Towards the use of Simplification Rules in Intuitionistic Tableaux

Mauro Ferrari[1] Camillo Fiorentini[2] Guido Fiorino[3]

[1]Dipartimento di Informatica e Comunicazione, Università degli Studi dell'Insubria,

[2]Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, Italy

[3]Dipartimento di Metodi Quantitativi per le Scienze Economiche Aziendali, Università degli Studi di Milano-Bicocca

Ferrara, 25 giugno 2009 - CILC'09

Outline

Work in progress: apply simplification techniques to tableaux calculi for Intuitionistic Propositional Logic **Int**

Outline

Work in progress: apply simplification techniques to tableaux calculi for Intuitionistic Propositional Logic **Int**

Known results

- The decidability problem for **Int** is PSPACE-complete [Statman, TCS 1979]
- Calculi for **Int** have been provided such that:
  - Proofs have linear depth
  - The related decision procedure requires $O(n \log n)$-space.

We introduce simplifications rules which can be applied to any complete calculus for **Int**

An intuitive introduction to simplifications

We consider two kinds of simplifications of proofs derivations, firstly introduced for Classical and Modal Logics:

An intuitive introduction to simplifications

We consider two kinds of simplifications of proofs derivations, firstly introduced for Classical and Modal Logics:

- Replacement
  If a formula A is "*proved to be true*", replace every occurrence of A with $\top$

  $$A \lor B \quad \rightsquigarrow_{A \text{ is true}} \quad \top \lor B$$

  Similarly, if A is "*proved to be false*", replace A with $\bot$

An intuitive introduction to simplifications

We consider two kinds of simplifications of proofs derivations, firstly introduced for Classical and Modal Logics:

- Replacement
  If a formula A is "*proved to be true*", replace every occurrence of A with $\top$

  $$A \vee B \quad \rightsquigarrow_{A \text{ is true}} \quad \top \vee B$$

  Similarly, if A is "*proved to be false*", replace A with $\bot$

- Boolean simplifications
  Simplification of formulas containing $\top$ and $\bot$ based on **Int** equivalences

  $$\top \vee B \quad \rightsquigarrow \quad \top$$

An intuitive introduction to simplifications

We consider two kinds of simplifications of proofs derivations, firstly introduced for Classical and Modal Logics:

- Replacement
  If a formula A is "*proved to be true*", replace every occurrence of A with $\top$

$$A \vee B \quad \rightsquigarrow_{A \text{ is true}} \quad \top \vee B$$

  Similarly, if A is "*proved to be false*", replace A with $\bot$

- Boolean simplifications
  Simplification of formulas containing $\top$ and $\bot$ based on **Int** equivalences

$$\top \vee B \quad \rightsquigarrow \quad \top$$

To sum up, we have rewritten $A \vee B$ as $\top$

Preliminary definitions and notation

The language $\mathcal{L}$:

$$p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \neg A \qquad \text{with } p \in \mathsf{Var}$$

Preliminary definitions and notation

The language $\mathcal{L}$:

$$p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \neg A \qquad \text{with } p \in \mathrm{Var}$$

A Kripke model for $\mathcal{L}$ is a structure $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$, where:

- $\langle P, \leq, \rho \rangle$ is a finite poset with minimum element ρ;
- $\Vdash$ is the forcing relation over $P \times \mathrm{Var}$ satisfying the monotonicity condition:

$$\alpha \Vdash p \quad \text{and} \quad \alpha \leq \beta \quad \text{implies} \quad \beta \Vdash p$$

The Intuitionistic Logic **Int**

The forcing relation is extended as follows:

- $\alpha \Vdash \top$
- $\alpha \nVdash \bot$
- $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$
- $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$
- $\alpha \Vdash A \rightarrow B$ iff, *for every* $\beta \geq \alpha$, $\beta \Vdash A$ implies $\beta \Vdash B$
- $\alpha \Vdash \neg A$ iff, *for every* $\beta \geq \alpha$, $\beta \nVdash A$.

The Intuitionistic Logic **Int**

The forcing relation is extended as follows:

- $\alpha \Vdash \top$
- $\alpha \nVdash \bot$
- $\alpha \Vdash A \land B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$
- $\alpha \Vdash A \lor B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$
- $\alpha \Vdash A \to B$ iff, *for every* $\beta \geq \alpha$, $\beta \Vdash A$ implies $\beta \Vdash B$
- $\alpha \Vdash \neg A$ iff, *for every* $\beta \geq \alpha$, $\beta \nVdash A$.

The monotonicity property holds for arbitrary formulas

The Intuitionistic Logic **Int**

The forcing relation is extended as follows:

- $\alpha \Vdash \top$
- $\alpha \nVdash \bot$
- $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$
- $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$
- $\alpha \Vdash A \rightarrow B$ iff, *for every* $\beta \geq \alpha$, $\beta \Vdash A$ implies $\beta \Vdash B$
- $\alpha \Vdash \neg A$ iff, *for every* $\beta \geq \alpha$, $\beta \nVdash A$.

The monotonicity property holds for arbitrary formulas

- A is valid in $\underline{K}$ iff, for every $\alpha \in P$, $\alpha \Vdash A$
- **Int** is the set of formulas valid in all Kripke models

Tableau calculi: the language

We use signed formulas of the form

$$\mathbf{T}A \mid \mathbf{F}A$$

Tableau calculi: the language

We use signed formulas of the form

$$\mathbf{T}A \mid \mathbf{F}A$$

The meaning is defined by the realizability relation $\rhd$.

