

# Assignment 7

Priya Bhatia

**Abstract—This document finds the coordinates of foot of perpendicular using Singular Value Decomposition**

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[https://github.com/priya6971/matrix\\_theory\\_EE5609/tree/master/Assignment7/codes](https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment7/codes)

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## 1 PROBLEM

Determine the distance from the Z-axis to the plane  $5x - 12y - 8 = 0$

## 2 SOLUTION

Equation of plane can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.1)$$

Rewriting given equation of plane in (2.0.1) form

$$(5 \quad -12 \quad 0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \quad (2.0.2)$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 5 \\ -12 \\ 0 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2.0.4)$$

$$c = 8 \quad (2.0.5)$$

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \quad (2.0.6)$$

Here  $p$  is any point on plane and  $\mathbf{q}, \mathbf{r}$  are two vectors parallel to plane and hence  $\perp$  to  $\mathbf{n}$ . Now, we need to find these two vectors  $\mathbf{q}$  and  $\mathbf{r}$  which are  $\perp$  to  $\mathbf{n}$

$$(5 \quad -12 \quad 0) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \implies 5a - 12b = 0 \quad (2.0.7)$$

Put  $a = 0$  and  $c = 1$  in (2.0.7),  $\implies b = 0$

Put  $a = 1$  and  $c = 0$  in (2.0.7),  $\implies b = \frac{5}{12}$

$$\text{Hence } \mathbf{q} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let us find point  $\mathbf{p}$  on the plane. Put  $x = 1, z = 0$  in

$$(2.0.2), \text{ we get } \mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Since given plane is parallel to Z-axis, we can use any point  $P$  on Z-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.8)$$

Let  $\mathbf{Q}$  be the point on plane with shortest distance to  $\mathbf{P}$ .  $\mathbf{Q}$  can be expressed in (2.0.7) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.0.9)$$

Computation of Pseudo Inverse using SVD in order to determine the value of  $\lambda_1$  and  $\lambda_2$  :

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.10)$$

$$\lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.0.11)$$

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.0.12)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.13)$$

$$\implies \mathbf{x} = \mathbf{M}^+ \mathbf{b} \quad (2.0.14)$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.15)$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.0.17)$$

Applying Singular Value Decomposition on  $\mathbf{M}$ ,

$$\mathbf{M} = \mathbf{USV}^T \quad (2.0.18)$$

Where the columns of  $\mathbf{V}$  are the eigenvectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigenvectors of  $\mathbf{MM}^T$  and  $\mathbf{S}$  is diagonal matrix of Singular values of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.20)$$

As we know that,

$$\begin{aligned} \mathbf{USV}^T \mathbf{x} &= \mathbf{b} \\ \Rightarrow \mathbf{x} &= \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \end{aligned} \quad (2.0.21)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Calculating eigenvalues of  $\mathbf{MM}^T$ ,

$$\begin{aligned} |\mathbf{MM}^T - \lambda \mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^3 - \frac{313}{144}\lambda^2 + \frac{169}{144}\lambda &= 0 \end{aligned}$$

Hence eigenvalues of  $\mathbf{MM}^T$  are,

$$\lambda_1 = \frac{169}{144}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \quad (2.0.22)$$

And the corresponding eigenvectors are,

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{5}{12} \\ 1 \\ 0 \end{pmatrix} \quad (2.0.23)$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.24)$$

Using values from (2.0.22),

$$\mathbf{S} = \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.25)$$

Calculating the eigenvalues of  $\mathbf{M}^T\mathbf{M}$ ,

$$\begin{aligned} |\mathbf{M}^T\mathbf{M} - \lambda \mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} \frac{169}{144} - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^2 - \frac{313}{144}\lambda + \frac{169}{144} &= 0 \end{aligned}$$

Hence, eigenvalues of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = \frac{169}{144}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.26)$$

From (2.0.26) we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.27)$$

Now, we can compute SVD of  $\mathbf{M}$  :

$$\mathbf{M} = \mathbf{USV}^T \quad (2.0.28)$$

$$= \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.29)$$

$$\mathbf{M}^+ = \mathbf{VS}^T \mathbf{U}^T \quad (2.0.30)$$

$$= \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.31)$$

Substitute (2.0.31) in (2.0.14),

$$\mathbf{x} = \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.0.32)$$

$$\mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \quad (2.0.33)$$

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \quad (2.0.34)$$

Substituting  $\lambda_1, \lambda_2$  in (2.0.9)

$$\mathbf{Q} = \begin{pmatrix} -\frac{204}{169} \\ -\frac{85}{169} \\ 0 \end{pmatrix} \quad (2.0.35)$$

Distance between point  $\mathbf{P}$  and  $\mathbf{Q}$  is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(-\frac{204}{169}\right)^2 + \left(-\frac{85}{169}\right)^2 + 0} \quad (2.0.36)$$

$$\|\mathbf{P} - \mathbf{Q}\| = \frac{17}{13} \quad (2.0.37)$$

Hence, the distance from the Z-axis to the plane  $5x - 12y - 8 = 0$  is  $\frac{17}{13}$ . Now, we can verify the solution using Least Squares Method,

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \quad (2.0.38)$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (2.0.39)$$

Substituting  $\mathbf{M}, \mathbf{b}$  from (2.0.12) in (2.0.39)

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.0.40)$$

$$\begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{17}{12} \\ 0 \end{pmatrix} \quad (2.0.41)$$

$$\implies \frac{169}{144}\lambda_1 = -\frac{17}{12} \quad (2.0.42)$$

$$\lambda_1 = -\frac{17}{12} \times \frac{144}{169} = -\frac{204}{169} \quad (2.0.43)$$

$$\text{and } \lambda_2 = 0 \quad (2.0.44)$$

$$\implies \mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \quad (2.0.45)$$

Comparing (2.0.32) and (2.0.45) solution is verified.