Submission 1

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Problem 1: Self avoiding polymer model:

a) Probability distribution of 4 monomer chain:

Possible values for chain length (found by Drawing): $|\boldsymbol{R}_3 - \boldsymbol{R}_0| = 1, \sqrt{5}, 3$.

Probabilities, determined by enumeration and symmetry arguments:

p	Value
$p(\|\boldsymbol{R}_3 - \boldsymbol{R}_0\| = 1)$	$\frac{2}{9}$
$p(\|\boldsymbol{R}_3 - \boldsymbol{R}_0\| = \sqrt{5})$	$\frac{6}{9}$
$p(\|\boldsymbol{R}_3 - \boldsymbol{R}_0\| = 3)$	$\frac{1}{9}$

The probability to find the chain in a straight line is that of $p(|\boldsymbol{R}_3 - \boldsymbol{R}_0| = 3) = \frac{1}{9}$.

b) Mean:

$$\langle|\boldsymbol{R}_3 - \boldsymbol{R}_0|\rangle = \frac{2}{9} + \sqrt{5}\frac{6}{9} + 3\frac{1}{9} \approx 2.04$$

Variance:

$$\sigma = \sqrt{\left\langle|\boldsymbol{R}_3 - \boldsymbol{R}_0|^2\right\rangle} = \sqrt{\frac{2}{9} + 5\frac{6}{9} + 9\frac{1}{9}} \approx 1.91$$

Problem 2: Random walk on a lattice without self-avoidance:

a) Probability distribution $p(X, Y|N)$:

Start with considering random walk in x-direction, assume $N_x = r + l$ steps in this direction, with r the steps in positive and l steps in negative direction, i.e. $X = r - l$.

$\implies r - l = x, N = r + l \implies r = \frac{N_x + x}{2}, l = \frac{N_x - x}{2}$.

Thus, we have

$$p(X|N_x) = \binom{N_x}{\frac{N_x+x}{2}} p^{N_x}$$

in one direction (of course analogously in y-direction), with $p = \frac{1}{2}$.

To get the probability $p(X, Y|N)$, we need to sum over all combinations of possible number of path-steps in each direction, where we know:

$$\begin{aligned} N &= N_x + N_y, \\ \forall N_x &\in \{X, \ldots, N - Y\}, \\ \forall N_y &\in \{Y, \ldots, N - X\}. \end{aligned}$$

Thus, overall, we have:

$$p(X, Y|N) = \sum_{N_x = X}^{N-Y} \binom{N_x}{\frac{N_x + X}{2}} \binom{N - N_x}{\frac{N - N_x + Y}{2}} \frac{1}{4}^N ,$$

with the condition that $X + Y$ is even (odd), if N is even (odd), otherwise we immediately know $p(X, Y|N) = 0$, because there is no possible path to that point.

I am not sure if there is a more elegant, close form solution to this probability distribution, because this is a bit ugly.

b) Mean value:

One can reason that there is nothing special about the x-direction, i.e. $\langle N_x \rangle = \frac{N}{2}$, thus, we may use the probability distribution $p(X|N_x)$ as defined above in a).

$$\langle X|N \rangle = \sum_X X p(X| \underbrace{N_x}_{=N/2}) = \sum_{X=0}^{N} X \binom{N/2}{\frac{N/2+X}{2}} \frac{1}{2}^{N/2}$$

The same obviously holds for $\langle Y|N \rangle$.

c) The path cannot return to (0,0) for odd number of jumps, as $X + Y = 2\mathbb{N}$ is only possible for even numbers of steps.

The probability function: for a return to the starting point can be determined like this: Consider that to end up at the same square one has to walk r steps to the right and $l = r$ steps to the left. Similarly, one has to walk the same number of steps upwards compared to downwards. Again, there is nothing special about each directions, so this will be the probability of one walker coming back to zero squared, with the average number of steps in one direction being $n := N/2$:

$$p(0, 0| \underbrace{2n}_{=N}) = \left(\binom{2n}{n} \frac{1}{2^{2n}} \right)^2 .$$

To see the asymptotic behavior of this formula, confer with fig. 1.

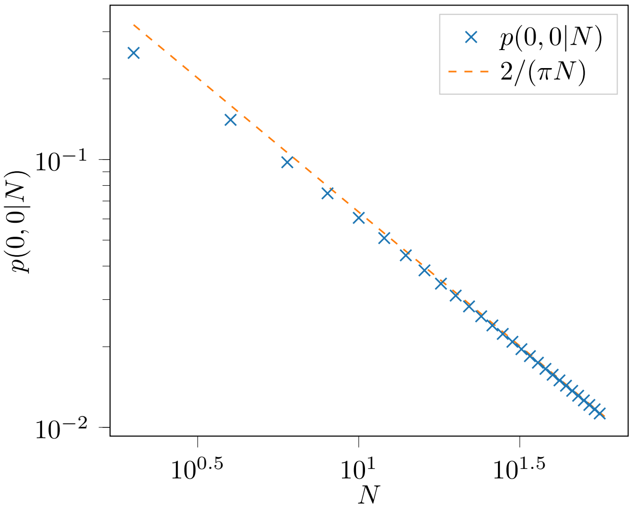

Figure 1: Exact probability for $p(0,0|N)$ plotted together with $2/(\pi N)$.

d) To prove the asymptotic behavior of this random walk, start with the probability function as above

$$p(0,0|N) = \left(\binom{N}{N/2} \frac{1}{2^{2n}} \right)^2 = \left(\frac{N!}{(N/2)!(N-N/2)!} \frac{1}{2^N} \right)^2 = \left(\frac{N!}{((N/2)!)^2} \frac{1}{2^N} \right)^2$$

Using Stirling's approximation $n! \to \sqrt{2\pi n}(n/e)^n (n \to \infty)$, we have

$$\begin{aligned} p(0,0|N) &\to \left(\frac{\sqrt{2\pi N}(N/e)^N}{(\sqrt{\pi N}(N/2e)^{N/2})^2} \frac{1}{2^N} \right)^2 \qquad (N \to \infty) \\ &\to \left(\frac{\sqrt{2}}{\sqrt{\pi N}} \frac{2^N}{2^N} \right)^2 = \frac{2}{\pi N}. \end{aligned}$$