Challenge Problem

Priya Bhatia

***Abstract*—This document finds the Determinant of a Vandermonde matrix**

Download latex-tikz codes from

https://github.com/priya6971/matrix_theory_EE5609/tree/master/Vandermonde_Matrix

1 Problem

Derive an expression for the determinant of Vandermonde matrix.

2 Solution

Consider matrix $\mathbf{V_n}$ described below :

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & ... & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & ... & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & ... & \alpha_3^{n-1} \\ 1 & \alpha_4 & \alpha_4^2 & ... & \alpha_4^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & ... & \alpha_{n-1}^{n-1} \\ 1 & \alpha_n & \alpha_n^2 & ... & \alpha_n^{n-1} \end{pmatrix} \tag{2.0.1}$$

We can subtract row 1 from each of the other rows and leave $\mathbf{V_n}$ unchanged:

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & ... & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha 1 & \alpha_2^2 - \alpha_1^2 & ... & \alpha_2^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_3 - \alpha 1 & \alpha_3^2 - \alpha 1^2 & ... & \alpha_3^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_4 - \alpha 1 & \alpha_4^2 - \alpha 1^2 & ... & \alpha_4^{n-1} - \alpha_1^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \alpha_{n-1} - \alpha 1 & \alpha_{n-1}^2 - \alpha 1^2 & ... & \alpha_{n-1}^{n-1} - \alpha_1^{n-1} \\ 0 & \alpha_n - \alpha 1 & \alpha_n^2 - \alpha 1^2 & ... & \alpha_n^{n-1} - \alpha_1^{n-1} \end{pmatrix} \tag{2.0.2}$$

Similarly without changing the value of $\mathbf{V_n}$, we can subtract:

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & \alpha_2 - \alpha 1 & (\alpha_2 - \alpha_1)\alpha_2 & ... & (\alpha_2 - \alpha_1)\alpha_2^{n-2} \\ 0 & \alpha_3 - \alpha 1 & (\alpha_3 - \alpha_1)\alpha_3 & ... & (\alpha_3 - \alpha_1)\alpha_3^{n-2} \\ 0 & \alpha_4 - \alpha 1 & (\alpha_4 - \alpha 1)\alpha_4 & ... & (\alpha_4 - \alpha_1)\alpha_4^{n-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \alpha_{n-1} - \alpha 1 & (\alpha_{n-1} - \alpha 1)\alpha_{n-1} & ... & (\alpha_{n-1} - \alpha_1)\alpha_{n-1}^{n-2} \\ 0 & \alpha_n - \alpha 1 & (\alpha_n - \alpha 1)\alpha_n & ... & (\alpha_n - \alpha_1)\alpha_n^{n-2} \end{pmatrix} \tag{2.0.3}$$

So we can extract all these as factors :

$$\mathbf{V_n} = \prod_{k=2}^{n}(x_k - x_1)\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & \alpha_2 & ... & \alpha_2^{n-2} \\ 0 & 1 & \alpha_3 & ... & \alpha_3^{n-2} \\ 0 & 1 & \alpha_4 & ... & \alpha_4^{n-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 1 & \alpha_{n-1} & ... & \alpha_{n-1}^{n-2} \\ 0 & 1 & \alpha_n & ... & \alpha_n^{n-2} \end{pmatrix} \tag{2.0.4}$$

From Determinant with Unit Element in Otherwise Zero Row, we can see that this directly gives us:

$$\mathbf{V_n} = \prod_{k=2}^{n}(x_k - x_1)\begin{pmatrix} 1 & \alpha_2 & ... & \alpha_2^{n-2} \\ 1 & \alpha_3 & ... & \alpha_3^{n-2} \\ 1 & \alpha_4 & ... & \alpha_4^{n-2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & \alpha_{n-1} & ... & \alpha_{n-1}^{n-2} \\ 1 & \alpha_n & ... & \alpha_n^{n-2} \end{pmatrix} \tag{2.0.5}$$

So, it can be seen that

$$\mathbf{V_n} = \prod_{k=2}^{n}(x_k - x_1)\mathbf{V_{n-1}} \tag{2.0.6}$$