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Unit · 1 : Partial Differential Eq^n

Forming PDE :-

1) Eliminating by arbitrary constant
2) Eliminating by arbitrary functions

Type 1:

Form PDE by eliminating arbitrary constant

$$z = ax + by + a^2 + b^2$$
$$z = f(x, y)$$

$\frac{\partial z}{\partial x} = p$ $\frac{\partial z}{\partial y} = q$ $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial y^2} = t$

$\frac{\partial^2 z}{\partial x \partial y} = s$

1) $z = ax + by + a^2 + b^2$ ------ (1)

Soln $\frac{\partial z}{\partial x} = p = a$ $\frac{\partial z}{\partial y} = q = b$

sub in eq (1)

$z = px + qy + p^2 + a^2$

---

2) $z = (x-a)^2 + (y-b)^2 + 1$

Soln $z = (x-a)^2 + (y-b)^2 + 1$ ------ (1)

diff w.r.t. x

$p = \frac{\partial z}{\partial x} = 2(x-a)(1-a)$

$\frac{p}{2} = z - a$

$q = \frac{\partial z}{\partial y} = 2(y-b) \Rightarrow \frac{q}{2} = (y-b)$

sub in eq (1)

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1$$

$4z = p^2 + q^2 + 4$ is reqd. eqⁿ

3) $\log(az-1) = x + ay + b$ ......(1)

Sol diff. w.r.t x

$$\frac{1}{(az-1)} \cdot \left(a \cdot \frac{dz}{dx}\right) = 1$$

$$\frac{a}{az-1} p = 1$$

$$ap = az - 1$$
$$1 = ap - az$$
$$a = \frac{1}{z-p}$$

diff w.r.t. y

$$\frac{1}{(az-1)} \left(a \cdot \frac{dz}{dy}\right) = a$$

$$\frac{q}{az-1} = 1$$

$$q + 1 = az$$
$$\frac{q+1}{z} = a$$

$$\frac{q+1}{z} = \frac{1}{z-p}$$

$$zq - pq = p$$
$$(z-p)q = p$$

Try it:

1) $z = (x^2+a)(y^2+b)$

$z = x^2y^2 + x^2b + ay^2 + ab$

diff w.r.t. x

$$\frac{dz}{dx} = 2xy^2 + 2xb$$

$$\frac{p}{2x} = y^2 + b$$

diff. w.r.t. y

$$\frac{dz}{dy} = 2x^2y + 2ay$$

$$\frac{q}{2y} = x^2 + a$$

$$z = \frac{p}{2x} \times \frac{q}{2x}$$

$$pq = 4xyz$$

2) $z = ax + by + cxy.$

Sol) $z = ax + by + cxy$

$$\frac{dz}{dx} = a + cy \qquad \frac{dz}{dy} = b + cx.$$

$$p = a + cy \qquad q = b + cx.$$

$$\frac{\partial^2 z}{\partial x \partial y} = c = s$$

$$a = (p - sy) \qquad b = (q - sx)$$

$$z = (p - sy)x + (q - sx)y + xys$$

Type 2: Form PDE by eliminating arbitrary functions.

1) $z = xy + f(x^2 + y^2 + z^2)$

diff w.r.t. x.

$$\frac{dz}{dx} = y + f'(x^2+y^2+z^2)\left(2x + 2z\frac{dz}{\partial x}\right)$$

$$p = y + f'(x^2+y^2+z^2)(2x + 2zp)$$

diff w.r.t. y.

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2+z^2)\left(2y + 2z\frac{dz}{\partial y}\right)$$

$$q = x + f'(x^2+y^2+z^2)(2y + 2zq)$$

$$\frac{q-x}{2y+2zq} = f'(x^2+y^2+z^2)$$

$$\frac{p-y}{2x+2zp} = f'(x^2+y^2+z^2)$$

$$\frac{q-x}{2y+2zq} = \frac{p-y}{2x+2zp}$$

$$2x^2 - 2y^2 + 2py - 2xy + 2xzp - 2zpy = 0$$

---

2) $xyz = \phi(x^2+y^2-z^2)$

Sol. diff w.r.t. x

$$y\left[x\frac{\partial z}{\partial x} + z\right] = \phi(x^2+y^2-z^2)\left[2x - 2z\frac{dz}{dx}\right]$$

$$y[xp+z] = \phi(x^2+y^2-z^2)(2x-2zp)$$

diff w.r.t. y

$$x[yq+z] = \phi(x^2+y^2-z^2)(2y-2zq)$$

$$xp(y^2+z^2) + yq(z^2-x^2) + z(y^2-x^2) = 0$$

3) $z = x^2 + 2g\left[\frac{1}{y} + \log x\right]$ . . . . (1)

