Unit - V Z-transforms

In Communication Engineering, two basic types of signals are encountered.

(i) Continuous time signals (L.T and F.T)

(ii) discrete time singnals (Z.T)

Z-transform is the discrete counter part of L.T.

Defn: Let $\{f(n)\}$ be a sequence defined for $n=0, \pm 1, \pm 2, \pm 3, \ldots$ Then the two sided z-transform of sequence $f(n)$ is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n} \longrightarrow (1)$$

where z is a complex variable.

Defn: If the function $f(n)$ is defined for $n=0,1,2,\ldots$ and $f(n)=0$ for $n<0$ then $f(0), f(1), f(2), \ldots$ is a causal sequence, denoted by $\{f(n)\}$. The z-transform of $\{f(n)\}$ is defined as

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} \longrightarrow (2)$$

This Z-transform is called one sided z-transform.

Note: The infinite series on the right hand side of (1) and (2) will be convergent only for certain values of z depending on the sequence $f(n)$.

Defn: If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, \ldots$, T being the sampling period, then z-transform of $f(t)$ is defined as

$$z[f(t)] = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

Z-transform of some standard sequence

1.) $Z(1) = \frac{z}{z-1}, \quad |z| > 1$

Proof:

By defn $\quad z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$z\{1\} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \quad \text{if } \left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$$

$$= \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

2.) $z\{a^n\} = \frac{z}{z-a}, \quad |z| > |a|$

Proof:

$$z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$z\{a^n\} = \left(1-\frac{a}{z}\right)^{-1} \quad \text{If } \left|\frac{a}{z}\right|<1 \Rightarrow |z|>|a|$$

$$= \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}$$

Cor: $z\{1\} = \frac{z}{z-1}$ If $a=1$, $z\{(-1)^n\} = \frac{z}{z+1}$ If $a=-1$

3) $z\{n\} = \frac{z}{(z-1)^2}$, $|z|>1$, $n\neq 0$

Proof:

$$z\{n\} = \sum_{n=1}^{\infty} n z^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right) \quad \because (1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2} \quad \text{If } \left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{z}{(z-1)^2}$$

4) $z\{\frac{1}{n}\} = \log_e\left(\frac{z}{z-1}\right)$, $|z|>1$, $n>0$

Proof:

$$z\left\{\frac{1}{n}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots$$

$$= -\log_e\left(1-\frac{1}{z}\right) \quad \because -\log_e(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

$$= -\log_e\left(\frac{z-1}{z}\right), \quad |z|>1$$

$$= \log_e\left(\frac{z}{z-1}\right)$$

5.) $z\left\{\frac{1}{n!}\right\} = e^{1/z}$

Proof:

$$z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$= e^{1/z} \qquad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

6.) $z\left\{\frac{1}{(n+1)!}\right\} = z(e^{1/z} - 1)$

Proof:

$$z\left\{\frac{1}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$$

$$= \frac{1}{1!} + \frac{1}{2!} z^{-1} + \frac{1}{3!} z^{-2} + \cdots$$

$$= z\left[\frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots\right]$$

$$= z\left[\frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right]$$

$$= z\left[1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots - 1\right]$$

$$= z\left[e^{1/z} - 1\right] \qquad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

7.) $z\{e^{an}\} = \frac{z}{z - e^a}$

Proof:

$$z\{e^{an}\} = \sum_{n=0}^{\infty} e^{an} z^{-n} = \sum_{n=0}^{\infty} (e^a z^{-1})^n$$

$$= 1 + (e^a z^{-1}) + (e^a z^{-1})^2 + (e^a z^{-1})^3 + \cdots$$

$$= (1 - e^a z^{-1})^{-1} = \left(1 - \frac{e^a}{z}\right)^{-1}$$

$$= \frac{z}{z - e^a}$$

8.) $z\left\{\frac{a^n}{n!}\right\} = e^{a/z}$

Proof:

$$z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$= 1 + \frac{(az^{-1})}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \cdots$$

$$= e^{az^{-1}} = e^{a/z} \qquad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

