Assignment 18

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Abstract

This document illustrates the concept of minimal polynomial, null space and basis.

Download the latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment18

1 Problem

Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \tag{1.0.1}$$

Express the minimal polynomial p for T in the form $p = p_1 p_2$, where p_1 and p_2 are monic and irreducible over the field of real numbers. Let W_i be the null space of $p_i(T)$. Find the basis B_i for the spaces W_1 and W_2. If T_i is the operator induced on W_i by T_i, find the matrix of T_i in the basis B_i above.

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left|x\mathbf{I} - \mathbf{A}\right|$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial and the minimal polynomial divides the charateristic polynomial.
Basis Theorem	Let V be a subspace of dimension m. Then: Any m linearly independent vectors in V forms a basis for V. Any m vectors that span V forms a basis for V.

TABLE 1: Definitions

3 Solution

Express Minimal Polynomial	$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ Characteristic Polynomial $= \lvert xI - A \rvert = \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix}$ By solving above determinant, we find out that $x^3 - 2x^2 + x - 2 = (x-2)\left(x^2+1\right)$ Since, $T - 2I \neq 0$ and the minimal polynomial divides the characteristic polynomial, thus minimal polynomial p for T is $p = m(x)$ $p = (x-2)\left(x^2+1\right)$ Put $p_1 = (x-2)$ and $p_2 = \left(x^2+1\right)$ Thus, $p = p_1 p_2$
Bases B_1 and matrix T_1	Let $W_1 = \{\alpha \in R^3 / p_1(T)\alpha = 0, (T-2I)\alpha = 0\}$ Therefore, $A - 2I = \begin{pmatrix} 4 & -3 & -2 \\ 4 & -3 & -2 \\ 10 & -5 & -5 \end{pmatrix} \to \begin{pmatrix} -4 & 3 & 2 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Rank of $A - 2I$ is 2 Nullity of $A - 2I$ = no of columns - Rank = 3 - 2 = 1 That means the dimension of W_1 is 1 Thus we can let, $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1$ (Basis theorem mentioned in Definitions) Therefore, $B_1 = \{\alpha_1\}$ is the basis for W_1 Let T_1 be the matrix induced by T on W_1 $T_1\alpha_1 = T\alpha_1 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 2\alpha_1$ $[T_1]_{B_1} = [2]$
Bases B_2 and matrix T_2	Let $W_2 = \{\alpha \in R^3 / p_2(T)\alpha = 0, \left(T^2 + I\right)\alpha = 0\}$ Therefore, $A^2 + I = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{pmatrix}$ Rank of $A^2 + I$ is 1 Nullity of $A^2 + I$ = no of columns - Rank = 3 - 1 = 2 That means the dimension of W_2 is 2 Thus we can let, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2$(Basis theorem in Definitions) Therefore, $B_2 = \{\alpha_2, \alpha_3\}$ is the basis for W_2 Let T_2 be the matrix induced by T on W_2

	$T_2\alpha_2 = T\alpha_2 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3\alpha_2 + 5\alpha_3$ $T_2\alpha_3 = T\alpha_3 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} = \text{-}2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \text{-}3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \text{-}2\alpha_2 \text{ - } 3\alpha_3$ $\begin{pmatrix} \alpha_2 & \alpha_3 \end{pmatrix} [T_2] = \begin{pmatrix} \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$ $\implies [T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$

TABLE 2: Finding of Basis and corresponding matrix

4 Summarization of Above Results

Express Minimal Polynomial	$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ We get, $p_1 = (x-2)$ and $p_2 = \left(x^2+1\right)$ Thus, $p = p_1 p_2$
W_i	$W_1 = \{\alpha \in R^3 / p_1(T)\alpha = 0, (T-2I)\alpha = 0\}$ $W_2 = \{\alpha \in R^3 / p_2(T)\alpha = 0, \left(T^2+I\right)\alpha = 0\}$
B_1	$B_1 = \{\alpha_1\}$ is the basis for W_1 where, $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1$
B_2	$B_2 = \{\alpha_2, \alpha_3\}$ is the basis for W_2 where, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2$
T_1	$[T_1]_{B_1} = \begin{pmatrix} 2 \end{pmatrix}$
T_2	$[T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$

TABLE 3: Conclusion of above Results