Tableau calculi: the language

We use signed formulas of the form

$$\mathbf{T}A \mid \mathbf{F}A$$

The meaning is defined by the realizability relation $\triangleright$.

Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ and $\alpha \in P$

- $\alpha \triangleright \mathbf{T}A$ iff $\alpha \Vdash A$
- $\alpha \triangleright \mathbf{F}A$ iff $\alpha \nVdash A$
- $\alpha \triangleright \Delta$ iff, for every $H \in \Delta$, $\alpha \triangleright H$.

$\underline{K} \triangleright \Delta$ (Δ is realized in $\underline{K}$) iff, for some $\alpha \in P$, $\alpha \triangleright \Delta$.

Tableau calculi: the rules

A tableau calculus is defined by rules of the form

$$\frac{\Delta}{\Delta_1 \mid \cdots \mid \Delta_n}\, r$$

where Δ (the premise of r) and $\Delta_1, \ldots, \Delta_n$ (the consequences of r) are sets of signed formulas.

- r is sound iff

$$\underline{K} \rhd \Delta \qquad \Longrightarrow \qquad \text{there exists } \underline{K}' \text{ and } 1 \leq k \leq n \text{ s.t. } \underline{K}' \rhd \Delta_k$$

- r is invertible iff r is sound and, for every $1 \leq k \leq n$

$$\underline{K} \rhd \Delta_k \qquad \Longrightarrow \qquad \text{there exists } \underline{K}' \text{ s.t. } \underline{K}' \rhd \Delta$$

Invertible rules *do not require backtracking* in proof search

Tableau calculi: the proofs

A proof for Δ is a tree τ such that:

- The root of τ is Δ
- If Δ' is a node of τ and there exists a rule

$$\frac{\Delta'}{\Delta_1 \mid \cdots \mid \Delta_k}\, r'$$

then $\Delta_1, \ldots, \Delta_k$ are the immediate successors of Δ'

Tableau calculi: closed proofs

A set of signed formulas Δ is contradictory iff one of the following conditions holds:

(i) $\{\mathbf{T}A, \mathbf{F}A\} \subseteq \Delta$;

(ii) $\mathbf{T}\bot \in \Delta$;

(iii) $\mathbf{F}\top \in \Delta$.

Tableau calculi: closed proofs

A set of signed formulas Δ is contradictory iff one of the following conditions holds:

(i) $\{\mathbf{T}A, \mathbf{F}A\} \subseteq \Delta$;

(ii) $\mathbf{T}\bot \in \Delta$;

(iii) $\mathbf{F}\top \in \Delta$.

A proof τ is closed if all its leaves are contradictory.

A is provable iff there exists a closed proof for $\mathbf{F}A$.

A calculus is complete for **Int** iff:

$$
\begin{array}{rcl}
A \in \mathbf{Int} & \Longleftrightarrow & A \text{ is provable} \\
 & \Longleftrightarrow & \text{there exists a closed proof for } \mathbf{F}A
\end{array}
$$

We apply simplification rules to complete calculi.

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?

A proof-search example

Is the formula

$$A \rightarrow (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\mathbf{F}(A \rightarrow (A \vee B) \wedge (A \vee C))$$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\to}$$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\dfrac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \color{red}{\mathbf{F}((A \vee B) \wedge (A \vee C))}}\ \mathbf{F}\!\to$$

We can only apply the branching rule $\color{red}{\mathbf{F}\wedge}$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\to}$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$\mathbf{T}A,\ \mathbf{F}(A \vee B)$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\to}$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$$\frac{\mathbf{T}A,\ \color{red}{\mathbf{F}(A \vee B)}}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}B}\ \color{red}{\mathbf{F}\vee}$$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?

We search for a proof of

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\to}$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$$\frac{\mathbf{T}A,\ \mathbf{F}(A \vee B)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}B}\ \mathbf{F}{\vee}$$

closed

A proof-search example

Is the formula

$$A \rightarrow (A \vee B) \wedge (A \vee C)$$

provable?

We search for a proof of

$$\frac{\mathbf{F}(A \rightarrow (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\rightarrow}$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$$\frac{\mathbf{T}A,\ \mathbf{F}(A \vee B)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}B}\ \mathbf{F}{\vee}$$

closed

Right-most branch

$$\mathbf{T}A,\ \mathbf{F}(A \vee C)$$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}\!\to$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$$\frac{\mathbf{T}A,\ \mathbf{F}(A \vee B)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}B}\ \mathbf{F}\vee$$

closed

Right-most branch

$$\frac{\mathbf{T}A,\ \mathbf{F}(A \vee C)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}C}\ \mathbf{F}\vee$$

A proof-search example

Is the formula

$$A \to (A \vee B) \wedge (A \vee C)$$

provable?
We search for a proof of

$$\dfrac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\to}$$

We can only apply the branching rule $\mathbf{F}\wedge$

Left-most branch

$$\dfrac{\mathbf{T}A,\ \mathbf{F}(A \vee B)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}B}\ \mathbf{F}\vee$$

closed

Right-most branch

$$\dfrac{\mathbf{T}A,\ \mathbf{F}(A \vee C)}{\mathbf{T}A,\ \mathbf{F}A,\ \mathbf{F}C}\ \mathbf{F}\vee$$

