Unit V: Z Transforms

- Introduction of Z-Transforms
- Properties of Z-transforms
- Z-Transforms-problems
- Inverse Z-Transforms
  - Long division method
  - Partial fraction method
  - Residue theorem method
- Convolution Theorem
- Application of Z-transform
  (Solutions of difference equations with constant coefficients using Z-transform)

Introduction

Z-transforms play an important role in Communication Engineering and Control Systems. It is a discrete analogue of Laplace transforms. The two basic types of signals in communication Engineering are continuous time signals and discrete time signals. Continuous time signals are defined for a continuum of values of the independent variable (that is) time denoted by $[f(t)]$. Discrete time signals are defined only at a discrete set of values of the independent variable and are denoted by a sequence $\{f(n)\}$.

Laplace and Fourier transforms play an important role in the study of continuous time signals, while Z-transforms plays an important role in the study of discrete time signals.

Z-transforms can be applied to solve difference equations which occur in systems with digital filters, found in Digital Signal Processing.

Definition: Two sided Z-transform

If $\{f(n)\}$ is a sequence defined for $n = 0, \pm1, \pm2, \pm3, \ldots$ then the two sided Z-transform of $\{f(n)\}$ is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n} \tag{1}$$

where z is a complex variable in general.

Definition: One sided Z-transform

If $\{f(n)\}$ is a causal sequence (that is) defined only for $n = 0, 1, 2, \ldots$ and $f(n) = 0$ for $n < 0$ then the on sided Z-transform is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \tag{2}$$

Note:

- The infinite series on the right hand side of (1) and (2) will be converge only for certain values of z depending on the sequence $f(n)$.
- This unit deals with one sided Z-transforms which shall be referred hereafter as Z-transform.
- The inverse Z-transform of $Z\{f(n)\} = F(z)$ is defined as $Z^{-1}[F(z)] = \{f(n)\}$.

Definition: Conversion of a continuous signal to a discrete signal

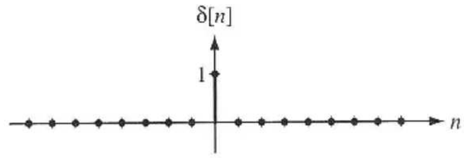
If $f(t)$ is a continuous function defines for discrete values of t where $t = nT, n = 0, 1, 2, \ldots,$

T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\left[f(t)\right] = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

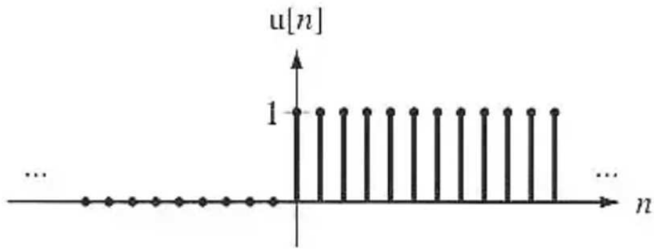
Unit impluse function (or) Unit sample sequence
The unit sample sequence $\delta(n)$ is defined as

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



Unit step sequence
The unit step sequence $u(n)$ is defined as

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



Relation between $\delta(n)$ and $u(n)$
$u(n) = \sum\limits_{n=-\infty}^{\infty} \delta(n)$ and $\delta(n) = u(n) - u(n-1)$
We have

$$\delta(n-k) = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \neq n \end{cases}$$

and

$$u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \text{ or } n \geq k \\ 0 & \text{for } (n-k) < 0 \text{ or } n < k \end{cases}$$

Also $f(n) = \sum\limits_{k=-\infty}^{\infty} f(k)\delta(n-k)$

Properties of Z-transform

Property 1: Linearity: Z-transform is linear

(i) $Z\left[a\left\{f(n)\right\} + b\left\{g(n)\right\}\right] = aZ\left\{f(n)\right\} + bZ\left\{g(n)\right\}$

(ii) $Z\left[af(t) + bg(t)\right] = aZ\left[f(t)\right] + bZ\left[g(t)\right]$

Proof:

$$
\begin{aligned}
(i) \quad Z\left[a\left\{f(n)\right\} + b\left\{g(n)\right\}\right] &= \sum_{n=0}^{\infty}\left[af(n) + bg(n)\right]z^{-n} \\
&= a\sum_{n=0}^{\infty} f(n)z^{-n} + b\sum_{n=0}^{\infty} g(n)z^{-n} \\
&= aZ\left\{f(n)\right\} + bZ\left\{g(n)\right\}
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad Z\left[af(t) + bg(t)\right] &= \sum_{n=0}^{\infty}\left[af(nT) + bg(nT)\right]z^{-n} \\
&= a\sum_{n=0}^{\infty} f(nT)z^{-n} + b\sum_{n=0}^{\infty} g(nT)z^{-n} \\
&= aZ\left[f(t)\right] + bZ\left[g(t)\right]
\end{aligned}
$$

Property 2: Frequency Shifting: (Dampling rule or scaling property)

(i) If $Z\left\{f(n)\right\} = F(z)$ then (a) $Z\left\{a^n f(n)\right\} = F\left(\frac{z}{a}\right)$ (b) $Z\left\{a^{-n} f(n)\right\} = F\left(az\right)$

(ii) If If $Z\left[f(t)\right] = F(z)$ then (a) $Z[a^n f(t)] = F\left(\frac{z}{a}\right)$ and (b) $Z[a^{-n} f(t)] = F\left(az\right)$

Proof:

$$
\begin{aligned}
(i)\ (a) \quad Z\left\{a^n f(n)\right\} &= \sum_{n=0}^{\infty} a^n f(n) z^{-n} \\
&= \sum_{n=0}^{\infty} f(n)(a^{-1}z)^{-n} \\
&= F\left(a^{-1}z\right) = F\left(\frac{z}{a}\right) \\
(b) \quad Z\left\{a^{-n} f(n)\right\} &= \sum_{n=0}^{\infty} a^{-n} f(n) z^{-n} \\
&= \sum_{n=0}^{\infty} f(n)(az)^{-n} \\
&= F\left(az\right)
\end{aligned}
$$

$$(ii)\ (a) \qquad Z\left[a^n f(t)\right] = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$$
$$= a \sum_{n=0}^{\infty} f(nT)(a^{-1}z)^{-n}$$
$$= F\left(a^{-1}z\right) = F\left(\frac{z}{a}\right)$$
$$(b) \qquad Z\left[a^{-n} f(t)\right] = \sum_{n=0}^{\infty} a^{-n} f(t) z^{-n}$$
$$= \sum_{n=0}^{\infty} f(nT)(az)^{-n}$$
$$= F\left(az\right)$$

Property 3: Differentiation in z domain or Multiplication by n

(i) If $Z\left\{f(n)\right\} = F(z)$ then $Z\left\{nf(n)\right\} = -z\dfrac{d}{dz}F(z)$

(ii) If $Z\left[f(t)\right] = F(z)$ then $Z\left[nf(t)\right] = -z\dfrac{d}{dz}F(z)$

Proof:

(i) W.K.T. $F(z) = Z\left\{f(n)\right\} = \sum\limits_{n=0}^{\infty} f(n)z^{-n}$

Differentiating both sides with respect to z,

$$\frac{d}{dz}F(z) = \frac{d}{dz}\left[\sum_{n=0}^{\infty} f(n)z^{-n}\right]$$
$$= \sum_{n=0}^{\infty} f(n)\left[-nz^{-n-1}\right]$$
$$= -z^{-1}\sum_{n=0}^{\infty} nf(n)z^{-n}$$
$$= -z^{-1} Z\left\{nf(n)\right\}$$

Therefore $Z\left\{nf(n)\right\} = -z\dfrac{d}{dz}F(z)$

(ii) W.K.T. $F(z) = Z\left\{f(t)\right\} = \sum\limits_{n=0}^{\infty} f(nT)z^{-n}$

Differentiating both sides with respect to z,

$$\begin{aligned}
\frac{d}{dz}F(z) &= \frac{d}{dz}\left[\sum_{n=0}^{\infty} f(nT)z^{-n}\right] \\
&= \sum_{n=0}^{\infty} f(nT)\left[-nz^{-n-1}\right] \\
&= -z^{-1}\sum_{n=0}^{\infty} nf(nT)z^{-n} \\
&= -z^{-1}Z\left[nf(nT)\right]
\end{aligned}$$

Therefore $Z\left[nf(t)\right] = -z\dfrac{d}{dz}F(z)$

Property 4: First shifting theorem

If $Z[f(t)] = F(z)$ then (i) $Z\left[e^{-at}f(t)\right] = F(ze^{aT})$ or $Z\left[e^{-at}f(t)\right] = \left[F(z)\right]_{z\to ze^{aT}}$

(ii) $Z\left[e^{at}f(t)\right] = F(ze^{-aT})$ or $Z\left[e^{at}f(t)\right] = \left[F(z)\right]_{z\to ze^{-aT}}$

Proof:

$$\begin{aligned}
(i)\quad Z\left[e^{-at}f(t)\right] &= \sum_{n=0}^{\infty} e^{-anT}f(nT)z^{-n} \\
&= \sum_{n=0}^{\infty} f(nT)(e^{aT}z)^{-n} \\
&= F[ze^{aT}] \text{ or } \left[F(z)\right]_{z\to ze^{aT}} \\
(ii)\quad Z\left[e^{at}f(t)\right] &= \sum_{n=0}^{\infty} e^{anT}f(nT)z^{-n} \\
&= \sum_{n=0}^{\infty} f(nT)(e^{-aT}z)^{-n} \\
&= F[ze^{-aT}] \text{ or } \left[F(z)\right]_{z\to ze^{-aT}}
\end{aligned}$$

Property 5: Time shifting theorem

(i) Shifting to the right: If $Z\left\{f(n)\right\} = F(z)$ then $Z\left\{f(n-k)\right\} = z^{-k}F(z)$ for $k>0$.

(ii) Shifting to the left:

If $Z\left\{f(n)\right\} = F(z)$ then $Z\left\{f(n+k)\right\} = z^{k}\left[F(z) - f(0) - \dfrac{f(1)}{z} - \dfrac{f(2)}{z^2} - \ldots - \dfrac{f(k-1)}{z^{k-1}}\right]$

for $k>0$.

(iii) If $Z[f(t)] = F(z)$ then

$Z\left[f(t+kT)\right] = z^{k}\left[F(z) - f(0.T) - \dfrac{f(1.T)}{z} - \dfrac{f(2.T)}{z^2} - \ldots - \dfrac{f(k-1).T}{z^{k-1}}\right]$

Proof:

(i) $Z\{f(n-k)\} = \sum\limits_{n=0}^{\infty} f(n-k)z^{-n}$

Put $m = n-k$

When $n = 0 \Rightarrow m = -k$ and $n = \infty \Rightarrow m = \infty$

$$\begin{aligned}
Z\{f(n-k)\} &= \sum_{m=-k}^{\infty} f(m)z^{-(m+k)} \\
&= z^{-k} \sum_{m=-k}^{\infty} f(m)z^{-m} \\
&= z^{-k} \sum_{m=0}^{\infty} f(m)z^{-m} \text{ since } f(m) = 0 \text{ for } m < 0 \\
&= z^{-k} F(z) \text{ if } k > 0
\end{aligned}$$

(ii) $Z\{f(n+k)\} = \sum\limits_{n=0}^{\infty} f(n+k)z^{-n}$

Put $m = n+k$

When $n = 0 \Rightarrow m = k$ and $n = \infty \Rightarrow m = \infty$

$$\begin{aligned}
Z\{f(n+k)\} &= \sum_{m=k}^{\infty} f(m)z^{-(m-k)} \\
&= z^{k} \sum_{m=k}^{\infty} f(m)z^{-m} \\
&= z^{k} \left[\sum_{m=0}^{\infty} f(m)z^{-m} - \sum_{m=0}^{k-1} f(m)z^{-m}\right] \\
&= z^{k} \left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} - \ldots - \frac{f(k-1)}{z^{k-1}}\right]
\end{aligned}$$

Corollary:

- $Z\{f(n+1)\} = z\left[F(z) - f(0)\right]$
- $Z\{f(n+2)\} = z^2\left[F(z) - f(0) - \dfrac{f(1)}{z}\right]$
- $Z\{f(n+3)\} = z^3\left[F(z) - f(0) - \dfrac{f(1)}{z} - \dfrac{f(2)}{z^2}\right]$

(iii) $Z\left[f(t+kT)\right] = Z\left\{f\left[(n+k)T\right]\right\} = \sum\limits_{n=0}^{\infty} f\left[(n+k)T\right]z^{-n}$

