Hands-on MR Physics with Pulseq

MR physics of RF pulses

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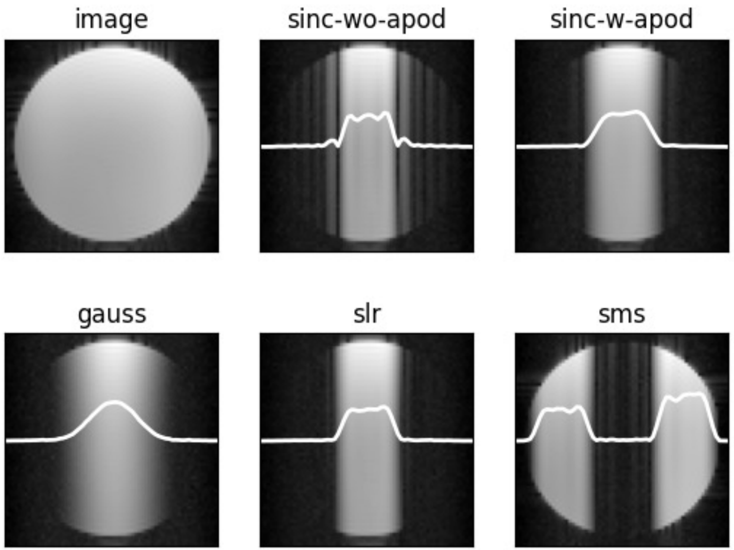


Outline

- *Bloch Equations with excitation*
- *On- and off-resonant excitation*
- *Some important RF pulses: sinc, SLR, VERSE, binomial, adiabatic*
- *Summary*
- *Hands-on Pulseq: Measurement of excitation slice profiles*

Equations of *classical* MR physics

Bloch Equation (with relaxation)

Classical spin ½ description: dynamic of the magnetization vector

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma\vec{B} - \begin{pmatrix} M_x/T_2 \\ M_y/T_2 \\ (M_z - M_0)/T_1 \end{pmatrix}$$

MR Signal Equation

(Complex) transverse magnetization induces voltage in the detector coils

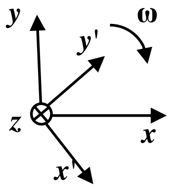
$$S_n(t) = c \int_V \sigma_n^*(\vec{r}) M_T(\vec{r}, t)\, dV$$

transverse magn.	M_T	$=$	$M_x + \mathrm{i} M_y$
coil sensitivity	σ_n	$=$	$\sigma_{n,x} + \mathrm{i}\sigma_{n,y}$
coil volume	V		

Solutions best described in the rotating frame of reference:

BE: *rotation around effective field*

$$\vec{B}_{\text{eff}} = \vec{B}_{1,\text{rot}} + \underbrace{\left(B_0 - \frac{\omega}{\gamma}\right)}_{\textit{off-resonance}:\ \Delta\omega/\gamma} \vec{e}_z$$



SE: *demodulation of the MR signal*

$$S(t) = c \int_V M_0(\vec{r})\, \mathrm{e}^{-2\pi \mathrm{i}\vec{k}(t)\cdot\vec{r}}\, \mathrm{e}^{-\mathrm{i}\Delta\omega(\vec{r})t}\, d^3r$$

The difficult part: solution of the Bloch Equation

Full Bloch Equation with matrix-vector product

$$\begin{pmatrix} \dot{M}_x \\ \dot{M}_y \\ \dot{M}_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{T_2} & \gamma B_z & -\gamma B_y \\ -\gamma B_z & -\frac{1}{T_2} & \gamma B_x \\ \gamma B_y & -\gamma B_x & -\frac{1}{T_1} \end{pmatrix} \cdot \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \end{pmatrix}$$

General driving magnetic field of MRI: RF transmitters, gradients, and off-resonance

$$\vec{B}(\vec{r},t) \ = \ B_{1x}(\vec{r},t)\vec{e}_x + B_{1y}(\vec{r},t)\vec{e}_y \ + \left[\vec{G}(t)\cdot\vec{r} + \frac{\Delta\omega(\vec{r},t)}{\gamma}\right]\vec{e}_z$$

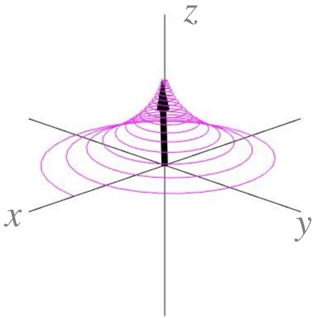
no analytic solution for general driving fields

Closed-form solutions of the Bloch Equations (1)

