Challenge Problem

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***Abstract*—This document proves that the eigenvalues of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ are positive**

Download latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/challengeProblem2

1 Problem

Prove that the eigenvalues of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ are positive.

2 Proof

Let, $\mathbf{A}$ is an arbitrary $m \times n$ matrix. Now consider the matrix $\mathbf{A}^{\mathbf{T}}\mathbf{A}$,

for any n dimensional vector $\mathbf{x}$,

$$\mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{x} = \mathbf{x}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}\mathbf{x} \tag{2.0.1}$$

$$\implies \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{x} = (\mathbf{A}\mathbf{x})^{\mathbf{T}}(\mathbf{A}\mathbf{x}) \tag{2.0.2}$$

$$\implies \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{x} = ||\mathbf{A}\mathbf{x}||^2 \geq 0 \tag{2.0.3}$$

If $\mathbf{x} \neq 0$, $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is positive definite, i.e

$$||\mathbf{A}\mathbf{x}||^2 > 0 \tag{2.0.4}$$

If $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is positive semi-definite, if $\mathbf{x} = 0$,

$$||\mathbf{A}\mathbf{x}||^2 = 0 \tag{2.0.5}$$

Hence, $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is positive semi-definite if the columns of $\mathbf{A}$ are linearly dependent and $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is positive definite if columns of $\mathbf{A}$ are linearly dependent.

$$(\mathbf{A}^{\mathbf{T}}\mathbf{A})^{\mathbf{T}} = (\mathbf{A}^{\mathbf{T}})(\mathbf{A}^{\mathbf{T}})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}}\mathbf{A} \tag{2.0.6}$$

Hence, $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is symmetric. As every eigen value of a Hermitian matrix is real and every symmetric matrix is Hermitian then $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ has real eigen values.
Let λ be a eigenvalue of $\mathbf{M} = \mathbf{A}^{\mathbf{T}}\mathbf{A}$ and let $\mathbf{x}$ be a corresponding real eigen-vector hence,

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.7}$$

Multiplying $\mathbf{x}^{\mathbf{T}}$ in (2.0.7),

$$\mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} = \lambda\mathbf{x}^{\mathbf{T}}\mathbf{x} \tag{2.0.8}$$

$$\implies \mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} = \lambda\,||\mathbf{x}||^2 \tag{2.0.9}$$

$$\implies \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{x} = \lambda\,||\mathbf{x}||^2 \tag{2.0.10}$$

When $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is positive definite (i.e columns of $\mathbf{A}$ are linearly independent) then $\mathbf{x}$ is a nonzero vector as it is an eigen-vector. Since $||\mathbf{x}||^2$ is positive, hence all eigen-values must be positive. Hence proved.