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Tropical Mathematics, Idempotent Analysis, Classical Mechanics and Geometry

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ABSTRACT. A very brief introduction to tropical and idempotent mathematics (including idempotent functional analysis) is presented. Applications to classical mechanics and geometry are especially examined.

To Mikhail Shubin with my admiration and gratitude

CONTENTS

1. Introduction

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in sections 2 and 3). For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$), see [**19**–**22**].

Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to the traditional algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [**17**, **41**, **42**, **47**–**49**]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of N. Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar results over idempotent semirings. A systematic application of this correspondence principle leads to a variety of theoretical and applied results [**19**–**23**], see Fig. 1.

The history of the subject is discussed, e.g., in [**19**]. There is a large list of references.
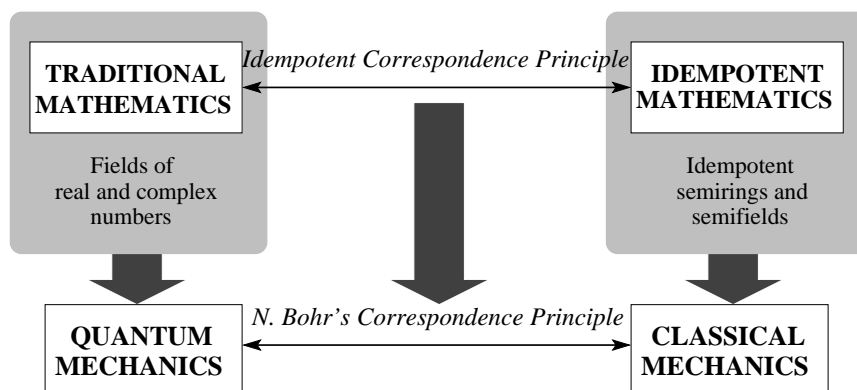
FIGURE 1. Relations between idempotent and traditional mathematics.

2. The Maslov dequantization

Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real and complex numbers. The so-called max-plus algebra $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be treated as a result of the *Maslov dequantization* of the semifield $\mathbf{R}_+$ of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x,$$

where $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$, see Fig. 2. Let the addition and multiplication operations be mapped from $\mathbf{R}_+$ to $\mathbf{R} \cup \{-\infty\}$ by Φ_h, i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,$$
$$\mathbf{0} = -\infty = \Phi_h(0), \quad \mathbf{1} = 0 = \Phi_h(1).$$

It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. Thus we get the semifield $\mathbf{R}_{\max}$ (i.e. the max-plus algebra) with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$ as a result of this deformation of the algebraic structure in $\mathbf{R}_+$.

The semifield $\mathbf{R}_{\max}$ is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element x of this semiring.
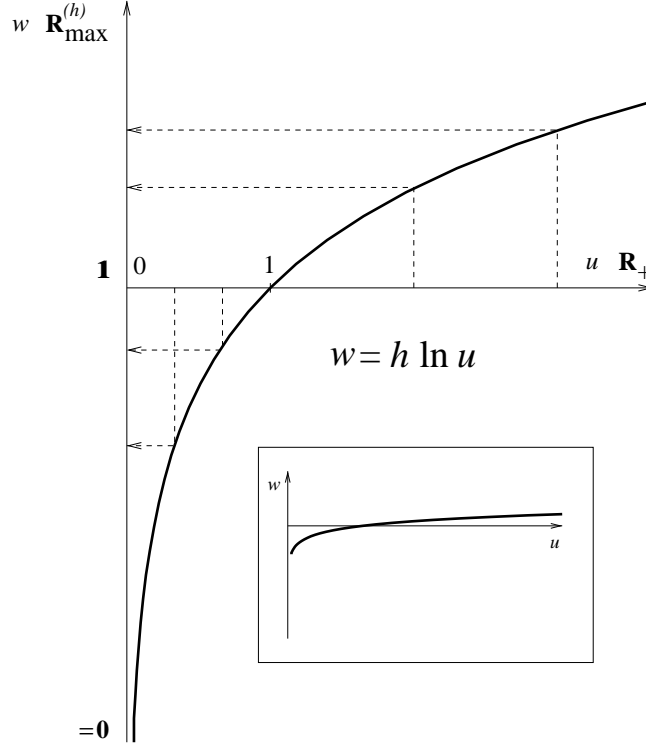
FIGURE 2. Deformation of $\mathbf{R}_+$ to $\mathbf{R}^{(h)}$. Inset: the same
for a small value of h.

The semifield $\mathbf{R}_{\max}$ is also called a *tropical algebra*. The semifield
$\mathbf{R}^{(h)} = \Phi_h(\mathbf{R}_+)$ with operations $\oplus_h$ and $\odot$ (i.e. $+$) is called a *subtropical
algebra*.

The semifield $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with operations $\oplus = \min$ and
$\odot = +$ ($\mathbf{0} = +\infty, \mathbf{1} = 0$) is isomorphic to $\mathbf{R}_{\max}$.

The analogy with quantization is obvious; the parameter h plays
the role of the Planck constant. The map $x \mapsto |x|$ and the Maslov
dequantization for $\mathbf{R}_+$ give us a natural transition from the field $\mathbf{C}$ (or
$\mathbf{R}$) to the max-plus algebra $\mathbf{R}_{\max}$. *We will also call this transition the
Maslov dequantization.* In fact the Maslov dequantization corresponds
to the usual Schrödinger dequantization but for imaginary values of
the Planck constant (see below). The transition from numerical fields

to the max-plus algebra $\mathbf{R}_{\max}$ (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics*. The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is the transition from basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called *idempotent mathematics*, i.e. mathematics over idempotent semifields and semirings.

Remark. The term 'tropical' appeared in [**45**] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand V. P. Maslov used this term in 80s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, 'tropical' means 'over $\mathbf{R}_{\max}$ (or $\mathbf{R}_{\min}$)' and tropical algebras are $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. The terms 'max-plus', 'max-algebra' and 'min-plus' are often used in the same sense.

3. Semirings and semifields

Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$:
$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$
  and
$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$
  for all $x, y, z \in S$.

A *unity* (we suppose that it exists) of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* (if it exists) of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral element

1 is called a *semifield* if every nonzero element of S is invertible with respect to the multiplication. The theory of semirings and semifields is treated, e.g., in [**13**].

4. Idempotent analysis

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject and applications are presented in the book of V. N. Kolokoltsov and V. P. Maslov [**18**] (a version of this book in Russian was published in 1994).

Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, and unit **1**. The set S is supplied with the *standard partial order* $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. If S contains a zero element **0**, then all elements of S are nonnegative: $\mathbf{0} \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic "topological" concepts and results of idempotent analysis at the purely algebraic level; this line of reasoning was examined systematically in [**19**]– [**32**] and [**8**].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $B(X, S)$ of all functions $X \to S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\, dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus_i \varphi(x_i) \odot \sigma_i = \max_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An *idempotent* (or *Maslov*) *measure* on X is defined by the formula $m_\psi(Y) = \sup_{x \in Y} \psi(x)$, where $\psi \in B(X, S)$ is a fixed function. The integral with respect to this measure is defined by the formula

$$I_\psi(\varphi) = \int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X}(\varphi(x) \odot \psi(x)). \tag{2}$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order is opposite to the conventional order $\leq$, so in this case equation (2) assumes the form

$$\int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \inf_{x \in X}(\varphi(x) \odot \psi(x)),$$

where inf is understood in the sense of the conventional order $\leq$.

5. The superposition principle and linear problems

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings. This is V. P. Maslov's idempotent superposition principle, see [**36**–**38**]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is

determined by H and F [**2**, **5**, **6**, **9**, **10**, **14**, **15**]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [**5**, **6**].

The linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$, which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),$$

where $x = (x_1, \ldots, x_N)$ are generalized coordinates, $p = (p_1, \ldots, p_N)$ are generalized momenta, m_i are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

$$S = \int_{t_0}^{t} L(x(t), \dot{x}(t), t) \, dt,$$

where the integration is performed along the factual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of t and t_0 and arbitrary trajectories $x(t)$, the action functional $S = S(x(t))$ can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as

elements of $\mathbf{R}_{\min}$. In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of e^{-S}, i.e. idempotent integral $\int_{\{paths\}}^{\oplus} e^{-S(x(t))} D\{x(t)\}$ with respect to the max-plus algebra $\mathbf{R}_{\max}$. Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleǐnik solution formula for the Hamilton–Jacobi equation.

Since $\partial S/\partial x_i = p_i$, $\partial S/\partial t = -H(p, x)$, the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \tag{3}$$

Quantization leads to the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \widehat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi, \tag{4}$$

where $\psi = \psi(x, t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbf{R}^N)$, and $\widehat{H}$ is the energy operator obtained by substitution of the momentum operators $\widehat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$ and the coordinate operators $\widehat{x}_i\colon \psi \mapsto x_i\psi$ for the variables p_i and x_i in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x, t) = a(x, t)e^{iS(x,t)/\hbar}$, and to keep only the leading order as $\hbar \to 0$ (the 'semiclassical' limit).

