



## Unit IV - Lecture notes unit 4

Transforms And Boundary Value Problems (SRM Institute of Science and Technology)

## UNIT – IV FOURIER TRANSFORMS

### INTRODUCTION

The development of mathematical representation of periodic phenomena using complex numbers leads to complex form of the Fourier series representation of periodic function. The representation of periodic signals as a linear combination of harmonically related complex exponentials can be extended to develop a representation of aperiodic signals as linear combination of complex exponentials. This leads to Fourier Transforms.

Fourier Transform is widely used in the theory of communication engineering, wave propagation and other branches of applied mathematics.

### FOURIER INTEGRAL THEOREM (AU 2005,2008,2009,2010,2011)

**If  $f(x)$  is piecewise continuous, has piecewise continuous derivatives in every finite interval in  $(-\infty, \infty)$  and absolutely integrable in  $(-\infty, \infty)$ , then**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(x-t)} dt ds \quad \text{-----(i)}$$

R.H.S is called the Fourier Complex integral or Fourier Complex integral representation of  $f(x)$ . OR  $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos\{s(x-t)\} dt ds$  ---(ii)

R.H.S is called the Fourier integral or Fourier integral representation of  $f(x)$

**RESULTS:** 1.  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$  if  $f(-x)=f(x)$  that is  $f(x)$  is an even function

$= 0$  if  $f(-x)=-f(x)$  that is  $f(x)$  is an odd function

2.  $e^{i\theta} = \cos \theta + i \sin \theta$ ,  $e^{-\infty} = 0$ ,  $e^0 = 1$ ,  $e^{\infty} = \infty$  &  $\cos(a-b) = \cos a \cos b + \sin a \sin b$  and so on

3.  $\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$  Bernoulli's theorem where  $u$  &  $v$  are function of  $x$ .

4.  $\sin 0 = 0$ ,  $\cos 0 = 1$ ,  $\cos n\pi = (-1)^n$  where  $n=0,1,2,3, \dots$   $\sin n\pi = 0$ ,  $\cos \frac{(2n-1)\pi}{2}$   
 $= 0$  for all  $n$  &  $\sin \frac{(2n-1)\pi}{2} = (-1)^{(n+1)}$  for  $n=1,2,3, \dots$

$$5. \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} \{a \cos bx + b \sin bx\} \quad \text{also} \quad \int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}$$

$$\int e^{-ax} \sin bx dx = \frac{e^{-ax}}{a^2+b^2} \{-a \sin bx - b \cos bx\} \quad \text{also}$$

$$\int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2}$$

**NOTE :**

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos\{s(x-t)\} dt ds \\ &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \{\cos sx \cos st + \sin sx \sin st\} dt ds \\ &= \frac{1}{\pi} \int_0^\infty \cos sx \left( \int_{-\infty}^\infty f(t) \cos st dt \right) ds + \frac{1}{\pi} \int_0^\infty \sin sx \left( \int_{-\infty}^\infty f(t) \sin st dt \right) ds \end{aligned}$$

**NOTE :**

1. If  $f(x)$  (or  $f(t)$ ) is even, then  $f(t) \cos st$  is an even function of  $t$  and  $f(t) \sin st$  is an odd function of  $t$ .

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos sx \cos st dt ds \quad \text{R.H.S is called the Fourier Cosine integral of } f(x)$$

2. If  $f(x)$  (or  $f(t)$ ) is odd, then  $f(t) \cos st$  is an odd function of  $t$  and  $f(t) \sin st$  is an even function of  $t$ .

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin sx \sin st dt ds \quad \text{R.H.S is called the Fourier sine integral of } f(x)$$

**Problems:**

1. Show that  $f(x)=1$ ,  $0 < x < \infty$  can not be represented by a Fourier integral.

**Sol:**  $\int_0^\infty |f(x)| dx = \int_0^\infty 1 dx = (X)_0^\infty$  and this value tends to  $\infty$  as  $x \rightarrow \infty$ . that is  $\int_0^\infty f(x) dx$  is not convergent. Hence  $f(x)=1$  can not be represented by a Fourier integral.

2. Using Fourier integral formula, prove that  $e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{(\lambda^2+2) \cos \lambda x}{\lambda^4+4} d\lambda$

**Sol:** Since the right hand side integrand contains Cosine term, We shall use cosine integral formula.

$$\begin{aligned}
f(x) &= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left( \int_0^{\infty} f(t) \cos \lambda t dt \right) d\lambda \quad \text{Here } f(t) = e^{-t} \cos t \\
&= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left( \int_0^{\infty} e^{-t} \cos t \cos \lambda t dt \right) d\lambda \\
&= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left( \int_0^{\infty} e^{-t} \left\{ \frac{1}{2} [\cos(1+\lambda)t + \cos(1-\lambda)t] \right\} dt \right) d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \left( \cos \lambda x \left\{ \frac{e^{-t}}{1+(1+\lambda)^2} [-\cos(1+\lambda)t + (1+\lambda)\sin(1+\lambda)t] \right\} + \right. \\
&\quad \left. \frac{e^{-t}}{1+(1-\lambda)^2} [-\cos(1-\lambda)t + (1-\lambda)\sin(1-\lambda)t] \right) d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \left\{ 0 - \frac{1}{1+(1+\lambda)^2} (-1) + \left\{ 0 - \frac{1}{1+(1-\lambda)^2} (-1) \right\} \right\} d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \left\{ \frac{1+(1-\lambda)^2 + 1+(1+\lambda)^2}{(1+(1-\lambda)^2)(1+(1+\lambda)^2)} \right\} d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \left\{ \frac{\lambda^2 - 2\lambda + 2 + \lambda^2 + 2\lambda + 2}{(\lambda^2 - 2\lambda + 2)(\lambda^2 + 2\lambda + 2)} \right\} d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \left\{ \frac{2\lambda^2 + 4}{(\lambda^2 + 2) - 4\lambda^2} \right\} d\lambda \\
&= \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \left\{ \frac{2(\lambda^2 + 2)}{\lambda^4 + 4\lambda^2 - 4\lambda^2 + 4} \right\} d\lambda \\
f(x) &= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left\{ \frac{(\lambda^2 + 2)}{\lambda^2 + 4} \right\} d\lambda
\end{aligned}$$

3. Using the Fourier integral representation, show that

$$\begin{aligned}
\text{(i)} \int_0^{\infty} \frac{w \sin xw}{1+w^2} dw &= \frac{\pi}{2} e^{-x}, x > 0 \quad \text{(ii)} \int_0^{\infty} \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x}, x > 0 \quad \text{(ii)} \int_0^{\infty} \frac{\sin w \cos xw}{w} dw \\
&= \frac{\pi}{2} \quad 0 \leq x < 1
\end{aligned}$$

**Sol:** (i). Fourier sine integral for  $f(x)$  is  $f(x) = \int_0^{\infty} \sin wx \left\{ \int_0^{\infty} f(t) \sin wt dt \right\} dw$  Here  $f(t) = e^{-t}$

$$f(x) = \int_0^{\infty} \sin wx \left\{ \int_0^{\infty} e^{-t} \sin wt dt \right\} dw$$

$$\begin{aligned}
&= \frac{2}{\pi} \int_0^{\infty} \sin wx \left\{ \frac{e^{-t}}{1+w^2} [-\sin wt - w \cos wt] \right\}_0^{\infty} dw \\
&= \frac{2}{\pi} \int_0^{\infty} \sin wx \frac{w}{1+w^2} dw
\end{aligned}$$

$$\int_0^{\infty} \frac{w \sin wx}{1+w^2} dw = \frac{\pi}{2} f(x) = f(x) = \begin{cases} \frac{\pi}{2} e^{-x}, & x > 0 \end{cases} \quad \text{where } f(x) = e^{-x}, x > 0$$

(ii). Fourier Cosine integral for  $f(x)$  is  $f(x) = \frac{2}{\pi} \int_0^{\infty} \cos wx \left\{ \int_0^{\infty} f(t) \cos wtdt \right\} dw$  Here  $f(t) = e^{-t}$

$$\begin{aligned}
f(x) &= \frac{2}{\pi} \int_0^{\infty} \cos wx \left\{ \int_0^{\infty} f(t) \cos wtdt \right\} dw \\
&= \frac{2}{\pi} \int_0^{\infty} \cos wx \left\{ \int_0^{\infty} e^{-t} \cos wtdt \right\} dw \\
&= \frac{2}{\pi} \int_0^{\infty} \cos wx \left\{ \frac{e^{-t}}{1+w^2} (-\cos wt + w \sin wt) \right\}_0^{\infty} dw \\
&= \frac{2}{\pi} \int_0^{\infty} \cos wx \frac{1}{1+w^2} dw \\
\int_0^{\infty} \cos wx \frac{1}{1+w^2} dw &= \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-x}, \quad x \geq 0
\end{aligned}$$

(iii). Fourier integral formula for  $f(x)$  is

$$\begin{aligned}
f(x) &= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos w(t-x) dt dw \quad \text{where } f(x) = 1 \text{ for } 0 < x < 1 \\
&= \frac{1}{\pi} \int_0^{\infty} \left\{ \int_{-\infty}^0 0 + \int_0^1 1 \cos w(t-x) dt + \int_1^{\infty} 0 \right\} dw \\
&= \frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\sin w(t-x)}{w} \right\}_0^1 dw \\
&= \frac{1}{\pi} \int_0^{\infty} \left[ \frac{\sin w(1-x)}{w} - \frac{\sin w(-x)}{w} \right] dw \\
&= \frac{1}{\pi} \int_0^{\infty} \left[ \frac{\sin w(1-x) + \sin wx}{w} \right] dw
\end{aligned}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{2 \sin \frac{w}{2} \cos \frac{(w-2wx)}{2}}{w} dw$$

$$\int_0^{\infty} \frac{\sin \frac{w}{2} \cos \frac{(w-2wx)}{2}}{w} dw = \frac{\pi}{2} f(x)$$

## FOURIER TRANSFORMS

### DEFINITION

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$  is called the **Fourier transform** of  $f(x)$  and  $F$  is the Fourier transform operator, where  $s$  is used as the transform variable. Also it is denoted as  $F(s)$

NOTE: Sometimes the letter  $p$  or  $w$  is used as the transform variable and it is obtained from Fourier complex integral representation of  $f(x)$

### DEFINITION

(AU 2001, 2007, 2010, 2011)

The  $f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$  is called the **inverse Fourier transform** of  $F(s)$ .

NOTE:  $F[f(x)]$  &  $f(x) = F^{-1}[F(s)]$  together is called as Fourier transform pair.