Bloch Equation in absence of RF pulse

- If $\vec{B}||\vec{e}_z \;\; (\Rightarrow \;\; B_{1x} = B_{1y} = 0)$ the Bloch equations *decouple*: simple solutions
- These solutions are important for the encoding step (after excitation)

$$\frac{d\vec{M}}{dt} = \begin{pmatrix} -\frac{1}{T_2} & \gamma\vec{G}\cdot\vec{r} & 0 \\ -\gamma\vec{G}\cdot\vec{r} & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{pmatrix} \cdot \vec{M} + \begin{pmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \end{pmatrix}$$



Solution *after* $90°_y$ pulse: $$M_x(t) + \mathrm{i}M_y(t) \;=\; M_0\,\mathrm{e}^{-t/T_2}\,\mathrm{e}^{-2\pi\mathrm{i}\vec{k}(t)\cdot\vec{r}} \quad , \quad \vec{k}(t) \equiv \frac{\gamma}{2\pi}\int_0^t \vec{G}(\tau)d\tau$$

$$M_z(t) \;=\; M_0\left(1 - \mathrm{e}^{-t/T_1}\right)$$

Closed-form solutions of the Bloch Equations (2)

Bloch Equation with RF pulse, $\vec{\omega}_1 = \gamma \vec{B}_1$, and neglecting relaxation (short RF pulses)

$$\frac{d\vec{M}}{dt} = \begin{pmatrix} 0 & \Delta\omega & -\omega_{1y} \\ -\Delta\omega & 0 & \omega_{1x} \\ -\omega_{1y} & -\omega_{1x} & 0 \end{pmatrix} \cdot \vec{M} \Leftrightarrow \frac{d\vec{M}}{dt} = \vec{M} \times \vec{\omega}_{\text{eff}} \quad , \quad \vec{\omega}_{\text{eff}} = \vec{\omega}_1 + \Delta\omega \vec{e}_z$$

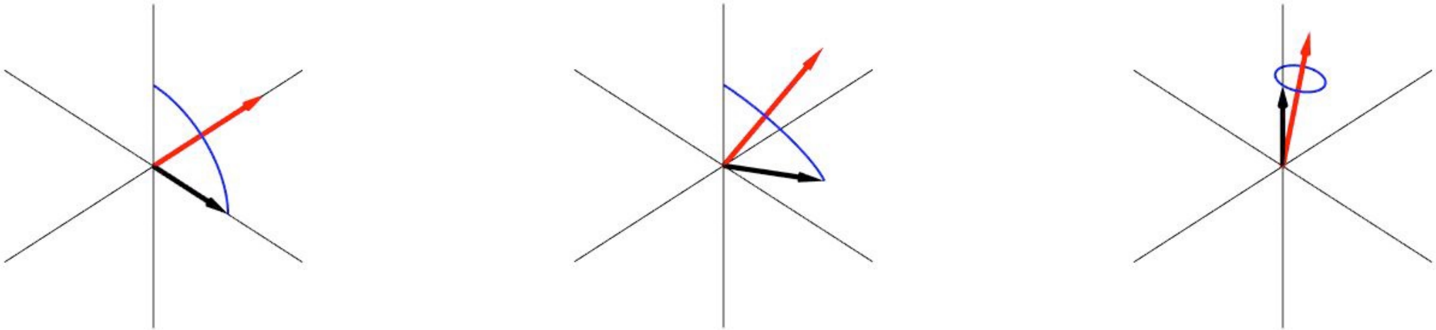
Analytical solutions exist only in few cases:

1. **$\Delta\omega$=0** ⇒ "on-resonant" rotation around **ω_1**: $\vec{M}(t) = \mathbf{R}_{\vec{n}(\omega_1)}(\alpha(t)) \cdot \vec{M}_0 \quad , \quad \alpha(t) \equiv \int_0^t \omega_1(\tau) d\tau$

2. **$\Delta\omega\neq 0$,** assuming small α (STA) ⇒ **$M_z \approx 0$**: $M_x(t,\Delta\omega) + \mathrm{i}M_y(t,\Delta\omega) \approx \mathrm{i}M_0\, \mathrm{e}^{-\Delta\omega t} \int_0^t \mathrm{e}^{\Delta\omega\tau} \omega_1(\tau) d\tau$

3. **$\Delta\omega\neq 0$** for some specific RF pulse shapes, e.g.
   ω_1=const. ⇒ *rotation around effective field ω_{eff}*

On- and off-resonance excitation



on-resonant ($\Delta\omega = 0$) | ***Medium* off-resonance** ($\Delta\omega > 0$) | ***Large* off-resonance** ($\Delta\omega >> 0$)

(*Red effective field axis, Blue magnetization trajectory*)

small tip angle approximation and slice-selection

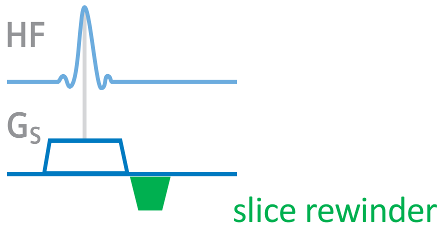
- small tip angle approximation $M_z(t) \approx M_0$
- Off-resonance from slice selection gradient $\Delta\omega = \gamma G z$