Instead of doing this, we switch to imaginary values of the Planck constant $\hbar$ by the substitution $h = i\hbar$, assuming $h > 0$. Thus the Schrödinger equation (4) turns to an analog of the heat equation:

$$h\frac{\partial u}{\partial t} = H\left(-h\frac{\partial}{\partial x_i}, \hat{x}_i\right)u, \tag{5}$$

where the real-valued function u corresponds to the wave function ψ. A similar idea (the switch to imaginary time) is used in the Euclidean

quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of equation (5). Thus if u_1 and u_2 are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \tag{6}$$

Let $S = h \ln u$ or $u = e^{S/h}$ as in Section 2 above. It can easily be checked that equation (5) thus turns to

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N} \frac{1}{2m_i} \left(\frac{\partial S}{\partial x_i} \right)^2 + h \sum_{i=1}^{n} \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}. \tag{7}$$

Thus we have a transition from (4) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{ReS/h}$, where $\mathrm{Re} S$ is the real part of S. Now let us consider S as a real variable. The equation (7) is nonlinear in the conventional sense. However, if S_1 and S_2 are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$

obtained from (6) by means of our substitution $S = h \ln u$. Here the generalized multiplication $\odot$ coincides with the ordinary addition and the generalized addition $\oplus_h$ is the image of the conventional addition under the above change of variables. As $h \to 0$, we obtain the operations of the idempotent semiring $\mathbf{R}_{\max}$, i.e., $\oplus = \max$ and $\odot = +$, and equation (7) turns to the Hamilton–Jacobi equation (3), since the third term in the right-hand side of equation (7) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over $\mathbf{R}_{\max}$. This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see, e.g., [**18**, **23**, **43**]. Notice that if h is changed to $-h$, then we have that the resulting Hamilton–Jacobi equation is linear over $\mathbf{R}_{\min}$.

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. The idempotent linear functional analysis

(see below) is a natural tool for investigation of those nonlinear infinite-dimensional problems that possess this property.

6. Convolution and the Fourier–Legendre transform

Let G be a group. Then the space $\mathcal{B}(G, \mathbf{R}_{\max})$ of all bounded functions $G \to \mathbf{R}_{\max}$ (see above) is an idempotent semiring with respect to the following analog $\circledast$ of the usual convolution:

$$(\varphi(x) \circledast \psi)(g) == \int_G^\oplus \varphi(x) \odot \psi(x^{-1} \cdot g)\, dx = \sup_{x \in G}(\varphi(x) + \psi(x^{-1} \cdot g)).$$

Of course, it is possible to consider other "function spaces" (and other basic semirings instead of $\mathbf{R}_{\max}$).

Let $G = \mathbf{R}^n$, where $\mathbf{R}^n$ is considered as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x)\, dx, \tag{8}$$

where $e^{i\xi \cdot x}$ is a character of the group G, i.e., a solution of the following functional equation:

$$f(x + y) = f(x)f(y).$$

The idempotent analog of this equation is

$$f(x + y) = f(x) \odot f(y) = f(x) + f(y),$$

so "continuous idempotent characters" are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (8) assumes the form

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^\oplus \xi \cdot x \odot \varphi(x)\, dx = \sup_{x \in G}(\xi \cdot x + \varphi(x)). \tag{9}$$

The transform in (9) is nothing but the *Legendre transform* (up to some notation) [**38**]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics. The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [**34**].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the

generalized convolution ⊛ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform.

The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory, see, e.g., [**28**]. In particular, "idempotent" representations of groups can be examined as representations of the corresponding convolution semirings (i.e. idempotent group semirings) in semimodules.

7. Idempotent functional analysis

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity take $S = \mathbf{R}_{\max}$ and let X be an arbitrary set. The idempotent integration can be defined by the formula (1), see above. The functional $I(\varphi)$ is linear over S and its values correspond to limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions φ and ψ is defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$

So it is natural to construct idempotent analogs of integral operators in the form

$$\varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x,y) \odot \varphi(y)\, dy = \sup_{y \in Y} \{K(x,y) + \varphi(y)\}, \ (10)$$

where $\varphi(y)$ is an element of a space of functions defined on a set Y, and $K(x,y)$ is an S-valued function on $X \times Y$. Of course, expressions of this type are standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings which are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces it was proved that every

'good' linear operator (in the idempotent sense) can be presented in the form (10); this is an idempotent version of the kernel theorem of L. Schwartz; results of this type were proved by V. N. Kolokoltsov, P. S. Dudnikov and S. N. Samborskiĭ, I. Singer, M. A. Shubin and others. So every 'good' linear functional can be presented in the form $\varphi \mapsto \langle \varphi, \psi \rangle$, where $\langle , \rangle$ is an idempotent scalar product.

In the framework of idempotent functional analysis results of this type can be proved in a very general situation. In [**25**–**28**, **30**, **32**] an algebraic version of the idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms (see below). The treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known theorems of Hahn-Banach, Riesz, and Riesz-Fisher) to idempotent analogs of A. Grothendieck's concepts and results on topological tensor products, nuclear spaces and operators. Abstract idempotent versions of the kernel theorem are formulated. Note that the transition from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many non-isomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) were obtained by G. Cohen, S. Gaubert, and J.-P. Quadrat. Idempotent functional analysis has received much attention in the last years, see, e.g., [**1**, **8**, **14**–**16**, **40**, **46**], [**18**]– [**32**] and works cited in [**19**]. Elements of "tropical" functional analysis are presented in [**18**]. All the results presented in this section are proved in [**27**] (subsections 7.1 – 7.4) and in [**32**] (subsections 7.5 – 7.10)

7.1. Idempotent semimodules and idempotent linear spaces. An additive semigroup S with commutative addition $\oplus$ is called an *idempotent semigroup* if the relation $x \oplus x = x$ is fulfilled for all elements $x \in S$. If S contains a neutral element, this element is denoted by the symbol **0**. Any idempotent semigroup is a partially ordered set with respect to the following standard order: $x \preceq y$ if and only if $x \oplus y = y$. It is obvious that this order is well defined and

$x \oplus y = \sup\{x, y\}$. Thus, any idempotent semigroup is an upper semi-lattice; moreover, the concepts of idempotent semigroup and upper semilattice coincide, see [**3**]. An idempotent semigroup S is called *a-complete* (or *algebraically complete*) if it is complete as an ordered set, i.e., if any subset X in S has the least upper bound $\sup(X)$ denoted by $\oplus X$ and the greatest lower bound $\inf(X)$ denoted by $\wedge X$. This semigroup is called *b-complete* (or *boundedly complete*), if any bounded above subset X of this semigroup (including the empty subset) has the least upper bound $\oplus X$ (in this case, any nonempty subset Y in S has the greatest lower bound $\wedge Y$ and S in a lattice). Note that any *a*-complete or *b*-complete idempotent semiring has the zero element **0** that coincides with $\oplus\emptyset$, where $\emptyset$ is the empty set. Certainly, *a*-completeness implies the *b*-completeness. Completion by means of cuts [**3**] yields an embedding $S \to \widehat{S}$ of an arbitrary idempotent semigroup S into an *a*-complete idempotent semigroup $\widehat{S}$ (which is called a *normal completion* of S); in addition, $\widehat{\widehat{S}} = S$. The *b*-completion procedure $S \to \widehat{S}_b$ is defined similarly: if $S \ni \infty = \sup S$, then $\widehat{S}_b = \widehat{S}$; otherwise, $\widehat{S} = \widehat{S}_b \cup \{\infty\}$. An arbitrary *b*-complete idempotent semigroup S also may differ from $\widehat{S}$ only by the element $\infty = \sup S$.

Let S and T be *b*-complete idempotent semigroups. Then, a homomorphism $f : S \to T$ is said to be a *b-homomorphism* if $f(\oplus X) = \oplus f(X)$ for any bounded subset X in S. If the *b*-homomorphism f is extended to a homomorphism $\widehat{S} \to \widehat{T}$ of the corresponding normal completions and $f(\oplus X) = \oplus f(X)$ for all $X \subset S$, then f is said to be an *a-homomorphism*. An idempotent semigroup S equipped with a topology such that the set $\{s \in S | s \preceq b\}$ is closed in this topology for any $b \in S$ is called a *topological idempotent semigroup S*.