closed

A proof table

The proof table corresponding to the previous example is

$$
\cfrac{\cfrac{\cfrac{\cfrac{\mathbf{F}(A \to (A \lor B) \land (A \lor C))}{\mathbf{T}A,\ \mathbf{F}((A \lor B) \land (A \lor C))}\ \mathbf{F}{\to}}{\mathbf{T}A,\ \mathbf{F}(A \lor B) \quad \Big| \quad \mathbf{T}A,\ \mathbf{F}(A \lor C)}\ \mathbf{F}{\land}}{\underline{\mathbf{T}A},\ \underline{\mathbf{F}A},\ \mathbf{F}B \quad \Big| \quad \mathbf{T}A,\ \mathbf{F}(A \lor C)}\ \mathbf{F}{\lor}}{\underline{\mathbf{T}A},\ \underline{\mathbf{F}A},\ \mathbf{F}B \quad \Big| \quad \underline{\mathbf{T}A},\ \underline{\mathbf{F}A},\ \mathbf{F}C}\ \mathbf{F}{\land}
$$

Simplifying proofs

$$\dfrac{\mathbf{F}(A \longrightarrow (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}{\longrightarrow}$$

Simplifying proofs

$$\dfrac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}\!\to$$

Step 1: Replacement

The formula $\mathbf{T}A$ can be understood as "*A is proved to be true*".
We can keep $\mathbf{T}A$ and replace all the other A with $\top$

$$\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C)) \quad \rightsquigarrow \quad \mathbf{T}A,\ \mathbf{F}((\top \vee B) \wedge (\top \vee C))$$

Simplifying proofs

$$\frac{\mathbf{F}(A \to (A \vee B) \wedge (A \vee C))}{\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))}\ \mathbf{F}\!\to$$

Step 1: Replacement

The formula $\mathbf{T}A$ can be understood as "*A is proved to be true*".
We can keep $\mathbf{T}A$ and replace all the other A with $\top$

$$\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C)) \quad \leadsto \quad \mathbf{T}A,\ \mathbf{F}((\top \vee B) \wedge (\top \vee C))$$

Step 2: Boolean simplifications

$$\top \vee B \equiv \top \qquad \top \vee C \equiv \top \qquad \top \wedge \top \equiv \top$$

$$\mathbf{T}A,\ \mathbf{F}((\top \vee B) \wedge (\top \vee C)) \quad \leadsto \quad \mathbf{T}A,\ \mathbf{F}\top$$

contradictory set

What do we gain?

We have reduced the search space:

- In the first derivation, the rule $\mathbf{F}\wedge$ applied to

  $$\mathbf{T}A,\ \mathbf{F}((A \vee B) \wedge (A \vee C))$$

  gives rise to two branches:

  $$\mathbf{T}A,\ \mathbf{F}(A \vee B) \quad \mid \quad \mathbf{T}A,\ \mathbf{F}(A \vee C)$$

- In the second proof, by the application of simplification rules, $\mathbf{F}((A \vee B) \wedge (A \vee C))$ is rewritten as $\mathbf{F}\top$, and the branch point is eliminated.
- We show that our simplification rules do not require backtracking

Replacement rules

- Firstly introduced in modal and classical calculi:

  F. Massacci. Simplification: A general constraint propagation technique for propositional and modal tableaux. TABLEAUX'98, LNCS, *vol. 1397, pp. 217–231.*

- In intuitionistic calculi

  A. Avellone, G. Fiorino, and U. Moscato. Optimization techniques for propositional intuitionistic logic and their implementation. TCS, 409(1):41–58, 2008.

$$\frac{\mathbf{T}A,\ \Delta}{\mathbf{T}A,\ \Delta[\top/A]}\,\mathit{Replace}-\mathbf{T}$$

$\Delta[\top/A]$:
replace all the occurrences of A in Δ with $\top$

$$\frac{\mathbf{T}\neg A,\ \Delta}{\mathbf{T}\neg A,\ \Delta[\bot/A]}\,\mathit{Replace}-\mathbf{T}\neg$$

$\Delta[\bot/A]$:
replace all the occurrences of A in Δ with $\bot$

Boolean simplification rules

Based on **Int** equivalences

$$\frac{\Delta}{\Delta[\bot/A\wedge\bot]}\text{S}\wedge\bot \qquad \frac{\Delta}{\Delta[\bot/\bot\wedge A]}\text{S}\bot\wedge$$

$$\frac{\Delta}{\Delta[A/A\wedge\top]}\text{S}\wedge\top \qquad \frac{\Delta}{\Delta[A/\top\wedge A]}\text{S}\top\wedge$$

$$\frac{\Delta}{\Delta[A/A\vee\bot]}\text{S}\vee\bot \qquad \frac{\Delta}{\Delta[A/\bot\vee A]}\text{S}\bot\vee$$

$$\frac{\Delta}{\Delta[\top/A\vee\top]}\text{S}\vee\top \qquad \frac{\Delta}{\Delta[\top/\top\vee A]}\text{S}\top\vee$$

$$\frac{\Delta}{\Delta[\top/\bot\to A]}\text{S}\bot\to \qquad \frac{\Delta}{\Delta[\neg A/A\to\bot]}\text{S}\to\bot$$

$$\frac{\Delta}{\Delta[A/\top\to A]}\text{S}\top\to \qquad \frac{\Delta}{\Delta[\top/A\to\top]}\text{S}\to\top$$

$$\frac{\Delta}{\Delta[\bot/\neg\top]}\text{S}\neg\top \qquad \frac{\Delta}{\Delta[\top/\neg\bot]}\text{S}\neg\bot$$

Invertibility = no backtracking

Theorem

The rules Replace-**T**, Replace-**T**$\neg$ *and the boolean simplification rules are invertible.*

Their addition to any complete calculus does not require backtracking.

