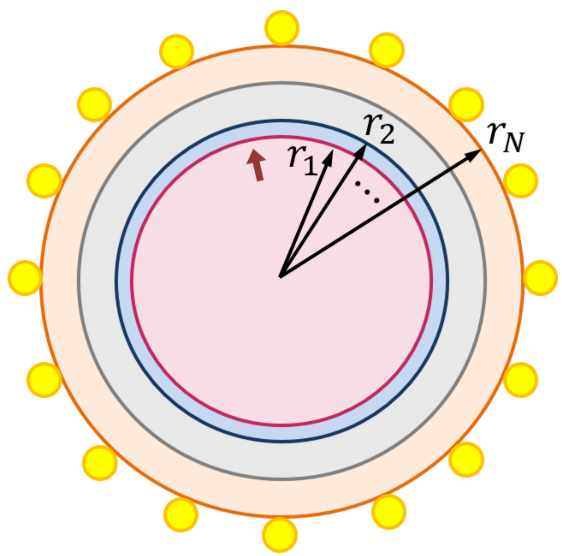
Source localization is the "inverse" problem, e.g.:

$$\left(\underline{\hat{q}}, \underline{\hat{\theta}}\right) = \arg\min_{\underline{\theta},\underline{q}} \left\|\underline{y} - \sum_{i=1}^{K} \boldsymbol{L}(\underline{\theta}_i) \cdot \underline{q}_i\right\|^2$$

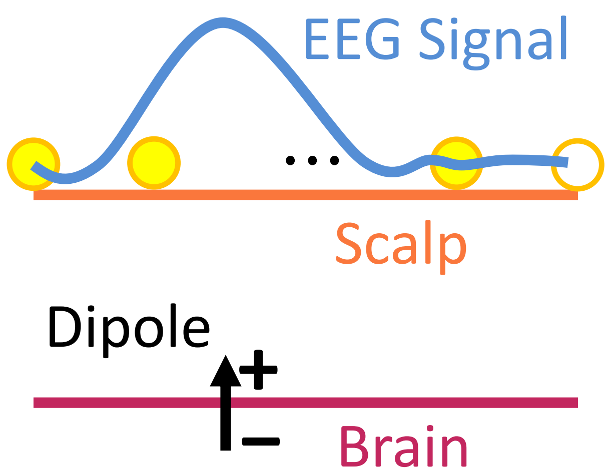
"Equivalent dipole fitting"