Put $m = n+k$

When $n = 0 \Rightarrow m = k$ and $n = \infty \Rightarrow m = \infty$

$$\begin{aligned}
Z\left\{f\left[(n+k)T\right]\right\} &= \sum_{m=k}^{\infty} f(mT) z^{-(m-k)} \\
&= z^k \sum_{m=k}^{\infty} f(mT) z^{-m} \\
&= z^k \left[\sum_{m=0}^{\infty} f(mT) z^{-m} - \sum_{m=0}^{k-1} f(mT) z^{-m}\right] \\
&= z^k \left[F(z) - f(0.T) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^2} - \ldots - \frac{f[(k-1).T]}{z^{k-1}}\right]
\end{aligned}$$

Corollary:

- $Z\left[f(t+T)\right] = z\left[F(z) - f(0)\right]$
- $Z\left[f(t+2T)\right] = z^2\left[F(z) - f(0) - \dfrac{f(1.T)}{z}\right]$
- $Z\left[f(t+3T)\right] = z^3\left[F(z) - f(0) - \dfrac{f(1.T)}{z} - \dfrac{f(2.T)}{z^2}\right]$

Note:
If $Z\left\{f\left[(n+k)T\right]\right\}$ is denoted by f_{n+k} then
$Z\left[f(t+kT)\right] = Z\left\{f_{n+k}\right\} = z^k\left[F(z) - f(0) - \dfrac{f_1}{z} - \dfrac{f_2}{z^2} - \ldots - \dfrac{f_{k-1}}{z^{k-1}}\right]$

Property 6: Second Shifting theorem
If $Z\left[f(t)\right] = F(z)$ then $Z\left[f(t+T)\right] = z\left[F(z) - f(0)\right]$
Proof:

$$\begin{aligned}
Z\left[f(t+T)\right] &= \sum_{n=0}^{\infty} f(nT+T) z^{-n} \\
&= \sum_{n=0}^{\infty} f[(n+1)T] z^{-n} \\
&= z \sum_{m=1}^{\infty} f(mT) z^{-m} \text{ put } m = n+1 \\
&= z\left[\sum_{m=0}^{\infty} f(mT) z^{-m} - f(0)\right] \\
&= z\left[F(z) - f(0)\right]
\end{aligned}$$

Property 7: Initial value theorem
(i) If $Z\{f(n)\} = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$
(ii) If $Z[f(t)] = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$
Proof:

$$\begin{aligned}
(i)\quad F(z) &= Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} \\
&= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \dots \\
\lim_{z\to\infty} F(z) &= f(0) \text{ since } \frac{1}{z^n} \to 0 \text{ as } z \to \infty \text{ for any integer } n \\
(ii)\quad F(z) &= Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \\
&= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \frac{f(3T)}{z^3} + \dots \\
\lim_{z\to\infty} F(z) &= f(0) \text{ since } \frac{1}{z^n} \to 0 \text{ as } z \to \infty \text{ for any integer } n
\end{aligned}$$

Corollary:

- $f(1) = \lim_{z\to\infty} z[F(z) - f(0)]$
- $f(2) = \lim_{z\to\infty} z^2 \left[F(z) - f(0) - \dfrac{f(1)}{z}\right]$

Property 8: Final value theorem
(i) If $Z\{f(n)\} = F(z)$ then $\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)F(z)$
(ii) If $Z[f(t)] = F(z)$ then $\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$
Proof:
(i) W.K.T. $Z\{f(n+1)\} = z[F(z) - f(0)]$
$\Rightarrow z[F(z) - f(0)] = Z\{f(n+1)\}$
Subtracting $Z\{f(n)\}$ on both sides

$$\begin{aligned}
zF(z) - zf(0) - Z\{f(n)\} &= Z\{f(n+1)\} - Z\{f(n)\} \\
(z-1)F(z) - zf(0) &= Z\{f(n+1) - f(n)\} \\
&= \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}
\end{aligned}$$

$$\begin{aligned}
\lim_{z\to 1}[(z-1)F(z)]-f(0) &= \sum_{n=0}^{\infty}[f(n+1)-f(n)] \\
&= f(1)-f(0)+f(2)-f(1)+f(3)-f(2)+\dots \\
&+ f(n+1)-f(n)+\dots+f(\infty) \\
&= f(\infty)-f(0) \\
\lim_{z\to 1}[(z-1)F(z)] &= \lim_{n\to\infty} f(n)
\end{aligned}$$

(ii) W.K.T. $Z[f(t+T)] = z[F(z)-f(0)]$
$\Rightarrow z[F(z)-f(0)] = Z[f(t+T)]$
Subtracting $Z[f(t)]$ on both sides

$$\begin{aligned}
zF(z)-zf(0)-Z[f(t)] &= Z[f(t+T)]-Z[f(t)] \\
(z-1)F(z)-zf(0) &= Z[f(t+T)-f(t)] \\
&= \sum_{n=0}^{\infty}[f(nT+T)-f(nT)]\,z^{-n} \\
\lim_{z\to 1}[(z-1)F(z)]-f(0) &= \sum_{n=0}^{\infty}[f[(n+1)T]-f(nT)] \\
&= f(T)-f(0)+f(2T)-f(T)+f(3T)-f(2T)+\dots \\
&+ f[(n+1)T]-f(nT)+\dots+f(\infty) \\
&= f(\infty)-f(0) \\
\lim_{z\to 1}[(z-1)F(z)] &= \lim_{t\to\infty} f(t)
\end{aligned}$$

Problems

1. $Z\{k\}$ where k is a constant.

   Sol: By definition $Z\{f(n)\} = \sum\limits_{n=0}^{\infty} f(n)z^{-n}$

$$\begin{aligned}
Z\{k\} &= \sum_{n=0}^{\infty} kz^{-n} \\
&= k\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right] \\
&= k\left(1-\frac{1}{z}\right)^{-1} \quad \text{if} \left|\frac{1}{z}\right|<1 \\
&= k\left(\frac{z-1}{z}\right)^{-1} = \frac{kz}{z-1}
\end{aligned}$$

Corollary: When $k=1 \Rightarrow \ Z\{1\} = \dfrac{z}{z-1}, |z|>1$

2. $Z\{a^n\} = \dfrac{z}{z-a}$ if $|z|>|a|$ or $Z\{a^n u(n)\} = \dfrac{z}{z-a}$ if $|z|>|a|$

Proof: By definition $Z\{f(n)\} = \sum\limits_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned}
Z\{a^n\} &= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \\
&= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \\
&= \left(1 - \frac{a}{z}\right)^{-1} \quad \text{if } \left|\frac{a}{z}\right| < 1 \\
&= \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}
\end{aligned}$$

Corollary: $Z\{1\} = \dfrac{z}{z-1}$ if $a=1$, $Z\{(-1)^n\} = \dfrac{z}{z+1}$ if $a=-1$.

3. $Z\{a^{n-1}\} = \dfrac{1}{z-a}$ if $|z|>|a|$

Proof: We know that $Z\{f(n-k)\} = z^{-k} Z\{f(n)\}$ by property 5

$$\begin{aligned}
Z\left\{a^{n-1}\right\} &= z^{-1}.Z\{a^n\} \\
&= z^{-1}.\left(\frac{z}{z-a}\right) \quad \text{by problem 2} \\
&= \frac{1}{z-a}
\end{aligned}$$

4. $Z\{n\} = \dfrac{z}{(z-1)^2}$ if $|z|>|1|$ or $Z\{nu(n)\} = \dfrac{z}{(z-1)^2}$ if $|z|>|1|$

$$\begin{aligned}
Z\{n\} &= \sum_{n=0}^{\infty} n z^{-n} \\
&= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots \\
&= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right) \\
&= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} \quad \text{if } \left|\frac{1}{z}\right| < 1 \\
&\text{since } (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \\
&= \frac{z}{(z-1)^2}; |z| > 1
\end{aligned}$$

5. $Z\left\{\frac{1}{n}\right\} = \log\left(\frac{z}{z-1}\right)$ if $|z| > 1$

$$\begin{aligned}
Z\left\{\frac{1}{n}\right\} &= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \\
&= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots \\
&= -\log\left(1 - \frac{1}{z}\right) \text{ if } \left|\frac{1}{z}\right| < 1 \\
&\text{since } -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \\
&= \log\left(\frac{z}{z-1}\right); |z| > 1
\end{aligned}$$

6. $Z\left\{\frac{1}{n!}\right\} = e^{1/z}$

$$\begin{aligned}
Z\left\{\frac{1}{n!}\right\} &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\
&= 1 + \frac{1}{1!.z} + \frac{1}{2!.z^2} + \frac{1}{3!.z^3} + \dots \\
&= e^{1/z}; \text{since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

7. $Z\left\{\frac{1}{(n+1)!}\right\} = z\left(e^{1/z} - 1\right)$

$$\begin{aligned}
Z\left\{\frac{1}{(n+1)!}\right\} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n} \\
&= \frac{1}{1!} + \frac{1}{2!.z} + \frac{1}{3!.z^2} + \dots \\
&= z\left[\frac{1}{1!.z} + \frac{1}{2!.z^2} + \frac{1}{3!.z^3} + \dots\right] \\
&= z(e^{1/z} - 1); \text{since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

8. $Z\left\{e^{an}\right\} = \dfrac{z}{z-e^a}$

$$\begin{aligned} Z\left\{e^{an}\right\} &= \sum_{n=0}^{\infty} e^{an} z^{-n} = \sum_{n=0}^{\infty} (e^a z^{-1})^n \\ &= 1 + (e^a z^{-1}) + (e^a z^{-1})^2 + (e^a z^{-1})^3 + \dots \\ &= \left(1 - e^a z^{-1}\right)^{-1} \\ &= \log\left(\frac{z}{z-e^a}\right) \end{aligned}$$

9. $Z\left\{\dfrac{a^n}{n!}\right\} = e^{1/z}$

$$\begin{aligned} Z\left\{\frac{a^n}{n!}\right\} &= \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} \\ &= 1 + \frac{a}{1!.z} + \frac{a^2}{2!.z^2} + \frac{a^3}{3!.z^3} + \dots \\ &= e^{a/z}; \text{since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

10. $Z\left\{\dfrac{1}{n+1}\right\} = z\log\left(\dfrac{z}{z-1}\right)$

$$\begin{aligned} Z\left\{\frac{1}{n+1}\right\} &= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} \\ &= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots \\ &= z\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots\right] \\ &= z\left[-\log\left(1 - \frac{1}{z}\right)\right] = z\log\left(\frac{z}{z-1}\right) \end{aligned}$$

11. $Z\left\{\dfrac{1}{n(n+1)}\right\}, n \geq 1$

Let $\dfrac{1}{n(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1}$

$A(n+1) + Bn = 1$

put $n = 0 \Rightarrow A = 1$ and $n = -1 \Rightarrow B = -1$

Therefore $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

$$\begin{aligned}
Z\left\{\frac{1}{n(n+1)}\right\} &= Z\left\{\frac{1}{n} - \frac{1}{n+1}\right\} \\
&= Z\left\{\frac{1}{n}\right\} - Z\left\{\frac{1}{n+1}\right\} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n} \\
&= \left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots\right] - \left[\frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots\right] \\
&= -\log\left(1 - \frac{1}{z}\right) - z\left[\frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots\right] \\
&= \log\left(\frac{z}{z-1}\right) - z\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots - \frac{1}{z}\right] \\
&= \log\left(\frac{z}{z-1}\right) - z\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right] \\
&= (1-z)\log\left(\frac{z}{z-1}\right) + 1
\end{aligned}$$

12. $Z\{na^n\}$

We know that $Z\{nf(n)\} = -z\dfrac{d}{dz}f(z)$ by property 4

$$\begin{aligned}
Z\{na^n\} &= -z\frac{d}{dz}\left[Z\{a^n\}\right] \\
&= -z\frac{d}{dz}\left(\frac{z}{z-a}\right) \\
&= \frac{az}{(z-a)^2}
\end{aligned}$$

13. $Z\{n^2\}$

$$\begin{aligned}
Z\{n.n\} &= -z\frac{d}{dz}Z\{n\} \\
&= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) \quad \text{by problem 4} \\
&= \frac{z(z+1)}{(z-1)^3}
\end{aligned}$$

14. $Z\{n(n-1)\}$

$$\begin{aligned} Z\{n(n-1)\} &= Z\left\{n^2\right\} - Z\{n\} \\ &= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} \text{ by problem 13 and 4} \\ &= \frac{2z}{(z-1)^3} \end{aligned}$$