9.) $z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right)$

Proof:

$$z\left\{\frac{1}{n+1}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n}$$

$$= 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \cdots$$

$$= z\left[z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{4}z^{-4} + \cdots\right]$$

$$= z\left[\log\left(1-\frac{1}{z}\right)\right] = z\log\left(\frac{z}{z-1}\right)$$

10.) $z\left\{\frac{1}{n(n+1)}\right\}, \; n \geq 1$

Sol: Let $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$\because 1 = A(n+1) + Bn$

$n=0 \Rightarrow A = 1$

$n=-1 \Rightarrow B = -1$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$z\left[\left\{\frac{1}{n(n+1)}\right\}\right] = z\left\{\frac{1}{n} - \frac{1}{n+1}\right\} = z\left\{\frac{1}{n}\right\} - z\left\{\frac{1}{n+1}\right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \left[z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3} + \cdots\right] - \left[\frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \frac{1}{4} z^{-3} + \cdots\right]$$

$$= \left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right] - \left[\frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots\right]$$

$$= -\log\left(1 - \frac{1}{z}\right) - z\left[\frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots\right]$$

$$= -\log\left(\frac{z-1}{z}\right) - z\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots - \frac{1}{z}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z \log\left(\frac{z}{z-1}\right) + 1$$

$$= (1-z)\log\left(\frac{z}{z-1}\right) + 1$$

11.) Find ~~$z[n^2 + a^{n+3}]$~~ $z\{na^n\}$

Sol:

~~$z[\{n^2 + a^{n+3}\}] = z[\{n^2\}] + z[\{a^{n+3}\}]$~~

We know that $z\{n f(n)\} = -z \frac{d}{dz} f(z)$ [By property]

$$\therefore z\{na^n\} = -z \frac{d}{dz}\left[z\{a^n\}\right]$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-a}\right) = -z\left[\frac{(z-a)\cdot 1 - z \cdot 1}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2}$$

12.) Find $z\{n^2\}$

Sol:

$$z\{n \cdot n\} = -z \frac{d}{dz} z\{n\}$$

$$= -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$= -z\left[\frac{(z-1)^2\cdot 1 - z\cdot 2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}$$

13) $z\{n(n-1)\}$

Sol: $z\{n(n-1)\} = z\{n^2-n\} = z\{n^2\} - z\{n\}$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} = \frac{z[(z+1)-(z-1)]}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

14) $z\{n^2 + a^{n+3}\}$

Sol: $z\{n^2 + a^{n+3}\} = z\{n^2\} + z\{a^{n+3}\}$

$$= z\{n^2\} + z\{a^n \cdot a^3\}$$

$$= \frac{z(z+1)}{(z-1)^3} + a^3\cdot z\{a^n\}$$

$$= \frac{z(z+1)}{(z-1)^3} + \frac{a^3 z}{z-a}$$

15) $z[\{(n+1)(n+2)\}]$

Sol:

$z[\{(n+1)(n+2)\}] = z\{n^2+3n+2\}$

$$= z\{n^2\} + 3z\{n\} + 2z\{1\} = \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$$

16) $Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\}$

Sol:

Let $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$\Rightarrow 2n+3 = A(n+2) + B(n+1)$

put $n=-1 \Rightarrow A=1$, Put $n=-2 \Rightarrow B=1$

$\therefore \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$

$$Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = Z\left\{\frac{1}{n+1} + \frac{1}{n+2}\right\} = Z\left\{\frac{1}{n+1}\right\} + Z\left\{\frac{1}{n+2}\right\}$$

Now,

$$Z\left\{\frac{1}{n+2}\right\} = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{5}z^{-3} + \cdots$$

$$= z^2\left[\frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{4}z^{-4} + \cdots\right]$$

$$= z^2\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots - \frac{1}{z}\right]$$

$$= z^2\left[-\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[\log\frac{z}{z-1} - \frac{1}{z}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$$

