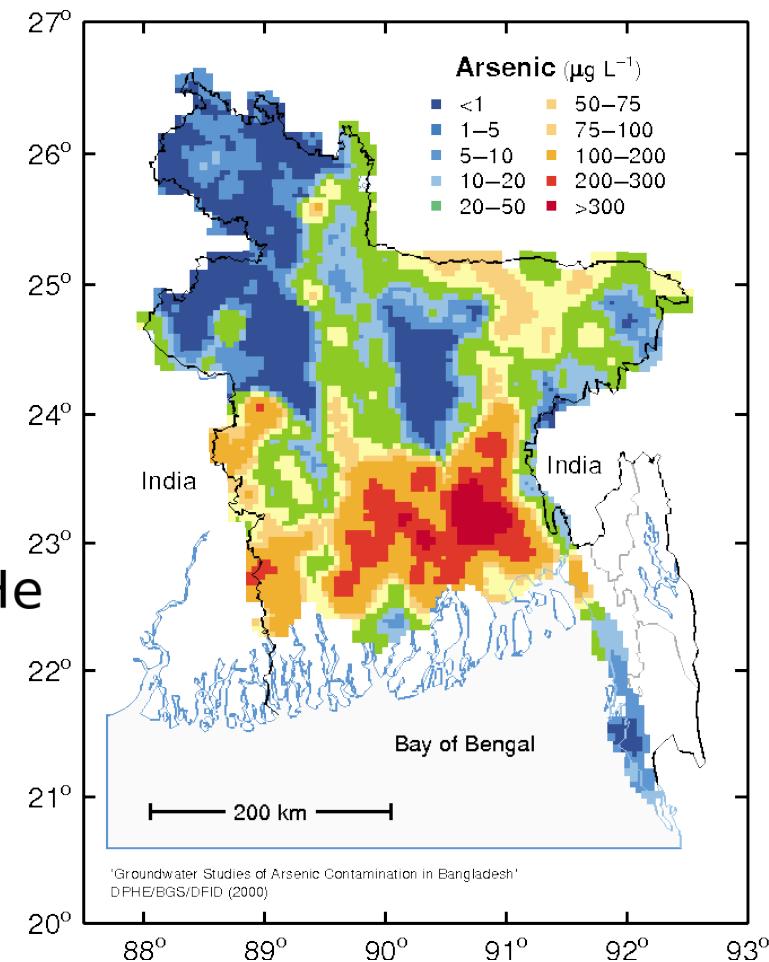
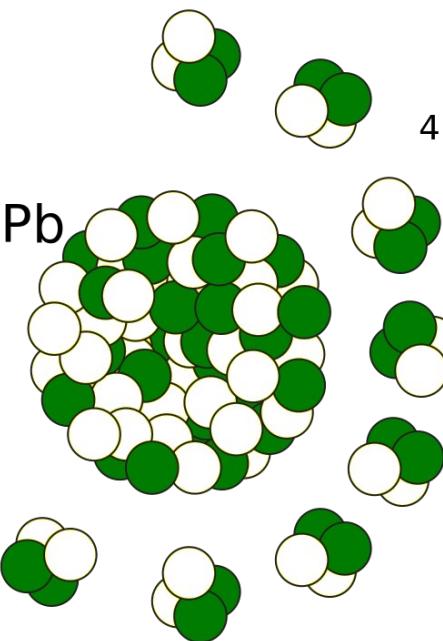
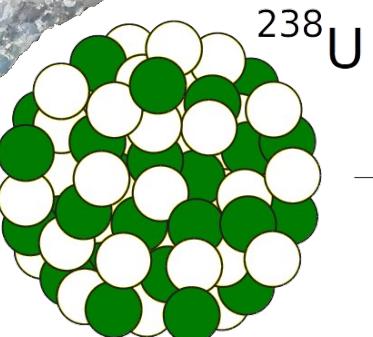


# Statistics for geoscientists

## Introduction

‘statistics’ is:

*The practice or science of collecting and analysing numerical data in large quantities, especially for the purpose of **inferring** proportions in a whole from those in a representative **sample**.*





Mathematical Statistics  
and Data Analysis

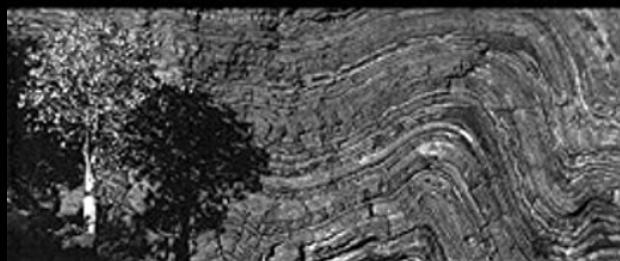
Second Edition

John A. Rice

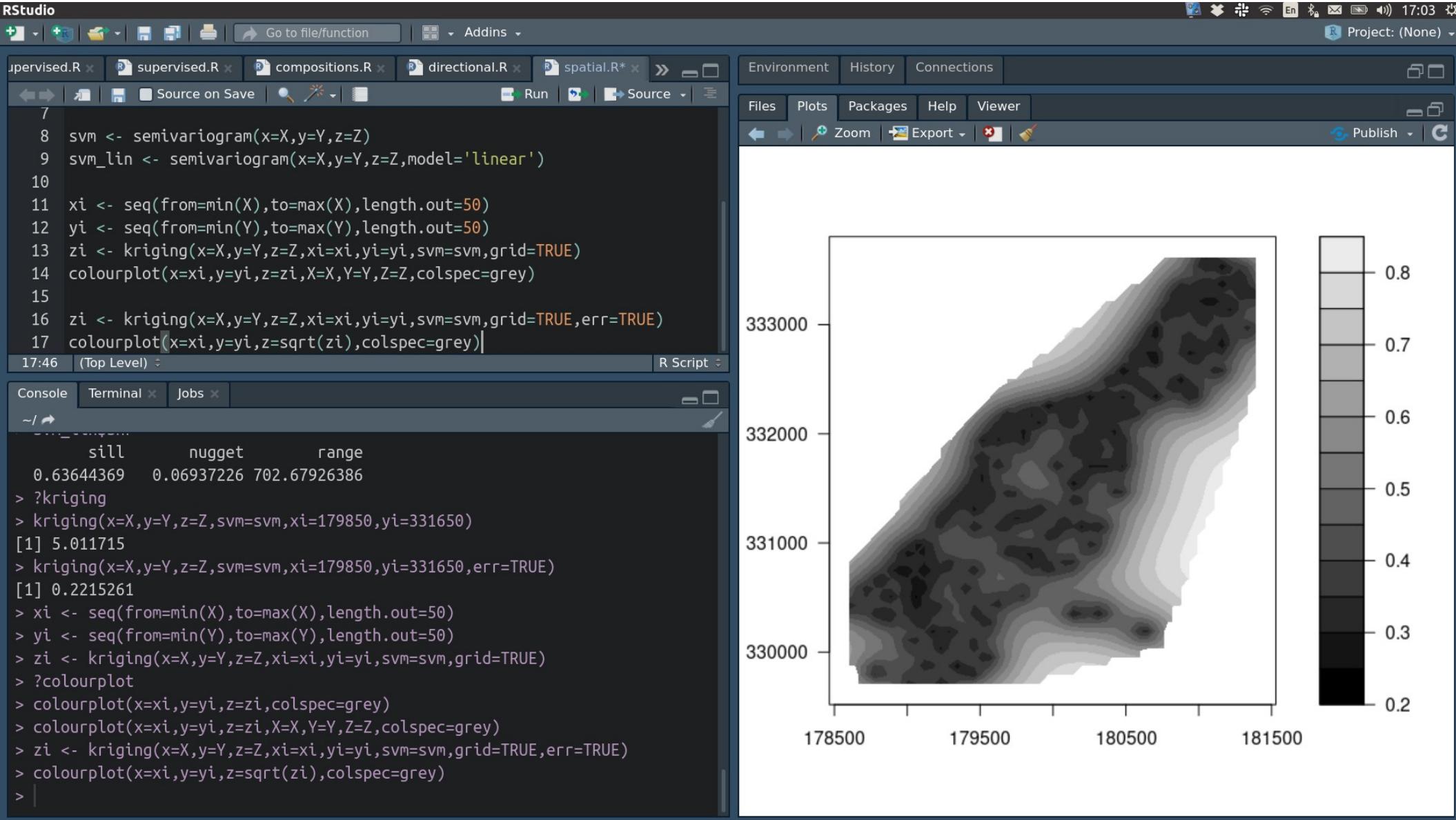
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STATISTICS  
AND  
DATA ANALYSIS  
IN  
**GEOLOGY**

THIRD EDITION



JOHN C. DAVIS

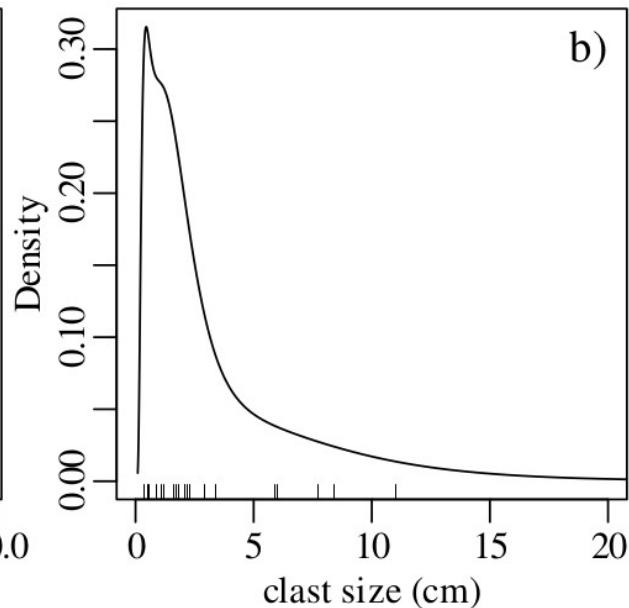
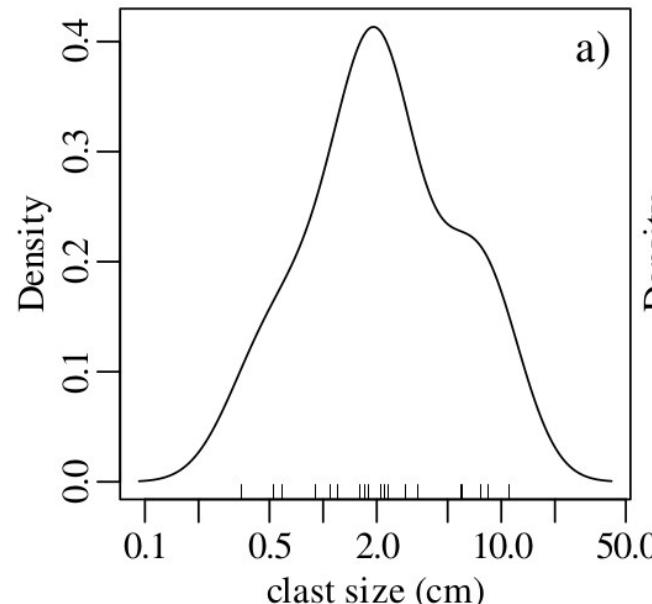
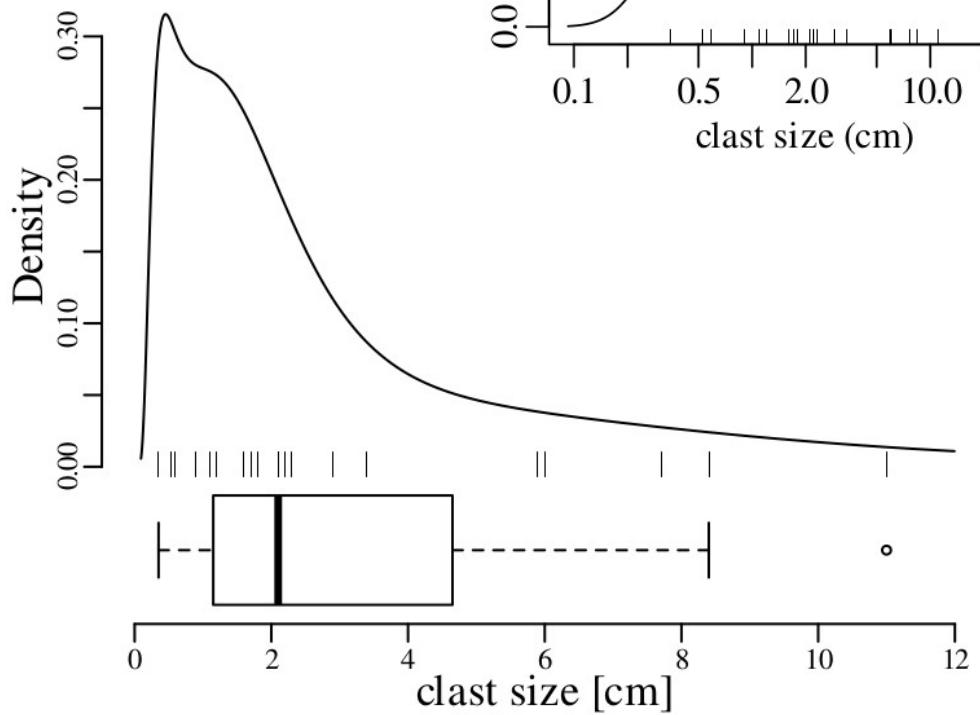