⇒ Decoupled Bloch equation: $\dot{M} = \dot{M}_x + \mathrm{i}\dot{M}_y = -\mathrm{i}\Delta\omega M + \mathrm{i}\omega_1 M_0$

Solution after RF-pulse of duration τ

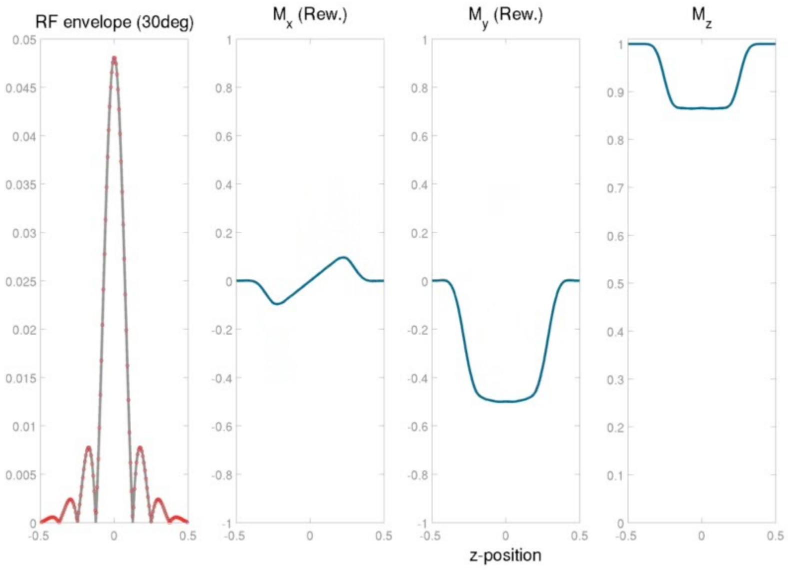
<u>the RF-pulse is the Fourier Transform of the slice profile</u>

$$M(\tau, \Delta\omega) = \mathrm{i}M_0 \mathrm{e}^{-\mathrm{i}\Delta\omega\tau/2} \int_{-\tau/2}^{\tau/2} \mathrm{e}^{\mathrm{i}\Delta\omega t} \omega_1(t + \tau/2) dt$$
$$= \mathrm{i}M_0 \boxed{\mathrm{e}^{-\mathrm{i}\Delta\omega\tau/2}} \mathcal{F}^{-1}_{t\to\Delta\omega} \left[\omega_1(t + \tfrac{\tau}{2})\right]$$

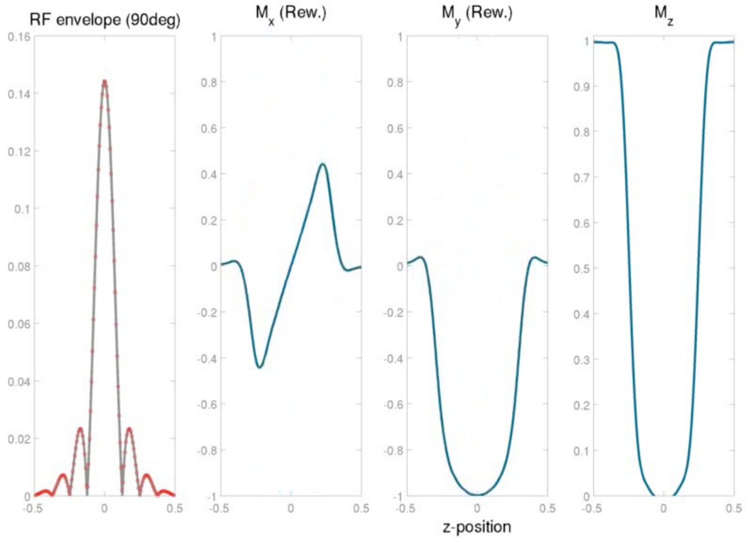


small tip angle approximation and slice-selection

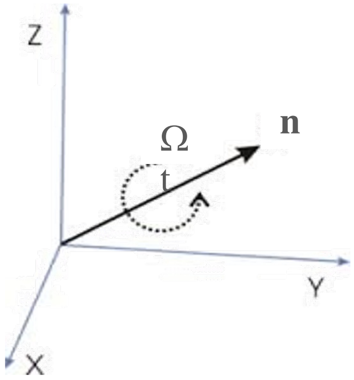
30° excitation



90° excitation



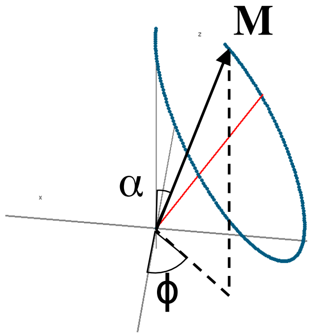
Closed-form solutions of the Bloch Equations (3)



