1) Find the F.T of $f(x)$ given by $f(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a>0 \end{cases}$

and hence evaluate (i) $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} ds$, (ii) $\int_0^{\infty} \frac{\sin x}{x} dx$

and prove that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

Sol:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} \cos sx \, dx + i \int_{-a}^{a} \sin sx \, dx \right\}$$

since the first integral is an even function and the 2nd integral is odd function.

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2 \cdot \int_0^a \cos sx \, dx + 0 \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s} \right]_0^a = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s} \rightarrow (*)$$

(i) Using inverse L.T

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s} e^{-isx} ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) ds$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) ds = \pi f(x)$$

$$= \begin{cases} \pi, & \text{for } |x|<a \\ 0, & \text{for } |x|>a>0 \end{cases}$$

Equating real part, we have

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s}\, ds = \pi} \longrightarrow (1)$$

(ii) To find $\int_0^{\infty} \frac{\sin x}{x}\, dx$

Put $x=0$ in eqn (1), we have

$$\int_{-\infty}^{\infty} \frac{\sin sa}{s}\, ds = \pi \Rightarrow 2\int_0^{\infty} \frac{\sin as}{s}\, ds = \pi$$

$$\boxed{\therefore \int_0^{\infty} \frac{\sin as}{s}\, ds = \pi/2} \longrightarrow (2)$$

$a=1$ in eqn (2), we get $\int_0^{\infty} \frac{\sin s}{s}\, ds = \pi/2$

$$\boxed{\therefore \int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2}$$

(iii) Using parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-a}^{a} 1 \cdot dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin as}{s}\right)^2 ds \quad \text{by } (*)$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\Rightarrow a = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\therefore \pi a = 2\int_0^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$ (since given fn is an even function)

$a=1$ in above

$$\frac{\pi}{2} = \int_0^{\infty} \left(\frac{\sin s}{s}\right)^2 ds$$

$$\therefore \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

2.) Show that the transformation of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the transform of $e^{-a^2x^2}$, $a>0$.

Sol:

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx$$

$2axb = isx$

$b = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx + \frac{i^2s^2}{4a^2} - \frac{i^2s^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, e^{-s^2/4a^2}\, dx \quad [\because i^2 = -1]$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, dx$$

put $t = ax - \frac{is}{2a} \Rightarrow dt = a\,dx$ | when $x = \infty \Rightarrow t = \infty$, $x = -\infty \Rightarrow t = -\infty$

$$F\{f(x)\} = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \frac{dt}{a}$$

$$= \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2}\, dt \quad \left[\because \int_{-\infty}^{\infty} e^{-t^2}\, dt \text{ is an even function}\right]$$

putting $t^2 = u \Rightarrow 2t\,dt = du$

$dt = \frac{du}{2t} \Rightarrow dt = \frac{du}{2\sqrt{u}}$

$$\therefore F\{f(x)\} = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-u} \cdot \frac{du}{2\sqrt{u}}$$

$$F\{f(x)\} = \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^\infty e^{-u} u^{-1/2}\, du$$

W.K.T $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$, $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

$$\therefore F\{e^{-a^2x^2}\} = \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_0^\infty e^{-u} u^{1/2-1}\, du$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot \Gamma \tfrac{1}{2}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}\sqrt{\pi}} \cdot \sqrt{\pi} = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$\therefore \boxed{F\{e^{-a^2x^2}\} = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}}$$

$a = 1/\sqrt{2}$

$$\Rightarrow \boxed{F\{e^{-x^2/2}\} = e^{-s^2/2}}$$

3) Find the F.T of f(x) given by $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ and

Hence prove that (i) $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \pi/4$ and

(ii) $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \pi/15$

Sol:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin sx)\, dx$$

$$= \left\{\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\cos sx\, dx\right\} + \left\{\frac{1}{\sqrt{2\pi}}\, i \int_{-a}^{a} (a^2 - x^2)\sin sx\, dx\right\}$$

$$F(s) = \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2)\cos sx \, dx$$ [Since the Second integral is an odd function]

$u = a^2 - x^2$ $\quad dv = \cos sx\, dx$

$u' = -2x$ $\quad v = \frac{\sin sx}{s}$

$u'' = -2$ $\quad v_1 = \frac{-\cos sx}{s^2}$

$v_2 = \frac{-\sin sx}{s^3}$

$\int u\,dv = uv - u'v_1 + u''v_2 + \ldots$

$$F(s) = \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2)\frac{\sin sx}{s} - \frac{2x\cos sx}{s^2} + 2\frac{\sin sx}{s^3} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{-2a\cos as}{s^2} + \frac{2\sin sa}{s^3} \right]$$

$$\boxed{F(s) = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - sa\cos as}{s^3} \right]}$$

(i) Using inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - sa\cos as}{s^3} \right] e^{-isx} \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as - sa\cos as}{s^3} \right) (\cos sx - i\sin sx) \, ds$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin as - sa\cos as}{s^3} \right) \cos sx \, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin as - sa\cos as}{s^3} \right) \sin sx \, ds \right\}$$

$$= \frac{2}{\pi} \left\{ 2\int_0^{\infty} \left(\frac{\sin as - sa\cos as}{s^3} \right) \cos sx \, ds - 0 \right\}$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin as - sa\cos as}{s^3}\right)\cos sx\, ds$$

Put $x=0$ in the above and $x=0$ is a point of continuity.