15. $Z\{n^2 + a^{n+3}\}$

$$\begin{aligned} Z\left\{n^2 + a^{n+3}\right\} &= Z\left\{n^2\right\} + Z\left\{a^{n+3}\right\} \\ &= \frac{z(z+1)}{(z-1)^3} + a^3 Z\{a^n\} \text{ by problem 13} \\ &= \frac{z(z+1)}{(z-1)^3} + \frac{a^3 z}{z-a} \text{ by problem 2} \end{aligned}$$

16. $Z\{(n+1)(n+2)\}$

$$\begin{aligned} Z\{(n+1)(n+2)\} &= Z\left\{n^2 + 3n + 2\right\} \\ &= Z\left\{n^2\right\} + 3Z\{n\} + 2Z\{1\} \\ &= \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1} \text{ by problem 12, 4, 1} \\ & \frac{2z^3}{(z-1)^3} \end{aligned}$$

17. $Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\}$

Let $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n++2}$

$A(n+2) + B(n+1) = 2n+3$

put $n = -1 \Rightarrow A = 1$ and $n = -2 \Rightarrow B = 1$

Therefore $\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$

$$\begin{aligned} Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} &= Z\left\{\frac{1}{n+1} + \frac{1}{n+2}\right\} \\ &= Z\left\{\frac{1}{n+1}\right\} + Z\left\{\frac{1}{n+2}\right\} \\ &= z\log\left(\frac{z}{z-1}\right) + Z\left\{\frac{1}{n+2}\right\} \text{by problem 10} \end{aligned}$$

Now

$$
\begin{aligned}
Z\left\{\frac{1}{n+2}\right\} &= \sum_{n=0}^{\infty}\frac{1}{n+2}z^{-n} \\
&= 1+\frac{1}{2}+\frac{1}{3z}+\frac{1}{4z^2}+\dots \\
&= z^2\left[\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\dots\right] \\
&= z^2\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\dots-\frac{1}{z}\right] \\
&= z^2\left[-\log\left(1-\frac{1}{z}\right)-\frac{1}{z}\right] = z^2\log\left(\frac{z}{z-1}\right)-z
\end{aligned}
$$

Therefore $Z\left\{\dfrac{2n+3}{(n+1)(n+2)}\right\} = (z^2+z)\log\left(\dfrac{z}{z-1}\right)-z$

18. $Z\left\{\dfrac{1}{n(n-1)}\right\}$

Let $\dfrac{1}{n(n-1)} = \dfrac{A}{n}+\dfrac{B}{n-1}$

$A(n-1)+Bn=1$

put $n=0 \Rightarrow A=-1$ and $n=1 \Rightarrow B=1$

Therefore $\dfrac{1}{n(n-1)} = -\dfrac{1}{n}+\dfrac{1}{n-1}$

$$
\begin{aligned}
Z\left\{\frac{1}{n(n-1)}\right\} &= Z\left\{-\frac{1}{n}+\frac{1}{n-1}\right\} \\
&= Z\left\{\frac{1}{n-1}\right\}-Z\left\{\frac{1}{n}\right\} \\
&= Z\left\{\frac{1}{n-1}\right\}-\log\left(\frac{z}{z-1}\right) \text{ by problem 5}
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } Z\left\{\frac{1}{n-1}\right\} &= \sum_{n=2}^{\infty}\frac{1}{n-1}z^{-n} \\
&= \frac{1}{z^2}+\frac{1}{2z^3}+\frac{1}{3z^4}+\frac{1}{4z^5}+\dots \\
&= \frac{1}{z}\left[\frac{1}{z}++\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\dots\right] \\
&= \frac{1}{z}\left[-\log\left(1-\frac{1}{z}\right)\right] \\
&= \frac{1}{z}\log\left(\frac{z}{z-1}\right)
\end{aligned}
$$

Therefore $Z\left\{\dfrac{1}{n(n-1)}\right\} = \left(\dfrac{1-z}{z}\right)\log\left(\dfrac{z}{z-1}\right)$

19. $Z\left\{r^n \cos n\theta\right\}$ and $Z\left\{r^n \sin n\theta\right\}$
We know that $Z\left\{a^n\right\} = \dfrac{z}{z-a}$

$$\begin{aligned}
Z\left\{(re^{i\theta})^n\right\} &= \frac{z}{z-re^{i\theta}}, |z| > |r| \\
Z\left\{r^n(\cos n\theta + i\sin n\theta)\right\} &= \frac{z}{z-r(\cos\theta + i\sin\theta)} \\
&= \frac{z}{z-r(\cos\theta + i\sin\theta)} \times \frac{[z+r(\cos\theta + i\sin\theta)]}{[z+r(\cos\theta + i\sin\theta)]} \\
&= \frac{z[z+r(\cos\theta + i\sin\theta)]}{[(z-r\cos\theta)^2 + r^2\sin^2\theta)]}
\end{aligned}$$

Equating real and imaginary part, we get
$Z\left\{r^n \cos n\theta\right\} = \dfrac{z(z-r\cos\theta)}{z^2-2z\cos\theta+1}$ and $Z\left\{r^n \sin n\theta\right\} = \dfrac{z}{z^2-2z\cos\theta+1}$

Corollary

(1) Put $r=1$ in above problem, we get
$Z\left\{\cos n\theta\right\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$ and $Z\left\{\sin n\theta\right\} = \dfrac{zr}{z^2-2z\cos\theta+1}$

(2) Put $\theta = \dfrac{\pi}{2}$ in above corollary
$Z\left\{\cos\dfrac{n\pi}{2}\right\} = \dfrac{z^2}{z^2+1}$ and $Z\left\{\sin\dfrac{n\pi}{2}\right\} = \dfrac{z}{z^2+1}$

20. Find the Z - transform of $\sin^3\dfrac{n\pi}{4}$
Sol.: $Z\left\{\sin^3\dfrac{n\pi}{4}\right\} = \dfrac{3}{4}Z\left\{\sin\dfrac{n\pi}{4}\right\} - \dfrac{1}{4}Z\left\{\sin\dfrac{3n\pi}{4}\right\}$
We know that $Z\left\{\sin n\theta\right\} = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$$\begin{aligned}
Z\left\{\sin n\theta\right\} &= \frac{3}{4}.\frac{z\sin\frac{\pi}{4}}{z^2-2z\cos\frac{\pi}{4}+1} - \frac{1}{4}.\frac{z\sin\frac{3\pi}{4}}{z^2-2z\cos\frac{3\pi}{4}+1} \\
&= \frac{3}{4}.\frac{\frac{z}{\sqrt{2}}}{4\sqrt{2}(z^2-\sqrt{2}z+1)} - \frac{1}{4}.\frac{\frac{z}{\sqrt{2}}}{4\sqrt{2}(z^2+\sqrt{2}z+1)} \\
&= \frac{3z}{4\sqrt{2}(z^2-\sqrt{2}z+1)} - \frac{z}{4\sqrt{2}(z^2-\sqrt{2}z+1)}
\end{aligned}$$

21. $Z\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\}$

$$\begin{aligned}
Z\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\} &= Z\left\{\cos\frac{n\pi}{2}\cos\frac{\pi}{2}-\sin\frac{n\pi}{2}\sin\frac{\pi}{2}\right\}\\
&= Z\left\{\cos\frac{n\pi}{2}.\frac{1}{\sqrt{2}}-\sin\frac{n\pi}{2}.\frac{1}{\sqrt{2}}\right\}\\
&= \frac{1}{\sqrt{2}}\left[Z\left\{\cos\frac{n\pi}{2}\right\}-Z\left\{\sin\frac{n\pi}{2}\right\}\right]\\
&= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right]=\frac{1}{\sqrt{2}}\frac{z(z-1)}{z^2+1}
\end{aligned}$$

22. $Z\{u(n)\}$

$$\begin{aligned}
Z[\{u(n)\}] &= \sum_{n=0}^{\infty} u(n)z^{-n}\\
&= 1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\\
&= \left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z-1}\text{ if }|z|>1
\end{aligned}$$

23. $Z\{\delta(n)\}$
$Z[\{\delta(n)\}]=\sum\limits_{n=0}^{\infty}\delta(n)z^{-n}=1.z^0=1$

24. $Z\{3^n\delta(n-1)\}=\sum\limits_{n=1}^{\infty}3^n\delta(n-1)z^{-n}=\frac{3}{z}$

25. $Z\{u(n-1)\}=\sum\limits_{n=1}^{\infty}u(n-1)z^{-n}=\frac{1}{z-1}$

26. $Z\{e^{at}\}=Z\left\{e^{anT}\right\}=Z\left\{(e^{aT})^n\right\}=\frac{z}{z-e^{aT}}$

27. $Z\{e^{-at}\}=Z\left\{e^{-anT}\right\}=Z\left\{(e^{-aT})^n\right\}=\frac{z}{z-e^{-aT}}$

28. $Z\{t\}$

$$\begin{aligned}
Z\{t\}=Z\{nT\} &= \sum_{n=0}^{\infty}(nT)z^{-n}\\
&= T\sum_{n=0}^{\infty}(n)z^{-n}=T\left[-z\frac{d}{dz}Z\{1\}\right]\\
&= -Tz\frac{d}{dz}\left(\frac{z}{z-1}\right)=\frac{Tz}{(z-1)^2}
\end{aligned}$$

29. $Z\{\sin \omega t\}$ and $Z\{\cos \omega t\}$

$Z\{\sin \omega t\} = Z\{\sin n(\omega T)\}$

W.K.T. $Z\{\sin n\theta\} = \dfrac{z}{z^2 - 2z\cos\theta + 1}$

Therefore $Z\{\sin n(\omega T)\} = \dfrac{z}{z^2 - 2z\cos\omega T + 1}$ if $|z| > 1$

$Z\{\cos \omega t\} = Z\{\cos n(\omega T)\}$

W.K.T. $Z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

Therefore $Z\{\cos n(\omega T)\} = \dfrac{z(z-\sin\omega T)}{z^2 - 2z\cos\omega T + 1}$ if $|z| > 1$

30. $Z\{\cos^3 t\}$

$$\begin{aligned}
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\
Z\left\{\cos^3 t\right\} &= Z\left\{\cos^3 nT\right\} \\
&= Z\left\{\frac{1}{4}\left(\cos 3nT + 3\cos nT\right)\right\} \\
&= \frac{1}{4}Z\{\cos 3nT\} + \frac{3}{4}Z\{\cos nT\} \\
&= \frac{1}{4}.\frac{z(z-\cos 3T)}{(z^2 - 2z\cos 3T + 1)} + \frac{3}{4}.\frac{z(z-\cos T)}{(z^2 - 2z\cos T + 1)}
\end{aligned}$$

31. $Z\left[e^{-at}\cos bt\right]$

$$\begin{aligned}
Z\left[e^{-at}\cos bt\right] &= Z\left[\cos bt\right]_{z\to ze^{aT}} \text{ by property 4} \\
&= \left(\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right)_{z\to ze^{aT}} \\
&= \frac{ze^{aT}[ze^{aT} - \cos bT]}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}
\end{aligned}$$

32. $Z\left[e^{-at}\sin bt\right]$

$$\begin{aligned}
Z\left[e^{-at}\sin bt\right] &= Z\left[\sin bt\right]_{z\to ze^{aT}} \text{ by property 4} \\
&= \left(\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right)_{z\to ze^{aT}} \\
&= \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}
\end{aligned}$$

33. Find $Z[te^{at}]$

$$\begin{aligned} Z[te^{at}] &= (Z[t])_{z\to ze^{-aT}} \\ &= \left(\frac{Tz}{(z-1)^2}\right)_{z\to ze^{-aT}} \\ &= \frac{Tze^{-aT}}{(ze^{-aT}-1)^2} \end{aligned}$$

34. Find $Z[t^2e^t]$

$$\begin{aligned} Z[t^2e^t] &= \left(Z[t^2]\right)_{z\to ze^{T}} \\ \text{Now}\quad Z[t^2] &= Z[(nT)^2] = T^2 Z\left\{n^2\right\} \\ Z[t^2e^t] &= \left(\frac{T^2z(z+1)}{(z-1)^3}\right)_{z\to ze^{T}} \\ &= \frac{Tze^{T}(ze^{T}+1)}{(ze^{T}-1)^3} \end{aligned}$$

35. $Z\left[e^t\sin 2t\right]$

$$\begin{aligned} Z\left[e^t\sin 2t\right] &= Z\left[\sin 2t\right]_{z\to ze^{-T}} \\ &= \left(\frac{z\sin 2T}{z^2-2z\cos 2T+1}\right)_{z\to ze^{-T}} \\ &= \frac{ze^{-T}\sin 2T}{z^2e^{-2T}-2ze^{-T}\cos 2T+1} \end{aligned}$$