$\Rightarrow px + qy = 2x^2$

diff. w.r.t. x

$$\frac{\partial z}{\partial x} = 2x + 2g'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right)$$

$$p = 2x + 2g'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right)$$

$$2g'\left(\frac{1}{y} + \log x\right) = (p-2x)x \quad \text{- - - - (2)}$$

diff. w.r.t. y

$$\frac{\partial z}{\partial y} = 2g'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right)$$

$$-qy^2 = 2g'\left(\frac{1}{y} + \log x\right) \quad \text{- - - - (3)}$$

from eq^n (3) and (2)

$$(p-2x)x = -qy^2$$

$$px - 2x^2 = -qy^2$$

$$\boxed{px + qy^2 = 2x^2}$$

Formation of the PDE by eliminating arbitrary function from $\phi(u, v) = 0$
where
u, v are functions of x, y, z
then we can form

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

Ques 1) Form PDE by eliminating
$f(xy + z^2, x + y + z) = 0$

Sol

$u = xy + z^2$ $\quad$ $v = x + y + z$

$\frac{\partial u}{\partial x} = y + 2zp$ $\quad$ $\frac{\partial v}{\partial x} = 1 + p$

$\frac{\partial u}{\partial y} = x + 2zq$ $\quad$ $\frac{\partial v}{\partial y} = 1 + q$

$$\begin{vmatrix} y + 2zp & 1 + p \\ x + 2zq & 1 + q \end{vmatrix} = 0$$

$x - y = 2zp - 2zq + qy - xp$

Try it:

1) $g\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$

$u = \frac{y}{x}$ $\quad$ $v = x^2 + y^2 + z^2$

$\frac{\partial u}{\partial x} = \frac{-1}{x^2}(y)$ $\quad$ $\frac{\partial v}{\partial x} = 2x + 2zp$

$\frac{\partial u}{\partial y} = \frac{1}{x}$ $\quad$ $\frac{\partial v}{\partial y} = 2y + 2zq.$

$\phi(u, v) = 0$

$$\begin{vmatrix} -y/x^2 & 2x + 2zp \\ 1/x & 2y + 2zp \end{vmatrix} = 0$$

$x^2 + y^2 + 2xp + 2yq = 0$

Solution of PDE:

Method of to solve Ist order partial differential equation:-

1) Complete solution:- The soln in which no. of arbitrary constants is equal to no. of independent variable is called complete solution.

2) Particular integral:- It is value of arbitrary constant.

3) Singular integral:- In complete solution $\phi(x,y,z,a,b)=0$

get $\frac{\partial\phi}{\partial a}=0$ $\quad$ $\frac{\partial\phi}{\partial b}=0$

then eliminate a and b.

Type 1 : $F(p,q)=0$

In this form we having only p, q then allows that $z=ax+by+c$ be the solution.

Note: There is no singular integral.

1) Solve : $p^2+q^2=4$

Sol) $p^2+q^2=4$ ----- Type(1) ----(1)

assumed soln is $z=ax+by+c$ ----(2)

differentiate eq(2) w.r.t. x

$$\frac{\partial z}{\partial x}=a=p$$

differentiate eq(2) w.r.t. y

$$\frac{\partial z}{\partial y}=b=q.$$

substituting p, q in(1)

$$a^2+b^2=4$$
$$a^2=4-b^2$$
$$a=\sqrt{4-b^2}$$

c soln is

$$z=\pm\sqrt{4-b^2}\;x+by+c$$

$\frac{\partial z}{\partial b}\neq 0$ $\quad$ $\frac{\partial z}{\partial c}\neq 0$

There is no singular integral

To find general solution:

$$z=\pm\sqrt{4-b^2}+by+c \quad f(a)$$

diff. w.r.t. b and a, we will find general soln.

2) Solve $\sqrt{p}+\sqrt{q}=1$ ------ Type (1)

Sol) soln is:

$$z=ax+by+c \quad ----(2)$$

differentiate eq (2) w.r.t. x

$$\frac{\partial z}{\partial x}=a=p$$

differentiate (1) w.r.t. y

$$\frac{\partial z}{\partial y} = b = q$$

Substitute a, b in eq (1)

$$\sqrt{a} + \sqrt{b} = 1$$

$$a = (1-\sqrt{b})^2$$

C.I is $z = (1-\sqrt{b})^2 x + by + c$

diff. w.r.t. b and c

$$\frac{\partial z}{\partial b} \neq 0, \quad \frac{\partial z}{\partial c} \neq 0$$

There is no SI

To get general solution.

$$z = (1-\sqrt{b})^2 x + by + f(a)$$

differentiate w.r.t. a, b and eliminate them to get general soln.

