Assignment 19

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Abstract

This document illustrates the concept of orthonormal basis.

Download the latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment19

1 Problem

Let $\{u_1, u_2, ..., u_n\}$ be an orthonormal basis of C^n as column vectors. Let $M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$ and P be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Then which of the following is true?

1. Rank($\mathbf{MPM}^*$) = $\mathbf{k}$ whenever $\alpha_i \neq \alpha_j$, $1 \leq i, \ j \leq k$
2. Trace($\mathbf{MPM}^*$) = $\sum_{i=1}^{k} \alpha_i$
3. Rank($\mathbf{M}^*\mathbf{N}$) = min(k,$n-k$)
4. Rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$

2 Definitions

Orthonormal Basis	$B = \{u_1, u_2, ..., u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $< u_i, u_j > = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $< u_i, u_j > = 1$ otherwise.
Trace	Trace of a square matrix A, denoted by $\mathbf{tr}(\mathbf{A})$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\mathbf{tr}(\mathbf{AB}) = \mathbf{tr}(\mathbf{BA})$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix
Basis Theorem	A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent,then it is invertible.

TABLE 1: Definitions

3 Solution

Rank($\mathbf{MPM^*}$) = $\mathbf{k}$	M and M^* vectors are linearly independent and thus it is invertible (Since the elementary matrices are invertible, such multiplication does not change the rank of a matrix) $\implies$ Rank($\mathbf{MPM^*}$) = Rank($\mathbf{P}$) Now $\mathbf{P}$ be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Rank($\mathbf{P}$) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0. Thus, Rank($\mathbf{MPM^*}$) $\neq$ $\mathbf{k}$ Thus, the given statement is false
Trace($\mathbf{MPM^*}$) $= \sum_{i=1}^{k} \alpha_i$	Trace($\mathbf{MPM^*}$) = Trace($\mathbf{M^*MP}$) (According to properties of trace mentioned in Definitions) $\mathbf{M} = \begin{pmatrix} u_1 & u_2 & u_3 & .... & u_k \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} \bar{u_1} \\ \bar{u_2} \\ \bar{u_3} \\ . \\ . \\ . \\ \bar{u_k} \end{pmatrix}$ $\mathbf{M^*M} = \begin{pmatrix} \bar{u_1}u_1 & 0 & 0 & ... & 0 \\ 0 & \bar{u_2}u_2 & 0 & ... & 0 \\ 0 & 0 & \bar{u_3}u_3 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \bar{u_k}u_k \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 0$ if $i \neq j$) $\mathbf{M^*M} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 1$ if $i = j$) $\mathbf{M^*M = I^k}$ $\mathbf{M^*MP = I^kP = P}$ Trace($\mathbf{M^*MP}$) = Trace($\mathbf{I^kP}$) = Trace($\mathbf{P}$) $= \sum_{i=1}^{k} \alpha_i$ (Refer Definition section of Trace, it is sum of elements on the main diagonal) So, the given statement is true
Rank($\mathbf{M^*N}$) = min(k,$n-k$)	$M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors,

	$u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $u_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $u_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $k = 2$, then $M = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $M^* = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$ $N = \begin{pmatrix}u_3 & u_4\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}$ $M^*N = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ $\text{Rank}(M^*N) = 0$ But, $\min(k,n-k) = (2,2) = 2$ And, this is clear from above that $\text{Rank}(\mathbf{M}^*\mathbf{N}) \neq \min(k,n-k)$ Thus, above statement is false
$\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$	$\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)$ $\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)$ $\text{Rank}(\mathbf{M}+\mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})$ $M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $u_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $u_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $k = 2$, then $M = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $\text{Rank}(\mathbf{M}) = 2 = k$ $N = \begin{pmatrix}u_3 & u_4\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}$

	$\text{Rank}(\mathbf{N}) = 2 = n - k$ Thus, $\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) = \text{Rank}(\mathbf{M} + \mathbf{N}) = 4 = n$ Thus, above statement is false

TABLE 2: Finding of True and False Statements

4 Conclusion

$\text{Rank}(\mathbf{MPM}^*) = \mathbf{k}$	False
$\text{Trace}(\mathbf{MPM}^*) = \sum_{i=1}^{k} \alpha_i$	True
$\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k,n - k)$	False
$\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$	False

TABLE 3: Conclusion of above Solutions