Duality for Convexity

Bart Jacobs,

Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen, The Netherlands.
Webaddress: www.cs.ru.nl/B.Jacobs

November 19, 2009

Abstract

This paper studies convex sets categorically, namely as algebras of a distribution monad. It is shown that convex sets occur in two dual adjunctions, namely one with preframes via the Boolean truth values $\{0, 1\}$ as dualising object, and one with effect algebras via the (real) unit interval $[0, 1]_{\mathbb{R}}$ as dualising object. These effect algebras are of interest in the foundations of quantum mechanics.

1 Introduction

A set X is commonly called convex if for each pair of elements $x, y \in X$ and each number $r \in [0, 1]_{\mathbb{R}}$ in the unit interval of real numbers the "convex" sum $rx + (1 - r)y$ is again in X. Informally this says that a whole line segment is contained in X as soon as the endpoints are in X. Convexity is of course a well-established notion that finds applications in for instance geometry, probability theory, optimisation, economics and quantum mechanics (with mixed states as convex combinations of pure states). The definition of convexity (as just given) assumes a monoidal structure $+$ on the set X and also a scalar multiplication $[0, 1]_{\mathbb{R}} \times X \to X$. People have tried to capture this notion of convexity with fewer assumptions, see for instance [20], [22] or [10]. We shall use the latter source that involves a ternary operation $\langle -, -, - \rangle \colon [0, 1]_{\mathbb{R}} \times X \times X \to X$ satisfying a couple of equations, see Definition 9. We first recall (see *e.g.* [23, 7, 15, 5]) that such convex structures can equivalently be described uniformly as algebras of a monad, namely of the distribution monad $\mathcal{D}$, see Theorem 10. Such an algebra map gives an interpretation of each convex combination $r_1x_1 + \cdots + r_nx_n$, where $r_1 + \cdots + r_n = 1$, as a single element of X. This algebraic description of convexity allows us to generalise it from scalars $[0, 1]_{\mathbb{R}}$ (or actually $\mathbb{R}_{\geq 0}$) to arbitrary semirings (or semifields) S as scalars, and yields an abstract description of a familiar embedding construction as an adjunction between S-convex sets and S-semimodules, see Proposition 8 below.

The main topic of this paper is duality for convex spaces. We shall describe two dual adjunctions:

$$\begin{tikzcd}[column sep=large]
\mathbf{PreFrm} \arrow[r, bend left, "Hom(-{,}\{0{,}1\})"] \arrow[r, phantom, "\perp"] & \mathbf{Conv}^{\mathrm{op}} \arrow[l, bend left, "Hom(-{,}\{0{,}1\})"] \arrow[r, bend right, "Hom(-{,}[0{,}1]_{\mathbb{R}})"'] \arrow[r, phantom, "\perp"] & \mathbf{EA} \arrow[l, bend right, "Hom(-{,}[0{,}1]_{\mathbb{R}})"']
\end{tikzcd} \tag{1}$$

namely in Theorems 13 and 23. This diagram involves the following structures.

- The category **Conv** of (real) convex sets, with as special objects the unit interval $[0,1]_{\mathbb{R}}$ and the two element set $\{0,1\}$. This unit interval captures probabilities, and $\{0,1\}$ the Boolean truth values.

- The category **PreFrm** of preframes: posets with directed joins and finite meets, distributing over these joins, see [14]. These preframes are slightly more general than frames (or complete Heyting algebras) that occur in the familiar duality with topological spaces, see [13].

- The category **EA** of effect algebras (from [8], see also [6] for an overview). Effect algebras have arisen in the foundations of quantum mechanics and are used to capture quantum effects, as studied in quantum statistics and quantum measurement theory, see *e.g.* [4].

The diagram (1) thus suggests that convex sets form a setting in which one can study both Boolean and probabilistic logics. It opens up new questions, like: can the adjunctions be refined further so that one actually obtains equivalences, like between Stone spaces and Boolean algebras (see [13] for an overview). This is left to future work. Dualities are important in algebra, topology and logic, for transferring results and techniques from one domain to another. They are used in the semantics of computation (see *e.g.* [1, 24]), but are relatively new in a quantum setting. They may become part of what is called in [2] an "extensive network of interlocking analogies between physics, topology, logic and computer science".

The paper starts with a preliminary section that recalls basic definitions and facts about monads and their algebras. It leads to an adjunction in Proposition 8 between two categories of algebras, namely of the multiset monad and the distribution monad. Section 3 recalls in Theorem 10 how (real) convex sets can be described as algebras of the distribution monad, giving us the freedom to generalise convexity to arbitrary semirings (as scalars). Subsequently, Section 4 describes the adjunction on the left in (1) between convex sets and preframes, via prime filters in convex sets and Scott-open filters in preframes. Both can be described via homomorphisms to the dualising object $\{0,1\}$. The adjunction on the right in (1) requires that we first sketch the basics of effect algebras. This is done in Section 5. The unit interval $[0,1]_{\mathbb{R}}$ now serves as dualising object, where we note that effect algebra maps $E \to [0,1]_{\mathbb{R}}$ are commonly studied as states in a quantum system. The paper concludes in Section 7 with a few remarks about Hilbert spaces in relation to the dual adjunctions (1).

2 Monads and their algebras

A monad is a key concept in the generic categorical description of algebraic structures. It is defined as a functor $T\colon \mathbf{C} \to \mathbf{C}$ from a category $\mathbf{C}$ to itself together with two natural transformations, called the "unit" and "multiplication", see below. In the present context we don't need the full generality and shall thus restrict ourselves to the case where $\mathbf{C}$ is the category $\mathbf{Sets}$ of sets and functions. Associated with a monad there is a category $\mathit{Alg}(T)$ of algebras. Many mathematical structures of interest arise in this uniform manner. Categories of algebras $\mathit{Alg}(T)$ satisfy certain useful properties by default, see Theorem 5. This section will review some standard definitions and results on monads and their algebras that will be usefull in the sequel. More information may be found in for instance [18, 3, 19].

Definition 1 *A monad (on* $\mathbf{Sets}$*) consists of an endofunctor* $T\colon \mathbf{Sets} \to \mathbf{Sets}$ *together with two natural transformations: a unit* $\eta\colon \mathrm{id} \Rightarrow T$ *and multiplication* $\mu\colon T^2 \Rightarrow T$*. These are required to make the following diagrams commute, for* $X \in \mathbf{Sets}$*.*

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\eta_X)"'] \arrow[dl, equal] \\
& T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\
T^2 \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

We mention a few instances of this definition, of which the last one (the distribution monad $\mathcal{D}$) will be most important.

Example 2 (1) *Let* $(M, +, 0)$ *be a monoid. It can be used to construct a monad* $\widehat{M}\colon \mathbf{Sets} \to \mathbf{Sets}$ *given by* $\widehat{M}(X) = M \times X$*. The unit* $\eta\colon X \to M \times X$ *is* $\eta(x) = (0, x)$ *and the multiplication* $\mu\colon \widehat{M}^2(X) = M \times (M \times X) \to M \times X = \widehat{M}(X)$ *is given by* $\mu(a, (b, x)) = (a + b, x)$*.*

(2) *The powerset operation* $\mathcal{P}$ *forms a functor* $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}$*, which on a function* $f\colon X \to Y$ *yields* $\mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ *by direct image:* $\mathcal{P}(f)(U \subseteq X) = \{f(x) \mid x \in U\}$*. Powerset is also a monad: the unit* $\eta\colon X \to \mathcal{P}(X)$ *is given by singleton* $\eta(x) = \{x\}$ *and multiplication* $\mu\colon \mathcal{P}^2(X) \to \mathcal{P}(X)$ *by union* $\mu(V \subseteq \mathcal{P}(X)) = \bigcup V$*.*

(3) *Let* S *be a semiring, consisting of an additive monoid* $(S, +, 0)$ *and a multiplicative monoid* $(S, \cdot, 1)$*, where multiplication distributes over addition. One can define a "multiset" functor* $\mathcal{M}_S\colon \mathbf{Sets} \to \mathbf{Sets}$ *by:*

$$\mathcal{M}_S(X) \quad = \quad \{\varphi\colon X \to S \mid \mathrm{supp}(\varphi) \text{ is finite}\},$$

where $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ *is the support of* φ*. For a function* $f\colon X \to Y$ *one defines* $\mathcal{M}_S(f)\colon \mathcal{M}_S(X) \to \mathcal{M}_S(Y)$ *by:*

$$\mathcal{M}_S(f)(\varphi)(y) \quad = \quad \textstyle\sum_{x \in f^{-1}(y)} \varphi(x).$$

Such a multiset $\varphi \in \mathcal{M}_s(X)$ may be written as formal sum $s_1x_1 + \cdots + s_kx_k$ where $\mathrm{supp}(\varphi) = \{x_1, \ldots, x_k\}$ and $s_i = \varphi(x_i) \in S$ describes the "multiplicity" of the element x_i. This formal sum notation might suggest an order $1, 2, \ldots k$ among the summands, but this is misleading. The sum is considered, up-to-permutation of the summands. Also, the same element $x \in X$ may be counted multiple times, but $s_1x + s_2x$ is considered to be the same as $(s_1 + s_2)x$ within such expressions. With this formal sum notation one can write the application of $\mathcal{M}_S$ on a map f as $\mathcal{M}_S(f)(\sum_i s_ix_i) = \sum_i s_if(x_i)$. Functoriality is then obvious.