Invertibility = no backtracking

Theorem

The rules $\mathrm{Replace}\text{-}\mathbf{T}$, $\mathrm{Replace}\text{-}\mathbf{T}\neg$ *and the boolean simplification rules are invertible.*

Their addition to any complete calculus does not require backtracking.

Example

$$\mathbf{T}p,\ \mathbf{T}((p \vee q) \to r),\ \mathbf{F}q$$

Invertibility = no backtracking

Theorem

The rules Replace-$\mathbf{T}$, Replace-$\mathbf{T}\neg$ *and the boolean simplification rules are invertible.*

Their addition to any complete calculus does not require backtracking.

Example

$$\mathbf{T}p,\ \mathbf{T}((p \vee q) \to r),\ \mathbf{F}q$$

1. Replace p with $\top$

$$\mathbf{T}p,\ \mathbf{T}((\top \vee q) \to r),\ \mathbf{F}q$$

Invertibility = no backtracking

Theorem

The rules Replace-$\mathbf{T}$, Replace-$\mathbf{T}\neg$ *and the boolean simplification rules are invertible.*

Their addition to any complete calculus does not require backtracking.

Example

$$\mathbf{T}p,\ \mathbf{T}((p \vee q) \to r),\ \mathbf{F}q$$

1. Replace p with $\top$

$$\mathbf{T}p,\ \mathbf{T}((\top \vee q) \to r),\ \mathbf{F}q$$

2. Simplify: $(\top \vee q) \to r \ \equiv\ \top \to r \ \equiv\ r$

$$\mathbf{T}p,\ \mathbf{T}r,\ \mathbf{F}q$$

Invertibility = no backtracking

Theorem

The rules Replace-$\mathbf{T}$, Replace-$\mathbf{T}\neg$ *and the boolean simplification rules are invertible.*

Their addition to any complete calculus does not require backtracking.

Example

$$\mathbf{T}p,\ \mathbf{T}((p \vee q) \to r),\ \mathbf{F}q$$

1. Replace p with $\top$

$$\mathbf{T}p,\ \mathbf{T}((\top \vee q) \to r),\ \mathbf{F}q$$

2. Simplify: $(\top \vee q) \to r \ \equiv\ \top \to r \ \equiv\ r$

$$\mathbf{T}p,\ \mathbf{T}r,\ \mathbf{F}q$$

Since the last set is non-contradictory, we conclude that the initial set is not provable (no need of backtracking).

Towards the rule $\mathbf{T}$-permanence

Let Δ be a set of signed formulas
Suppose that, for some p, we prove that:

$$\Delta \text{ is realizable} \iff Tp, \Delta \text{ is realizable}$$

Towards the rule $\mathbf{T}$-permanence

Let Δ be a set of signed formulas
Suppose that, for some p, we prove that:

$$\Delta \text{ is realizable} \iff Tp, \Delta \text{ is realizable}$$

Namely:

IF for some model $\underline{K}$ and state α of $\underline{K}$

$$\alpha \Vdash \Delta$$

THEN there exists a model $\underline{K}'$ and a state α' of $\underline{K}'$ such that:

$$\alpha' \Vdash' p \quad \text{and} \quad \alpha' \Vdash' \Delta$$

Towards the rule $\mathbf{T}$-permanence

Let Δ be a set of signed formulas
Suppose that, for some p, we prove that:

$$\Delta \text{ is realizable} \quad \Longleftrightarrow \quad Tp, \Delta \text{ is realizable}$$

Namely:

IF for some model $\underline{K}$ and state α of $\underline{K}$

$$\alpha \Vdash \Delta$$

THEN there exists a model $\underline{K}'$ and a state α' of $\underline{K}'$ such that:

$$\alpha' \Vdash' p \quad \text{and} \quad \alpha' \Vdash' \Delta$$

In this hypothesis the rule

$$\frac{\Delta}{Tp, \Delta}\, \mathbf{T}p$$

is invertible

Positive occurrence

Intuitively, a prop. variable p *positively occurs* in a signed formula H ($p \preceq^{+} H$) if in a tableau derivation for H the formula $\mathbf{F}p$ does not occur.

Positive occurrence

Intuitively, a prop. variable p *positively occurs* in a signed formula H ($p \preceq^+ H$) if in a tableau derivation for H the formula $\mathbf{F}p$ does not occur.

Formally:

- $p \preceq^+ \mathbf{T}p$
- $p \preceq^+ \mathcal{S}\top$ and $p \preceq^+ \mathcal{S}\bot \qquad \mathcal{S} \in \{\, \mathbf{T}, \mathbf{F} \,\}$
- $p \preceq^+ \mathcal{S}q$, where q is any propositional variable such that $q \neq p$
- $p \preceq^+ \mathcal{S}(A \odot B)$ iff $p \preceq^+ \mathcal{S}A$ and $p \preceq^+ \mathcal{S}B \qquad \odot \in \{\wedge, \vee\}$
- $p \preceq^+ \mathbf{F}(A \to B)$ iff $p \preceq^+ \mathbf{T}A$ and $p \preceq^+ \mathbf{F}B$
- $p \preceq^+ \mathbf{T}(A \to B)$ iff $p \preceq^+ \mathbf{F}A$ and $p \preceq^+ \mathbf{T}B$
- $p \preceq^+ \mathbf{F}\neg A$ iff $p \preceq^+ \mathbf{T}A$
- $p \preceq^+ \mathbf{T}\neg A$ iff $p \preceq^+ \mathbf{F}A$.