Type 2: Clairauts form.

$$z = px + qy + f(p,q)$$

Assume that $z = ax + by + f(a,b)$

Ques 1) Solve :

$$(1-x)p + (2-y)q = 3-z \quad \text{---} (1)$$

Assume CI is

$$z = ax + by + f(a,b)$$

$$z = 3 - (1-x)a - (2-y)q$$

$$z = ax - a + by - 2b + 3$$

d. w.r. to a

$$\frac{\partial z}{\partial a} = x - 1 = 0 \qquad \frac{\partial z}{\partial b} = y - 2 = 0$$

$$\boxed{x=1} \qquad \boxed{y=2}$$

$$z = a + 2b + f(a,b)$$

$$z = \cancel{a} - \cancel{a} + \cancel{2b} - \cancel{2b} + 3$$

$$\boxed{z = 3}$$

This is the general soln

2) $z = px + qy + \sqrt{1+p^2+q^2}$

$$CI = ax + by + f(a,b)$$

$$z = px + qy + \sqrt{1+p^2+q^2}$$

$$z = ax + by + \sqrt{1+a^2+b^2}$$

d.w.r. to a

$$\frac{\partial z}{\partial a} = x + \frac{1}{\sqrt{1+a^2+b^2}} \cdot 2a = 0$$

$$\frac{\partial z}{\partial b} = y + \frac{2b}{\sqrt{1+a^2+b^2}} = 0$$

$$x = \frac{-a}{\sqrt{a^2+b^2+1}} \qquad y = \frac{-b}{\sqrt{a^2+b^2+1}}$$

$$x^2 = \frac{a^2}{a^2+b^2+1} \qquad y^2 = \frac{b^2}{a^2+b^2+1}$$

$$x^2+y^2 = \frac{a^2+b^2}{a^2+b^2+1}$$

$$1-x^2+y^2 = \frac{\cancel{a^2}+\cancel{b^2}+1-\cancel{a^2}-\cancel{b^2}}{a^2+b^2+1}$$

$$a^2+b^2+1 = \frac{1}{1-x^2+y^2}$$

$$\sqrt{a^2+b^2+1} = \frac{1}{\sqrt{1-x^2+y^2}}$$

$$a = -x\sqrt{a^2+b^2+1} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$b = -y\sqrt{a^2+b^2+1} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{-x}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$z = \frac{-x^2-y^2+1}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$\boxed{z^2+x^2+y^2 = 1}$$

Type 3: $f(p,q,z)=0$

i) $F(p,q,x)=0$ ii) $F(p,q,y)=0$

soln is $z = f(x+ay)$

$u = x+ay \Rightarrow z = f(u)$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial z}$$

$$\boxed{p = \frac{\partial u}{\partial x} = 1}$$

diff w.r.t. to y

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

$$q = a\frac{\partial z}{\partial u}$$

$$f\left(\frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}, z\right) = 0$$

Note: SI is not guaranteed.

17 Solve :

$$pz = 1 + q^2$$

$$pz = 1 + q^2 \quad \text{-----} (1)$$

$$u + x + ay \Rightarrow z = f(u)$$

$$p = \frac{\partial z}{\partial u} \qquad q = a\frac{\partial z}{\partial u}$$

$$z\frac{\partial z}{\partial u} = 1 + a^2\left(\frac{\partial z}{\partial u}\right)^2$$

$$a^2\left(\frac{\partial z}{\partial u}\right)^2 - z\frac{\partial z}{\partial u} + 1 = 0$$

$$\frac{\partial z}{\partial u} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}$$

(Take +)

$$= \frac{z \pm \sqrt{z^2 - 4a}}{2a^2} = \frac{z + \sqrt{z^2 - 4a}}{2a^2}$$

$$\frac{\partial z}{\partial u} = \frac{z + \sqrt{z^2 - 4a}}{2a}$$

$$\frac{\partial z}{z + \sqrt{z^2 - 4a}} = \frac{\partial u}{2a}$$

Integrate from both sides :-

$$\int \frac{dz}{z + \sqrt{z^2 - 4a}} = \int \frac{\partial u}{2a}$$

$$\int \frac{z - \sqrt{z^2 - 4a^2}}{z^2 - z^2 + 4a^2} dz = \frac{1}{2a^2} u + c$$

$$\frac{1}{4a^2}\left[\int z\,dz - \int \sqrt{z^2 - 4a^2}\,dz\right] = \frac{1}{2a^2} u + c$$

$$\frac{1}{2}\left[\frac{z^2}{2} - \left(\frac{z}{2}\sqrt{z^2 - 4a^2} - \frac{4a^2}{2}\cos b^{-1}\left(\frac{z}{4a}\right)\right)\right] = a + c$$

$$\frac{z}{2} - \frac{z}{2}\sqrt{z^2 - 4a^2} + 2a^2 \cosh^{-1}\left(\frac{z}{2a}\right) = 2(x + ay) + z$$

is complete soln.