Chapter 2

Plotting data

Chapter 3

Summary statistics



## Chapter 4

Probability

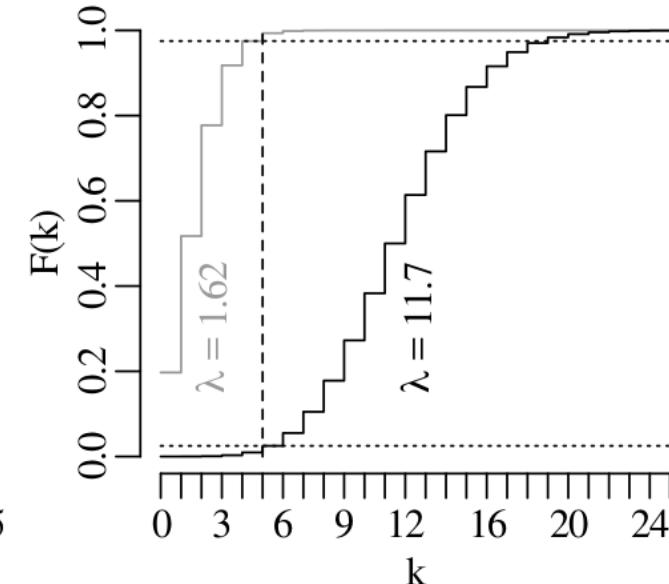
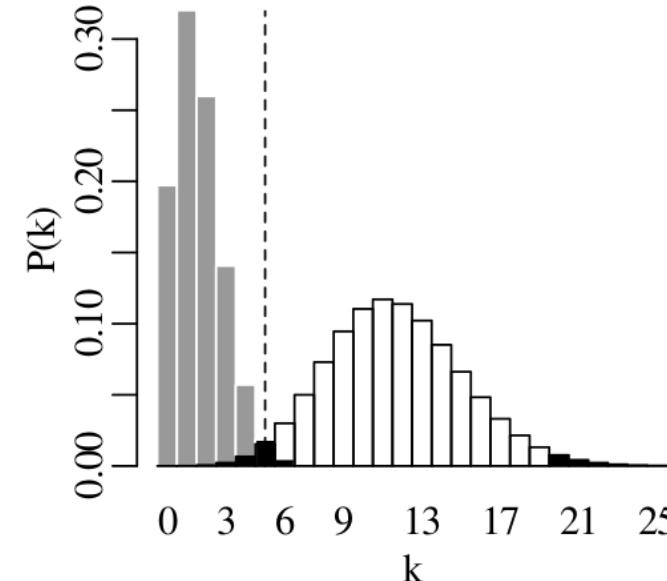
Chapter 5

The binomial distribution

Chapter 6

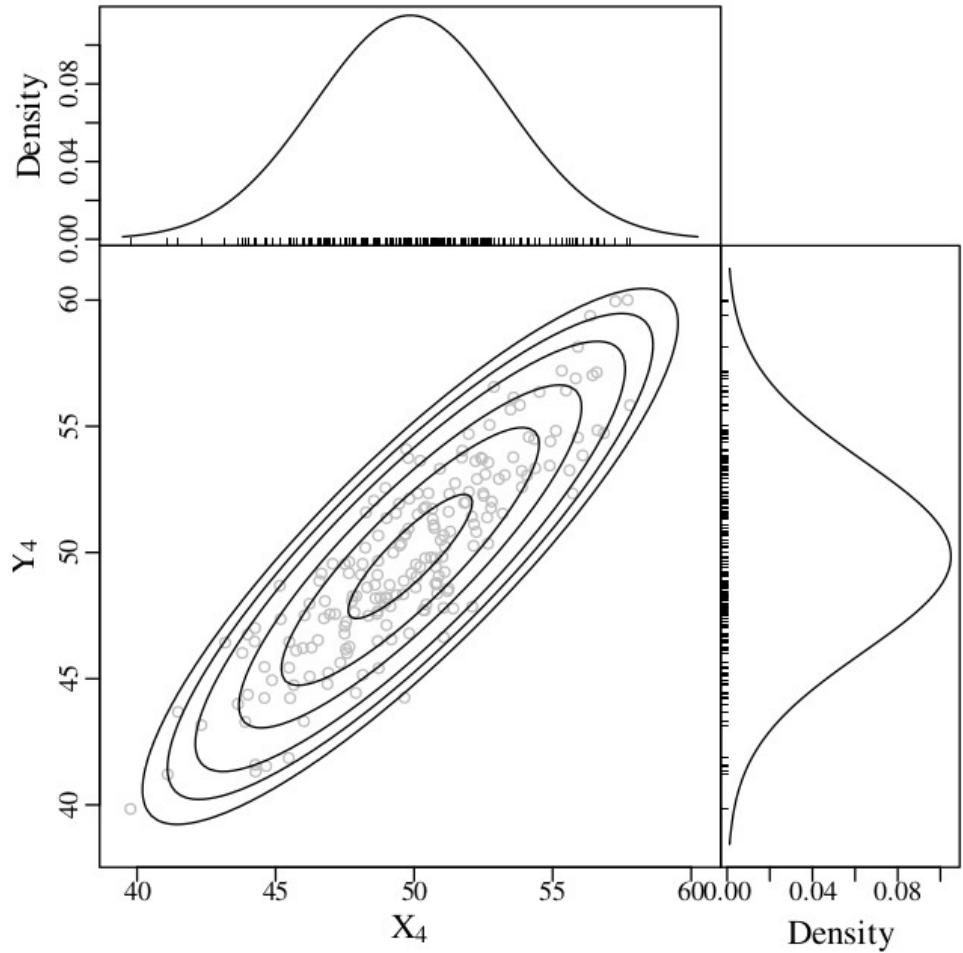
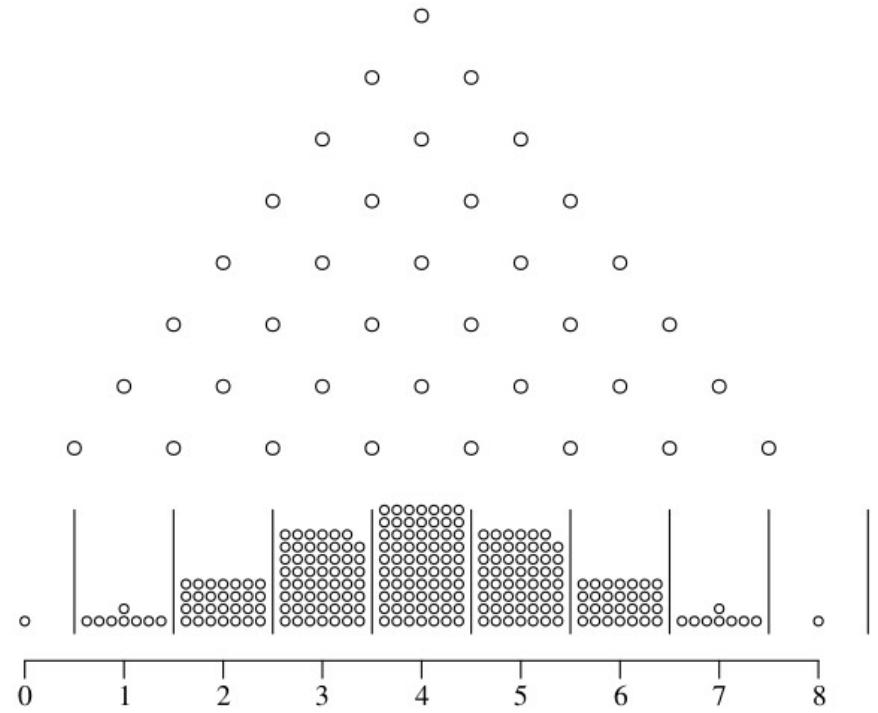
The Poisson distribution

$$P(1 \times \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \cap 1 \times \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array}) = \frac{\left\{ \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} \right\} \left\{ \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\}}{\left\{ \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \right\} \left\{ \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \right\}} = \frac{2}{36} = \frac{1}{18}$$



