Unit IV:Fourier Transforms

- Introduction of Fourier Transforms
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- Fourier cosine, sine Transforms problems
- Properties of Fourier cosine, sine Transforms
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Integral Transform:

If $f(x)$ is defined in (a,b), the integral transform of $f(x)$ with the Kernal $K(s,x)$ is defined by

$$F(s) = \overline{f}(s) = \int_a^b f(x)K(s,x)\,dx$$

if the integral exists.

Note: If a, b are finite, the transform is finite and if a, b are infinite, it is an infinite transform.

Fourier Integral Theorem

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i(x-t)s}\,dt\,ds$$

Complex Fourier Transform:

Let $f(x)$ be a function defined in $(-\infty,\infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty,\infty)$. Then the complex (or infinite) Fourier transform of $f(x)$ is given by

$$\overline{f}(s) = F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \tag{1}$$

Inversion theorem for Complex Fourier Transform:

If $f(x)$ satisfies the Dirichlet's conditions in every finite interval $(-l,l)$ and if it is absolutely integrable in the range and if $F(s)$ denotes the complex Fourier transform of $f(x)$ then at every point of continuity of $f(x)$, we have

$$f(x) = F^{-1}\{F(s)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds \tag{2}$$

Both equations (1) and (2) are called as Fourier Transforms pairs.

Properties of Fourier Transforms

Property 1: Linearity Property

Fourier transform is linear. i.e. $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$ where F stands for Fourier transform.

Proof:

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx$

$$\begin{aligned}
F[af(x)+bg(x)] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (af(x)+bg(x))\, e^{isx}\, dx \\
&= a.\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx + b.\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx \\
&= aF[f(x)] + bF[g(x)]
\end{aligned}$$

Property 2: Shifting property (in x)

If $F\{f(x)\} = F(s)$ then $F\{f(x-a)\} = e^{ias}F(s)$.

Proof:

$$\text{By definition} \quad F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$\Rightarrow F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x-a)e^{isx}\, dx$$

$$\text{Putting } x-a=t \Rightarrow dx = dt$$

$$\begin{aligned}
F\{f(t)\} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{is(t+a)}\, dt \\
&= e^{ias}F(s)
\end{aligned}$$

Property 3:

If $F\{f(x)\} = F(s)$ then $F\{e^{iax}f(x)\} = F(s+a)$.

Proof:

$$\text{By definition} \quad F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$\begin{aligned}
F\left\{e^{iax}f(x)\right\} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{iax}f(x)e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{i(s+a)x}\, dx \\
&= F(s+a)
\end{aligned}$$

Property 4: Change of scale property

If $F\{f(x)\} = F(s)$ then $F\{f(ax)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right)$ where $a \neq 0$.

Proof: $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(ax)e^{isx}\,dx$

Case (i): $a > 0$

Putting $ax = t \Rightarrow a\,dx = dt$

when $x = -\infty \Rightarrow t = -\infty$ and when $x = \infty \Rightarrow t = \infty$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{is\left(\frac{t}{a}\right)}\,\frac{dt}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{is\left(\frac{t}{a}\right)}\,dt$$

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \tag{3}$$

Case (ii): $a < 0$

Putting $ax = t \Rightarrow a\,dx = dt$

when $x = -\infty \Rightarrow t = \infty$ and when $x = \infty \Rightarrow t = -\infty$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\infty}^{-\infty} f(t)e^{is\left(\frac{t}{a}\right)}\,\frac{dt}{a}$$

$$= -\frac{1}{a\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{is\left(\frac{t}{a}\right)}\,dt$$

$$= -\frac{1}{a}F\left(\frac{s}{a}\right) \tag{4}$$

From (3) and (4), we get $F\{f(ax)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right)$.

Property 5: Modulation Theorem

If $F\{f(x)\} = F(s)$ then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$.

Proof:

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)\cos ax e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)\left[\frac{e^{iax} + e^{-iax}}{2}\right]e^{isx}\,dx$$

$$F\{f(x)\cos ax\} = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{i(s+a)}\,dx + \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{i(s-a)}\,dx\right]$$
$$= \frac{1}{2}\left[F(s-a)+F(s+a)\right].$$

Property 6: Derivative of transform

If $F\{f(x)\} = F(s)$ then $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n}F(s)$.

Proof: By definition $F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{isx}\,dx$

Differentiating with respect to s both sides n times, we get

$$\frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)(ix)^n e^{isx}\,dx$$
$$= i^n\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} x^n f(x)e^{isx}\,dx\right)$$
$$= i^n F\{x^n f(x)\}$$
$$F\{x^n f(x)\} = \frac{1}{i^n}\frac{d^n}{ds^n}F(s)$$
$$= \left(\frac{1}{i}\right)^n \frac{d^n}{ds^n}F(s)$$
$$= \left(\frac{i}{i\times i}\right)^n \frac{d^n}{ds^n}F(s)$$
$$= (-i)^n \frac{d^n}{ds^n}F(s)$$

Property 7: Fourier transform of Derivative

$F\left\{f^{'}(x)\right\} = -isF(s)$ if $f(x)\to 0$ as $x\to\pm\infty$

Proof:

$$F\left\{f^{'}(x)\right\} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{isx}f^{'}(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{isx}\,d\{f(x)\}$$
$$= \frac{1}{\sqrt{2\pi}}\left[\left\{e^{isx}f(x)\right\}_{-\infty}^{\infty} - is\int\limits_{-\infty}^{\infty} f(x)e^{isx}\,dx\right]$$
$$= -isF(s) \text{ if } f(x)\to 0 \text{ as } x\to\pm\infty$$

Property 8: Fourier transform of an integral function

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$$

Proof:

Let $\phi(x) = \int_a^x f(x)\,dx$ then $\phi^{'}(x) = f(x)$

$$\begin{aligned} F\left\{\phi^{'}(x)\right\} &= (-is)\overline{\phi}(s) \text{ by Property 7} \\ &= (-is)F\left(\phi(x)\right) \\ &= (-is)F\left\{\int_a^x f(x)\,dx\right\} \end{aligned}$$

$$\begin{aligned} \Rightarrow F\left\{\int_a^x f(x)\,dx\right\} &= \frac{1}{(-is)}F\left\{\phi^{'}(x)\right\} \\ &= \frac{1}{(-is)}F\left(f(x)\right) = \frac{F(s)}{(-is)} \end{aligned}$$

Property 9: $F\left\{\overline{f(-x)}\right\} = \overline{F(s)}$, where $\overline{F(s)}$ is the complex conjugate of $F(s)$.