Proposition 7.1. *Let S be an a-complete topological idempotent semigroup and T be a b-complete topological idempotent semigroup such that, for any nonempty subsemigroup X in T, the element $\oplus X$ is contained in the topological closure of X in T. Then, a homomorphism $f : T \to S$ that maps zero into zero is an a-homomorphism if and only if the mapping f is lower semicontinuous in the sense that the set $\{t \in T | f(t) \preceq s\}$ is closed in T for any $s \in S$.*

An idempotent semiring K is called *a-complete* (respectively *b-complete*) if K is an *a*-complete (respectively *b*-complete) idempotent semigroup and, for any subset (respectively, for any bounded subset) X in K and any $k \in K$, the generalized distributive laws $k \odot (\oplus X) = \oplus(k \odot X)$ and $(\oplus X) \odot k = \oplus(X \odot k)$ are fulfilled. Generalized distributivity implies that any *a*-complete or *b*-complete idempotent semiring has a zero element that coincides with $\oplus \varnothing$, where $\varnothing$ is the empty set.

The set $\mathbf{R}(\max, +)$ of real numbers equipped with the idempotent addition $\oplus = \max$ and multiplication $\odot = +$ is an idempotent semiring; in this case, $\mathbf{1} = 0$. Adding the element $\mathbf{0} = -\infty$ to this semiring, we obtain a *b*-complete semiring $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with the same operations and the zero element. Adding the element $+\infty$ to $\mathbf{R}_{\max}$ and assuming that $\mathbf{0} \odot (+\infty) = \mathbf{0}$ and $x \odot (+\infty) = +\infty$ for $x \neq \mathbf{0}$ and $x \oplus (+\infty) = +\infty$ for any x, we obtain the *a*-complete idempotent semiring $\widehat{\mathbf{R}}_{\max} = \mathbf{R}_{\max} \cup \{+\infty\}$. The standard order on $\mathbf{R}(\max, +)$, $\mathbf{R}_{\max}$ and $\widehat{\mathbf{R}}_{\max}$ coincides with the ordinary order. The semirings $\mathbf{R}(\max, +)$ and $\mathbf{R}_{\max}$ are semifields. On the contrary, an *a*-complete semiring that does not coincide with $\{\mathbf{0}, \mathbf{1}\}$ cannot be a semifield. An important class of examples is related to (topological) vector lattices (see, for example, [**3**] and [**44**], Chapter 5). Defining the sum $x \oplus y$ as $\sup\{x, y\}$ and the multiplication $\odot$ as the addition of vectors, we can interpret the vector lattices as idempotent semifields. Adding the zero element $\mathbf{0}$ to a complete vector lattice (in the sense of [**3**, **44**]), we obtain a *b*-complete semifield. If, in addition, we add the infinite element, we obtain an *a*-complete idempotent semiring (which, as an ordered set, coincides with the normal completion of the original lattice).

Important definitions. Let V be an idempotent semigroup and K be an idempotent semiring. Suppose that a multiplication $k, x \mapsto k \odot x$ of all elements from K by the elements from V is defined; moreover, this multiplication is associative and distributive with respect to the addition in V and $\mathbf{1} \odot x = x$, $\mathbf{0} \odot x = \mathbf{0}$ for all $x \in V$. In this case, the semigroup V is called an *idempotent semimodule* (or simply, a *semimodule*) over K. The element $\mathbf{0}_V \in V$ is called the *zero* of

the semimodule V if $k \odot \mathbf{0}_V = \mathbf{0}_V$ and $\mathbf{0}_V \oplus x = x$ for any $k \in K$ and $x \in V$. Let V be a semimodule over a b-complete idempotent semiring K. This semimodule is called b-complete if it is b-complete as an idempotent semiring and, for any bounded subsets Q in K and X in V, the generalized distributive laws $(\oplus Q) \odot x = \oplus(Q \odot x)$ and $k \odot (\oplus X) = \oplus(k \odot X)$ are fulfilled for all $k \in K$ and $x \in X$. This semimodule is called a-complete if it is b-complete and contains the element $\infty = \sup V$.

A semimodule V over a b-complete semifield K is said to be an *idempotent a-space* (*b-space*) if this semimodule is a-complete (respectively, b-complete) and the equality $(\wedge Q) \odot x = \wedge(Q \odot x)$ holds for any nonempty subset Q in K and any $x \in V$, $x \neq \infty = \sup V$. The normal completion $\widehat{V}$ of a b-space V (as an idempotent semigroup) has the structure of an idempotent a-space (and may differ from V only by the element $\infty = \sup V$).

Let V and W be idempotent semimodules over an idempotent semiring K. A mapping $p : V \to W$ is said to be *linear* (over K) if

$$p(x \oplus y) = p(x) \oplus p(y) \text{ and } p(k \odot x) = k \odot p(x)$$

for any $x, y \in V$ and $k \in K$. Let the semimodules V and W be b-complete. A linear mapping $p : V \to W$ is said to be *b-linear* if it is a b-homomorphism of the idempotent semigroup; this mapping is said to be *a-linear* if it can be extended to an a-homomorphism of the normal completions $\widehat{V}$ and $\widehat{W}$. Proposition 7.1 (see above) shows that a-linearity simulates (semi)continuity for linear mappings. The normal completion $\widehat{K}$ of the semifield K is a semimodule over K. If $W = \widehat{K}$, then the linear mapping p is called a *linear functional*.

Linear, a-linear and b-linear mappings are also called *linear, a-linear* and *b-linear operators* respectively.

Examples of idempotent semimodules and spaces that are the most important for analysis are either subsemimodules of topological vector lattices [**44**] (or coincide with them) or are dual to them, i.e., consist of linear functionals subject to some regularity condition, for example, consist of a-linear functionals. Concrete examples of idempotent

semimodules and spaces of functions (including spaces of bounded, continuous, semicontinuous, convex, concave and Lipschitz functions) see in $[\mathbf{18}, \mathbf{26}, \mathbf{27}, \mathbf{32}]$ and below.

7.2. Basic results. Let V be an idempotent b-space over a b-complete semifield K, $x \in \widehat{V}$. Denote by x^* the functional $V \to \widehat{K}$ defined by the formula $x^*(y) = \wedge\{k \in K | y \preceq k \odot x\}$, where y is an arbitrary fixed element from V.

Theorem 7.2. *For any $x \in \widehat{V}$ the functional x^* is a-linear. Any nonzero a-linear functional f on V is given by $f = x^*$ for a unique suitable element $x \in V$. If $K \neq \{\mathbf{0}, \mathbf{1}\}$, then $x = \oplus\{y \in V | f(y) \preceq \mathbf{1}\}$.*

Note that results of this type obtained earlier concerning the structure of linear functionals cannot be carried over to subspaces and sub-semimodules.

A subsemigroup W in V closed with respect to the multiplication by an arbitrary element from K is called a *b-subspace* in V if the imbedding $W \to V$ can be extended to a b-linear mapping. The following result is obtained from Theorem 7.2 and is the idempotent version of the Hahn–Banach theorem.

Theorem 7.3. *Any a-linear functional defined on a b-subspace W in V can be extended to an a-linear functional on V. If $x, y \in V$ and $x \neq y$, then there exists an a-linear functional f on V that separates the elements x and y, i.e., $f(x) \neq f(y)$.*

The following statements are easily derived from the definitions and can be regarded as the analogs of the well-known results of the traditional functional analysis (the Banach–Steinhaus and the closed-graph theorems).

Proposition 7.4. *Suppose that P is a family of a-linear mappings of an a-space V into an a-space W and the mapping $p : V \to W$ is the pointwise sum of the mappings of this family, i.e., $p(x) = \sup\{p_\alpha(x) | p_\alpha \in P\}$. Then the mapping p is a-linear.*

Proposition 7.5. *Let V and W be a-spaces. A linear mapping $p : V \to W$ is a-linear if and only if its graph Γ in $V \times W$ is closed with*

respect to passing to sums (i.e., to least upper bounds) of its arbitrary subsets.

In [**8**] the basic results were generalized for the case of semimodules over the so-called reflexive b-complete semirings.

7.3. Idempotent b-semialgebras. Let K be a b-complete semifield and A be an idempotent b-space over K equipped with the structure of a semiring compatible with the multiplication $K \times A \to A$ so that the associativity of the multiplication is preserved. In this case, A is called an *idempotent b-semialgebra* over K.

Proposition 7.6. *For any invertible element $x \in A$ from the b-semialgebra A and any element $y \in A$, the equality $x^*(y) = \mathbf{1}^*(y \odot x^{-1})$ holds, where $\mathbf{1} \in A$.*

The mapping $A \times A \to \widehat{K}$ defined by the formula $(x, y) \mapsto \langle x, y \rangle = \mathbf{1}^*(x \odot y)$ is called the *canonical scalar product* (or simply *scalar product*). The basic properties of the scalar product are easily derived from Proposition 7.6 (in particular, the scalar product is commutative if the b-semialgebra A is commutative). The following theorem is an idempotent version of the Riesz–Fisher theorem.