Bloch Equation with constant RF pulse and off-resonance

$$\frac{d\vec{M}}{dt} = \vec{M} \times \vec{\omega}_{\text{eff}} \quad , \quad \vec{\omega}_{\text{eff}} = \vec{\omega}_1 + \Delta\omega \vec{e}_z = \text{const}$$

- Solutions is a **rotation** around the **effective Field:** $\vec{n} = \frac{\omega_{1x}}{\Omega}\vec{e}_x + \frac{\omega_{1y}}{\Omega}\vec{e}_y + \frac{\Delta\omega}{\Omega}\vec{e}_z$
- *nutation frequency* $\Omega = \sqrt{\omega_{1x}^2 + \omega_{1y}^2 + \Delta\omega^2}$
- rotation angle after time τ is $\Omega\tau$ (this is not the flip angle!)
- resulting flip angle[1]: $\cos(\alpha) = \cos(\Omega\tau) + \left(\frac{\Delta\omega}{\Omega}\right)^2 (1 - \cos(\Omega\tau))$
- resulting phase[1]: $\tan(\phi) = \frac{\Delta\omega}{\Omega} \tan\left(\frac{\Omega\tau}{2}\right)$



[1] *Freed DE et al. JMR 2003;162:328–335*

numerical solutions of the Bloch equation

Two possible approaches

a) Use a generic ODE solver for numerical integration

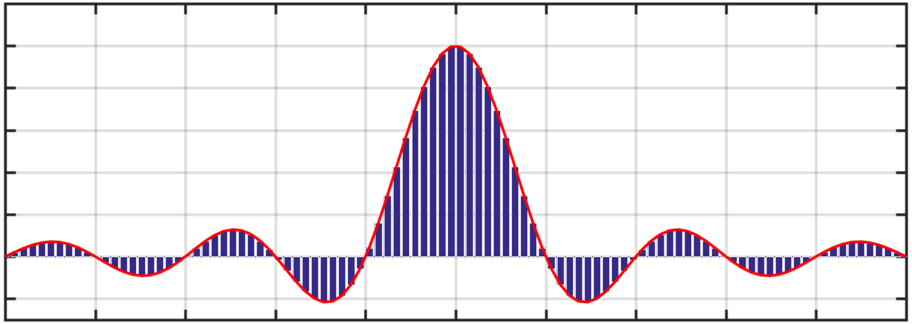
$$\vec{M}(t+\Delta t) \approx \vec{M}(t) + \Delta t(\vec{M} \times \omega)$$

b) Approximate the driving field to use analytical solutions:

$$\vec{M}(t+\Delta t) \approx \mathbf{R}(\omega, \Delta t) \cdot \vec{M}(t)$$

Numerical solutions of the Bloch Equation

Hard Pulse Approximation: Decompose RF pulse into short rectangular pulses



$$\omega_1(t) \approx \sum_{n=0}^{N-1} \omega_{1n}\, g(t - n\Delta t) \quad , \quad g(t) = \begin{cases} 1 & \text{if } 0 < t < \Delta t \\ 0 & \text{otherwise} \end{cases}$$

For each sub-pulse, the magnetization rotates around the effective field: $\vec{\omega}_{eff} = \vec{\omega}_1 + \Delta\omega \vec{e}_z$

Total action by chained rotation matrix:
(using 2x2 complex spinor rotation basis)

$$\mathbf{Q}_{\text{tot}} = \prod_i \mathbf{Q}_i \qquad \mathbf{Q} = \begin{pmatrix} \alpha & -\beta^* \\ \beta & -\alpha^* \end{pmatrix}$$

$$\alpha = \cos\varphi/2 \; - i\, n_z \sin\varphi/2$$
$$\beta = -i(n_x + i\, n_y) \sin\varphi/2$$

Cayley-Klein parameters from rotation axis, angle (**n**,ϕ)

⇒ total rotation of magnetization (with $M = M_x + iM_y$)

$$\begin{pmatrix} M \\ M^* \\ M_z \end{pmatrix}^{+} = \begin{pmatrix} (\alpha^*)^2 & \beta^2 & 2\alpha^*\beta \\ -(\beta^*)^2 & \alpha^2 & 2\alpha\beta^* \\ -\alpha^*\beta^* & -\alpha\beta & \alpha\alpha^* - \beta\beta^* \end{pmatrix} \cdot \begin{pmatrix} M \\ M^* \\ M_z \end{pmatrix}^{-}$$

[1] Pauly et al. 1991. “Parameter Relations for the Shinnar-Le Roux Selective Excitation Pulse Design Algorithm.” *IEEE Transactions on Medical Imaging* 10 (1): 53–65. https://doi.org/10.1109/42.75611.