A necessary condition for $\top$ substitution

- If $p \preceq^+ \Delta$ (i.e., for every $H \in \Delta$, $p \preceq^+ H$), then:

$$\Delta \text{ is realizable} \quad \Longleftrightarrow \quad \mathbf{T}p, \Delta \text{ is realizable}$$

- As a consequence, the rule

$$\frac{\Delta}{Tp,\ \Delta}\ Tp$$

is invertible

An example (1)

Let us build a closed proof table (depth-first search) for

An example (1)

Let us build a closed proof table (depth-first search) for

$$\mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)$$

An example (1)

Let us build a closed proof table (depth-first search) for

$$\frac{\mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}{\mathbf{T}q_1,\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p) \quad \mid \quad \mathbf{T}p,\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}\ \mathbf{T}\vee$$

An example (1)

Let us build a closed proof table (depth-first search) for

$$\cfrac{\cfrac{\mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}{\mathbf{T}q_1,\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p) \ \mid\ \mathbf{T}p,\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}\ \mathbf{T}\vee}{\mathbf{T}q_1,\ \mathbf{T}q_2,\ \mathbf{T}(q_3 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p) \ \mid\ \mathbf{T}q_1,\ \mathbf{T}p,\ \mathbf{T}(q_3 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p) \ \mid\ \ldots}\ \mathbf{T}\vee$$

$$\vdots$$

After n steps we get the non-contradictory set

$$\mathbf{T}q_1,\ \mathbf{T}q_2,\ \mathbf{T}q_3,\ \ldots,\ \mathbf{T}q_n$$

An example (2)

1. By the above definition, we have:

$$p \preceq^{+} \{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)\}$$

An example (2)

1. By the above definition, we have:

$$p \preceq^{+} \quad \{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)\}$$

2. We can apply the invertible rule

$$\frac{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}{Tp, \mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}\ \mathbf{T}p$$

An example (2)

1. By the above definition, we have:

$$p \preceq^{+} \{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)\}$$

2. We can apply the invertible rule

$$\frac{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}{Tp, \mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}\ \mathbf{T}p$$

3. We can replace p with $\top$

$$\mathbf{T}p, \mathbf{T}(q_1 \vee \top), \mathbf{T}(q_2 \vee \top), \ldots, \mathbf{T}(q_n \vee \top)$$

An example (2)

1. By the above definition, we have:

$$p \ \preceq^{+} \quad \{\mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)\}$$

2. We can apply the invertible rule

$$\frac{\mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}{Tp,\ \mathbf{T}(q_1 \vee p),\ \mathbf{T}(q_2 \vee p),\ \ldots,\ \mathbf{T}(q_n \vee p)}\ \mathbf{T}p$$

3. We can replace p with $\top$

$$\mathbf{T}p,\ \mathbf{T}(q_1 \vee \top),\ \mathbf{T}(q_2 \vee \top),\ \ldots,\ \mathbf{T}(q_n \vee \top)$$

4. By the equivalences $q_1 \vee \top \equiv \top, \ldots,\ q_n \vee \top \equiv \top$, we can simplify:

$$\mathbf{T}p,\ \mathbf{T}\top$$

which is not contradictory.

An example (2)

1. By the above definition, we have:

$$p \preceq^{+} \{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)\}$$

2. We can apply the invertible rule

$$\frac{\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}{Tp, \mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)}\ \mathbf{T}p$$

3. We can replace p with $\top$

$$\mathbf{T}p, \mathbf{T}(q_1 \vee \top), \mathbf{T}(q_2 \vee \top), \ldots, \mathbf{T}(q_n \vee \top)$$

4. By the equivalences $q_1 \vee \top \equiv \top, \ldots, q_n \vee \top \equiv \top$, we can simplify:

$$\mathbf{T}p, \mathbf{T}\top$$

which is not contradictory.

5. We conclude that $\mathbf{T}(q_1 \vee p), \mathbf{T}(q_2 \vee p), \ldots, \mathbf{T}(q_n \vee p)$ is not provable.

The rule $\mathbf{T}$-permanence

$$\frac{\Delta}{\Delta[\top/p]}\ \mathbf{T}\text{-permanence} \qquad p \preceq^{+} \Delta$$

$\Delta[\top/p]$: replace all the occurrences of p in Δ with $\top$

Note that in the conclusion we can leave $\mathbf{T}p$ implicit

The rule $\mathbf{T}$-permanence

$$\frac{\Delta}{\Delta[\top/p]}\ \mathbf{T}\text{-permanence} \qquad p \preceq^{+} \Delta$$

$\Delta[\top/p]$: replace all the occurrences of p in Δ with $\top$

Note that in the conclusion we can leave $\mathbf{T}p$ implicit

Theorem

The rule $\mathbf{T}$-permanence *is invertible.*

Negative occurrence

p *negatively occurs* in H ($p \preceq^- H$) iff:

- $p \preceq^- \mathbf{F}p$
- $p \preceq^- \mathcal{S}\top$ and $p \preceq^- \mathcal{S}\bot$ $\qquad \mathcal{S} \in \{\, \mathbf{T}, \mathbf{F} \,\}$
- $p \preceq^- \mathcal{S}q$, where q is any propositional variable such that $q \neq p$
- $p \preceq^- \mathcal{S}(A \odot B)$ iff $p \preceq^- \mathcal{S}A$ and $p \preceq^- \mathcal{S}B$ $\qquad \odot \in \{\wedge, \vee\}$
- $p \preceq^- \mathbf{F}(A \to B)$ iff $p \preceq^- \mathbf{T}A$ and $p \preceq^- \mathbf{F}B$
- $p \preceq^- \mathbf{T}(A \to B)$ iff $p \preceq^- \mathbf{F}A$ and $p \preceq^- \mathbf{T}B$
- $p \preceq^- \mathbf{F}\neg A$ iff $p \preceq^- \mathbf{T}A$
- $p \preceq^- \mathbf{T}\neg A$ iff $p \preceq^- \mathbf{F}A$.