**OR IF**  $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$  **THEN**  $f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

### DEFINITION

$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$  is called as the **Fourier cosine transform** of  $f(x)$  and it is

also denoted as  $F_c(s)$

$f(x) = F_c^{-1}[F_c(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(f(x)) \cos sx ds$  is called as the **Inverse Fourier cosine transform** of  $f(x)$

NOTE:  $F_c[f(x)]$  &  $f(x) = F_c^{-1}[F_c[f(x)]]$  together is called as **Fourier Cosine transform pair**. Obtained from Fourier Cosine integral representation of  $f(x)$ .

**OR** **IF**  $F_c[f(x)] = \int_0^\infty f(x) \cos sx \, dx$  **THEN**  $f(x) = F_c^{-1}[F_c(f(x))] = \frac{2}{\pi} \int_0^\infty F_c(f(x)) \cos sx \, ds$

#### DEFINITION

$F_s[f(x)] = \sqrt{2} \int_0^\infty f(x) \sin sx \, dx$  is called as the **Fourier sine transform** of  $f(x)$  and it is also denoted as  $F_s(s)$

$f(x) = F_s^{-1}[F_s(f(x))] = \sqrt{2} \int_0^\infty F_s(f(x)) \sin sx \, ds$  is called as the **Inverse Fourier sine transform** of  $f(x)$

**NOTE:**  $F_s[f(x)]$  &  $f(x) = F_s^{-1}[F_s[f(x)]]$  Together is called as **Fourier sine transform pair**. Obtained from Fourier sine integral representation of  $f(x)$ . **OR** **IF**  $F_s[f(x)] = \int_0^\infty f(x) \sin sx \, dx$  **THEN**  $f(x) = F_s^{-1}[F_s(f(x))] = \frac{2}{\pi} \int_0^\infty F_s(f(x)) \sin sx \, ds$

#### PROPERTIES OF FOURIER TRANSFORMS

**1. LINEARITY PROPERTY:**  $F$  is a linear operator.  
 $F[(c_1 f_1(x) + c_2 f_2(x))] = c_1 F[f_1(x)] + c_2 F[f_2(x)]$ , where  $c_1$  and  $c_2$  are constants.

$$\begin{aligned} \text{Proof: } F[(c_1 f_1(x) + c_2 f_2(x))] &= \int_{-\infty}^{\infty} (c_1 f_1(x) + c_2 f_2(x)) e^{isx} dx \\ &= c_1 \int_{-\infty}^{\infty} f_1(x) e^{isx} dx + c_2 \int_{-\infty}^{\infty} f_2(x) e^{isx} dx \\ &= c_1 F[f_1(x)] + c_2 F[f_2(x)] \end{aligned}$$

**2. CHANGE OF SCALE PROPERTY:** If  $F[f(x)] = F[s]$ , then  $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$

(AU 2006, 2009, 2010, 2011)

$$F[f(ax)] = \int_{-\infty}^{\infty} f(ax) e^{isx} dx \text{ put } ax = t \text{ and assuming that } a > 0$$

$$F[f(ax)] = \int_{-\infty}^{\infty} f(t) e^{is \frac{t}{a}} \frac{dt}{a} = \frac{1}{a} F\left[\frac{s}{a}\right]$$

$$\text{But } F[f(ax)] = \int_{\infty}^{-\infty} f(t) e^{is \frac{t}{a}} \frac{dt}{a} = -\frac{1}{a} F\left[\frac{s}{a}\right], \text{ if } a < 0$$

$$\text{Therefore } F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$$

### 3. Shifting property (AU 1999,2000,2001,2006,2007,2008,2010,2011,2012)

$$\text{i. } F[f(x-a)] = e^{ias} F[s] \qquad \text{ii. } F[e^{iax} f(x)] = F[s+a]$$

Proof: we know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{ias} dx$$

$$\text{Put } t = x-a, \qquad x \rightarrow -\infty \text{ implies } t \rightarrow -\infty$$

$$dt = dx, \qquad x \rightarrow \infty \text{ implies } t \rightarrow \infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{ist} e^{isa} dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{ist} dt$$

$$= e^{isa} F[f(t)]$$

$$F[f(x-a)] = e^{isa} F[s] \quad \text{Hence proved (i)}$$

ii. We know that



$$\begin{aligned}
 F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\
 F[e^{iax} f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx \\
 F[e^{iax} f(x)] &= F(s+a)
 \end{aligned}$$

Hence proved (ii).

**4. MODULATION THEOREM:** If  $F[f(x)] = F[s]$ , then F

$$[f(x) \cos ax] = \frac{1}{2} \{ F[s+a] + F[s-a] \} \quad (\text{AU 2001, 2003, 2010})$$

Proof:  $F[f(x) \cos ax] = \int_{-\infty}^{\infty} f(x) \cos ax e^{isx} dx$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} f(x) \frac{e^{iax} + e^{-iax}}{2} e^{isx} dx \\
 &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} f(x) e^{iax} e^{isx} dx + \int_{-\infty}^{\infty} f(x) e^{-iax} e^{isx} dx \right] \\
 &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} f(x) e^{(a+s)ix} dx + \int_{-\infty}^{\infty} f(x) e^{(s-a)ix} dx \right]
 \end{aligned}$$

$$F[f(x) \cos ax] = \frac{1}{2} \{ F[s+a] + F[s-a] \}$$

**5. TRANSFORM OF DERIVATIVES:** If  $f(x)$  is continuous,  $f'(x)$  is piecewise continuously differentiable and  $f(x)$  and  $f'(x)$  are absolutely integrable in  $(-\infty, \infty)$  and limit  $x$  tends to  $-\infty$  and  $\infty$  of  $f(x) = 0$ , then  $F[f'(x)] = isF[s]$

Proof: By the first three conditions given,  $F[f(x)]$  and  $F[f'(x)]$  exist.

$$\begin{aligned}
 F[f'(x)] &= \int_{-\infty}^{\infty} f'(x) e^{isx} dx \\
 &= \left\{ e^{-isx} f(x) \right\}_{-\infty}^{\infty} + is \int_{-\infty}^{\infty} e^{-isx} f(x) dx, \text{ on integration by parts.}
 \end{aligned}$$

$= 0 + is F[f(x)]$ , by the given condition

$$F[f'(x)] = isF[s]$$

NOTE:  $F\{f^{(n)}(x)\} = [is]^n F[s]$  where  $F[s] = F[f(x)]$

6. If  $F[s] = F[f(x)]$  then

$$(i). F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$$

$$(ii). F\left[\frac{d^n}{ds^n} \{f(x)\}\right] = (-is)^n F[s] \text{ if } f, f', f'', \dots, f^{n-1} \rightarrow 0 \text{ as } x \rightarrow +\infty \text{ or } -\infty$$

$$(iii) F[x^k f^{(m)}(x)] = (-1)^{k+m} \frac{d^k}{ds^k} [s^m F\{f(x)\}]$$

Proof:

$$(i) \frac{d^n}{ds^n} [F(s)] = \frac{d^n}{ds^n} \left[ \int_{-\infty}^{\infty} f(x) e^{isx} dx \right]$$

$$= \int_{-\infty}^{\infty} f(x) \left[ \frac{\partial^n}{\partial s^n} e^{isx} \right] dx \quad \text{by Leibnitz's rule for constants limits}$$

$$= \int_{-\infty}^{\infty} f(x) [(ix)^n e^{isx}] dx$$

$$\frac{d^n}{ds^n} [F(s)] = (i)^n \int_{-\infty}^{\infty} f(x) [(x)^n e^{isx}] dx$$

$$\frac{d^n}{ds^n} [F(s)] = (i)^n F[x^n f(x)]$$

$$F[x^n f(x)] = \frac{1}{i^n} \frac{d^n}{ds^n} [F(s)] \quad \text{since } \frac{1}{i} = -i \text{ therefore } \left(\frac{1}{i}\right)^n = (-i)^n$$

$$F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$$

$$(ii) F[f'(x)] = \int_{-\infty}^{\infty} f'(x) e^{isx} dx$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} e^{isx} d(f(x)) \\
&= \left\{ e^{-isx} f(x) \right\}_{-\infty}^{\infty} - \text{is} \int_{-\infty}^{\infty} e^{isx} f(x) dx, \text{ on integration by parts.} \\
&= 0 - \text{is} \int_{-\infty}^{\infty} e^{isx} f(x) dx
\end{aligned}$$

$$F[f'(x)] = -is F(s) \text{ similarly } F\left[\frac{d^n}{ds^n}\{f(x)\}\right] = (-is)^n F[s]$$

$$(iii) F[x^k f^m(x)] = (-i)^k \frac{d^k}{ds^k} \{F[f^m(x)]\} \text{ since } F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$$

$$= (-i)^k \frac{d^k}{ds^k} \{(-is)^m F[s]\} \text{ since } F\left[\frac{d^n}{ds^n}\{f(x)\}\right] = (-is)^n F[s]$$

$$F[x^k f^m(x)] = (-i)^{k+m} \frac{d^k}{ds^k} \{(s)^m F[s]\}$$

**CONVOLUTION: (AU 2000, 2003, 2008 )** The convolution of two functions  $f(x)$  and  $g(x)$  over the interval  $(-\infty, \infty)$  is defined as  $f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du$

**Convolution Theorem for Fourier Transforms:** (or Faltung theorem): The Fourier Transform of the convolution of  $f(x)$  and  $g(x)$  is the product of their Fourier Transforms

$$F[f(x) * g(x)] = F[f(x)] \cdot G[g(x)]$$

**Proof:**

$$\begin{aligned}
F[f(x) * g(x)] &= F\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right] \\
&= \int_{-\infty}^{\infty} e^{isx} \left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right] dx \\
&= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{isx} g(x-u)dx\right] du \text{ (By changing the order of} \\
&\text{integration where both variables } x \text{ and } u \text{ limits are constants)}
\end{aligned}$$