frequency response of 5 ms sinc-pulse

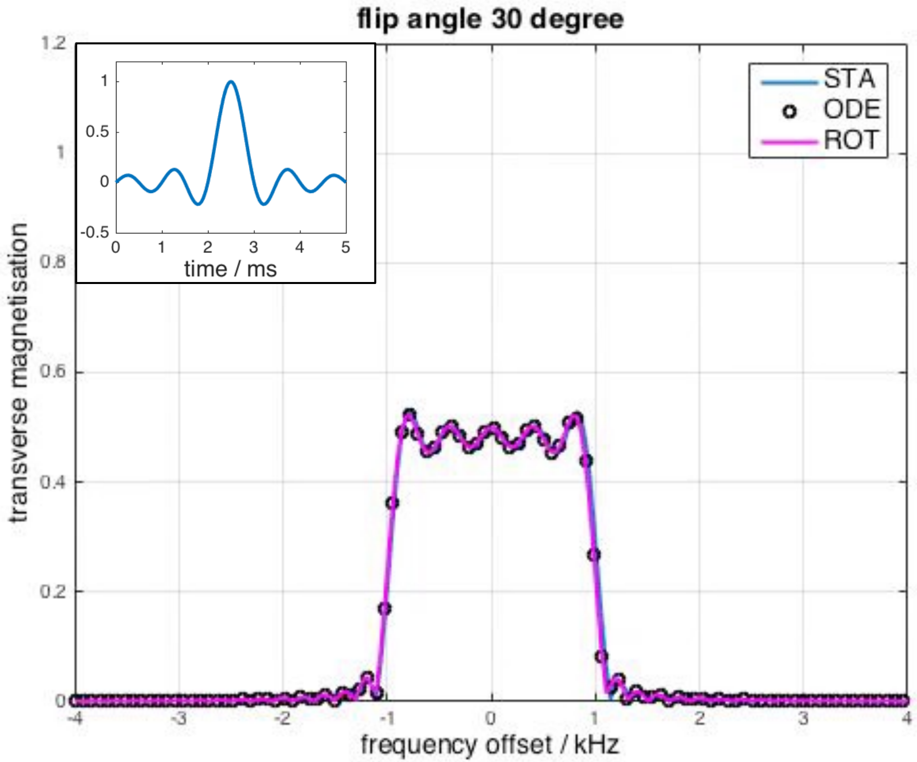
<u>Comparison of three methods:</u>

a) Small Tip-Angle Approximation (STA)

b) Numerical ODE integration
(`scipy odeint`)

c) Rotation matrices (spinor)

b) and c) provide exact solutions



see notebook `sinc-excitation.ipynb`

The pulse design problem

Pulse design does not search for *M*, but for $\omega_1 = \gamma B_1$

$$\frac{d\vec{M}}{dt} = \begin{pmatrix} 0 & \Delta\omega & -\omega_{1y} \\ -\Delta\omega & 0 & \omega_{1x} \\ -\omega_{1y} & -\omega_{1x} & 0 \end{pmatrix} \cdot \vec{M}$$

<u>Given</u>

- range of off-resonances $\omega_{min} < \Delta\omega < \omega_{max}$
- pulse duration *T*
- *desired* transverse magnetization $\{M_x(T, \Delta\omega), M_y(T, \Delta\omega)\}$ as a function of $\Delta\omega$ (i.e. the slice profile where $\Delta\omega = G_z \cdot z$)

<u>Wanted</u>

- $\omega_1(t)$ which creates this solution at time *T*

(This is a strange question to ask to an ODE)

Shinnar-Le-Roux Pulses

- Slice-Selective RF Pulses are needed up to 180deg (e.g. spin echo refocusing)
- The SLR Transform is one prominent realization (many other methods exist)

Basic Idea

1. Solve the *forward problem*: "$\omega_1(t)$ given, $M_{xy}(T, \Delta\omega)$ wanted"
2. Solve the *inverse problem*: "$M_{xy}(T, \Delta\omega)$ given, $\omega_1(t)$ wanted"

Forward SLR - express net action of RF pulse in terms of a complex polynomial:

$$A_N(z) = \sum_{n=0}^{N} a_n \, z^{-n} = z^{-N/2}\alpha \quad , \quad B_N(z) = \sum_{n=0}^{N} b_n \, z^{-n} = z^{-N/2}\beta$$

$$\text{with} \quad z = \exp(i\Delta\omega\Delta t) \quad \text{and Cayley-Klein Parmeters} \ \ (a_n, b_n), \ (\alpha, \beta)$$

Pauly et al. 1991. IEEE TMI 10 (1): 53–65.

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Inverse SLR - this process is equivalent to optimal FIR filter design in signal processing!