This multiset functor is a monad, with unit $\eta\colon X \to \mathcal{M}_S(X)$ is $\eta(x) = 1x$, and multiplication $\mu\colon \mathcal{M}_S(\mathcal{M}_S(X)) \to \mathcal{M}_S(X)$ given by $\mu(\sum_i s_i\varphi_i) = \lambda x.\, \sum_i s_i \cdot \varphi_i(x)$, where the "lambda" notation $\lambda x.\,\cdots$ is used for the function $x \mapsto \cdots$.

For the semiring $S = \mathbb{N}$ of natural numbers one gets the free commutative monoid $\mathcal{M}_{\mathbb{N}}(X)$ on a set X. And if $S = \mathbb{Z}$ one obtains the free Abelian group $\mathcal{M}_{\mathbb{Z}}(X)$ on X. The Boolean semiring $2 = \{0, 1\}$ yields the finite powerset monad $\mathcal{P}_{fin} = \mathcal{M}_2$.

(4) *Analogously to the previous example one defines the distribution monad $\mathcal{D}_S$ for a semiring S by:*

$$\mathcal{D}_S(X) \quad = \quad \{\varphi\colon X \to S \mid \mathrm{supp}(\varphi) \textit{ is finite and } \textstyle\sum_{x\in X} \varphi(x) = 1\},$$

Elements of $\mathcal{D}_S(X)$ are convex combinations $s_1x_1 + \cdots + s_kx_k$ where $\sum_i s_i = 1$. Unit and multiplication can be defined as before. This multiplication is well-defined since:

$$\textstyle\sum_x \mu(\sum_i s_i\varphi_i)(x) = \sum_x \sum_i s_i \cdot \varphi_i(x) = \sum_i s_i \cdot \big(\sum_x \varphi_i(x)\big) = \sum_i s_i = 1.$$

For the semiring $\mathbb{R}_{\geq 0}$ of non-negative real numbers one obtains the familiar distribution monad $\mathcal{D}_{\mathbb{R}_{\geq 0}}$ with elements $\sum_i r_ix_i$ containing probabilities $r_i \in [0, 1]_{\mathbb{R}}$ summing up to 1. Whenever we write $\mathcal{D}$ without semiring subscript we refer to this $\mathcal{D}_{\mathbb{R}_{\geq 0}}$. For the two-element semiring $2 = \{0, 1\}$—with join $\vee$ as sum and meet $\wedge$ as multiplication—the monad $\mathcal{D}_2$ is the non-empty finite powerset monad $\mathcal{P}^+_{fin}$.

The inclusion maps $\mathcal{D}_S(X) \hookrightarrow \mathcal{M}_S(X)$ are natural and commute with the units and multiplications of the two monads, and thus form an example of a "map of monads".

Definition 3 *Given a monad $T = (T, \eta, \mu)$ as in the previous definition, one defines an algebra of this monad as a map $\alpha\colon T(X) \to X$ satisfying two requirements, expressed via the diagrams:*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & T(X) \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2(X) \arrow[r, "\mu_X"] \arrow[d, "T(\alpha)"'] & T(X) \arrow[d, "\alpha"] \\ T(X) \arrow[r, "\alpha"'] & X \end{tikzcd}$$

We shall write Alg(T) *for the category with such algebras as objects. A morphism* $\big(T(X) \xrightarrow{\alpha} X\big) \xrightarrow{f} \big(T(Y) \xrightarrow{\beta} Y\big)$ *in* Alg(T) *is a map* $f\colon X \to Y$ *between the underlying sets satisfying* $f \circ \alpha = \beta \circ T(f)$.

There is an obvious forgetful functor $U\colon Alg(T) \to \mathbf{Sets}$ that maps an algebra to its underlying set: $U\big(T(X) \xrightarrow{\alpha} X\big) = X$. It has a left adjoint mapping a set Y to the multiplication μ_Y, as algebra $T(T(Y)) \to T(Y)$ on $T(Y)$.

We shall briefly review what algebras are of the monads (1)–(3) in Example 2. Elaborating all details requires some amount of work. The algebras of the fourth (distribution) monad will be characterised in the next section.

Example 4 (1) *The category of algebras of the monad* $\widehat{M} = M \times (-)$ *for a monoid* M *is precisely the category of* M*-actions and their morphisms. Such an action consists of a scalar multiplication map* $\bullet\colon M \times X \to X$ *satisfying two equations,* $0 \bullet x = x$ *and* $(a + b) \bullet x = a \bullet (b \bullet x)$, *corresponding to the two diagrams in Definition 3.*

(2) *Algebras* $\alpha\colon \mathcal{P}(X) \to X$ *for the powerset monad* $\mathcal{P}$ *correspond to a join operation of a complete lattice. Such* α *yields an partial order* $x \leq y \Leftrightarrow \alpha(\{x, y\}) = y$, *with* $\alpha(U)$ *as least upperbound of the elements in* U. *Algebra homomorphisms correspond to "linear" functions that preserve all joins.*

(3) *An algebra* $\alpha\colon \mathcal{M}_S(X) \to X$ *for the multiset monad corresponds to a monoid structure on* X*—given by* $x{+}y = \alpha(1x{+}1y)$*—together with a scalar multiplication* $\bullet\colon S{\times}X \to X$ *given by* $s \bullet x = \alpha(sx)$. *It preserves the additive structure (of* S *and of* X*) in each coordinate separately. This makes* X *a semimodule, for the semiring* S. *Conversely, such an* S*-semimodule structure on a commutative monoid* M *yields an algebra* $\mathcal{M}_S(M) \to M$ *by* $\sum_i s_i x_i \mapsto \sum_i s_i \bullet x_i$. *Thus the category of algebras* $Alg(\mathcal{M}_S)$ *is equivalent to the category* $\mathbf{SMod}_S$ *of* S*-semimodules.*

We continue this section with two basic results, which are stated without proof, but with a few subsequent pointers.

Theorem 5 *For a monad* T *on* **Sets***, the category* $Alg(T)$ *of algebras is:*

1. *both complete and cocomplete, so has all limits and colimits;*
2. *symmetric monoidal/tensorial closed in case the monad* T *is "commutative".* $\square$

A category of algebras is always "as complete" as its underlying category, see *e.g.* [19, 3]. Since **Sets** is complete, so is $Alg(T)$. Cocompleteness is special for algebras over **Sets** and follows from a result of Linton's, see [3, § 9.3, Prop. 4].

Monoidal structure in categories of algebras goes back to [17, 16]. Each monad on **Sets** is strong, via a "strength" map $\mathsf{st}\colon X \times T(Y) \to T(X \times Y)$ given as $\mathsf{st}(x, v) = T(\lambda y.\, \langle x, y\rangle)(v)$. There is also a swapped version $\mathsf{st}'\colon T(X) \times Y \to T(X \times Y)$ given by $\mathsf{st}'(u, y) = T(\lambda x.\, \langle x, y\rangle)(u)$. There are now in principle two maps $T(X) \times T(Y) \rightrightarrows T(X \times Y)$, namely:

$$\begin{tikzcd}[row sep=small] & T(T(X) \times Y) \arrow[r, "T(\mathsf{st}')"] & T^2(X \times Y) \arrow[dr, "\mu"] & \\ T(X) \times T(Y) \arrow[ur, "\mathsf{st}"] \arrow[dr, "\mathsf{st}'"'] & & & T(X \times Y) \\ & T(X \times T(Y)) \arrow[r, "T(\mathsf{st})"'] & T^2(X \times Y) \arrow[ur, "\mu"'] & \end{tikzcd}$$

The monad T is called commutative if these two composites $T(X) \times T(Y) \rightrightarrows T(X \times Y)$ are the same.