36. $Z\left[e^{3t}\cos 3t\right]$

$$\begin{aligned} Z\left[e^{3t}\cos 3t\right] &= Z\left[\cos t\right]_{z\to ze^{-3T}} \\ &= \left(\frac{z(z-\cos T)}{z^2-2z\cos T+1}\right)_{z\to ze^{-3T}} \\ &= \frac{ze^{-3T}[ze^{-3T}-\cos T]}{z^2e^{-6T}-2ze^{-3T}\cos T+1} \end{aligned}$$

37. $Z[e^{3t+7}]$

$$Z[e^{3t+7}] = e^7 Z(e^{3t}) = \frac{ze^7}{z-e^{3T}}$$

38. $Z[e^{-2t}t^3]$

$$Z[e^{-2t}t^3] = [Z(t^3)]_{z\to ze^{2T}}$$

$$\begin{aligned} Z(t^3) &= Z\left\{n^3T^3\right\} = T^3 Z\left\{n^3\right\} \\ &= T^3 Z\left\{n.n^2\right\} \\ &= T^3\left[-z\frac{z(z+1)}{(z-1)^3}\right] \\ &= T^3\left[\frac{z(z^2+4z+1)}{()^4}\right] \\ \left[Z(t^3)\right]_{z\to ze^{2T}} &= \frac{T^3 ze^{2T}[z^2e^{4T}+4ze^{2T}+1]}{(ze^{2T}-1)^4} \end{aligned}$$

39. Find the Z-transform of (i) $e^{2(t+T)}$ (ii) $\sin(t+T)$ and (iii) $(t+T)e^{-(t+T)}$

$$\begin{aligned} (i)\quad Z\left[e^{2(t+T)}\right] &= Z\left[f(t+T)\right] \text{ where } f(t) = e^{2t} \\ &= z\left[F(z) - f(0)\right] \\ &= z\left[\frac{z}{z-e^{2T}} - 1\right] \\ &= \frac{ze^{2T}}{z-e^{2T}} \\ (ii)\quad Z\left[\sin(t+T)\right] &= Z\left[f(t+T)\right] \text{ where } f(t) = \sin t \\ &= z\left[F(z) - f(0)\right] \\ &= z\left[\frac{z\sin T}{z^2-2z\cos T+1} - 0\right] \\ &= \frac{z^2\sin T}{z^2-2z\cos T+1} \\ (iii)\quad Z\left[(t+T)e^{-(t+T)}\right] &= Z\left[f(t+T)\right] \text{ where } f(t) = te^{-t} \\ &= z\left[F(z) - f(0)\right] \\ &= z\left[\frac{Tze^{T}}{(ze^{T}-1)^2} - 0\right] \\ &= \frac{Tz^2e^{T}}{(ze^{T}-1)^2} \end{aligned}$$

40. Use final value theorem to find $f(\infty)$ where $F(z) = \dfrac{Tze^{aT}}{(ze^{aT}-1)^2}$

$$\begin{aligned} f(\infty) &= \lim_{z\to 1}(z-1)F(z) \text{ by final value theorem} \\ &= \lim_{z\to 1}\left[(z-1).\frac{Tze^{aT}}{(ze^{aT}-1)^2}\right] = 0 \end{aligned}$$

41. Use initial value theorem to find $f(0)$ when $F(z) = \dfrac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$

$$
\begin{aligned}
f(0) &= \lim_{z\to\infty} F(z) \text{ by initial value theorem} \\
&= \lim_{z\to\infty}\left[\frac{z^2e^{aT}\left(e^{aT} - \frac{1}{z}\cos bT\right)}{z^2\left(e^{2aT} - \frac{2}{z}e^{aT}\cos bT + \frac{1}{z^2}\right)}\right] \\
&= \frac{e^{aT}.e^{aT}}{e^{2aT}} = 1
\end{aligned}
$$

42. Verify initial value theorem for $f(n) = \dfrac{2^{n+1}}{n!}$

Initial value theorem: $f(0) = \lim_{z\to\infty} F(z)$

$$
\begin{aligned}
\text{L.H.S } f(0) &= \frac{2^{0+1}}{0!} = 2 \\
\text{R.H.S } F(z) &= Z\{f(n)\} = Z\left\{\frac{2^{n+1}}{n!}\right\} \\
&= 2.Z\left\{\frac{2^n}{n!}\right\} = 2e^{2/z} \text{ since } Z\left\{\frac{a^n}{n!}\right\} = e^{a/z}
\end{aligned}
$$

$$\lim_{z\to\infty} F(z) = 2$$

Therefore L.H.S=R.H.S.
Hence the initial value theorem is verified.

43. Verify initial value theorem for $f(t) = t^2$
Initial value theorem: $f(0) = \lim_{z\to\infty} Z[f(t)]$
Given $f(t) = t^2$

$$\text{L.H.S } f(0) = 0$$

$$\text{R.H.S } Z[f(t)] = Z\left[t^2\right] = \frac{T^2z(z+1)}{(z-1)^3}$$

$$\lim_{z\to\infty} Z[f(t)] = \lim_{z\to\infty}\left[\frac{T^2z^2\left(1+\frac{1}{z}\right)}{z^3\left(1-\frac{1}{z}\right)^3}\right] = 0$$

Therefore L.H.S=R.H.S.
Hence the initial value theorem is verified.

44. Verify final value theorem for $f(t) = e^{-at}\cos bt$
    By final value theorem: $f(\infty) = \lim_{z\to 1}(z-1)Z\left[f(t)\right]$
    Given $f(t) = e^{-at}\cos bt$

$$\begin{aligned}
\text{L.H.S } f(\infty) &= 0 \\
\text{R.H.S } Z\left[f(t)\right] &= Z\left[e^{-at}\cos bt\right] \\
&= \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1} \\
\lim_{z\to 1}(z-1)Z\left[f(t)\right] &= \lim_{z\to 1}\left[\frac{(z-1)ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}\right] = 0
\end{aligned}$$

Therefore L.H.S=R.H.S.
Hence the final value theorem is verified.

Inverse Z-transform

As $Z\left\{f(n)\right\} = F(z)$, the inverse Z-transform of $F(z)$ is defined as

$$Z^{-1}[F(z)] = \{f(n)\}$$

Examples: (i) $Z\left\{a^n\right\} = \dfrac{z}{z-a} \Rightarrow Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$
(ii) $Z\left\{n\right\} = \dfrac{z}{(z-1)^2} \Rightarrow Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$

Methods to find $\{f(n)\}$ given $F(z)$

1. Long division method
2. Partial fraction method
3. Residue method or Inverse integral method
4. Convolution method

1. Long division method
Since Z-transform is defined by the series $F(z) = \sum\limits_{n=0}^{\infty} f(n)z^{-n}$, to find the inverse Z-transform of $F(z)$, expand $F(z)$ in the proper power series and collect the coefficient of z^{-n} to get $f(n)$.

1. Find the inverse Z-transform of $\dfrac{1+2z^{-1}}{1-z^{-1}}$ by long division method

   Let $F(z) = \dfrac{1+2z^{-1}}{1-z^{-1}}$
   By actual division,

$$
\begin{array}{r|lllll}
 & 1 & +3z^{-1} & +3z^{-2} & +3z^{-2} & +\dots \\
\hline
1-z^{-1} & 1 & +2z^{-1} & & & \\
 & 1 & -z^{-1} & & & \\
\hline
 & & 3z^{-1} & & & \\
 & & 3z^{-1} & -3z^{-2} & & \\
\hline
 & & & 3z^{-2} & & \\
 & & & 3z^{-2} & -3z^{-3} & \\
\hline
 & & & & 3z^{-3} &
\end{array}
$$

   Therefore $F(z) = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$
   $\Rightarrow \sum\limits_{n=0}^{\infty} f(n) z^{-n} = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$
   $\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$
   Equating the like terms, we get
   $f(0) = 1, f(1) = 3, f(2) = 3, f(3) = 3, \dots$

$$\text{Hence } f(n) = \begin{cases} 1, & n = 0 \\ 3, & n \geq 1 \end{cases}$$

2. Find $Z^{-1}\left\{\dfrac{z^2+z}{(z-1)^3}\right\}$ by long division

$$
\begin{aligned}
\text{Let } F(z) &= \frac{z^2+z}{(z-1)^3} \\
&= \frac{z^2+z}{z^3-3z^2+3z-1} \\
&= \frac{z^{-1}+z^{-2}}{1-3z^{-1}+3z^{-2}-z^{-3}}
\end{aligned}
$$

   By actual division,

$$
\begin{array}{r|llllll}
 & z^{-1} & +4z^{-2} & +9z^{-3} & +16z^{-4} & +\ldots & \\
\hline
1-3z^{-1}+3z^{-2}-z^{-3} & z^{-1} & +z^{-2} & & & & \\
 & z^{-1} & -3z^{-2} & +3z^{-3} & -z^{-4} & & \\
\hline
 & & 4z^{-2} & -3z^{-3} & +z^{-4} & & \\
 & & 4z^{-2} & -12z^{-3} & +12z^{-4} & -4z^{-5} & \\
\hline
 & & & 9z^{-3} & -11z^{-4} & +4z^{-5} & \\
 & & & 9z^{-3} & -27z^{-4} & +27z^{-5} & -9z^{-6} \\
\hline
 & & & & 16z^{-4} & -23z^{-5} & +9z^{-6} \\
 & & & & 16z^{-4} & -48z^{-5} & +48z^{-6} \quad -16z^{-7} \\
\hline
 & & & & & 25z^{-5} & -39z^{-6} \quad +16z^{-7}
\end{array}
$$

Therefore $F(z) = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \ldots$

$\Rightarrow \sum\limits_{n=0}^{\infty} f(n)z^{-n} = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \ldots$

$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \ldots = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \ldots$

Equating the like terms, we get

$f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 9, f(4) = 16, \ldots$

$$\text{Hence } f(n) = \begin{cases} 0, & n = 0 \\ n^2, & n \geq 1 \end{cases}$$

3. Find $Z^{-1}\left\{\dfrac{1}{1+4z^{-2}}\right\}$ by the long division method

Let $F(z) = \dfrac{1}{1+4z^{-2}}$ By actual division,

$$
\begin{array}{r|llllll}
 & 1 & -4z^{-2} & +16z^{-4} & -64z^{-6} & +256z^{-8} & +\ldots \\
\hline
1+4z^{-2} & 1 & & & & & \\
 & 1 & +4z^{-2} & & & & \\
\hline
 & & -4z^{-2} & & & & \\
 & & -4z^{-2} & -16z^{-4} & & & \\
\hline
 & & & 16z^{-4} & & & \\
 & & & 16z^{-4} & +64z^{-6} & & \\
\hline
 & & & & -64z^{-6} & & \\
 & & & & -64z^{-6} & -256z^{-8} & \\
\hline
 & & & & & 256z^{-8} &
\end{array}
$$

Therefore $F(z) = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \ldots$

$\Rightarrow \sum\limits_{n=0}^{\infty} f(n)z^{-n} = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \ldots$

$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \ldots = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \ldots$

Equating the like terms, we get

$f(0) = 1, f(1) = 0, f(2) = -4, f(3) = 0, f(4) = 16, f(5) = 0, f(6) = -6, \ldots$

Hence $f(n) = 2^n \cos \dfrac{n\pi}{2}$

4. Find the inverse Z-transform of $\dfrac{10z}{(z-1)(z-2)}$ by long division method

Let $F(z) = \dfrac{10z}{(z-1)(z-2)} = \dfrac{10z^{-1}}{1 - 3z^{-1} + 2z^{-2}}$

By actual division

$$\begin{array}{r|lllll}
 & 10z^{-1} & +30z^{-2} & +70z^{-3} & +150z^{-4} & + \ldots \\ \hline
1 - 3z^{-1} + 2z^{-2} & 10z^{-1} & & & & \\
 & 10z^{-1} & -30z^{-2} & +20z^{-3} & & \\ \hline
 & & 30z^{-2} & -20z^{-3} & & \\
 & & 30z^{-2} & -90z^{-3} & +60z^{-4} & \\ \hline
 & & & 70z^{-3} & -60z^{-4} & \\
 & & & 70z^{-3} & -210z^{-4} & +40z^{-5} \\ \hline
 & & & & 150z^{-4} & -140z^{-5}
\end{array}$$

Therefore $F(z) = 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \ldots$

$\Rightarrow \sum_{n=0}^{\infty} f(n) z^{-n} = 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \ldots$