Simplifying the brain model

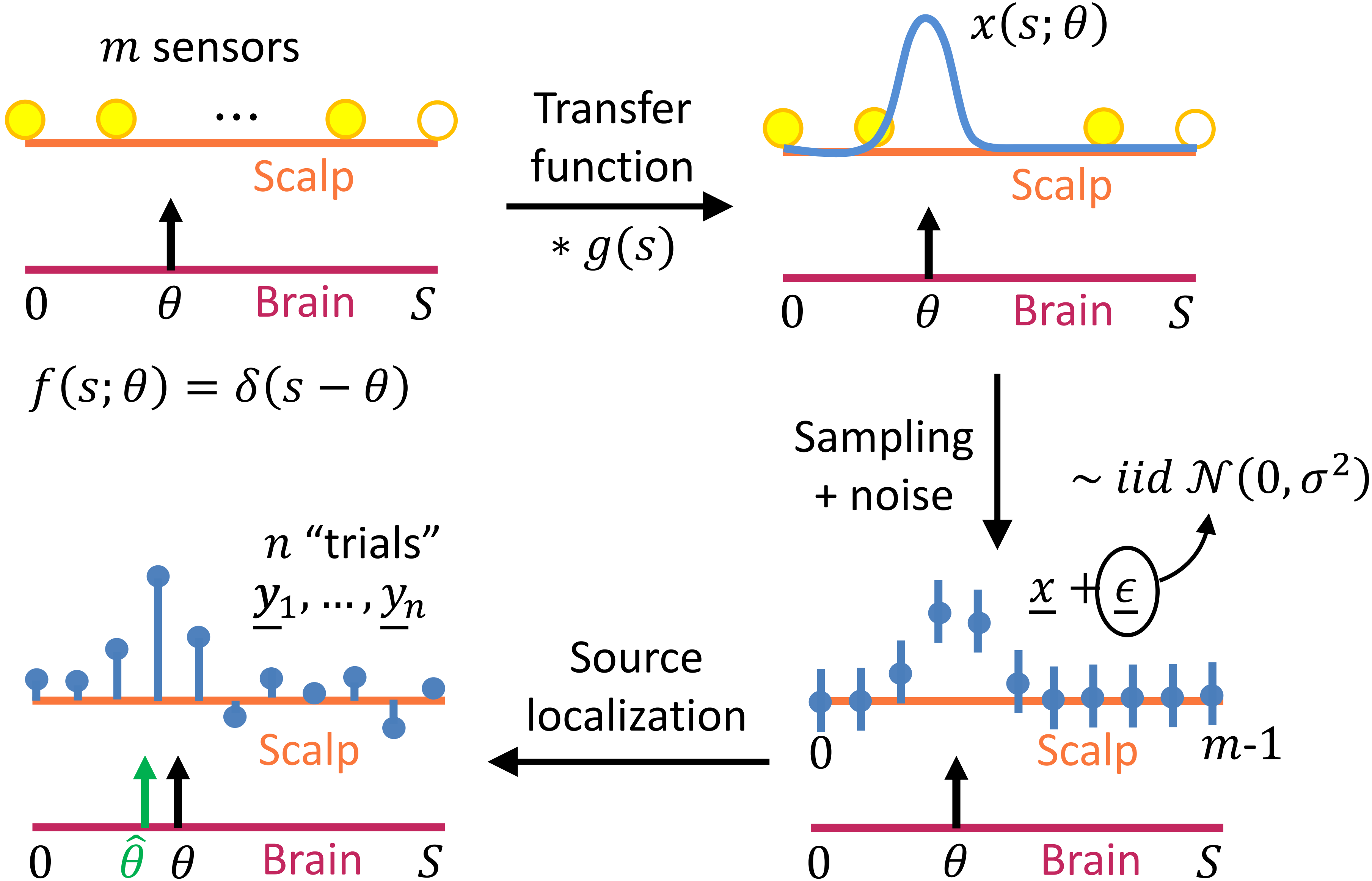


1. Spherical head model
   - ✘ Shifts are rotations
   - ✘ Spherical Harmonics
   - ✘ Uniform sampling



2. Linear head model
   - "Circular" domain
   - Known transfer function

Problem setup



Previous work on fundamental limits

- Source localization literature
  - Mosher et. al. (1993): Cramer Rao lower bounds
  - Gro ethod
- Mini blems
  - Ibra
  - Efromovich, '97
  - Cavalier and Tsybakov, '02

Did not address:

How does error scale with number of sensors?

Minimax Lower Bound: Le Cam's method

$$\theta \longrightarrow p_\theta \longrightarrow \underline{y} \longrightarrow \hat{\theta}$$

Dipole location	Data-generating distribution	Observed sensor values	Location estimate

Loss function:

$$\Phi\left(\rho\left(\hat{\theta}, \theta\right)\right) = \left\|\hat{\theta} - \theta\right\|_2^2$$

Minimax risk:

$$\inf_{\underline{\hat{\theta}}} \sup_{\underline{\theta} \in \Theta} \mathbb{E}_{\underline{y}}\left[\Phi\left(\rho\left(\hat{\theta}(\underline{y}), \theta\right)\right)\right]$$

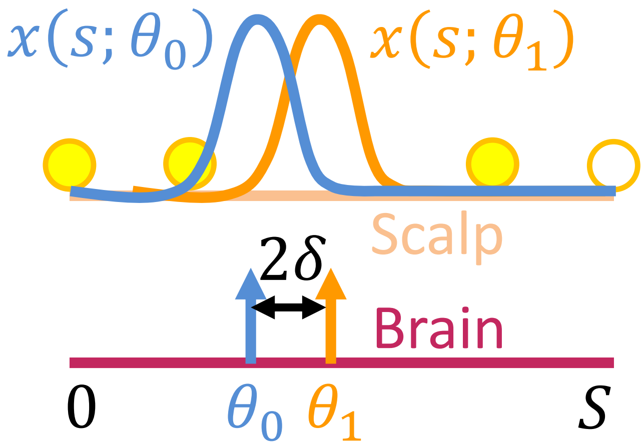
Best possible estimator

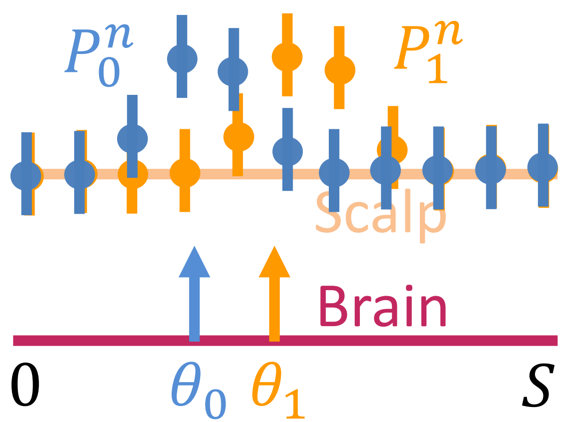
Worst-case parameters

Choose the estimator with the *best* worst-case performance

Le Cam's method

$$\Phi\left(\rho(\hat{\theta},\theta)\right) = \|\hat{\theta}-\theta\|_2^2$$





$$\inf_{\hat{\theta}} \sup_{\theta\in\Theta} \mathbb{E}_{\underline{y}}\left[\Phi\left(\rho(\hat{\theta}(\underline{y}),\theta)\right)\right]$$

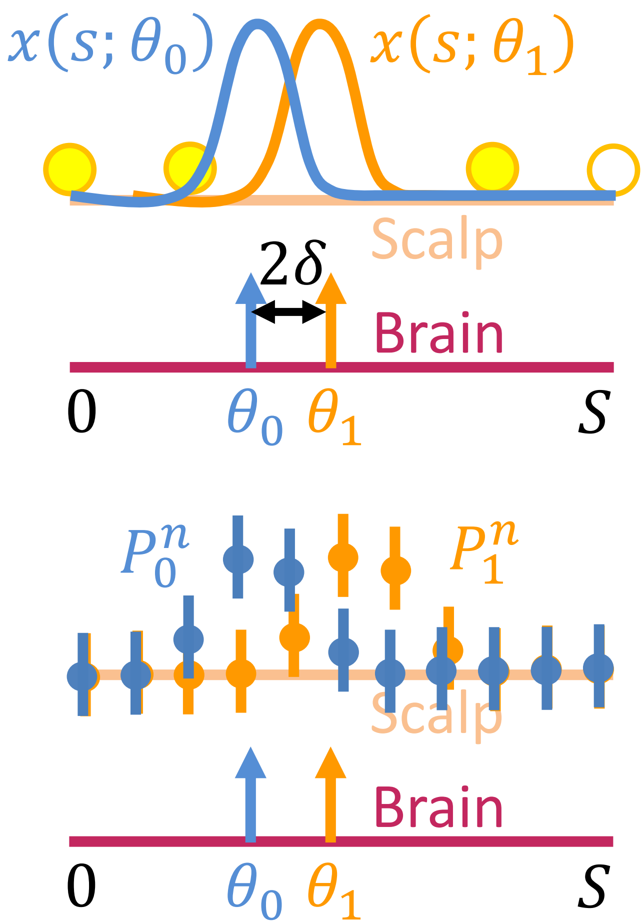
$$\geq \Phi(\delta) \inf_{\hat{V}} \mathbb{P}[\theta_{\hat{V}} \neq \theta_V]$$

$$= \frac{\delta^2}{2}[1 - \|P_1^n - P_0^n\|_{TV}]$$

$$\geq \frac{\delta^2}{2}\left[1 - \sqrt{\frac{n}{2} D_{KL}(P_0 \parallel P_1)}\right]$$

$$\geq \frac{\delta^2}{2}\left[1 - \sqrt{\frac{n}{4\sigma^2}\|\underline{x}(\theta_0) - \underline{x}(\theta_1)\|^2}\right]$$

