Submission 4

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Problem 1: Mean-square displacement of a free Brownian particle

a) Start with the solution to the Langevin equation

$$p(t) = -e^{-\Gamma t}\left(p^0 + \int_0^t \mathrm{d}t' e^{\Gamma t'} f(t')\right) =: p_0(t) + \delta p(t),$$

which then, inserted into $\langle p(t)p(t')\rangle$ leads us to

$$\begin{aligned}\langle p(t)p(t')\rangle &= \langle (p_0(t) + \delta p(t))(p_0(t') + \delta p(t'))\rangle \\ &= \left\langle \underbrace{p_0(t)p_0(t')}_{=e^{-\Gamma(t+t')}(p^0)^2} + \underbrace{\delta p(t)p_0(t)}_{=0} + \underbrace{p_0(t)\delta p(t')}_{=0} + \underbrace{\delta p(t)\delta p(t')}_{=:A}\right\rangle,\end{aligned}$$

where A can be calculated by

$$\begin{aligned}A &= e^{-\Gamma(t-t')}\int_0^t \mathrm{d}\tau e^{\Gamma\tau}\int_0^{t'} \mathrm{d}\tau' e^{\Gamma\tau'} \underbrace{\langle f(t)f(t')\rangle}_{=2\Gamma M/\beta\delta(\tau-\tau')} \\ &= e^{-\Gamma(t+t')}\int_0^t \mathrm{d}\tau e^{2\Gamma\tau} 2B \\ &= \frac{2M}{\beta}\left(e^{-\Gamma(t'-t)} + e^{-\Gamma(t+t')}\right).\end{aligned}$$

overall, we thus have

$$\langle p(t)p(t')\rangle = e^{-\Gamma(t+t')}(p^0)^2 + \frac{2M}{\beta}\left(e^{-\Gamma(t'-t)} + e^{-\Gamma(t+t')}\right).$$

b) Next, consider the mean square displacement $\langle [x(t)-x(0)]^2 \rangle$. As per definition (4), we have

$$
\begin{aligned}
\left\langle (x(t)-x(0))^2 \right\rangle &= \frac{1}{M^2} \left\langle \left(\int\limits_0^t \mathrm{d}t' p(t') \right)^2 \right\rangle = \left\langle \left(\int\limits_0^t \mathrm{d}t' p_0(t') + \delta p(t') \right)^2 \right\rangle \\
&= \underbrace{\frac{1}{M^2} (p^0)^2 \frac{(1-e^{\Gamma t})^2}{t}}_{=:A} + \frac{1}{M^2} \int\limits_0^t \mathrm{d}\tau \int\limits_0^t \mathrm{d}\tau' \left\langle \delta p(\tau) \delta p(\tau') \right\rangle \\
&= A + \frac{1}{M^2} \int\limits_0^t \mathrm{d}\tau \int\limits_0^t \mathrm{d}\tau' \frac{B}{\Gamma} \left(e^{-\Gamma(\tau'-\tau)} + e^{-\Gamma(\tau+\tau')} \right) \\
&= A + \frac{B}{\Gamma M^2} \left[\frac{1}{\Gamma^2} \left(1 - e^{-2\Gamma t} \right) + \left(\frac{e^{-\Gamma t}-1}{\Gamma} \right)^2 \right]
\end{aligned}
$$