$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \ldots = 10z^{-1} + 30z^{-2} + 70z^{-3} + \ldots$

Equating the like terms, we get

$f(0) = 0, f(1) = 10, f(2) = 30, f(3) = 70, \ldots$

Hence $f(n) = 10(2^n - 1), n \geq 0$

5. Find $Z^{-1}\left\{\dfrac{z^2 + 2z}{z^2 + 2z + 4}\right\}$ by long division

$$\begin{aligned}
\text{Let } F(z) &= \frac{z^2 + 2z}{z^2 + 2z + 4} \\
&= \frac{1 + \frac{2}{z}}{1 + \frac{2}{z} + \frac{4}{z^2}} \\
&= \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}}
\end{aligned}$$

By actual division,

$$
\begin{array}{r|lllll}
 & 1 & -4z^{-2} & +8z^{-3} & -32z^{-5} & +\ldots \\
\hline
1+2z^{-1}+4z^{-2} & 1 & +2z^{-1} & & & \\
 & 1 & +2z^{-1} & +4z^{-2} & & \\
\hline
 & & -4z^{-2} & & & \\
 & & -4z^{-2} & -8z^{-3} & -16z^{-4} & \\
\hline
 & & & 8z^{-3} & +16z^{-4} & \\
 & & & 8z^{-3} & +16z^{-4} & +32z^{-5} \\
\hline
 & & & & & -32z^{-5}
\end{array}
$$

Therefore $F(z) = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \ldots$

$\Rightarrow \sum\limits_{n=0}^{\infty} f(n)z^{-n} = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \ldots$

$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \ldots = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \ldots$

Equating the like terms, we get

$f(0) = 1, f(1) = 0, f(2) = -4, f(3) = 8, f(4) = 0, f(5) = -32, \ldots$

Therefore the sequence is 1, 0, -4, 8, 0, -32, $\ldots$

2. Partial fraction method

Step 1: When $F(z)$ is a rational function in which the denominator can be factorized, resolve $F(z)$ into partial fractions.

Step 2: $Z^{-1}[F(z)]$ is the sum of the inverse Z-transforms of the partial fractions declare the result.

Note: (i) The degree of z in the numerator should be atleast one less than the degree of z in the denominator of $F(z)$

(ii) Wherever possible rewrite the given functions as $\dfrac{F(z)}{z}$ and apply the above steps.

1. Find the inverse Z-transform of (i) $\dfrac{z}{z^2+7z+10}$ (ii) $\dfrac{z^2+z}{(z-1)(z^2+1)}$ and (iii) $\dfrac{z}{(z-1)^2(z+1)}$

   (i) Given $F(z) = \dfrac{z}{z^2+7z+10}$

$$
\begin{aligned}
\Rightarrow \frac{F(z)}{z} &= \frac{1}{z^2+7z+10} \\
&= \frac{1}{(z+2)(z+5)} \\
\text{Now } \frac{1}{(z+2)(z+5)} &= \frac{A}{z+2} + \frac{B}{z+5} \\
1 &= A(z+5) + B(z+2)
\end{aligned}
$$

put $z=-2 \Rightarrow A=\frac{1}{3}$ and $z=-5 \Rightarrow B=\frac{-1}{3}$

$$\text{Therefore } \frac{F(z)}{z} = \frac{1/3}{z+2} + \frac{-1/3}{z+5}$$
$$F(z) = \frac{1}{3}\left[\frac{z}{z+2} - \frac{z}{z+5}\right]$$

Taking inverse on both sides, we have

$$Z^{-1}[F(z)] = f(n) = \frac{1}{3}\left[Z^{-1}\left(\frac{z}{z+2}\right) - Z^{-1}\left(\frac{z}{z+5}\right)\right]$$
$$= \frac{1}{3}[(-2)^n - (-5)^n],\ n=0,1,2,....$$

(ii) Given $F(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

$$\Rightarrow \frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{(Bz+C)}{z^2+1}$$
$$\Rightarrow z+1 = A(z^2+1) + (Bz+c)(z-1)$$

Put $z=1 \Rightarrow A=1$

Equating the coefficients of z^2 and constant term, we have

$$0 = A+B \tag{3}$$
$$1 = A-C \tag{4}$$

$A=1$ in equations (3) and (4) we get $B=-1,\ C=0$

$$\text{Therefore } \frac{F(z)}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$$
$$\Rightarrow F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

Taking inverse on both sides, we have

$$Z^{-1}[F(z)] = f(n) = Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z^2}{z^2+1}\right)$$
$$= 1^n - \cos\frac{n\pi}{2}$$

(iii) Let $F(z) = \frac{z}{(z-1)^2(z+1)}$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z-1)^2(z+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1}$$
$$\Rightarrow 1 = A(z-1)(z+1) + B(z+1) + C(z-1)^2$$

Put $z=1 \Rightarrow B=\frac{1}{2}$, $z=-1 \Rightarrow C=\frac{1}{4}$ and $z=0 \Rightarrow A=-\frac{1}{4}$

$$\text{Therefore } \frac{F(z)}{z}=\frac{-1/4}{z-1}+\frac{1/2}{(z-1)^2}+\frac{1/4}{z+1}$$
$$\Rightarrow F(z)=-\frac{1}{4}.\frac{z}{z-1}+\frac{1}{2}.\frac{z}{(z-1)^2}+\frac{1}{4}.\frac{z}{z+1}$$

Taking inverse on both sides, we get
$f(n)=-\frac{1}{4}.(1)^n+\frac{1}{2}.n+\frac{1}{4}(-1)^n$

2. Find $Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\}$ by the partial fraction method

Consider $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}=\frac{A}{(z-2)}+\frac{B}{(z-3)}+\frac{C}{(z-4)}$

$\Rightarrow 3z^2-18z+26=A(z-3)(z-4)+B(z-2)(z-4)+C(z-2)(z-3)$
Put $z=2 \Rightarrow A=1, z=3 \Rightarrow B=1$ and $z=4 \Rightarrow C=1$

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}=\frac{1}{(z-2)}+\frac{1}{(z-3)}+\frac{1}{(z-4)}$$
Taking inverse on both sides, we have

$$\begin{aligned} Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\} &= Z^{-1}\left(\frac{1}{z-2}\right)+Z^{-1}\left(\frac{1}{z-3}\right)+Z^{-1}\left(\frac{1}{z-4}\right) \\ &= 2^{n-1}+3^{n-1}+4^{n-1} \end{aligned}$$

3. Find $Z^{-1}\left\{\frac{4z^3}{(2z-1)^2(z-1)}\right\}$ by the method of partial fraction

Let $F(z)=\frac{4z^3}{(2z-1)^2(z-1)}$

$$\Rightarrow \frac{F(z)}{z}=\frac{4z^2}{(2z-1)^2(z-1)}=\frac{A}{(2z-1)}+\frac{B}{(2z-1)^2}+\frac{C}{*z-1)}$$
$$\Rightarrow 4z^2=A(2z-1)(z-1)+B(z-1)+C(2z-1)^2$$

Put $z=1 \Rightarrow C=4$, $z=\frac{1}{2} \Rightarrow B=-2$ and $z=0 \Rightarrow A=-6$

$$\text{Therefore } \frac{F(z)}{z}=\frac{-6}{2z-1}+\frac{(-2)}{(2z-1)^2}+\frac{4}{(z-1)}$$
$$\Rightarrow F(z)=\frac{-6}{2}\frac{z}{z-\frac{1}{2}}-\frac{-2}{4}\frac{z}{\left(z-\frac{1}{2}\right)^2}+4\frac{z}{z-1}$$

Taking inverse on both sides, we have

$$Z^{-1}[F(z)] = -3Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{-1}{2}Z^{-1}\left(\frac{z}{\left(z-\frac{1}{2}\right)^2}\right) + 4Z^{-1}\left(\frac{z}{z-1}\right)$$

$$f(n) = -3\left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n + 4(1)$$

$$= 4 - (n+3)\left(\frac{1}{2}\right)^n$$

4. Find $Z^{-1}\left\{\frac{z^2}{(z+2)(z^2+4)}\right\}$ by method of partial fraction.

Let $F(z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\Rightarrow \frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{(3z+C)}{(z^2+4)}$$

$$\Rightarrow z = A(z^2+4) + (Bz+4) + (Bz+C)(z+2)$$

Put $z=-2 \Rightarrow A = -\frac{1}{4}$

Equating the coefficient of z^2 , constant term, we get

$$0 = A + B \tag{5}$$

$$0 = 4A + 2C \tag{6}$$

Sub. $A = \frac{1}{4}$ in equations (5) and (6) we have $B = \frac{1}{4}$, C= $\frac{1}{2}$

$$\text{Therefore } \frac{F(z)}{z} = \frac{-1/4}{z+2} + \frac{(1/4z+1/2)}{z^2+4}$$

$$\Rightarrow F(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+1} + \frac{1}{4}\frac{2z}{z^2+4}$$

$$Z^{-1}[F(z)] = -\frac{1}{4}Z^{-1}\left(\frac{z}{z+2}\right) + \frac{1}{4}Z^{-1}\left(\frac{z^2}{z^2+1}\right) + \frac{1}{4}Z^{-1}\left(\frac{2z}{z^2+4}\right)$$

$$f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4}.2^n\cos\frac{n\pi}{2} + \frac{1}{2}.2^n\sin\frac{n\pi}{2}$$

5. Find $Z^{-1}\left\{\dfrac{4-8z^{-1}+6z^{-2}}{(1+z^{-1})(1-2z^{-1})^2}\right\}$ by method of partial fraction.

$$\text{Let } F(z) = \frac{4-8z^{-1}+6z^{-2}}{(1+z^{-1})(1-2z^{-1})^2}$$
$$= \frac{4-\frac{8}{z}+\frac{6}{z^2}}{\left(1+\frac{1}{z}\right)\left(1-\frac{2}{z}\right)^2}$$
$$= \frac{\left(\frac{1}{z^2}\right)(4z^2-8z+6)}{\left(\frac{1}{z^3}\right)(z+1)(z-2)^2}$$
$$= \frac{4z^3-8z^2+6z}{(z+1)(z-2)^2}$$
$$\Rightarrow \frac{F(z)}{z} = \frac{4z^2-8z+6}{(z+1)(z-2)^2} = \frac{A}{z+1}+\frac{B}{z-2}+\frac{C}{(z-2)^2}$$
$$\Rightarrow 4z^2-8z+6 = A(z-2)^2+B(z-2)(z+1)+C(z+1)$$

Put $z=2 \Rightarrow C=2,\ \ z=-1 \Rightarrow A=2$
Equating the Coefficient of z^2, we have
$4 = A+B \Rightarrow B=2$

$$\text{Therefore } \frac{F(z)}{z} = \frac{2}{z+1}+\frac{2}{z-2}+\frac{2}{(z-2)^2}$$
$$\Rightarrow F(z) = 2\left[\frac{z}{z+1}+\frac{z}{z-2}+\frac{z}{(z-2)^2}\right]$$
$$Z^{-1}[F(z)] = 2Z^{-1}\left(\frac{z}{(z+1)}\right)+2Z^{-1}\left(\frac{z}{(z-2)}\right)+2Z^{-1}\left(\frac{z}{(z-2)^2}\right)$$
$$= 2(-1)^n+2.(2)^n+n.2^n$$

3. Residue method or Inverse integral method

By using the relation between the Z-transform and Fourier transform of a sequence, if can be proved that $f(n) = \dfrac{1}{2\pi i}\displaystyle\int_C F(z)z^{n-1}\,dz$ where C is a circle whose centre is the origin and radius is sufficiently large to include all the isolated singularities of $F(z)$, C may also be a closed contour including the origin and the isolated singularities of $F(z)$.
By Cauchy's residue theorem
$\displaystyle\int_C F(z)z^{n-1}\,dz = 2\pi i\times$ sum of the residues of $F(z)z^{n-1}$ at the isolated singularities.

Therefore $f(n)$ =Sum of the residues of $F(z)z^{n-1}$ at the isolated singularities.