The monad $\widehat{M} = M \times (-)$ in Example 2 is commutative if and only if M is a commutative monoid. The other three examples $\mathcal{P}, \mathcal{M}_S$ and $\mathcal{D}$ are commutative.

We proceed with some elementary observations about the functoriality in the semiring S of the monad constructions $\mathcal{M}_S$ and $\mathcal{D}_S$ from Example 2.

Lemma 6 *Let $h\colon S \to S'$ be a homomorphism of semirings (preserving both $0, +$ and $1, \cdot$). It yields:*

1. *homomorphisms of monads $\mathcal{M}_S \to \mathcal{M}_{S'}$ and $\mathcal{D}_S \to \mathcal{D}_{S'}$ by post-composition: $\varphi \mapsto f \circ \varphi$, or equivalently, $\sum_i s_i x_i \mapsto \sum_i h(s_i) x_i$;*
2. *functors, in the opposite direction, between the associated categories of algebras $\mathrm{Alg}(\mathcal{M}_{S'}) \to \mathrm{Alg}(\mathcal{M}_S)$ and $\mathrm{Alg}(\mathcal{D}_{S'}) \to \mathrm{Alg}(\mathcal{D}_S)$, via pre-composition with the monad map from the previous point.* $\square$

Definition 7 *A semiring S is called zerosumfree if $x + y = 0$ implies both $x = 0$ and $y = 0$. It is integral if it has no zero divisors: $x \cdot y = 0$ implies either $x = 0$ or $y = 0$. And it is called a semifield if it is non-trivial (i.e. $0 \neq 1$), zerosumfree, integral and each non-zero element $s \in S$ has a multiplicative inverse $s^{-1} = \frac{1}{s} \in S$.*

The semiring $\mathbb{N}$ of natural numbers is non-trivial, zerosumfree and integral. The semirings $\mathbb{Q}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ of nonnegative rational and real numbers are examples of semifields. As is well-known, the quotient ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of integers modulo n is integral if n is prime.

For a non-trivial, zerosumfree, integral semiring S there is a homomorphism of semirings $h\colon S \to 2$ given by $h(x) = 0$ iff $x = 0$. For such a semiring Lemma 6 yields functors:

$$\begin{aligned} \mathbf{FJL} = \mathit{Alg}(\mathcal{P}_{\mathit{fin}}) = \mathit{Alg}(\mathcal{M}_2) &\longrightarrow \mathit{Alg}(\mathcal{M}_S) \\ \mathbf{BJL} = \mathit{Alg}(\mathcal{P}^+_{\mathit{fin}}) = \mathit{Alg}(\mathcal{D}_2) &\longrightarrow \mathit{Alg}(\mathcal{D}_S) \end{aligned} \tag{2}$$

where **FJL** is the category of finite join lattices, with finite joins $(0, \vee)$, and **BJL** the category of binary join lattices, with join $\vee$ only (and thus joins over all non-empty finite subsets). This construction uses for a lattice L the scalar multiplication:

$$S \times L \longrightarrow L \quad \text{given by} \quad (s, x) \longmapsto \begin{cases} 0 & \text{if } s = 0 \\ x & \text{otherwise.} \end{cases}$$

and thus the interpretation $s_1x_1+\cdots+s_nx_n \longmapsto x_1 \vee \cdots \vee x_n$, assuming $s_i \neq 0$ for each i. For the distribution monad $\mathcal{D}$ this involves a non-empty join, since the s_i must add up to 1. We can do the same for a meet semilattice $(K, \wedge, 1)$ since K^{op} with order reserved is a join semilattice. The scalar multiplication becomes $(0, x) \mapsto 1$ and $(s, x) \mapsto x$ if $s \neq 0$, so that the induced semimodule structure is, assuming $s_i \neq 0$,

$$s_1x_1 + \cdots + s_nx_n \quad \longmapsto \quad x_1 \wedge \cdots \wedge x_n. \tag{3}$$

The next construction goes back to [22] and occurs in many places (see *e.g.* [21, 15]) but is usually not formulated in the following way. It can be understood as a representation theorem turning a convex set into a semimodule.

Proposition 8 *Let S be a semifield. The functor $U\colon \mathit{Alg}(\mathcal{M}_S) \to \mathit{Alg}(\mathcal{D}_S)$ induced by the map of monads $\mathcal{D}_S \Rightarrow \mathcal{M}_S$ has a left adjoint.*

Proof Assume an algebra $\alpha\colon \mathcal{D}_S(X) \to X$ and write $S_{\neq 0} = \{s \in S \mid s \neq 0\}$ for the set of non-zero elements. We shall turn it into a semimodule $F(X)$, where:

$$F(X) \quad = \quad \{0\} + S_{\neq 0} \times X,$$

with addition for $u, v \in F(X)$,

$$u + v \quad = \quad \begin{cases} 0 & \text{if } u = 0 \text{ and } v = 0 \\ u & \text{if } v = 0 \\ v & \text{if } u = 0 \\ (s+t, \alpha(\frac{s}{s+t}x + \frac{t}{s+t}y)) & \text{if } u = (s, x) \text{ and } v = (t, y). \end{cases}$$

It is well-defined by zerosumfreeness of S. By construction, $0 \in F(X)$ is the neutral element for this $+$. A scalar multiplication $\bullet\colon S \times F(X) \to F(X)$ is defined as:

$$s \bullet u \quad = \quad \begin{cases} 0 & \text{if } u = 0 \text{ or } s = 0 \\ (s \cdot t, x) & \text{if } u = (t, x) \text{ and } s \neq 0. \end{cases}$$

Well-definedness follows because S is integral. Obviously, $1 \bullet u = u$ (because S is non-trivial) and $r \bullet (s \bullet u) = (r \cdot s) \bullet u$. This makes $F(X)$ a semimodule over S.

Next we show that F yields a left adjoint to $U\colon \mathit{Alg}(\mathcal{M}_S) \to \mathit{Alg}(\mathcal{D}_S)$, via the following bijective correspondence. For a semimodule Y,

$$\begin{array}{cc} X \xrightarrow{f} U(Y) & \text{in } \mathit{Alg}(\mathcal{D}_S) \\ \hline\hline F(X) \xrightarrow[g]{} Y & \text{in } \mathit{Alg}(\mathcal{M}_S) \end{array}$$

It works as follows.

- Given $f\colon X \to U(Y)$ in $Alg(\mathcal{D}_S)$ define $\overline{f}\colon F(X) \to Y$ by $\overline{f}(0) = 0$ and $\overline{f}(r,x) = r \bullet f(x)$ where $\bullet$ is scalar multiplication in Y. This yields a homomorphism of semimodules, *i.e.* a homomorphism of $\mathcal{M}_S$-algebras.

- Conversely, given $g\colon F(X) \to Y$ take $\overline{g}\colon X \to U(Y)$ to be $\overline{g}(x) = g(1,x)$. This yields a map of $\mathcal{D}_S$-algebras.

Finally we check that we actually have a bijective correspondence:

$$\overline{\overline{f}}(x) = \overline{f}(1,x) = 1 \bullet f(x) = f(x).$$

Similarly, $\overline{\overline{g}}(0) = 0$ and:

$$\overline{\overline{g}}(r,x) = r \bullet \overline{g}(x) = r \bullet g(1,x) = g(r \bullet (1,x)) = g(r,x). \qquad \square$$

3 Convex Sets

This section introduces convex structures—or simply, convex sets—as described in [10] and recalls that such structures can also be described as algebras of the distribution monad $\mathcal{D}$ from Example 2 (4).

Definition 9 *A convex set consists of a set X together with a ternary operation $\langle -,-,- \rangle\colon [0,1]_{\mathbb{R}} \times X \times X \to X$ satisfying the following four requirements, for all $r \in [0,1]_{\mathbb{R}}$ and $x,y,z \in X$.*

1. $\langle r,x,y \rangle = \langle 1-r,y,x \rangle$
2. $\langle r,x,x \rangle = x$
3. $\langle 0,x,y \rangle = y$
4. $\langle r,x,\langle s,y,z \rangle \rangle = \langle r+(1-r)s,\ \langle \frac{r}{(r+(1-r)s)},x,y \rangle,\ z \rangle$, *assuming that* $(r+(1-r)s) \neq 0$.