A necessary condition for $\perp$ substitution

- If $p \preceq^- \Delta$ (i.e., for every $H \in \Delta$, $p \preceq^- H$), then:

$$\Delta \text{ is realizable} \quad \Longleftrightarrow \quad \mathbf{T}\neg p, \Delta \text{ is realizable}$$

- As a consequence, the rule

$$\frac{\Delta}{T\neg p,\ \Delta}\ T\neg p$$

is invertible.

We can replace every occurrence of p in Δ with $\perp$.

Remark: $\mathbf{T}\neg p$ is a stronger condition than $\mathbf{F}p$
(in Kripke semantics: *local* falsity vs. *global* falsity)

The rule $\mathbf{T}\neg$-permanence

It allows the substitution of a propositional variable with $\bot$

$$\frac{\Delta}{\Delta[\bot/p]}\ \mathbf{T}\neg\text{-permanence} \qquad p \preceq^{-} \Delta$$

$\Delta[\bot/p]$: replace all the occurrences of p in Δ with $\bot$

Theorem

The rule $\mathbf{T}\neg$-permanence *is invertible.*

Replacing **F**-formulas

$$\frac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\ \mathit{Replace}{-}\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Replacing **F**-formulas

$$\frac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\,\mathit{Replace}-\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Example

$$\mathbf{F}(A \vee \neg A)$$

Replacing **F**-formulas

$$\frac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\,\mathit{Replace}{-}\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Example

$$\frac{\mathbf{F}(A \vee \neg A)}{\mathbf{F}A,\ \mathbf{F}\neg A}\,\mathbf{F}\vee$$

Replacing **F**-formulas

$$\frac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\ \textit{Replace}-\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Example

$$\dfrac{\dfrac{\mathbf{F}(A \vee \neg A)}{\mathbf{F}A,\ \mathbf{F}\neg A}\ \mathbf{F}\vee}{\mathbf{F}A,\ \mathbf{F}\neg\bot}\ \textit{Replace}-\mathbf{F}$$

Replacing **F**-formulas

$$\frac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\ \mathit{Replace}{-}\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Example

$$\cfrac{\cfrac{\cfrac{\mathbf{F}(A \vee \neg A)}{\mathbf{F}A,\ \mathbf{F}\neg A}\ \mathbf{F}\vee}{\mathbf{F}A,\ \mathbf{F}\neg\bot}\ \mathit{Replace}{-}\mathbf{F}}{\mathbf{F}A,\ \mathbf{F}\top}\ \mathit{simpl}.$$

Replacing **F**-formulas

$$\dfrac{\mathbf{F}A,\ \Delta}{\mathbf{F}A,\ \Delta[\bot/A]}\ \textit{Replace}-\mathbf{F}$$

Sound in Classical Logic, but not in **Int**.

Example

$$\dfrac{\dfrac{\dfrac{\mathbf{F}(A \vee \neg A)}{\mathbf{F}A,\ \mathbf{F}\neg A}\ \mathbf{F}\vee}{\mathbf{F}A,\ \mathbf{F}\neg\bot}\ \textit{Replace}-\mathbf{F}}{\mathbf{F}A,\ \mathbf{F}\top}\ \textit{simpl}.$$

This closed proof of $A \vee \neg A$ is sound in Classical Logic, but not in **Int**

Problem

Differently from $\mathbf{T}\neg A$, $\mathbf{F}A$ *does not mean "A is proved to be false"*
We need a weaker notion of substitution.

Partial substitution

Partial substitution $Z\{\bot/A\}$:
do not substitute the A under the scope of $\rightarrow$ or $\neg$

Partial substitution

Partial substitution $Z\{\perp/A\}$:
do not substitute the A under the scope of $\rightarrow$ or $\neg$

$$Z\{\perp/A\} = \begin{cases} \perp & \text{if } Z = A \\ X\{\perp/A\} \odot Y\{\perp/A\} & \text{if } Z = (X \odot Y) \quad \odot \in \{\wedge, \vee\} \\ Z & \text{if } Z = X \rightarrow Y \text{ or } Z = \neg X \\ & \text{or } Z \text{ is a prop. variable s.t. } Z \neq A \end{cases}$$

Partial substitution

Partial substitution $Z\{\perp/A\}$:
do not substitute the A under the scope of $\to$ or $\neg$

$$Z\{\perp/A\} = \begin{cases} \perp & \text{if } Z = A \\ X\{\perp/A\} \odot Y\{\perp/A\} & \text{if } Z = (X \odot Y) \quad \odot \in \{\wedge, \vee\} \\ Z & \text{if } Z = X \to Y \text{ or } Z = \neg X \\ & \text{or } Z \text{ is a prop. variable s.t. } Z \neq A \end{cases}$$