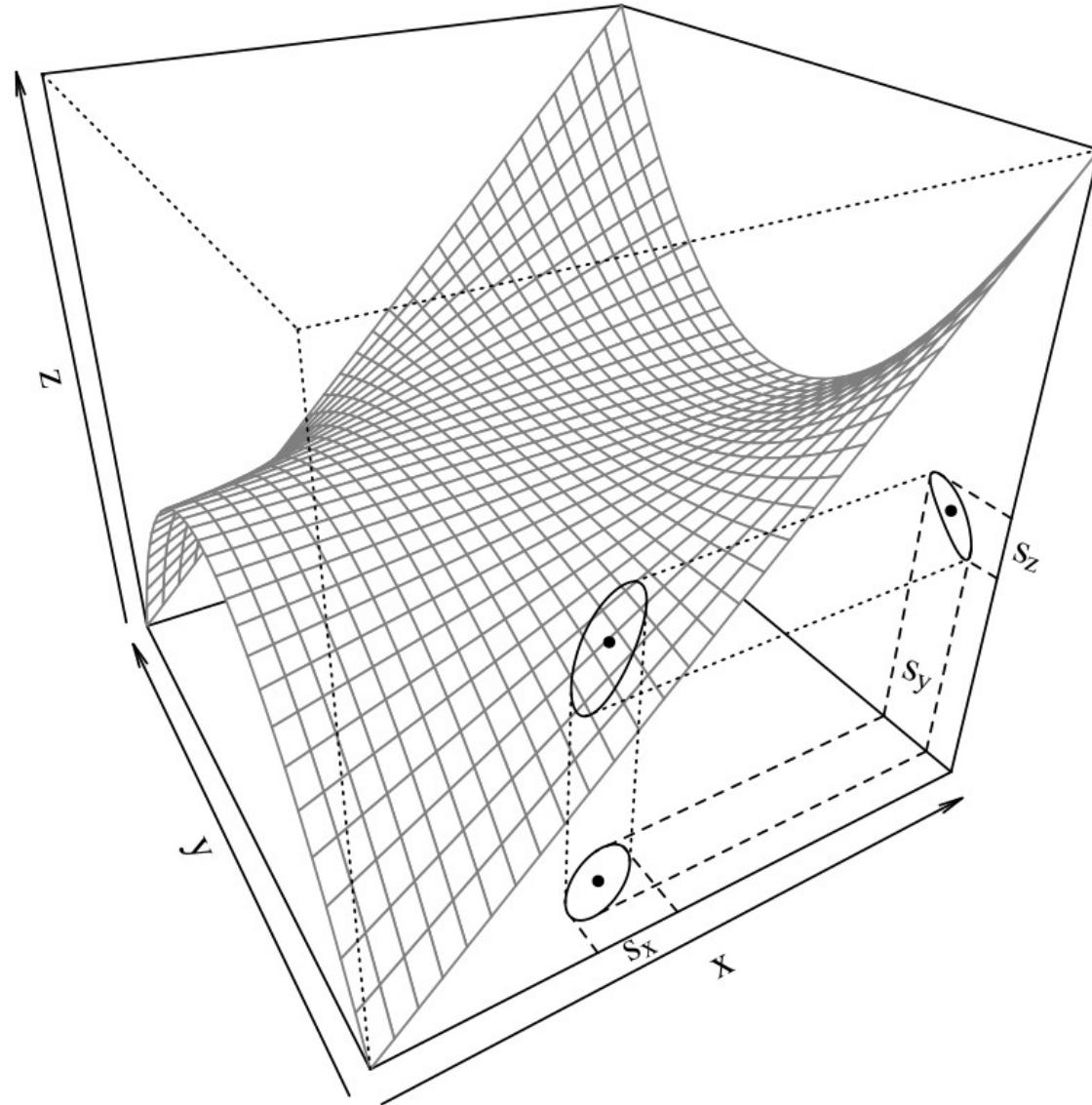
# Chapter 7

## The normal distribution



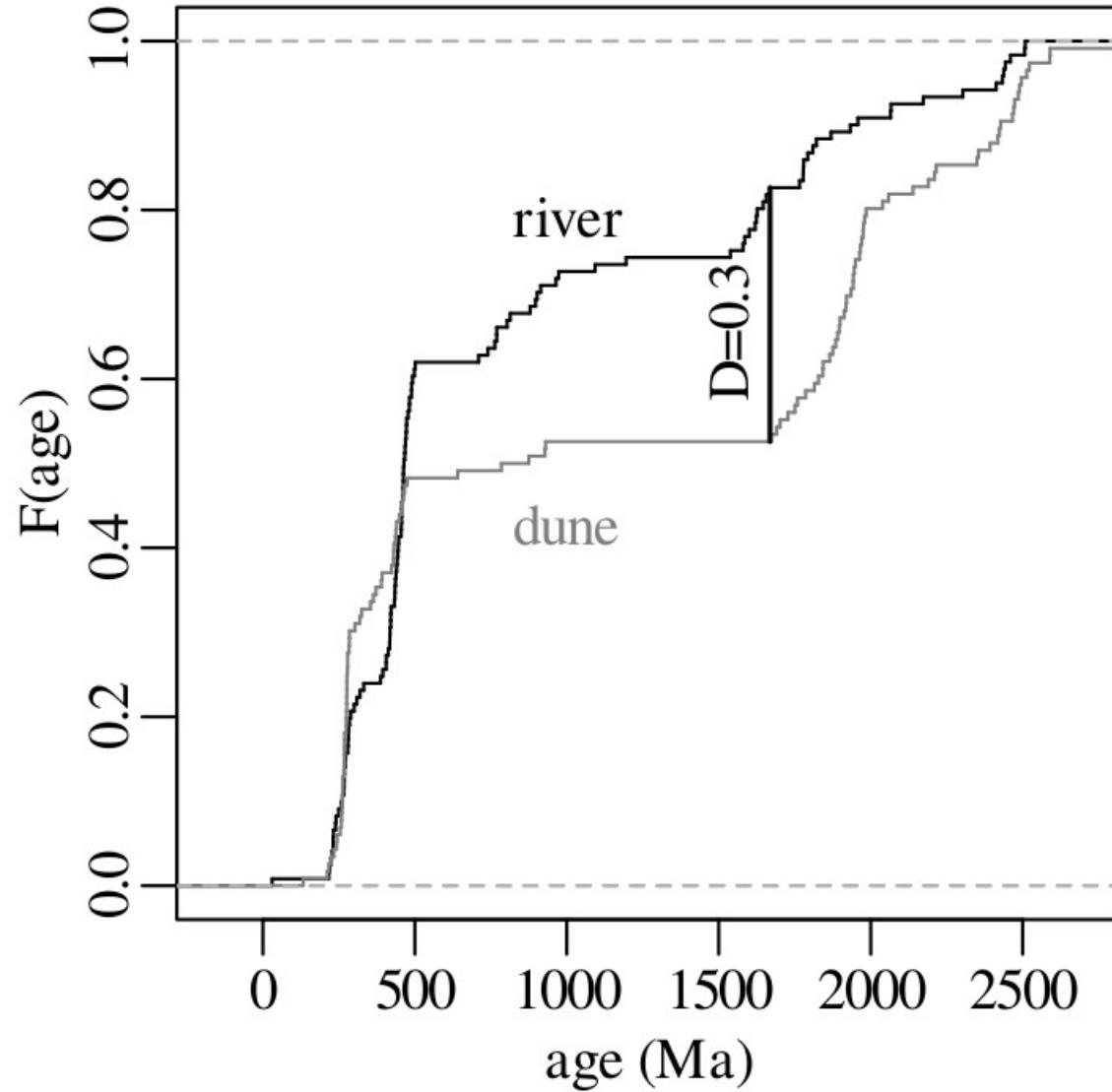
# Chapter 8

## Error propagation



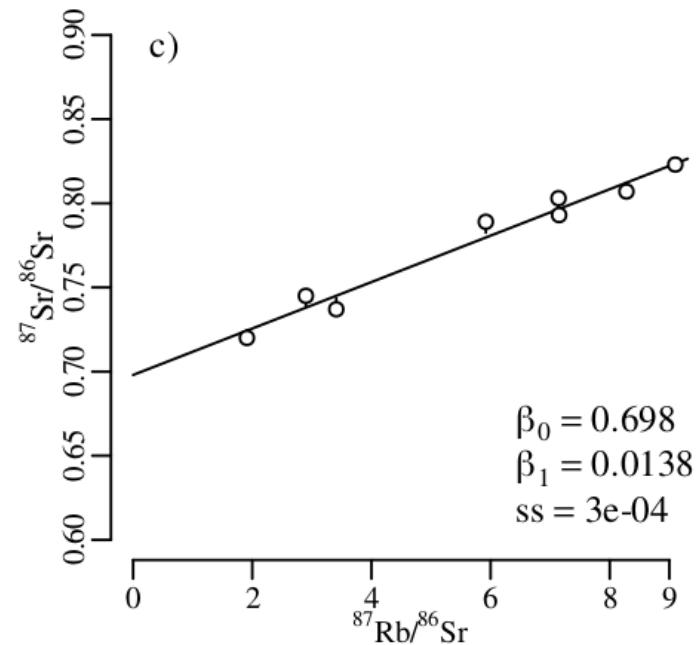
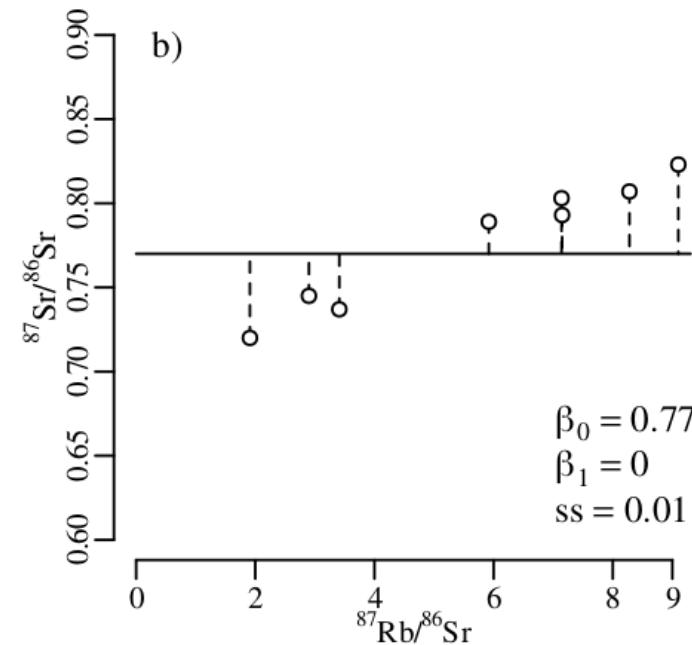
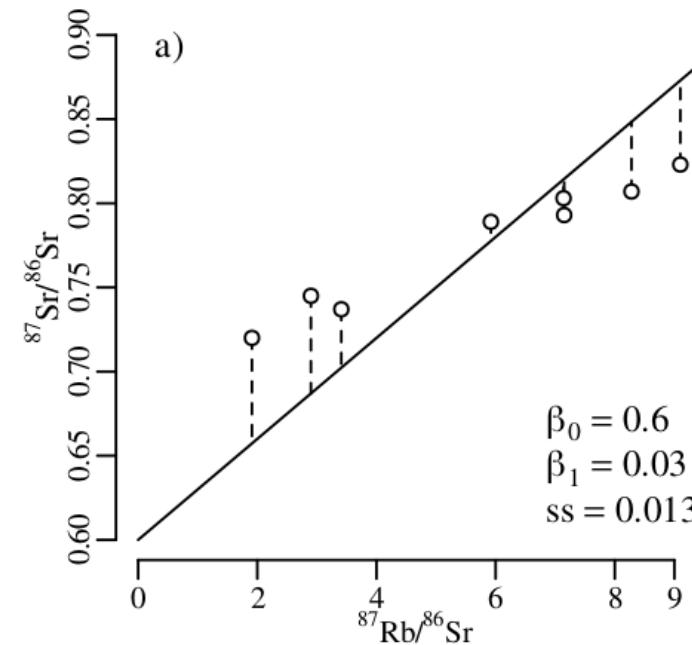
## Chapter 9

### Comparing distributions



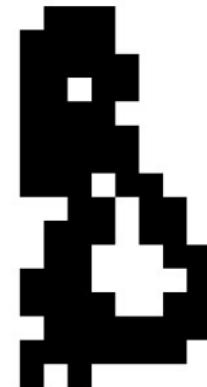
# Chapter 10

## Regression

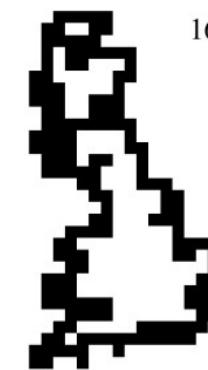


# Chapter 11

## Fractals and chaos



74/256



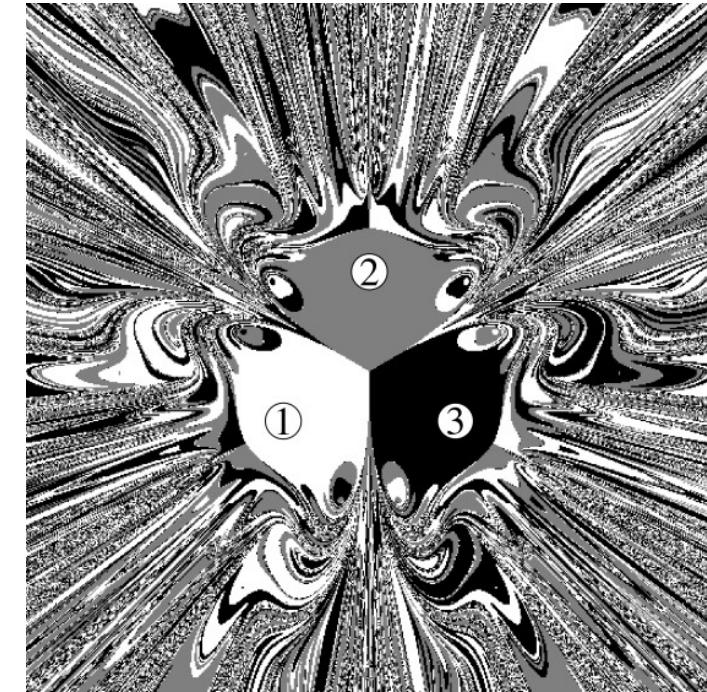
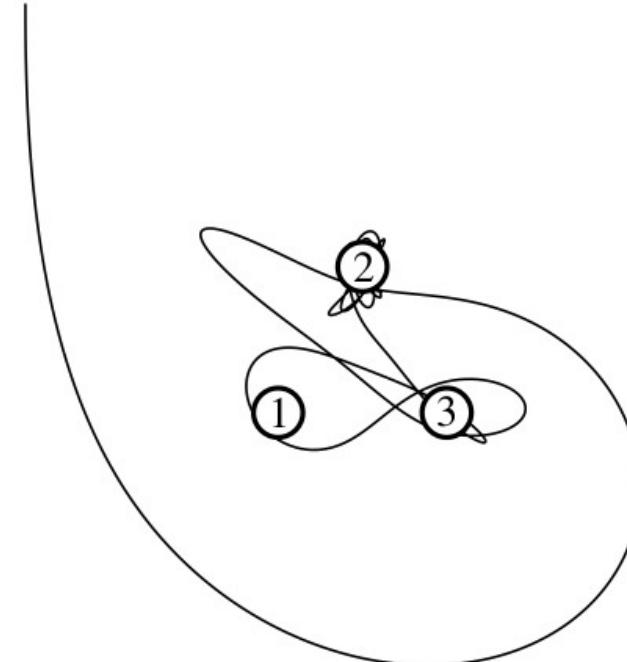
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819/16384



3058/262144

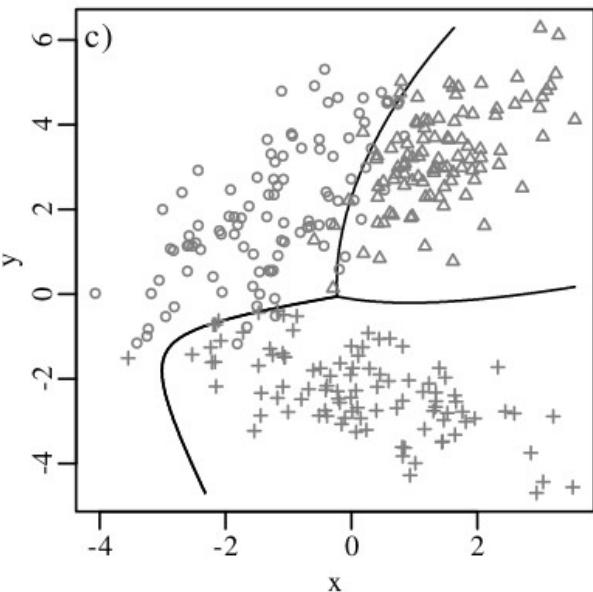
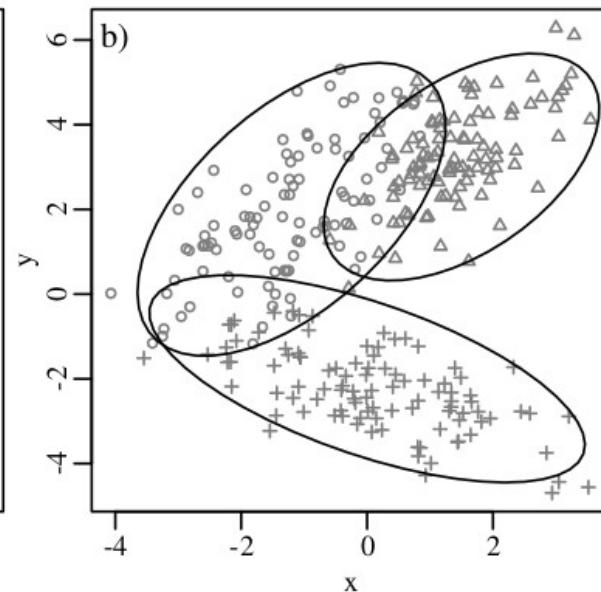
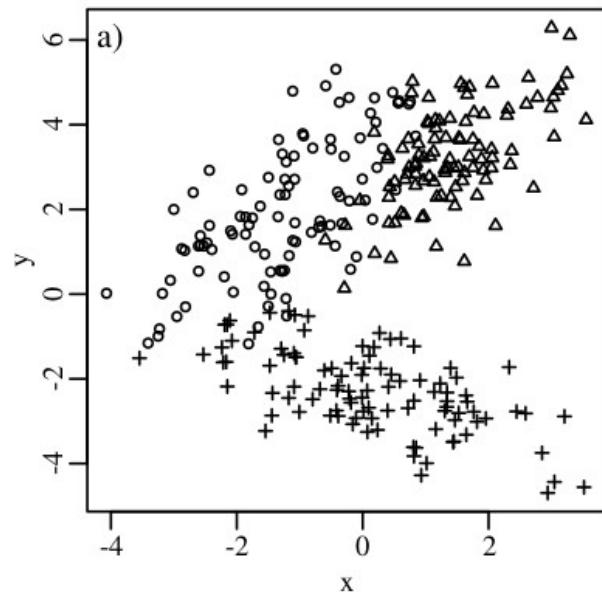
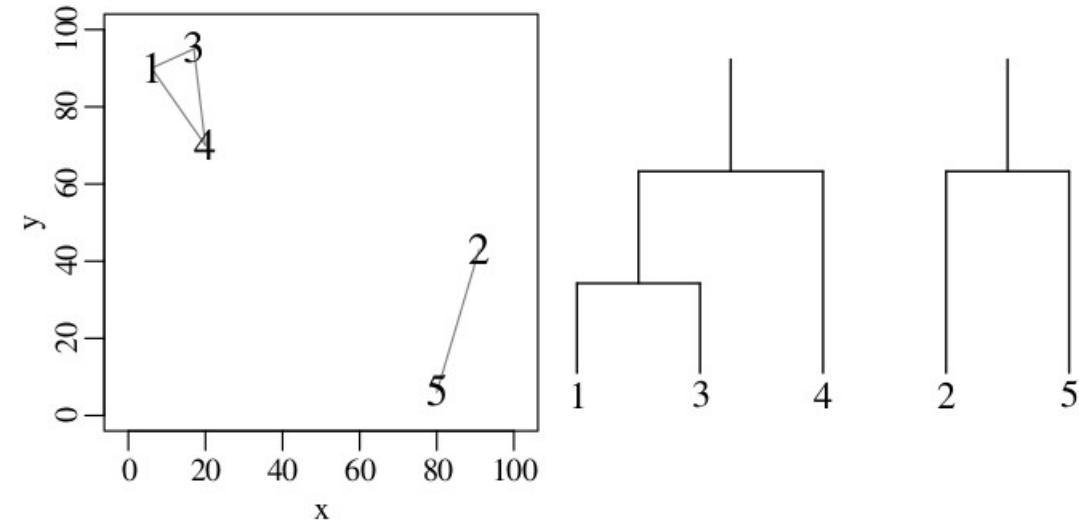