Le Cam's method



$$\inf_{\hat{\theta}} \sup_{\theta\in\Theta} \mathbb{E}_{\underline{y}}\left[\left\|\hat{\theta}(\underline{y}) - \theta\right\|^2\right]$$

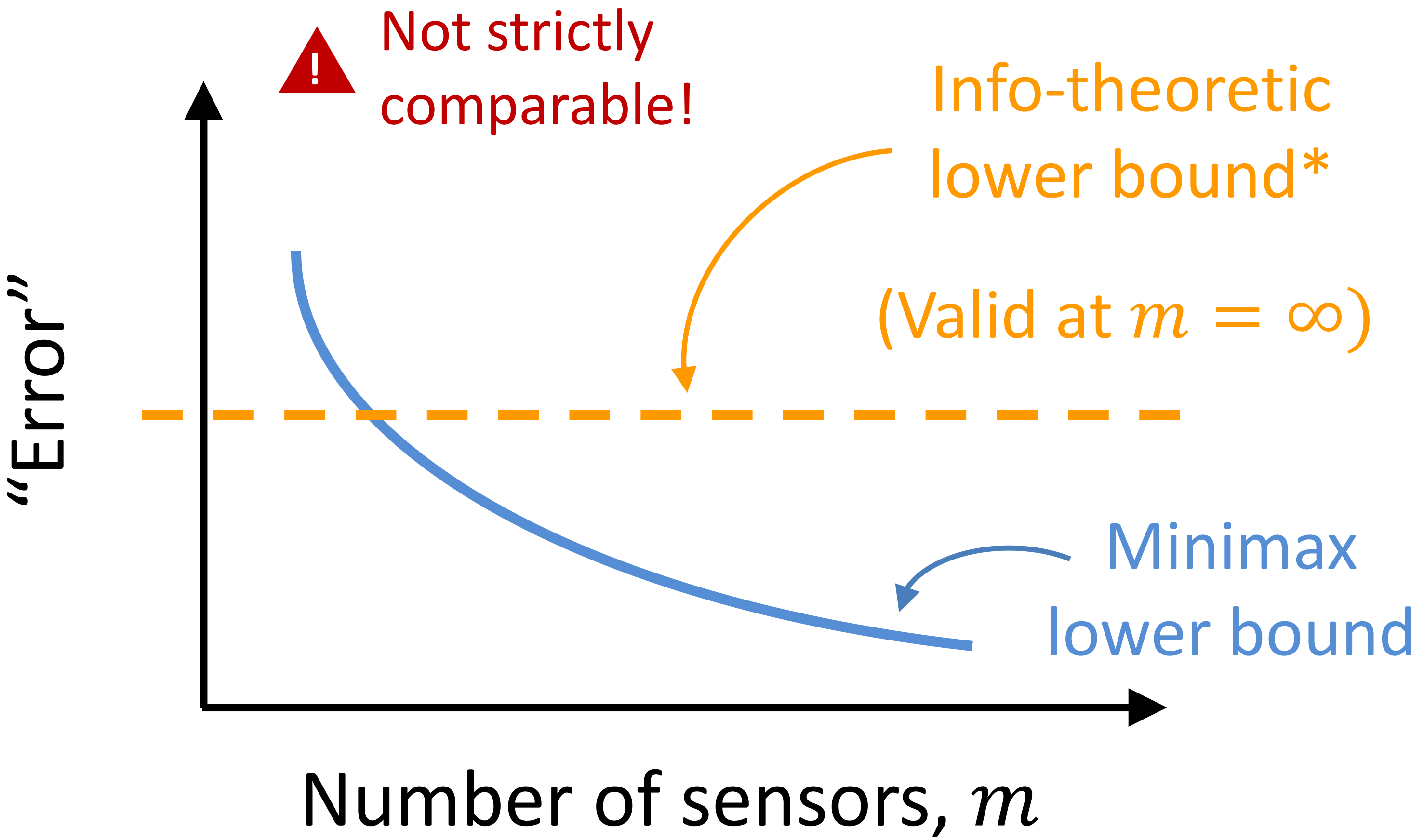
$$\geq \frac{\delta^2}{2}\left[1 - \sqrt{\frac{n}{4\sigma^2}\left\|\underline{x}(\theta_0) - \underline{x}(\theta_1)\right\|^2}\right]$$

$$\geq \frac{\delta^2}{2}\left[1 - \sqrt{\frac{n}{4\sigma^2}\kappa^2 4\delta^2 \cdot m}\right]$$

$$\approx \frac{1}{32}\,\frac{\sigma^2/n}{m\,\kappa^2}\,\frac{S}{w} \xrightarrow{m\to\infty} 0$$

So what's wrong with this?

This bound is loose!