Theorem 7.7. *Let a b-semialgebra A be a semifield. Then any nonzero a-linear functional f on A can be represented as $f(y) = \langle y, x \rangle$, where $x \in A$, $x \neq \mathbf{0}$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product on A.*

Remark 7.8. Using the completion precedures, one can extend all the results obtained to the case of incomplete semirings, spaces, and semimodules, see [**27**].

Example 7.9. Let $B(X)$ be a set of all bounded functions with values belonging to $\mathbf{R}(\max, +)$ on an arbitrary set X and let $\widehat{B}(X) = B(X) \cup \{\mathbf{0}\}$. The pointwise idempotent addition of functions $(\varphi_1 \oplus \varphi_2)(x) = \varphi_1(x) \oplus \varphi_2(x)$ and the multiplication $(\varphi_1 \odot \varphi_2)(x) = (\varphi_1(x)) \odot (\varphi_2(x))$ define on $\widehat{B}(X)$ the structure of a b-semialgebra over the b-complete semifield $\mathbf{R}_{\max}$. In this case, $\mathbf{1}^*(\varphi) = \sup_{x \in X} \varphi(x)$ and the scalar product is expressed in terms of idempotent integration: $\langle \varphi_1, \varphi_2 \rangle =$

$$\sup_{x\in X}(\varphi_1(x)\odot\varphi_2(x)) = \sup_{x\in X}(\varphi_1(x)+\varphi_2(x)) = \int_X^{\oplus}(\varphi_1(x)\odot\varphi_2(x))\ dx.$$

Scalar products of this type were systematically used in idempotent analysis. Using Theorems 7.2 and 7.7, one can easily describe a-linear functionals on idempotent spaces in terms of idempotent measures and integrals.

Example 7.10. Let X be a linear space in the traditional sense. The idempotent semiring (and linear space over $\mathbf{R}(\max,+)$) of convex functions $\mathrm{Conv}(X,\mathbf{R})$ is b-complete but it is not a b-semialgebra over the semifield $K = \mathbf{R}(\max,+)$. Any nonzero a-linear functional f on $\mathrm{Conv}(X,\mathbf{R})$ has the form

$$\varphi \mapsto f(\varphi) = \sup_x\{\varphi(x) + \psi(x)\} = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx,$$

where ψ is a concave function, i.e., an element of the idempotent space $\mathrm{Conc}(X,\mathbf{R}) = - \mathrm{Conv}(X,\mathbf{R})$.

7.4. Linear operator, b-semimodules and subsemimodules.

In what follows, we suppose that all semigroups, semirings, semifields, semimodules, and spaces are idempotent unless otherwise specified. We fix a basic semiring K and examine semimodules and subsemimodules over K. We suppose that every linear functional takes it values in the basic semiring.

Let V and W be b-complete semimodules over a b-complete semiring K. Denote by $L_b(V,W)$ the set of all b-linear mappings from V to W. It is easy to check that $L_b(V,W)$ is an idempotent semigroup with respect to the pointwise addition of operators; the composition (product) of b-linear operators is also a b-linear operator, and therefore the set $L_b(V,V)$ is an idempotent semiring with respect to these operations, see, e.g., [**27**]. The following proposition can be treated as a version of the Banach–Steinhaus theorem in idempotent analysis (as well as Proposition 7.4 above).

Proposition 7.11. *Assume that S is a subset in $L_b(V,W)$ and the set $\{g(v) \mid g \in S\}$ is bounded in W for every element $v \in V$; thus the element $f(v) = \sup_{g\in S} g(v)$ exists, because the semimodule W is*

b-complete. Then the mapping $v \mapsto f(v)$ is a b-linear operator, i.e., an element of $L_b(V, W)$. The subset S is bounded; moreover, $\sup S = f$.

Corollary 7.12. *The set $L_b(V, W)$ is a b-complete idempotent semi-group with respect to the (idempotent) pointwise addition of operators. If $V = W$, then $L_b(V, V)$ is a b-complete idempotent semiring with respect to the operations of pointwise addition and composition of operators.*

Corollary 7.13. *A subset S is bounded in $L_b(V, W)$ if and only if the set $\{g(v) \mid g \in S\}$ is bounded in the semimodule W for every element $v \in V$.*

A subset of an idempotent semimodule is called a *subsemimodule* if it is closed under addition and multiplication by scalar coefficients. A subsemimodule V of a *b*-complete semimodule W is *b-closed* if V is closed under sums of any subsets of V that are bounded in W. A subsemimodule of a *b*-complete semimodule is called a *b-subsemimodule* if the corresponding embedding is a *b*-homomorphism. It is easy to see that each *b*-closed subsemimodule is a *b*-subsemimodule, but the converse is not true. The main feature of *b*-subsemimodules is that restrictions of *b*-linear operators and functionals to these semimodules are *b*-linear.

The following definitions are very important for our purposes. Assume that W is an idempotent *b*-complete semimodule over a *b*-complete idempotent semiring K and V is a subset of W such that V is closed under multiplication by scalar coefficients and is an upper semilattice with respect to the order induced from W. Let us define an addition operation in V by the formula $x \oplus y = \sup\{x, y\}$, where sup means the least upper bound in V. If K is a semifield, then V is a semimodule over K with respect to this addition.

For an arbitrary *b*-complete semiring K, we will say that V is a *quasisubsemimodule* of W if V is a semimodule with respect to this addition (this means that the corresponding distribution laws hold).

Recall that the simbol $\wedge$ means the greatest lower bound (see Subsection 7.1 above). A quasisubsemimodule V of an idempotent

b-complete semimodule W is called a $\wedge$-*subsemimodule* if it contains **0** and is closed under the operations of taking infima (greatest lower bounds) in W. It is easy to check that *each $\wedge$-subsemimodule is a b-complete semimodule.*

Note that quasisubsemimodules and $\wedge$-subsemimodules may fail to be subsemimodules, because only the order is induced and not the corresponding addition (see Example 7.18 below).

Recall that idempotent semimodules over semifields are *idempotent spaces*. In idempotent mathematics, such spaces are analogs of traditional linear (vector) spaces over fields. In a similar way we use the corresponding terms like *b-spaces, b-subspaces, b-closed subspaces, $\wedge$-subspaces*, etc.

Some examples are presented below.

7.5. Functional semimodules. Let X be an arbitrary nonempty set and K be an idempotent semiring. By $K(X)$ denote the semimodule of all mappings (functions) $X \to K$ endowed with the pointwise operations. By $K_b(X)$ denote the subsemimodule of $K(X)$ consisting of all bounded mappings. If K is a b-complete semiring, then $K(X)$ and $K_b(X)$ are b-complete semimodules. Note that $K_b(X)$ is a b-subsemimodule but not a b-closed subsemimodule of $K(X)$. Given a point $x \in X$, by δ_x denote the functional on $K(X)$ that maps f to $f(x)$. It can easily be checked that the functional δ_x is b-linear on $K(X)$.

We say that a quasisubsemimodule of $K(X)$ is an (idempotent) *functional semimodule* on the set X. An idempotent functional semimodule in $K(X)$ is called *b-complete* if it is a b-complete semimodule.

A functional semimodule $V \subset K(X)$ is called a *functional b-semimodule* if it is a b-subsemimodule of $K(X)$; a functional semimodule $V \subset K(X)$ is called a *functional $\wedge$-semimodule* if it is a $\wedge$-subsemimodule of $K(X)$.

In general, a functional of the form δ_x on a functional semimodule is not even linear, much less b-linear (see Example 7.18 below). However, the following proposition holds, which is a direct consequence of our definitions.

Proposition 7.14. *An arbitrary b-complete functional semimodule W on a set X is a b-subsemimodule of $K(X)$ if and only if each functional of the form δ_x (where $x \in X$) is b-linear on W.*

Example 7.15. The semimodule $K_b(X)$ (consisting of all bounded mappings from an arbitrary set X to a b-complete idempotent semiring K) is a functional $\wedge$-semimodule. Hence it is a b-complete semimodule over K. Moreover, $K_b(X)$ is a b-subsemimodule of the semimodule $K(X)$ consisting of all mappings $X \to K$.

Example 7.16. If X is a finite set consisting of n elements ($n > 0$), then $K_b(X) = K(X)$ is an "n-dimensional" semimodule over K; it is denoted by K^n. In particular, $\mathbf{R}^n_{max}$ is an idempotent space over the semifield $\mathbf{R}_{max}$, and $\widehat{\mathbf{R}}^n_{\max}$ is a semimodule over the semiring $\widehat{\mathbf{R}}_{\max}$. Note that $\widehat{\mathbf{R}}^n_{\max}$ can be treated as a space over the semifield $\mathbf{R}_{max}$. For example, the semiring $\widehat{\mathbf{R}}_{\max}$ can be treated as a space (semimodule) over $\mathbf{R}_{\max}$.