$\therefore$ The $\Leftarrow$. $f(0) = a^2$

Hence $a^2 = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin as - sa\cos as}{s^3}\right) ds$

$a=1 \Rightarrow \frac{\pi}{4} = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) ds$

$$\therefore \boxed{\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \pi/4}$$

(ii) Using parseval's identity

$$\int_{-\infty}^\infty |f(x)|^2 dx = \int_{-\infty}^\infty |F(s)|^2 ds$$

$$\Rightarrow \int_{-a}^a (a^2-x^2)^2 dx = \int_{-\infty}^\infty \frac{16}{2\pi}\left(\frac{\sin as - sa\cos as}{s^3}\right)^2 ds$$

$$2\int_0^a (a^2-x^2)^2 dx = \frac{8}{\pi}\cdot 2\int_0^\infty \left(\frac{\sin as - sa\cos as}{s^3}\right)^2 ds$$

$$\therefore \int_0^\infty \left(\frac{\sin as - sa\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}\int_0^a (a^2-x^2)^2 dx$$

$$= \frac{\pi}{8}\int_0^a (a^4 - 2a^2x^2 + x^4)dx$$

$$= \frac{\pi}{8}\left[a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right]_0^a$$

$$= \frac{\pi}{8}a^5\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{\pi}{8}a^5 \cdot \frac{8}{15} = \frac{\pi a^5}{15}$$

put $a=1$

$$\Rightarrow \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

$$\boxed{\therefore \int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}}$$

4) Find the F.T of $f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a>0 \end{cases}$ and hence deduce that (i) $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$ and (ii) $\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \pi/3$.

Sol:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a-|x|)\cos sx\, dx + i\int_{-a}^{a} (a-|x|)\sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x)\cos sx\, dx$$

$u = a-x \longrightarrow$ $dv = \cos sx\, dx$, $v = \frac{\sin sx}{s}$

$u' = -1 \longrightarrow$ $v_1 = \frac{-\cos sx}{s^2}$

$$= \frac{2}{\sqrt{2\pi}} \left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[-\frac{\cos as}{s^2} + \frac{1}{s^2} \right] = \frac{2}{\sqrt{2\pi}} \left(\frac{1-\cos as}{s^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos as}{s^2} \right)$$

$$\therefore F(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin^2 as/2}{s^2}$$

By Fourier inverse formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin^2 \frac{sa}{2}}{s^2} e^{-isx}\, ds$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \left\{ \int_{-\infty}^{\infty} \frac{\sin^2 sa/2}{s^2} (\cos sx - i \sin sx)\, ds \right\}$$

$$= \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin^2 sa/2}{s^2} \cos sx\, ds$$

Put $t = s/2 \Rightarrow ds = 2dt$, when $s=0 \Rightarrow t=0$, $s=\infty \Rightarrow t=\infty$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin ta}{2t}\right)^2 \cos 2tx \cdot 2\, dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin at}{t}\right)^2 \cos 2tx\, dt$$

Put $x=0$. But $x=0$ is a point of continuity of $f(x)$.

$$\therefore f(0) = a$$

$$\therefore a = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin at}{t}\right)^2 dt$$

$$\Rightarrow \frac{\pi a}{2} = \int_0^{\infty} \left(\frac{\sin at}{t}\right)^2 dt$$

put $a=1$. $\Rightarrow \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

$$\boxed{\therefore \int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi/2}$$

(ii) Using parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\Rightarrow \int_{-a}^{a} (a-|x|)^2\, dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \cdot 4\left(\frac{\sin as/2}{s}\right)^4 ds$$

$$\therefore 2\int_{0}^{a} (a-x)^2\, dx = 2 \cdot \frac{8}{\pi} \int_{0}^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^4 ds$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin as/2}{s}\right)^4 ds = \frac{\pi}{8} \int_{0}^{a} (a^2 - 2ax + x^2)\, dx$$

$$= \frac{\pi}{8}\left[a^2x - \frac{2ax^2}{2} + \frac{x^3}{3}\right]_0^a$$

$$= \frac{\pi}{8} \cdot a^3 \left(1 - 1 + 1/3\right)$$

$$= \frac{\pi a^3}{24}$$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin as/2}{s}\right)^4 ds = \frac{\pi a^3}{24} \Rightarrow \int_{0}^{\infty} \left(\frac{\sin s/2}{s}\right)^4 ds = \frac{\pi}{24} \quad [\because a = 1]$$

put $s/2 = t \Rightarrow ds = 2dt$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin t}{2t}\right)^4 2dt = \frac{\pi}{24}$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

$$\boxed{\therefore \int_{0}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \pi/3}$$