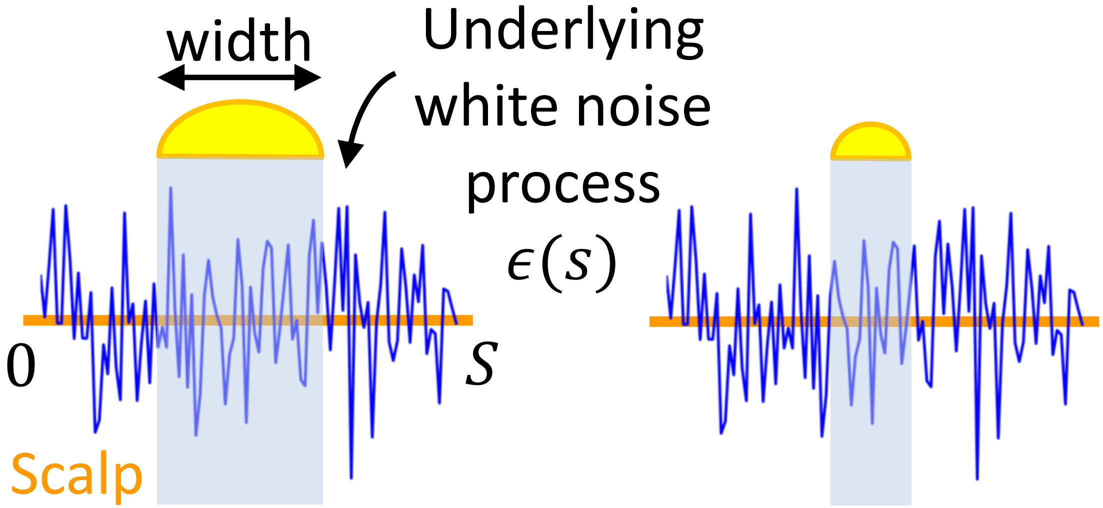
*Pulkit Grover, ISIT '16

Sensor model!

Noise $\sim iid\ \mathcal{N}(0, \sigma^2)$,
independent of m

SNR *decreases* as
# of sensors increases

$$y_k = \int_{\text{width}} \big(x(s;\theta) + \epsilon(s)\big)\, ds$$



More averaging

Less averaging

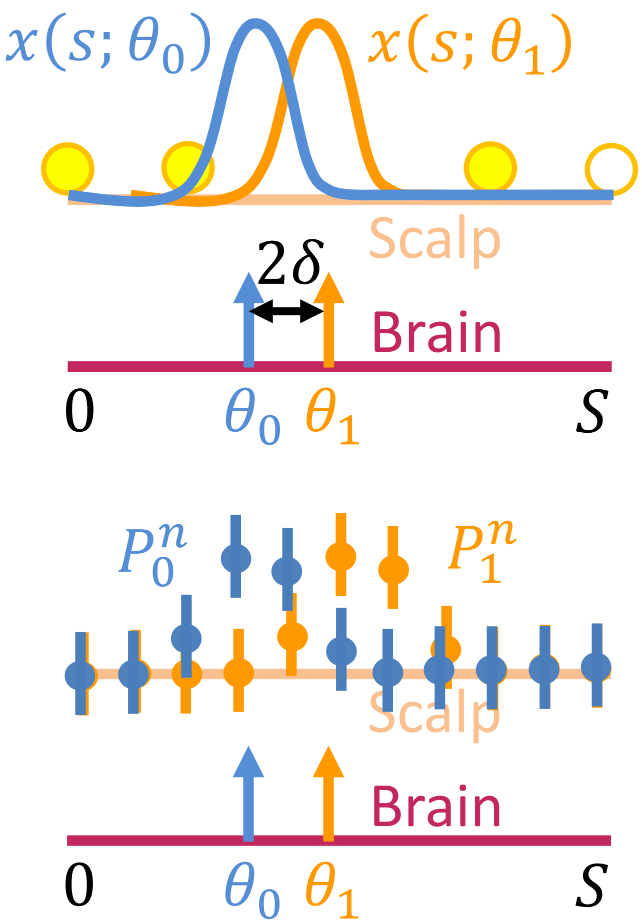
Noise var. ∝ 1 / width

width ∝ 1 / #sensors

Noise var. ∝ #sensors

SNR ∝ 1 / #sensors

Bounds for the “integrator sensor” model



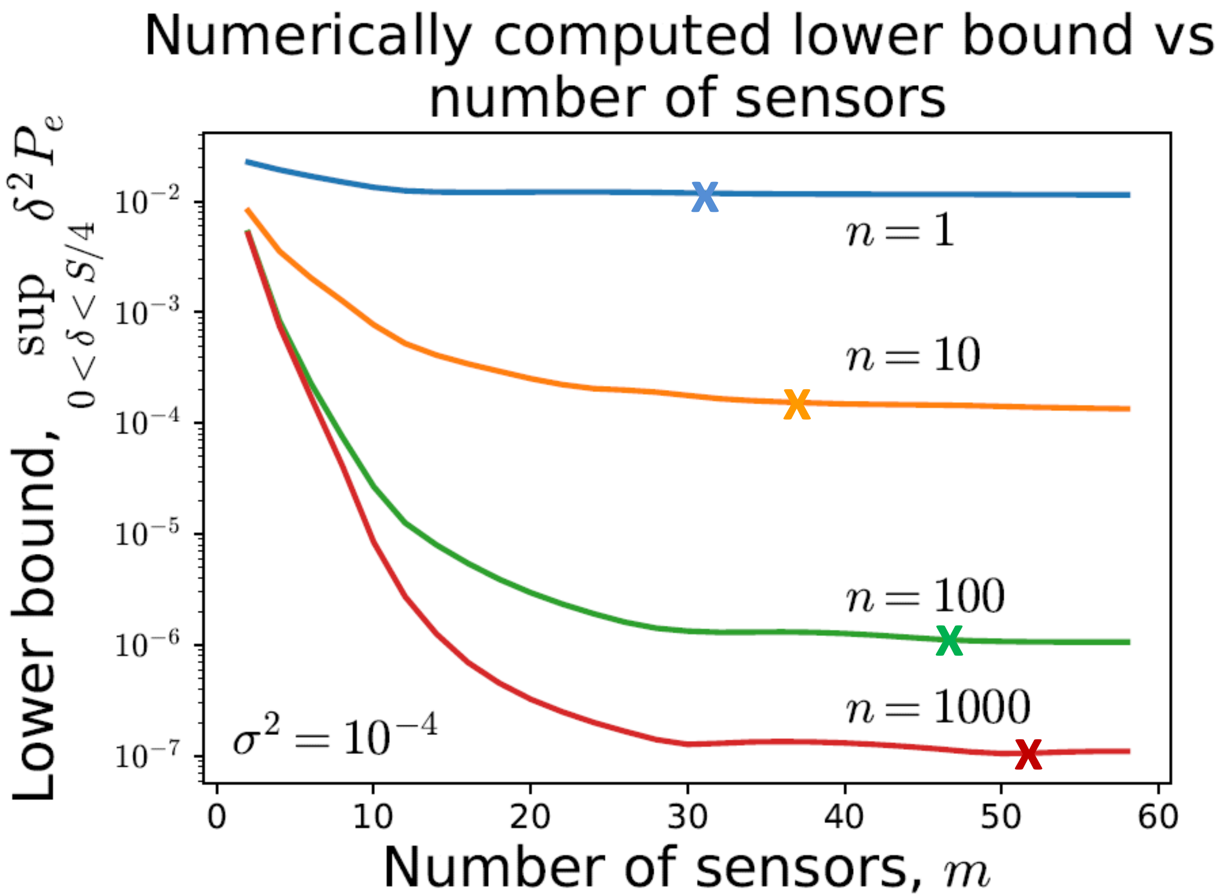
$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_{\underline{y}} \left[\left\| \hat{\theta}(\underline{y}) - \theta \right\|^2 \right]$$