# Chapter 12

## Unsupervised learning

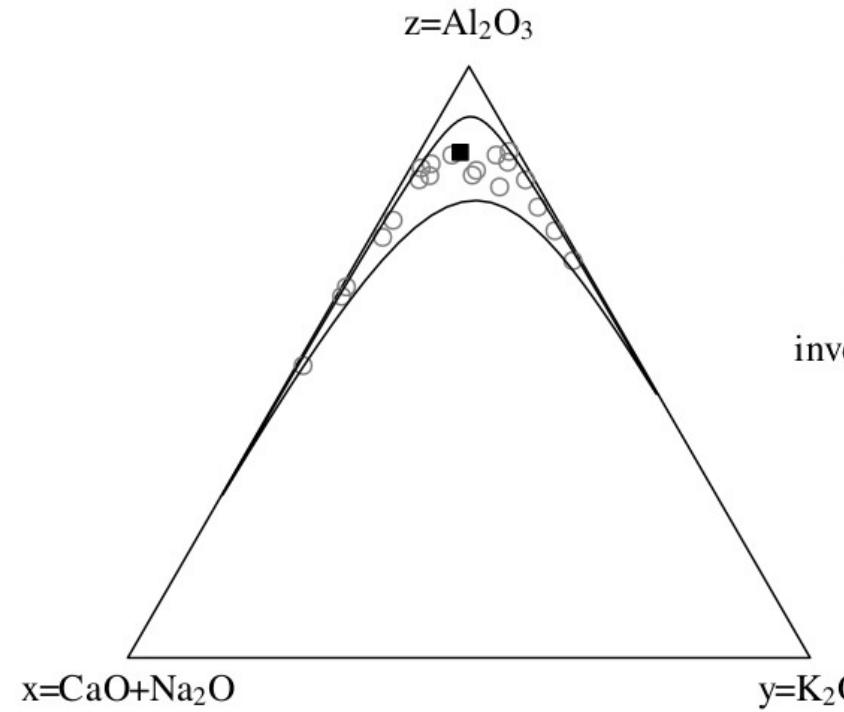
### Chapter 13

## Supervised learning

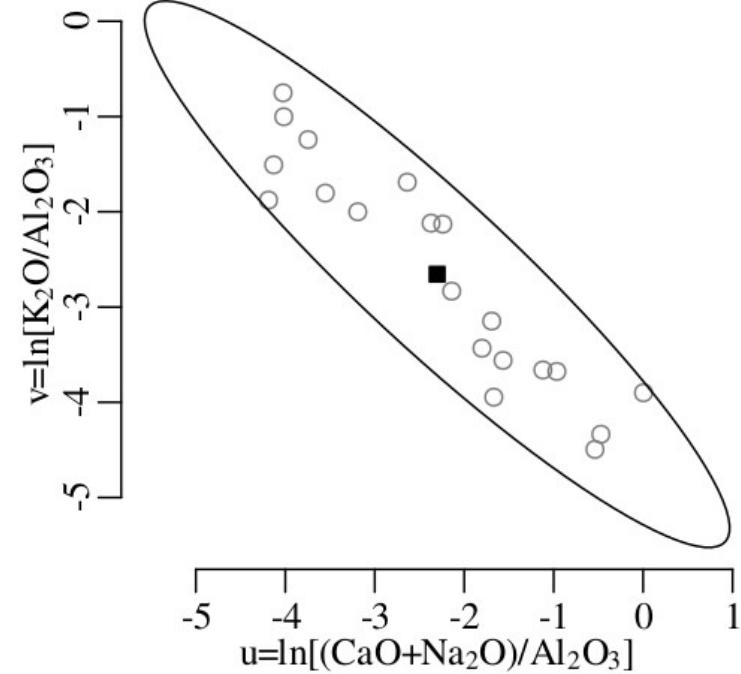


# Chapter 14

## Compositional data

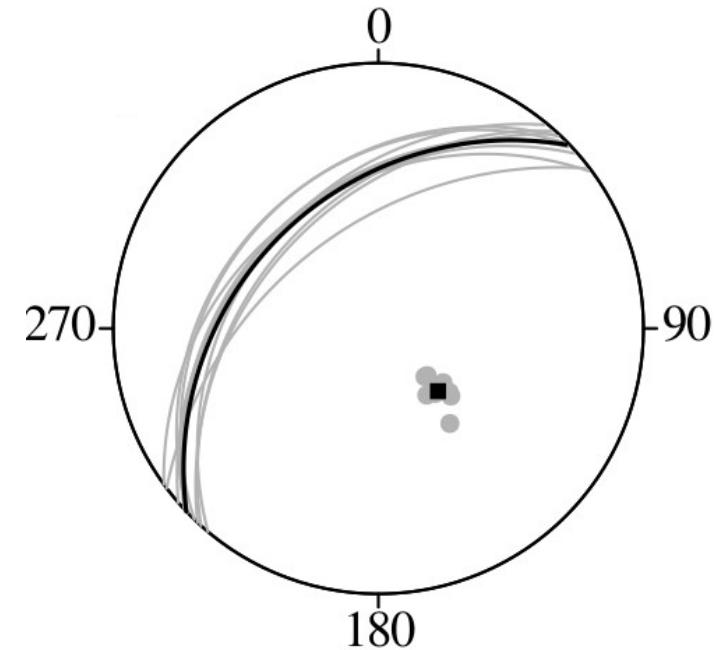
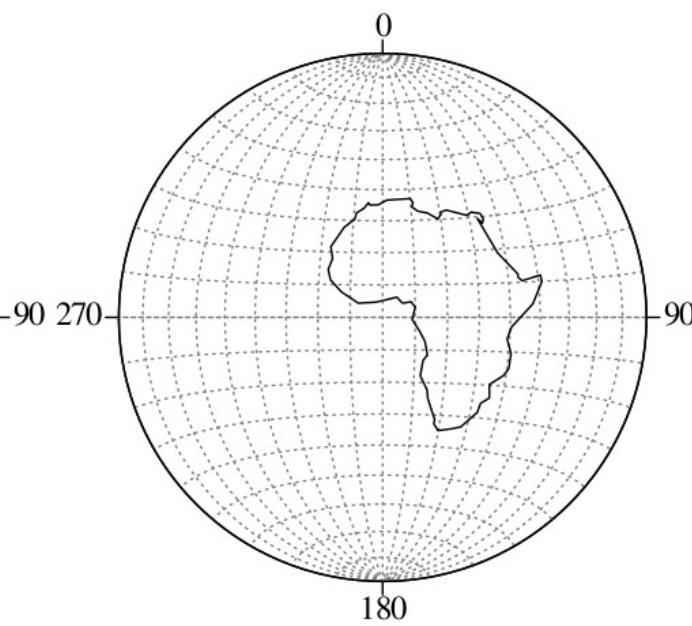
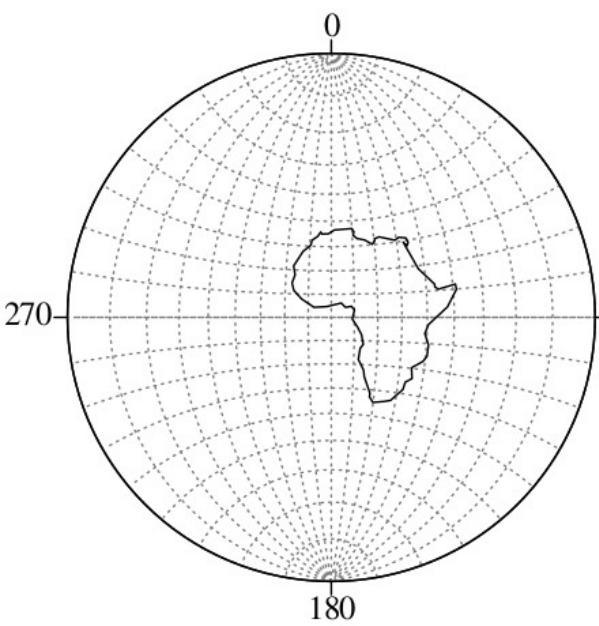
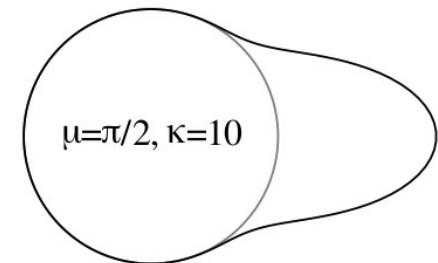
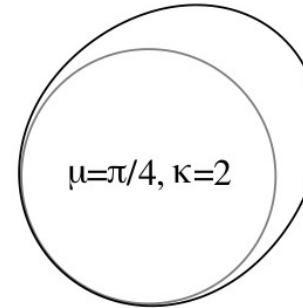
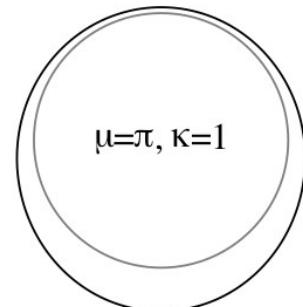
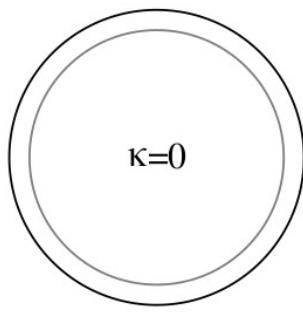


logratio transformation  
↔  
inverse logratio transformation



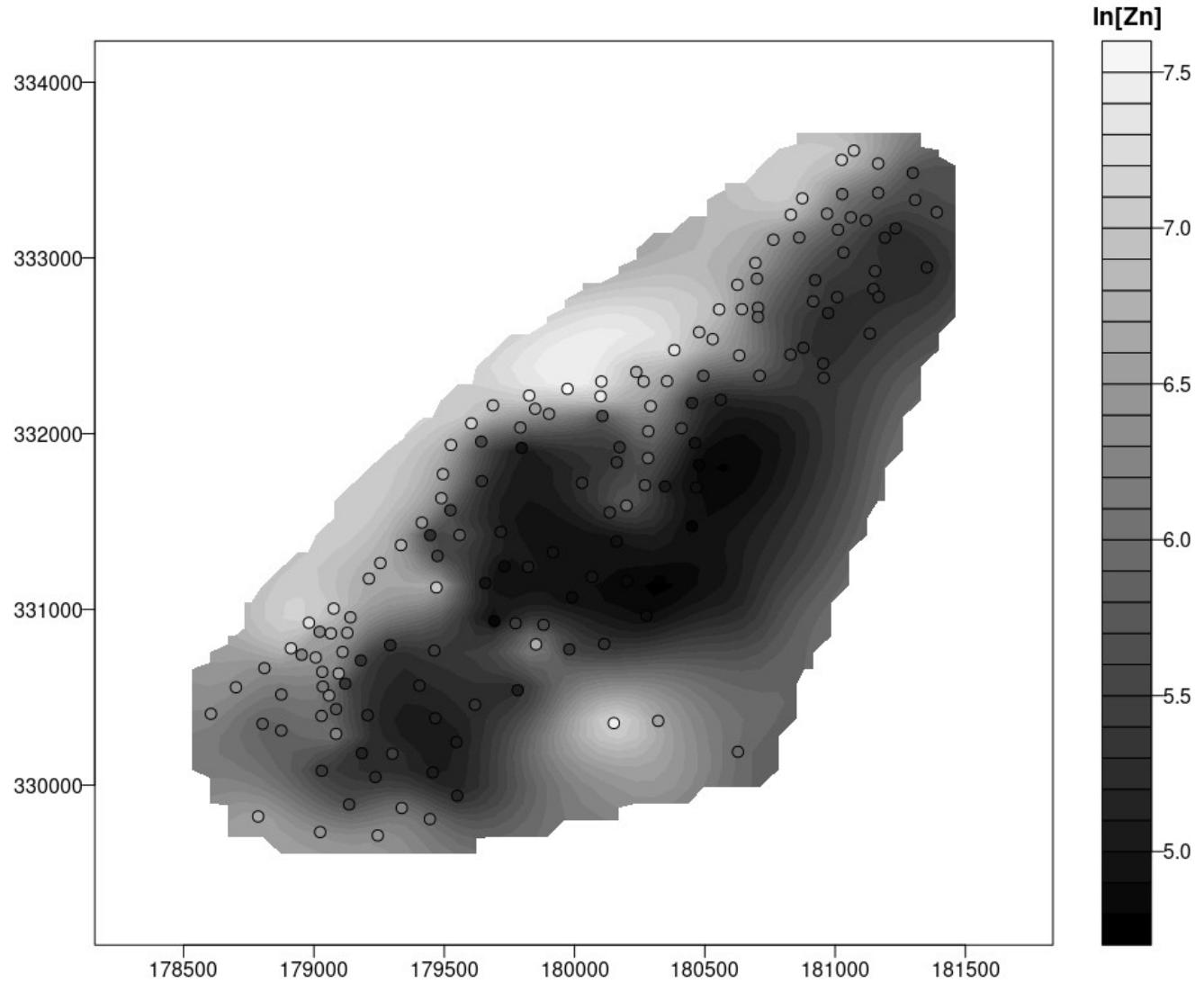
# Chapter 15

## Directional data



# Chapter 16

## Spatial data



## **17 An introduction to R**

17.1 The basics . . . . .
17.2 Plotting data . . . . .
17.3 Summary Statistics . . . . .
17.4 Probability . . . . .
17.5 The binomial distribution . . . . .
17.6 The Poisson distribution . . . . .
17.7 The normal distribution . . . . .
17.8 Error propagation . . . . .
17.9 Comparing distributions . . . . .
17.10 Regression . . . . .
17.11 Fractals and chaos . . . . .
17.12 Unsupervised learning . . . . .
17.13 Supervised learning . . . . .
17.14 Compositional data . . . . .
17.15 Directional data . . . . .
17.16 Spatial data . . . . .