Proof:

By definition $F(s) = F\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

Taking complex conjugate, we get $\overline{F(s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}e^{-isx}\,dx$

Put $x = -y \Rightarrow dx = -\,dy$; When $x \to -\infty \Rightarrow y \to \infty$ and $x \to \infty \Rightarrow y \to -\infty$

$$\begin{aligned} \overline{F(s)} &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} \overline{f(-y)}e^{isy}(-\,dy) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(-y)}e^{isy}\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(-x)}e^{isx}\,dx, \text{ by changing the dummy variable} \\ &= F\left\{\overline{f(-x)}\right\} \end{aligned}$$

Convolution of two function: The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

Theorem: Convolution Theorem or Faltung Theorem

The Fourier transforms of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

That is $F\{f(x) * g(x)\} = F(s).G(s) = F\{f(x)\}.F\{g(x)\}.$

Proof:

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$$\begin{aligned}
\Rightarrow F\{f * g\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(x) * g(x))\, e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt \right) e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t) e^{isx}\, dx \right) dt \\
&\text{by changing the order of integration} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F\{g(x-t)\}\, dt
\end{aligned}$$

$$\begin{aligned}
F\{f(x) * g(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} G(s)\, dt \\
&\text{by shifting theorem} \\
&= G(s).\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its}\, dt \\
&= G(s).F(s) \\
&= F(s).G(s) = F\{f(x)\}.F\{g(x)\}
\end{aligned}$$

Note:

By inversion, $f * g = F^{-1}\{F(s)G(s)\} = F^{-1}\{F(s)\} * F^{-1}\{G(s)\}.$

Parseval's Identity

If $F(s)$ is the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Proof:

By convolution theorem, $F\{f(x) * g(x)\} = F(s)G(s)$

$$\Rightarrow f * g = F^{-1}\{F(s)G(s)\}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)G(s)e^{-isx}\,ds$$

Put $x = 0$, we get

$$\int_{-\infty}^{\infty} f(t)g(-t)\,dt = \int_{-\infty}^{\infty} F(s)G(s)\,ds \tag{5}$$

Take $g(-t) = \overline{f(t)} \Rightarrow g(t) = \overline{f(-t)}$

Therefore $G(s) = F\{g(t)\} = F\left\{\overline{f(-t)}\right\} = \overline{F(s)}$ by property 9

Hence equation (5) becomes

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Example 1: Find the complex Fourier transform of $f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$.

Solution:

$$\begin{aligned} F\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos sx + i \sin sx)\,dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} x \cos sx\,dx + i \int_{-a}^{a} x \sin sx\,dx \right\} \end{aligned}$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left\{0 + 2i\int_0^a x\sin sx\,dx\right\}$$

Since the first integral is an odd function and
the second integral is an even function.

$$= \frac{2i}{\sqrt{2\pi}}\left[x\left(-\frac{\cos sx}{s}\right) - 1\left(-\frac{\sin sx}{s^2}\right)\right]_0^a$$

$$= \frac{2i}{\sqrt{2\pi}}\left[\frac{-a\cos sa}{s} + \frac{\sin sa}{s^2}\right]$$

$$= \frac{2i}{\sqrt{2\pi}}\left[\frac{\sin sa - as\cos sa}{s^2}\right]$$

Example 2:

Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$. Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\,dx$.

Sol: By definition $F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1}(1-x^2)\cos sx dx + i\int_{-1}^{1}(1-x^2)\sin sx\,dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2\int_0^1(1-x^2)\cos sx\,dx + 0\right\}$$

Since the first integral is an even function and
the second integral is an odd function.

$$= \frac{2}{\sqrt{2\pi}}\left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right)\right.$$

$$\left. + (-2)\left(-\frac{\sin sx}{s^3}\right)\right]_0^1 = \frac{2}{\sqrt{2\pi}}\left[\frac{-2\cos s}{s^2} + \frac{2\sin s}{s^3}\right]$$

$$F(s) = \frac{-4}{\sqrt{2\pi}} \left[\frac{s\cos s - \sin s}{s^3}\right]$$

To find $\int\limits_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\, dx$

Using inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(s) e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{-4}{\sqrt{2\pi}} \left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{-2}{\pi}\left\{ \int\limits_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\, ds \right.$$

$$\left. - i\int\limits_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3}\right)\sin sx\, ds \right\}$$

$$f(x) = \frac{-2}{\pi}\left\{ 2\int\limits_0^{\infty} \left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\, ds - 0 \right\}$$

Since the first integral is an even function and
the second integral is an odd function.

Put $x = \frac{1}{2}$ in the above integral. But $x = \frac{1}{2}$ is a point of continuity of $f(x)$.
Therefore value of the integral when $x = \frac{1}{2}$ is

$$f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Therefore $\frac{3}{4} = -\frac{4}{\pi}\int\limits_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos\left(\frac{s}{2}\right)\, ds$

$\Rightarrow \int\limits_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos\left(\frac{s}{2}\right)\, ds = -\frac{3\pi}{16}$

Hence $\int\limits_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)\, dx = -\frac{3\pi}{16}.$

Example 3:

Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evaluate

(i) $\int\limits_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$, (ii) $\int\limits_{0}^{\infty} \frac{\sin x}{x}\, dx$ and prove that $\int\limits_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Sol:

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-a}^{a} 1.(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int\limits_{-a}^{a} \cos sx\, dx + i \int\limits_{-a}^{a} \sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \int\limits_{0}^{a} \cos sx\, dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function.

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} . \frac{\sin as}{s}$$

To find (i) $\int\limits_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$

Using inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(s) e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} . \frac{\sin as}{s} (\cos sx - i \sin sx)\, ds$$

$$1 = \frac{1}{\pi} \left\{ \int\limits_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - i \int\limits_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \sin sx\, ds \right\}$$

Equating the real part, we have $1 = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right) \cos sx \, ds$

Hence

$$\int\limits_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} \, ds = \pi. \tag{6}$$

To find (ii) $\int\limits_{0}^{\infty} \frac{\sin x}{x} \, dx$

Put $x = 0$ in equation (6), we have $\int\limits_{-\infty}^{\infty} \frac{\sin as}{s} \, ds = \pi.$

$\Rightarrow 2 \int\limits_{0}^{\infty} \frac{\sin as}{s} \, ds = \pi.$ Since the given integral is an even.