$$\geq \Phi(\delta) \inf_{\hat{V}} \mathbb{P}\left[\hat{V} \neq V\right]$$

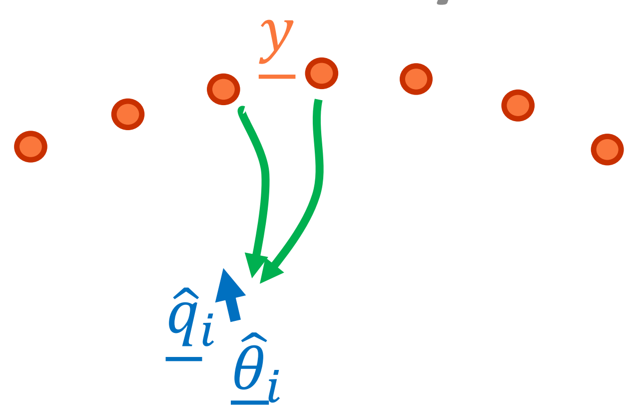
$$\geq \delta^2 Q\left(\frac{\left\| \underline{x}(\theta_0) - \underline{x}(\theta_1) \right\|}{\sigma(m)} \right)$$

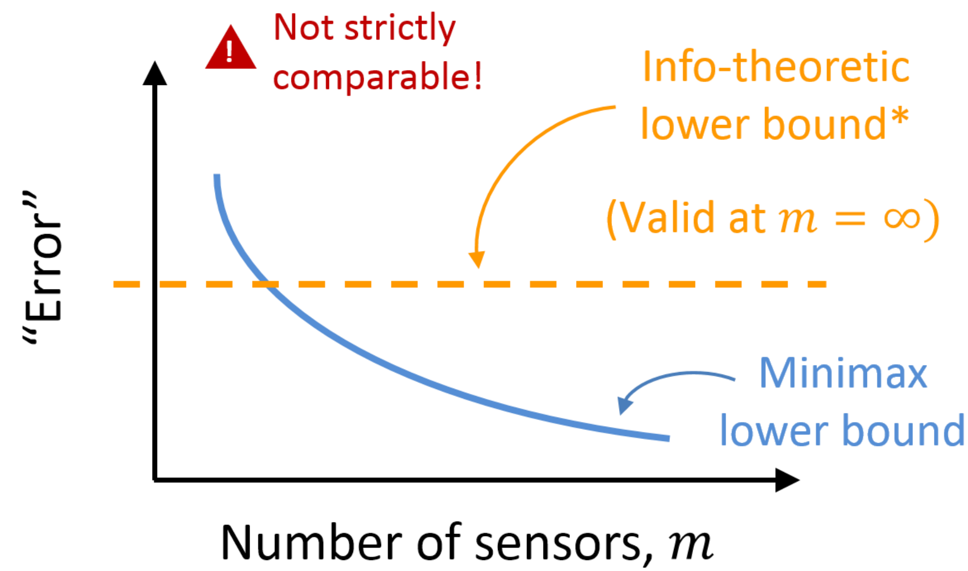
(For gaussian noise)

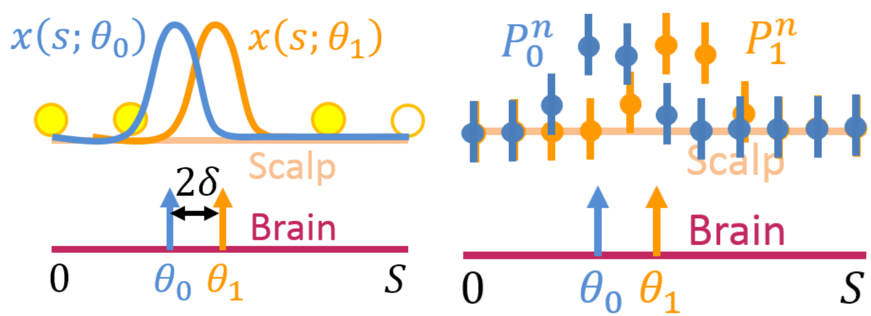
Bounds for the “integrator sensor” model