Put  $x-u=t$ ,  $dx=dt$ ,  $t$  tends to  $-\infty$  to  $\infty$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} e^{is(u+t)} g(t) dt \right] du \\
&= \int_{-\infty}^{\infty} f(u) e^{isu} \left[ \int_{-\infty}^{\infty} e^{ist} g(t) dt \right] du \\
&= \int_{-\infty}^{\infty} f(u) e^{isu} F[g(t)] du \text{ Treating } u \text{ and } t \text{ as dummy variable}
\end{aligned}$$

$$F[f(x) * g(x)] = F[f(x)] F[g(x)] \text{ that is } F[f(x) * g(x)] = F(s).G(s)$$

**NOTE:** (i)  $[f(x) * g(x)] = F^{-1} [F(s).G(s)]$

**CONJUGATE SYMMETRY PROPERTY:** If  $F[f(x)] = F[s]$ , then  $F[\overline{f(-x)}] = \overline{F(s)}$

**Proof:**  $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$

$$\overline{F(s)} = \int_{-\infty}^{\infty} f(x) e^{-isx} dx \text{ Putting } -x=t, dx=-dt \text{ and } t \text{ varies from } \infty \text{ to } -\infty$$

$$= \int_{\infty}^{-\infty} f(-t) e^{ist} (-dt)$$

$$= \int_{-\infty}^{\infty} f(-t) e^{ist} dt \quad \text{by definite integral property}$$

$$\overline{F(s)} = F[\overline{f(-x)}] \quad \text{treating } t \text{ as the dummy variable}$$

**PARSEVAL'S IDENTITY THEOREM:** If  $F[f(x)] = F[s]$ , then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds \quad (\text{AU 2002,2007,2008,2010, 2011 Statement})$$

**Proof:** By convolution theorem we know that  $F[f(x) * g(x)] = F(s).G(s)$

That is  $[f(x) * g(x)] = F^{-1} [F(s).G(s)]$

$$\int_{-\infty}^{\infty} f(u) g(x-u) du = \int_{-\infty}^{\infty} [F(s).G(s)] e^{-isx} ds$$

Put  $x=0$ , we get  $\int_{-\infty}^{\infty} f(u) g(-u) du = \int_{-\infty}^{\infty} [F(s).G(s)] ds$

Now taking  $g(u) = \overline{f(-u)}$  then  $g(-u) = \overline{f(u)}$  also  $f(u)\overline{f(u)} = |f(u)|^2$

$F[g(u)] = F[\overline{f(-u)}] = \overline{F(s)}$  implies that  $G(s) = \overline{F(s)}$

$$\int_{-\infty}^{\infty} f(u)\overline{f(u)}du = \int_{-\infty}^{\infty} [F(s) \cdot \overline{F(s)}] ds$$

Hence

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

## PROPERTIES OF FOURIER SINE AND COSINE TRANSFORMS

### 1. CHANGE OF SCALE PROPERTY :

$$F_c\{f(ax)\} = \frac{1}{a} F_c\left\{\frac{s}{a}\right\} \text{ and } F_s\{f(ax)\} = \frac{1}{a} F_s\left\{\frac{s}{a}\right\}$$

**2. MODULATION THEOREM:** If  $F_c[f(x)] = F_c[s]$ , and  $F_s[f(x)] = F_s[s]$  then

$$(i) F_c\{f(x)\cos ax\} = \frac{1}{2} [F_c[s+a] + F_c[s-a]]$$

$$(i) F_c\{f(x)\sin ax\} = \frac{1}{2} [F_s[s+a] + F_s[a-s]]$$

$$(i) F_s\{f(x)\cos ax\} = \frac{1}{2} [F_s[s+a] + F_s[s-a]]$$

$$(i) F_s\{f(x)\sin ax\} = \frac{1}{2} [F_c[s-a] - F_c[s+a]]$$

### 3. PARSEVAL'S IDENTITY FOR FOURIER SINE AND COSINE TRANSFORMS:

If  $F_c[f(x)] = F_c[s]$ , and  $F_s[f(x)] = F_s[s]$  then

$$(i) \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds = \int_{-\infty}^{\infty} |F_s(s)|^2 ds$$

$$(ii) \int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds = \int_0^{\infty} F_s[s]G_s[s]ds$$

### 4. DERIVATIVES FOR FOURIER SINE AND COSINE TRANSFORM:

$$(i) F_c\{f'(x)\} = sF_s\{f(x)\} - f(0)$$

$$(ii) \quad F_c\{f''(x)\} = -s^2 F_c\{f(x)\} - f(0)$$

$$(iii) \quad F_s\{f'(x)\} = -s F_c\{f(x)\}$$

$$(iv) \quad F_s\{f''(x)\} = -s^2 F_s\{f(x)\} + sf(0)$$

NOTE:  $F_c\{f'(x)\} = s F_s\{f(x)\}$  and  $F_s\{f'(x)\} = -s F_c\{f(x)\}$  where  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$

5. If  $F_c[f(x)] = F_c[s]$ , and  $F_s[f(x)] = F_s[s]$  then (i)  $\frac{d}{ds}\{F_c[s]\} = -F_s\{xf(x)\}$

$$(ii) \quad \frac{d}{ds}\{F_s[s]\} = F_c\{xf(x)\}$$

**Proof:** (i)  $F_c[f(x)] = \int_0^\infty f(x) \cos sx \, dx$

$$\frac{d}{ds}\{F_c[s]\} = \int_0^\infty f(x) \left\{ \frac{\partial}{\partial s} \cos sx \right\} dx \quad \text{By Leibnit's rule for constants limits}$$

$$= \int_0^\infty f(x) \{-x \sin sx\} dx$$

$$= - \int_0^\infty f(x) \{x \sin sx\} dx$$

$$= - \int_0^\infty [xf(x)] \sin sx \, dx$$

$$\frac{d}{ds}\{F_c[s]\} = -F_s\{xf(x)\}$$

$$(ii) \quad F_s[f(x)] = \int_0^\infty f(x) \sin sx \, dx$$

$$\frac{d}{ds}\{F_s[s]\} = \int_0^\infty f(x) \left\{ \frac{\partial}{\partial s} \sin sx \right\} dx \quad \text{By Leibnit's rule for constants limits}$$

$$= \int_0^\infty f(x) \{x \cos sx\} dx$$

$$= \int_0^\infty [xf(x)] \cos sx \, dx$$

$$\frac{d}{ds}\{F_s[s]\} = F_c\{xf(x)\}$$

## PROBLEMS

1. Find the Fourier transform of the unit step function and unit impulse function.

**SOL:** The unit step function is defined as  $u_a(x) = u(a-x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$

$$F[u_a(x)] = \int_a^{\infty} e^{-isx} dx = \left\{ \frac{e^{-isx}}{-is} \right\}_a^{\infty} = \frac{e^{-ias}}{is} \quad \text{since } e^{-\infty} = 0$$

**NOTE:**  $F[u_a(x)] = \frac{e^{-ias}}{is}$  &  $F[u_0(x)] = \frac{1}{is} = -\frac{i}{s}$

The unit impulse function or Dirac Delta function  $\delta_a(x)$  is  $\lim_{h \rightarrow 0} f(x)$  where

$$f(x) = \begin{cases} \frac{1}{h}, & \text{for } a - \frac{h}{2} \leq x \leq a + \frac{h}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} F\{f(x)\} &= \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} \frac{1}{h} e^{-isx} dx \\ &= \frac{1}{h} \left\{ \frac{e^{-isx}}{-is} \right\}_{a-\frac{h}{2}}^{a+\frac{h}{2}} \\ &= \frac{1}{ihs} \left\{ e^{-is\left(a-\frac{h}{2}\right)} - e^{-is\left(a+\frac{h}{2}\right)} \right\} \end{aligned}$$

$$F[f(x)] = e^{-ias} \frac{\sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)}$$

$$F[\delta_a(x)] = \lim_{h \rightarrow 0} e^{-ias} \frac{\sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)}$$

$$F[\delta_a(x)] = \left\{ \text{since } \lim_{h \rightarrow 0} \frac{\sin\left(\frac{hs}{2}\right)}{\left(\frac{hs}{2}\right)} = 1 \right\} e^{-ias} \quad \& \quad F[\delta_0(1)] =$$

2. Find the Fourier transform of  $f(x)$  if  $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$  Where  $a$  is a positive real number, hence deduce that (i)  $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$  (ii)  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt = \frac{\pi}{2}$  (AU 2003, 2004, 2005)

$$\text{SOL: } F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (\cos sx + i \sin sx) dx \quad \{f(x) = 1 \text{ in } (-a, a) \text{ } \cos sx \text{ is even \& } \sin sx \text{ is odd} \}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (\cos sx) dx$$

$$\left\{ \begin{array}{l} \text{since } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is an even function} \\ = 0 \text{ if } f(x) \text{ is an odd function} \end{array} \right\}$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin sx}{s} \right\}_0^a = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin sa}{s} \right\} \quad [\text{since } \sin 0 = 0]$$

$$\text{By Fourier inversion Formula } f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin sa}{s} \right\} (\cos sx - i \sin sx) ds \quad (\cos sx \text{ is even \& } \sin sx \text{ is odd})$$

$$= \frac{2}{\pi} \int_0^\infty \left\{ \frac{\sin sa}{s} \right\} (\cos sx) ds$$

$$\left\{ \begin{array}{l} \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is an even function} \\ = 0 \text{ if } f(x) \text{ is an odd function} \end{array} \right\}$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \frac{2 \sin sa \cos sx}{s} \right\} ds \quad \{\text{put } x = 0 \Rightarrow f(0) = 1\}$$

$$1 = \frac{2}{\pi} \int_0^\infty \left\{ \frac{\sin sa}{s} \right\} ds \quad \{\text{put } as = t, a ds = dt \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty\}$$

$$\int_0^\infty \left\{ \frac{\sin t}{\frac{t}{a}} \right\} \frac{dt}{a} = \frac{\pi}{2} \Rightarrow \int_0^\infty \left\{ \frac{\sin t}{t} \right\} dt = \frac{\pi}{2}$$

$$\text{By Parseval's identity theorem } \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$



$$\int_{-a}^a 1^2 dx = \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin sa}{s} \right\} \right]^2 ds$$

$$\{x\}_{-a}^a = \frac{2}{\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin as}{s} \right]^2 ds \begin{cases} \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx & \text{if } f(x) \text{ is an even function} \\ = 0 & \text{if } f(x) \text{ is an Odd function} \end{cases}$$

$$2a = \frac{4}{\pi} \int_0^{\infty} \left[ \frac{\sin as}{s} \right]^2 ds \quad \{ \text{put } as = t, ads = dt \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \}$$

$$\int_0^{\infty} \left[ \frac{\sin t}{\frac{t}{a}} \right]^2 \frac{dt}{a} = \left[ \frac{\pi}{4} \right] \Rightarrow \int_0^{\infty} \left[ \frac{\sin t}{t} \right]^2 dt \left[ 2a \left[ \frac{\pi}{4} \right] \left[ \frac{2a}{a} \right] \right] = \frac{\pi}{2}$$

3. Find the Fourier transform of  $f(x)$  given by  $f(x) = \begin{cases} 1 - x^2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$