## **18 Exercises**

18.1 The basics . . . . .
18.2 Plotting data . . . . .
18.3 Summary statistics . . . . .
18.4 Probability . . . . .
18.5 The binomial distribution . . . . .
18.6 The Poisson distribution . . . . .
18.7 The normal distribution . . . . .
18.8 Error propagation . . . . .
18.9 Comparing distributions . . . . .
18.10 Regression . . . . .
18.11 Fractals and chaos . . . . .
18.12 Unsupervised learning . . . . .
18.13 Supervised learning . . . . .
18.14 Compositional data . . . . .
18.15 Directional data . . . . .
18.16 Spatial data . . . . .

## **19 Solutions**

19.1 The basics . . . . .
19.2 Plotting data . . . . .
19.3 Summary statistics . . . . .
19.4 Probability . . . . .
19.5 The binomial distribution . . . . .
19.6 The Poisson distribution . . . . .
19.7 The normal distribution . . . . .
19.8 Error propagation . . . . .
19.9 Comparing distributions . . . . .
19.10 Regression . . . . .
19.11 Fractals and chaos . . . . .
19.12 Unsupervised learning . . . . .
19.13 Supervised learning . . . . .
19.14 Compositional data . . . . .
19.15 Directional data . . . . .
19.16 Spatial data . . . . .

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latex Further corrections made whilst recording final practical. 1 hour ago  
package Further corrections made whilst recording final practical. 1 hour ago  
slides Further corrections made whilst recording final practical. 1 hour ago  
.Rbuildignore Added notes 2 months ago  
.gitignore Removed temporary files, added README file 2 months ago  
README.md Updated documentation, removed github.io page 2 months ago

README.md

# geostats

Lecture notes, example datasets and code for the introductory statistics module for geoscientists at University College

About

No description, website, or topics provided.

Readme

Releases

No releases published

Create a new release

Packages

No packages published

Publish your first package

Environments 1

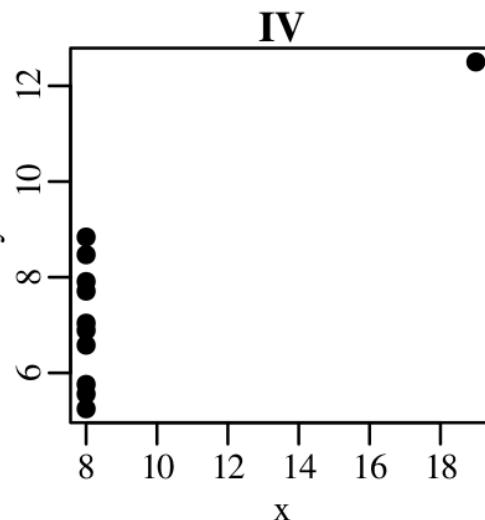
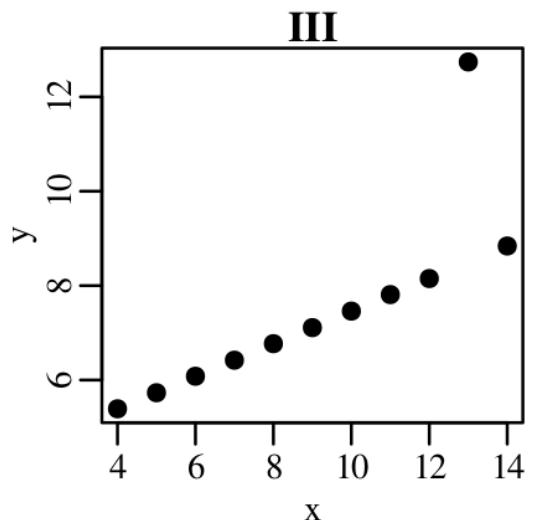
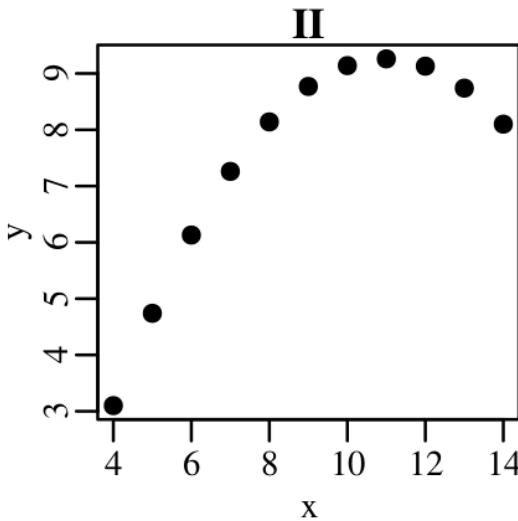
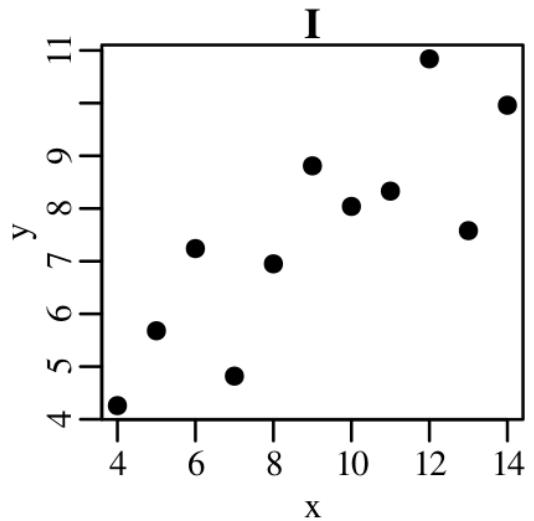
github-pages Active

Languages

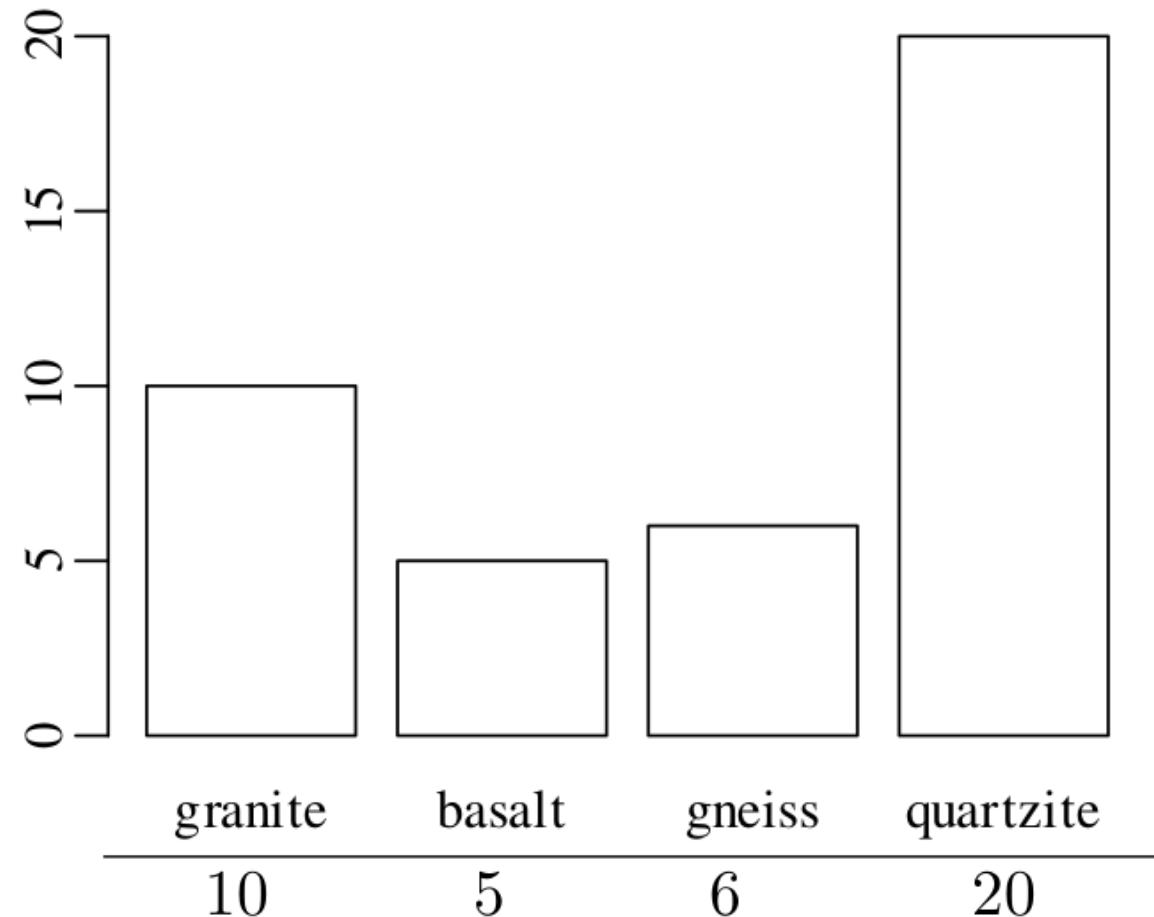
# Statistics for geoscientists

Plotting data

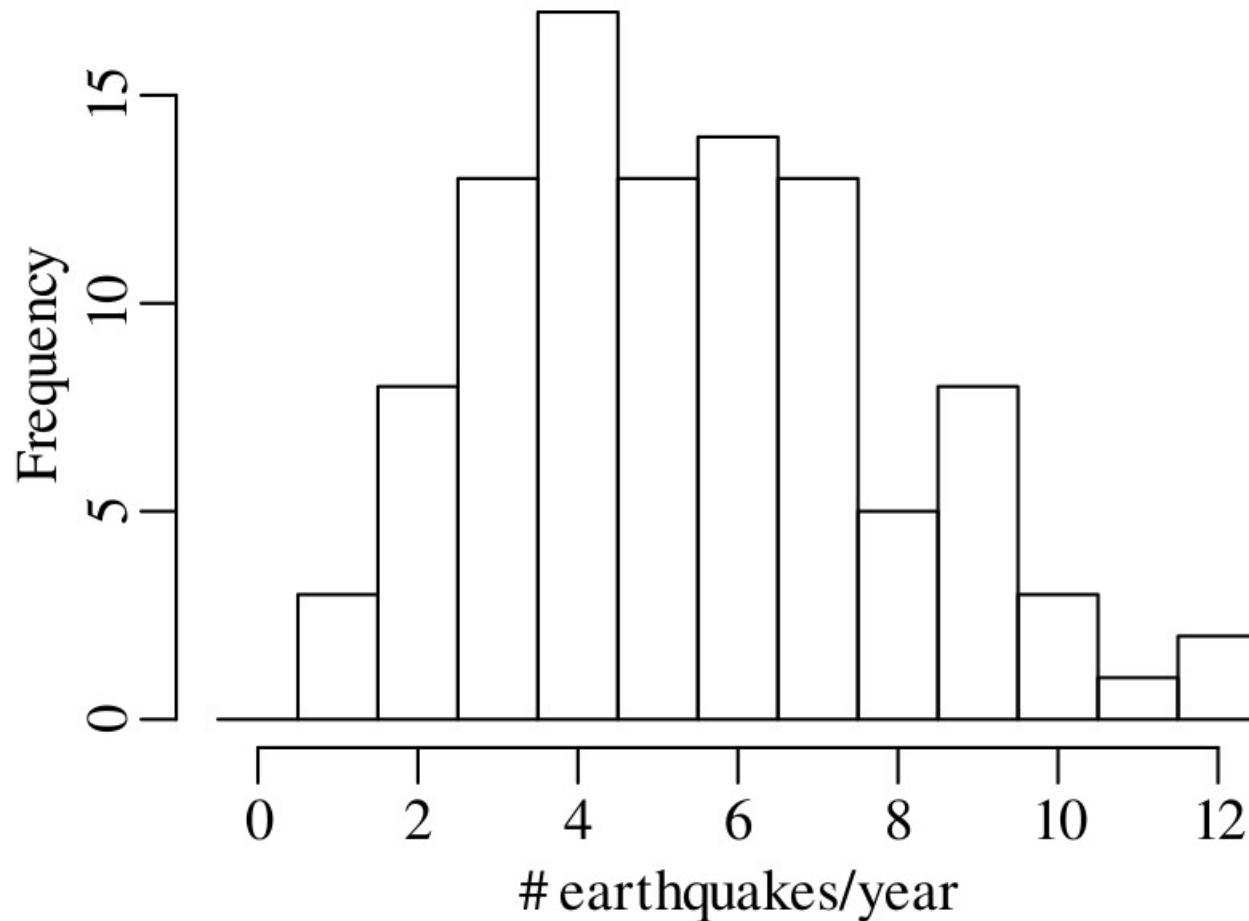
I	II		III		IV			
$x$	$y$	$x$	$y$	$x$	$y$	$x$	$y$	
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58	– the mean of $x$ is 9
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76	
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71	– the variance of $x$ is 11
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84	
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47	– the mean of $y$ is 7.50
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04	
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25	– the variance of $y$ is 4.125
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50	
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56	– the correlation coefficient of $x$ and $y$ is 0.816
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91	
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89	– the best fit line is given by $y = 3.00 + 0.500x$