III^{ly}

$$Z\left\{\frac{1}{n+1}\right\} = z\log\frac{z}{z-1}$$

$$\therefore Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = z\log\frac{z}{z-1} + z^2\log\frac{z}{z-1} - z$$

$$= (z^2+z)\log\left(\frac{z}{z-1}\right) - z$$

17) $z\left\{\frac{1}{n(n-1)}\right\}$

Sol:

let $\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$

$1 = A(n-1) + Bn \Rightarrow$ put $n=1 \Rightarrow B=1$

$n=0 \Rightarrow A=-1$

$\therefore \frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$

$$z\left\{\frac{1}{n(n-1)}\right\} = z\left\{-\frac{1}{n} + \frac{1}{n-1}\right\} = z\left\{\frac{1}{n-1}\right\} - z\left\{\frac{1}{n}\right\}$$

Now

$$z\left\{\frac{1}{n-1}\right\} = \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n}$$

$$= z^{-2} + \frac{1}{2} z^{-3} + \frac{1}{3} z^{-4} + \cdots$$

$$= \frac{1}{z^2} + \frac{1}{2}\cdot\frac{1}{z^3} + \frac{1}{3}\cdot\frac{1}{z^4} + \cdots$$

$$= \frac{1}{z}\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[-\log(1 - 1/z)\right] = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

18) $z\{r^n \cos n\theta\}$ and $z\{r^n \sin n\theta\}$

Sol:

We know that $z\{a^n\} = \frac{z}{z-a}$, $|z| > |a|$

put $a = re^{i\theta}$

$$\therefore z\{(re^{i\theta})^n\} = \frac{z}{z - re^{i\theta}}, \quad |z| > |r|$$

$$\Rightarrow z\{r^n(\cos n\theta + i \sin n\theta)\} = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$z\{r^n[\cos n\theta + i\sin n\theta]\} = \frac{z}{z-r(\cos\theta + i\sin\theta)} \times \frac{\{(z-r(\cos\theta)) + ir\sin\theta\}}{\{(z-r\cos\theta) + ir\sin\theta\}}$$

$$= \frac{z\{z-r\cos\theta + ir\sin\theta\}}{(z-r\cos\theta)^2 + r^2\sin^2\theta}$$

Equating real and imaginary parts, we get

$$z\{r^n\cos n\theta\} = \frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2} \quad \text{and} \quad \rightarrow (1)$$

$$z\{r^n\sin n\theta\} = \frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2} \quad \rightarrow (2)$$

Cor 1: $r=1$ in (1) and (2)

$$z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Cor 2: $\theta = \pi/2$ in above

$$z\{\cos\frac{n\pi}{2}\} = \frac{z^2}{z^2+1}, \quad z\{\sin\frac{n\pi}{2}\} = \frac{z}{z^2+1}$$

19) Find the z-transforms of $\sin^3\frac{n\pi}{4}$

Sol:

$$z\left\{\sin^3\frac{n\pi}{4}\right\} = \frac{3}{4}z\left\{\sin\frac{n\pi}{4}\right\} - \frac{1}{4}z\left\{\sin\frac{3n\pi}{4}\right\}$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

W.K.T $z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$

$$z\left\{\sin^3\frac{n\pi}{4}\right\} = \frac{3}{4}\cdot\frac{z\sin\frac{\pi}{4}}{z^2-2z\cos\frac{\pi}{4}+1} - \frac{1}{4}\,\frac{z\sin 3\pi/4}{z^2-2z\cos\frac{3\pi}{4}+1}$$

$$= \frac{3}{4}\cdot\frac{\frac{z}{\sqrt{2}}}{z^2-2z\cdot\frac{1}{\sqrt{2}}+1} - \frac{1}{4}\cdot\frac{z\cdot\frac{1}{\sqrt{2}}}{z^2+2z\cdot\frac{1}{\sqrt{2}}+1}$$

$$= \frac{3z}{4\sqrt{2}(z^2-\sqrt{2}z+1)} - \frac{z}{4\sqrt{2}(z^2-\sqrt{2}z+1)}$$