Hence evaluate (i)  $\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \left( \frac{s}{2} \right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$  show that (ii)

$$\int_0^{\infty} \frac{(s \cos s - \sin s)^2}{s^6} ds = \frac{\pi}{15} \quad (\text{AU 2000, 2001, 2004, 2005, 2006, 2007, 2010, 2011})$$

$$\text{Sol: } F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1 - x^2) [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) [\cos sx] dx \quad \{ \text{since } [(1 - x^2) \sin sx] \text{ is an odd function} \}$$

=

$$\sqrt{\frac{2}{\pi}} \left\{ (1 - x^2) \frac{\sin sx}{s} - (-2x) \left( \frac{-\cos sx}{s^2} \right) + (-2) \left( \frac{-\sin sx}{s^3} \right) \right\}_0^1 \quad \{ \text{By Bernoulli's theorem} \}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{2 \cos s}{s^2} + \frac{2 \sin s}{s^3} - 0 \right\}$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left\{ \frac{2 \sin s - 2 \cos s}{s^3} \right\} = F(s)$$

By inversion formula  $f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \frac{2\sin s - 2s \cos s}{s^3} \right\} [\cos sx - i \sin sx] ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} 2 \int_0^{\infty} \left\{ \frac{2\sin s - 2s \cos s}{s^3} \right\} [\cos sx] ds \quad (\text{since } \frac{2\sin s - 2s \cos s}{s^3} \text{ is an}$$

even function therefore  $\frac{2\sin s - 2s \cos s}{s^3} \cos sx$  is an even function and  $\frac{2\sin s - 2s \cos s}{s^3} \sin sx$  is an odd function )

$$\int_0^{\infty} \frac{2\sin s - 2s \cos s}{s^3} \cos sx ds = \frac{\pi}{2} f(x) \text{ ---(i) put } x=1/2 \text{ in (i) so } f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4} \text{ and}$$

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right) = \left(\frac{\pi}{4}\right) \left(\frac{3}{4}\right) = \frac{3\pi}{16}$$

**Sol:**(ii) By Parseval's identity theorem  $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

$$\int_{-1}^1 (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \frac{2(\sin s - s \cos s)}{s^3} \right]^2 ds$$

$$2 \int_0^1 (1-x^2+x^4) dx = \frac{8}{\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin s - s \cos s}{s^3} \right]^2 ds$$

$$\left[ x - 2\frac{x^3}{3} + \frac{x^5}{5} \right]_0^1 = \frac{4}{\pi} 2 \int_0^{\infty} \left[ \frac{\sin s - s \cos s}{s^3} \right]^2 ds$$

$$\left[ 1 - \frac{2}{3} + \frac{1}{5} \right] = \frac{8}{\pi} \int_0^{\infty} \left[ \frac{\sin s - s \cos s}{s^3} \right]^2 ds$$

$$\text{Therefore } \int_0^{\infty} \left[ \frac{\sin s - s \cos s}{s^3} \right]^2 ds = \left(\frac{\pi}{8}\right) \left(\frac{15-10+3}{15}\right) = \left(\frac{\pi}{8}\right) \left(\frac{8}{15}\right) = \frac{\pi}{15}$$

4. Find the Fourier transform of  $f(x)$  given by

$$f(x) = \begin{cases} 1 - |x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence evaluate (i)  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt = \frac{\pi}{2}$  (ii)  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^4 dt = \frac{3\pi}{8}$

(AU 2001,2005,2006,2007,2008,2009,2010,2011,2012)

$$\text{Sol: } F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

=

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1 - |x|) [\cos sx + i \sin sx] dx \quad \left\{ |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases} \text{ is an even function} \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x) [\cos sx] dx \quad \{ \text{since } [(1 - |x|) \sin sx] \text{ is an odd function} \}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (1 - x) \frac{\sin sx}{s} - (-1) \left( \frac{-\cos sx}{s^2} \right) \right\}_0^1 \quad \{ \text{By Bernoulli's theorem} \}$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 - \frac{\cos s}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right] \quad \{ \text{since } \sin 0 = 0, \cos 0 = 1 \}$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{1 - \cos s}{s^2} \right\} \right] = F[s]$$

$$\text{By using inverse Fourier transform } f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \cos s}{s^2} \right\} [\cos sx - i \sin sx] ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} 2 \int_0^\infty \left\{ \frac{1 - \cos s}{s^2} \right\} [\cos sx] ds \quad \left( \text{since } \left\{ \frac{1 - \cos s}{s^2} \right\} \text{ is an even function} \right)$$

$$\text{therefore } \left\{ \frac{1 - \cos s}{s^2} \right\} \cos sx \text{ is an even function and } \left\{ \frac{1 - \cos s}{s^2} \right\} \sin sx \text{ is an odd function )}$$

$$\int_0^\infty \left\{ \frac{1-\cos s}{s^2} \right\} \cos sx \, ds = \frac{\pi}{2} f(x) \quad \{ \text{Put } x=0 \text{ then } f(0)=1 \}$$

$$\int_0^\infty \left\{ \frac{2\sin^2\left(\frac{s}{2}\right)}{s^2} \right\} ds = \frac{\pi}{2}$$

$$\frac{1}{2} \int_0^\infty \left\{ \frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2} \right\} ds = \frac{\pi}{2} \quad \left\{ \text{Put } t = \frac{s}{2} \text{ then } dt = \frac{ds}{2} \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \right\}$$

$$\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt = \frac{\pi}{2}$$

By Parseval's identity theorem  $\int_{-\infty}^\infty |f(x)|^2 dx = \int_{-\infty}^\infty |F(s)|^2 ds$

$$\int_{-1}^1 [1 - |x|]^2 dx = \int_{-\infty}^\infty \left[ \sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \cos s}{s^2} \right\} \right]^2 ds$$

$$2 \int_0^1 (1 - 2x + x^2) dx = \frac{2}{\pi} \int_{-\infty}^\infty \left[ \frac{1 - \cos s}{s^2} \right]^2 ds$$

$$\left[ x - 2 \frac{x^2}{2} + \frac{x^3}{3} \right]_0^1 = \frac{1}{\pi} 2 \int_0^\infty \left[ \frac{1 - \cos s}{s^2} \right]^2 ds$$

$$\left[ 1 - 1 + \frac{1}{3} \right] = \frac{2}{\pi} \int_0^\infty \left[ \frac{2\sin^2\left(\frac{s}{2}\right)}{s^2} \right]^2 ds$$

$$\int_0^\infty \left[ \left( \frac{1}{2} \right) \frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2} \right]^2 ds = \left( \frac{\pi}{2} \right) \left( \frac{1}{3} \right) \quad \left\{ \text{Put } t = \frac{s}{2} \text{ then } dt = \frac{ds}{2} \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \right\}$$

$$\int_0^\infty \left( \frac{1}{4} \right) \left[ \frac{\sin^2(t)}{(t)^2} \right]^2 2dt = \left( \frac{\pi}{2} \right) \left( \frac{1}{3} \right)$$

$$\text{Therefore } \int_0^\infty \left[ \frac{\sin t}{t} \right]^4 dt = \frac{\pi}{3}$$

5. Find the Fourier transform of  $f(x)$  given by

$$f(x) = \begin{cases} a^2 - x^2, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Show that (i)  $F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[ \frac{\sin as - a \cos as}{s^3} \right]$  (ii)  $\int_0^\infty \left[ \frac{\sin t - t \cos t}{t^3} \right] dt = \frac{\pi}{4}$  (iii)

$$\int_0^\infty \frac{(\sin t - t \cos t)^2}{t^6} dt = \frac{\pi}{15} \quad (\text{AU 1996, 2001, 2004, 2008, 2009, 2010, 2011, 2012})$$

$$\text{Sol: } F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a^2 - x^2) [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2) [\cos sx] dx \quad \{\text{since } [(a^2 - x^2) \sin sx] \text{ is an odd function}\}$$

=

$$\sqrt{\frac{2}{\pi}} \left\{ (a^2 - x^2) \frac{\sin sx}{s} - (-2x) \left( \frac{-\cos sx}{s^2} \right) + (-2) \left( \frac{-\sin sx}{s^3} \right) \right\}_0^a \quad \{\text{By Bernoulli's theorem}\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{2a \cos sa}{s^2} + \frac{2 \sin sa}{s^3} - 0 \right\}$$

$$F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left\{ \frac{\sin as - a \cos as}{s^3} \right\} = F(s)$$

$$\text{By inversion formula } f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \frac{2 \sin as - 2a \cos as}{s^3} \right\} [\cos sx - i \sin sx] ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} 2 \int_0^\infty \left\{ \frac{2 \sin as - 2a \cos as}{s^3} \right\} [\cos sx] ds \quad \left( \text{since } \frac{2 \sin as - 2a \cos as}{s^3} \right)$$

is an even function therefore  $\frac{2 \sin as - 2a \cos as}{s^3} \cos sx$  is an even function and

$\frac{2 \sin as - 2a \cos as}{s^3} \sin sx$  is an odd function )

$$\int_0^{\infty} \frac{2\sin as - 2s \cos as}{s^3} \cos sx ds = \frac{\pi}{2} f(x)$$

Put  $x = 0$  then  $f(0) = a^2$

$$\int_0^{\infty} \frac{\sin as - as \cos sa}{s^3} ds = \frac{\pi}{4} f(0) = \frac{\pi a^2}{4}$$

{Put  $t = as \Rightarrow dt = ads$  and  $s \rightarrow 0$  to  $\infty \Rightarrow t \rightarrow 0$  to  $\infty$ }

$$\int_0^{\infty} \frac{\sin t - t \cos t}{\left(\frac{t}{a}\right)^3} \frac{dt}{a} = \frac{\pi a^2}{4} \Rightarrow \int_0^{\infty} \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$$

By Parseval's identity theorem  $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

$$\int_{-a}^a (a^2 - x^2)^2 dx = \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \frac{2(\sin as - s \cos as)}{s^3} \right]^2 ds$$

$$2 \int_0^a (a^4 - 2a^2 x^2 + x^4) dx = \frac{8}{\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin as - s \cos as}{s^3} \right]^2 ds$$

$$\left[ a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right]_0^a = \frac{4}{\pi} 2 \int_0^{\infty} \left[ \frac{\sin as - s \cos as}{s^3} \right]^2 ds$$

$$\left[ a^5 - \frac{2a^5}{3} + \frac{a^5}{5} \right] = \frac{8}{\pi} \int_0^{\infty} \left[ \frac{\sin as - s \cos as}{s^3} \right]^2 ds$$

$$\int_0^{\infty} \left[ \frac{\sin as - s \cos as}{s^3} \right]^2 ds = a^5 \left( \frac{\pi}{8} \right) \left( \frac{15 - 10 + 3}{15} \right) = \left( \frac{\pi}{8} \right) \left( \frac{8a^5}{15} \right) = \frac{\pi a^5}{15}$$

$$\int_0^{\infty} \left[ \frac{\sin t - t \cos t}{\left(\frac{t}{a}\right)^3} \right]^2 \frac{dt}{a} = \frac{\pi a^5}{15} \{ \text{substituting } t = as \Rightarrow dt = ads \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \}$$

$$\int_0^{\infty} \left[ \frac{\sin t - t \cos t}{t^3} \right]^2 dt = \frac{\pi}{15}$$