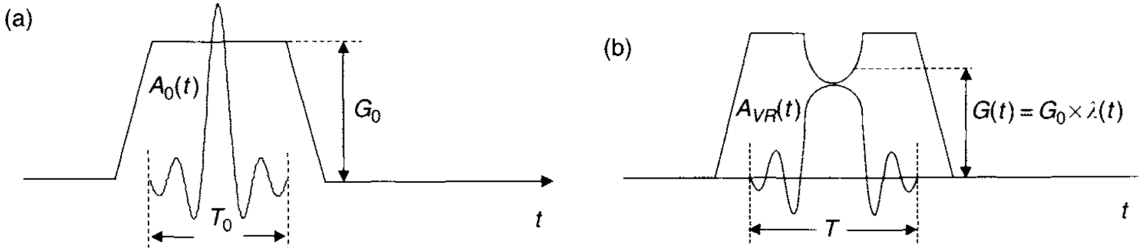
- Express $M_{xy}(T, \Delta\omega)$ in terms of polynomials (e.g. the Parks–McClellan algorithm)
- Invert the chain of rotations by means of the spinor calculus
- Obtain RF pulse that *exactly* fulfills the Bloch equation

Pauly et al. 1991. IEEE TMI 10 (1): 53–65.

VERSE Pulses

Variable Excitation Rate Selective Excitation (VERSE)* pulses execute the RF field along with a **non-constant gradient**

Theory: time-dilated version of the RF Pulse, conserving area under the curve



From: Bernstein et al. - Handbook of MRI pulse sequences

$\Rightarrow$ Reduce SAR $SAR \propto \int B_1^2 \, dt$

keep flip angle $\alpha \propto \int B_1 \, dt$

1. Conservation of flip angle under time stretching λ(t)<1

$$B_1^{VR}(t) = \lambda(t) \, B_1(u(t)) \quad \text{with} \quad u(t) = \int_0^t \lambda(t') \, dt'$$

$$\alpha = \gamma \int_0^{T_0 = u(T)} B_1(u) \, du \; = \; \gamma \int_0^T B_1^{VR}(t) \, dt$$

2. Maintain slice thickness by accordingly reduced gradient:

$$G(t) = \frac{2\pi \Delta f \lambda(t)}{\Delta z} = G_0 \lambda(t)$$

with *instantaneous* RF bandwidth: $\Delta f \lambda(t)$

Binomial Pulses – Excitation of spectral components

Binomial pulses

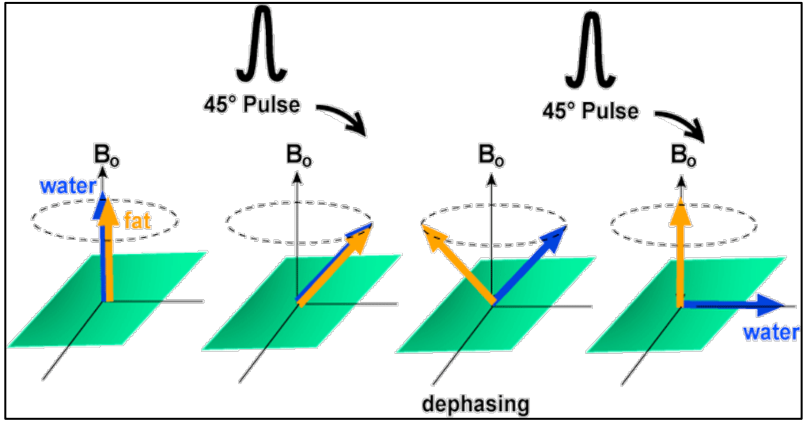
have flip angles that follow the pattern of coefficients of the binomial expansion of $(a+b)^n$

1-1, 1-2-1, 1-3-3-1, etc.

Thus, a 90-pulse could be constructed as a

- [45°, 45°] pair
- [22.5°, 45°, 22.5°] triplet
- [11.25°, 33.75°, 33.75°, 11.25°] quadruplet.

1-1 Pulse for water excitation (fat suppression)



http://mri-q.com/water-excitation.html

Time between pulses depends on off-resonance to be suppressed: $$\tau = \frac{\pi}{\Delta\omega} = \frac{1}{2\Delta\nu}$$

General formula (assuming instantaneous pulses): $$P(t) = \sum_{k=0}^{n} \binom{n}{k} \delta\left(t - \frac{n\tau}{2} + k\tau\right)$$

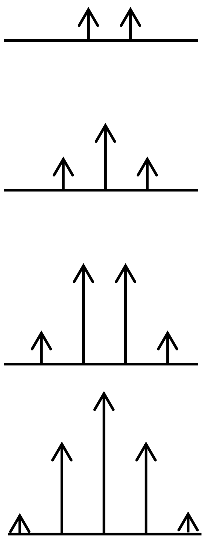
Binomial Pulses – Excitation of spectral components

