BTVN (tuần 1 + 2)

Bài 1.1

a, $E(Y-c)^2 = E(Y^2) - 2c\mu + c^2$

$= E(Y^2) + (c-\mu)^2 - \mu^2 \geq E(Y^2) - \mu^2$

Do đó $\mu = \arg\min_{c \in \mathbb{R}} E(Y-c)^2$.

b, $E((Y-g(X))^2 \mid X) = E(Y^2 \mid X) - 2g(X).E(Y\mid X) + g^2(X)$

$= E(Y^2 \mid X) + (g(X) - E(Y\mid X))^2 - E(Y\mid X)^2$

$\geq E(Y^2 \mid X) - E(Y \mid X)$

Do đó, $E(Y\mid X) = \arg\min_{g} E((Y-g(X))^2 \mid X)$

c, $E(Y-g(X))^2 = E(E((Y-g(X))^2 \mid X))$

$\geq E(E(Y^2 \mid X) - E(Y\mid X))$

Dấu "=" xảy ra $\Leftrightarrow g(X) = E(Y\mid X)$ theo như ý (b).

Bài 1.2.

a, Tương tự 1.1b, ta có

$E(X_{n+1} \mid X_1, \dots X_n) = \arg\min_{g} E((X_{n+1} - g(X_1, \dots X_n))^2 \mid X_1, \dots X_n)$

b, $E(X_{n+1} - g(X_1, \dots X_n))^2 = E_{X_1,\dots,X_n}[\underbrace{E((X_{n+1} - g(X_1,\dots X_n))^2 \mid X_1, \dots X_n)}]$

đạt min khi $g(X_1; \dots X_n)$

$= E(X_{n+1} \mid X_1, \dots, X_n)$

c, $\hat{X}_{n+1} = E_{X_1,\dots X_n}(E(X_{n+1} \mid X_1, \dots X_n)) = \mu$

d, Ta tìm $c = (c_1, c_2 \dots, c_n)^T \in \mathbb{R}^n$ sao cho $\sum_{k=1}^{n} c_k X_k$ là ước lg tuyến tính ko chệch tốt nhất theo nghĩa phương sai cực tiểu.

Đặt $\underline{x} = (X_1; X_2 \dots X_n)^T \in \mathbb{R}^n$. Khi đó $\sum_{k=1}^{n} c_k X_k = \underline{c}^T \underline{x}$.

$$Var(\underline{c}^T \underline{x}) = E[(\underline{c}^T \underline{x} - E(\underline{c}^T \underline{x}))^2] = E[(\underline{c}^T(\underline{x} - E\underline{x}))^2]$$

$$= E(\underline{c}^T (\underline{x} - E\underline{x}).(\underline{x} - E\underline{x})^T \underline{c}]$$

$$= \underline{c}^T E[(\underline{x} - E\underline{x}).(\underline{x} - E\underline{x})^T].\underline{c}$$

$$= \underline{c}^T \Sigma \underline{c}.$$

Mặt khác: $\Sigma := E[(\underline{x} - E\underline{x})(\underline{x} - E\underline{x})^T]$

$$= diag\{Var(X_1); Var(X_2); \dots; Var(X_n)\}$$

$$= \sigma^2 I_{n \times n} \ (\text{do } \{X_t\} \text{ là iid})$$

Hơn nữa $\underline{c}^T \underline{x}$ là ước lg o chệch nên.

$$E[\underline{c}^T \underline{x}] = \mu \Leftrightarrow \underline{c}^T . E[\underline{x}] = \mu$$

$$\Leftrightarrow \underline{c}^T . \mathbb{1} = 1$$ ($\mathbb{1}$ là vectơ cột gồm toàn số 1)

Ta tìm $\underline{c}$ để cực tiểu hoá hàm lagrange $\mathcal{L}(c;\lambda)$ xđ bởi:

$$\mathcal{L}(\underline{c};\lambda) = Var(\underline{c}^T \underline{x}) + \lambda(\underline{c}^T \mathbb{1} - 1)$$

$$= \sigma^2 \|\underline{c}\|_2^2 + \lambda.(\underline{c}^T . \mathbb{1} - 1)$$

$$\frac{\partial \mathcal{L}}{\partial \underline{c}}(\underline{c};\lambda) = 2\sigma^2 \underline{c} + \lambda \mathbb{1} = \mathbb{0}$$ ($\mathbb{0}$ là vectơ không)

$$\Leftrightarrow \underline{c} = \frac{-\lambda}{2\sigma^2} . \mathbb{1}$$

Lại có: $\underline{c}^T \mathbb{1} = 1$. Do đó $\frac{-\lambda}{2\sigma^2} \mathbb{1}^T \mathbb{1} = 1 \Leftrightarrow \lambda = \frac{-2\sigma^2}{n}$

Như vậy, $\underline{c} = \frac{-\lambda}{2\sigma^2} . \mathbb{1} = \frac{1}{n} . \mathbb{1} = \left(\frac{1}{n}; \dots; \frac{1}{n}\right)^T$

hay $\underline{c}^T \underline{x} = \frac{1}{n} \sum_{k=1}^{n} X_k$ là ước lg TT o chệch tốt nhất.

