Assignment 19

Mtech in AI Department
Priya Bhatia
AI20MTECH14015

Abstract

This document illustrates the concept of orthonormal basis.

Download the latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment19

1 Problem

Let $\{u_1, u_2, ..., u_n\}$ be an orthonormal basis of C^n as column vectors. Let $M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$ and P be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Then which of the following is true?
1. Rank($\mathbf{MPM}^*$) = $\mathbf{k}$ whenever $\alpha_i \neq \alpha_j,\ 1 \leq i,\ j \leq k$
2. Trace($\mathbf{MPM}^*$) = $\sum_{i=1}^{k} \alpha_i$
3. Rank($\mathbf{M}^*\mathbf{N}$) = min(k,$n - k$)
4. Rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$

2 Definitions

Orthonormal Basis	$B = \{u_1, u_2, ..., u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $< u_i, u_j > = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $< u_i, u_j > = 1$ otherwise.
Trace	Trace of a square matrix A, denoted by $\mathbf{tr}(\mathbf{A})$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\mathbf{tr}(\mathbf{AB}) = \mathbf{tr}(\mathbf{BA})$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix
Basis Theorem	A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent,then it is invertible.

TABLE 1: Definitions

3 Solution

Rank($\mathbf{MPM}^*$) = $\mathbf{k}$	Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $\mathbf{u_2} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $\mathbf{u_4} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $\mathbf{k} = 2$, then $\mathbf{M} = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$ $\mathbf{P} = \begin{pmatrix}\alpha_1 & 0\\0 & \alpha_2\end{pmatrix}$ $\mathbf{MPM}^* = \begin{pmatrix}\alpha_1 & 0 & 0 & 0\\0 & \alpha_2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$ $\implies$ Rank($\mathbf{MPM}^*$) ≤ 2 (which is the value of k) (It depends on diagonal values α_1 and α_2) Rank($\mathbf{MPM}^*$) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0. Thus, Rank($\mathbf{MPM}^*$) $\neq \mathbf{k}$ Thus, the given statement is false
Trace($\mathbf{MPM}^*$) = $\sum_{i=1}^{k} \alpha_i$	Consider $\mathbf{MP} = \mathbf{A}$ and $\mathbf{M}^* = \mathbf{B}$ Using Properties, Trace($\mathbf{AB}$) = Trace($\mathbf{BA}$) We can say, Trace($\mathbf{MPM}^*$) = Trace($\mathbf{M}^*\mathbf{MP}$) $\mathbf{M} = \begin{pmatrix}u_1 & u_2 & u_3 & .... & u_k\end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix}\bar{u_1}\\\bar{u_2}\\\bar{u_3}\\.\\.\\.\\\bar{u_k}\end{pmatrix}$ $\mathbf{M}^*\mathbf{M} = \begin{pmatrix}\bar{u_1}u_1 & 0 & 0 & ... & 0\\0 & \bar{u_2}u_2 & 0 & ... & 0\\0 & 0 & \bar{u_3}u_3 & ... & 0\\. & . & . & ... & .\\0 & 0 & 0 & ... & \bar{u_k}u_k\end{pmatrix}$

	(Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 0$ if $i \neq j$) $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 1$ if $i = j$) $\mathbf{M}^*\mathbf{M} = \mathbf{I^k}$ $\mathbf{M}^*\mathbf{MP} = \mathbf{I^k P} = \mathbf{P}$ $\text{Trace}(\mathbf{M}^*\mathbf{MP}) = \text{Trace}(\mathbf{I^k P}) = \text{Trace}(\mathbf{P}) = \sum_{i=1}^{k} \alpha_i$ (Refer Definition section of Trace, it is sum of elements on the main diagonal) So, the given statement is true
$\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n-k)$	$\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Consider $k = 2$, then $\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{N} = \begin{pmatrix} u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^*\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{Rank}(\mathbf{M}^*\mathbf{N}) = 0$ But, $\min(k, n-k) = (2, 2) = 2$ And, this is clear from above that $\text{Rank}(\mathbf{M}^*\mathbf{N}) \neq \min(k, n-k)$ Thus, above statement is false
$\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$	$\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)$ $\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)$ $\text{Rank}(\mathbf{M}+\mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})$

	$\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \mathbf{u_2} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \mathbf{u_4} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $k = 2$, then $\mathbf{M} = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ Rank($\mathbf{M}$) $= 2 = k$ $\mathbf{N} = \begin{pmatrix}u_3 & u_4\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}$ Rank($\mathbf{N}$) $= 2 = n - k$ Thus, Rank($\mathbf{MM}^* + \mathbf{NN}^*$) = Rank($\mathbf{M} + \mathbf{N}$) $= 4 = n$ Thus, above statement is false

TABLE 2: Finding of True and False Statements

4 Conclusion

Rank($\mathbf{MPM}^*$) $= \mathbf{k}$	False
Trace($\mathbf{MPM}^*$) $= \sum_{i=1}^{k} \alpha_i$	True
Rank($\mathbf{M}^*\mathbf{N}$) $= \min(k, n - k)$	False
Rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$	False

TABLE 3: Conclusion of above Solutions