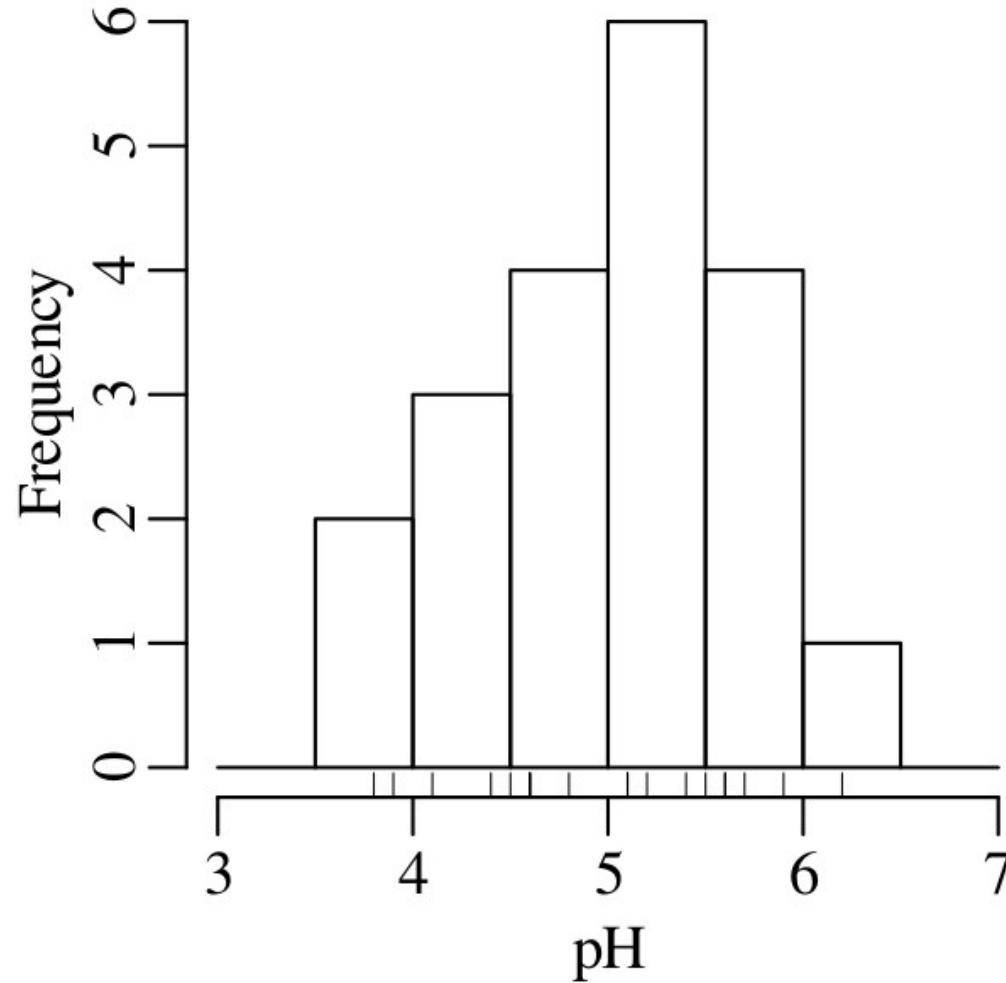
# Categorical data



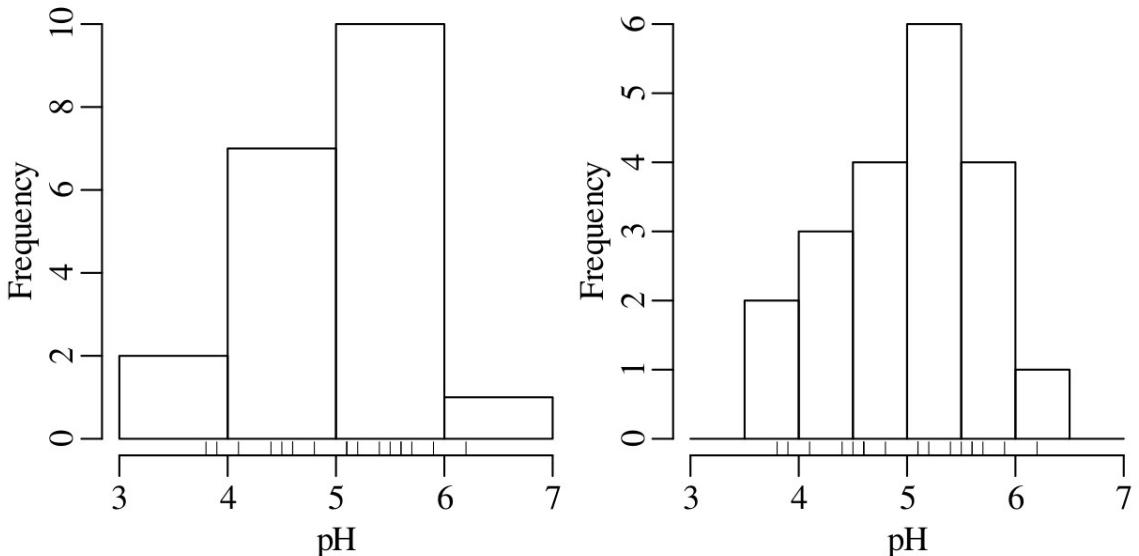
# Count data



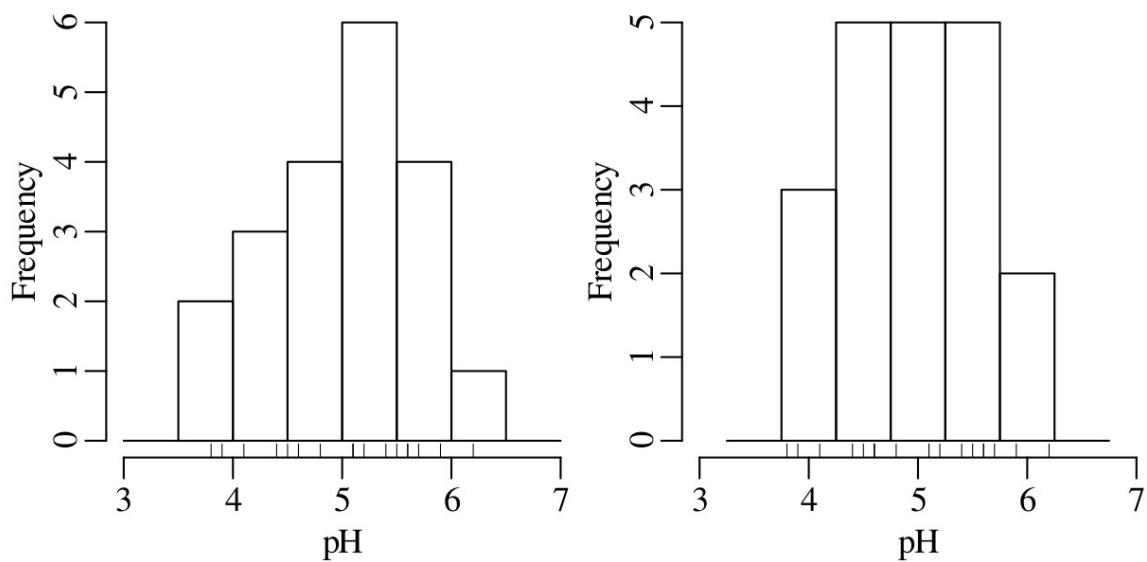
# Continuous data



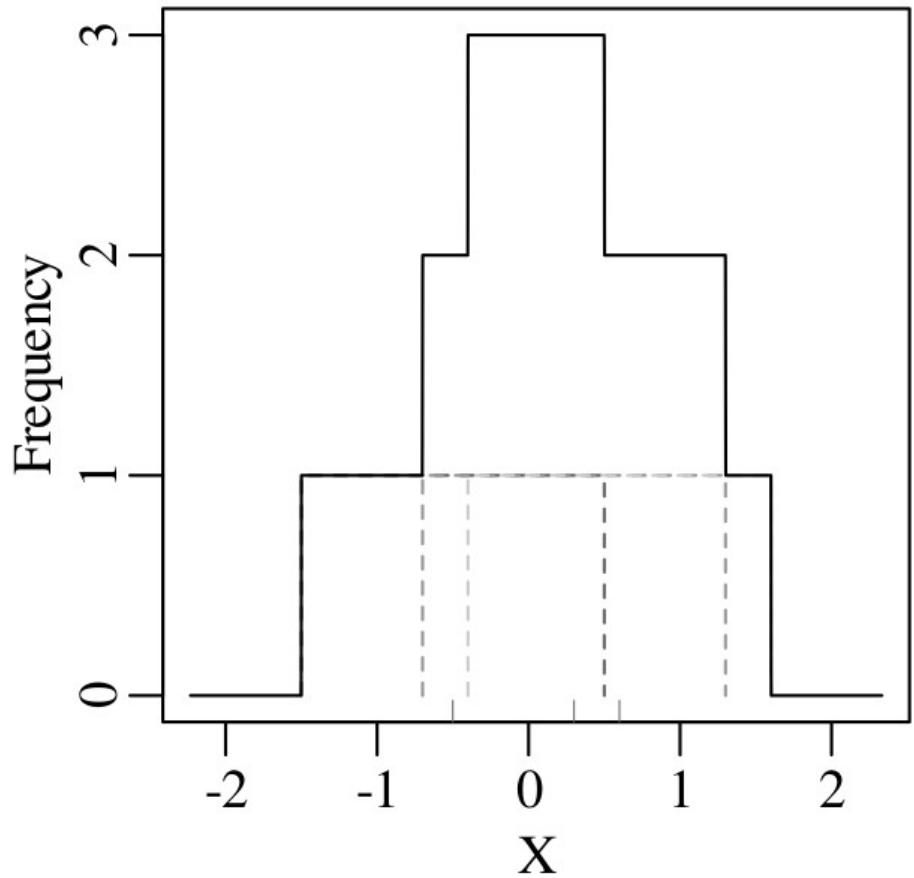
i. How many bins?



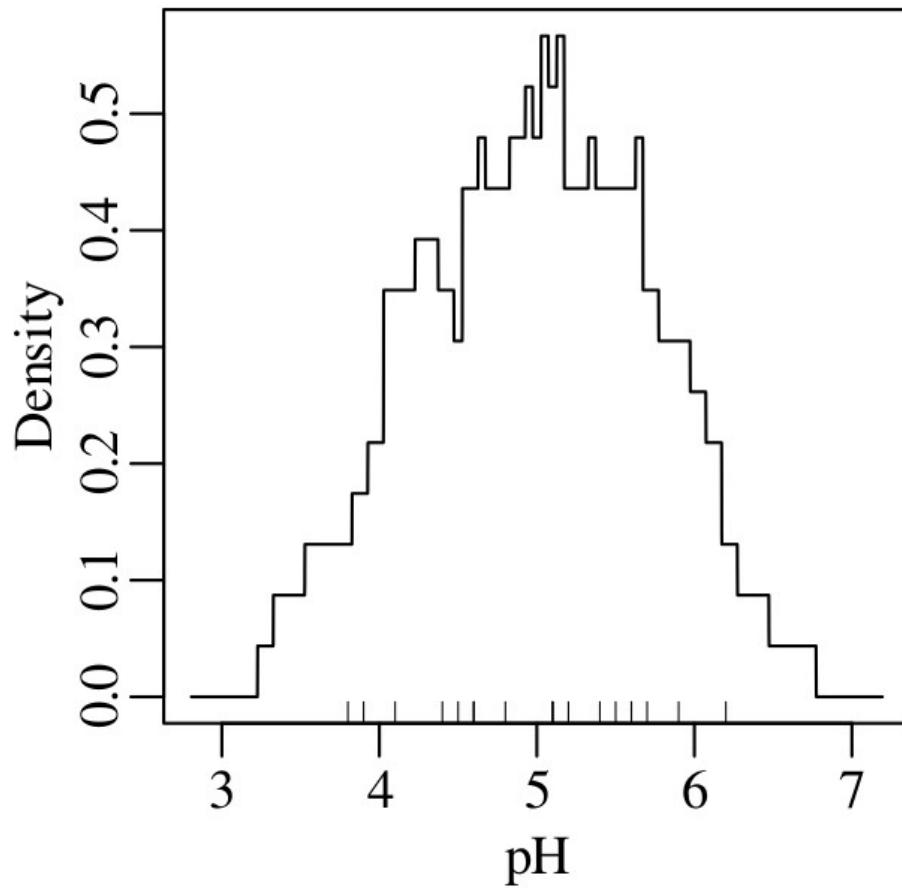
ii. Where to place the bins?