Example 7.17. Let X be a topological space. Denote by $USC(X)$ the set of all upper semicontinuous functions with values in $\mathbf{R}_{\max}$. By definition, a function $f(x)$ is upper semicontinuous if the set $X_s = \{x \in X \mid f(x) \geq s\}$ is closed in X for every element $s \in \mathbf{R}_{\max}$ (see, e.g., [**27**], Sec. 2.8). If a family $\{f_\alpha\}$ consists of upper semicontinuous (e.g., continuous) functions and $f(x) = \inf_\alpha f_\alpha(x)$, then $f(x) \in USC(X)$. It is easy to check that $USC(X)$ has a natural structure of an idempotent space over $\mathbf{R}_{\max}$. Moreover, $USC(X)$ is a functional $\wedge$-space on X and a b-space. The subspace $USC(X) \cap K_b(X)$ of $USC(X)$ consisting of bounded (from above) functions has the same properties.

Example 7.18. Note that an idempotent functional semimodule (and even a functional $\wedge$-semimodule) on a set X is not necessarily a subsemimodule of $K(X)$. The simplest example is the functional space (over $K = \mathbf{R}_{\max}$) $\mathrm{Conc}(\mathbf{R})$ consisting of all concave functions on $\mathbf{R}$ with values in $\mathbf{R}_{\max}$. Recall that a function f belongs to $\mathrm{Conc}(\mathbf{R})$ if and only if the subgraph of this function is convex, i.e., the formula $f(ax + (1-a)y) \geq af(x) + (1-a)f(y)$ is valid for $0 \leq a \leq 1$. The basic operations with $\mathbf{0} \in \mathbf{R}_{\max}$ can be defined in an obvious way. If

$f, g \in \mathrm{Conc}(\mathbf{R})$, then denote by $f \oplus g$ the sum of these functions in $\mathrm{Conc}(\mathbf{R})$. The subgraph of $f \oplus g$ is the convex hull of the subgraphs of f and g. Thus $f \oplus g$ does not coincide with the pointwise sum (i.e., $\max\{f(x), g(x)\}$).

Example 7.19. Let X be a nonempty metric space with a fixed metric r. Denote by $\mathrm{Lip}(X)$ the set of all functions defined on X with values in $\mathbf{R}_{\max}$ satisfying the following *Lipschitz condition*:

$$| f(x) \odot (f(y))^{-1} | = | f(x) - f(y) | \leq r(x, y),$$

where x, y are arbitrary elements of X. The set $\mathrm{Lip}(X)$ consists of continuous real-valued functions (but not all of them!) and (by definition) the function equal to $-\infty = \mathbf{0}$ at every point $x \in X$. The set $\mathrm{Lip}(X)$ has the structure of an idempotent space over the semifield $\mathbf{R}_{\max}$. Spaces of the form $\mathrm{Lip}(X)$ are said to be *Lipschitz spaces*. These spaces are b-subsemimodules in $K(X)$.

7.6. Integral representations of linear operators in functional semimodules.

Let W be an idempotent b-complete semimodule over a b-complete semiring K and $V \subset K(X)$ be a b-complete functional semimodule on X. A mapping $A : V \to W$ is called an *integral operator* or an operator with an *integral representation* if there exists a mapping $k : X \to W$, called the *integral kernel* (or *kernel*) *of the operator A*, such that

$$Af = \sup_{x \in X} (f(x) \odot k(x)). \tag{11}$$

In idempotent analysis, the right-hand side of formula (11) is often written as $\int_X^{\oplus} f(x) \odot k(x) dx$. Regarding the kernel k, it is assumed that the set $\{f(x) \odot k(x) | x \in X\}$ is bounded in W for all $f \in V$ and $x \in X$. We denote the set of all functions with this property by $\mathrm{kern}_{V,W}(X)$. In particular, if $W = K$ and A is a functional, then this functional is called *integral*. Thus each integral functional can be presented in the form of a "scalar product" $f \mapsto \int_X^{\oplus} f(x) \odot k(x) \, dx$, where $k(x) \in K(X)$; in idempotent analysis, this situation is standard.

Note that a functional of the form δ_y (where $y \in X$) is a typical integral functional; in this case, $k(x) = \mathbf{1}$ if $x = y$ and $k(x) = \mathbf{0}$ otherwise.

We call a functional semimodule $V \subset K(X)$ *nondegenerate* if for every point $x \in X$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$, and *admissible* if for every function $f \in V$ and every point $x \in X$ such that $f(x) \neq \mathbf{0}$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$ and $f(x) \odot g < f$.

Note that all idempotent functional semimodules over semifields are admissible (it is sufficient to set $g = f(x)^{-1} \odot f$).

Proposition 7.20. *Denote by X_V the subset of X defined by the formula $X_V = \{x \in X \mid \exists f \in V : f(x) = \mathbf{1}\}$. If the semimodule V is admissible, then the restriction to X_V defines an embedding $i : V \to K(X_V)$ and its image $i(V)$ is admissible and nondegenerate.*

If a mapping $k : X \to W$ is a kernel of a mapping $A : V \to W$, then the mapping $k_V : X \to W$ that is equal to k on X_V and equal to $\mathbf{0}$ on $X \smallsetminus X_V$ is also a kernel of A.

A mapping $A : V \to W$ is integral if and only if the mapping $i_{-1}A : i(A) \to W$ is integral.

In what follows, K always denotes a fixed *b*-complete idempotent (basic) semiring. If an operator has an integral representation, this representation may not be unique. However, if the semimodule V is nondegenerate, then the set of all kernels of a fixed integral operator is bounded with respect to the natural order in the set of all kernels and is closed under the supremum operation applied to its arbitrary subsets. In particular, *any integral operator defined on a nondegenerate functional semimodule has a unique maximal kernel.*

An important point is that an integral operator is not necessarily *b*-linear and even linear except when V is a *b*-subsemimodule of $K(X)$ (see Proposition 7.21 below).

If W is a functional semimodule on a nonempty set Y, then an integral kernel k of an operator A can be naturally identified with the function on $X \times Y$ defined by the formula $k(x, y) = (k(x))(y)$. This function will also be called an *integral kernel* (or *kernel*) of the

operator A. As a result, the set $\mathrm{kern}_{V,W}(X)$ is identified with the set $\mathrm{kern}_{V,W}(X,Y)$ of all mappings $k : X \times Y \to K$ such that for every point $x \in X$ the mapping $k_x : y \mapsto k(x,y)$ lies in W and for every $v \in V$ the set $\{v(x) \odot k_x | x \in X\}$ is bounded in W. Accordingly, the set of all integral kernels of b-linear operators can be embedded into $\mathrm{kern}_{V,W}(X,Y)$.

If V and W are functional b-semimodules on X and Y, respectively, then the set of all kernels of b-linear operators can be identified with $\mathrm{kern}_{V,W}(X,Y)$ and the following formula holds:

$$Af(y) = \sup_{x \in X} \left(f(x) \odot k(x,y) \right) = \int_X^{\oplus} f(x) \odot k(x,y) dx. \qquad (12)$$

This formula coincides with the usual definition of an integral representation of an operator. Note that formula (11) can be rewritten in the form

$$Af = \sup_{x \in X} \left(\delta_x(f) \odot k(x) \right). \qquad (13)$$

Proposition 7.21. *An arbitrary b-complete functional semimodule V on a nonempty set X is a functional b-semimodule on X (i.e., a b-sub-semimodule of $K(X)$) if and only if all integral operators defined on V are b-linear.*

The following notion (definition) is especially important for our purposes. Let $V \subset K(X)$ be a b-complete functional semimodule over a b-complete idempotent semiring K. We say that the *kernel theorem* holds for the semimodule V if every b-linear mapping from V into an arbitrary b-complete semimodule over K has an integral representation.

Theorem 7.22. *Assume that a b-complete semimodule W over a b-complete semiring K and an admissible functional $\wedge$-semimodule $V \subset K(X)$ are given. Then every b-linear operator $A : V \to W$ has an integral representation of the form (11). In particular, if W is a functional b-semimodule on a set Y, then the operator A has an integral representation of the form (12). Thus for the semimodule V the kernel theorem holds.*

Remark 7.23. Examples of admissible functional $\wedge$-semimodules (and $\wedge$-spaces) appearing in Theorem 7.22 are presented above, see, e.g., examples 7.15 –7.17. Thus for these functional semimodules and spaces V over K, the kernel theorem holds and every b-linear mapping V into an arbitrary b-complete semimodule W over K has an integral representation (12). Recall that every functional space over a b-complete semifield is admissible, see above.