A morphism of convex structures $(X, \langle -,-,- \rangle_X) \to (Y, \langle -,-,- \rangle_Y)$ consists of an "affine" function $f\colon X \to Y$ satisfying $f(\langle r,x,x' \rangle_X) = \langle r, f(x), f(x') \rangle_Y$, for all $r \in [0,1]_{\mathbb{R}}$ and $x,x' \in X$. This yields a category **Conv**.

A convex set is sometimes called a barycentric algebra, using terminology from [22]. The tuple $\langle r,x,y \rangle$ can also be written as labeled sum $x +_r y$, like in [15], but the fourth condition becomes a bit difficult to read with this notation.

The next result recalls an alternative description of convex structures and their homomorphisms, namely as algebras of a monad. It goes back to [23] and also applies to compact Hausdorff spaces [15] or Polish spaces [5]. For convenience, a proof sketch is included. Recall that the notation $\mathcal{D}$ without subscript refers to the distribution monad $\mathcal{D}_{\mathbb{R}_{\geq 0}}$ for the semiring $\mathbb{R}_{\geq 0}$ of non-negative real numbers.

Theorem 10 *The category* **Conv** *of (real) convex structures is isomorphic to the category* $\mathrm{Alg}(\mathcal{D})$ *of Eilenberg-Moore algebras of the distribution monad. Hence convex sets are algebraic over sets.*

Proof Given an algebra $\alpha\colon \mathcal{D}(X) \to X$ on a set X one defines an operation $\langle -,-,-\rangle\colon [0,1]_{\mathbb{R}} \times X \times X \to X$ by:

$$\langle r, x, y\rangle \quad = \quad \alpha(rx + (1-r)y). \tag{4}$$

It is not hard to show that the four requirements from Definition 9 hold.

Conversely, given a convex set X with operation $\langle -,-,-\rangle$ one defines a function $\alpha\colon \mathcal{D}(X) \to X$ inductively by:

$$\begin{aligned} &\alpha(r_1x_1 + \cdots + r_nx_n) \\ &\quad = \begin{cases} x_1 & \text{if } r_1 = 1, \text{ so } r_2 = \cdots = r_n = 0 \\ \langle r_1, x_1, \alpha(\frac{r_2}{1-r_1}x_2 + \cdots + \frac{r_n}{1-r_1}x_n)\rangle & \text{otherwise, } \textit{i.e. } r_1 < 1. \end{cases} \end{aligned} \tag{5}$$

Repeated application of this definition yields:

$$\begin{aligned} &\alpha(r_1x_1 + \cdots + r_nx_n) \\ &\quad = \langle r_1, x_1, \langle \tfrac{r_2}{1-r_1}, x_2, \langle \tfrac{r_3}{1-r_1-r_2}, x_3, \langle \ldots, \langle \tfrac{r_{n-1}}{1-r_1-\cdots-r_{n-2}}, x_{n-1}, x_n\rangle \ldots\rangle\rangle\rangle\rangle. \end{aligned} \tag{6}$$

One first has to show that the function α in (5) is well-defined, in the sense that it does not depend on permutations of summands, see also [22, Lemma 2]. Via some elementary calculations one checks that exchanging the summands r_ix_i and $r_{i+1}x_{i+1}$ produces the same result. In a next step one proves the algebra equations: $\alpha \circ \eta = \mathrm{id}$ and $\alpha \circ \mu = \alpha \circ \mathcal{D}(\alpha)$. The first one is easy, since $\alpha(\eta(a)) = \alpha(1a) = a$, directly by applying (5). The second one requires more work. Explicitly, it amounts to:

$$\alpha\big(\textstyle\sum_{i\leq n} r_i\alpha(\sum_{j\leq m_i} s_{ij}x_{ij})\big) \quad = \quad \alpha\big(\textstyle\sum_{i\leq n}\sum_{j\leq m_i}(r_is_{ij})x_{ij}\big). \tag{7}$$

For the proof the following auxiliary result is convenient. It handles nested tuples in the second argument of a triple $\langle -,-,-\rangle$, just like condition (4) in Definition 9 deals with nested structure in the third argument. In a general convex structure one has:

$$\langle r, \langle s, x, y\rangle, z\rangle \quad = \quad \langle rs, x, \langle \tfrac{r(1-s)}{1-rs}, y, z\rangle\rangle. \tag{8}$$

assuming $rs \neq 1$. The rest is then left to the reader. $\square$

This theorem now allows us to apply Theorem 5 to the category **Conv** of (real) convex structures. First we may conclude that it is both complete and cocomplete; also, that the forgetful functor **Conv** $\to$ **Sets** has a left adjoint, giving free convex structures of the form $\mathcal{D}(X)$. And since $\mathcal{D}$ is a commutative monad, the category **Conv** is symmetric monoidal closed: maps $X \otimes Y \to Z$

in **Conv** correspond to functions $X \times Y \to Z$ that are "bi-homomorphisms", *i.e.* homomorphisms of convex structures in each variable separately. Closedness means that the functors $(-) \otimes Y$ have a right adjoint, given by $Y \multimap (-)$. Moreover, $\mathcal{D}(A \times B) \cong \mathcal{D}(A) \otimes \mathcal{D}(B)$, for set A, B.

Theorem 10 only applies to the particular monad $\mathcal{D} = \mathcal{D}_{\mathbb{R}_{\geq 0}}$ from our family of monad $\mathcal{D}_S$, for the special case where the semiring S is given by the non-negative real numbers $\mathbb{R}_{\geq 0}$. Of course, one may try to formulate a notion of "convex set", like in Definition 9 but more generally, with respect to a semiring S, possibly with some additional properties. But there is really no need to do so if we are willing to work in terms of algebras of the monad $\mathcal{D}_S$. In light of Theorem 10 one may consider such algebras as a generalised form of "S-convex set", and write $\mathbf{Conv}_S = \mathit{Alg}(\mathcal{D}_S)$. The only equations we thus have for such convex sets are the algebra equations, see Definition 3, with multiplication equation written explicitly in (7). Proposition 8 then describes an adjunction between S-convex sets and S-modules. This line of thinking will be pursued in the next section.

4 Prime filters in convex sets

The following definition generalises some familiar notions to S-convex sets, *i.e.* to $\mathcal{D}_S$-algebras. In [7] ideals instead of filters are used.

Definition 11 *Let S be a semiring and $\alpha\colon \mathcal{D}_S(X) \to X$ be an algebra of the monad $\mathcal{D}_S$, making X convex. We write $(\sum_{i \leq n} s_i x_i) \in \mathcal{D}_S(X)$ for an arbitrary convex combination. A subset $U \subseteq X$ is called a:*

- *subalgebra if $\forall_{i \leq n}. \, x_i \in U$ implies $\alpha(\sum_i s_i x_i) \in U$;*
- *filter if $\alpha(\sum_i s_i x_i) \in U$ implies $x_i \in U$, for each i with $s_i \neq 0$;*
- *prime filter if it is both a subalgebra and a filter.*

An element $x \in X$ is called extreme, or a boundary point, if $\{x\}$ is a prime filter. Often one writes ∂X for the set of extreme points.

It is not hard to see that subalgebras are closed under arbitrary intersections and under directed joins. Hence one can form the least subalgebra $\overline{V} \subseteq X$ containing an arbitrary set $V \subseteq X$, by intersection. Explicitly,

$$\overline{V} \quad = \quad \{\alpha(\textstyle\sum_i s_i x_i) \mid \forall_i. \, x_i \in V\}.$$

Filters are closed under arbitrary intersections and joins, hence also prime filters are closed under arbitrary intersections and directed joins. We shall write $\mathit{pFil}(X)$ for the set of prime filters in a convex set X, ordered by inclusion.