$\therefore \int\limits_{0}^{\infty} \frac{\sin as}{s} \, ds = \frac{\pi}{2}.$ Putting $as = t \Rightarrow a \, ds = dt$

$\int\limits_{0}^{\infty} \frac{\sin t}{(t/a)} . \frac{dt}{a} = \frac{\pi}{2} \Rightarrow \int\limits_{0}^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}.$

Hence $\int\limits_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$

(iii) To prove that $\int\limits_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2 \, ds = \int\limits_{-\infty}^{\infty} |f(x)|^2 \, dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin as}{s} \right)^2 ds = \int\limits_{-a}^{a} 1. \, dx$$

$$\Rightarrow \frac{2}{\pi} .2 \int\limits_{0}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds = (x)_{-a}^{a}$$

$$\Rightarrow \frac{4}{\pi} \int\limits_{0}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds = 2a$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = \frac{a\pi}{2}$$

Putting $as = t \Rightarrow a\,ds = dt$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{(t/a)}\right)^2 . \frac{dt}{a} = \frac{a\pi}{2}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Example 4:

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a \end{cases}$. Hence evaluate (i) $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \frac{\pi}{4}$ and

(ii) $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$

Sol:

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin sx)\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a^2 - x^2) \cos sx dx + i \int_{-a}^{a} (a^2 - x^2) \sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_0^a (a^2 - x^2) \cos sx\, dx + 0 \right\}$$

Since the first integral is an even function and
the second integral is an odd function

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2) \left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$\begin{aligned} F(s) &= \frac{2}{\sqrt{2\pi}}\left[\frac{-2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right] \\ &= \frac{4}{\sqrt{2\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \end{aligned}$$

To find (i) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \frac{\pi}{4}$

Using Inverse Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(s)e^{-isx}\, ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$\begin{aligned} f(x) &= \frac{2}{\pi}\left\{ \int\limits_{-\infty}^{\infty} \left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds \right. \\ &\quad \left. - i\int\limits_{-\infty}^{\infty} \left(\frac{\sin as - as\cos as}{s^3}\right)\sin sx\, ds \right\} \\ &= \frac{2}{\pi}\left\{ 2\int\limits_{0}^{\infty} \left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds - 0 \right\} \end{aligned}$$

Since the first integral is an even function and
the second integral is an odd function

Put $x = 0$ in the above integral. But $x = 0$ is a point of continuity of $f(x)$.

$\therefore$ the value of the integral when $x = 0$ is $f(0) = a^2 - 0 = a^2$.

$$\therefore a^2 = \frac{4}{\pi} \int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right) ds$$

$$\Rightarrow \int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right) ds = \frac{\pi a^2}{4}$$

Putting $as = t \Rightarrow a\,ds = dt$, we get

$$\int\limits_0^\infty \left(\frac{\sin t - t\cos t}{(t/a)^3}\right)\frac{dt}{a} = \frac{\pi a^2}{4} \Rightarrow \int\limits_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$$

Hence $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \frac{\pi}{4}.$

To find (ii) $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2\, ds = \int\limits_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{16}{2\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \int\limits_{-a}^{a} (a^2 - x^2)^2\, dx$$

$$\Rightarrow \frac{8}{\pi}.2\int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = 2.\int\limits_0^a (a^2 - x^2)^2\, dx$$

$$\Rightarrow \int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}.\int\limits_0^a (a^4 - 2a^2x^2 + x^4)\, dx$$

$$\Rightarrow \int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}\left[a^4x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^a$$

$$\Rightarrow \int\limits_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi a^5}{15}$$

Putting $as = t \Rightarrow a\,ds = dt$, we get

$$\int\limits_0^\infty \left(\frac{\sin t - t\cos t}{(t/a)^3}\right)^2 \frac{dt}{a} = \frac{\pi a^5}{15} \Rightarrow \int\limits_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$

Hence $\int\limits_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$

Example 5: Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence deduce that (i) $\int\limits_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ and (ii) $\int\limits_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}.$

Sol:

$$
\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i \sin sx)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a-|x|) \cos sx dx + i \int_{-a}^{a} (a-|x|) \sin sx\, dx \right\}
\end{aligned}
$$

$$
F(s) = \frac{1}{\sqrt{2\pi}} \left\{ 2 \int_{0}^{a} (a-|x|) \cos sx\, dx + 0 \right\}
$$

Since the first integral is an even function and the second integral is an odd function

$$
\begin{aligned}
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a-x) \cos sx\, dx \\
&= \frac{2}{\sqrt{2\pi}} \left[(a-x) \left(\frac{\sin sx}{s} \right) - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^a \\
&= \frac{2}{\sqrt{2\pi}} \left[\frac{-\cos as}{s^2} + \frac{1}{s^2} \right]
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos as}{s^2} \right) \\
&= \sqrt{\frac{2}{\pi}} . \left(\frac{2 \sin^2(as/2)}{s^2} \right)
\end{aligned}
$$

To find (i) $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}$

Using Inverse Fourier Transform $\displaystyle f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$

$$
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{2 \sin^2(as/2)}{s^2} \right) (\cos sx - i \sin sx)\, ds
$$

$$f(x) = \frac{2}{\pi}\left\{\int\limits_{-\infty}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\cos sx\,ds - i\int\limits_{-\infty}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\sin sx\,ds\right\}$$

Since the first integral is an even function and

the second integral is an odd function

$$= \frac{2}{\pi}\left\{2\int\limits_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\cos sx\,ds - 0\right\}$$

Put $x = 0$ in the above integral. But $x = 0$ is a point of continuity of $f(x)$.

Therefore value of the integral when $x = 0$ is $f(0) = a - 0 = a$.

$$\therefore\ a = \frac{4}{\pi}\int\limits_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\,ds$$

$$\Rightarrow \int\limits_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)\,ds = \frac{\pi a}{4}$$

Putting $\dfrac{as}{2} = t$. Therefore $a\,ds = 2\,dt$

$$\int\limits_{0}^{\infty}\left(\frac{\sin^2 t}{(2t/a)^2}\right).\frac{2\,dt}{a} = \frac{\pi a^2}{4} \Rightarrow \int\limits_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2\,dt = \frac{\pi}{2}$$

Hence $\displaystyle\int\limits_{0}^{\infty}\left(\frac{\sin x}{x}\right)^2\,dx = \frac{\pi}{2}.$

To prove that (ii) $\displaystyle\int\limits_{0}^{\infty}\left(\frac{\sin x}{x}\right)^4\,dx = \frac{\pi}{3}$

Using Parseval's identity $\displaystyle\int\limits_{-\infty}^{\infty}|F(s)|^2\,ds = \int\limits_{-\infty}^{\infty}|f(x)|^2\,dx$

$$\Rightarrow \int\limits_{-\infty}^{\infty}\frac{2}{\pi}\left(\frac{2\sin^2(as/2)}{s^2}\right)^2\,ds = \int\limits_{-a}^{a}(a-|x|)^2\,dx$$

$$\Rightarrow \frac{8}{\pi}.2\int\limits_{0}^{\infty}\left(\frac{\sin^2(as/2)}{s^2}\right)^2\,ds = 2\int\limits_{0}^{a}(a-x)^2\,dx$$

$$\Rightarrow \frac{8}{\pi}\int\limits_{0}^{\infty}\frac{\sin^4(as/2)}{s^4}\,ds = \int\limits_{0}^{a}(a^2-2ax+x^2)\,dx$$

$$\Rightarrow \frac{8}{\pi}\int_0^{\infty} \frac{\sin^4(as/2)}{s^4}\,ds = \left(a^2x - 2a\frac{x^2}{2} + \frac{x^3}{3}\right)_0^a$$

$$\Rightarrow \frac{8}{\pi}\int_0^{\infty} \frac{\sin^4(as/2)}{s^4}\,ds = \frac{a^3}{3}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^4(as/2)}{s^4}\,ds = \frac{\pi a^3}{24}$$