e, Suy ra trực tiếp từ b, và d,

f, Trước hết, ta tìm hàm f cực tiểu hoá

$$E\left[(S_{n+1} - f(S_1, \dots S_n))^2 \mid S_1; \dots S_n\right]$$

$$= E\left[S_{n+1}^2 \mid S_1; \dots S_n\right] - 2f(S_1; \dots; S_n) E[S_{n+1} \mid S_1; \dots S_n] + f(S_1; \dots S_n)^2$$

$$= Var(S_{n+1} \mid S_1; \dots S_n) + \left(f(S_1; \dots S_n) - E[S_{n+1} \mid S_1; \dots S_n]\right)^2$$

$$\geq Var(S_{n+1} \mid S_1, \dots S_n)$$

Như vậy $E[S_{n+1} \mid S_1, \dots S_n] \in$

$$= \underset{f}{\operatorname{argmin}} \; E\left[(S_{n+1} - f(S_1; \dots; S_n))^2 \mid S_1; \dots; S_n\right]$$

Chọn $\hat{S}_{n+1} = E_{S_1, \dots, S_n}\left(E(S_{n+1} \mid S_1, \dots S_n)\right)$

$$= E_{X_1, \dots, X_n}\left(E(X_1 + \dots + X_{n+1} \mid X_1, \dots; X_n)\right)$$

$$= \mu(n+1).$$

Bài 1.3.

Xét chuỗi tgian dừng chặt $\{X_t\}_{t>0}$ với $\forall h \in R$ và $\forall t_k > 0$.

Ta có: $P(X_{t_k} \leq x_{t_k}) = P(X_{t_k + h} \leq x_{t_k})$

$\Rightarrow X_{t_k}$ và $X_{t_k + h}$ có cùng pp $\forall h \in R$ $\forall t_k > 0$

Giả sử chúng có cùng pp vs bnn X.

Đặt $\mu_X = E(X)$, khi đó $E(X_t) = E(X) = \mu_X$

không phụ $\in$ vào t.

Do đó $\{X_t\}$ là chuỗi TG dừng yếu.

Bài 1.5.

a, ACVF:

$$\gamma_x(t+h;t) = Cov(X_{t+h}; X_t)$$
$$= Cov(Z_{t+h} + \theta Z_{t+h-2}; Z_t + \theta Z_{t-2})$$
$$= Cov(Z_{t+h}; Z_t) + \theta[Cov(Z_{t+h-2}; Z_t) + Cov(Z_{t+h}; Z_{t-2})] + \theta^2 Cov(Z_{t+h-2}; Z_{t-2})$$
$$= \chi_{\{0\}}(h) + \theta\chi_{\{2\}}(h) + \theta\chi_{\{-2\}}(h) + \theta^2\chi_{\{0\}}(h)$$
$$= \begin{cases} 1+\theta^2 ; \text{ nếu } h=0 \\ \theta ; \text{ nếu } h=\pm 2 \\ 0 ; \text{ trái lại} \end{cases} \xrightarrow{\theta=0,8} \begin{cases} 1,64 ; \text{ nếu } h=0 \\ 0,8 ; \text{ nếu } h=\pm 2 \\ 0 ; \text{ trái lại} \end{cases}$$

ACF:

$$\rho(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 1 & \text{nếu } h=0 \\ 20/41 & \text{nếu } h=\pm 2 \end{cases}$$

b, $$Var\left(\frac{1}{4}\sum_{k=1}^{4} X_k\right) = \frac{1}{16} Var\left(\sum_{k=1}^{4} X_k\right)$$
$$= \frac{1}{16}\left(\sum_{k=1}^{4} Var(X_k) + 2\sum_{1\le k<l\le 4} Cov(X_k; X_l)\right)$$
$$= \frac{1}{16}\left(\sum_{k=1}^{4}\gamma_x(0) + 2Cov(X_1;X_3) + 2Cov(X_2;X_4)\right)$$
$$= \frac{1}{16}(4\gamma_x(0) + 4\gamma_x(2))$$
$$= \frac{1}{4}(\gamma_x(0) + \gamma_x(2)) = 0,61.$$

c, Với $\theta = -0,8$ thì $\gamma_x(\pm 2) = 0,8$ và $\gamma_x(0) = 1,64$. Khi đó:

$$Var\left(\frac{1}{4}\sum_{k=1}^{4} X_k\right) = \frac{1}{4}(\gamma_x(0) + \gamma_x(2)) = 0,21 < 0,61.$$

Bài 1.7.

Đặt $\mu_X = E(X_t)$, $\mu_Y = E(Y_t)$

$E(X_t + Y_t) = \mu_X + \mu_Y$ không phụ $\in t$.

$\gamma_{X+Y}(t+h; t) = Cov(X_{t+h} + Y_{t+h}; X_t + Y_t)$

$= Cov(X_{t+h}; X_t) + Cov(Y_{t+h}; X_t) + Cov(X_{t+h}; X_t)$

$+ Cov(Y_{t+h}; Y_t)$

$= \gamma_X(t+h; t) + \gamma_Y(t+h; t)$

$= \underbrace{\gamma_X(h) + \gamma_Y(h)}_{\text{không phụ} \in t}$ do $\{X_t\}; \{Y_t\}$ là 2 chuỗi dừng.

Do đó $\{X_t + Y_t\}$ là chuỗi dừng.