Notice that $\partial[0,1]_{\mathbb{R}} = \{0,1\}$ and $\overline{\{0,1\}} = [0,1]_{\mathbb{R}}$. Hence the unit interval is generated by its boundary points. In a free convex set $\mathcal{D}_S(A)$ the elements $\eta(a) = 1a \in \mathcal{D}_S(A)$, for $a \in A$, are the only boundary points. They also generate the whole convex set $\mathcal{D}_S(A)$. In a quantum context a state is called pure if it is a boundary point in the convex set of states, see Section 6. The set of mixed states is the closure of the set of pure states, given by convex combinations of these pure states.

Lemma 12 *Assume S is a non-trivial, zerosumfree and integral semiring and X is an S-convex set. A subset $U \subseteq X$ is a prime filter if and only if it is the "true kernel" $f^{-1}(1)$ of a homomorphism of convex sets $f\colon X \to \{0,1\}$. It yields an order isomomorphism:*

$$pFil(X) \quad\cong\quad Hom(X, \{0,1\}).$$

(Here we consider $\{0,1\}$ as meet semilattice, with the S-semimodule, and hence convex, structure described in (3)*.)*

Proof Let $\alpha\colon \mathcal{D}_S(X) \to X$ be an algebra on X. Given a prime filter $U \subseteq X$, define $f_U(x) = 1$ iff $x \in U$. This yields a homormophism of algebras/convex sets, since for a convex sum $\sum_i s_i x_i$ with $s_i \neq 0$,

$$\begin{array}{rcll}
(f_U \circ \alpha)(\sum_i s_i x_i) = 1 & \Longleftrightarrow & \alpha(\sum_i s_i x_i) \in U & \\
& \Longleftrightarrow & \forall_i.\, x_i \in U & \text{since } U \text{ is a prime filter} \\
& \Longleftrightarrow & \forall_i.\, f_U(x_i) = 1 & \\
& \Longleftrightarrow & \sum_i s_i f_U(x) = \bigwedge_i f_U(x_i) = 1 & \text{as in (3)} \\
& \Longleftrightarrow & (\beta \circ \mathcal{D}_S(f_U))(\sum_i s_i x_i) = 1, &
\end{array}$$

where $\beta\colon \mathcal{D}_S(\{0,1\}) \to \{0,1\}$ is the convex structure induced by the meet semilattice structure of $\{0,1\}$. Similarly one shows that such homomorphisms induce prime filters as their true-kernels. $\square$

We write **PreFrm** for the category of preframes. They consist of a poset L with directed joins $\bigvee^{\uparrow}$ and finite meets $(1, \wedge)$ distributing over these joins: $x \wedge \bigvee^{\uparrow}_i y_i = \bigvee^{\uparrow}_i x \wedge y_i$. Morphisms in **PreFrm** preserve both finite meets and directed joins. The two-element set $\{0,1\}$ is obviously a preframe. Homomorphisms of preframes $L \to \{0,1\}$ correspond (as true-kernels) to Scott-open filters $U \subseteq L$, see [24]. They are upsets, closed under finite meets, with the property that if $\bigvee^{\uparrow}_i x_i \in U$ then $x_i \in U$ for some i.

We have seen so far that taking prime filters yields a contravariant functor $pFil = Hom(-, \{0,1\})\colon \mathbf{Conv}_S = Alg(\mathcal{D}_S) \to \mathbf{PreFrm}$. The main result of this section shows that this forms actually a (dual) adjunction.

Theorem 13 *For each non-trivial zerosumfree and integral semiring S there is a dual adjunction between S-convex sets and preframes:*

$$\begin{tikzcd}
(\mathbf{Conv}_S)^{\mathrm{op}} \arrow[r, bend right, "Hom(-,\{0,1\})"'] \arrow[r, phantom, "\perp"] & \mathbf{PreFrm} \arrow[l, bend right, "Hom(-,\{0,1\})"']
\end{tikzcd}$$

Proof For a preframe L the homset $Hom(L, \{0,1\})$ of Scott-open filters is closed under finite intersections: if $\bigvee_i^{\uparrow} x_i \in U_1 \cap \cdots \cap U_m$, then for each $j \leq m$ there is an i_j with $x_j \in U_{i_j}$. By directedness there is an i with $x_i \geq x_{i_j}$ for each j, so that x_i is in each U_j. Hence, $Hom(L, \{0,1\})$ carries a $\mathcal{D}_S$-algebra structure as in (3). We shall write it as $\beta\colon \mathcal{D}_S(Hom(L, \{0,1\})) \to Hom(L, \{0,1\})$.

For an S-convex set X we need to construct a bijective correspondence:

$$\begin{array}{cl} \underline{X \xrightarrow{f} Hom(L, \{0,1\})} & \text{in } \mathbf{Conv}_S \\ \overline{L \xrightarrow[g]{} Hom(X, \{0,1\})} & \text{in } \mathbf{PreFrm} \end{array}$$

The correspondence between these f and g is given in the usual way by swapping arguments. $\square$

Homomorphisms from convex sets to the set of Boolean values $\{0,1\}$ capture only a part of what is going on. Richer structures arise via homomorphisms to the unit interval $[0,1]_{\mathbb{R}}$. They give rise to effect algebras, instead of preframes, as will be shown in the next two sections.

5 Effect algebras

This section recalls the basic definition, examples and result of effect algebras. To start, we need the notion of partial commutative monoid. It consists of a set M with a zero element $0 \in M$ and a partial binary operation $\varovee\colon M \times M \to M$ satisfying the three requirements below—involving the notation $x \perp y$ for: $x \varovee y$ is defined.

1. Commutativity: $x \perp y$ implies $y \perp x$ and $x \varovee y = y \varovee x$;
2. Associativity: $y \perp z$ and $x \perp (y \varovee z)$ implies $x \perp y$ and $(x \varovee y) \perp z$ and also $x \varovee (y \varovee z) = (x \varovee y) \varovee z$;
3. Zero: $0 \perp x$ and $0 \varovee x = x$;

An example of a partially commutative monoid is the unit interval $[0,1]_{\mathbb{R}}$ of real numbers, where $\varovee$ is the partially defined sum $+$. The notation $\varovee$ for the sum might suggest a join, but this is not intended, as the example $[0,1]_{\mathbb{R}}$ shows. We wish to avoid the notation $\oplus$ (and its dual $\otimes$) that is more common in the context of effect algebras because we like to reserve these operations $\oplus, \otimes$ for tensors on categories.

As an aside, for the more categorically minded, a partial commutative monoid may also be understood as a monoid in the category $\mathbf{Sets}_{\bullet}$ of pointed sets (or sets and partial functions). However, morphisms of partially commutative monoids are mostly total maps.

The notion of effect algebra is due to [8], see also [6] for an overview.

Definition 14 *An effect algebra is a partial commutative monoid $(E, 0, \varovee)$ with an orthosupplement. The latter is a unary operation $(-)^{\perp}\colon E \to E$ satisfying:*

1. $x^\perp \in E$ *is the unique element in* E *with* $x \ovee x^\perp = 1$*, where* $1 = 0^\perp$*;*

2. $x \perp 1 \Rightarrow x = 0$*.*

When writing $x \ovee y$ we shall implicitly assume that $x \ovee y$ is defined, *i.e.* that $x \perp y$ holds.

Example 15 *We briefly discuss several classes of examples.* (1) *A singleton set forms an example of a degenerate effect algebra, with* $0 = 1$*. A two element set* $2 = \{0, 1\}$ *is also an example.*

(2) *A more interesting example is the unit interval* $[0, 1]_{\mathbb{R}} \subseteq \mathbb{R}$ *of real numbers, with* $r^\perp = 1 - r$ *and* $r \ovee s$ *is defined as* $r + s$ *in case this sum is in* $[0, 1]_{\mathbb{R}}$*. In fact, for each positive number* $M \in \mathbb{R}$ *the interval* $[0, M]_{\mathbb{R}} = \{r \in \mathbb{R} \mid 0 \leq r \leq M\}$ *is an example of an effect algebra, with* $r^\perp = M - r$*.*

Also the interval $[0, M]_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid 0 \leq q \leq M\}$ *of rational numbers, for positive* $M \in \mathbb{Q}$*, is an effect algebra. And so is the interval* $[0, M]_{\mathbb{N}}$ *of natural numbers, for* $M \in \mathbb{N}$*.*

