

Challenge Problem

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Abstract—This document finds the Determinant of a Vandermonde matrix

Download latex-tikz codes from

[https://github.com/priya6971/
matrix_theory_EE5609/tree/master/
Vandermonde_Matrix](https://github.com/priya6971/matrix_theory_EE5609/tree/master/Vandermonde_Matrix)

1 PROBLEM

Derive an expression for the determinant of Vandermonde matrix.

2 SOLUTION

Consider matrix \mathbf{V}_n described below :

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ 1 & \alpha_4 & \alpha_4^2 & \dots & \alpha_4^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \dots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{pmatrix} \quad (2.0.1)$$

We can subtract row 1 from each of the other rows and leave \mathbf{V}_n unchanged:

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \dots & \alpha_2^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_3 - \alpha_1 & \alpha_3^2 - \alpha_1^2 & \dots & \alpha_3^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_4 - \alpha_1 & \alpha_4^2 - \alpha_1^2 & \dots & \alpha_4^{n-1} - \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n-1} - \alpha_1 & \alpha_{n-1}^2 - \alpha_1^2 & \dots & \alpha_{n-1}^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & \dots & \alpha_n^{n-1} - \alpha_1^{n-1} \end{pmatrix} \quad (2.0.2)$$

Similarly without changing the value of \mathbf{V}_n , we can subtract α_1 times column n1 from column n, α_1

times column n2 from column n1 and so on, till we subtract α_1 times column 1 from column 2.

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 - \alpha_1 & (\alpha_2 - \alpha_1)\alpha_2 & \dots & (\alpha_2 - \alpha_1)\alpha_2^{n-2} \\ 0 & \alpha_3 - \alpha_1 & (\alpha_3 - \alpha_1)\alpha_3 & \dots & (\alpha_3 - \alpha_1)\alpha_3^{n-2} \\ 0 & \alpha_4 - \alpha_1 & (\alpha_4 - \alpha_1)\alpha_4 & \dots & (\alpha_4 - \alpha_1)\alpha_4^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n-1} - \alpha_1 & (\alpha_{n-1} - \alpha_1)\alpha_{n-1} & \dots & (\alpha_{n-1} - \alpha_1)\alpha_{n-1}^{n-2} \\ 0 & \alpha_n - \alpha_1 & (\alpha_n - \alpha_1)\alpha_n & \dots & (\alpha_n - \alpha_1)\alpha_n^{n-2} \end{pmatrix} \quad (2.0.3)$$

So we can extract all these as factors :

$$\mathbf{V}_n = \prod_{k=2}^n (\alpha_k - \alpha_1) \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \alpha_2 & \dots & \alpha_2^{n-2} \\ 0 & 1 & \alpha_3 & \dots & \alpha_3^{n-2} \\ 0 & 1 & \alpha_4 & \dots & \alpha_4^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \alpha_{n-1} & \dots & \alpha_{n-1}^{n-2} \\ 0 & 1 & \alpha_n & \dots & \alpha_n^{n-2} \end{pmatrix} \quad (2.0.4)$$

From Determinant with Unit Element in Otherwise Zero Row, we can see that this directly gives us:

$$\mathbf{V}_n = \prod_{k=2}^n (\alpha_k - \alpha_1) \begin{pmatrix} 1 & \alpha_2 & \dots & \alpha_2^{n-2} \\ 1 & \alpha_3 & \dots & \alpha_3^{n-2} \\ 1 & \alpha_4 & \dots & \alpha_4^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n-1} & \dots & \alpha_{n-1}^{n-2} \\ 1 & \alpha_n & \dots & \alpha_n^{n-2} \end{pmatrix} \quad (2.0.5)$$

So, it can be seen that

$$\mathbf{V}_n = \prod_{k=2}^n (\alpha_k - \alpha_1) \mathbf{V}_{n-1} \quad (2.0.6)$$