Calculation of residue:

(i) When $z = a$ is a simple pole or a pole of order one, then the residue is given by $\text{Res}\,[F(z), z = a] = \lim_{z\to a}(z-a)F(z)$

(ii) When $z = a$ is a pole of order m, then the residue is given by

$\text{Res}\,[F(z), z = a] = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-a)F(z)\right]$

1. Find $Z^{-1}\left\{\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right\}$ by using Residue theorem

   Let $F(z) = \dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}$

   $\Rightarrow F(z)z^{n-1} = \dfrac{z^n(z^2-z+2)}{(z+1)(z-1)^2}$

   The poles are given by $z = -1, z = 1$

   $z = -1$ is a simple pole and $z = 1$ is a pole of order 2

$$\begin{aligned}
R_1 &= \text{Residue at } z = -1\\
&= \lim_{z\to -1}(z+1).\frac{z^n(z^2-z+2)}{(z+1)(z-1)^2}\\
&= \lim_{z\to -1}\frac{z^n(z^2-z+2)}{(z-1)^2} = (-1)^n\\
R_2 &= \text{Residue at } z = 1\\
&= \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2.\frac{(z+1)z^n(z^2-z+2)}{(z+1)(z-1)^2}\right]\\
&= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^n(z^2-z+2)}{(z+1)}\right]\\
&= \lim_{z\to 1}\left\{\frac{(z+1)[nz^{n-1}(z^2-z+2)+z^n(2z-1)]-z^n(z^2-z+2).1}{(z+1)^2}\right\}\\
&= \frac{2[n.1^{n-1}(1-1+2)+1^n(2-1)]-1^n(1-1+2)}{4}\\
&= \frac{2(2n+2)-2}{4} = n
\end{aligned}$$

   Therefore $f(n)$ =sum of the residues of $F(z)z^{n-1}$ at poles

   $\Rightarrow f(n) = R_1 + R_2 = (-1)^n + n$

2. Find $Z^{-1}\left\{\dfrac{2z^2+4z}{(z-2)^3}\right\}$ by using Residue theorem

   Let $F(z) = \dfrac{2z^2+4z}{(z-2)^3}$

   $\Rightarrow F(z)z^{n-1} = \dfrac{2z^{n+1}+4z^n}{(z-2)^3}$

   The poles are $z = 2$ is pole of order 3

$$\begin{aligned}
R &= \text{Residue at } z = 3 \\
&= \frac{1}{(3-1)!}\lim_{z\to 2}\frac{d^2}{dz^2}\left[(z-2)^3.\frac{[2z^{n+1}+4z^n]}{(z-2)^3}\right] \\
&= \frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}[2z^{n+1}+4z^n] \\
&= \frac{1}{2}\lim_{z\to 2}\frac{d}{dz}[2(n+1)z^n+4nz^{n-1}] \\
&= \frac{1}{2}\lim_{z\to 2}[2(n+1)nz^{n-1}+4n(n-1)z^{n-2}] \\
&= \frac{1}{2}[2.(n+1).n2^{n-1}+4n(n-1)2^{n-2}] \\
&= \frac{1}{2}[(n+1).n.2^n+n(n-1)2^n] \\
&= \frac{1}{2}.n2^n[n+1+n-1] \\
&= n2^n.
\end{aligned}$$

   Hence $f(n) =$ sum of the Residue of $F(z)z^{n-1}$ at poles inside $C = n2^n$

3. Find the inverse z-transform of $\dfrac{z}{(z-1)(z-2)}$

   Let $F(z) = \dfrac{z}{(z-1)(z-2)}$

   $\Rightarrow F(z)z^{n-1} = \dfrac{z^n}{(z-1)(z-2)}$

   The poles are $z = 1, 2$, each simple pole

$$\begin{aligned}
R_1 &= \text{Residue at } z = 1 \\
&= \lim_{z\to 1}(z-1).\frac{z^n}{(z-1)(z-2)} \\
&= \lim_{z\to 1}\frac{z^n}{(z-2)} = -1
\end{aligned}$$

$$\begin{aligned}
R_2 &= \text{Residue at } z = 2 \\
&= \lim_{z\to 2}(z-2).\frac{z^n}{(z-1)(z-2)} \\
&= \lim_{z\to 2}\frac{z^n}{(z-1)} = 2^n
\end{aligned}$$

Hence $f(n) = R_1 + R_2 = 2^n - 1$

4. Find $Z^{-1}\left\{\dfrac{z^2}{(z+2)(z^2+4)}\right\}$ by the method of residues.

$$\begin{aligned}
\text{Let } F(z) &= \frac{z^2}{(z+2)(z^2+4)} \\
\Rightarrow z^{n-1}F(z) &= \frac{z^{n+1}}{(z+2)(z^2+4)} \\
&= \frac{z^{n+1}}{(z+2)(z+2i)(z-2i)}
\end{aligned}$$

The poles are given by $z = -1, -2i, 2i$, each of simple poles.

$$\begin{aligned}
R_1 &= \text{Residue at } z = -2 \\
&= \lim_{z\to -2}(z+2)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)} \\
&= \lim_{z\to -2}\frac{z^{n+1}}{(z+2i)(z-2i)} \\
&= \frac{(-2)^{n+1}}{(z+2i)(z-2i)} \\
&= \frac{(-2)^{n+1}}{8}
\end{aligned}$$

$$\begin{aligned}
R_2 &= \text{Residue at } z = -2i \\
&= \lim_{z\to -2i}(z+2i)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)} \\
&= \lim_{z\to -2i}\frac{z^{n+1}}{(z+2)(z-2i)} \\
&= \frac{(-2i)^{n+1}}{(-2i+2)(-2i-2i)} \\
&= \frac{(-2)^n(-2)(-i)(-i)^n}{2(1-i)(-4i)} \\
&= \frac{(-2)^n(-1)(-i)^n}{4(1-i)} \times \frac{(1+i)}{(1+i)} = \frac{(2)^n}{8}(-i)^n(1+i)
\end{aligned}$$

$$\begin{aligned}
R_3 &= \text{Residue at } z = 2i \\
&= \lim_{z\to 2i}(z-2i)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)} \\
&= \lim_{z\to 2i}\frac{z^{n+1}}{(z+2)(z+2i)} \\
&= \frac{(2i)^{n+1}}{(2i+2)(4i)} \\
&= \frac{(2)^n(i)^n}{4(1+i)}\times\frac{(1-i)}{(1-i)} \\
&= \frac{(2)^n}{8}(i)^n(1-i)
\end{aligned}$$

Therefore $f(n) = \sum R = R_1 + R_2 + R_3$

$$\begin{aligned}
f(n) &= \frac{(-2)^{n+1}}{8} + \frac{(2)^n}{8}(-i)^n(1+i) + \frac{(-2)^n}{8}(i)^n(1-i) \\
&= \frac{(-2)^{n+1}}{8} + \frac{(2)^n(1+i)}{8}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) \\
&+ \frac{(2)^n(1-i)}{8}\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) \\
&= \frac{(-2)^{n+1}}{8} + \frac{2^n}{8}\left\{\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2} + i\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right. \\
&\left. + \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} - i\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right\} \\
&= \frac{(-2)^{n+1}}{8} + \frac{2^n}{4}\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)
\end{aligned}$$

5. Find $Z^{-1}\left\{\dfrac{z}{z^2+2z+2}\right\}$ by the method of residues

Let $F(z) = \dfrac{z}{z^2+2z+2}$

$\Rightarrow z^{n-1}F(z) = \dfrac{z^n}{z^2+2z+2}$

The poles are given by $z^2+2z+2=0$

$$\begin{aligned}
\Rightarrow z &= \frac{-2\pm\sqrt{4-8}}{2} \\
&= \frac{-1\pm 2i}{2} \\
&= -1\pm i \text{ which are simple}
\end{aligned}$$

$$\begin{aligned}
R_1 =& \text{Residue at } z=-1+i\\
=& \lim_{z\to(-1+i)}[z-(-1+i)].\frac{z^n}{z^2+2z+2}\\
=& \lim_{z\to(-1+i)}[z-(-1+i)].\frac{z^n}{[z-(-1+i)][z-(-1+i)]}\\
=& \lim_{z\to(-1+i)}\frac{z^n}{z-(-1-i)}\\
=& \frac{(-1+i)^n}{-1+i+1+i}=\frac{(-1+i)^n}{2i}\\
R_2 =& \text{Residue at } z=-1-i\\
=& \lim_{z\to(-1-i)}[z-(-1-i)].\frac{z^n}{[z-(-1+i)][z-(-1-i)]}\\
=& \lim_{z\to(-1-i)}\frac{z^n}{z-(-1+i)}\\
=& \frac{(-1-i)^n}{-1-i+1-i}=\frac{-(-1-i)^n}{2i}
\end{aligned}$$

Therefore $f(n)=\sum R=R_1+R_2$

$f(n)=\dfrac{(-1+i)^n}{2i}-\dfrac{(-1-i)^n}{2i}$

Let $-1+i=r(\cos\theta+i\sin\theta)$

Equating real and imaginary parts, we have

$r\cos\theta=-1,\ r\sin\theta=1$

$r^2\cos^2\theta+r^2\sin^2\theta=1+1$

$r^2=2\implies r=\sqrt{2}$

Therefore $\cos\theta=-\dfrac{1}{\sqrt{2}}$ and $\sin\theta=-\dfrac{1}{\sqrt{2}}\Rightarrow\theta=\dfrac{3\pi}{4}$

Therefore $-1+i=\sqrt{2}\left[\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right]$ and $-1-i=\sqrt{2}\left[\cos\dfrac{3\pi}{4}-i\sin\dfrac{3\pi}{4}\right]$

$$\begin{aligned}
\text{Hence } f(n) =& \frac{1}{2i}\left\{(\sqrt{2})^n\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)^n-(\sqrt{2})^n\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)^n\right\}\\
=& \frac{(\sqrt{2})^n}{2i}\left\{\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4}-\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4}\right\}\\
=& \frac{(\sqrt{2})^n}{2i}.2i\sin\frac{3n\pi}{4}=(\sqrt{2})^n\sin\frac{3n\pi}{4}, n\geq 0
\end{aligned}$$

4. Convolution method

Convolution of sequence:

The Convolution of two sequence $\{f(n)\}$ and $\{g(n)\}$ is defined as

$\{f(n)*g(n)\}=\sum\limits_{r=0}^{n}f(r)g(n-r)$

Convolution theorem:
If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$ then

$$\begin{aligned} Z\{f(n) * g(n)\} &= Z\{f(n)\}.Z\{g(n)\} \\ &= F(z).G(z) \end{aligned}$$

That is Z-transform of Convolution of two sequence is equal to the product of the Z-transform.
Note:

- If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$ then $Z^{-1}[F(z)] = f(n)$, $Z^{-1}[G(z)] = g(n)$ and $Z^{-1}[F(z)G(z)] = f(n) * g(n) = \sum_{r=0}^{n} f(r)g(n-r)$
- If $Z[f(t)] = F(z)$ and $Z[g(t)] = G(z)$ then $Z^{-1}[F(z)G(z)] = f(t) * g(t) = \sum_{k=0}^{n} f(kT)g[(n-k)T]$

1. Find $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$ using convolution theorem

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] =& Z^{-1}\left[\frac{z}{(z+a)}.\frac{z}{(z+a)}\right] \\ &= Z^{-1}\left(\frac{z}{z+a}\right) Z^{-1}\left(\frac{z}{z+a}\right) \\ &= (-a)^n * (-a)^n \\ &= \sum_{r=0}^{n} (-a)^r(-a)^{n-r} \\ &= (-a)^n \sum_{r=0}^{n} 1 = (n+1)(-a)^n \end{aligned}$$

2. Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ using convolution theorem

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{z-a}.\frac{z}{z-b}\right] \\ &= Z^{-1}\left(\frac{z}{z-a}\right) * Z^{-1}\left(\frac{z}{z-b}\right) \end{aligned}$$

$$
\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= a^n * b^n = \sum_{r=0}^{n} a^r b^{n-r} \\
&= b^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r \\
&= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + ... + \left(\frac{a}{b}\right)^n\right] \\
&= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right] \\
&= \frac{a^{n+1} - b^{n+1}}{a-b}, n \geq 0
\end{aligned}
$$

3. Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

$$
\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] &= Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right).4\left(z+\frac{1}{4}\right)}\right] \\
&= Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right] \\
&= Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) * Z^{-1}\left(\frac{z}{z+\frac{1}{4}}\right) \\
&= \left(\frac{1}{2}\right)^n * \left(-\frac{1}{4}\right)^n \\
&= \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r * \left(-\frac{1}{4}\right)^{n-r} \\
&= \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n} (-2)^r \\
&= \left(\frac{1}{4}\right)^n [1 + (-2) + (-2)^2 + ... + (-2)^n] \\
&= \left(\frac{1}{4}\right)^n [\frac{(-2)^{n+1} - 1}{(-2) - 1}] \\
&= \left(\frac{1}{4}\right)^n \left(-\frac{1}{3}\right) [(-2)^{n+1} - 1]
\end{aligned}
$$