6. Find the Fourier transform of  $f(x)$  given by  $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$

Hence evaluate (i)  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt$  (ii)  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^4 dt$  (AU 1996, 2008, 2011)

$$\text{Sol: } F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

=

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^a (a - |x|) [\cos sx + i \sin sx] dx \quad \left\{ |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases} \text{ is an even function} \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a - x) [\cos sx] dx \quad \{ \text{since } [(1 - |x|) \sin sx] \text{ is an odd function} \}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (a - x) \frac{\sin sx}{s} - (-1) \left( \frac{-\cos sx}{s^2} \right) \right\}_0^a \quad \{ \text{By Bernoulli's theorem} \}$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 - \frac{\cos sa}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right] \quad \{ \text{since } \sin 0 = 0, \cos 0 = 1 \}$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{1 - \cos sa}{s^2} \right\} \right] = F[s]$$

$$\text{By using inverse Fourier transform } f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \cos as}{s^2} \right\} [\cos sx - i \sin sx] ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} 2 \int_0^\infty \left\{ \frac{1 - \cos as}{s^2} \right\} [\cos sx] ds \quad \left( \text{since } \left\{ \frac{1 - \cos as}{s^2} \right\} \text{ is an even function} \right)$$

therefore  $\left\{ \frac{1 - \cos as}{s^2} \right\} \cos sx$  is an even function and  $\left\{ \frac{1 - \cos as}{s^2} \right\} \sin sx$  is an odd function

)

$$\int_0^\infty \left\{ \frac{1 - \cos as}{s^2} \right\} \cos sx ds = \frac{\pi}{2} f(x) \quad \{ \text{Put } x = 0 \text{ then } f(0) = a \}$$

$$\int_0^\infty \left\{ \frac{2\sin^2\left(\frac{as}{2}\right)}{s^2} \right\} ds = \frac{\pi}{2} a$$

$$\frac{a^2}{2} \int_0^\infty \left\{ \frac{\sin^2\left(\frac{as}{2}\right)}{\left(\frac{as}{2}\right)^2} \right\} ds = \frac{\pi a}{2}$$

$$\left\{ \text{Put } t = \frac{as}{2} \text{ then } dt = \frac{ads}{2} \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \right\}$$

$$\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 \frac{2dt}{a} = \frac{\pi}{a} \Rightarrow \int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt = \frac{\pi}{2}$$

$$\text{By Parseval's identity theorem } \int_{-\infty}^\infty |f(x)|^2 dx = \int_{-\infty}^\infty |F(s)|^2 ds$$

$$\int_{-a}^a [a - |x|]^2 dx = \int_{-\infty}^\infty \left[ \sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \cos as}{s^2} \right\} \right]^2 ds$$

$$2 \int_0^a (a^2 - 2ax + x^2) dx = \frac{2}{\pi} \int_{-\infty}^\infty \left[ \frac{1 - \cos as}{s^2} \right]^2 ds$$

$$\left[ a^2 x - 2 \frac{ax^2}{2} + \frac{x^3}{3} \right]_0^a = \frac{1}{\pi} 2 \int_0^\infty \left[ \frac{1 - \cos as}{s^2} \right]^2 ds$$

$$\left[ a^3 - a^3 + \frac{a^3}{3} \right] = \frac{2}{\pi} \int_0^\infty \left[ \frac{2\sin^2\left(\frac{as}{2}\right)}{s^2} \right]^2 ds$$

$$\int_0^\infty \left[ \left( \frac{a^2}{2} \right) \frac{\sin^2\left(\frac{as}{2}\right)}{\left(\frac{sa}{2}\right)^2} \right]^2 ds = \left( \frac{\pi}{2} \right) \left( \frac{a^3}{3} \right) \left\{ \text{Put } t = \frac{as}{2} \text{ then } dt = \frac{ads}{2} \text{ and } s \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty \right\}$$

$$\int_0^\infty \left( \frac{a^4}{4} \right) \left[ \frac{\sin^2(t)}{(t)^2} \right]^2 \frac{2dt}{a} = \left( \frac{\pi}{2} \right) \left( \frac{a^3}{3} \right)$$

$$\text{Therefore } \int_0^\infty \left[ \frac{\sin t}{t} \right]^4 dt = \frac{\pi}{3}$$



7. Find the Fourier transform of  $e^{-a^2 x^2}$ . Hence (i) Prove that  $e^{-\frac{x^2}{2}}$  is self reciprocal with respect to Fourier transforms and (ii) Find the Fourier cosine transform of  $e^{-x^2}$  (AU 2007 2009)

**Sol:**  $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

F

$$\begin{aligned} \left[ e^{-a^2 x^2} \right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2 + isx} dx \end{aligned}$$

$$F\left[ e^{-a^2 x^2} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ a^2 x^2 - isx + \left( \frac{is}{2a} \right)^2 - \left( \frac{is}{2a} \right)^2 \right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ a^2 x^2 - isx + \left( \frac{is}{2a} \right)^2 \right] - \frac{s^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left[ ax - \frac{is}{2a} \right]^2} dx$$

$$\left\{ \text{put } ax - \frac{is}{2a} = t \Rightarrow adx = dt \text{ and } x \rightarrow -\infty \text{ to } \infty \Rightarrow t \rightarrow -\infty \text{ to } \infty \right\}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-[t]^2} \frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \frac{2}{a} \int_0^{\infty} e^{-t^2} dt$$

$$\begin{aligned} \left\{ \text{put } u = t^2 \Rightarrow du = 2t dt, \text{ and } t \rightarrow 0 \text{ to } \infty \Rightarrow u \rightarrow 0 \text{ to } \infty \therefore \int_0^{\infty} e^{-t^2} dt = \int_0^{\infty} e^{-u} \frac{du}{2\sqrt{u}} = \right. \\ \left. \frac{1}{2} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du = \frac{\sqrt{\pi}}{2} \right\} \end{aligned}$$

$$= \frac{2}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \frac{\sqrt{\pi}}{2}$$

$$\therefore F[e^{-a^2 x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \quad (\text{put } a = \frac{1}{\sqrt{2}})$$

$$\therefore F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}} \quad (\because e^{-\frac{x^2}{2}} \text{ is self reciprocal with respect to Fourier transforms})$$

(ii) From (i) we have  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$

$$\int_{-\infty}^{\infty} e^{-a^2 x^2} [\cos sx + i \sin sx] dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}} \quad \text{Equating the real parts on}$$

both sides, we get  $\int_{-\infty}^{\infty} e^{-a^2 x^2} [\cos sx] dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}}$

$$2 \int_0^{\infty} e^{-a^2 x^2} [\cos sx] dx = \frac{\sqrt{\pi}}{a} e^{-\frac{s^2}{4a^2}} \Rightarrow F_c \left[ e^{-a^2 x^2} \right] = \frac{\sqrt{\pi}}{2a} e^{-\frac{s^2}{4a^2}}$$

8. Find the Fourier cosine transform of  $e^{-2x} + 3e^{-x}$

**Sol:**  $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$  where  $f(x) = \{e^{-2x} + 3e^{-x}\}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \{e^{-2x} + 3e^{-x}\} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \{e^{-2x}\} \cos sx dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \{3e^{-x}\} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-2x}}{4+s^2} (-2\cos sx + s \sin sx) \right\}_0^{\infty} + 3 \left\{ \frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) \right\}_0^{\infty} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{2}{4+s^2} \right\} + \left\{ 0 + \frac{1}{1+s^2} \right\} \right]$$

$$\{\sin 0 = 0, \cos 0 = 1, e^{-\infty} = 0 \text{ \& } e^0 = 1\}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{2}{4+s^2} + \frac{1}{1+s^2} \right\} \right]$$

9. Find the Fourier sine transform of  $f(x) = e^{-ax}$ ,  $a > 0$  hence deduce that

$$\int_0^{\infty} \frac{s}{a^2+s^2} \sin sx \, ds = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-ax} \quad (\text{AU 2007, 2009, 2010, 2012})$$

$$\text{Sol: } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx \quad \text{where } f(x) = \{e^{-ax}\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right\}_0^{\infty} \right]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{s}{a^2+s^2} \right\} \right]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2+s^2} \right\} \right] \quad a > 0$$

$$\text{By inversion formula for sine transform } f(x) = F_s^{-1}[F_s(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(f(x)) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2+s^2} \right\} \right] \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{s}{a^2+s^2} \sin sx \, ds$$

$$\int_0^{\infty} \frac{s}{a^2+s^2} \sin sx \, ds = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-ax}, \quad a > 0$$

$$\int_0^\infty \frac{x}{1^2+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m} \quad \text{By replacing } s=x, x=m \text{ \& } a=1$$

10. Find the Fourier sine transform of  $f(x) = \frac{1}{x}$  (AU 2008,2009)

$$\begin{aligned} \text{Sol: } F_s[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \quad \text{where } f(x) = \frac{1}{x} \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx \quad \{ \text{Let } sx = \theta, sdx = d\theta, \theta \rightarrow 0 \text{ to } \infty \} \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \theta}{\frac{\theta}{s}} \frac{d\theta}{s} \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \theta}{\theta} d\theta \end{aligned}$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2}\right] \quad \left\{ \text{since } \int_0^\infty \frac{\sin \theta}{\theta} d\theta = \frac{\pi}{2} \right\}$$

