Solving the 1D Heat equation
Modelling the heat evolution inside a wall

GP3 - 2018

The Heat equation is a Partial Differential Equation (PDE) modelling the evolution of the heat inside a solid body:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C} \nabla . \left(k \nabla T \right)$$

With the thermal conductivity k, the density ρ and the specific heat capacity C.
In the 1D case and with k constant, it can be reduced to:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C} \frac{\partial^2 T}{\partial x^2}$$

We will study a wall of width $e = 35cm$ of concrete ($k = 0.28W/m/K$, $\rho = 600kg.m^{-3}$ and $C = 1kJ/K/kg$) with two different domains (exterior on the left and interior on the right) of respective temperature T_{ext} and T_{int} on each side :

- $T_{int} = 300K$
- $T_{ext} = 290 - 5\sin(\frac{2\pi t}{24 \times 3600})$

In order to keep thing simple for today, we will use finite difference, and more precisely:

- Forward first order finite difference in time (also called explicit Euler)
- Central second order finite difference in space
- First order finite difference in space at the boundaries

In time we will need an initial condition, wich is chosen to be constant in all the wall : $T(x, t = 0) = T_{ext}$.
Moreover, the time step Δt is constrained. We will take $\Delta t = \frac{\Delta x^2}{4a}$ with $a = \frac{k}{\rho C}$

In space we will need two boundary conditions. In order to impose them, the first point and the last point will answer to a specific equation, one of:

- Dirichlet: $T = T_{\text{bnd}}$
- Neuman: $-k\nabla T = \phi_{\text{bnd}}$
- Robin: $-k\nabla T + hT = \phi_{\text{bnd}} + hT_{\text{bnd}}$

with a solar flux $\phi_{ext} = 10W.m^{-2}$, an interior flux $\phi_{int} = 100W.m^{-2}$ and a convective coefficient $h = 10W.m^{-2}.K^{-1}$.

The objective of your work is to test the following sets of boundary conditions:

1. Dirichlet on both sides
2. Neumann on the left and Dirichlet on the right
3. Robin on the left and Dirichlet on the right
4. Robin on both sides

The complete program should be similar to that:

```
Initialize physical constants ;
Initialize parameters (tmax, precision, nx, etc.) ;
Initialize numerical constants (Δx, Δt, etc.));
Initialize variables: t, T, etc.;
while t < tmax
    Compute T_ext ;
    Compute T inside (except boundary conditions) ;
    Compute T at the boundaries ;
    Advance time ;
end
Print valuable infos ;
Plot the evolution of the temperature norm in time along with the exterior temperature;
Plot the final temperature in space;
```

The temperature norm will be computed as $\left(\frac{1}{e}\int_0^e T^2 dx\right)^{\frac{1}{2}}$

Complete the following tasks (in any order you want)

- Write a file tools.py containing:
  - a function norm(T, Δx, e) that return the norm of the current temperature
- Write a file finite.py containing:
  - a function dder(T, Δx) that return the second derivative of T inside
  - the functions apply_bnd_i(T, T_{int}, T_{ext}, k, h, ϕ_{int}, ϕ_{ext}, Δx) that apply the ith boundary condition to T and return nothing
- Write a file equation.py containing:
  - a function heat(T, Δt, Δx, k, ρ, C) that compute a new T inside (and zeros at the boundaries) and return this new T
- Write a file program.py containing:
  - a function T_ext(t) that compute the exterior temperature
  - the main program

Bellow your program add a large comment, answering the following questions:

1. How much time does it take to converge toward a “stationnary” state ?
2. Does the temperature of the wall and the exterior temperature have the same frequency ? Why ?
3. Does the temperature of the wall and the exterior temperature are synchronized ? Why ?
4. Does the number of points have an influence on the final temperature ?
5. What is the order of convergence of our discretisation ?
6. Why the time step is constrained ?
7. Can you give a physical interpretation of the 3 type of boundary conditions