$$\frac{\partial \phi}{\partial a} = 0 \quad , \quad \frac{\partial \phi}{\partial c} = 1 \neq 0 \quad \therefore \text{no ST}$$

Case I

Type 3: (i): $f(x, p, q) = 0$

$dz = p\,dx + q\,dy$

$q = a$

1) Solve: $q = px + p^2$

$q = px + p^2 \quad \text{---- (1)}$

$q = a \qquad \partial z = p\,\partial x + q\,\partial y$

Sub a in (1)

$a = px + p^2$

$p^2 + px - a = 0$

$p = \frac{-x \pm \sqrt{x^2 - 4(1)(-a)}}{2(1)}$

$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$

Sub in (2)

$\partial z = \frac{-x \pm \sqrt{x^2+4a}}{2}\,dx + a\,dy$

By integrating on both sides

$\int dz = \frac{1}{2}\int -x\,dx + \int \sqrt{x^2+4a}\,dx + a\int dy$

$= \frac{1}{2}\left[\frac{-x^2}{2} + \frac{4a}{2}\log\left(x \pm \sqrt{x^2+4a}\right)\right] + ay + c$

$x = \frac{-x^2}{4} + a\log\left(x \pm \sqrt{x^2+4a} + ay + c\right)$

is CI

$\frac{\partial z}{\partial c} = 1 \neq 0 \quad \therefore$ no SI

$c = f(a)$ Diff. w.r.t a and c and eliminate a and c to get General solution.

Case II

Type 3:

$F(y, p, q) = 0$

$dz = p\,dx + q\,dy \qquad p = a.$

Solve: $pq = y.$

$aq = y$

$q = \frac{y}{a}$

$dz = a\,dx + \frac{y}{a}\,dy$

$\int dz = a\int dx + \frac{1}{a}\int y\,dy$

$z = ax + \frac{y^2}{2a} + c$

CI: $\frac{\partial z}{\partial c} \neq 0 = 1 \quad \therefore$ no SI

$c = f(a)$ diff. w.r.t. a and c

Thus eliminate a and c to get general solution.

Type 4: $F(x,p) = F(y,q) = K$

$$\partial z = p\,dx + q\,dy$$

Ques-1) Solve:

$$q^2 - p = y - x$$

$$a = q^2 - y = p - x \quad \text{-----} \ (1)$$

$p - x = a \qquad p = a + x$

$q^2 - y = a \qquad q = \sqrt{a+y}$

$$\int dz = \int (a+x)\,dx + \int \sqrt{a+y}\,dy$$

$$z = \frac{(x+a)^2}{2(1)} + \frac{(a+4)^{1/2+1}}{1/2+1} + c$$

$$z = \frac{(x+a)^2}{2} + \frac{2}{3}(a+y)^{3/2} + c$$

$\frac{\partial z}{\partial c} \neq 0 \quad \therefore$ no SI

$c = f(a)$ D w.r.t. a and c eliminate a and c thus general solution.

Try it :-

2) $yp = 2yx + \log q$

3) $p^2 + q^2 = x + y$

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Lagranges Linear Eqⁿ

$Pp + Qq = R$ is known as Lagranges linear eqⁿ and P, Q, R are the functions of x, y, z.

I: Method of grouping

II: Method of multipliers

I: Method of grouping

(i) A·E $\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$

(ii) $\phi(u, v) = 0$

II: Method of multipliers

(i) AE $\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$ $= 0$

(ii) $\phi(u, v) = 0$

Q.) Solve: $xp + yq = zx$

$P = x$ $\quad Q = y$ $\quad R = xz$

A·E. is $\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$

$\frac{\partial x}{x} = \frac{\partial y}{y} = \frac{\partial z}{xz}$

$$\frac{\partial x}{x} = \frac{\partial z}{xz}$$

$$\int \partial x = \int \frac{dz}{z}$$

$$x = \log z + C_1$$

~~$x = \log$~~

$$e^x = zC_1$$

$$\frac{e^x}{z} = C_1$$

$$\int \frac{\partial x}{x} = \int \frac{\partial y}{y}$$

$$\log x - \log y = \log C_2$$

$$\log\left(\frac{x}{y}\right) = \log C_2$$

$$C_2 = \frac{x}{y}$$

$$\phi(u,v) = \phi\left(\frac{e^x}{z}, \frac{x}{y}\right) = 0$$

Q.7 Solve $\frac{y^2 z}{x} P + xzq = y^2$

Sol.7 $P = \frac{y^2 z}{x}$ $Q = xz$ $R = y^2$

A.E is $\frac{\partial x}{y^2 z/x} = \frac{\partial y}{x^2} = \frac{\partial z}{y^2}$