The general situation involves so-called "interval effect algebras", see e.g. [9] or [6, 1.4]. An Abelian group $(G, 0, -, +)$ *is called ordered if it carries a partial order* $\leq$ *such that* $a \leq b$ *implies* $a + c \leq b + c$*, for all* $a, b, c \in G$*. A positive point is an element* $p \in G$ *with* $p \geq 0$*. For such a point we write* $[0, p]_G \subseteq G$ *for the "interval"* $[0, p] = \{a \in G \mid 0 \leq a \leq p\}$*. It forms an effect algebra with* p *as top, orthosupplement* $a^\perp = p - a$*, and sum* $a + b$*, which is considered to be defined in case* $a + b \leq p$*.*

(3) *A separate class of examples has a join as sum* $\ovee$*. Let* $(L, \vee, 0, (-)^\perp)$ *be an ortholattice:* $\vee, 0$ *are finite joins and complementation* $(-)^\perp$ *satisfies* $x \leq y \Rightarrow y^\perp \leq x^\perp$*,* $x^{\perp\perp} = x$ *and* $x \vee x^\perp = 1 = 0^\perp$*. This* L *is called an orthomodular lattice if* $x \leq y$ *implies* $y = x \vee (x^\perp \wedge y)$*. Such an orthomodular lattice forms an effect algebra in which* $x \ovee y$ *is defined if and only if* $x \perp y$ *(i.e.* $x \leq y^\perp$*, or equivalently,* $y \leq x^\perp$*); and in that case* $x \ovee y = x \vee y$*. This restriction of* $\vee$ *is needed for the validity of requirements (1) and (2) in Definition 14:*

- *suppose* $x \ovee y = 1$*, where* $x \perp y$*, i.e.* $x \leq y^\perp$*. Then, by the orthomodularity property,*

$$y^\perp = x \vee (x^\perp \wedge y^\perp) = x \vee (x \vee y)^\perp = x \vee 1^\perp = x \vee 0 = x.$$

  Hence $y = x^\perp$*, making orthosupplements unique.*

- $x \perp 1$ *means* $x \leq 1^\perp = 0$*, so that* $x = 0$*.*

In particular, the lattice $\mathrm{KSub}(H)$ *of closed subsets of a Hilbert space* H *is an orthomodular lattice and thus an effect algebra. This applies more generally to the kernel subobjects of an object in a dagger kernel category [12]. These kernels can also be described as self-adjoint endomaps below the identity, see [12, Prop. 12]—in group-representation style, like in the above point 2.*

(4) *Since Boolean algebras are (distributive) orthomodular lattices, they are also effect algebras. By distributivity, elements in a Boolean algebra are orthogonal if and only if they are disjoint, i.e.* $x \perp y$ *iff* $x \wedge y = 0$*. In particular,*

the Boolean algebra of measurable subsets of a measurable space forms an effect algebra, where $U \ovee V$ is defined if $U \cap V = \emptyset$, and is then equal to $U \cup V$.

An obvious next step is to organise effect algebras into a category **EA**.

Definition 16 *A homomorphism $E \to D$ of effect algebras is given by a function $f\colon E \to D$ between the underlying sets satisfying:*

- *$x \perp x'$ in E implies both $f(x) \perp f(x')$ in D and $f(x \ovee x') = f(x) \ovee f(x')$;*
- *$f(1) = 1$.*

Effect algebras and their homomorphisms form a category, which we call **EA**.

Homomorphisms are like measurable maps. Indeed, for the effect algebra Σ associated in Example 15 (4) with a measureable space (X, Σ), effect algebra homomorphisms $f\colon \Sigma \to [0,1]_{\mathbb{R}}$ satisfy $f(U \cup V) = f(U) + f(V)$ in case U, V are disjoint—because then $U \ovee V$ is defined and equals $U \cup V$. In general, effect algebra homomorphisms $E \to [0,1]_{\mathbb{R}}$ to the unit interval are often called states. They form a convex subset, see Section 6.

Homomorphisms of effect algebras preserve all the relevant structure.

Lemma 17 *Let $f\colon E \to D$ be a homomorphism of effect algebras. Then:*

$$f(x^{\perp}) = f(x)^{\perp} \quad \textit{and thus} \quad f(0) = 0.$$

Proof From $1 = f(1) = f(x \ovee x^{\perp}) = f(x) \ovee f(x^{\perp})$ we obtain $f(x^{\perp}) = f(x)^{\perp}$ by uniqueness of orthosupplements. In particular, $f(0) = f(1^{\perp}) = f(1)^{\perp} = 1^{\perp} = 0$. $\square$

Example 18 *It is not hard to see that the one-element effect algebra 1 is final, and the two-element effect algebra 2 is initial.*

Orthosupplement $(-)^{\perp}$ is a homomorphism $E \to E^{\mathrm{op}}$ in **EA***, namely from $(E, 0, \ovee, (-)^{\perp})$ to $E^{\mathrm{op}} = (E, 1, \owedge, (-)^{\perp})$, where $x \owedge y = (x^{\perp} \ovee y^{\perp})^{\perp}$.*

An element (or point) $x \in E$ of an effect algebra E can be identified with a homomorphism $2 \times 2 \to E$ in **EA***, as in:*

$$2 \times 2 = \mathsf{MO}(2) = \left(\begin{array}{ccc} & 1 & \\ \bullet & & \bullet^{\perp} \\ & 0 & \end{array}\right) \xrightarrow{\quad x \quad} E$$

On the homset $\mathrm{Hom}(E, D)$ of homomorphisms $E \to D$ in **EA** *one may define $(-)^{\perp}$ and $\ovee$ pointwise, as in $f^{\perp}(x) = f(x)^{\perp}$. But this does not yield a homomorphism $E \to D$, since for instance $f^{\perp}(1) = f(1)^{\perp} = 1^{\perp} = 0$. Hence these homsets do not form effect algebras.*

Example 19 *Recall from Example 15.(2) the effect algebra $[0,1]_{\mathbb{Q}}$ given by the unit interval of rational numbers. We claim that it has precisely one state: there is precisely one morphism of effect algebras $f\colon [0,1]_{\mathbb{Q}} \to [0,1]_{\mathbb{R}}$, namely* *the inclusion. To see this we first prove that* $f(\frac{1}{n}) = \frac{1}{n}$ *for each positive* $n \in \mathbb{N}$. *Since the* n*-fold sum* $\frac{1}{n} + \cdots + \frac{1}{n}$ *equals* 1 *this follows from:*

$$1 = f(1) = f(\tfrac{1}{n} + \cdots + \tfrac{1}{n}) = f(\tfrac{1}{n}) + \cdots + f(\tfrac{1}{n}).$$

Similarly we get $f(\frac{m}{n}) = \frac{m}{n}$, *for* $m \leq n$, *via an* m*-fold sum:*

$$f(\tfrac{m}{n}) = f(\tfrac{1}{n} + \cdots + \tfrac{1}{n}) = f(\tfrac{1}{n}) + \cdots + f(\tfrac{1}{n}) = \tfrac{1}{n} + \cdots + \tfrac{1}{n} = \tfrac{m}{n}.$$

We briefly mention some basic structure in the category of effect algebras.

Proposition 20 *The category* **EA** *of effect algebras is complete, where products and equalisers are constructed as in* **Sets** *and equipped with the appropriate effect algebra structure.*

The category **EA** *also has set-indexed coproducts, given by identifying top and bottom elements, as in: the coproduct* $E + D = \big((E - \{0,1\}) + (D - \{0,1\})\big) + \{0,1\}$, *where* $+$ *on the right-hand-side of the equality is disjoint union (or coproduct) of sets.* $\square$

Coequalisers in **EA** are more complicated, but are not needed here.

6 Effect algebras and convex sets

Our aim in this section is to establish the dual adjunction between convex sets and effect algebras on the right in the diagram (1) in the introduction. From now on we restrict ourselves to the semiring $\mathbb{R}_{\geq 0}$ of non-negative real numbers. As before, we omit it as subscript and write $\mathcal{D}$ for $\mathcal{D}_{\mathbb{R}_{\geq 0}}$ and **Conv** for $\mathbf{Conv}_{\mathbb{R}_{\geq 0}} = \mathit{Alg}(\mathcal{D}_{\mathbb{R}_{\geq 0}})$.

We already mentioned that the unit interval $[0,1]_{\mathbb{R}}$ of real numbers is a convex set. The set of states of an effect algebra is also convex, as noticed for instance in [9].