Put $\dfrac{as}{2} = t$. Therefore $a\,ds = 2\,dt$

Hence $\displaystyle\int_0^{\infty} \frac{\sin^4 t}{(2t/a)^4}.\frac{2\,dt}{a} = \frac{\pi a^3}{24} \Rightarrow \int_0^{\infty} \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.$

Example 6:

Show that the transformation of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the transform of $e^{-a^2x^2}, a > 0.$

Sol:

By the Fourier transform $F(s) = F\{f(x)\} = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx + \frac{i^2s^2}{4a^2} - \frac{i^2s^2}{4a^2}\right]}\,dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}.e^{\frac{i^2s^2}{4a^2}}\,dx$$

$$= \frac{e^{\frac{-s^2}{4a^2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\,dx$$

Putting $t = ax - \dfrac{is}{2a} \Rightarrow dt = a\,dx$

when $x = \infty \Rightarrow t = \infty$ and when $x = -\infty \Rightarrow t = -\infty$

$\therefore F\{f(x)\} = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \Rightarrow F\{f(x)\} = \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} .2 \int\limits_{0}^{\infty} e^{-t^2}\, dt$; Since the integral is an even function.

Putting $t^2 = u \Rightarrow 2t\, dt = du \Rightarrow dt = \frac{du}{2\sqrt{u}}$

$\therefore F\{f(x)\} = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int\limits_{0}^{\infty} e^{-u} \frac{du}{2\sqrt{u}}$

We know that (Gamma definition) $\Gamma n = \int\limits_0^\infty e^{-x} x^{n-1}\, dx$ and $\Gamma(1/2) = \sqrt{\pi}$

$\Rightarrow F\{f(x)\} = \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int\limits_0^\infty e^{-u} u^{1/2-1}\, du$

$$\begin{aligned} F\left\{e^{-a^2x^2}\right\} &= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \Gamma(1/2) \\ &= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi} \\ &= \frac{e^{-s^2/4a^2}}{a\sqrt{2}} \end{aligned}$$

Substituting $a = \frac{1}{\sqrt{2}}$ in above, we get

$$F\left\{e^{-x^2/2}\right\} = e^{-s^2/2}.$$

Example 7:

Find the Fourier transform of $f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ and hence find the value of $\int\limits_0^\infty \frac{\sin^4 t}{t^4}\, dt.$

Sol: By the Fourier transform $F(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-1}^{1} (1 - |x|)(\cos sx + i \sin sx)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int\limits_{-1}^{1} (1 - |x|) \cos sx dx + i \int\limits_{-1}^{1} (1 - |x|) \sin sx\, dx \right\} \end{aligned}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^1 (1-|x|)\cos sx\,dx + 0 \right\}$$

Since the first integral is an even function and the second integral is an odd function

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x)\cos sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{-\cos s}{s^2} + \frac{1}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right)$$

$$= \sqrt{\frac{2}{\pi}}. \left(\frac{2\sin^2(s/2)}{s^2} \right)$$

To find $\int\limits_0^\infty \frac{\sin^4 t}{t^4}\,dt$

Using Parseval's identity $\int\limits_{-\infty}^{\infty} |F(s)|^2\,ds = \int\limits_{-\infty}^{\infty} |f(x)|^2\,dx$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{2\sin^2(s/2)}{s^2} \right)^2 ds = \int_{-1}^{1} (1-|x|)^2\,dx$$

$$\Rightarrow \frac{8}{\pi}.2\int_0^\infty \left(\frac{\sin^2(s/2)}{s^2} \right)^2 ds = 2\int_0^1 (1-x)^2\,dx$$

$$\Rightarrow \frac{8}{\pi} \int_0^\infty \frac{(\sin^2(s/2))^2}{s^4}\,ds = \int_0^1 (1-2x+x^2)\,dx$$

$$\Rightarrow \frac{8}{\pi} \int_0^\infty \frac{\sin^4(s/2)}{s^4}\,ds = \left(x - x^2 + \frac{x^3}{3} \right)_0^1$$

$$\Rightarrow \frac{8}{\pi} \int_0^\infty \frac{\sin^4(s/2)}{s^4}\,ds = \frac{1}{3}$$

$$\Rightarrow \int_0^\infty \frac{\sin^4(s/2)}{s^4}\, ds = \frac{\pi}{24}.$$

Put $\dfrac{s}{2} = t \Rightarrow s = 2t$. Therefore $ds = 2\,dt$

Hence $\displaystyle\int_0^\infty \frac{\sin^4 t}{(2t)^4} .2\,dt = \frac{\pi}{24}$

$$\Rightarrow \int_0^\infty \frac{\sin^4 t}{t^4}\, dt = \frac{\pi}{3}.$$

Fourier cosine and Sine Transform

The Fourier Cosine and Sine transform of $f(x)$ in $(0, \infty)$ is defined to be

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

The inverse Fourier Cosine and Sine transform is defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx\, ds$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds$$

Properties of Fourier Cosine and Sine Transforms
Property 1: Linearity Property

(i) $F_c\left[af(x) + bg(x)\right] = aF_c\left[f(x)\right] + bF_c\left[g(x)\right]$
(ii) $F_s\left[af(x) + bg(x)\right] = aF_s\left[f(x)\right] + bF_s\left[g(x)\right]$
Proof:

(i) By definition $F_c\left\{f(x)\right\} = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \cos sx\, dx$

$$
\begin{aligned}
F_c\left[af(x)+bg(x)\right] &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty (af(x)+bg(x))\cos sx\,dx \\
&= a.\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx\,dx + b.\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx\,dx \\
&= aF_c\left[f(x)\right] + bF_c\left[g(x)\right]
\end{aligned}
$$

(ii) By definition $F_s\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx$

$$
\begin{aligned}
F_s\left[af(x)+bg(x)\right] &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty (af(x)+bg(x))\sin sx\,dx \\
&= a.\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx + b.\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx \\
&= aF_s\left[f(x)\right] + bF_s\left[g(x)\right]
\end{aligned}
$$