(i) $\frac{\partial x}{y^2 \cancel{z}/x} = \frac{\partial y}{x\cancel{z}}$

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + C_1$$

$$C_1 = \frac{x^3}{3} - \frac{y^3}{3}$$

$$C_1 = x^3 - y^3$$

(ii) $\frac{\partial x}{y^2 z/x} = \frac{\partial z}{y^2}$

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$$C_2 = \frac{x^2}{2} - \frac{z^2}{2}$$

$$C_2 = x^2 - z^2$$

$$\phi(u,v) = \phi(x^3 - y^3, x^2 - z^2) = 0$$

Q.7 Solve:
$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

Sol

$$P = x(z^2 - y^2)$$
$$Q = y(x^2 - z^2)$$
$$R = z(y^2 - x^2)$$

A.E is $\frac{\partial x}{P} = \frac{dy}{Q} = \frac{\partial z}{R}$

$$\frac{\partial x}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{\partial z}{z(y^2 - x^2)}$$

(i) By Lagranges multiplier x, y, z

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \frac{x\,dx + y\,dy + z\,dz}{x(xz^2 - xy^2) + y(x^2 y - yz^2) + z(y^2 z - x^2 z)}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = 0 \Rightarrow \boxed{x^2 + y^2 + z^2 = 0}$$

(ii) By Lagranges multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\boxed{xyz = c_2}$$

$$\therefore \phi(u, v) = (x^2+y^2+z^2,\ xyz) = 0$$

Try it:

1) Solve: $y^2p - xyq = x(z-2y)$

2) Solve: $(y+z)p + (z+x)q = x+y$

1) $y^2p - xyq = x(z-2y)$

$P = y^2 \qquad Q = -xy \qquad R = x(z-2y)$

$$A.E = \frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$$

$$= \frac{\partial x}{y^2} = \frac{\partial y}{-xy} = \frac{\partial z}{x(z-2y)}$$

$$\frac{\partial x}{y^2} = \frac{\partial y}{-xy}$$

$$-\int x\,dx = \int y\,dy$$

$$\frac{-x^2}{2} = \frac{y^2}{2} + c_1$$

$$c_1 = y^2 - x^2$$

$$\frac{\partial y}{-xy} = \frac{\partial z}{x(z-2y)}$$

$$(z-2y)\,dy = -y\,dz$$

2) $(y+z)p + (z+x)q = x+y$

$P = y+z \qquad Q = z+x \qquad R = x+y$

$$A.E \Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{\partial z}{R}$$

$$\frac{\partial x}{y+z} = \frac{\partial y}{z+x} = \frac{\partial z}{x+y}$$

$$\frac{\partial x}{y+z} = \frac{\partial z}{x+y}$$

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Homogeneous Linear Partial Differential Eqn.

$$a_n \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial^2 y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

$\Rightarrow \boxed{D = \frac{\partial}{\partial x} \quad D' = \frac{\partial}{\partial y}}$

if $F(x,y) = 0 \Rightarrow z = CF$

if $F(x,y) \neq 0 \Rightarrow z = CF + PI$

To find CF:

AE $\Rightarrow$ roots $\Rightarrow m_1, m_2$ ------

Case 1: Real And Imaginary, unequal ($m_1 \neq m_2 \neq m_3 \neq m_4$)

CF is $z = f_1(y + m_1 x) + f_2(y + m_2 x)$

Case 2: Roots are equal ($m_1 = m_2$)

CF is $z = f_1(y + m_1 x) + x f_2(y + m_1 x)$ --- so on.