4. $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$

$$\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] &= Z^{-1}\left[\frac{z}{z-1}.\frac{z}{z-3}\right]\\
&= Z^{-1}\left(\frac{z}{z-1}\right) * Z^{-1}\left(\frac{z}{z-3}\right)\\
&= 1^n * 3^n\\
&= \sum_{r=0}^{n} 1^r.3^{n-r}\\
&= \frac{1}{2}(3^{n+1}-1)
\end{aligned}$$

Application of Z-transform to solve linear difference equation

We know that:
$F(z) = Z\{y_n\}$, $Z\{y_{n+1}\} = z[F(z)-y_0]$, $Z\{y_{n+2}\} = z^2\left[F(z)-y_0-\dfrac{y_1}{z}\right]$ and
$Z\{y_{n+3}\} = z^3\left[F(z)-y_0-\dfrac{y_1}{z}-\dfrac{y_2}{z^2}\right]$

1. Solve $y_{n+1}-2y_n=0$ given $y_0=3$. (OR) $y(n+1)-2y(n)=0$ given $y_0=3$.
   Solution:
   Given $y_{n+1}-2y_n=0$
   Taking Z-transform on both sides, we have
   $Z\{y_{n+1}\}-2Z\{y_n\}=0$
   $z[F(z)-y_0]-2F(z)=0$
   $(z-2)F(z)-z.y_0=0$
   $(z-2)F(z)-3z=0$
   $\Rightarrow F(z)=\dfrac{3z}{z-2}$
   Taking inverse Z-transform on both sides, we have
   $Z^{-1}[F(z)] = 3Z^{-1}\left(\dfrac{z}{z-2}\right)$
   $\Rightarrow f(n) = 3.2^n$

2. Solve $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$, given $y_0 = y_1 = 0$.
Solution:
Given $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$
Taking inverse Z-transform on both sides, we have

$$\begin{aligned}
Z\{y_{n+2}\} - 7Z\{y_{n+1}\} + 12Z\{y_n\} &= Z\{2^n\} \\
z^2\left[F(z) - y_0 - \frac{y_1}{z}\right] - 7z[F(z) - y_0] + 12F(z) &= \frac{z}{z-2} \\
(z^2 - 7z + 12)F(z) &= \frac{z}{z-2} \\
F(z) &= \frac{z}{(z^2 - 7z + 12)(z-2)} \\
Z^{-1}[F(z)] &= Z^{-1}\left[\frac{z}{(z-3)(z-4)(z-2)}\right] \\
f(n) &= Z^{-1}\left[\frac{z}{(z-3)(z-4)(z-2)}\right] \\
&= Z^{-1}[\phi(z)]
\end{aligned}$$

$$\begin{aligned}
\text{where } \phi(z) &= \frac{z}{(z-3)(z-4)(z-2)} \\
\Rightarrow z^{n-1}\phi(z) &= \frac{z^n}{(z-3)(z-4)(z-2)}
\end{aligned}$$

The poles are given by $z = 2, 3, 4$; each are simple pole

$$\begin{aligned}
R_1 &= \text{Residue at } z = 2 \\
&= \lim_{z\to 2}(z-2).\frac{z^n}{(z-3)(z-4)(z-2)} \\
&= \lim_{z\to 2}\frac{z^n}{(z-3)(z-4)} = \frac{2^n}{(-1)(-2)} = \frac{2^n}{2} \\
R_2 &= \text{Residue at } z = 3 \\
&= \lim_{z\to 3}(z-3).\frac{z^n}{(z-3)(z-4)(z-2)} \\
&= \lim_{z\to 3}\frac{z^n}{(z-2)(z-4)} = \frac{3^n}{(-1)(1)} = 3^n \\
R_3 &= \text{Residue at } z = 4 \\
&= \lim_{z\to 4}(z-4).\frac{z^n}{(z-3)(z-4)(z-2)} \\
&= \lim_{z\to 4}\frac{z^n}{(z-2)(z-3)} = \frac{4^n}{2}
\end{aligned}$$

Therefore $f(n) = R_1 + R_2 + R_3 = \dfrac{2^n}{2} - 3^n + \dfrac{4^n}{2}$

3. Solve $y(n+2)+4y(n+1)+4y(n)=n$, given that $y(0)=0$ and $y(1)=1$

Solution:

Given $y(n+2)+4y(n+1)+4y(n)=n$

Taking Z-transform on both sides, we get

$$Z\{y(n+2)\}+4Z\{y(n+1)\}+4Z\{y_n\}=Z\{n\}$$

$$z^2\left[F(z)-y(0)-\frac{y(1)}{z}\right]+4z[F(z)-y(0)]+4F(z)=\frac{z}{(z-1)^2}$$

$$(z^2+4z+4)F(z)-z=\frac{z}{(z-1)^2}$$

$$(z^2+4z+4)F(z)=\frac{(z^3-2z^2+2z)}{(z-1)^2}$$

$$F(z)=\frac{(z^3-2z^2+2z)}{(z-1)^2(z^2+4z+4)}$$

Taking inverse on both sides, we have

$$Z^{-1}[F(z)]=Z^{-1}\left[\frac{(z^3-2z^2+2z)}{(z-1)^2(z^2+4z+4)}\right]$$

$$f(n)=Z^{-1}\left[\frac{(z^3-2z^2+2z)}{(z-1)^2.(z+2)^2}\right]$$

$$=Z^{-1}[\phi(z)]$$

where $\phi(z)=\dfrac{(z^3-2z^2+2z)}{(z-1)^2.(z+2)^2}=\dfrac{z(z^2-2z+2)}{(z-1)^2.(z+2)^2}$

$\Rightarrow \dfrac{\phi(z)}{z}=\dfrac{(z^3-2z^2+2z)}{(z-1)^2.(z+2)^2}=\dfrac{A}{(z-1)}+\dfrac{B}{(z-1)^2}+\dfrac{C}{(z+2)}+\dfrac{D}{(z+2)^2}$

$(z^3-2z^2+2z)=A(z-1)(z+2)^2+B(z+2)^2+C(z+2)(z-1)^2+D(z-1)^2$

Put $z=1\Rightarrow B=\dfrac{1}{9}$, $z=-2\Rightarrow D=\dfrac{10}{9}$

Equating the coefficient of z^3 and constant term, we have

$A+C=0$ and $-4A+4B+2C+D=2$

Solving above equations, we get $C=\dfrac{2}{27}$ and $A=\dfrac{-2}{27}$

$$\begin{aligned}
\text{Therefore}\frac{\phi(z)}{z} &= \frac{\frac{-2}{27}}{z-1} + \frac{\frac{1}{9}}{(z-1)^2} + \frac{\frac{2}{27}}{(z+2)} + \frac{\frac{10}{9}}{(z+2)^2} \\
\phi(z) &= \frac{-2}{27}.\frac{z}{z-1} + \frac{1}{9}.\frac{z}{(z-1)^2} + \frac{2}{27}.\frac{z}{z+2} + \frac{10}{9}.\frac{z}{(z+2)^2} \\
Z^{-1}\left[\phi(z)\right] &= \frac{-2}{27}Z^{-1}\left(\frac{z}{z-1}\right) + \frac{1}{9}Z^{-1}\left(\frac{z}{(z-1)^2}\right) + \frac{2}{27}Z^{-1}\left(\frac{z}{(z+2)}\right) \\
&\quad + \frac{10}{9}Z^{-1}\left(\frac{z}{(z+2)^2}\right) \\
f(n) &= \frac{-2}{27} + \frac{1}{9}.n + \frac{2}{27}(-2)^n + \frac{10}{9}\left[\frac{-1}{2}.n(-2)^n\right] \\
&= \frac{-2}{27} + \frac{1}{9}.n + \frac{2}{27}(-2)^n - \frac{5}{9}n(-2)^n
\end{aligned}$$

4. Solve $x(n+1) - 2x(n) = 1$, given $x(0) = 0$

   Solution:

   Given $x(n+1) - 2x(n) = 1$

   Taking Z-transform on both sides, we have

$$\begin{aligned}
Z\left\{x(n+1)\right\} - 2Z\left\{x(n)\right\} &= Z\left\{1\right\} \\
z[X(z) - x(0)] - 2X(z) &= \frac{z}{z-1} \\
(z-2)X(z) &= \frac{z}{z-1} \\
\Rightarrow X(z) &= \frac{z}{(z-1)(z-2)} \\
\Rightarrow Z^{-1}\left\{X(z)\right\} &= Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] \\
\Rightarrow x(n) &= Z^{-1}[\phi(z)] \\
\text{where } \phi(z) &= \frac{z}{(z-1)(z-2)} \\
\Rightarrow z^{n-1}\phi(z) &= \frac{z^n}{(z-1)(z-2)}
\end{aligned}$$

   The poles are given by $z = 1, 2;$ each of simple poles.

$$\begin{aligned}
R_1 &= \text{Residue at } z = 1 \\
&= \lim_{z\to 1}(z-1).\frac{z^n}{(z-1)(z-2)} \\
&= \lim_{z\to 1}\frac{z^n}{(z-2)} = -1
\end{aligned}$$

$$
\begin{aligned}
R_2 &= \text{Residue at } z=2 \\
&= \lim_{z\to 2}(z-2).\frac{z^n}{(z-1)(z-2)} \\
&= \lim_{z\to 2}\frac{z^n}{(z-1)} = 2^n
\end{aligned}
$$

Therefore $x(n) = R_1 + R_2 = -1 + 2^n$

5. Solve $y_{n+2} + y_n = 2$ given $y_0 = y_1 = 0$
Solution:
Given $y_{n+2} + y_n = 2$
Taking Z-transform on both sides, we have $Z[y_{n+2}] + Z[y_n] = Z(2)$
$z^2\left[Y(z) - y_0 - \frac{y_1}{z}\right] + Y(z) = \frac{2z}{(z-1)}$
$z^2[Y(z) - 0 - 0] + y(z) = \frac{2z}{(z-1)}$
$(z^2+1)Y(z) = \frac{2z}{(z-1)}$
$\Rightarrow Y(z) = \frac{2z}{(z-1)(z^2+1)}$
$\frac{Y(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$
Now $2 = A(z^2+1) + (Bz+C)(z-1)$
Put $z = 1 \Rightarrow A = 1$
Equating Co-efficient of z^2 and constant term.

$$A + B = 0 \tag{7}$$

$$A - C = 2 \tag{8}$$

$A = 1$ in equations (7) and (8), we have $B = 1$ and $C = -1$
Therefore $\frac{Y(z)}{z} = \frac{1}{z-1} - \frac{z+1}{z^2+1}$
$Y(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$
Taking inverse on both sides, we get
$y(n) = 1 - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$

6. Solve $y(n) - ay(n-1) = u(n)$
Solution
Given $y(n) - ay(n-1) = u(n)$
Taking Z-transform on both sides
$Z[y(n)] - aZ[y(n-1)] = Z[u(n)]$

$Y(z) - a.z^{-1}Y(z) = \dfrac{z}{z-1}$ since $Z\left[x(n-m)\right] = z^{-m}X(z)$
$(1 - az^{-1})Y(z) = \dfrac{z}{z-1}$
$\Rightarrow \dfrac{z-a}{z}Y(z) = \dfrac{z}{z-1}$
Therefore $Y(z) = \dfrac{z^2}{(z-a)(z-1)}$
$\Rightarrow \dfrac{Y(z)}{z} = \dfrac{z}{(z-a)(z-1)} = \dfrac{A}{z-a} + \dfrac{B}{z-1}$
Therefore $A(z-1) + B(z-a) = z$
put $z = 1 \;\Rightarrow\; B = \dfrac{1}{1-a}$ and $z = a \;\Rightarrow\; A = \dfrac{-a}{1-a}$
Therefore $\dfrac{y(z)}{z} = \dfrac{\frac{-a}{1-a}}{z-a} + \dfrac{\frac{1}{1-a}}{z-1}$
$y(z) = \dfrac{1}{1-a}\left[\dfrac{-az}{z-a} + \dfrac{z}{z-1}\right]$

Taking inverse Z-transform, we have

$$\begin{aligned}
y(n) &= \frac{1}{1-a}\left[-Z^{-1}\left(\frac{az}{z-a}\right) + Z^{-1}\left(\frac{z}{z-1}\right)\right] \\
&= \frac{1}{1-a}\left[Z^{-1}\left(\frac{z}{z-1}\right) - aZ^{-1}\left(\frac{z}{z-a}\right)\right] \\
&= \frac{1}{1-a}[1 - a.a^n]
\end{aligned}$$

7. Solve $y(n) = y(n-1) = u(n) + u(n-1)$
Solution:
Given $y(n) = y(n-1) = u(n) + u(n-1)$
Taking Z-Transform on both sides,
$Z[y(n)] - Z[y(n-1)] = Z[u(n)] + Z[u(n-1)]$
$Y(z) - z^{-1}Y(z) = \dfrac{z}{z-1} + z^{-1}.\dfrac{z}{z-1}$ since $Z\left[x(n-m)\right] = z^{-m}X(z)$
$(1 - z^{-1})Y(z) = \dfrac{z+1}{z-1}$
$\Rightarrow Y(z) = \dfrac{z(z+1)}{(z-1)^2}$
$\dfrac{Y(z)}{z} = \dfrac{(z+1)}{(z-1)^2} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-1)^2}$
Now $z + 1 = A(z-1) + B$
Put $z = 1 \Rightarrow\; B = 2$ and equating co.eff. of $z,$ we get $A = 1$
Therefore $\dfrac{Y(z)}{z} = \dfrac{1}{(z-1)} + \dfrac{2}{(z-1)^2}$

$\Rightarrow Y(z) = \dfrac{z}{(z-1)} + \dfrac{2z}{(z-1)^2}$

Taking inverse Z-transform on both sides, we get

$y(n) = Z^{-1}[Y(z)] = Z^{-1}\left(\dfrac{z}{z-1}\right) + 2Z^{-1}\left(\dfrac{z}{(z-1)^2}\right) = 1 + 2n$

8. Solve $x(k+2) - 3x(k+1) + 2x(k) = u(k)$ given $x(k) = 0$ for $k \leq 0$ and $u(0) = 1$, $u(k) = 0$ for $k < 0$ and $k > 0$.