7.7. Nuclear operators and their integral representations.
Let us introduce some important definitions. Assume that V and W are b-complete semimodules. A mapping $g : V \to W$ is called *one-dimensional* (or a *mapping of rank* 1) if it is of the form $v \mapsto \phi(v) \odot w$, where ϕ is a b-linear functional on V and $w \in W$. A mapping g is called *b-nuclear* if it is the sum (i.e., supremum) of a bounded set of one-dimensional mappings. Since every one-dimensional mapping is b-linear (because the functional ϕ is b-linear), *every b-nuclear operator is b-linear* (see Corollary 7.12 above). Of course, b-nuclear mappings are closely related to tensor products of idempotent semimodules, see [**26**].

By $\phi \odot w$ we denote the one-dimensional operator $v \mapsto \phi(v) \odot w$. In fact, this is an element of the corresponding tensor product.

Proposition 7.24. *The composition (product) of a b-nuclear and a b-linear mapping or of a b-linear and a b-nuclear mapping is a b-nuclear operator.*

Theorem 7.25. *Assume that W is a b-complete semimodule over a b-complete semiring K and $V \subset K(X)$ is a functional b-semimodule. If every b-linear functional on V is integral, then a b-linear operator $A : V \to W$ has an integral representation if and only if it is b-nuclear.*

7.8. The b-approximation property and b-nuclear semimodules and spaces.
We say that a b-complete semimodule V has the *b-approximation property* if the identity operator id:$V \to V$ is b-nuclear (for a treatment of the approximation property for locally convex spaces in the traditional functional analysis, see [**44**]).

Let V be an arbitrary b-complete semimodule over a b-complete idempotent semiring K. We call this semimodule a *b-nuclear semimodule* if any b-linear mapping of V to an arbitrary b-complete semimodule W over K is a b-nuclear operator. Recall that, in the traditional functional analysis, a locally convex space is nuclear if and only if all continuous linear mappings of this space to any Banach space are nuclear operators, see [**44**].

Proposition 7.26. *Let V be an arbitrary b-complete semimodule over a b-complete semiring K. The following statements are equivalent:*

1) the semimodule V has the b-approximation property;

2) every b-linear mapping from V to an arbitrary b-complete semimodule W over K is b-nuclear;

3) every b-linear mapping from an arbitrary b-complete semimodule W over K to the semimodule V is b-nuclear.

Corollary 7.27. *An arbitrary b-complete semimodule over a b-complete semiring K is b-nuclear if and only if this semimodule has the b-approximation property.*

Recall that, in the traditional functional analysis, any nuclear space has the approximation property but the converse is not true.

Concrete examples of b-nuclear spaces and semimodules are described in Examples 7.15, 7.16 and 7.19 (see above). Important b-nuclear spaces and semimodules (e.g., the so-called Lipschitz spaces and semi-Lipschitz semimodules) are described in [**32**]. In this paper there is a description of all functional b-semimodules for which the kernel theorem holds (as semi-Lipschitz semimodules); this result is due to G. B. Shpiz.

It is easy to show that the idempotent spaces $USC(X)$ and $\mathrm{Conc}(\mathbf{R})$ (see Examples 7.17 and 7.18) are not b-nuclear (however, for these spaces the kernel theorem is true). The reason is that these spaces are not functional b-spaces and the corresponding δ-functionals are not b-linear (and even linear).

7.9. Kernel theorems for functional b-semimodules. Let $V \subset K(X)$ be a b-complete functional semimodule over a b-complete semiring K. Recall that for V the *kernel theorem* holds if every b-linear mapping of this semimodule to an arbitrary b-complete semimodule over K has an integral representation.

Theorem 7.28. *Assume that a b-complete semiring K and a nonempty set X are given. The kernel theorem holds for any functional b-semimodule $V \subset K(X)$ if and only if every b-linear functional on V is integral and the semimodule V is b-nuclear, i.e., has the b-approximation property.*

Corollary 7.29. *If for a functional b-semimodule the kernel theorem holds, then this semimodule is b-nuclear.*

Note that the possibility to obtain an integral representation of a functional means that one can decompose it into a sum of functionals of the form δ_x.

Corollary 7.30. *Assume that a b-complete semiring K and a nonempty set X are given. The kernel theorem holds for a functional b-semimodule $V \subset K(X)$ if and only if the identity operator id: $V \to V$ is integral.*

7.10. Integral representations of operators in abstract idempotent semimodules. In this subsection, we examine the following problem: when a b-complete idempotent semimodule V over a b-complete semiring is isomorphic to a functional b-semimodule W such that the kernel theorem holds for W.

Assume that V is a b-complete idempotent semimodule over a b-complete semiring K and ϕ is a b-linear functional defined on V. We call this functional a δ-*functional* if there exists an element $v \in V$ such that

$$\phi(w) \odot v < w$$

for every element $w \in V$. It is easy to see that every functional of the form δ_x is a δ-functional in this sense (but the converse is not true in general).

Denote by $\Delta(V)$ the set of all δ-functionals on V. Denote by i_Δ the natural mapping $V \to K(\Delta(V))$ defined by the formula

$$(i_\Delta(v))(\phi) = \phi(v)$$

for all $\phi \in \Delta(V)$. We say that an element $v \in V$ is *pointlike* if there exists a b-linear functional ϕ such that $\phi(w) \odot v < w$ for all $w \in V$. The set of all pointlike elements of V will be denoted by $P(V)$. Recall that by $\phi \odot v$ we denote the one-dimensional operator $w \mapsto \phi(w) \odot v$.

The following assertion is an obvious consequence of our definitions (including the definition of the standard order) and the idempotency of our addition.

Remark 7.31. If a one-dimensional operator $\phi \odot v$ appears in the decomposition of the identity operator on V into a sum of one-dimensional operators, then $\phi \in \Delta(V)$ and $v \in P(V)$.

Denote by id and Id the identity operators on V and $i_\Delta(V)$, respectively.

Proposition 7.32. *If the operator id is b-nuclear, then i_Δ is an embedding and the operator Id is integral.*

If the operator i_Δ is an embedding and the operator Id is integral, then the operator id is b-nuclear.

Theorem 7.33. *A b-complete idempotent semimodule V over a b-complete idempotent semiring K is isomorphic to a functional b-semimodule for which the kernel theorem holds if and only if the identity mapping on V is a b-nuclear operator, i.e., V is a b-nuclear semimodule.*

The following proposition shows that, in a certain sense, the embedding i_Δ is a universal representation of a b-nuclear semimodule in the form of a functional b-semimodule for which the kernel theorem holds.

Proposition 7.34. *Let K be a b-complete idempotent semiring, X be a nonempty set, and $V \subset K(X)$ be a functional b-semimodule on X for which the kernel theorem holds. Then there exists a natural mapping*

$i : X \to \Delta(V)$ *such that the corresponding mapping* $i_* : K(\Delta(V)) \to$ $K(X)$ *is an isomorphism of* $i_\Delta(V)$ *onto* V.

8. The dequantization transform, convex geometry and the Newton polytopes

Let X be a topological space. For functions $f(x)$ defined on X we shall say that a certain property is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X. Suppose X is $\mathbf{C}^n$ or $\mathbf{R}^n$; denote by $\mathbf{R}^n_+$ the set $x = \{ (x_1, \ldots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \ldots, n$. For $x = (x_1, \ldots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \ldots, \exp(x_n))$; so if $x \in \mathbf{R}^n$, then $\exp(x) \in \mathbf{R}^n_+$.

Denote by $\mathcal{F}(\mathbf{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbf{C}^n$ such that $U \supset \mathbf{R}^n_+$. It is clear that $\mathcal{F}(\mathbf{C}^n)$ is a ring (and an algebra over $\mathbf{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbf{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|, \tag{14}$$

where h is a (small) real positive parameter and $x \in \mathbf{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to +0} \hat{f}_h(x), \tag{15}$$

if the right-hand side of (15) exists almost everywhere.

We shall say that the function $\hat{f}(x)$ is a *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbf{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to $\mathbf{R}^n_+$ only; so in fact the dequantization transform is constructed for functions defined on $\mathbf{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$.

Of course, similar definitions can be given for functions defined on $\mathbf{R}^n$ and $\mathbf{R}^n_+$. If $s = 1/h$, then we have the following version of (14) and (15):

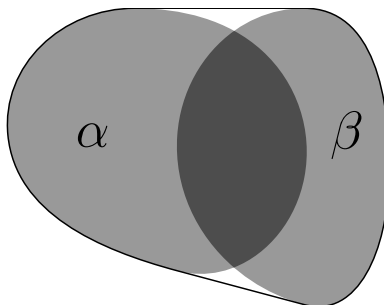$$\hat{f}(x) = \lim_{s \to \infty} (1/s) \log |f(e^{sx})|. \tag{15'}$$

FIGURE 3. Algebra of convex subsets.

