

Assignment 18

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Abstract

This document illustrates the concept of minimal polynomial, null space and basis.

Download the latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment18

1 PROBLEM

Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \quad (1.0.1)$$

Express the minimal polynomial p for T in the form $p = p_1 p_2$, where p_1 and p_2 are monic and irreducible over the field of real numbers. Let W_i be the null space of $p_i(T)$. Find the basis B_i for the spaces W_1 and W_2 . If T_i is the operator induced on W_i by T , find the matrix of T_i in the basis B_i above.

2 DEFINITIONS

Characteristic Polynomial	For an $n \times n$ matrix \mathbf{A} , characteristic polynomial is defined by, $p(x) = x\mathbf{I} - \mathbf{A} $
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = 0$ Every root of characteristic polynomial should be the root of minimal polynomial and the minimal polynomial divides the characteristic polynomial.
Basis Theorem	Let V be a subspace of dimension m . Then: Any m linearly independent vectors in V forms a basis for V . Any m vectors that span V forms a basis for V .

TABLE 1: Definitions

3 SOLUTION

Express Minimal Polynomial	$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ <p>Characteristic Polynomial = $xI - A = \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix}$</p> <p>By solving above determinant, we find out that</p> $x^3 - 2x^2 + x - 2 = (x-2)(x^2 + 1)$ <p>Since, $T - 2I \neq 0$ and the minimal polynomial divides the characteristic polynomial, thus minimal polynomial p for T is $p = m(x)$</p> $p = (x-2)(x^2 + 1)$ <p>Put $p_1 = (x-2)$ and $p_2 = (x^2 + 1)$</p> <p>Thus, $p = p_1 p_2$</p>
Bases B_1 and matrix T_1	<p>Let $W_1 = \{\alpha \in R^3 / p_1(T)\alpha = 0, (T - 2I)\alpha = 0\}$</p> <p>Therefore, $A - 2I = \begin{pmatrix} 4 & -3 & -2 \\ 4 & -3 & -2 \\ 10 & -5 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 3 & 2 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Rank of $A - 2I$ is 2</p> <p>Nullity of $A - 2I$ = no of columns - Rank = $3 - 2 = 1$</p> <p>That means the dimension of W_1 is 1</p> <p>Thus we can let, $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1$ (Basis theorem mentioned in Definitions)</p> <p>Therefore, $B_1 = \{\alpha_1\}$ is the basis for W_1</p> <p>Let T_1 be the matrix induced by T on W_1</p> $T_1 \alpha_1 = T \alpha_1 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 2\alpha_1$ <p>$[T_1]_{B_1} = [2]$</p>
Bases B_2 and matrix T_2	<p>Let $W_2 = \{\alpha \in R^3 / p_2(T)\alpha = 0, (T^2 + I)\alpha = 0\}$</p> <p>Therefore, $A^2 + I = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{pmatrix}$</p> <p>Rank of $A^2 + I$ is 1</p> <p>Nullity of $A^2 + I$ = no of columns - Rank = $3 - 1 = 2$</p> <p>That means the dimension of W_2 is 2</p> <p>Thus we can let, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2$ (Basis theorem in Definitions)</p> <p>Therefore, $B_2 = \{\alpha_2, \alpha_3\}$ is the basis for W_2</p> <p>Let T_2 be the matrix induced by T on W_2</p>

$T_2\alpha_2 = T\alpha_2 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3\alpha_2 + 5\alpha_3$ $T_2\alpha_3 = T\alpha_3 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + -3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2\alpha_2 - 3\alpha_3$ $(\alpha_2 \ \alpha_3)[T_2] = (\alpha_2 \ \alpha_3) \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$ $\Rightarrow [T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$
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TABLE 2: Finding of Basis and corresponding matrix

4 SUMMARIZATION OF ABOVE RESULTS

Express Minimal Polynomial	$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ <p>We get, $p_1 = (x - 2)$ and $p_2 = (x^2 + 1)$ Thus, $p = p_1 p_2$</p>
W_i	$W_1 = \{\alpha \in R^3 / p_1(T)\alpha = 0, (T - 2I)\alpha = 0\}$ $W_2 = \{\alpha \in R^3 / p_2(T)\alpha = 0, (T^2 + I)\alpha = 0\}$
B_i	$B_1 = \{\alpha_1\}$ is the basis for W_1 where, $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1$ $B_2 = \{\alpha_2, \alpha_3\}$ is the basis for W_2 where, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2$
T_i	$[T_1]_{B_1} = \begin{pmatrix} 2 \end{pmatrix}$ $[T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$

TABLE 3: Conclusion of above Results