Ques 1 solve $(D^2 - 6DD' + 9D'^2)z = 0$

sol AE $D = m \quad D' = 1$

$m^2 - 6m + 9 = 0$

$m = 3, 3$

CF is $z = f_1(y+3x) + x f_2(y+3x)$

z is CF

$z = f_1(y+3x) + x f_2(y+3x)$

2) solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

sol $D = m \quad D' = 1$

$m^3 - 4m^2 + 4 = 0$

$m = 0, 2, 2$

CF is $z = f_1(y) + f_2(y+2x) + xf_3(y+2x)$

$z = CF$

$z = f_1(y) + f_2(y+2x) + xf_3(y+2x)$

3) solve $(D^4 - D'^4)z = 0$

sol A.E is $D = m \quad D' = 1$

$m^4 - 1 = 0$

$(m^2-1)(m^2+1) = 0$

$m = \pm 1 \qquad m = \pm i$

CF is $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$

If $F(x,y) \neq 0$

Case I: $F(x,y) = e^{ax+by}$

$$PI = \frac{1}{f(D,D')} \cdot e^{ax+by}$$

$D = a, \quad D' = b$

Ques) solve $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

sol A.E is $m^2 - 5m + 6 = 0$

$m = 3, 2$

CF is $z = f_1(y+2x) + f_2(y+3x)$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

$$= \frac{1}{D^2 - 5DD' + 6D'} \cdot e^{x+y}$$

Substitute $D = a = 1 \quad D' = b = 1$

$$= \frac{1}{1-5+6} e^{x+y}$$

$$P.I = \frac{1}{2} e^{x+y}$$

$z = CF + PI$

$$z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2} e^{x+y}$$

Ques) solve $(D^2 - 2DD' + D'^2)z = e^{x+2y}$

sol A.E is $m^2 - 2m + 1 = 0$

$m = 1, 1$

CF is $z = f_1(y+x) + xf_2(y+x)$

$$P.I = \frac{1}{f(D, D')} \cdot e^{ax+by}$$

$$= \frac{1}{D^2 - 2DD' + D'^2} \cdot e^{x+2y}$$

Put $D = a = 1$ $D' = b = 2$

$$= \frac{1}{1-4+4} \cdot e^{x+2y}$$

$$P.I = e^{x+2y}$$

$$z = CF + PI$$

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y}$$

Ques) Solve $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

Sol) A.E is $m^2 - 4m + 4 = 0$

$m = 2, 2$

CF is $z = f_1(y+2x) + x f_2(y+2x)$

$$P.I = \frac{1}{f(D, D')} \cdot e^{ax+by}$$

$$= \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+y}$$

Put $D = a = 2$ $D' = b = 1$

$$= \frac{1}{4-8+4} \cdot e^{2x+y} \quad (=0)$$

$$= \frac{x}{2D - 4D'} e^{2x+y}$$

$$= x \cdot \frac{x}{2} e^{2x+y}$$

$$P.I = \frac{x^2}{2} \cdot e^{2x+y}$$

$$z = CF + PI$$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} \cdot e^{2x+y}$$

Case II: $f(x, y) = x^m y^n$

$\frac{1}{D}$ = Integrate w.r.t. x.

$\frac{1}{D'}$ = Integrate w.r.t. y

$$PI = \frac{1}{f(D, D')} x^m y^n$$

if $m > n \Rightarrow f\left(\frac{D'}{D}\right)$

if $m < n \Rightarrow f\left(\frac{D}{D'}\right)$

Ques Solve $(D^2 - 2DD')z = e^{2x} + x^3y$

Sol.

A.E. is $m^2 - 2m = 0$

$m = 0, 2.$

CF is $z = f_1(y) + f_2(y+2x)$

$P.I_1 = \dfrac{1}{D^2 - 2DD'} \cdot e^{2x}$

Put $a = 2$ $\quad b = 0$

$= \dfrac{1}{4} \cdot e^{2x}$

$P.I_2 = \dfrac{1}{D^2 - 2DD'} x^3y$

$= \dfrac{1}{D^2\left(1 - \frac{2D'}{D}\right)} \cdot x^3y$

$= \dfrac{1}{D^2}\left[1 - \dfrac{2D'}{D}\right]^{-1} \cdot x^3y$

$= \dfrac{1}{D^2}\left[1 + \dfrac{2D'}{D} + \dfrac{4D'^2}{D^2} + \dfrac{8D'^3}{D^3} \cdots\right] x^3y$

$= \dfrac{1}{D^2}\left[1 + \dfrac{2D'}{D}\right] x^3y$

$= \dfrac{1}{D^2}(x^3y) + \dfrac{2}{D^3}(D'(x^3y))$

$= \dfrac{1}{D^2}(x^3y) + \dfrac{2}{D^3}(x^3)$

$\boxed{\begin{aligned} \frac{1}{D}(x^3y) &= \frac{x^4y}{4} \\ \frac{1}{D^2}(x^3y) &= \frac{x^5y}{20} \\ \frac{1}{D^3}(x^3y) &= \frac{x^6y}{120} \end{aligned}}$