20) $z\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\}$

Sol:

$$z\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\} = z\left\{\cos\frac{n\pi}{2}\cdot\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\cdot\sin\frac{\pi}{4}\right\}$$

$$= z\left\{\cos\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\,z\left\{\cos\frac{n\pi}{2}\right\} - \frac{1}{\sqrt{2}}\,z\left\{\sin\frac{n\pi}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right\} = \frac{1}{\sqrt{2}}\left\{\frac{z(z-1)}{z^2+1}\right\}$$

1.) $z\{e^{at}\} = z\{e^{anT}\} = z\{(e^{aT})^n\} = \frac{z}{z-e^{aT}}$ $\quad\because z\{e^{an}\} = \frac{z}{z-e^{a}}$

2.) $z\{e^{-at}\} = z\{e^{-anT}\} = z\{(e^{-aT})^n\} = \frac{z}{z-e^{-aT}}$

3.) $z\{t\} = z\{nT\} = \sum_{n=0}^{\infty}(nT)z^{-n}$

$$= T\sum_{n=0}^{\infty} n z^{-n}$$

$$= T\left[-z\frac{d}{dz}z\{1\}\right] = -Tz\frac{d}{dz}\left\{\frac{z}{z-1}\right\} = \frac{Tz}{(z-1)^2}$$

4) $z\{\sin\omega t\}$ and $z\{\cos\omega t\}$

Sol:

$$z\{\sin\omega t\} = z\{\sin n(\omega T)\}$$

W.K.T $\quad z\{\sin n\theta\} = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$$\therefore z\{\sin n(\omega T)\} = \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}, \quad \text{If } |z|>1$$

$$z\{\cos\omega t\} = z\{\cos n(\omega T)\}$$

W.K.T $\quad z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$$\therefore z\{\cos n(\omega T)\} = \frac{z(z-\sin\omega T)}{z^2-2z\cos\omega T+1}, \quad \text{If } |z|>1$$

5.) $z[\cos^3 t]$

Sol: $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\Rightarrow \cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$$

$$z[\cos^3 t] = z\left\{\frac{1}{4}(\cos 3t + 3\cos t)\right\}$$

$$= \frac{1}{4} z\{\cos 3t\} + \frac{3}{4} z\{\cos t\}$$

$$= \frac{1}{4} z\{\cos 3nT\} + \frac{3}{4} z\{\cos nT\}$$

$$= \frac{1}{4} \cdot \frac{z(z-\cos 3T)}{(z^2-2z\cos 3T+1)} + \frac{3}{4} \cdot \frac{z(z-\cos T)}{(z^2-2z\cos T+1)}$$

Unit impulse function (or) Unit sample sequence $\delta(n)$

$$\delta(n) = \begin{cases} 1, & \text{If } n=0 \\ 0, & \text{If } n\neq 0 \end{cases}$$

Unit step sequence $u(n)$

$$u(n) = \begin{cases} 1, & \text{If } n\geq 0 \\ 0 & \text{If } n<0 \end{cases}$$

1.) $z[\{u(n)\}]$ and $z[\{\delta(n)\}]$

Sol:

$$z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} = \frac{z}{z-1} \quad \text{If } |z|>1$$

$$z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 + 0 + 0 + \cdots = 1$$

2.) $$z\{3^n \delta(n-1)\} = \sum_{n=0}^{\infty} 3^n \delta(n-1) z^{-n} = 3\delta(0) z^{-1} = \frac{3}{z}$$

3.) $$z\{u(n-1)\} = \sum_{n=0}^{\infty} u(n-1) z^{-n} = \sum_{n=1}^{\infty} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{1}{z-1}$$

Scanned by CamScanner