11. Find  $F_c[xe^{-ax}]$  and  $F_s[xe^{-ax}]$  (AU 2011)

Sol: we know that (i)  $\frac{d}{ds} \{F_c[s]\} = -F_s\{xf(x)\}$  & (ii)  $\frac{d}{ds} \{F_s[s]\} = F_c\{xf(x)\}$

$$\text{Also } F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2+s^2} \right\} \right] \quad \text{and } F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{a}{a^2+s^2} \right\} \right]$$

$$\begin{aligned} \therefore F_c[xe^{-ax}] &= \frac{d}{ds} \{F_s[e^{-ax}]\} \\ &= \frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2+s^2} \right\} \right] \right\} \end{aligned}$$

$$\therefore F_c[xe^{-ax}] = \sqrt{\frac{2}{\pi}} \left\{ \frac{s2s - (s^2+a^2)}{[s^2+a^2]^2} \right\} = \sqrt{\frac{2}{\pi}} \left\{ \frac{(s^2-a^2)}{[s^2+a^2]^2} \right\}$$

$$\begin{aligned}
 F_s\{xf(x)\} &= -\frac{d}{ds}\{F_c[f(x)]\} \\
 &= -\frac{d}{ds}\{F_c[e^{-ax}]\} \\
 &= -\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\left\{\frac{a}{a^2+s^2}\right\}\right]\right\}
 \end{aligned}$$

$$F_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\left\{\frac{2as}{[s^2+a^2]^2}\right\}$$

12. Find the Fourier Sine transform of  $xe^{-\frac{x^2}{2}}$

**Sol:**  $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$F_c\left[e^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \cos sx \, dx = I \quad (\text{say})$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \left\{ \frac{\partial}{\partial s} \cos sx \right\} dx \quad (\text{By Leibnit's rule for constants limits})$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \{-\sin sx\} x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \{\sin sx\} d\left[e^{-\frac{x^2}{2}}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \sin sx \left( e^{-\frac{x^2}{2}} \right) \right\}_0^\infty - \int_0^\infty \left( e^{-\frac{x^2}{2}} \right) s \cos s x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[ 0 - \int_0^\infty \left( e^{-\frac{x^2}{2}} \right) s \cos s x \, dx \right]$$

$$\frac{dI}{ds} = s \sqrt{\frac{2}{\pi}} \left[ \int_0^\infty \left( e^{-\frac{x^2}{2}} \right) \cos s x \, dx \right]$$

$$\frac{dl}{ds} = -sI \Rightarrow \frac{dl}{l} = s ds \Rightarrow \int \frac{dl}{l} = \int s ds$$

$$\log l = -\frac{s^2}{2} + \log c \Rightarrow l = e^{-\frac{s^2}{2}} e^{\log c} \quad \text{when } s=0, \quad \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx = I$$

$$(\text{put } t = \frac{x^2}{2} \text{ then } dt = x dx \text{ and } x \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty)$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \frac{dt}{\sqrt{2t}} = \sqrt{\frac{1}{\pi}} \int_0^\infty e^{-t} t^{\left(\frac{1}{2}-1\right)} dt = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$$F_c \left[ e^{-\frac{x^2}{2}} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-\frac{x^2}{2}}) \cos sx dx = I \quad I = e^{-\frac{s^2}{2}} e^{\log c} \Rightarrow c=1$$

$$\therefore I = e^{-\frac{s^2}{2}} F_c \left[ e^{-\frac{x^2}{2}} \right] = I = e^{-\frac{s^2}{2}}$$

$$F_s \{xf(x)\} = -\frac{d}{ds} \{F_c[f(x)]\}$$

$$F_s \left\{ x e^{-\frac{x^2}{2}} \right\} = -\frac{d}{ds} \left\{ F_c \left[ e^{-\frac{x^2}{2}} \right] \right\}$$

$$\therefore F_s \left\{ x e^{-\frac{x^2}{2}} \right\} = -\frac{d}{ds} \left\{ e^{-\frac{s^2}{2}} \right\} = s e^{-\left(\frac{s^2}{2}\right)} \quad \text{that is } F_s \left\{ x e^{-\frac{x^2}{2}} \right\} = s e^{-\left(\frac{s^2}{2}\right)}$$

13. Find the Fourier cosine transform of  $e^{-x^2}$  (AU 2006, 2008, 2010, 2011)

$$\text{Sol: } F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$F_c[e^{-x^2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx dx$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-x^2} \text{Real part } e^{isx} dx$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \text{Real part} \int_{-\infty}^{\infty} e^{-x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part} \int_{-\infty}^{\infty} e^{-x^2 + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part} \int_{-\infty}^{\infty} e^{-\left[x^2 - isx + \left(\frac{is}{2}\right)^2 - \left(\frac{is}{2}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part} \int_{-\infty}^{\infty} e^{-\left[x^2 - isx + \left(\frac{is}{2}\right)^2\right] - \frac{s^2}{4}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part} e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left[x - \frac{is}{2}\right]^2} dx$$

$$\left\{ \text{put } x - \frac{is}{2} = t \Rightarrow dx = dt \text{ and } x \rightarrow -\infty \text{ to } \infty \Rightarrow t \rightarrow -\infty \text{ to } \infty \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \text{Real part} e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-[t]^2} dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} 2 \int_0^{\infty} e^{-t^2} dt$$

$$\left\{ \text{put } u = t^2 \Rightarrow du = 2t dt, \text{ and } t \rightarrow 0 \text{ to } \infty \Rightarrow u \rightarrow 0 \text{ to } \infty \therefore \int_0^{\infty} e^{-t^2} dt = \int_0^{\infty} e^{-u} \frac{du}{2\sqrt{u}} = \frac{1}{2} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du = \frac{\sqrt{\pi}}{2} \right\}$$

$$\text{Fc} \left[ e^{-x^2} \right] = \text{Real part} \frac{2}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \frac{\sqrt{\pi}}{2}$$

$$\therefore \text{Fc} \left[ e^{-x^2} \right] = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

14. Find the Fourier sine transform of  $e^{-|x|}$ . Hence show that  $\int_0^{\infty} \frac{x \sin ax}{1+x^2} dx = \frac{\pi}{2} e^{-a}$

,  $a > 0$

$$\text{Sol: } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-|x|} \sin sx \, dx$$

$$F_s[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right\}_0^{\infty} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{s}{1+s^2} \right\} \right] \quad \{ \sin 0 = 0 \text{ and } \cos 0 = 1 \}$$

$$F_s[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{1+s^2} \right\} \right]$$

$$\text{By inversion formula } f(x) = F_s^{-1}[F_s(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(f(x)) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{1+s^2} \right\} \right] \sin sx \, ds$$

$$e^{-|x|} = \frac{2}{\pi} \int_0^{\infty} \left[ \left\{ \frac{s}{1+s^2} \right\} \right] \sin sx \, ds$$

$$\int_0^{\infty} \left[ \left\{ \frac{s}{1+s^2} \right\} \right] \sin sx \, ds = \frac{\pi}{2} e^{-x} \quad (\text{Replacing } x=a \text{ \& } s=x \text{ then we get})$$

$$\int_0^{\infty} \left[ \left\{ \frac{x \sin ax}{1+x^2} \right\} \right] dx = \frac{\pi}{2} e^{-a}$$

15. Find the Fourier sine transform of  $\frac{e^{-ax}}{x}, a > 0$  and deduce that

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin s x \, dx = \tan^{-1} \left( \frac{b}{s} \right) - \tan^{-1} \left( \frac{a}{s} \right) \quad (\text{AU 2005, 2007, 2009})$$

$$\text{Sol: } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$



$$F_s \left[ \frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I \quad (\text{say})$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \left\{ \frac{\partial}{\partial s} \sin sx \right\} dx \quad (\text{By Leibnit's rule for constants limits})$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} x \{ \cos sx \} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx) \right\}_0^\infty \right]$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{a}{a^2 + s^2} \right\} \right]$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{a}{a^2 + s^2} \right\} \right] \quad a > 0 \quad \text{Integrating bothsides with respect to s we get}$$

$$I = \sqrt{\frac{2}{\pi}} \frac{1}{a} \tan^{-1} \left( \frac{s}{a} \right) + c$$

$$\text{To show that } \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx = \left[ \tan^{-1} \left( \frac{b}{s} \right) - \tan^{-1} \left( \frac{a}{s} \right) \right]$$

$$\text{We know that } F_s \left[ \frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I = \sqrt{\frac{2}{\pi}} \tan^{-1} \left( \frac{s}{a} \right)$$

$$F_s [e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2 + s^2} \right\} \right] \Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{s}{a^2 + s^2} \right\} \right]$$

Integrating with respect to 'a' on bothsides we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{-x} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \tan^{-1} \left( \frac{a}{s} \right)$$

$$\Rightarrow F_s\left[\frac{e^{-ax}}{x}\right] = -\sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{a}{s}\right) \text{ similarly } F_s\left[\frac{e^{-bx}}{x}\right] = -\sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{b}{s}\right)$$

$$\therefore F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}} \left[ \tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right) \right]$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left[ \tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right) \right]$$

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx = \left[ \tan^{-1}\left(\frac{b}{s}\right) - \tan^{-1}\left(\frac{a}{s}\right) \right]$$

16. Find the Fourier sine and cosine transform of  $x^{n-1}$  and prove that  $\frac{1}{\sqrt{x}}$  is self reciprocal under Fourier sine and cosine transforms.

Sol:  $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$  Replacing  $x$  by  $ax$  and changing the limits,

$$\Gamma n = \int_0^{\infty} e^{-ax} a^{n-1} x^{n-1} a \, dx$$

$$= a^n \int_0^{\infty} e^{-ax} x^{n-1} \, dx$$

$$\Rightarrow \int_0^{\infty} e^{-ax} x^{n-1} \, dx = \frac{\Gamma n}{a^n}, \quad n > 0 \text{ Putting } a = is \text{ then}$$

$$\int_0^{\infty} e^{-isx} x^{n-1} \, dx = \frac{\Gamma n}{(is)^n}, \quad n > 0$$

$$\int_0^{\infty} e^{-isx} x^{n-1} \, dx = \frac{(-i)^n \Gamma n}{(s)^n}, \quad n > 0$$

$$\left\{ (-i)^n = \left[ \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) \right]^n = e^{-\frac{in\pi}{2}} = \cos n\left(\frac{\pi}{2}\right) - i \sin n\left(\frac{\pi}{2}\right) \right\}$$

$$\int_0^{\infty} e^{-isx} x^{n-1} \, dx = \frac{e^{-\frac{in\pi}{2}} \Gamma n}{(s)^n}, \quad n > 0$$

$$\int_0^{\infty} e^{-isx} x^{n-1} dx = \frac{\Gamma n}{(s)^n} \left[ \cos n \left( \frac{\pi}{2} \right) - i \sin n \left( \frac{\pi}{2} \right) \right]$$

$$\int_0^{\infty} [\cos sx - i \sin sx] x^{n-1} dx = \frac{\Gamma n}{(s)^n} (\cos n \left( \frac{\pi}{2} \right) - i \sin n \left( \frac{\pi}{2} \right))$$

