The importance of generating series in Algebraic Geometry

I asked this question on MSE but got no answer. I was advised to put it here. I am sorry if it is not suitable for MO.

What follows is a very nebulous question. I just seek for some help in understanding a "technique" which has proven itself very powerful.

I noticed that very often generating series appear in Algebraic Geometry. I am referring to enumerative geometry, for instance Gromov-Witten theory or Donaldson-Thomas theory. I am wondering: why can they be a so powerful tool? What is their secret, which makes them so helpful even if they look, at a glance, so intractable?

What is actually happening when we have computed some numbers, enumerative invariants, classes in a Grothendieck ring... whatever data we are interested in, and we put them all together in a generating series?

Example. Think about Witten's conjecture. One can compute Gromov-Witten invariants

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cup \cdots \cup \psi_n^{a_n}$$

and put them all together in the generating series

$$F_g = \sum_{n\geq 0} \frac{1}{n!} \sum_{a_1,\ldots,a_n} \left(\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cup \cdots \cup \psi_n^{a_n} \right) t_{a_1} \ldots t_{a_n}.$$

Not yet satisfied, one takes the generating series over all genera $F = \sum_g F_g h^{2g-2}$ and then the exponential e^F of F. Witten's conjecture is equivalent to $L_n e^F = 0$ for all n, where L_n are certain differential operators.

I am sorry for the vagueness of the above. Any insight and concrete example is very welcome.

ag.algebraic-geometry

asked Sep 20 '13 at 20:32
Brenin
585 4 13

1 The obvious guess is that the entire series has some meaning, e.g. maybe it is the asymptotic expansion of some path integral or something. – Qiaochu Yuan Sep 20 '13 at 21:56

1 Plainly, to paraphrase a line from an article (*Geometry and physics*, Phil. Trans. R. Soc. A 2010, highly recommended) by Atiyah, Dijkgraaf, and Hitchin, dualities in physics, including the one that underlies the enumeration formula you mention, are often [always?] "captured by a generating function that allows two different expansions." This is true of other mathematical formulas as well. In this way generating series are more than just a formal book keeping device for recurrence relations among coefficients. In any case they fully deserve to be called functions rather than power series. – Vesselin Dimitrov Sep 21 '13 at 0:44

@VesselinDimitrov: the article you indicated to me is amazing. Thank you, I did not know it and it enlightened my morning. – Brenin Sep 21 '13 at 17:26

1 I think the crucial sentence in that article is this one (p. 917/(6 of 15)): "Within quantum theory it makes perfect sense to combine all the numbers into a single generating function In fact, this function has a straightforward physical interpretation. It can be seen as a probability amplitude for a string" – Urs Schreiber Oct 31 '13 at 22:49

2 Answers

Your example from Witten makes one point: generating functions can make differential operators summarize combinatoric/algebraic information -- basically by making differential operators express recurrence relations on the series of coefficients. And generating functions often have nice closed forms, as for example the long known expression of the power generating function for Fibonacci numbers as $x/(1-x-x^2)$. A closed form captures the series as a whole. Closed forms are at least concise information. They can reveal the effect of differential operators, or algebraic relations. Wilf's free book Generatingfunctionology http://www.math.upenn.edu/~wilf/DownldGF.html gives many examples of all this. Because you can use power generating functions, exponential generating functions, Dirichlet series, and more, you have some choice in tailoring the series to meet the application.

Treating generating functions as functions in the usual sense can be useful but is not always possible since the series are not always convergent. As noted Jeff Harvey's reply to Does Physics need non-analytic smooth functions? an important kind of power generating function in physics can have radius of convergence 0.

I would be surprised (and delighted) to see a concise yet comprehensive explanation of all the reasons generating functions work so well. For a brilliant, concise, avowedly not comprehensive effort at this see the preface of Wilf's book. He gives exceptionally clear exposition and motivation throughout that book.

edited Sep 21 '13 at 20:17 answered Sep 20 '13 at 22:41
Colin McLarty
3,862 12 36

Thanks for your answer. I need more life experience and examples to digest the subject but your answer will help. Regards. – Brenin Sep 23 '13 at 21:12

It is certainly the case for enumerative invariants that the series as a whole is important. For example, I believe the famous GW/DT correspondence by MNOP goes via a change of variables of one generating series, plus analytic continuation over the so-called stringy Kähler moduli space. Also, comparison formulae for these invariants tend to be neater by considering the whole series (even when considered as purely formal gadgets), for example the correspondence between DT invariants and the stable-pairs invariants of Pandharipande-Thomas.

But I for one would like to see some examples outside the realm of (virtual) enumerative geometry.

answered Sep 20 '13 at 21:36
John Salvatierrez
469 2 12

1 Local zeta-functions in number theory are functions where the logarithmic derivative is generating function for the number of solutions of a set of equations on a finite field. And the generating function of the Riemann zeta function for nonnegative even integers, or for nonnegative odd integers, are both fairly simply expressed in terms of the cotangent function (oeis.org/wiki/Riemann_zeta_function). – Colin McLarty Sep 22 '13 at 1:41