Example

$$\text{Global substitution } \perp/A : \quad A \vee \neg A \quad \rightsquigarrow \quad \perp \vee \neg \perp$$

$$\text{Partial substitution } \perp/A : \quad A \vee \neg A \quad \rightsquigarrow \quad \perp \vee \neg A$$

Replacement of **F**-formulas

The invertible rule for replacing **F**-formulas is:

$$\dfrac{\Delta,\ \mathbf{F}A}{\Delta\{\perp/A\},\ \mathbf{F}A}\,\text{Replace-}\mathbf{F}$$

Replacement of $\mathbf{F}$-formulas

The invertible rule for replacing **F**-formulas is:

$$\frac{\Delta,\ \mathbf{F}A}{\Delta\{\perp/A\},\ \mathbf{F}A}\,\text{Replace-}\mathbf{F}$$

In the previous example:

$$\frac{\mathbf{F}(A \vee \neg A)}{\mathbf{F}A,\ \mathbf{F}\neg A}\,\mathbf{F}\vee$$

The rule Replace-**F** has no effect and we cannot build a closed proof.

Weak negative occurrence

We give a necessary condition to safely add $\mathbf{F}p$ to a set Δ.

Weak negative occurrence

We give a necessary condition to safely add $\mathbf{F}p$ to a set Δ.

$p \preceq_w^- H$ (p weakly negatively occurs in H) iff:

- $p \preceq_w^- \mathcal{S}\top$ and $p \preceq_w^- \mathcal{S}\bot$
- $p \preceq_w^- \mathbf{F}A$ and $p \preceq_w^- \mathbf{T}\neg A$, for every A
- $p \preceq_w^- \mathbf{T}q$ if $q \neq p$
- $p \preceq_w^- \mathbf{T}(A \odot B)$ iff $p \preceq_w^- \mathbf{T}A$ and $p \preceq_w^- \mathbf{T}B$ $\qquad \odot \in \{\wedge, \vee\}$
- $p \preceq_w^- \mathbf{T}(A \to B)$ iff $p \preceq_w^- \mathbf{T}B$.

Weak negative occurrence

We give a necessary condition to safely add $\mathbf{F}p$ to a set Δ.

$p \preceq_w^- H$ (p weakly negatively occurs in H) iff:

- $p \preceq_w^- \mathcal{S}\top$ and $p \preceq_w^- \mathcal{S}\bot$
- $p \preceq_w^- \mathbf{F}A$ and $p \preceq_w^- \mathbf{T}\neg A$, for every A
- $p \preceq_w^- \mathbf{T}q$ if $q \neq p$
- $p \preceq_w^- \mathbf{T}(A \odot B)$ iff $p \preceq_w^- \mathbf{T}A$ and $p \preceq_w^- \mathbf{T}B$ $\qquad \odot \in \{\wedge, \vee\}$
- $p \preceq_w^- \mathbf{T}(A \to B)$ iff $p \preceq_w^- \mathbf{T}B$.

$p \preceq_w^- \Delta$ iff, for every $H \in \Delta$, $p \preceq_w^- H$.

Weak negative occurrence

We give a necessary condition to safely add $\mathbf{F}p$ to a set Δ.

$p \preceq_w^- H$ (p weakly negatively occurs in H) iff:

- $p \preceq_w^- \mathcal{S}\top$ and $p \preceq_w^- \mathcal{S}\bot$
- $p \preceq_w^- \mathbf{F}A$ and $p \preceq_w^- \mathbf{T}\neg A$, for every A
- $p \preceq_w^- \mathbf{T}q$ if $q \neq p$
- $p \preceq_w^- \mathbf{T}(A \odot B)$ iff $p \preceq_w^- \mathbf{T}A$ and $p \preceq_w^- \mathbf{T}B$ $\qquad \odot \in \{\wedge, \vee\}$
- $p \preceq_w^- \mathbf{T}(A \to B)$ iff $p \preceq_w^- \mathbf{T}B$.

$p \preceq_w^- \Delta$ iff, for every $H \in \Delta$, $p \preceq_w^- H$.

- $p \preceq^- H$ implies $p \preceq_w^- H$
- $\preceq_w^-$ relation permits weaker simplifications than $\preceq^-$

The rule $\mathbf{F}$-permanence

If $p \preceq_w^- \Delta$ then:

$$\Delta \text{ is realizable} \quad \Longleftrightarrow \quad \mathbf{F}p, \Delta \text{ is realizable}$$

Now, let us consider the rule:

$$\frac{\Delta}{\Delta\{\bot/p\}}\ \mathbf{F}\text{-permanence} \qquad p \preceq_w^- \Delta$$

Theorem

The rule $\mathbf{F}$-permanence *is invertible.*

An example

$$\Delta \;=\; \{\, \mathbf{T}(p \vee q),\ \mathbf{F}(q \wedge r),\ \mathbf{F}(p \wedge r),\ \mathbf{F}(r \rightarrow q) \,\}$$

An example

$$\Delta \;=\; \{\, \mathbf{T}(p \vee q),\, \mathbf{F}(q \wedge r),\, \mathbf{F}(p \wedge r),\, \mathbf{F}(r \to q) \,\}$$

- We cannot apply $\mathbf{T}$-permanence and $\mathbf{T}\neg$-permanence since

$$p \not\preceq^{+} \Delta,\;\; p \not\preceq^{-} \Delta,\;\; q \not\preceq^{+} \Delta,\;\; q \not\preceq^{-} \Delta,\;\; r \not\preceq^{+} \Delta,\;\; r \not\preceq^{-} \Delta$$