Lemma 21 *The state functor* $\mathcal{S} = \mathit{Hom}(-, [0,1]_{\mathbb{R}}) \colon \mathbf{EA} \to \mathbf{Sets}^{\mathrm{op}}$ *restricts to* $\mathbf{EA} \to \mathbf{Conv}^{\mathrm{op}}$.

Proof Let E be an effect algebra with states $f_i \colon E \to [0,1]_{\mathbb{R}}$ and $r_i \in [0,1]_{\mathbb{R}}$ with $\sum_i r_i = 1$, then we can form a new state $f = r_1 f_1 + \cdots + r_n f_n$ by $f(x) = \sum_i r_i \cdot f_i(x)$, using multiplication $\cdot$ in $[0,1]_{\mathbb{R}}$. This yields a homomorphism of effect algebras $E \to [0,1]_{\mathbb{R}}$, since:

- $f(1) = \sum_i r_i \cdot f_i(1) = \sum_i r_i \cdot 1 = \sum_i r_i = 1$;
- if $x \perp x'$ in E, then in $[0,1]_{\mathbb{R}}$:

$$\begin{array}{rclcl} f(x \ovee x') & = & \sum_i r_i \cdot f_i(x \ovee x') & = & \sum_i r_i \cdot (f_i(x) + f_i(x')) \\ & & & = & \sum_i r_i \cdot f_i(x) + r_i \cdot f_i(x') \\ & & & = & \sum_i r_i \cdot f_i(x) + \sum_i r_i \cdot f_i(x') \\ & & & = & f(x) + f(x'). \end{array}$$

Further, for a map of effect algebras $g\colon E \to D$ the induced function $\mathcal{S}(g) = (-) \circ g\colon \mathit{Hom}(D, [0,1]_{\mathbb{R}}) \to \mathit{Hom}(E, [0,1]_{\mathbb{R}})$ is a map of convex sets:

$$\begin{array}{rcl} \mathcal{S}(g)(\sum_i r_i f_i) & = & \lambda x.\, (\sum_i r_i f_i)(g(x)) \\ & = & \lambda x.\, \sum_i r_i \cdot f_i(g(x)) \\ & = & \lambda x.\, \sum_i r_i \cdot \mathcal{S}(g)(f_i)(x) \\ & = & \sum_i r_i(\mathcal{S}(g)(f_i)). \end{array} \qquad \square$$

Interestingly, there is also a Hom functor in the other direction.

Lemma 22 *For each convex set X the homset $\mathit{Hom}(X, [0,1]_{\mathbb{R}})$ of homomorphisms of convex sets is an effect algebra. In this way one gets a functor $\mathit{Hom}(-, [0,1]_{\mathbb{R}})\colon \mathbf{Conv}^{\mathrm{op}} \to \mathbf{EA}$.*

Proof Let X be a convex set. We have to define effect algebra structure on the homset $\mathit{Hom}(X, [0,1]_{\mathbb{R}})$. There is an obvious zero element, namely the zero function $\lambda x.\, 0$. A partial sum $f + f'$ is defined as $(f + f')(x) = f(x) + f'(x)$, provided the sum $f(x) + f'(x) \leq 1$ for all $x \in X$. It is easy to see that this $f + f'$ is again a map of convex sets. Similarly, one defines $f^{\perp} = \lambda x.\, 1 - f(x)$, which is again a homomorphism since:

$$\begin{array}{rcl} f^{\perp}(r_1x_1 + \cdots + r_nx_n) & = & 1 - f(r_1x_1 + \cdots + r_nx_n) \\ & = & (r_1 + \cdots + r_n) - (r_1 \cdot f(x_1) + \cdots + r_n \cdot f(x_n)) \\ & = & r_1 \cdot (1 - f(x_1)) + \cdots + r_n \cdot (1 - f(x_n)) \\ & = & r_1 \cdot f^{\perp}(x_1) + \cdots + r_n \cdot f^{\perp}(x_n). \end{array}$$

Functoriality is easy: for a map $g\colon X \to Y$ of convex sets we obtain a map of effect algebras $(-) \circ g\colon \mathit{Hom}(Y, [0,1]_{\mathbb{R}}) \to \mathit{Hom}(X, [0,1]_{\mathbb{R}})$ since:

- $1 \circ g = \lambda x.\, 1(g(x)) = \lambda x.\, 1 = 1$;
- $(f + f') \circ g = \lambda x.\, (f + f')(g(x)) = \lambda x.\, f(g(x)) + f'(g(x)) = \lambda x.\, (f \circ g)(x) + (f' \circ g)(x) = (f \circ g) + (f' \circ g)$. $\square$

The next result is now an easy combination of the previous two lemmas.

Theorem 23 *There is a dual adjunction between convex sets and effect algebras:*

$$\begin{tikzcd} \mathbf{Conv}^{\mathrm{op}} \arrow[r, bend right, "\mathit{Hom}(-,[0,1]_{\mathbb{R}})"'] \arrow[r, phantom, "\perp"] & \mathbf{EA} \arrow[l, bend right, "\mathcal{S}=\mathit{Hom}(-,[0,1]_{\mathbb{R}})"'] \end{tikzcd}$$

Proof We need to check that the unit and counit

$$\begin{array}{ccc} E \xrightarrow{\eta} \mathit{Hom}(\mathcal{S}(E), [0,1]_{\mathbb{R}}) & \qquad & X \xrightarrow{\varepsilon} \mathcal{S}(\mathit{Hom}(X, [0,1]_{\mathbb{R}})) \\ x \longmapsto \lambda f.\, f(x) & & x \longmapsto \lambda f.\, f(x) \end{array}$$

are appropriate homomorphisms. First we check that η is a map of effect algebras:

- $\eta(1) = \lambda f.\, f(1) = \lambda f.\, 1 = 1$;
- and if $x \perp x'$ in E, then:

$$\begin{array}{rcl} \eta(x \ovee x') \;=\; \lambda f.\, f(x \ovee x') & = & \lambda f.\, f(x) + f(x') \\ & = & \lambda f.\, \eta(x)(f) + \eta(x')(f) \\ & = & \eta(x) + \eta(x'). \end{array}$$

Similarly ε is a map of convex sets:

$$\begin{array}{rcl} \varepsilon(r_1 x_1 + \cdots + r_n x_n) & = & \lambda f.\, f(r_1 x_1 + \cdots + r_n x_n) \\ & = & \lambda f.\, r_1 \cdot f(x_1) + \cdots + r_n \cdot f(x_n) \\ & = & \lambda f.\, r_1 \cdot \varepsilon(x_1)(f) + \cdots + r_n \cdot \varepsilon(x_n)(f) \\ & = & r_1 \varepsilon(x_1) + \cdots + r_n \varepsilon(x_n). \end{array}$$

□

Recall that the forgetful functor $U \colon \mathbf{Conv} = \mathit{Alg}(\mathcal{D}) \to \mathbf{Sets}$ has a left adjoint, also written as $\mathcal{D}$. By taking opposites $\mathcal{D}$ becomes a right adjoint to U, so that we can compose adjoint, as in the following result.

Proposition 24 *By composition of adjoints, as in:*

$$\begin{tikzcd}
\mathbf{Conv}^{\mathrm{op}} \arrow[rr, bend right=10, "\mathit{Hom}(-,[0,1]_{\mathbb{R}})"'] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right=15, "U"'] \arrow[dr, phantom, "\dashv"] & & \mathbf{EA} \arrow[ll, bend right=10, "\mathit{Hom}(-,[0,1]_{\mathbb{R}})"'] \arrow[dl, bend right=15] \arrow[dl, phantom, "\dashv"] \\
& \mathbf{Sets}^{\mathrm{op}} \arrow[ul, bend right=15, "\mathcal{D}"'] \arrow[ur, bend right=15, "([0,1]_{\mathbb{R}})^{(-)}"'] &
\end{tikzcd}$$

one obtains in a standard way a dual adjunction between effect algebras and sets.