Denote by $\partial \hat{f}$ the subdifferential of the function $\hat{f}$ at the origin. If f is a polynomial we have

$$\partial \hat{f} = \{\, v \in \mathbf{R}^n \mid (v, x) \le \hat{f}(x) \; \forall x \in \mathbf{R}^n \,\}.$$

It is well known that all the convex compact subsets in $\mathbf{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $\alpha, \beta \in \mathcal{S}$ the sum $\alpha \oplus \beta$ is the convex hull of the union $\alpha \cup \beta$; the product $\alpha \odot \beta$ is defined in the following way: $\alpha \odot \beta = \{\, x \mid x = a + b,$ where $a \in \alpha, b \in \beta$, see Fig.3. In fact $\mathcal{S}$ is an idempotent linear space over $\mathbf{R}_{\max}$.

Of course, the Newton polytopes of polynomials in n variables form a subsemiring $\mathcal{N}$ in $\mathcal{S}$. If f, g are polynomials, then $\partial(\widehat{fg}) = \partial \hat{f} \odot \partial \hat{g}$; moreover, if f and g are "in general position", then $\partial(\widehat{f + g}) = \partial \hat{f} \oplus \partial \hat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this "traditional" semiring to the idempotent semiring $\mathcal{N}$.

Theorem 8.1. *If f is a polynomial, then the subdifferential $\partial \hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of f. For the semiring of polynomials with nonnegative coefficients, the transform $f \mapsto \partial \hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the Minkowski operations (see above).*

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see below and [**31**].

8.1. Dequantization transform: algebraic properties. Denote by V the set $\mathbf{R}^n$ treated as a linear Euclidean space (with the scalar product $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$) and set $V_+ = \mathbf{R}^n_+$. We shall say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *dequantizable* whenever its dequantization $\hat{f}(x)$ exists (and is defined on an open dense subset of V). By $\mathcal{D}(\mathbf{C}^n)$ denote the set of all dequantizable functions and by $\widehat{\mathcal{D}}(V)$ denote the set $\{\, \hat{f} \mid f \in \mathcal{D}(\mathbf{C}^n) \,\}$. Recall that functions from $\mathcal{D}(\mathbf{C}^n)$ (and $\widehat{\mathcal{D}}(V)$) are defined almost everywhere and $f = g$ means that $f(x) = g(x)$ a.e., i.e., for x ranging over an open dense subset of $\mathbf{C}^n$ (resp., of V). Denote by $\mathcal{D}_+(\mathbf{C}^n)$ the set of all functions $f \in \mathcal{D}(\mathbf{C}^n)$ such that $f(x_1, \ldots, x_n) \geq 0$ if $x_i \geq 0$ for $i = 1, \ldots, n$; so $f \in \mathcal{D}_+(\mathbf{C}^n)$ if the restriction of f to $V_+ = \mathbf{R}^n_+$ is a nonnegative function. By $\widehat{\mathcal{D}}_+(V)$ denote the image of $\mathcal{D}_+(\mathbf{C}^n)$ under the dequantization transform. We shall say that functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ are in *general position* whenever $\hat{f}(x) \neq \hat{g}(x)$ for x running an open dense subset of V.

Theorem 8.2. *For functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ and any nonzero constant c, the following equations are valid:*

1) $\widehat{fg} = \hat{f} + \hat{g}$;
2) $|\hat{f}| = \hat{f}$; $\widehat{cf} = f$; $\hat{c} = 0$;
3) $(\widehat{f + g})(x) = \max\{\hat{f}(x), \hat{g}(x)\}$ *a.e. if f and g are nonnegative on V_+ (i.e., $f, g \in \mathcal{D}_+(\mathbf{C}^n)$) or f and g are in general position.*

Left-hand sides of these equations are well-defined automatically.

Corollary 8.3. *The set $\mathcal{D}_+(\mathbf{C}^n)$ has a natural structure of a semiring with respect to the usual addition and multiplication of functions taking their values in $\mathbf{C}$. The set $\widehat{\mathcal{D}}_+(V)$ has a natural structure of an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$; elements of $\widehat{\mathcal{D}}_+(V)$ can be naturally treated as functions taking their values in $\mathbf{R}_{\max}$. The dequantization transform generates a homomorphism from $\mathcal{D}_+(\mathbf{C}^n)$ to $\widehat{\mathcal{D}}_+(V)$.*

8.2. Generalized polynomials and simple functions. For any nonzero number $a \in \mathbf{C}$ and any vector $d = (d_1, \ldots, d_n) \in V = \mathbf{R}^n$ we set $m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$; functions of this kind we shall call *generalized monomials*. Generalized monomials are defined a.e. on $\mathbf{C}^n$ and on

V_+, but not on V unless the numbers d_i take integer or suitable rational values. We shall say that a function f is a *generalized polynomial* whenever it is a finite sum of linearly independent generalized monomials. For instance, Laurent polynomials and Puiseax polynomials are examples of generalized polynomials.

As usual, for $x, y \in V$ we set $(x, y) = x_1 y_1 + \cdots + x_n y_n$. The following proposition is a result of a trivial calculation.

Proposition 8.4. *For any nonzero number $a \in V = \mathbf{C}$ and any vector $d \in V = \mathbf{R}^n$ we have $(\widehat{m_{a,d}})_h(x) = (d, x) + h \log |a|$.*

Corollary 8.5. *If f is a generalized monomial, then $\hat{f}$ is a linear function.*

Recall that a real function p defined on $V = \mathbf{R}^n$ is *sublinear* if $p = \sup_\alpha p_\alpha$, where $\{p_\alpha\}$ is a collection of linear functions. Sublinear functions defined everywhere on $V = \mathbf{R}^n$ are convex; thus these functions are continuous, see [**34**]. We discuss sublinear functions of this kind only. Suppose p is a continuous function defined on V, then p is sublinear whenever

    1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$;

    2) $p(cx) = cp(x)$ for all $x \in V$, $c \in \mathbf{R}_+$.

So if p_1, p_2 are sublinear functions, then $p_1 + p_2$ is a sublinear function.

We shall say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *simple*, if its dequantization $\hat{f}$ exists and a.e. coincides with a sublinear function; by misuse of language, we shall denote this (uniquely defined everywhere on V) sublinear function by the same symbol $\hat{f}$.

Recall that simple functions f and g are *in general position* if $\hat{f}(x) \neq \hat{g}(x)$ for all x belonging to an open dense subset of V. In particular, generalized monomials are in general position whenever they are linearly independent.

Denote by $Sim(\mathbf{C}^n)$ the set of all simple functions defined on V and denote by $Sim_+(\mathbf{C}^n)$ the set $Sim(\mathbf{C}^n) \cap \mathcal{D}_+(\mathbf{C}^n)$. By $Sbl(V)$ denote the set of all (continuous) sublinear functions defined on $V = \mathbf{R}^n$ and by $Sbl_+(V)$ denote the image $\widehat{Sim_+}(\mathbf{C}^n)$ of $Sim_+(\mathbf{C}^n)$ under the dequantization transform.

The following statements can be easily deduced from Theorem 8.2 and definitions.

Corollary 8.6. *The set $Sim_+(\mathbf{C}^n)$ is a subsemiring of $\mathcal{D}_+(\mathbf{C}^n)$ and $Sbl_+(V)$ is an idempotent subsemiring of $\widehat{\mathcal{D}}_+(V)$. The dequantization transform generates an epimorphism of $Sim_+(\mathbf{C}^n)$ onto $Sbl_+(V)$. The set $Sbl(V)$ is an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$.*

Corollary 8.7. *Polynomials and generalized polynomials are simple functions.*

We shall say that functions $f, g \in \mathcal{D}(V)$ are *asymptotically equivalent* whenever $\hat{f} = \widehat{g}$; any simple function f is an *asymptotic monomial* whenever $\hat{f}$ is a linear function. A simple function f will be called an *asymptotic polynomial* whenever $\hat{f}$ is a sum of a finite collection of nonequivalent asymptotic monomials.

Corollary 8.8. *Every asymptotic polynomial is a simple function.*

Example 8.9. Generalized polynomials, logarithmic functions of (generalized) polynomials, and products of polynomials and logarithmic functions are asymptotic polynomials. This follows from our definitions and formula (15).

8.3. Subdifferentials of sublinear functions. We shall use some elementary results from convex analysis. These results can be found, e.g., in [**34**], ch. 1, §1.

For any function $p \in Sbl(V)$ we set

$$\partial p = \{\, v \in V \mid (v, x) \leq p(x) \;\forall x \in V \,\}.$$

It is well known from convex analysis that for any sublinear function p the set ∂p is exactly the *subdifferential* of p at the origin. The following propositions are also known in convex analysis.

Proposition 8.10. *Suppose $p_1, p_2 \in Sbl(V)$, then*

    1) $\partial(p_1 + p_2) = \partial p_1 \odot \partial p_2 = \{v \in V \mid v = v_1 + v_2,\ \text{where } v_1 \in \partial p_1, v_2 \in \partial p_2\}$;

2) $\partial(\max\{p_1(x), p_2(x)\}) = \partial p_1 \oplus \partial p_2$.