Property 2: Modulation property

(i) $F_c\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right]$

(ii) $F_s\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$

(iii) $F_c\left[f(x)\sin ax\right] = \frac{1}{2}\left[F_s(a+s)+F_s(a-s)\right]$

(iv) $F_s\left[f(x)\sin ax\right] = \frac{1}{2}\left[F_c(s-a)-F_c(s+a)\right]$

Proof:

(i) By definition $F_c\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx\,dx$

$$
\begin{aligned}
F_c\left[f(x)\cos ax\right] &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos ax\cos sx\,dx \\
&= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\cos(s+a)x+\cos(s-a)x\right]\,dx \\
&= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos(s-a)x\,dx\right\}
\end{aligned}
$$

$$\Rightarrow F_c\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$$

(ii) By definition $F_S\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx$

$$\begin{aligned}
F_s\left[f(x)\cos ax\right] &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos ax\sin sx\,dx \\
&= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\sin(a+s)x - \sin(a-s)x\right]\,dx \\
&= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\sin(s+a)x + \sin(s-a)x\right]\,dx \\
&= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin(s-a)x\,dx\right\} \\
&= \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]
\end{aligned}$$

(iii) By definition $F_c\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx\,dx$

$$\begin{aligned}
F_c\left[f(x)\sin ax\right] &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin ax\cos sx\,dx \\
&= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\sin(a+s)x + \sin(a-s)x\right]\,dx \\
&= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin(a-s)x\,dx\right\} \\
&= \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right]
\end{aligned}$$

(iv) By definition $F_s\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx$

$$F_s\left[f(x)\sin ax\right] = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin ax\sin sx\,dx$$

$$F_s\left[f(x)\sin ax\right] = \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\cos(a-s)x - \cos(a+s)x\right]\,dx$$
$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int\limits_0^\infty f(x)\left[\cos(s-a)x - \cos(s+a)x\right]\,dx$$
$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos(s+a)x\,dx\right\}$$
$$= \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$$

Property 3: Change of Scale property

(i) $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{s}{a}\right)$ if $a > 0$

(ii) $F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right)$ if $a > 0$

Proof:

(i) $F_c\{f(ax)\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(ax)\cos sx\,dx$

Put $ax = t \Rightarrow a\,dx = dt$

When $x = 0 \Rightarrow t = 0$ and When $x = \infty \Rightarrow t = \infty$

$$F_c\{f(ax)\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(t)\cos s\left(\frac{t}{a}\right)\frac{dt}{a}$$
$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(t)\cos\left(\frac{s}{a}\right)t\,dt$$
$$= \frac{1}{a}F_c\left(\frac{s}{a}\right)$$

(ii) $F_s\{f(ax)\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(ax)\sin sx\,dx$

Put $ax = t \Rightarrow a\,dx = dt$

When $x = 0 \Rightarrow t = 0$ and When $x = \infty \Rightarrow t = \infty$

$$F_s\{f(ax)\} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(t)\sin s\left(\frac{t}{a}\right)\frac{dt}{a}$$
$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(t)\sin\left(\frac{s}{a}\right)t\,dt = \frac{1}{a}F_s\left(\frac{s}{a}\right)$$

Property 4: Differentiation of Cosine and Sine Transform

(i) $F_c\left[xf(x)\right] = \dfrac{d}{ds}\left[F_s(s)\right]$

(ii) $F_s\left[xf(x)\right] = -\dfrac{d}{ds}\left[F_c(s)\right]$

Proof:

(i) By definition $F_s\left\{f(x)\right\} = \sqrt{\dfrac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx$

Differentiate both sides with respect to s, we get

$$\begin{aligned}\frac{d}{ds}F_s(s) &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx.x\,dx\\ &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty [xf(x)]\cos sx\,dx\\ &= F_c\left[xf(x)\right]\end{aligned}$$

(ii) By definition $F_c\left\{f(x)\right\} = \sqrt{\dfrac{2}{\pi}}\int\limits_0^\infty f(x)\cos sx\,dx$

Differentiate both sides with respect to s, we get

$$\begin{aligned}\frac{d}{ds}F_c(s) &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)(-\sin sx).x\,dx\\ &= -\sqrt{\frac{2}{\pi}}\int\limits_0^\infty [xf(x)[\sin sx\,dx\\ &= -F_s\left[xf(x)\right]\end{aligned}$$

$\Rightarrow F_s\left[xf(x)\right] = -\frac{d}{ds}F_c(s)$

Property 5: Cosine and Sine transforms of derivative:

If $f(x)$ is continuous and absolutely integrable in $(-\infty,\infty)$ and if $f(x)\to\infty$ as $x\to\infty$, then

(i) $F_c\left[f'(x)\right] = sF_s(s) - \sqrt{\dfrac{2}{\pi}}f(0)$

(ii) $F_s\left[f'(x)\right] = -sF_c(s)$

(iii) $F_c\left[f''(x)\right] = -s^2F_c(s) - \sqrt{\dfrac{2}{\pi}}f'(0)$

(iv) $F_s\left[f''(x)\right] = -s^2F_s(s) + \sqrt{\dfrac{2}{\pi}}sf(0)$

Parseval's identities:

If $F_c(s)$ and $G_c(s)$ are the Fourier Cosine transforms and $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

1. $\displaystyle\int_0^\infty f(x).g(x)\,dx = \int_0^\infty F_c(s).G_c(s)\,ds$

2. $\displaystyle\int_0^\infty f(x).g(x)\,dx = \int_0^\infty F_s(s).G_s(s)\,ds$

3. $\displaystyle\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds$

4. $\displaystyle\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(s)|^2\,ds$

Formula:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

Example 1:

Find the Fourier Cosine and Sine transform of $f(x) = e^{-ax}, a > 0$ and hence deduce that $\displaystyle\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}e^{-ax}$ and $\displaystyle\int_0^\infty \frac{s\sin sx}{a^2+s^2}\,ds = \frac{\pi}{2}e^{-ax}$. Also using Parseval's identity evaluate $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ and $\displaystyle\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)^2}$ if $a > 0$.

Sol:

By definition
$$\begin{aligned} F_c(s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^\infty \end{aligned}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left[0 - \left(\frac{1}{a^2+s^2}\right)(-a)\right]$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)$$

By definition $F_s(s) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin sx\,dx$

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-ax}\sin sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$
$$= \sqrt{\frac{2}{\pi}}\left[0 - \left(\frac{1}{a^2+s^2}\right)(-s)\right]$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$$

To find $\int\limits_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}e^{-ax}$ **and** $\int\limits_0^\infty \frac{s\sin sx}{a^2+s^2}\,ds = \frac{\pi}{2}e^{-ax}$.