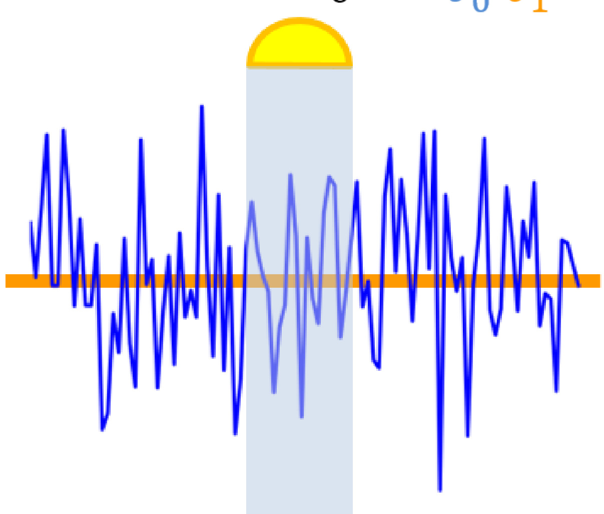


Summary

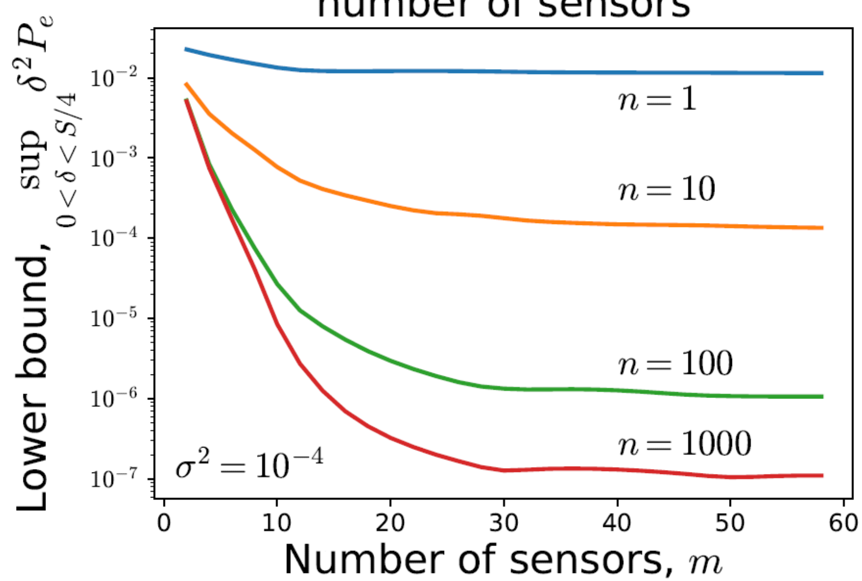




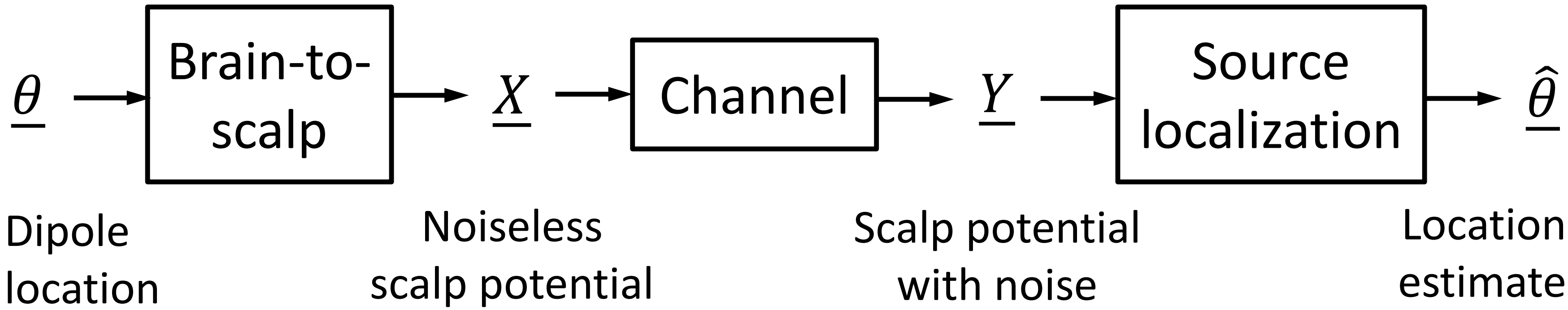




Numerically computed lower bound vs. number of sensors



Shortcomings of the info-theory bound



$$\mathbb{E}\left\|\underline{\hat{\theta}} - \underline{\theta}\right\|^2 \sim I\left(\underline{\theta}; \underline{\hat{\theta}}\right) \leq I\left(\underline{X}; \underline{Y}\right) \leq C \text{ (channel capacity)}$$