$= \dfrac{x^5y}{20} + \dfrac{2}{D^3}(x^3)$

$= \dfrac{x^5y}{20} + 2\left(\dfrac{x^6}{120}\right)$

$= \dfrac{x^5y}{20} + \dfrac{x^6}{60}$

$= \dfrac{x^5y}{20} + \dfrac{x^6}{60}$

$Z = CF + PI_1 + PI_2$

$z = f_1(y) + f_2(y+2x) + \dfrac{1}{4} \cdot e^{2x} + \dfrac{x^5y}{20} + \dfrac{x^6}{60}$

Ques Solve $(D^2 - DD' - 2D'^2)z = (2x+3y) + e^{3x+4y}$

sol A.E is $m^2 - m - 2 = 0$

$m = -1, 2$

CF is $z = f_1(y-x) + f_2(y+2x)$

$PI_2 = \dfrac{1}{D^2 - DD' - 2D'^2} \cdot e^{3x+4y}$

Put $a = 3,\ b = 4.$

$16 \times 2 = 32$

$= \dfrac{1}{9 - 12 - 32} e^{3x+4y} = \dfrac{1}{9 - 44} \cdot e^{3x+4y}$

$= -\dfrac{1}{35} \cdot e^{3x+4y}$

$$P.I_2 = \frac{1}{D^2 - DD' - 2D'^2}(2x+3y)$$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D} + \frac{2D'^2}{D^2}\right]}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)^2 + \dots\right](2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D}\right](2x+3y)$$

$$= \frac{1}{D^2}(2x+3y) + \frac{1}{D^3}\left(D'(2x+3y)\right)$$

$$= \frac{1}{D^2}(2x+3y) + \frac{1}{D^3}(3)$$

$$\frac{1}{D}(2x+3y) = \frac{2x^2}{2} + 3xy$$

$$\frac{1}{D^2} = \left(\frac{x^3}{3} + \frac{3x^2y}{2}\right)$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{3}{D^3}$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{3x^3}{6}$$

$$= \frac{2x^3 + 3x^2}{6} + \frac{3x^2y}{2}$$

$$PI_1 = \frac{5x^3}{6} + \frac{3x^2y}{2}$$

$$z = CF + PI_1 + PI_2$$

$$\boxed{z = f_1(y-x) + f_2(y+2x) + \frac{5x^3}{6} + \frac{3x^2y}{2} - \frac{1}{35}e^{3x+4y}}$$

Try it:

Tutorial 3 :- Part A : 1, 2, 5

26/8/22

Type 3: $F(x,y) = \cos(ax+by)$
or
$\sin(ax+by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

$D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$

Ques 1) Solve $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

Sol: Cf is $z = f_1(y+x) + x f_2(y+x)$

$$PI = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$$

$D^2 = -a^2 = 1$, $DD' = -ab = 3$, $D'^2 = -b^2 = -9$

$$PI = \frac{1}{-1-2(3)-9} \cos(x-3y)$$

$$= \frac{1}{-16} \cos(x-3y)$$

Ques 2) Solve $(D^3 - 2D^2D')z = e^{x+2y} + 4\sin(x+y)$

Sol:

C.F. is $z = f_1(y) + x f_2(y) + f_3(y+2x)$

$$P.I_1 = \frac{-1}{3} e^{x+2y}$$

$$P.I_2 = \frac{1}{D^3 - 2D^2D'} \cdot 4\sin(x+y)$$

$$= \frac{1}{D(D^2 - 2DD')} \cdot 4\sin(x+y)$$

$D^2 = -a^2 = -1$, $DD' = -ab = -1$, $D'^2 = -b^2 = -1$

$$= \frac{4}{D(-1+2)} \cdot \sin(x+y)$$

$$= 4 \cdot \frac{1}{D} \sin(x+y)$$

$= 4 \cdot \frac{1}{D} (\sin(x+y))$

$= 4 \int \frac{1}{D} (\sin(x+y))$

$PI_2 = -4\cos(x+y)$

$z = Cf + PI_1 + PI_2$

Type 4: Shifting Exponential

$f(x,y) = e^{ax+by}\,\phi(x,y)$

→ poly

→ Trig

$PI = e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x,y)$

$D = D+a$

$D' = D'+b$

Ques- Solve: $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cdot \cos 2y$

sol.