By inversion formula $f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty F_c(s)\cos sx\,ds$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\cos sx\,ds$$
$$\Rightarrow e^{-ax} = \frac{2a}{\pi}\int\limits_0^\infty \frac{\cos sx}{a^2+s^2}\,ds$$
$$\therefore \int\limits_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}e^{-ax}.$$

By inversion formula $f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty F_s(s)\sin sx\,ds$

$$f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\sin sx\,ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds$$

$$\therefore \int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds = \frac{\pi}{2} e^{-ax}.$$

To find $\displaystyle\int_0^\infty \frac{dx}{(a^2 + x^2)^2}$ **and** $\displaystyle\int_0^\infty \frac{x^2\, dx}{(a^2 + x^2)^2}$

Using Parseval's identity $\displaystyle\int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |f(x)|^2\, dx$

$$\Rightarrow \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + s^2}\right)\right]^2 ds = \int_0^\infty (e^{-ax})^2\, dx$$

$$\Rightarrow \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(a^2 + s^2)^2} = \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$\Rightarrow \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(a^2 + s^2)^2} = \frac{1}{2a}$$

$$\Rightarrow \int_0^\infty \frac{ds}{(a^2 + s^2)^2} = \frac{\pi}{4a^3}$$

$$\therefore \int_0^\infty \frac{dx}{(a^2 + x^2)^2} = \frac{\pi}{4a^3}.$$

Using Parseval's identity $\displaystyle\int_0^\infty |F_s(s)|^2\, ds = \int_0^\infty |f(x)|^2\, dx$

$$\Rightarrow \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left(\frac{s}{a^2 + s^2}\right)\right]^2 ds = \int_0^\infty (e^{-ax})^2\, dx$$

$$\Rightarrow \frac{2}{\pi} \int_0^\infty \frac{s^2\, ds}{(a^2 + s^2)^2} = \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$\Rightarrow \frac{2}{\pi} \int_0^\infty \frac{s^2\, ds}{(a^2 + s^2)^2} = \frac{1}{2a}$$

$$\Rightarrow \int_0^\infty \frac{s^2\,ds}{(a^2+s^2)^2} = \frac{\pi}{4a}$$

$$\therefore \int_0^\infty \frac{x^2\,dx}{(a^2+x^2)^2} = \frac{\pi}{4a}.$$

Example 2:

Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transform methods.

Sol:

Given $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty \frac{1}{(x^2+a^2)}.\frac{1}{(x^2+b^2)}\,dx$

Consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$, $a > 0, b > 0$.

Then $F_c(s) = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{a}{a^2+s^2}\right)$ and $G_c(s) = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{b}{b^2+s^2}\right)$ by problem 1.

By Parseval's identity $\displaystyle\int_0^\infty F_c(s)G_c(s)\,ds = \int_0^\infty f(x)g(x)\,dx$

$$\Rightarrow \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right).\sqrt{\frac{2}{\pi}}\left(\frac{b}{b^2+s^2}\right)ds = \int_0^\infty e^{-ax}.e^{-bx}\,dx$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\Rightarrow \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab}.\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\Rightarrow \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)}$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}.$$

Example 3:

Prove that $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}.$

Sol:

Given $\int\limits_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \int\limits_0^\infty \frac{x}{(x^2+a^2)}.\frac{x}{(x^2+b^2)}\,dx$

Consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$.

Then $F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$ and $G_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{b^2+s^2}\right)$ by problem 1.

By Parseval's identity $\int\limits_0^\infty F_s(s)G_s(s)\,ds = \int\limits_0^\infty f(x)g(x)\,dx$

$$\Rightarrow \int\limits_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right).\sqrt{\frac{2}{\pi}}\left(\frac{s}{b^2+s^2}\right)\,ds = \int\limits_0^\infty e^{-ax}.e^{-bx}\,dx$$

$$\Rightarrow \frac{2}{\pi}\int\limits_0^\infty \frac{s^2\,ds}{(a^2+s^2)(b^2+s^2)} = \int\limits_0^\infty e^{-(a+b)x}\,dx$$

$$\Rightarrow \int\limits_0^\infty \frac{s^2\,ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2}.\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\Rightarrow \int\limits_0^\infty \frac{s^2\,ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2(a+b)}$$

$$\therefore \int\limits_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}.$$

Example 4:

Using Parseval's identity evaluate $\int\limits_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$.

Sol:

Let $f(x) = e^{-ax}$. We know that $F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$

By Parseval's identity $\int\limits_0^\infty |F_s(s)|^2\,ds = \int\limits_0^\infty |f(x)|^2\,dx$

$$\Rightarrow \int\limits_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\right]^2\,ds = \int\limits_0^\infty (e^{-ax})^2\,dx$$

$$\Rightarrow \frac{2}{\pi}\int\limits_0^\infty \frac{s^2\,ds}{(a^2+s^2)^2} = \int\limits_0^\infty e^{-2ax}\,dx$$

$$\Rightarrow \int_0^\infty \frac{s^2\,ds}{(a^2+s^2)^2} = \frac{\pi}{2}.\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\Rightarrow \int_0^\infty \frac{s^2\,ds}{(a^2+s^2)^2} = \frac{\pi}{4a}$$

$$\therefore \int_0^\infty \frac{x^2\,dx}{(a^2+x^2)^2} = \frac{\pi}{4a}.$$

Example 5:

Find the Fourier Cosine transform of $\dfrac{1}{x^2+a^2}$.

Sol: By definition

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x^2+a^2}\cos sx\,dx \tag{7}$$

We know that $F_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}.\frac{a}{a^2+s^2}$

Taking Inverse Fourier Cosine transform, we get

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}.\frac{a}{a^2+s^2}\cos sx\,ds$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{1}{a^2+s^2}\cos sx\,ds$$

$$\Rightarrow \int_0^\infty \frac{1}{a^2+s^2}\cos sx\,ds = \frac{\pi}{2a}e^{-ax}$$

Interchanging x and s, we get

$$\int_0^\infty \frac{1}{x^2+a^2}\cos sx\,dx = \frac{\pi}{2a}e^{-as} \tag{8}$$

Substituting (8) in (7), we get

$$F_c\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2a}e^{-as}$$

Therefore $F_c\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}.\frac{e^{-as}}{a}$.

Example 6:

Find the Fourier sine transform of $\dfrac{x}{x^2+a^2}$.

Sol: By definition

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{x^2+a^2}\sin sx\,dx \tag{9}$$

We know that $F_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}.\frac{s}{a^2+s^2}$

Taking Inverse Fourier sine transform, we get

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}.\frac{s}{a^2+s^2}\sin sx\,ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s}{a^2+s^2}\sin sx\,ds$$

$$\Rightarrow \int_0^\infty \frac{s}{a^2+s^2}\sin x\,ds = \frac{\pi}{2}e^{-ax}$$

Interchanging x and s, we get

$$\int_0^\infty \frac{x}{x^2+a^2}\sin sx\,dx = \frac{\pi}{2}e^{-as} \tag{10}$$

Substituting (10) in (9), we get

$$F_s\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2}e^{-as}$$

Therefore $F_s\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}.e^{-as}$.