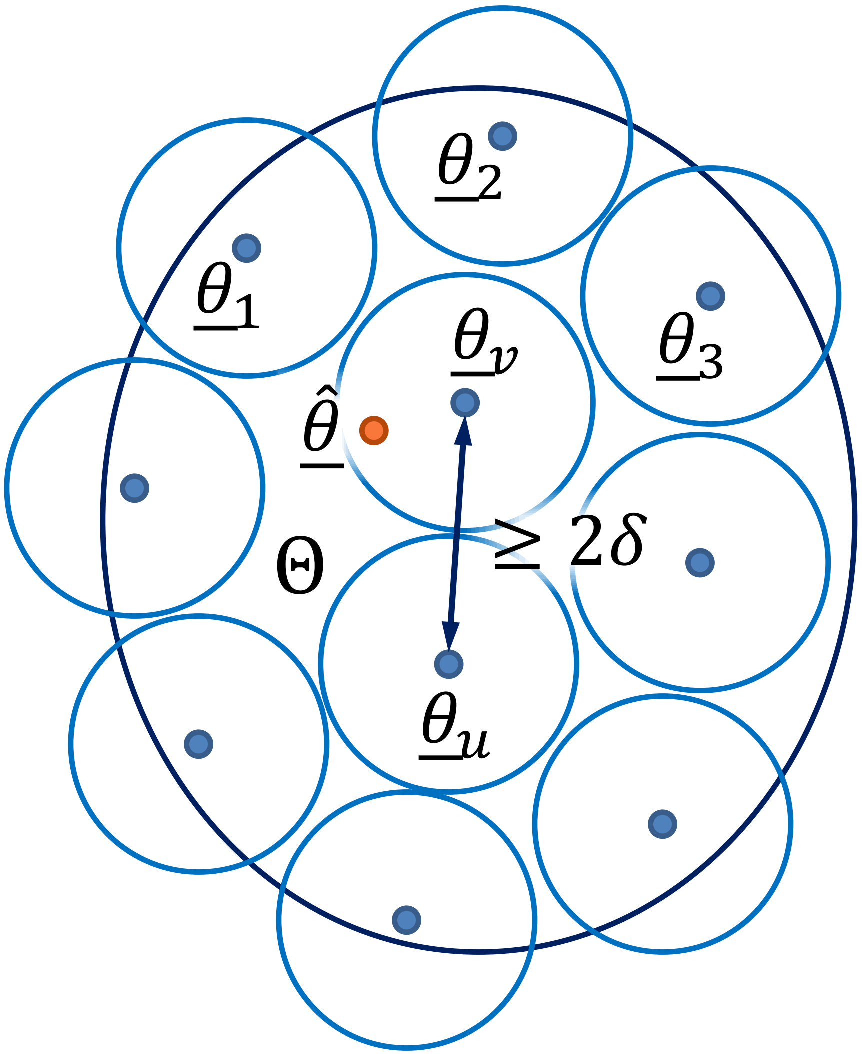
$\underline{X}$ and $\underline{Y}$ are "continuous-space" signals

$\Rightarrow$ Assumes an infinite number of sensors

Can severely underestimate the lower bound!

e.g. imagine if you had only 10 sensors

Lower bounding estimation error with hypothesis testing error

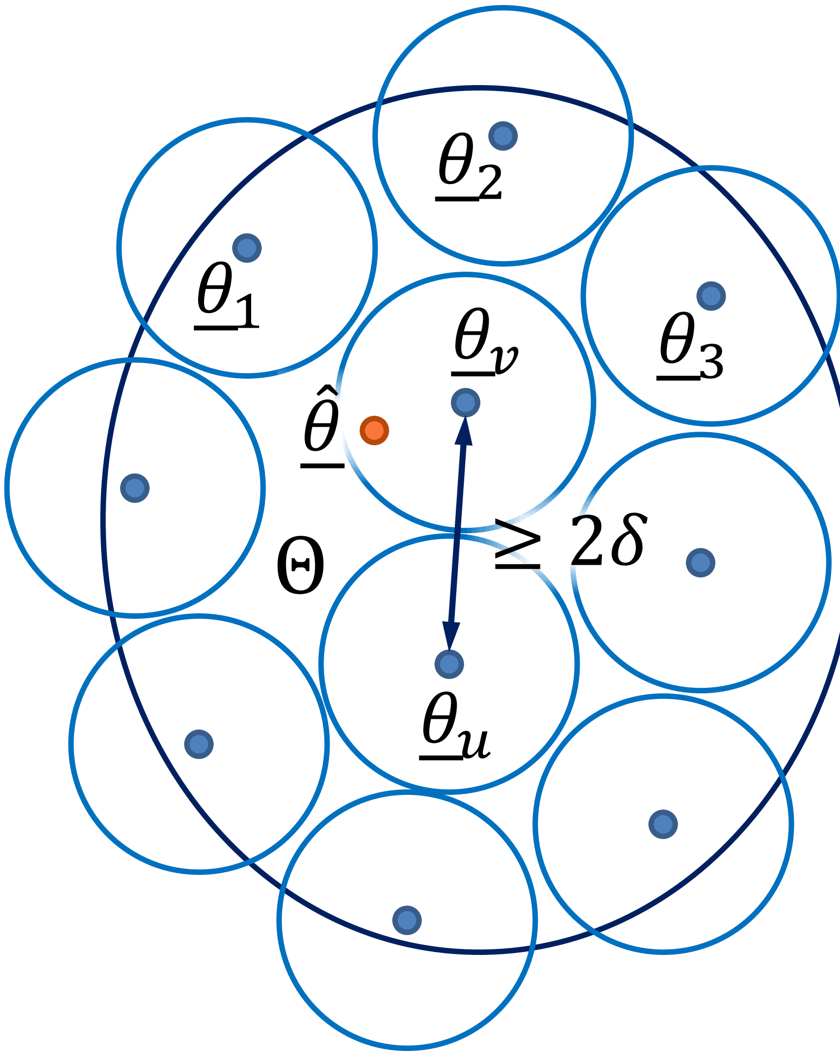


- Assume $\underline{\theta} \in \{\underline{\theta}_v\}_{v=1}^{\mathcal{V}}$
- V = unknown index
- $\hat{V}$ = Estimator of V