Recall that $\partial p_1 \oplus \partial p_2$ is a convex hull of the set $\partial p_1 \cup \partial p_2$.

Proposition 8.11. *Suppose $p \in Sbl(V)$. Then ∂p is a nonempty convex compact subset of V.*

Corollary 8.12. *The map $p \mapsto \partial p$ is a homomorphism of the idempotent semiring $Sbl(V)$ (see Corollary 8.3) to the idempotent semiring $\mathcal{S}$ of all convex compact subsets of V (see Subsection 8.1 above).*

8.4. Newton sets for simple functions. For any simple function $f \in Sim(\mathbf{C}^n)$ let us denote by $N(f)$ the set $\partial(\hat{f})$. We shall call $N(f)$ the *Newton set* of the function f.

Proposition 8.13. *For any simple function f, its Newton set $N(f)$ is a nonempty convex compact subset of V.*

This proposition follows from Proposition 8.11 and definitions.

Theorem 8.14. *Suppose that f and g are simple functions. Then*

1) $N(fg) = N(f) \odot N(g) = \{v \in V \mid v = v_1 + v_2 \text{ with } v_1 \in N(f), v_2 \in N(g)\}$;
2) $N(f + g) = N(f) \oplus N(g)$, *if f_1 and f_2 are in general position or $f_1, f_2 \in Sim_+(\mathbf{C}^n)$ (recall that $N(f) \oplus N(g)$ is the convex hull of $N(f) \cup N(g)$).*

This theorem follows from Theorem 8.2, Proposition 8.10 and definitions.

Corollary 8.15. *The map $f \mapsto N(f)$ generates a homomorphism from $Sim_+(\mathbf{C}^n)$ to $\mathcal{S}$.*

Proposition 8.16. *Let $f = m_{a,d}(x) = a \prod_{i=1}^{n} x_i^{d_i}$ be a monomial; here $d = (d_1, \ldots, d_n) \in V = \mathbf{R}^n$ and a is a nonzero complex number. Then $N(f) = \{d\}$.*

This follows from Proposition 8.4, Corollary 8.5 and definitions.

Corollary 8.17. *Let $f = \sum_{d \in D} m_{a_d,d}$ be a polynomial. Then $N(f)$ is the polytope $\oplus_{d \in D}\{d\}$, i.e. the convex hull of the finite set D.*

This statement follows from Theorem 8.14 and Proposition 8.16. Thus in this case $N(f)$ is the well-known classical Newton polytope of the polynomial f.

Now the following corollary is obvious.

Corollary 8.18. *Let f be a generalized or asymptotic polynomial. Then its Newton set $N(f)$ is a convex polytope.*

Example 8.19. Consider the one dimensional case, i.e., $V = \mathbf{R}$ and suppose $f_1 = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $f_2 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$, where $a_n \neq 0$, $b_m \neq 0$, $a_0 \neq 0$, $b_0 \neq 0$. Then $N(f_1)$ is the segment $[0, n]$ and $N(f_2)$ is the segment $[0, m]$. So the map $f \mapsto N(f)$ corresponds to the map $f \mapsto \deg(f)$, where $\deg(f)$ is a degree of the polynomial f. In this case Theorem 2 means that $\deg(fg) = \deg f + \deg g$ and $\deg(f + g) = \max\{\deg f, \deg g\} = \max\{n, m\}$ if $a_i \geq 0$, $b_i \geq 0$ or f and g are in general position.

9. Dequantization of set functions and measures on metric spaces

The following results are presented in [**33**].

Example 9.1. Let M be a metric space, S its arbitrary subset with a compact closure. It is well-known that a Euclidean d-dimensional ball B_ρ of radius ρ has volume

$$\mathrm{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1 + d/2)} \rho^d,$$

where d is a natural parameter. By means of this formula it is possible to define a volume of B_ρ for any *real d*. Cover S by a finite number of balls of radii ρ_m. Set

$$v_d(S) := \lim_{\rho \to 0} \inf_{\rho_m < \rho} \sum_m \mathrm{vol}_d(B_{\rho_m}).$$

Then there exists a number D such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number D is called the *Hausdorff-Besicovich dimension* (or *HB-dimension*) of S, see, e.g., [**35**]. Note

that a set of non-integral HB-dimension is called a fractal in the sense of B. Mandelbrot.

Theorem 9.2. *Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius ρ covering S. Then*

$$D(S) = \lim_{\rho \to +0} \log_\rho(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of S. Set $\rho = e^{-s}$, then

$$D(S) = \lim_{s \to +\infty} (1/s) \cdot \log \mathcal{N}_{exp(-s)}(S).$$

So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $\mathcal{N}_\rho(S)$.

Example 9.3. Let μ be a set function on M (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball B_ρ. Let $B_{x,\rho}$ be a ball of radius ρ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let $\rho = e^{-s}$ and

$$D_{x,\mu} := \lim_{s \to +\infty} -(1/s) \cdot \log(|\mu_x(e^{-s})|).$$

This number could be treated as a dimension of M at the point x with respect to the set function μ. So this dimension is a result of a dequantization of the function $\mu_x(\rho)$, where x is fixed. There are many dequantization procedures of this type in different mathematical areas. In particular, V.P. Maslov's negative dimension (see [**39**]) can be treated similarly.

10. Dequantization of geometry

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [**47**]. Starting from the idempotent correspondence principle O. Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problem for constructing real algebraic varieties with prescribed properties and

parameters) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, see [**12**, **48**]. Then complex algebraic geometry was dequantized by G. Mikhalkin and the result turned out to be the same; this new 'idempotent' (or asymptotic) geometry is now often called the *tropical algebraic geometry*, see, e.g., [**17**, **23**, **24**, **29**, **41**, **42**].

There is a natural relation between the Maslov dequantization and amoebas.

Suppose $(\mathbf{C}^*)^n$ is a complex torus, where $\mathbf{C}^* = \mathbf{C} \backslash \{0\}$ is the group of nonzero complex numbers under multiplication. For $z = (z_1, \ldots, z_n) \in (\mathbf{C}^*)^n$ and a positive real number h denote by $\mathrm{Log}_h(z) = h \log(|z|)$ the element

$$(h \log |z_1|, h \log |z_2|, \ldots, h \log |z_n|) \in \mathbf{R}^n.$$

Suppose $V \subset (\mathbf{C}^*)^n$ is a complex algebraic variety; denote by $\mathcal{A}_h(V)$ the set $\mathrm{Log}_h(V)$. If $h = 1$, then the set $\mathcal{A}(V) = \mathcal{A}_1(V)$ is called the *amoeba* of V; the amoeba $\mathcal{A}(V)$ is a closed subset of $\mathbf{R}^n$ with a non-empty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity suppose V is a hypersurface in $(\mathbf{C}^*)^n$ defined by a polynomial f; then there is a deformation $h \mapsto f_h$ of this polynomial generated by the Maslov dequantization and $f_h = f$ for $h = 1$. Let $V_h \subset (\mathbf{C}^*)^n$ be the zero set of f_h and set $\mathcal{A}_h(V_h) = \mathrm{Log}_h(V_h)$. Then there exists a tropical variety $Tro(V)$ such that the subsets $\mathcal{A}_h(V_h) \subset \mathbf{R}^n$ tend to $Tro(V)$ in the Hausdorff metric as $h \to 0$. The tropical variety $Tro(V)$ is a result of a deformation of the amoeba $\mathcal{A}(V)$ and the Maslov dequantization of the variety V. The set $Tro(V)$ is called the *skeleton* of $\mathcal{A}(V)$.

Example 10.1. For the line $V = \{ (x, y) \in (\mathbf{C}^*)^2 \mid x + y + 1 = 0 \}$ the piecewise-linear graph $Tro(V)$ is a tropical line, see Fig.4(a). The amoeba $\mathcal{A}(V)$ is represented in Fig.4(b), while Fig.4(c) demonstrates the corresponding deformation of the amoeba.
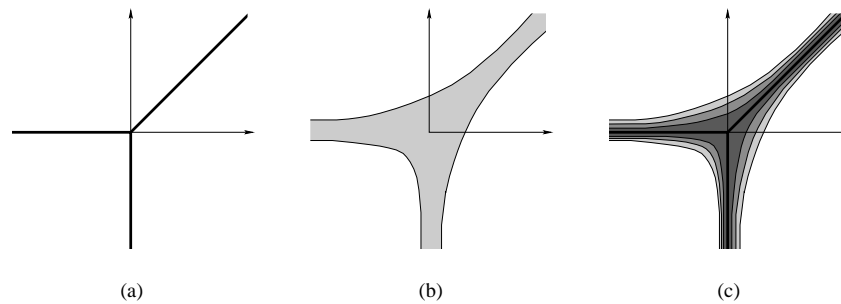
FIGURE 4. Tropical line and deformations of an amoeba.

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