Paramesh
Anisotropic calculations 2D and 3D

May 3, 2012

1 Anisotropic calculations

Given

$$G = \int g(\nabla\phi) \equiv \int \tfrac{1}{2} A^2 |\nabla\phi|^2$$

First we assume the identity

$$-\frac{\delta G}{\delta\phi} = \nabla \cdot \frac{\partial g}{\partial \nabla\phi} \equiv \phi_{,ij} \frac{\partial}{\partial\phi_i} \frac{\partial g}{\partial\phi_j}$$

where

$$\phi_{,ij} \equiv \frac{\partial^2\phi}{\partial x^i \partial x^j}$$

We also denote (without comma)

$$g_i \equiv \frac{\partial g}{\partial\phi_i}, \quad g_{ij} \equiv \frac{\partial^2 g}{\partial\phi_i \partial\phi_j}$$

so that

$$g_i = A A_i |\nabla\phi|^2 + A^2 \phi_i$$

and

$$g_{ij} = A_i A_j |\nabla\phi|^2 + 2A(A_i\phi_j + A_j\phi_i) + A|\nabla\phi|^2 A_{ij} + A^2\phi_{ij}$$

where we use the notation

1.1 2 D

Given

$$A \equiv A_0 \left[1 + \epsilon \left(\frac{\phi_{,1}^4 + \phi_{,2}^4}{|\nabla\phi|^4}\right)\right] \equiv A_0 \left[1 + \epsilon \left(1 - \frac{2\phi_{,1}^2\phi_{,2}^2}{|\nabla\phi|^4}\right)\right]$$

we find using $X = (\phi_{,x}/|\nabla\phi|)^2, Y = (\phi_{,y}/|\nabla\phi|)^2$ and $A_i \equiv \dfrac{\partial A}{\partial \phi_{,,i}}$, that

$$A_1 = 4\,\frac{A_0\,\epsilon\,\phi_{,1} Y\,(X - Y)}{|\nabla\phi|^2}$$

$$A_2 = 4\,\frac{A_0\,\epsilon\,X\phi_{,2}\,(Y - X)}{|\nabla\phi|^2}$$

$$A_{11} = -4\,\frac{A_0\,\epsilon\,Y\left(3\,X^2 - 8\,XY + Y^2\right)}{|\nabla\phi|^2}$$

$$A_{22} = -4\,\frac{A_0\,\epsilon\,X\left(X^2 - 8\,XY + 3\,Y^2\right)}{|\nabla\phi|^2}$$

$$A_{12} = 8\,\frac{A_0\,\epsilon\,\phi_{,1}\phi_{,2}\left(X^2 - 4\,XY + Y^2\right)}{|\nabla\phi|^4}$$

To avoid floating point errors define

$$\tilde{A}_i = A_i|\nabla\phi|^2 \text{ and } \tilde{A}_{ij} = A_{ij}|\nabla\phi|^2$$

so that

$$g_{ij} = \frac{1}{|\nabla\phi|^2}\left[\tilde{A}_i\tilde{A}_j + 2A(\tilde{A}_i\phi_{,j} + \tilde{A}_j\phi_{,i})\right] + A\tilde{A}_{ij} + A^2\delta_{ij}$$

1.2 3D

$$\begin{aligned}
A &\equiv A_0\left[1 + \epsilon\left(\frac{\phi_1^4 + \phi_2^4 + \phi_3^4}{|\nabla\phi|^4}\right)\right] \\
A_1 &= \frac{4A_0\epsilon\phi_{,1}}{|\nabla\phi|^2}(X(Y+Z) - Y^2 - Z^2) \\
A_{12} &= \frac{8A_0\epsilon\phi_{,1}\phi_{,2}}{|\nabla\phi|^4}(X^2 + Y^2 - 4XY - 2Z(X+Y) + 3Z^2) \\
A_{11} &= \frac{-4A_0\epsilon}{|\nabla\phi|^2}(3X^2(Y+Z) - 8X(Y^2+Z^2) - 6XYZ + Y^3 + Y^2Z + Z^2Y + Z^3)
\end{aligned}$$

and we can find the other A_{ij} by cyclic permutation.

1.3 Isolating the Laplacian

in 2D

$$\begin{aligned}
\phi_{,ij}g_{ij} &= g_{11}\phi_{,11} + g_{22}\phi_{,22} + 2g_{12}\phi_{,12} \\
&= \tfrac{1}{2}(g_{11} + g_{22})(\phi_{,11} + \phi_{,22}) + \tfrac{1}{2}(g_{11} - g_{22})(\phi_{,11} - \phi_{,22}) + 2g_{12}\phi_{,12} \\
&= \tfrac{1}{2}(g_{11} + g_{22})\nabla^2\phi + \tfrac{1}{2}(g_{11} - g_{22})(\phi_{,11} - \phi_{,22}) + 2g_{12}\phi_{,12}
\end{aligned}$$

For the purposes of the pointwise Newton-Raphson in the multigrid smoother, the functions, g_{ij}, are discretised without using the central node. So the only contributions to the Newton method are from the discretisation of the Laplacian. Example, the 5 point stencil has a contribution from the central node, $-4/\Delta x^2$, giving net contribution to the smoother of

$$-\tfrac{1}{2}(g_{11} + g_{22})4/\Delta x^2$$

Moreover, different stencils may be used for the Laplacian making it advantageous to isolate this term.

The procedure above is equivalent to writing the matrix $\mathbf{g}$ as

$$\mathbf{g} = \tfrac{1}{2}\mathrm{tr}(\mathbf{g})\mathbf{I} + \tilde{\mathbf{g}},$$

where $\mathbf{I}$ is the identity, so that

$$\mathrm{tr}(\tilde{\mathbf{g}}) = 0$$

and taking advantage of $I_{ij}\phi_{ij} = \nabla^2\phi$. Similarly in 3D

$$\mathbf{g} = \tfrac{1}{3}\mathrm{tr}(\mathbf{g})\mathbf{I} + \tilde{\mathbf{g}}$$

guarantees $\mathrm{tr}(\tilde{\mathbf{g}}) = 0$ and thus

$$\phi_{,ij}g_{ij} = \tfrac{1}{3}\mathrm{tr}(\mathbf{g})\nabla^2\phi + \phi_{,ij}\tilde{g}_{ij}$$