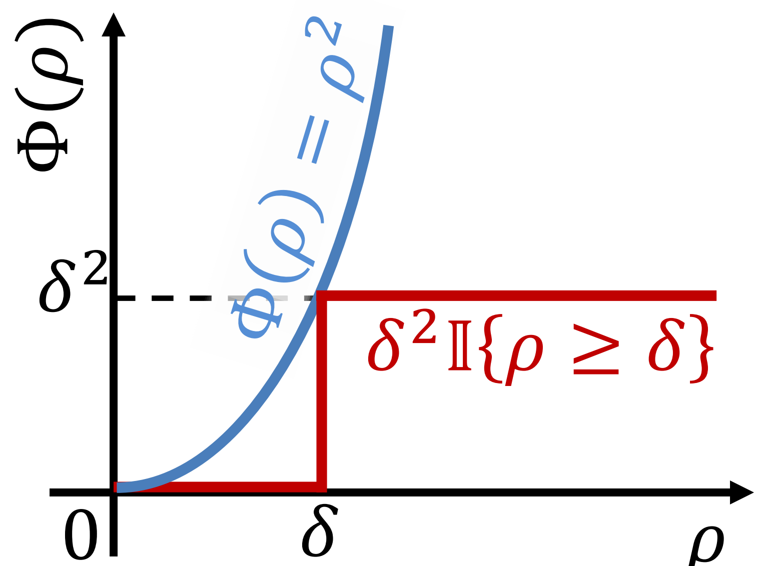
$$\hat{V}(Y) = \arg\min_{v \in \mathcal{V}} \rho\left(\underline{\hat{\theta}}(Y), \underline{\theta}_v\right)$$

Construct a "2δ-packing" of Θ

Relating error in estimation and hypothesis testing



$$\mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta}_v)\right)\right]$$
$$\geq \mathbb{E}\left[\Phi(\delta)\,\mathbb{I}\{\rho(\underline{\hat{\theta}}, \underline{\theta}_v) \geq \delta\}\right]$$
$$= \Phi(\delta)\,\mathbb{P}\left[\rho(\underline{\hat{\theta}}, \underline{\theta}_v) \geq \delta\right]$$
$$= \Phi(\delta)\,\mathbb{P}\left[\hat{V} \neq v \mid V = v\right]$$



From estimation to testing

Need to bound: $\inf_{\underline{\hat{\theta}}} \sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right]$

$$\sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right] \geq \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta_v})\right)\right]$$

$$\mathcal{V} \cdot \sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right] \geq \sum_{v=1}^{\mathcal{V}} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta_v})\right)\right]$$

$$\sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right] \geq \frac{1}{\mathcal{V}} \sum_{v=1}^{\mathcal{V}} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta_v})\right)\right]$$

Putting it together

$$\inf_{\underline{\hat{\theta}}} \sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right]$$

$$\geq \inf_{\underline{\hat{\theta}}} \frac{1}{\mathcal{V}} \sum_{v=1}^{\mathcal{V}} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta}_v)\right)\right]$$

$$\sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta})\right)\right] \geq \frac{1}{\mathcal{V}} \sum_{v=1}^{\mathcal{V}} \mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta}_v)\right)\right]$$

$$\geq \inf_{\underline{\hat{V}}} \frac{1}{\mathcal{V}} \sum_{v=1}^{\mathcal{V}} \Phi(\delta)\, \mathbb{P}\left[\hat{V} \neq v \mid V = v\right]$$

$$\mathbb{E}\left[\Phi\left(\rho(\underline{\hat{\theta}}, \underline{\theta}_v)\right)\right] \geq \Phi(\delta)\, \mathbb{P}\left[\hat{V} \neq v | V = v\right]$$

$$= \Phi(\delta) \inf_{\hat{V}} \mathbb{P}\left[\hat{V} \neq V\right]$$

Need to bound this next

V takes values $\{1, \dots, \mathcal{V}\}$ uniformly

Le Cam's method

Binary hypothesis: $V \in \{0,1\}$ (uniformly)

$$\mathbb{P}\left[\hat{V} \neq V\right] = \frac{1}{2} P_{V=0}\left(\hat{V} \neq 0\right) + \frac{1}{2} P_{V=1}\left(\hat{V} \neq 1\right)$$

Define $A = \left\{\underline{Y} : \hat{V}\left(\underline{Y}\right) = 1\right\}$, "acceptance region"

$$\frac{1}{2}\left[P_0\left(\hat{V} \neq 0\right) + P_1\left(\hat{V} \neq 1\right)\right] = \frac{1}{2}\left[P_0(A) + P_1(A^c)\right]$$

$$= \frac{1}{2}\left[P_0(A) + 1 - P_1(A)\right]$$

Taking infimum:

$$\inf_{\hat{V}} \mathbb{P}\left[\hat{V} \neq V\right] = \frac{1}{2} \inf_{A} \left\{1 - \left(P_1(A) - P_0(A)\right)\right\}$$

$$= \frac{1}{2}\left[1 - \sup_{A} \left\{P_1(A) - P_0(A)\right\}\right]$$

$$= \frac{1}{2}\left[1 - \|P_1 - P_0\|_{TV}\right]$$

Le Cam's method for source localization

$$\|P_1^n - P_0^n\|_{TV}^2 \leq \frac{1}{2} D_{KL}(P_1^n \| P_0^n) = \frac{n}{2} D_{KL}(P_1 \| P_0)$$

For normal distributions of equal variance,

$$D_{KL}(P_1 \| P_0) = \frac{1}{2\sigma^2} \left\| \boldsymbol{L}(\underline{\theta}_0)\underline{q} - \boldsymbol{L}(\underline{\theta}_1)\underline{q} \right\|^2 = \frac{1}{2\sigma^2} d(\underline{\theta}_0, \underline{\theta}_1)$$

$$\inf_{\hat{\underline{\theta}}} \sup_{\underline{\theta} \in \Theta} \mathbb{E}\left[\Phi\left(\rho(\hat{\underline{\theta}}, \underline{\theta})\right)\right] \geq \sup_{\underline{\theta}_0, \underline{\theta}_1} \frac{\delta^2}{2}\left[1 - \sqrt{\frac{n}{4\sigma^2} d(\underline{\theta}_0, \underline{\theta}_1)}\right]$$