An example

$$\Delta = \{\, \mathbf{T}(p \vee q),\, \mathbf{F}(q \wedge r),\, \mathbf{F}(p \wedge r),\, \mathbf{F}(r \to q) \,\}$$

- We cannot apply $\mathbf{T}$-permanence and $\mathbf{T}\neg$-permanence since

$$p \not\preceq^{+} \Delta,\;\; p \not\preceq^{-} \Delta,\;\; q \not\preceq^{+} \Delta,\;\; q \not\preceq^{-} \Delta,\;\; r \not\preceq^{+} \Delta,\;\; r \not\preceq^{-} \Delta$$

- Since $r \preceq^{-}_{w} \Delta$, we can apply $\mathbf{F}$-permanence (partial subst. $\bot/r$)

$$\frac{\mathbf{T}(p \vee q),\; \mathbf{F}(q \wedge \bot),\; \mathbf{F}(p \wedge \bot),\; \mathbf{F}(r \to q)}{\mathbf{T}(p \vee q),\; \mathbf{F}\bot,\; \mathbf{F}(r \to q)}\ \text{simpl.}$$

An example

$$\Delta \;=\; \{\, \mathbf{T}(p \vee q),\, \mathbf{F}(q \wedge r),\, \mathbf{F}(p \wedge r),\, \mathbf{F}(r \to q) \,\}$$

- We cannot apply **T**-permanence and **T**¬-permanence since

$$p \not\preceq^{+} \Delta,\;\; p \not\preceq^{-} \Delta,\;\; q \not\preceq^{+} \Delta,\;\; q \not\preceq^{-} \Delta,\;\; r \not\preceq^{+} \Delta,\;\; r \not\preceq^{-} \Delta$$

- Since $r \preceq_w^{-} \Delta$, we can apply **F**-permanence (partial subst. $\bot / r$)

$$\frac{\mathbf{T}(p \vee q),\ \mathbf{F}(q \wedge \bot),\ \mathbf{F}(p \wedge \bot),\ \mathbf{F}(r \to q)}{\mathbf{T}(p \vee q),\ \mathbf{F}\bot,\ \mathbf{F}(r \to q)}\ \text{simpl.}$$

- Now we can apply **T**-permanence (subst. $\top / p$)

$$\frac{\mathbf{T}(\top \vee q),\ \mathbf{F}\bot,\ \mathbf{F}(r \to q)}{\mathbf{T}\top,\ \mathbf{F}\bot,\ \mathbf{F}(r \to q)}\ \text{simpl.}$$

Since the last set is not provable and there are no branch points, we conclude that Δ is not provable.

Preliminary experimental results

- Experiments have been carried out along the lines of [Raths et al., JAR 2007]
- 2000 random generated formulas with 1024 connectives and a number of variables ranging from 1 to 1024.

	0-1s	1-10s	10-100s	100-600s	>600s
PITP	1085	7	3	2	3
BPPI	1025	51	11	4	9(n.a.)
IPPI	856	227	12	4	1
EPPI	859	226	11	4	0

PIPT : Efficient **Int** prover [Avellone et al., TCS 2008] (C++)

BPPI : PIPT calculus plus Replace-**T**, Replace-**T**$\neg$ and simpl. (Prolog)

IPPI : BPPI plus **T**-permanence and **T**$\neg$-permanence (Prolog)

EPPI : IPPI plus **F**-permanence (Prolog)

Future improvements: substitutions inside contexts

Investigate substitutions inside a formula.

Future improvements: substitutions inside contexts

Investigate substitutions inside a formula.

For instance, in

$$\mathbf{F}(\ \neg q \vee ((q \rightarrow \neg p) \rightarrow ((\neg q \wedge p) \vee (p \rightarrow q)))\ )$$

none of the previous simplifications can be applied

Future improvements: substitutions inside contexts

Investigate substitutions inside a formula.

For instance, in

$$\mathbf{F}(\ \neg q \vee ((q \rightarrow \neg p) \rightarrow ((\neg q \wedge p) \vee (p \rightarrow q)))\ )$$

none of the previous simplifications can be applied

On the other hand, the simplification step

$$\frac{\mathbf{F}(\ \neg q \vee ((q \rightarrow \neg p) \rightarrow ((\neg q \wedge p) \vee (p \rightarrow q)))\ )}{\mathbf{F}(\ \neg q \vee ((q \rightarrow \neg p) \rightarrow ((\neg q \wedge \bot) \vee (p \rightarrow q)))\ )}$$

is invertible.

The last formula simplifies to

$$\mathbf{F}(\ \neg q \vee ((q \rightarrow \neg p) \rightarrow (p \rightarrow q)\ )$$

Future improvements: substitutions inside contexts

Investigate substitutions inside a formula.

For instance, in

$$\mathbf{F}(\ \neg q \vee ((q \to \neg p) \to ((\neg q \wedge p) \vee (p \to q)))\ )$$

none of the previous simplifications can be applied

On the other hand, the simplification step

$$\frac{\mathbf{F}(\ \neg q \vee ((q \to \neg p) \to ((\neg q \wedge p) \vee (p \to q)))\ )}{\mathbf{F}(\ \neg q \vee ((q \to \neg p) \to ((\neg q \wedge \bot) \vee (p \to q)))\ )}$$

is invertible.

The last formula simplifies to

$$\mathbf{F}(\ \neg q \vee ((q \to \neg p) \to (p \to q)\ )$$

Main idea: substitution *inside a context*