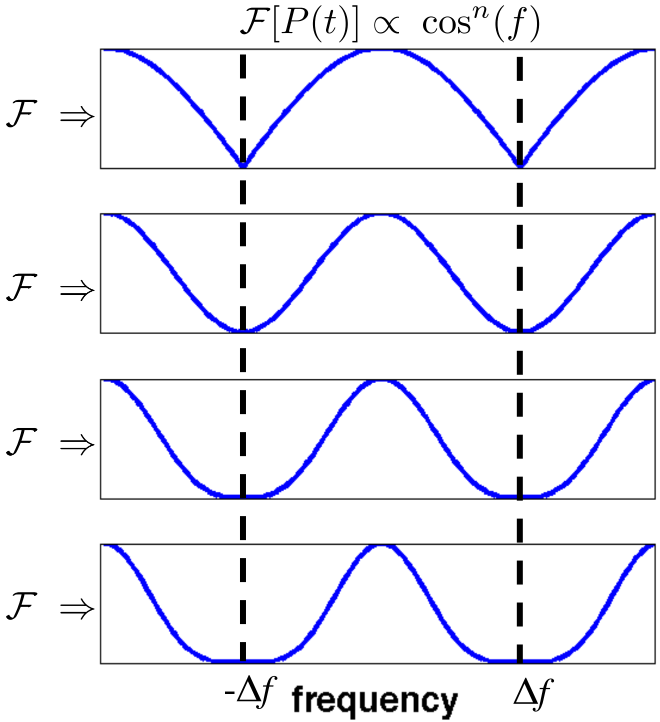
The higher the order, the better is the suppression of frequencies.

STA spectral response of binomial pulses:

order n	binomial
1	1-1
2	1-2-1
3	1-3-3-1
4	1-4-6-4-1

pulse scheme



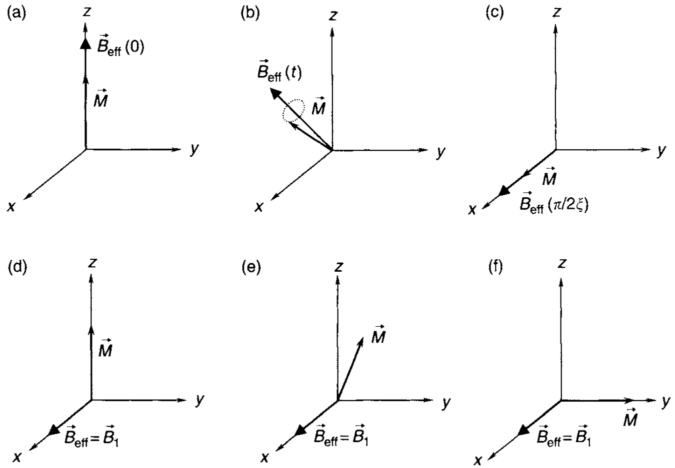


Adiabatic Pulses

- Frequency modulated (off-resonant!) pulses for <u>spatially homogenous excitation</u>
- Effective field changes due to frequency modulation
- *Adiabatic condition*: Magnetization follows the effective field
- Flip angle independent on B1 amplitude!

adiabatic excitation

on-resonant excitation



Bernstein et al, Handbook of MRI pulse sequences

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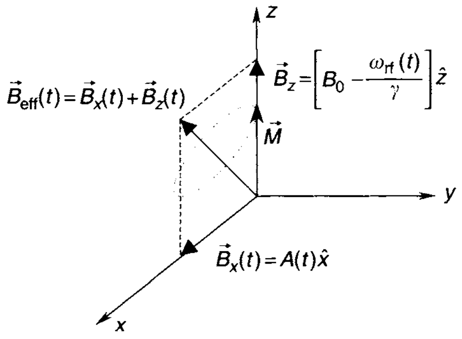


Bernstein et al, Handbook of MRI pulse sequences

$$B_1(t) = A(t)e^{-i\phi(t)}$$
$$\omega_{rf}(t) = \dot{\phi}(t) \quad \text{(freq. mod.)}$$
$$B_z(t) = B_0 - \omega_{rf}(t)/\gamma$$
$$|\vec{B}_{eff}| = \sqrt{A^2 + B_z^2}$$
$$\tan\Psi = \frac{A}{B_z}$$
$$\gamma|\vec{B}_{eff}| \gg \left|\frac{d\Psi}{dt}\right|$$

<u>Adiabatic passage principle:</u>
The magnetization vector follows the effective field, $\boldsymbol{B}_{eff}$, if its directional change, $|d\Psi/dt|$, is slow against the nutation frequency $\gamma|\boldsymbol{B}_{eff}|$

Closed-form solutions of the Bloch Equations (4)

Nonlinearity of the Bloch Equation

Bloch equation without relaxation in complex notation:

$$\dot{M} = -\mathrm{i}\Delta\omega M + \mathrm{i}\omega_1 M_z \qquad (M = M_x + \mathrm{i}M_y)$$

$$\dot{M}_z = \tfrac{\mathrm{i}}{2}\omega_1^* M - \tfrac{\mathrm{i}}{2}\omega_1 M^* \qquad (\omega_1 = \omega_{1x} + \mathrm{i}\omega_{1y})$$