Equating the real and imaginary parts on both sides,

$$\int_0^{\infty} [\cos sx] x^{n-1} dx = \frac{\Gamma n}{(s)^n} (\cos n \left( \frac{\pi}{2} \right))$$

$$\int_0^{\infty} [\sin sx] x^{n-1} dx = \frac{\Gamma n}{(s)^n} (\sin n \left( \frac{\pi}{2} \right))$$

$$\therefore F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \left( \frac{n\pi}{2} \right) \text{ and } F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \left( \frac{n\pi}{2} \right)$$

Taking  $n = \frac{1}{2}$  we get  $F_c \left[ x^{-\left(\frac{1}{2}\right)} \right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos \left( \frac{\pi}{4} \right)$

$$F_c \left[ \frac{1}{\sqrt{x}} \right] = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} \Rightarrow F_c \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}} \text{ since } \left\{ \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \right\}$$

Similarly  $n = \frac{1}{2}$  substitution gives  $F_s \left[ x^{-\left(\frac{1}{2}\right)} \right] = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \sin \left( \frac{\pi}{4} \right)$

$$\Rightarrow F_s \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$$

$$\frac{F_s \left[ \frac{1}{\sqrt{x}} \right]}{F_c \left[ \frac{1}{\sqrt{x}} \right]} = \frac{\sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{s}}} \text{ and } F_s \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}} \Rightarrow \text{that } \frac{1}{\sqrt{x}} \text{ is self reciprocal under Fourier sine and cosine transforms.}$$

17. Evaluate (i)  $\int_0^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$  (ii)  $\int_0^{\infty} \frac{dx}{(a^2+x^2)^2}$  (iii)  $\int_0^{\infty} \frac{x^2 dx}{(a^2+x^2)^2}$

Using convolution theorem. (AU 2001,2005,2006,2009,2010,2011),

Sol: Let  $f(x) = e^{-ax}$  and  $g(x) = e^{-bx}$  then  $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$  and

$$F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$

By convolution theorem

By convolution theorem If  $F_c[f(x)] = F_c[s]$ , and  $F_s[f(x)] = F_s[s]$  then

$$(i) \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds = \int_{-\infty}^{\infty} |F_s(s)|^2 ds$$

$$(ii) \quad \int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds = \int_0^{\infty} F_s[s]G_s[s]ds$$

**Sol (i)**  $\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c[s]G_c[s]ds$

$$\Rightarrow \int_0^{\infty} e^{-ax}e^{-bx}dx = \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2} ds$$

$$\int_0^{\infty} e^{-(a+b)x} dx = \frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds$$

$$\int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds = \frac{\pi}{2} \left\{ \frac{e^{-(a+b)x}}{-(a+b)} \right\}_0^{\infty}$$

$$\int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)} \quad \text{where } a>0 \text{ and } b>0.$$

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)(b+x^2)} = \frac{\pi}{2ab(a+b)}$$

**Sol(ii)**  $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds$

$$\int_{-\infty}^{\infty} e^{-2ax} dx = \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \right]^2 ds$$

$$2 \int_0^{\infty} e^{-2ax} dx = 2 \int_0^{\infty} \frac{2}{\pi} \left\{ \frac{a^2}{[a^2+s^2]^2} \right\} ds$$

$$\left\{ \frac{e^{-2ax}}{-2a} \right\}_0^\infty = \frac{2}{\pi} \int_0^\infty \left\{ \frac{a^2}{[a^2 + s^2]^2} \right\} ds$$

$$\int_0^\infty \left\{ \frac{1}{[a^2 + s^2]^2} \right\} ds = \left[ \frac{\pi}{2a^2} \right] \left[ \frac{1}{2a} \right]$$

$$\int_0^\infty \left\{ \frac{1}{[a^2 + s^2]^2} \right\} ds = \left[ \frac{\pi}{4a^3} \right]$$

$$\int_0^\infty \left\{ \frac{1}{[a^2 + x^2]^2} \right\} dx = \left[ \frac{\pi}{4a^3} \right]$$

$$\text{Sol(iii)} \int_{-\infty}^\infty |f(x)|^2 dx = \int_{-\infty}^\infty |F_s(s)|^2 ds$$

$$\int_{-\infty}^\infty e^{-2ax} dx = \int_{-\infty}^\infty \left[ \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \right]^2 ds$$

$$2 \int_0^\infty e^{-2ax} dx = 2 \int_0^\infty \frac{2}{\pi} \left\{ \frac{s^2}{[a^2 + s^2]^2} \right\} ds$$

$$\left\{ \frac{e^{-2ax}}{-2a} \right\}_0^\infty = \frac{2}{\pi} \int_0^\infty \left\{ \frac{s^2}{[a^2 + s^2]^2} \right\} ds$$

$$\int_0^\infty \left\{ \frac{1}{[a^2 + s^2]^2} \right\} ds = \left[ \frac{\pi}{2} \right] \left[ \frac{1}{2a} \right]$$

$$\int_0^\infty \left\{ \frac{s^2}{[a^2 + s^2]^2} \right\} ds = \left[ \frac{\pi}{4a} \right]$$

$$\int_0^\infty \left\{ \frac{x^2}{[a^2 + x^2]^2} \right\} dx = \left[ \frac{\pi}{4a} \right]$$

18. Find the function whose sine transform is  $\frac{e^{-as}}{s}$ ,  $a > 0$  (AU 2010, 2011)

Sol:  $F_s[f(x)] = \frac{e^{-as}}{s}$  The inverse Fourier sine transform of  $F_s[s]$  is

$$f(x) = F_s^{-1}[F_s(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin sx \, ds \quad \dots (i) \text{ differentiating both sides with respect to } x \text{ we get}$$

$$f'(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} s \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-as} \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-as}}{a^2 + x^2} (-a \cos sx + x \sin sx) \right\}_0^{\infty} \right]$$

$$f'(x) = \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{a}{a^2 + x^2} \right\} \right]$$

$$f'(x) = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{a}{a^2 + x^2} \right\} \right] \quad a > 0 \quad \text{Integrating both sides with respect to } x \text{ we get}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \left( \frac{x}{a} \right) + c \quad \dots (ii) \text{ putting } x=0 \text{ in (i) gives } f(0)=0 \text{ and (ii)} \Rightarrow c=0$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \left( \frac{x}{a} \right)$$

19. Find the Fourier cosine transform of  $e^{-4x}$ . Deduce

$$\int_0^{\infty} \frac{\cos 2x}{x^2 + 16} \, dx = \left[ \frac{\pi}{8} \right] e^{-8} \quad \text{and} \quad \int_0^{\infty} \frac{x \sin 2x}{x^2 + 16} \, dx = \left[ \frac{\pi}{2} \right] e^{-8}$$

$$\text{Sol: } F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-4x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-4x}}{16+s^2} (-4 \cos sx + s \sin sx) \right\}_0^\infty \right]$$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ 0 + \frac{4}{16+s^2} \right\} \right]$$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{4}{16+s^2} \right\} \right] \text{ By the inverse Fourier cosine transform}$$

$$f(x) = F_c^{-1}[F_c(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{4}{16+s^2} \right\} \right] \cos sx \, ds$$

$$e^{-4x} = \frac{2}{\pi} \int_0^\infty \left[ \left\{ \frac{4}{16+s^2} \right\} \right] \cos sx \, ds$$

$$\int_0^\infty \left[ \left\{ \frac{1}{16+s^2} \right\} \right] \cos sx \, ds = \left[ \frac{\pi}{2} \right] \left[ \frac{e^{-4x}}{4} \right] \text{ Interchanging } x \text{ as } s \text{ implies that}$$

$$\int_0^\infty \left[ \left\{ \frac{\cos sx}{16+s^2} \right\} \right] ds = \left[ \frac{\pi}{8} \right] e^{-4s} \dots (i) \text{ putting } s=2 \text{ in (i) we get}$$

$$\int_0^\infty \left[ \left\{ \frac{\cos 2x}{16+x^2} \right\} \right] dx = \left[ \frac{\pi}{8} \right] e^{-8}$$

$$\text{Differentiating (i) with respect to } s \text{ we get } \int_0^\infty \left[ \left\{ \frac{-x \sin sx}{16+x^2} \right\} \right] dx = \left[ \frac{\pi}{8} \right] e^{-4s} (-4)$$

$$\Rightarrow \int_0^\infty \left[ \left\{ \frac{x \sin sx}{16+x^2} \right\} \right] dx = \left[ \frac{\pi}{2} \right] e^{-4s} \text{ putting } s=2 \text{ we get } \int_0^\infty \left[ \left\{ \frac{x \sin 2x}{16+x^2} \right\} \right] dx = \left[ \frac{\pi}{2} \right] e^{-8}$$

20. Find the Fourier sine transform of  $\frac{x}{a^2+x^2}$  and Fourier cosine transform of  $\frac{1}{a^2+x^2}$

$$\text{Sol: } F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\text{Fc} \left[ \frac{1}{a^2+x^2} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{a^2+x^2} \cos sx \, dx = I \quad (\text{say}) \dots (i)$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{a^2+x^2} \left\{ \frac{\partial}{\partial s} \cos sx \right\} dx \quad (\text{By Leibnit's rule for constants limits})$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{-x \sin sx}{a^2+x^2} \right] dx \dots (ii)$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{x^2}{x} \right] \left[ \frac{\sin sx}{a^2+x^2} \right] dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{a^2+x^2-a^2}{x[a^2+x^2]} \right] [\sin sx] dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{a^2+x^2}{x[a^2+x^2]} \right] [\sin sx] dx + a^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{\sin sx}{x[a^2+x^2]} \right] dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{\sin sx}{x} \right] dx + a^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \frac{\sin sx}{x[a^2+x^2]} \right] dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{\pi}{2} \right\} - a^2 \left\{ \int_0^\infty \left[ \frac{\sin sx}{x[a^2+x^2]} \right] dx \right\} \right] \quad \text{Differentiating}$$

again with respect to s

$$\frac{d^2 I}{ds^2} = a^2 \left\{ \int_0^\infty \left[ \frac{x \cos sx}{x[a^2+x^2]} \right] dx \right\}$$

$$\frac{d^2 I}{ds^2} = a^2 I \Rightarrow \frac{d^2 I}{ds^2} - a^2 I = 0 \quad \text{that is } [D^2 - a^2]I = 0$$

$$\Rightarrow I = Ae^{-as} + Be^{as} \quad \text{and} \quad \frac{dI}{ds} = -aAe^{-as} + aBe^{as}$$



Put  $s=0$ ,  $\Rightarrow A + B = \frac{1}{a} \sqrt{\frac{\pi}{2}}$  ... (i) and  $-aA + aB = -\sqrt{\frac{\pi}{2}}$  ... (ii) since  $s=0$  gives