Example 7:

Find Fourier Sine transform of $\dfrac{1}{x}$.

Sol: By definition

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left\{\frac{1}{x}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

Putting $sx = t \Rightarrow s\,dx = dt$

$$
\begin{aligned}
F_s\left\{\frac{1}{x}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{(t/s)}.\frac{dt}{s} \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}\,dt \\
&= \sqrt{\frac{2}{\pi}}.\frac{\pi}{2} \text{ Refer problem 3; see page no.11} \\
&= \sqrt{\frac{\pi}{2}}.
\end{aligned}
$$

Example 8:

Find the Fourier Sine and Cosine transform of x^{n-1}. Deduce taht $\frac{1}{\sqrt{x}}$ is self reciprocal under fourier sine and cosine transform.

Sol:

$$
F_c\left\{x^{n-1}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\,dx \tag{11}
$$

$$
F_s\left\{x^{n-1}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx \tag{12}
$$

We know that $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx; n > 0$

Putting $x = ist \Rightarrow dx = is\,dt$

$$
\begin{aligned}
\Gamma n &= \int_0^\infty e^{-ist}(ist)^{n-1} is\,dt \\
&= i^n s^n \int_0^\infty e^{-ist}t^{n-1}\,dt
\end{aligned}
$$

$$
\Rightarrow \int_0^\infty e^{-ist}t^{n-1}\,dt = \frac{\Gamma n}{i^n s^n}
$$

$$
\Rightarrow \int_0^\infty e^{-isx}x^{n-1}\,dx = \left(\frac{1}{i}\right)^n \frac{\Gamma n}{s^n}
$$

$$\Rightarrow \int_0^\infty e^{-isx} x^{n-1}\, dx = \left(\frac{i}{i \times i}\right)^n \frac{\Gamma n}{s^n}$$
$$= (-i)^n \frac{\Gamma n}{s^n}$$
$$= \left(e^{-\pi i/2}\right)^n \frac{\Gamma n}{s^n}$$

$$\int_0^\infty (\cos sx - i \sin sx) x^{n-1}\, dx = \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right) . \frac{\Gamma n}{s^n}$$

Equating real and imaginary parts, we get

$$\int_0^\infty \cos sx x^{n-1}\, dx = \frac{\cos \frac{n\pi}{2} . \Gamma n}{s^n} \text{ and } \int_0^\infty \sin sx x^{n-1}\, dx = \frac{\sin \frac{n\pi}{2} . \Gamma n}{s^n}$$

Hence equations (11) and (12) becomes

$$F_c\left\{x^{n-1}\right\} = \sqrt{\frac{2}{\pi}} \left(\frac{\cos \frac{n\pi}{2} . \Gamma n}{s^n}\right)$$
$$F_s\left\{x^{n-1}\right\} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin \frac{n\pi}{2} . \Gamma n}{s^n}\right)$$

Taking $n = 1/2$, we get

$$F_c\left(x^{-1/2}\right) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos \frac{\pi}{4} . \Gamma(1/2)}{s^{1/2}}\right)$$
$$= \sqrt{\frac{2}{\pi}} . \frac{1}{\sqrt{2}} . \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}}$$

Similarly $F_s\left(x^{-1/2}\right) = \frac{1}{\sqrt{s}}$

Therefore $\frac{1}{\sqrt{x}}$ is self reciprocal under fourier sine and cosine transform.

Example 9:
Find fourier cosine and sine transform of xe^{-ax}.
Sol:

$$\begin{aligned}
F_c\left[xe^{-ax}\right] &= \frac{d}{ds}F_s\left(e^{-ax}\right) \text{ by property 4}\\
&= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right] \text{ by example 1}\\
&= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2).1-s.2s}{(s^2+a^2)^2}\right]\\
&= \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]\\
F_s\left[xe^{-ax}\right] &= -\frac{d}{ds}F_c\left(e^{-ax}\right) \text{ by property 4}\\
&= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\right] \text{ by example 1}\\
&= \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]
\end{aligned}$$

Example 10:
Find Fourier Cosine transform of $e^{-a^2x^2}$ and hence evaluate fourier sine transform of $xe^{-a^2x^2}$.
Sol:
The Fourier Cosine transform

$$\begin{aligned}
F_c\left\{e^{-a^2x^2}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos sx\,dx\\
&= \sqrt{\frac{2}{\pi}} \text{ Real part of} \int_0^\infty e^{-a^2x^2}e^{isx}\,dx\\
&= \sqrt{\frac{2}{\pi}} \text{ Real part of} \int_0^\infty e^{-(a^2x^2-isx)}\,dx\\
&= \sqrt{\frac{2}{\pi}} \text{ Real part of} \int_0^\infty e^{-\left[a^2x^2-isx+\frac{i^2s^2}{4a^2}-\frac{i^2s^2}{4a^2}\right]}\,dx\\
&= \sqrt{\frac{2}{\pi}} \text{ Real part of} \int_0^\infty e^{-\left[\left(ax-\frac{is}{2a}\right)^2-\frac{i^2s^2}{4a^2}\right]}\,dx
\end{aligned}$$

$$F_c\left\{e^{-a^2x^2}\right\} = \text{Real part of } \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\left(ax-\frac{is}{2a}\right)^2}.e^{\frac{i^2s^2}{4a^2}}\,dx$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}}e^{-s^2/4a^2}\int_0^\infty e^{-\left(ax-\frac{is}{2a}\right)^2}\,dx$$

Putting $t = ax - \dfrac{is}{2a} \Rightarrow dt = a\,dx$
when $x = \infty \Rightarrow t = \infty$ and when $x = -\infty \Rightarrow t = -\infty$

Therefore $F_c\left\{e^{-a^2x^2}\right\} = \text{Real part of } \sqrt{\dfrac{2}{\pi}}e^{-s^2/4a^2}\displaystyle\int_0^\infty e^{-t^2}\frac{dt}{a}$

Putting $t^2 = u \Rightarrow 2t\,dt = du \Rightarrow dt = \dfrac{du}{2\sqrt{u}}$

Therefore $F_c\left\{e^{-a^2x^2}\right\} = \text{Real part of } \sqrt{\dfrac{2}{\pi}}\dfrac{e^{-s^2/4a^2}}{a}\displaystyle\int_0^\infty e^{-u}\frac{du}{2\sqrt{u}}$