Transformation of the linear coupled system into a

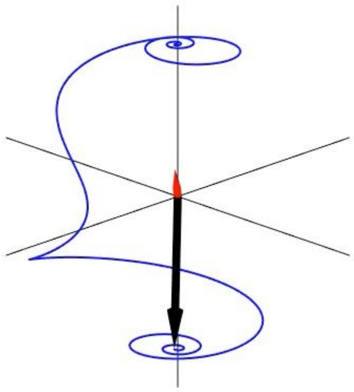
single quadratic differential equation (aka Ricatti Equation):

$$\dot{f} = -\mathrm{i}\Delta\omega f - \tfrac{\mathrm{i}}{2}\omega_1^* f^2 + \tfrac{\mathrm{i}}{2}\omega_1 \quad , \qquad f = \frac{M}{M_0 + M_z}$$

where $M_0 = \sqrt{|M_z(t)|^2 + |M(t)|^2} = \mathrm{const}$ (no relaxation!)

Analytic solution exists in case of the *Hyperbolic Secant RF-pulse* [1]

$$\omega_1(t) = \omega_1^{\max}\mathrm{sech}(\beta t)^{(1+i\mu)}$$



Adiabatic Inversion

Red = effective field

Blue = magnetization trajectory

[1] Singer et al, Phys. Rev. A, 1985

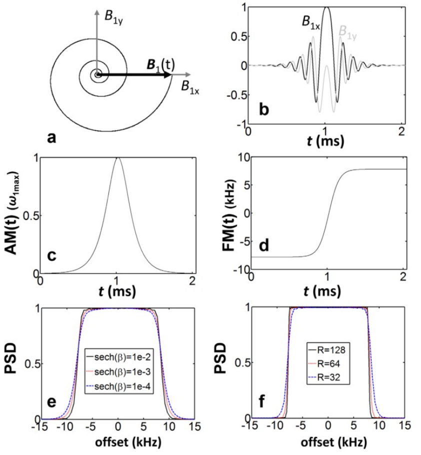
Adiabatic Pulses

The hyperbolic secant (sech) pulse

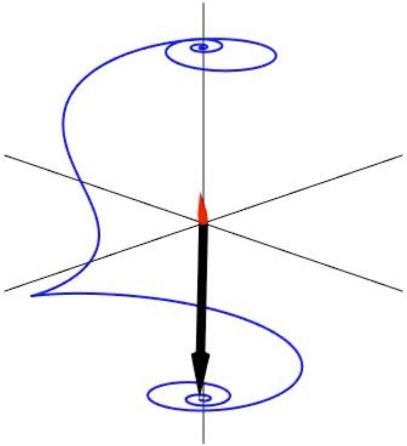
- Historically the first adiabatic pulse (Singer et al 1985)
- Homogenous inversion in inhomogeneous B_1-field, insensitive to off-resonance

$$\omega_1(t) = \omega_1^{\max}\mathrm{sech}(\beta t)^{(1+i\mu)} \;= A(t)\,\mathrm{e}^{-i\phi(t)} \;= A(t)\,\mathrm{e}^{-i\int \Omega(t)\,dt}$$

$$\text{where}\quad A(t) = \omega_1^{\max}\mathrm{sech}(\beta t)\;,\;\Omega(t) = \dot{\phi}(t) = -\mu\beta\tanh(\beta t)$$



Zhang et al, MRM 77, 2017



More complicated: adiabatic pulses for excitation of arbitrary flip angles

- can be achieved with so-called BIR pulses (B1-Independent-Rotation)
- generally, BIR-design through numerical solutions of Bloch Equation
- many BIR pulses exist with different sensitivity to off-resonance

Summary

MR Physics of RF pulses

- Fully described by Bloch equations
- Analytic solutions only in special cases (constant RF pulse ➞ rotation around effective field)
- Numerical solution with hard-pulse approximation (e.g. spinor calculus)
- Pulse design for selective excitation (slice selection): SLR pulses
- Other topics briefly covered: VERSE, binomial, and adiabatic pulses

Important topics not covered in this lecture

- Simultaneous-multi-slice (SMS) pulses
- Multidimensional selective pulses (2D or 3D)
- Parallel transmit (pTx) pulses (for B1 homogenization)
- Pulse design with optimal control theory (e.g. GRAPE pulses)
- …

Measurement of the pulse profile: Hands-on Pypulseq Example

Idea of this exercise: measure the pulse profile of various RF pulses with Pulseq

Given: a basic single-slice 2D spin echo imaging sequence (URL)

Your Tasks

- Change the sequence to measure the slice profile of the excitation pulse (how?)
- Investigate the slice profile of the sinc-pulse with and without apodization
- Change the excitation pulse and compare its slice profile with the previous results
  a) Possible pulses available in PyPulseq: Gauss, SLR, SMS (the last two use the interface to sigpy-rf)
  b) Implement your own pulse (?)