Assignment 7

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***Abstract*—This document finds the coordinates of foot of perpendicular using Singular Value Decomposition**

Download python codes from

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https://github.com/priya6971/
    matrix_theory_EE5609/tree/master/
    Assignment7/codes
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Download latex-tikz codes from

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https://github.com/priya6971/
    matrix_theory_EE5609/tree/master/
    Assignment7
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1 Problem

Determine the distance from the Z-axis to the plane 5x - 12y - 8 = 0

2 Solution

Equation of plane can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.1}$$

Rewriting given equation of plane in (2.0.1) form

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \tag{2.0.2}$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 5 \\ -12 \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.4}$$

$$c = 8 \tag{2.0.5}$$

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{2.0.6}$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Now, we need to find these two vectors $\mathbf{q}$ and $\mathbf{r}$ which are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \implies 5a - 12b = 0 \tag{2.0.7}$$

Put $a = 0$ and $c = 1$ in (2.0.7), $\implies b = 0$
Put $a = 1$ and $c = 0$ in (2.0.7), $\implies b = \frac{5}{12}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, z = 0$ in (2.0.2), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Since given plane is parallel to Z-axis, we can use any point P on Z-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.8}$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (2.0.7) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.9}$$

Computation of Pseudo Inverse using SVD in order to determine the value of λ_1 and λ_2 :

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.10}$$

$$\lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.12}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.13}$$

$$\implies \mathbf{x} = \mathbf{M}^+\mathbf{b} \tag{2.0.14}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.17}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.18}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of Singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.20}$$

As we know that,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{2.0.21}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 1-\lambda & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^3 - \frac{313}{144}\lambda^2 + \frac{169}{144}\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{169}{144}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{2.0.22}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -\frac{5}{12} \\ 1 \\ 0 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.24}$$

Using values from (2.0.22),

$$\mathbf{S} = \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.25}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} \frac{169}{144}-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - \frac{313}{144}\lambda + \frac{169}{144} = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{169}{144}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.26}$$

From (2.0.26) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.27}$$

Now, we can compute *SVD* of $\mathbf{M}$:

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.28}$$

$$= \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{M}^+ = \mathbf{V}\mathbf{S}^T\mathbf{U}^T \tag{2.0.30}$$

$$= \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.31}$$

Substitute (2.0.31) in (2.0.14),

$$\mathbf{x} = \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{2.0.34}$$

Substituting λ_1, λ_2 in (2.0.9)

$$\mathbf{Q} = \begin{pmatrix} -\frac{204}{169} \\ -\frac{85}{169} \\ 0 \end{pmatrix} \tag{2.0.35}$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(-\frac{204}{169}\right)^2 + \left(-\frac{85}{169}\right)^2 + 0} \tag{2.0.36}$$

$$\|\mathbf{P} - \mathbf{Q}\| = \frac{17}{13} \tag{2.0.37}$$

Hence, the distance from the Z-axis to the plane $5x - 12y - 8 = 0$ is $\frac{17}{13}$. Now, we can verify the solution using Least Squares Method,

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{2.0.38}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.39}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (2.0.12) in (2.0.39)

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.40}$$

$$\begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{17}{12} \\ 0 \end{pmatrix} \tag{2.0.41}$$

$$\implies \frac{169}{144}\lambda_1 = -\frac{17}{12} \tag{2.0.42}$$

$$\lambda_1 = -\frac{17}{12} \times \frac{144}{169} = -\frac{204}{169} \tag{2.0.43}$$

$$\text{and } \lambda_2 = 0 \tag{2.0.44}$$

$$\implies \mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{2.0.45}$$

Comparing (2.0.32) and (2.0.45) solution is verified.