Assignment 19

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Abstract

This document illustrates the concept of orthonormal basis.

Download the latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Assignment19

1 PROBLEM

Let $\{u_1, u_2, ..., u_n\}$ be an orthonormal basis of C^n as column vectors. Let $M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$ and P be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Then which of the following is true?

1. Rank($\mathbf{MPM^*}$) = $\mathbf{k}$ whenever $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq k$
2. Trace($\mathbf{MPM^*}$) = $\sum_{i=1}^{k} \alpha_i$
3. Rank($\mathbf{M^*N}$) = min($k, n-k$)
4. Rank($\mathbf{MM^* + NN^*}$) $< n$

2 DEFINITIONS

Orthonormal Basis	$B = \{u_1, u_2, ..., u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $<u_i, u_j> = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $<u_i, u_j> = 1$ otherwise.
Trace	Trace of a square matrix A, denoted by $\mathbf{tr(A)}$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\mathbf{tr(AB) = tr(BA)}$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix
Basis Theorem	A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent, then it is invertible.

TABLE 1: Definitions

3 Solution

Rank(**MPM***) = **k**	M and M^* vectors are linearly independent and thus it is invertible (Since the elementary matrices are invertible, such multiplication does not change the rank of a matrix) $\implies$ Rank(**MPM***) = Rank(**P**) Now **P** be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Rank(**P**) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0. Thus, Rank(**MPM***) $\neq$ **k** Thus, the given statement is false
Trace(**MPM***) = $\sum_{i=1}^{k} \alpha_i$	Trace(**MPM***) = Trace(**M*MP**) (According to properties of trace mentioned in Definitions) $\mathbf{M} = \begin{pmatrix} u_1 & u_2 & u_3 & .... & u_k \end{pmatrix}$ $\mathbf{M^*} = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \\ . \\ . \\ . \\ u_k^T \end{pmatrix}$ $\mathbf{M^*M} = \begin{pmatrix} u_1^T u_1 & 0 & 0 & ... & 0 \\ 0 & u_2^T u_2 & 0 & ... & 0 \\ 0 & 0 & u_3^T u_3 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & u_k^T u_k \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 0$ if $i \neq j$) $\mathbf{M^*M} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 1$ if $i = j$) $\mathbf{M^*M} = \mathbf{I^k}$ $\mathbf{M^*MP} = \mathbf{I^k P} = \mathbf{P}$ Trace(**M*MP**) = Trace($\mathbf{I^k P}$) = Trace(**P**) = $\sum_{i=1}^{k} \alpha_i$ (Refer Definition section of Trace, it is sum of elements on the main diagonal) So, the given statement is true
Rank(**M*N**) = min(k,n − k)	$M = \{u_1, u_2, ..., u_k\}$ and $N = \{u_{k+1}, u_{k+2}, ..., u_n\}$

$$\mathbf{M}^*\mathbf{N} = \begin{pmatrix} u_{11}^T & u_{21}^T & . & . & u_{n1}^T \\ u_{12}^T & u_{22}^T & . & . & u_{n2}^T \\ . & . & . & . & . \\ u_{1K}^T & u_{2K}^T & . & . & u_{nK}^T \end{pmatrix} \begin{pmatrix} u_{(k+1)1} & u_{(k+1)2} & . & . & u_{(k+1)n} \\ u_{(k+2)1} & u_{(k+2)2} & . & . & u_{(k+2)n} \\ . & . & . & . & . \\ u_{K1} & u_{K2} & . & . & u_{Kn} \end{pmatrix}$$

Rank($\mathbf{MM}^*$ + $\mathbf{NN}^*$) < n	M and M^* vectors are linearly independent and thus it is invertible (Since the elementary matrices are invertible, such multiplication does not change the rank of a matrix) $\implies$ Rank($\mathbf{MPM}^*$) = Rank($\mathbf{P}$) Now $\mathbf{P}$ be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in R$. Rank($\mathbf{P}$) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0. Thus, Rank($\mathbf{MPM}^*$) $\neq$ **k** Thus, the given statement is false

TABLE 2: Finding of True and False Statements

4 CONCLUSION

Rank($\mathbf{MPM}^*$) = **k**	False
Trace($\mathbf{MPM}^*$) = $\sum_{i=1}^{k} \alpha_i$	True
Rank($\mathbf{M}^*\mathbf{N}$) = min(k, $n - k$)	False
Rank($\mathbf{MM}^*$ + $\mathbf{NN}^*$) < n	False

TABLE 3: Conclusion of above Solutions