$\Rightarrow$

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{a^2 + x^2} dx = \left\{ \sqrt{\frac{2}{\pi}} \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right\}_0^{\infty} = \frac{1}{a} \sqrt{\frac{2}{\pi}} \tan^{-1}[\infty] = \left[ \frac{\pi}{2} \right] \frac{1}{a} \sqrt{\frac{2}{\pi}} = \frac{1}{a} \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{\pi}{2} \right\} \right] = -\sqrt{\frac{\pi}{2}}$$

Solving (i) and (ii)  $A + B = \frac{1}{a} \sqrt{\frac{\pi}{2}}$  &  $A - B = -\frac{1}{a} \sqrt{\frac{\pi}{2}}$  we get

$$A = 0 \text{ and } B = \frac{2}{a} \sqrt{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{a}$$

$$\therefore I = \frac{\sqrt{2\pi}}{a} e^{as} = \text{Fc} \left[ \frac{1}{a^2 + x^2} \right]$$

F

$$s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$\text{Fs} \left[ \frac{x}{a^2 + x^2} \right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{x \sin sx}{a^2 + x^2} dx \text{ ) But } \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[ \frac{-x \sin sx}{a^2 + x^2} \right] dx$$

$$\therefore \text{Fs} \left[ \frac{x}{a^2 + x^2} \right] = -\frac{dI}{ds} \Rightarrow \text{Fs} \left[ \frac{x}{a^2 + x^2} \right] = \sqrt{2\pi} e^{as}$$

21. Find the Fourier sine transform of  $f(x) = \begin{cases} \sin x & \text{in } 0 < x < \pi \\ 0 & \text{in } \pi \leq x < \infty \end{cases}$

$$\text{Sol: Fs}[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$\text{Fs}[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin x \sin sx dx$$

$$\begin{aligned}
&= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\pi [\cos(s-1)x - \cos(s+1)x] dx \\
&= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1} \right\}_0^\pi \right] \\
&= \sqrt{\frac{1}{2\pi}} \left[ \left\{ \frac{\sin(s-1)\pi}{s-1} - \frac{\sin(s+1)\pi}{s+1} \right\} \right] \\
&= \sqrt{\frac{1}{2\pi}} \left[ \left\{ \frac{\sin(s\pi - \pi)}{s-1} - \frac{\sin(s\pi + \pi)}{s+1} \right\} \right] \\
&= \sqrt{\frac{1}{2\pi}} \left[ \left\{ \frac{-\sin(\pi - s\pi)}{s-1} - \frac{\sin(\pi + s\pi)}{s+1} \right\} \right] \\
&= \sqrt{\frac{1}{2\pi}} \left[ \left\{ \frac{-\sin(s\pi)}{s-1} - \frac{\sin(s\pi)}{s+1} \right\} \right] \\
&= \frac{\sin(s\pi)}{\sqrt{2\pi}} \left[ \left\{ \frac{-s-1+s-1}{s^2-1} \right\} \right]
\end{aligned}$$

$$F_s[f(x)] = \frac{\sin(s\pi)}{\sqrt{2\pi}} \left[ \left\{ \frac{-2}{s^2-1} \right\} \right] \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin(s\pi)}{1-s^2}$$

22. Find the Fourier cosine transform of  $f(x) = \begin{cases} \cos x & \text{in } 0 < x < 1 \\ 0 & \text{in } 1 \leq x < \infty \end{cases}$

$$\begin{aligned}
\text{Sol: } F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 \cos x \cos sx \, dx
\end{aligned}$$

$$F_c[f(x)] = \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^1 [\cos(s+1)x + \cos(1-s)x] dx$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{\sin(s+1)x}{s+1} - \frac{\sin(1-s)x}{1-s} \right\}_0^1 \right]$$

$$\therefore F_c[f(x)] = \sqrt{\frac{1}{2\pi}} \left[ \left\{ \frac{\sin(s+1)}{s+1} - \frac{\sin(1-s)}{1-s} \right\} \right]$$

23. Solve the integral equation  $\int_0^\infty f(x) \cos \lambda x dx = \begin{cases} 1-\lambda & \text{in } 0 \leq \lambda \leq 1 \\ 0 & \text{in } \lambda > 1 \end{cases}$

Hence evaluate  $\int_0^\infty \left[ \frac{\sin t}{t} \right]^2 dt$

Sol:  $\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1-\lambda & \text{in } 0 \leq \lambda \leq 1 \\ 0 & \text{in } \lambda > 1 \end{cases}$

$$\therefore F_c[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1-\lambda & \text{in } 0 \leq \lambda \leq 1 \\ 0 & \text{in } \lambda > 1 \end{cases} \dots (i)$$

By the inverse Fourier cosine transform

$$f(x) = F_c^{-1}[F_c(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos \lambda x d\lambda$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} [1-\lambda] \cos \lambda x d\lambda$$

$$= \frac{2}{\pi} \int_0^1 [1-\lambda] \cos \lambda x d\lambda$$

$$= \frac{2}{\pi} \left[ \left\{ [1-\lambda] \frac{\sin \lambda x}{x} - (-1) \left[ \frac{-\cos \lambda x}{x^2} \right] \right\}_0^1 \right]$$

$$= \frac{2}{\pi} \left[ \left\{ 0 - \frac{\cos x}{x^2} + \frac{1}{x^2} \right\} \right]$$

$$\therefore f(x) = \frac{2}{\pi} \left[ \left\{ \frac{1-\cos x}{x^2} \right\} \right] = \frac{2}{\pi x^2} 2 \sin^2 \left( \frac{x}{2} \right)$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x dx = \pi r^2$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{4}{\pi x^2} \sin^2\left(\frac{x}{2}\right) \cos \lambda x dx \quad \dots (ii)$$

$$\text{From (i) and (ii)} \quad \sqrt{\frac{2}{\pi}} \begin{cases} 1 - \lambda & \text{in } 0 \leq \lambda \leq 1 \\ 0 & \text{in } \lambda > 1 \end{cases} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{4}{\pi x^2} \sin^2\left(\frac{x}{2}\right) \cos \lambda x dx$$

$$\text{Now putting } \lambda = 0 \Rightarrow \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2\left(\frac{x}{2}\right)}{x^2} dx = 1$$

$$\int_0^{\infty} \frac{\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}} dx = \pi \Rightarrow \int_0^{\infty} \left[ \frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)} \right]^2 dx = \pi$$

$$(\text{put } t = \frac{x}{2} \text{ then } dt = \frac{dx}{2} \text{ and } x \rightarrow 0 \text{ to } \infty \Rightarrow t \rightarrow 0 \text{ to } \infty)$$

$$\int_0^{\infty} \left[ \frac{\sin(t)}{t} \right]^2 2dt = \pi \Rightarrow \int_0^{\infty} \left[ \frac{\sin(t)}{t} \right]^2 dt = \frac{\pi}{2}$$

24. Solve the integral equation  $\int_0^{\infty} f(x) \cos \lambda x dx = e^{-\lambda}$  Also show that

$$\int_0^{\infty} \frac{\cos \lambda x}{\lambda^2 + 1} d\lambda = \left[ \frac{\pi}{2} \right] e^{-x}$$

$$\text{Sol: Given } \int_0^{\infty} f(x) \cos \lambda x dx = e^{-\lambda}$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x dx = \sqrt{\frac{2}{\pi}} e^{-\lambda}$$

$$\therefore F_c[f(x)] = \sqrt{\frac{2}{\pi}} e^{-\lambda} \dots (i)$$

By the inverse Fourier cosine transform

$$f(x) = F_c^{-1}[F_c(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos \lambda x d\lambda$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} e^{-\lambda} \cos \lambda x d\lambda$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-\lambda} \cos \lambda x d\lambda$$

$$= \frac{2}{\pi} \left[ \left\{ \frac{e^{-\lambda}}{1+x^2} (-\cos \lambda x + x \sin \lambda x) \right\}_0^{\infty} \right]$$

$$= \frac{2}{\pi} \left[ \left\{ 0 + \frac{1}{1+x^2} \right\} \right]$$

$$\therefore f(x) = \frac{2}{\pi} \left[ \left\{ \frac{1}{1+x^2} \right\} \right]$$

25. Solve the integral equation  $\int_0^{\infty} f(x) \sin sx dx = \begin{cases} 1 & \text{for } 0 \leq s < 1 \\ 2 & \text{for } 1 \leq s < 2 \\ 0 & \text{for } s \geq 2 \end{cases}$

Sol: Given  $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1 & \text{for } 0 \leq s < 1 \\ 2 & \text{for } 1 \leq s < 2 \\ 0 & \text{for } s \geq 2 \end{cases}$

$$\therefore F_s[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1 & \text{for } 0 \leq s < 1 \\ 2 & \text{for } 1 \leq s < 2 \\ 0 & \text{for } s \geq 2 \end{cases} \dots\dots(i)$$

By the inverse Fourier sine transform  $f(x) = F_s^{-1}[F_s(f(x))] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx ds$

$$f(x) = \left[ \left\{ \int_0^1 \sin s x ds \right\} + \left\{ \int_1^2 2 \sin s x ds \right\} \right]$$

$$\begin{aligned}
&= \frac{2}{\pi} \left[ \left\{ \frac{-\cos x}{x} \right\}_0^1 + 2 \left\{ \frac{-\cos x}{x} \right\}_1^2 \right] \\
&= \frac{2}{\pi} \left[ \left\{ \frac{1 - \cos x}{x} - 2 \frac{\cos 2x - \cos x}{x} \right\} \right] \\
\therefore f(x) &= \frac{2}{\pi x} [1 + \cos x - 2 \cos 2x]
\end{aligned}$$

### Shifting property

$$\text{iii. } F[f(x - a)] = e^{ias} F[s] \qquad \text{ii. } F[e^{iax} f(x)] = F[s + a]$$

Proof: we know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x - a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - a) e^{ias} dx$$

Put  $t = x - a$ ,  $x \rightarrow -\infty$  implies  $t \rightarrow -\infty$

$dt = dx$ ,  $x \rightarrow \infty$  implies  $t \rightarrow \infty$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - a) e^{is(t+a)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - a) e^{ist} e^{isa} dt \\
&= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - a) e^{ist} dt \\
&= e^{isa} F[f(t)]
\end{aligned}$$

$$F[f(x - a)] = e^{isa} F[s] \quad \text{Hence proved (i)}$$

iv. We know that

$$\begin{aligned}
 F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\
 F[e^{iax} f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx \\
 F[e^{iax} f(x)] &= F(s+a)
 \end{aligned}$$

Hence proved (ii).