Solution:

Given $x(k+2) - 3x(k+1) + 2x(k) = u(k)$

Taking Z-transform on both sides, we have

$Z[x(k+2)] - 3Z[x(k+1)] + 2Z[x(k)] = Z[u(k)]$

$$z^2\left[X(z) - x(0) - \frac{x(1)}{z}\right] - 3z[X(z) - x(0)] + 2X(z) = 1 \tag{9}$$

putting $k = -1$ in given equation, we have

$x(1) - 3x(0) + 2x(-1) = u(-1)$

$x(1) - 3.0 + 2.0 = 0$ since $x(0) = 0, x(-1) = 0, u(-1) = 0$

Therefore $x(1) = 0$

From equation (9), becomes

$z^2X(z) - 3zX(z) + 2X(z) = 1$

$(z^2 - 3z + 2)X(z) = 1$

$\Rightarrow X(z) = \dfrac{1}{z^2 - 3z + 2}$

$X(z) = \dfrac{1}{(z-1)(z-2)}$

Taking inverse Z-transform, we have

$x(k) = Z^{-1}[X(z)] = Z^{-1}\left[\dfrac{1}{(z-1)(z-2)}\right]$

Let $\phi(z) = \dfrac{1}{(z-1)(z-2)}$

$\Rightarrow \phi(z)z^{k-1} = \dfrac{z^{k-1}}{(z-1)(z-2)}$

The poles are given by $z = 1, 2;$ each are simple pole

$$\begin{aligned} R_1 &= \text{Residue at } z = 1 \\ &= \lim_{z\to 1}(z-1).\frac{z^{k-1}}{(z-1)(z-2)} = -1 \\ R_2 &= \text{Residue at } z = 2 \\ &= \lim_{z\to 2}(z-2).\frac{z^{k-1}}{(z-1)(z-2)} = 2^{k-1} \end{aligned}$$

Hence $x(k) = R_1 + R_2 = -1 + 2^{k+1}$

9. Solve $y_{n+2} - 4y_n = 0$ using z-transform.

Solution:

Here, the conditions y_0 and y_1 are not given.

Take $y_0 = A$, $y_1 = B$

Given $y_{n+2} - 4y_n = 0$

Taking Z-Transform on both sides, we get

$\Rightarrow Z[y_{n+2}] - 4Z[y_n] = 0$

$\Rightarrow z^2\left[Y(z) - y_0 - \dfrac{y_1}{z}\right] - 4Y(z) = 0$

$\Rightarrow (z^2-4)Y(z) - Az^2 - Bz = 0$

$\Rightarrow Y(z) = \dfrac{Az^2+Bz}{z^2-4}$

$$\frac{Y(z)}{z} = \frac{Az}{z^2-4} + \frac{B}{z^2-4} \tag{10}$$

Now $\dfrac{z}{z^2-4} = \dfrac{z}{(z+2)(z-2)} = \dfrac{A}{z+2} + \dfrac{B}{z-2}$

$z = A(z-2) + B(z+2)$

Put $z = 2 \Rightarrow B = \dfrac{1}{2}$ and $z = -2 \Rightarrow A = \dfrac{1}{2}$

$$\text{Therefore } \frac{z}{z^2-4} = \frac{1}{2}.\frac{1}{z+2} + \frac{1}{2}.\frac{1}{z-2} \tag{11}$$

and $\dfrac{1}{z^2-4} = \dfrac{1}{(z+2)(z-2)} = \dfrac{A}{z+2} + \dfrac{B}{z-2}$

$\Rightarrow 1 = A(z-2) + B(z+2)$

Put $z = 2 \Rightarrow B = \dfrac{1}{4}$ and $z = -2 \Rightarrow A = -\dfrac{1}{4}$

$$\text{Therefore } \frac{1}{z^2-4} = \frac{-1}{4}.\frac{1}{z+2} + \frac{1}{4}.\frac{1}{z-2} \tag{12}$$

Substituting (11) and (12) in (10), we get

$$\begin{aligned}
\frac{Y(z)}{z} &= \frac{A}{2}\left[\frac{1}{z+2} + \frac{1}{z-2}\right] + \frac{B}{4}\left[\frac{1}{z-2} - \frac{1}{z+2}\right] \\
\Rightarrow Y(z) &= \frac{A}{2}\left[\frac{z}{z+2} + \frac{z}{z-2}\right] + \frac{B}{4}\left[\frac{z}{z-2} - \frac{z}{z+2}\right]
\end{aligned}$$

Taking inverse z-transform on both sides, we get

$$\begin{aligned}
y(n) &= \frac{A}{2}\left[(-2)^n + 2^n\right] + \frac{B}{4}\left[2^n - (-2)^n\right] \\
&= \left(\frac{A}{2} + \frac{B}{4}\right)2^n + \left(\frac{A}{2} - \frac{B}{4}\right)(-2)^n \\
&= C.2^n + D(-2)^n \text{ where } C = \frac{A}{2} + \frac{B}{4} \text{ and } D = \frac{A}{2} - \frac{B}{4}
\end{aligned}$$

SHIFTING PROPERTY

We know that $Z\{x(n-m)\} = z^{-m}x(z)$

Corollary: $x(n-m) = Z^{-1}[z^{-m}X(z)] = (Z^{-1}[X(z)])_{n\to n-m}$

1. Find $Z^{-1}\left[\dfrac{1}{z-\frac{1}{2}}\right]$

   Solution:

$$\begin{aligned} Z^{-1}\left[\frac{1}{z-\frac{1}{2}}\right] &= Z^{-1}\left[z^{-1}\left(z-\frac{1}{2}\right)\right] \\ &= Z^{-1}\left(\frac{z}{z-1/2}\right)_{n\to n-1} \\ &= \left(\frac{1}{2}\right)^{n-1} \text{ or } \left(\frac{1}{2}\right)^{n-1} u(n-1) \end{aligned}$$

2. Evaluate $Z^{-1}\left(\dfrac{1}{z+1}\right)$ given $Z^{-1}\left(\dfrac{z}{z+1}\right) = (-1)^n$

   Solution:

$$\begin{aligned} Z^{-1}\left(\frac{1}{z+1}\right) &= Z^{-1}\left[z^{-1}\left(\frac{z}{z+1}\right)\right] \\ &= Z^{-1}\left(\frac{z}{z+1}\right) n \to n-1 \\ &= (-1)^{n-1} \end{aligned}$$

3. Find $Z^{-1}(X(z))$ where $X(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$

   Solution:

   Given $X(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$

$$\begin{aligned} \Rightarrow X(z)z^{n-1} &= \frac{z^{n-1}.2z(2z-1)}{z^3-5z^2+8z-4} \\ &= \frac{2z^n(2z-1)}{z^3-5z^2+8z-4} \end{aligned}$$

   The poles are given by $z^3-5z^2+8z-4=0$

   $\Rightarrow (z-1)(z^2-4z+4)=0$

$\Rightarrow (z-1)(z-2)^2 = 0$
$\Rightarrow z = 1$ is a simple pole and $z = 2$ is a pole of order 2

$$R_1 = \text{Residue at } z = 1$$
$$= \lim_{z\to 1}(z-1).\frac{2z^n(2z-1)}{(z-1)(z-2)^2} = 2$$
$$R_2 = \text{Residue at } z = 2$$
$$= \frac{1}{(2-1)!}\lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2.\frac{2z^n(2z-1)}{(z-1)(z-2^2)}\right]$$
$$= \lim_{z\to 2}\frac{d}{dz}\left(\frac{2z^2(2z-1)}{z-1}\right)$$
$$= \lim_{z\to 2} 2\left[\frac{(z-1)[nz^{n-1}(2z-1)+z^n.2]-z^n(2z-1).1}{(z-1)^2}\right]$$
$$= 2[n.2^{n-1}.3 + 2.2^n - 3.2^n] = 2.2^n\left[\frac{3}{2}n - 1\right]$$
$$= 2^n(3n-2)$$

Hence $x(n) = R_1 + R_2 = 2 + 2^n(3n-2)$

4. Find the inverse Z-transform of $\dfrac{z(z+1)}{(z-1)^3}$

Solution:

Let $F(z) = \dfrac{z(z+1)}{(z-1)^3}$

$\Rightarrow F(z).z^{n-1} = \dfrac{z^n(z+1)}{(z-1)^3}$

$\Rightarrow z = 1$ is a pole of order 3

$$R_1 = \text{Residue at } z = 1$$
$$= \frac{1}{(3-1)!}\lim_{z\to 1}\left[\frac{d^2}{dz^2}(z-1)^3.\frac{z^n(z+1)}{(z-1)^3}\right]$$
$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(z^n(z+1))$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[nz^{n-1}(z+1) + z^n.1\right]$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[n(n-1)z^{n-2}(z+1) + nz^{n-1} + nz^{n-1}\right]$$
$$= \frac{1}{2}\left[n(n-1).2 + n + n\right] = n^2$$

Hence $f(n) = n^2$

5. If $X(z) = (z-1)\log\left(1 - \frac{1}{z}\right) + 1$, find $x(n)$

Solution:

$$\begin{aligned}
x(n) &= Z^{-1}\left[(z-1)\log\left(1-\frac{1}{z}\right)+1\right] \\
&= Z^{-1}\left[(z-1)\left(-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{3z^3}-...\right)+1\right] \\
&= Z^{-1}\left[z\left(-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{3z^3}-...\right)-1\left(-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{3z^3}-...\right)+1\right] \\
&= Z^{-1}\left[\left(-1-\frac{1}{2z^2}-\frac{1}{3z^3}-...\right)+\left(\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}-...\right)+1\right] \\
&= Z^{-1}\left[\left(1-\frac{1}{2}\right)\frac{1}{z}+\left(\frac{1}{2}-\frac{1}{3}\right)\frac{1}{z^2}+...\right] \\
&= Z^{-1}\left[\frac{1}{1.2}\frac{1}{z}+\frac{1}{2.3}\frac{1}{z^2}+...\right] \\
&= Z^{-1}\left[\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.\frac{1}{z^n}\right] \\
&= \begin{cases} \frac{1}{n(n+1)}, & \text{if } n = 0 \\ 0, & \text{if } n \geq 1 \end{cases}
\end{aligned}$$

6. If $X(z) = (1-az^{-1})^{-2}$, find $x(n)$

Solution:

$$\begin{aligned}
X(z) = (1-az^{-1})^{-2} &= \frac{1}{(1-az^{-1})^2} \\
&= \frac{z^2}{(z-a)^2} \\
\Rightarrow z^{n-1}X(z) &= \frac{z^{n+1}}{(z-a)^2}
\end{aligned}$$

$z = a$ is a pole of order 2

$$\begin{aligned}
R &= \text{Residue at } z = a \\
&= \lim_{z\to a}\frac{d}{dz}\left((z-a)^2\frac{z^{n+1}}{(z-a)^2}\right) \\
&= \lim_{z\to a}\frac{d}{dz}z^{n+1} = (n+1)a^n
\end{aligned}$$

Hence $f(n) = (n+1)a^n$