Cf is $z = f_1(y-x) + x f_2(y-x) + f_3(y+x)$

$P.I = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} \cdot (e^x \cdot \cos 2y)$

$= e^x \frac{1}{(D+1)^3 + (D+1)^2D' - (D+1)D'^2 - D'^3} \cos 2y \quad \left[\begin{matrix} D = D+a, & a=1 \\ D' = D+b, & b=0 \end{matrix}\right]$

$= e^x \frac{1}{D^3 + 1 + 3D(D+1) + D^2\cancel{D'} + D' + 2DD' - DD' - D'^2 - D'^3} \cos 2y$

$= e^x \frac{1}{\cancel{D^3} + 1 + 3D^2 + 3D + D^2 + D' + 2DD' - DD' - D'^2 - \cancel{D'^3}}$

$$= e^x \, R.P. \left[\frac{e^{i2y}}{(D+1)^3 + (D+1)^2 D' - (D+1) D'^2 - D'^3} \right]$$

$D = a = 0, \; D' = b = 2i$

$$= e^x \, R.P. \left[\frac{e^{i2y}}{(1)^3 + (1)^2 2i - (1) 4i^2 - 8i^3} \right]$$

$$= e^x \, R.P. \left[\frac{e^{i2y}}{1 + 2i + 4 + 8i} \right]$$

$$= e^x \, R.P. \left[\frac{e^{i2y}}{5 + 10i} \right]$$

$$= \frac{e^x}{5} \, R.P. \; \frac{e^{i2y}}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{e^x}{5} \, R.P. \; \frac{(1-2i) e^{i2y}}{1 - 4i^2}$$

$$= \frac{e^x}{5} \, R.P. \; \frac{(1-2i) e^{i2y}}{5}$$

$$= \frac{e^x}{25} \, R.P. (1-2i)(\cos 2y + i \sin 2y)$$

$$= \frac{e^x}{25} \, R.P. (\cos 2y + i \sin 2y - 4i \cos 2y + 2 \sin 2y)$$

$$= \frac{e^x}{25} (\cos 2y + 2 \sin 2y)$$

Ques-7 Solve: $(D^2 - 3DD' + 2D'^2) z = (4x+2) e^{x+2y}$

sol

CF is $z = f_1(y+x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 - 3DD' + 2D'^2} (4x+2) e^{x+2y}$$

$$= e^{x+2y} \frac{(4x+2)}{(D+1)^2 - 3(D+1)(D'+2) + 2(D'+2)^2} \qquad \begin{pmatrix} D = D+a = D+1 \\ D' = D'+b = D'+2 \end{pmatrix}$$

$$= e^{x+2y} \frac{(4x+2)}{D^2+1+2D - 3(DD'+2D+D'+2) + 2D'^2+8+8D'}$$

$$= e^{x+2y} \frac{(4x+2)}{D^2+1+2D - 3DD' - 6D - 3D' - 6 + 2D'^2 + 8 + 8D'}$$

$$= e^{x+2y} \frac{4x+2}{D^2+3-4D-3DD'+5D'}$$

complete

$$= \frac{e^{x+2y}}{3} \frac{1}{(1-D+D')\left(1-\frac{D}{3}+\frac{2D'}{3}\right)} (4x+2)$$

$$= \frac{e^{x+2y}}{3} \frac{1}{[1-(D-D')]\left[1-\left(\frac{D+2D'}{3}\right)\right]} (4x+2)$$

$$= \frac{e^{x+2y}}{3} [1-(D-D')]^{-1} \left[1-\left(\frac{D}{3}-\frac{2D'}{3}\right)\right]^{-1} (4x+2)$$

$$= \frac{e^{x+2y}}{3} [1+(D-D') + (D-D')^2 + \dots] \left[1 + \left(\frac{D}{3} - \frac{2D'}{3}\right) + \left(\frac{D}{3} - \frac{2D'}{3}\right)^2 - \dots\right] (4x+2)$$

$$= \frac{e^{x+2y}}{3} \Big[1+D-D' + \underbrace{D^2 + D'^2 - 2DD'}_{\text{neglect}}\Big] \left[1+\frac{D}{3} - \frac{2D'}{3}\right] (4x+2)$$

$$= \frac{e^{x+2y}}{3} [1+D-D'] \left[1+\frac{D}{3} - \frac{2D}{3}\right] (4x+2)$$

$$= \frac{e^{x+2y}}{3} \left[(4x+2) + \frac{D}{3}(4x+2) + D(4x+2) + \frac{D^2}{3}(4x+2)\right]$$

$$= \frac{e^{x+2y}}{3}$$

my answer

$$= \frac{e^{x+2y}}{3} \left(4x + \frac{22}{3}\right)$$