We know that (Gamma definition) $\Gamma n = \displaystyle\int_0^\infty e^{-x}x^{n-1}\,dx$ and $\Gamma(1/2) = \sqrt{\pi}$

$$F_c\left\{e^{-a^2x^2}\right\} = \text{Real part of } \sqrt{\frac{2}{\pi}}\frac{e^{-s^2/4a^2}}{2a}\int_0^\infty e^{-u}u^{1/2-1}\,du$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}}\frac{e^{-s^2/4a^2}}{2a}\Gamma(1/2)$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}}\frac{e^{-s^2/4a^2}}{2a}\sqrt{\pi} = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}.$$

To find $F_s\left(xe^{-a^2x^2}\right)$

$$F_s\left\{e^{-a^2x^2}\right\} = -\frac{d}{ds}F_c\left[e^{-a^2x^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{e^{-s^2/4a^2}}{a\sqrt{2}}\right]$$

$$= -\frac{1}{a\sqrt{2}}e^{-s^2/4a^2}.\left(\frac{-2s}{4a^2}\right)$$

$$= \frac{s}{2\sqrt{2}a^3}e^{-s^2/4a^2}$$

Example 11: Find the function if its sine transform is $\dfrac{e^{-as}}{s}$.

Solution: Given $F_s(s) = \dfrac{e^{-as}}{s}$.

By inverse Fourier sine transform $f(x) = \sqrt{\dfrac{2}{\pi}} \int\limits_0^\infty F_s(s) \sin sx \, ds$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \frac{e^{-as}}{s} \sin sx \, ds \tag{13}$$

Differentiating w.r.to x, we get

$$\begin{aligned} \frac{df}{dx} &= \sqrt{\frac{2}{\pi}} \int\limits_0^\infty e^{-as} \cos sx \, ds \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-as}}{a^2 + x^2} (-a \cos sx + x \sin sx) \right]_0^\infty \\ &= \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} \end{aligned}$$

Integrating w. r. to x, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a} + c \tag{14}$$

At $x = 0$, in (13), we get $f(0) = 0$.
Put $x = 0$ and $f(0) = 0$ in (14), we get $c = 0$.
Hence $f(x) = \sqrt{\dfrac{2}{\pi}} \tan^{-1} \dfrac{x}{a}$.

Example 12: Solve for $f(x)$ from the integral equation $\int\limits_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$.

Solution: Given $\int\limits_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$.

Multiplying both sides by $\sqrt{\dfrac{2}{\pi}}$, we get

$$\sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(x) \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} e^{-\alpha}$$

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} e^{-\alpha} \text{ by definition}$$

$$\Rightarrow f(x) = F_c^{-1}\left(\sqrt{\frac{2}{\pi}}e^{-\alpha}\right)$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}e^{-\alpha}\cos\alpha x\, d\alpha$$
$$= \frac{2}{\pi}\left[\frac{e^{-\alpha}}{1+x^2}\left(-\cos\alpha x + x\sin\alpha x\right)\right]_0^\infty$$
$$= \frac{2}{\pi}.\frac{1}{1+x^2}.$$

Example 13: Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x)\sin sx\, dx = \begin{cases} 1 & \text{for } 0 \le s < 1 \\ 2 & \text{for } 1 \le s < 2 \\ 0 & \text{for } \ s \ge 2 \end{cases}$$

Solution: Multiplying both sides by $\sqrt{\frac{2}{\pi}}$, we get

$$F_s\{f(x)\} = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{for } 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}} & \text{for } 1 \le s < 2 \\ 0 & \text{for } \ s \ge 2 \end{cases}$$

$$\Rightarrow f(x) = F_s^{-1}\left(\begin{cases} \sqrt{\frac{2}{\pi}} & \text{for } 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}} & \text{for } 1 \le s < 2 \\ 0 & \text{for} s \ge 2 \end{cases}\right)$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, ds$$
$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 \sqrt{\frac{2}{\pi}}\sin sx\, ds + \int_1^2 2\sqrt{\frac{2}{\pi}}\sin sx\, ds\right\}$$
$$= \frac{2}{\pi}\left(-\frac{\cos sx}{x}\right)_0^1 + \frac{4}{\pi}\left(-\frac{\cos sx}{x}\right)_1^2$$
$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x}\right) + \frac{4}{\pi}\left(\frac{\cos x - \cos 2x}{x}\right)$$
$$= \frac{2}{\pi x}\left(1 + \cos x - 2\cos 2x\right).$$

Dirac delta function

Dirac delta function $\delta(t-a)$ is defined as $\delta(t-a) = \lim\limits_{h\to 0} I(h, t-a)$ where

$$I(h,t-a) = \begin{cases} \dfrac{1}{h} & \text{for } a < t < a+h \\ 0 & \text{for } t < a \ \text{ and } \ t > a+h \end{cases}$$

Example 14: Find the complex Fourier transform of dirac delta function $\delta(t-a)$.

Solution:

By definition $F\{f(x)\} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{ist}\, dt$

$$\begin{aligned}
\Rightarrow F\{\delta(t-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t-a) e^{ist}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \lim_{h\to 0} \int_{a}^{a+h} \frac{1}{h} e^{ist}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \lim_{h\to 0} \frac{1}{h} \left(\frac{e^{ist}}{is}\right)_a^{a+h} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{h\to 0} \frac{1}{h} \left(\frac{e^{is(a+h)} - e^{isa}}{is}\right) \\
&= \frac{e^{isa}}{\sqrt{2\pi}} \lim_{h\to 0} \left(\frac{e^{ish} - 1}{ish}\right) \\
&= \frac{e^{isa}}{\sqrt{2\pi}} \lim_{h\to 0} \frac{1}{ish} \left[1 + \frac{(ish)}{1!} + \frac{(ish)^2}{2!} + \frac{(ish)^3}{3!} + \ldots - 1\right] \\
&= \frac{e^{isa}}{\sqrt{2\pi}} \lim_{h\to 0} \left[\frac{1}{1!} + \frac{(ish)}{2!} + \frac{(ish)^2}{3!} + \ldots\right] = \frac{e^{isa}}{\sqrt{2\pi}}
\end{aligned}$$

Relationship between Fourier and Laplace Transform

Consider $f(t) = \begin{cases} e^{-xt} g(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$

Then the Fourier transform of $f(t)$ is given by

$$\begin{aligned}
F\{f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ist} e^{-xt} g(t)\, dt
\end{aligned}$$

$$
\begin{aligned}
F\left\{f(t)\right\} &= \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} e^{-(x-is)t} g(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} e^{-pt} g(t)\, dt \text{ where } \ p = x - is \\
&= \frac{1}{\sqrt{2\pi}} L\left\{g(t)\right\}
\end{aligned}
$$

Therefore Fourier transform of $f(t) = \dfrac{1}{\sqrt{2\pi}} \times$ Laplace transform of $g(t)$