Proof Because by the adjunction $\mathcal{D} \dashv U$, for $X \in \mathbf{Sets}$,

$$\mathit{Hom}_{\mathbf{Conv}}(\mathcal{D}(X), [0,1]_{\mathbb{R}}) \cong \mathit{Hom}_{\mathbf{Sets}}(X, U([0,1]_{\mathbb{R}})) \cong \big([0,1]_{\mathbb{R}}\big)^X$$

which, yields an effect algebra because by Proposition 20 effect algebras are closed under products (and hence under powers). Its right adjoint is $E \mapsto \mathit{Hom}_{\mathbf{EA}}(E, [0,1]_{\mathbb{R}}) = U\big(\mathit{Hom}_{\mathbf{EA}}(E, [0,1]_{\mathbb{R}})\big)$, where this last homset is considered as object of the category $\mathbf{Conv}$. □

7 Hilbert spaces

In the end one may ask: how does the standard way of modeling quantum phenomena in Hilbert spaces fit in the picture (1) provided by the dual adjunctions? The answer is: only partially. There is a contravariant functor from

Hilbert spaces to effect algebras, mapping a Hilbert space to its orthomodular lattice (and hence effect algebra) of closed subspaces. The unit ball H_1 in each Hilbert space H—with H_1 consisting of points $a \in H$ with $\|a\| \leq 1$—is convex. However, this mapping $H \mapsto H_1$ is not functorial in an obvious way. Nevertheless, each unit element does give rise to a state, as described in the following lemma. It uses notation and terminology from [12, 11].

Lemma 25 *Let H be a Hilbert space, with a unit element $a \in H$ (so that $\|a\| = 1$). It gives rise to a map of effect algebras:*

$$\mathrm{KSub}(H) \xrightarrow{\varepsilon(a)} [0,1]_{\mathbb{R}}, \quad \textit{namely} \quad k \longmapsto \langle k^{\dagger}(a) \,|\, k^{\dagger}(a)\rangle = \|k^{\dagger}(a)\|^2$$

where $\mathrm{KSub}(H)$ *is the orthomodular lattice of closed subspaces* $k\colon K \rightarrowtail H$.

Proof Clearly $\varepsilon(a)(k) \geq 0$. Further, $\varepsilon(a)$ preserves the top element:

$$\varepsilon(a)(1) = \varepsilon(a)(\mathrm{id}_H) = \|\mathrm{id}^{\dagger}(a)\|^2 = \|\mathrm{id}(a)\|^2 = \|a\|^2 = 1^2 = 1.$$

We show that $\varepsilon(a)$ is monotone. Assume therefor $k \leq k'$ in $\mathrm{KSub}(H)$. Then we can write k' as cotuple $[k, m]\colon K \oplus M \rightarrowtail H$, where $K \oplus M = K \times M$ describes the biproduct in **Hilb**, so that:

$$\begin{array}{rcl}
\varepsilon(a)(k') \;=\; \varepsilon(a)([k,m]) & = & \langle [k,m]^{\dagger}(a) \,|\, [k,m]^{\dagger}(a)\rangle \\
& = & \langle \langle k^{\dagger}, m^{\dagger}\rangle(a) \,|\, \langle k^{\dagger}, m^{\dagger}\rangle(a)\rangle \\
& = & \langle \langle k^{\dagger}(a), m^{\dagger}(a)\rangle \,|\, \langle k^{\dagger}(a), m^{\dagger}(a)\rangle\rangle \\
& = & \langle k^{\dagger}(a) \,|\, k^{\dagger}(a)\rangle + \langle m^{\dagger}(a) \,|\, m^{\dagger}(a)\rangle \\
& = & \varepsilon(a)(k) + \varepsilon(a)(m).
\end{array}$$

Hence $\varepsilon(a)(k) \leq \varepsilon(a)(k')$ in $\mathbb{R}$. In particular, since each $k \in \mathrm{KSub}(H)$ satisfies $k \leq 1$ we get $\varepsilon(a)(k) \leq \varepsilon(a)(1) = 1$. Therefor $\varepsilon(a)(k) \in [0,1]_{\mathbb{R}}$.

Finally we show that $\varepsilon(a)$ is a map of effect algebras. Assume $k \perp m$ for $k, m \in \mathrm{KSub}(H)$. This means $k \leq m^{\dagger}$ so that $k^{\dagger} \circ m = 0$ and also $m^{\dagger} \circ k = 0$. Hence the cotuple $[k, m]$ is a dagger mono, and thus the join $k \otimes m$. Then, like in the previous computation:

$$\varepsilon(a)(k \otimes m) = \varepsilon(a)([k,m]) = \varepsilon(a)(k) + \varepsilon(a)(m). \qquad \square$$

The resulting mapping $\varepsilon\colon H_1 \to \mathcal{S}(\mathrm{KSub}(H)) = \mathbf{EA}(\mathrm{KSub}(H), [0,1]_{\mathbb{R}})$ need not preserve convex sums. Hence Hilbert spaces do not fit nicely in the dual adjunctions diagram (1). More research is needed to clarify the situation. In particular, it seems worthwhile to bring compact and Hausdorff spaces into the picture, like in [15], and to look for restrictions of the dual adjunction in Theorem 23, possibly involving C^*-algebras (instead of Hilbert spaces).

Acknowledgements

Thanks to Dion Coumans, Chris Heunen, Bas Spitters and Jorik Mandemaker for feedback and/or helpful discussions.

References

[1] S. Abramsky. Domain theory in logical form. *Ann. Pure & Appl. Logic*, 51(1/2):1–77, 1991.

[2] J.C. Baez and M. Stay. Physics, topology, logic and computation: A rosetta stone. 2009. `arXiv/0903.0340`.

[3] M. Barr and Ch. Wells. *Toposes, Triples and Theories*. Springer, Berlin, 1985. Revised and corrected version available from URL: `www.cwru.edu/artsci/math/wells/pub/ttt.html`.

[4] P. Busch, M. Grabowski, and P. Lahti. *Operational Quantum Physics*. Springer Verlag, Berlin, 1995.

[5] E.-E. Doberkat. Eilenberg-Moore algebras for stochastic relations. *Inf. & Comp.*, 204(12):1756–1781, 2006. Erratum and addendum in: 206(12):1476–1484, 2008.

[6] A. Dvurečenskij and S. Pulmannová. *New Trends in Quantum Structures*. Kluwer Acad. Publ., Dordrecht, 2000.

[7] J. Flood. Semiconvex geometry. *Journ. Austr. Math. Soc.*, Series A 30:496–510, 1981.

[8] D. J. Foulis and M.K. Bennett. Effect algebras and unsharp quantum logics. *Found. Physics*, 24(10):1331–1352, 1994.

[9] D. J. Foulis, R.J. Greechie, and M.K. Bennett. The transition to unigroups. *Int. Journ. Theor. Physics*, 37(1):45–63, 1998.

[10] S. Gudder. A general theory of convexity. *Milan Journal of Mathematics*, 49(1):89–96, 1979.

[11] C. Heunen. *Categorical Quantum Models and Logics*. PhD thesis, Univ. Nijmegen, 2010, to appear.

[12] C. Heunen and B. Jacobs. Quantum logic in dagger kernel categories. In B. Coecke, P. Panangaden, and P. Selinger, editors, *Proceedings of the 6th International Workshop on Quantum Programming Languages (QPL 2009)*, Elect. Notes in Theor. Comp. Sci. Elsevier, Amsterdam, 2010. Available from `http://arxiv.org/abs/0902.2355`.

[13] P.T. Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1982.

[14] P.T. Johnstone and S. Vickers. Preframe presentations present. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Como Conference on Category Theory*, number 1488 in Lect. Notes Math., pages 193–212. Springer, Berlin, 1991.

[15] K. Keimel. The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras. *Topology and its Applications*, 156:227–239, 2008.

[16] A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, 29:161–174, 1971.

[17] A. Kock. Closed categories generated by commutative monads. *Journ. Austr. Math. Soc.*, XII:405–424, 1971.

[18] S. Mac Lane. *Categories for the Working Mathematician*. Springer, Berlin, 1971.

[19] E.G. Manes. *Algebraic Theories*. Springer, Berlin, 1974.

[20] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

[21] S. Pulmannová and S. Gudder. Representation theorem for convex effect algebras. *Commentationes Mathematicae Universitatis Carolinae*, 39(4):645–659, 1998. Available from `http://dml.cz/dmlcz/119041`.

[22] M.H. Stone. Postulates for the barycentric calculus. *Ann. Math.*, 29:25–30, 1949.

[23] T. Swirszcz. Monadic functors and convexity. *Bull. de l'Acad. Polonaise des Sciences. Sér. des sciences math., astr. et phys.*, 22:39–42, 1974.

[24] S. Vickers. *Topology Via Logic*. Number 5 in Tracts in Theor. Comp. Sci. Cambridge Univ. Press, 1989.