# Kernel Density Estimate

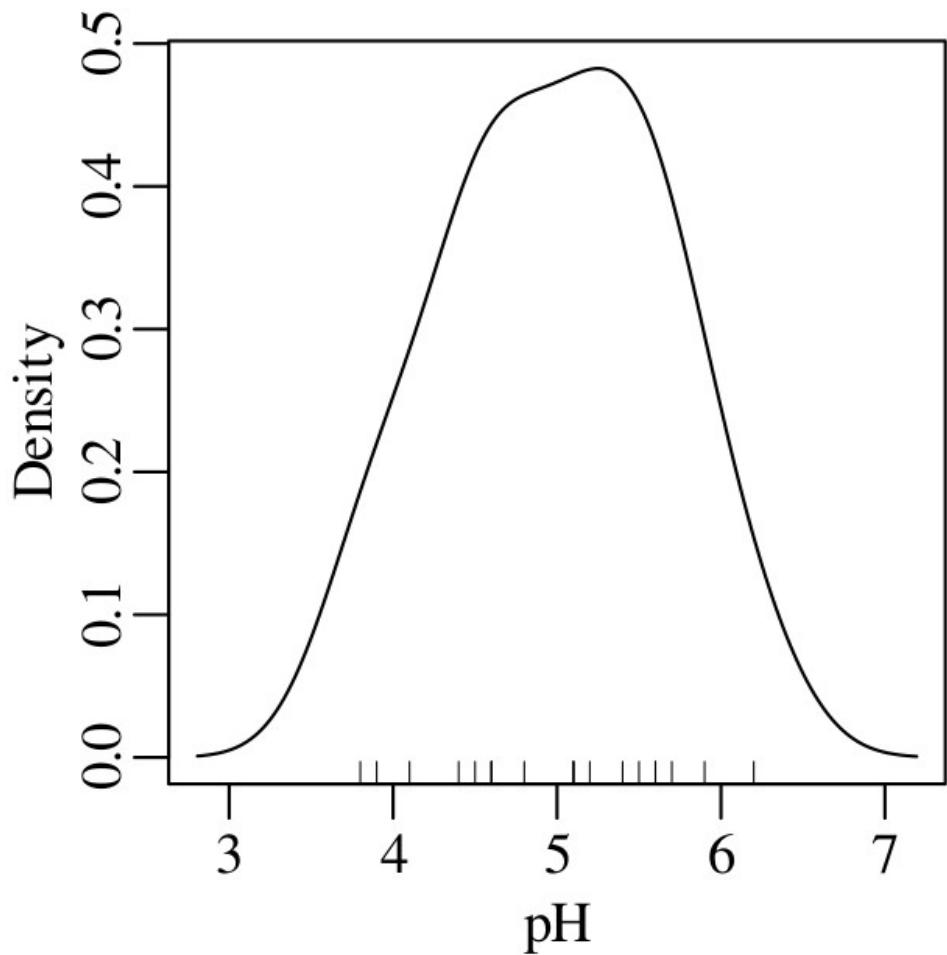
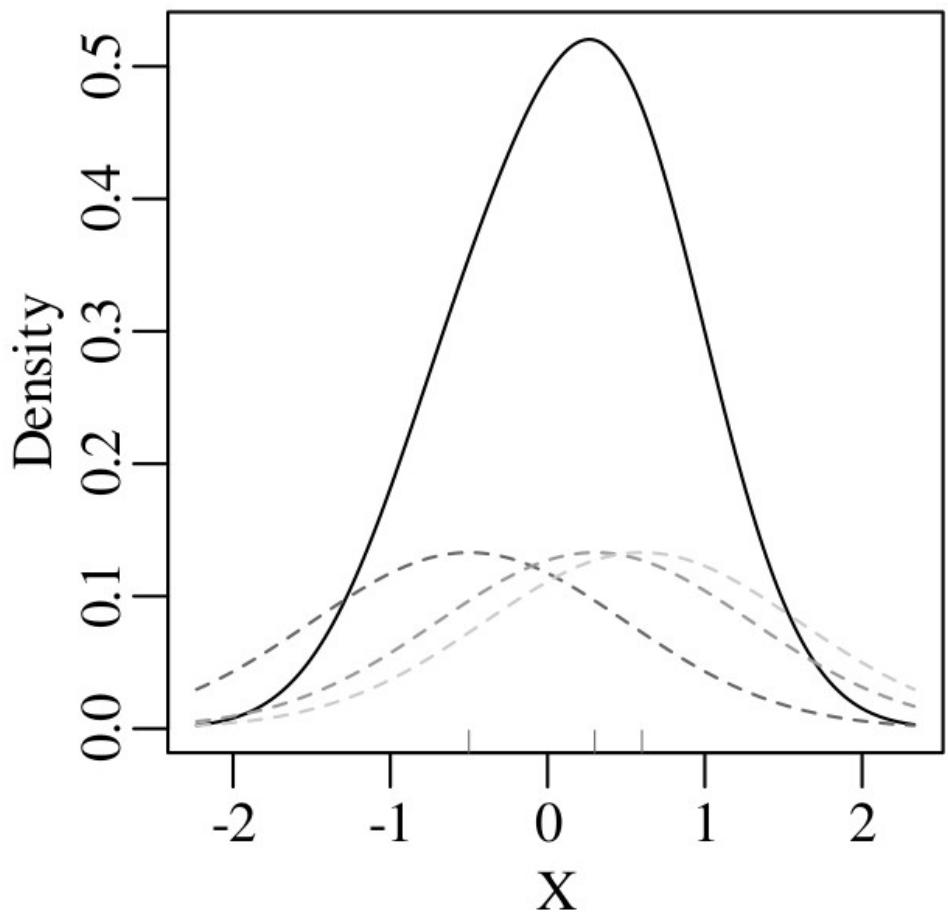


$$KDE(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - x_i}{h}\right)$$

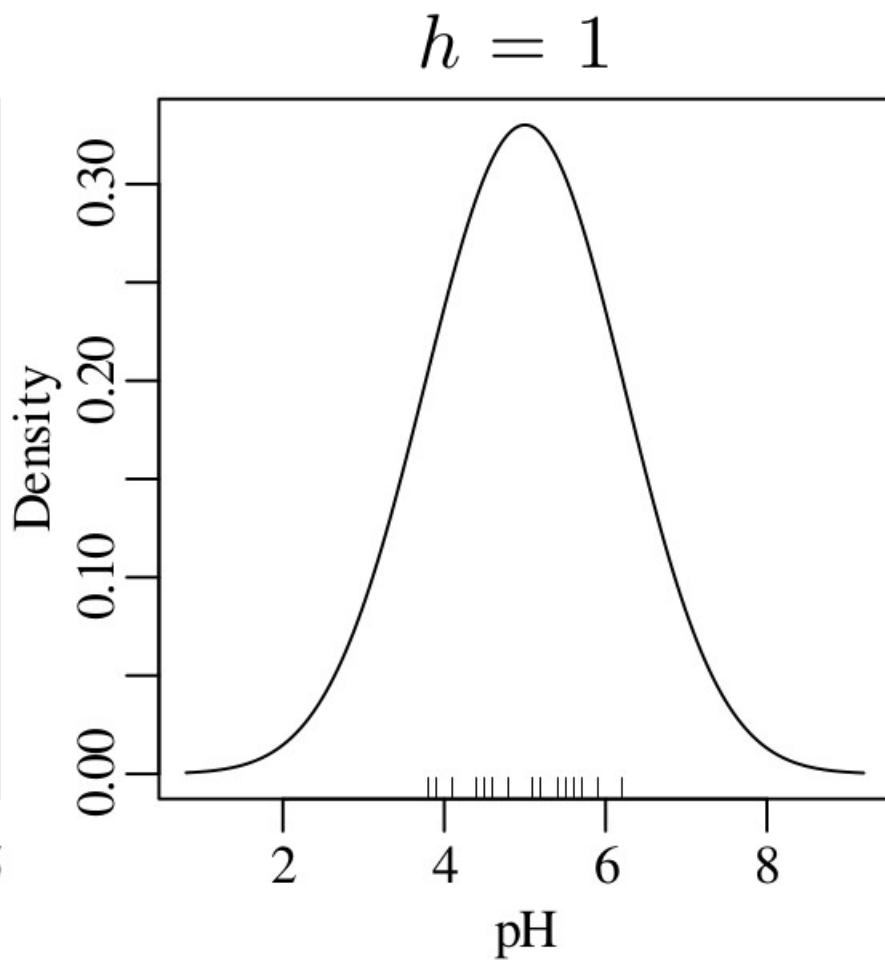
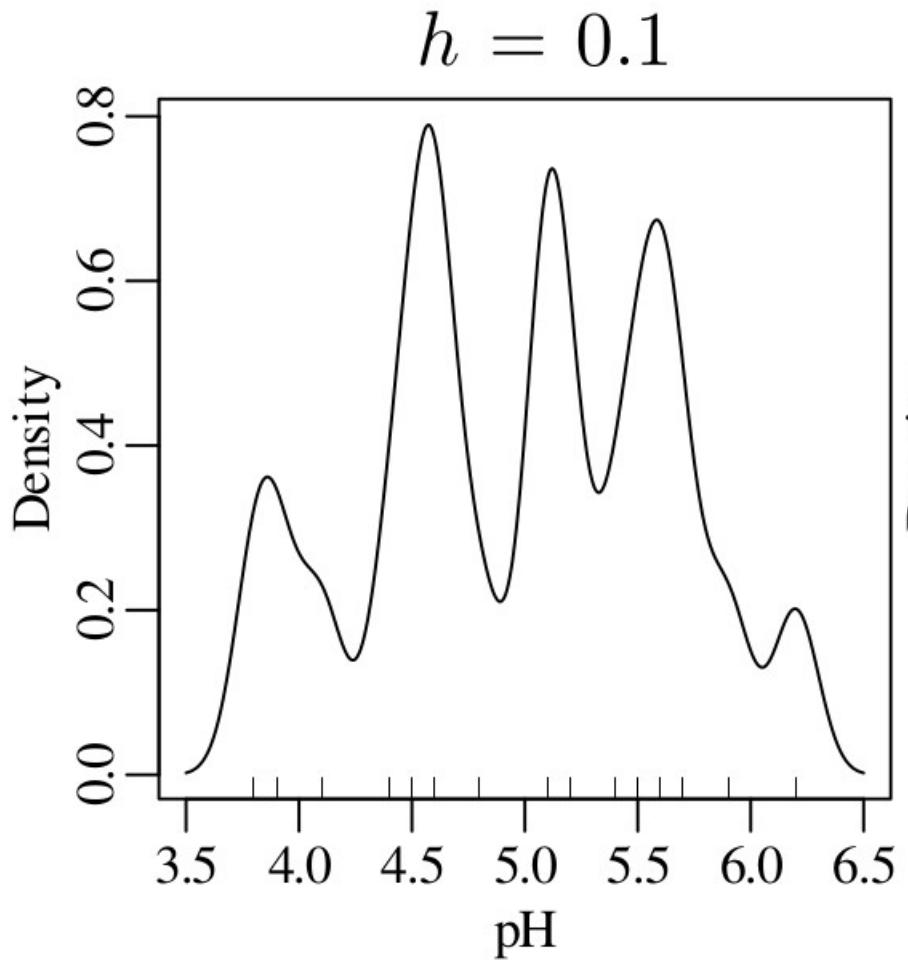


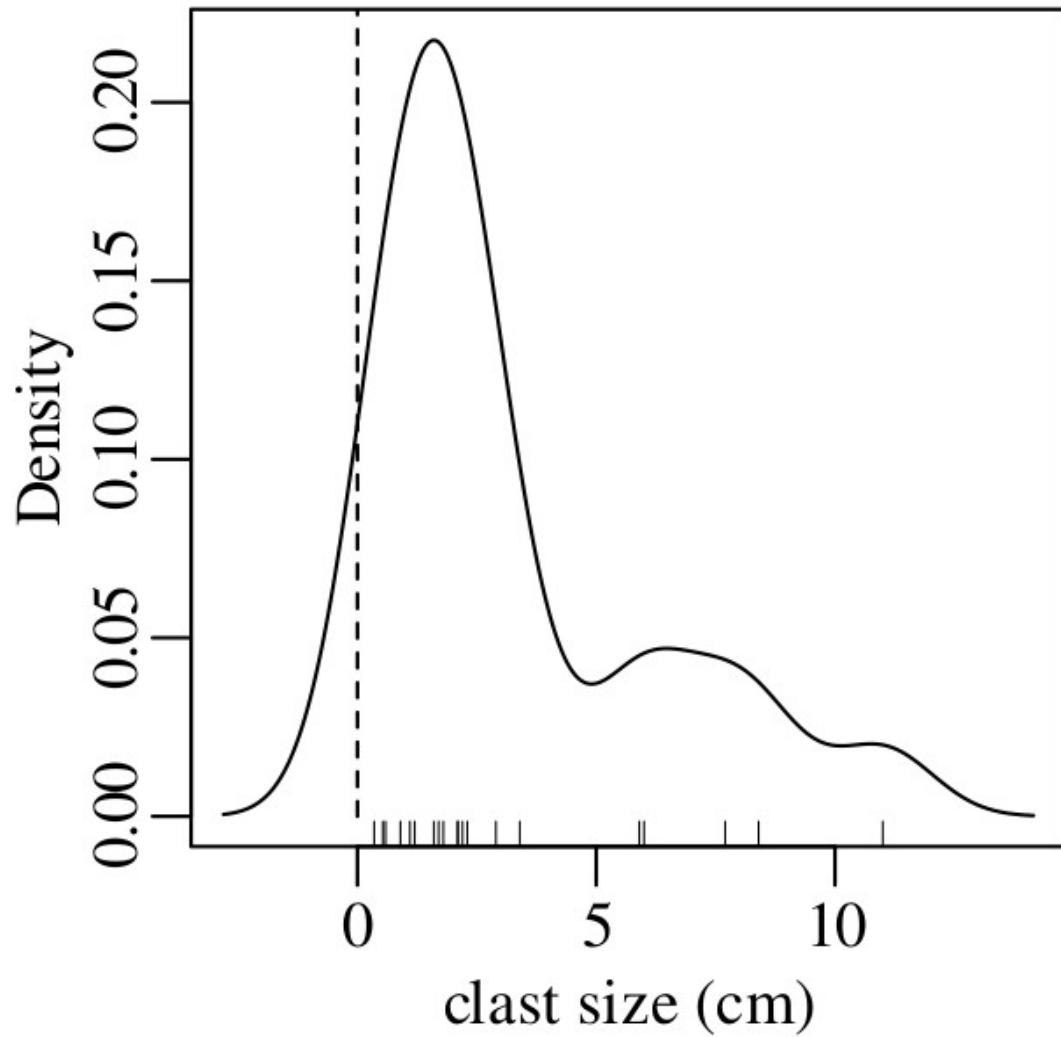
Gaussian kernel:

$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right]$$

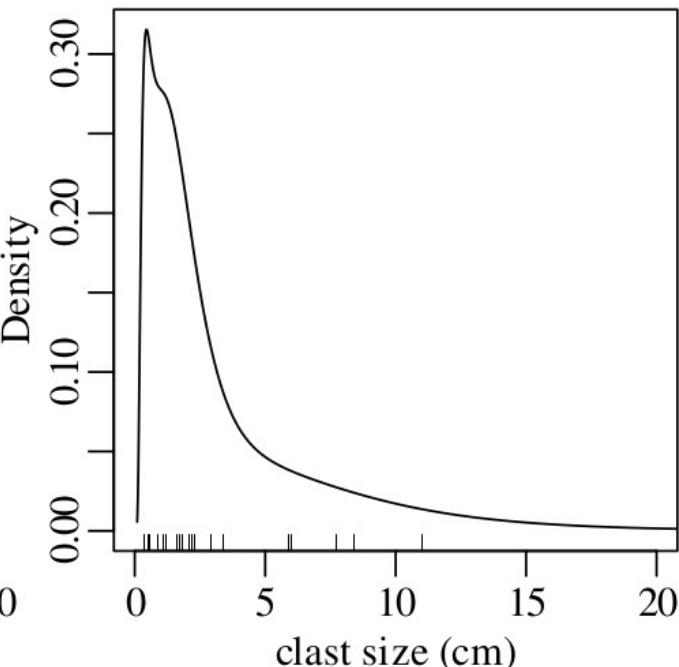
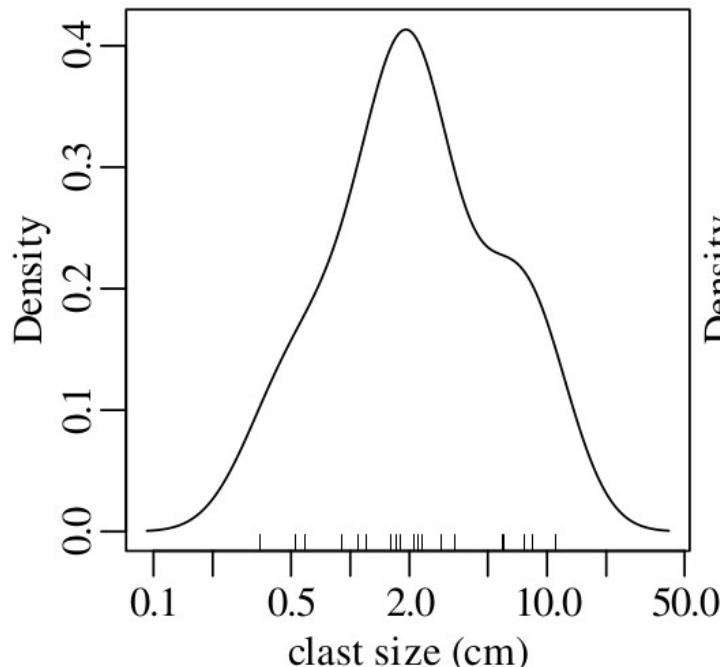
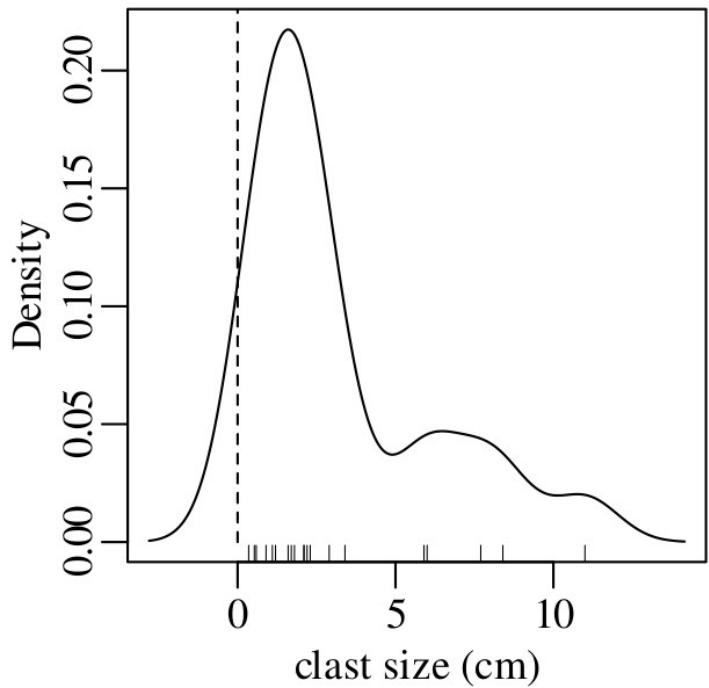


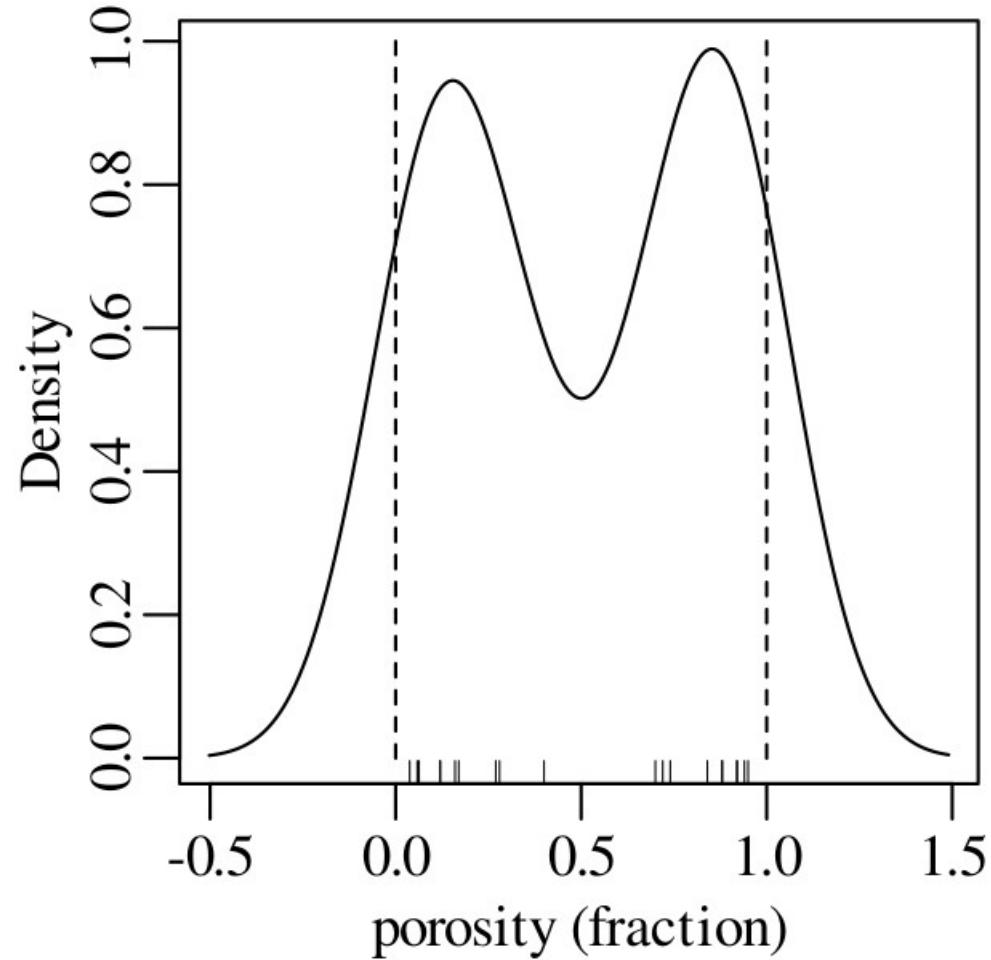
bandwidth





# logarithmic transformation

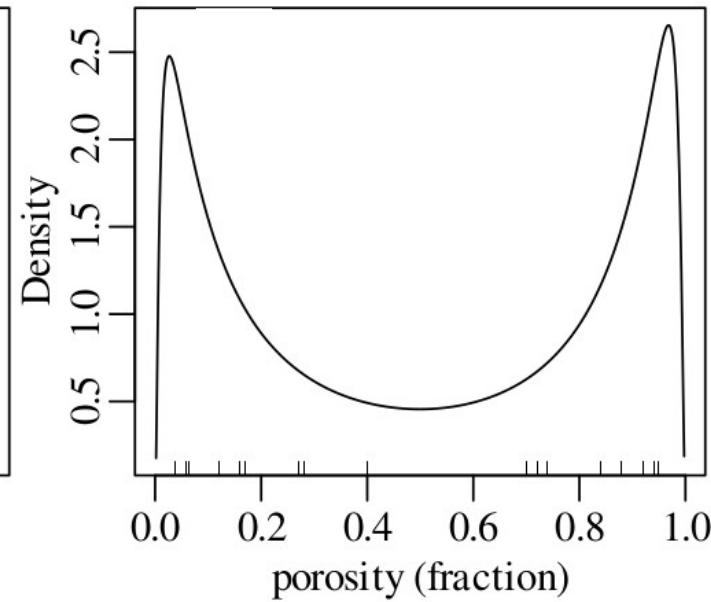
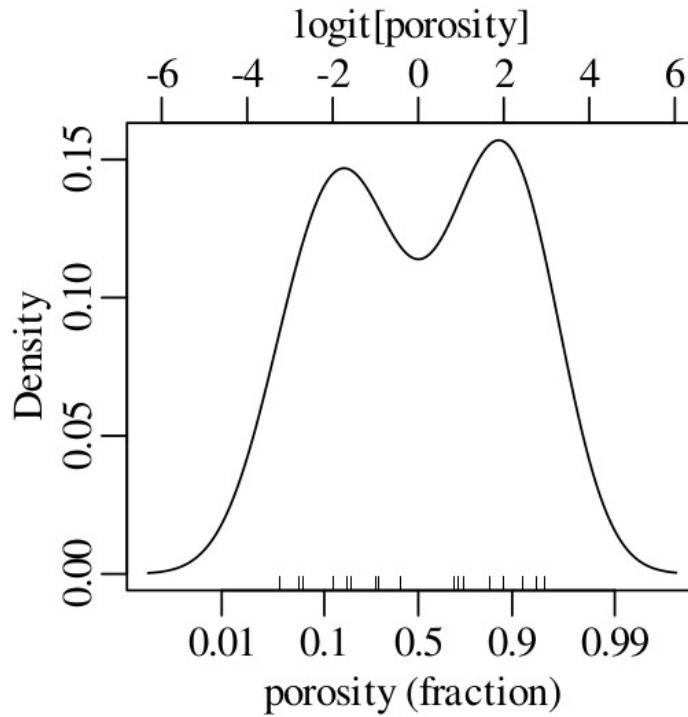
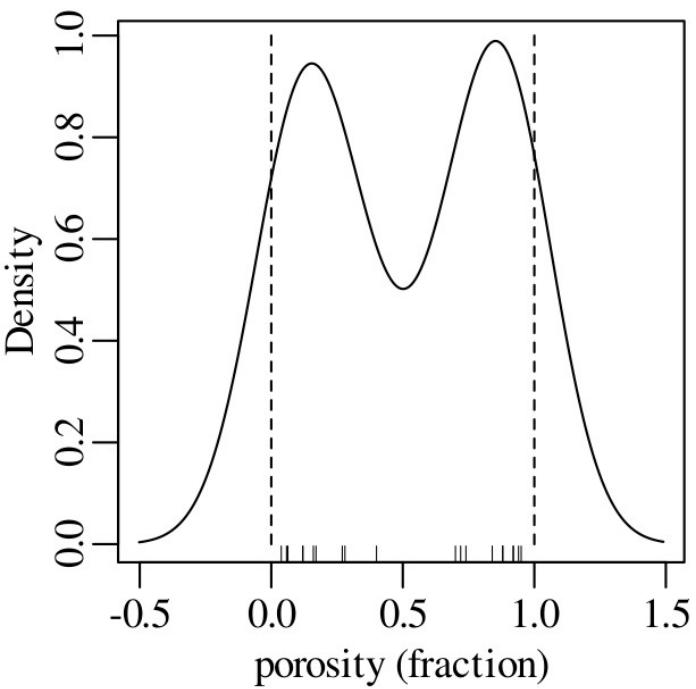




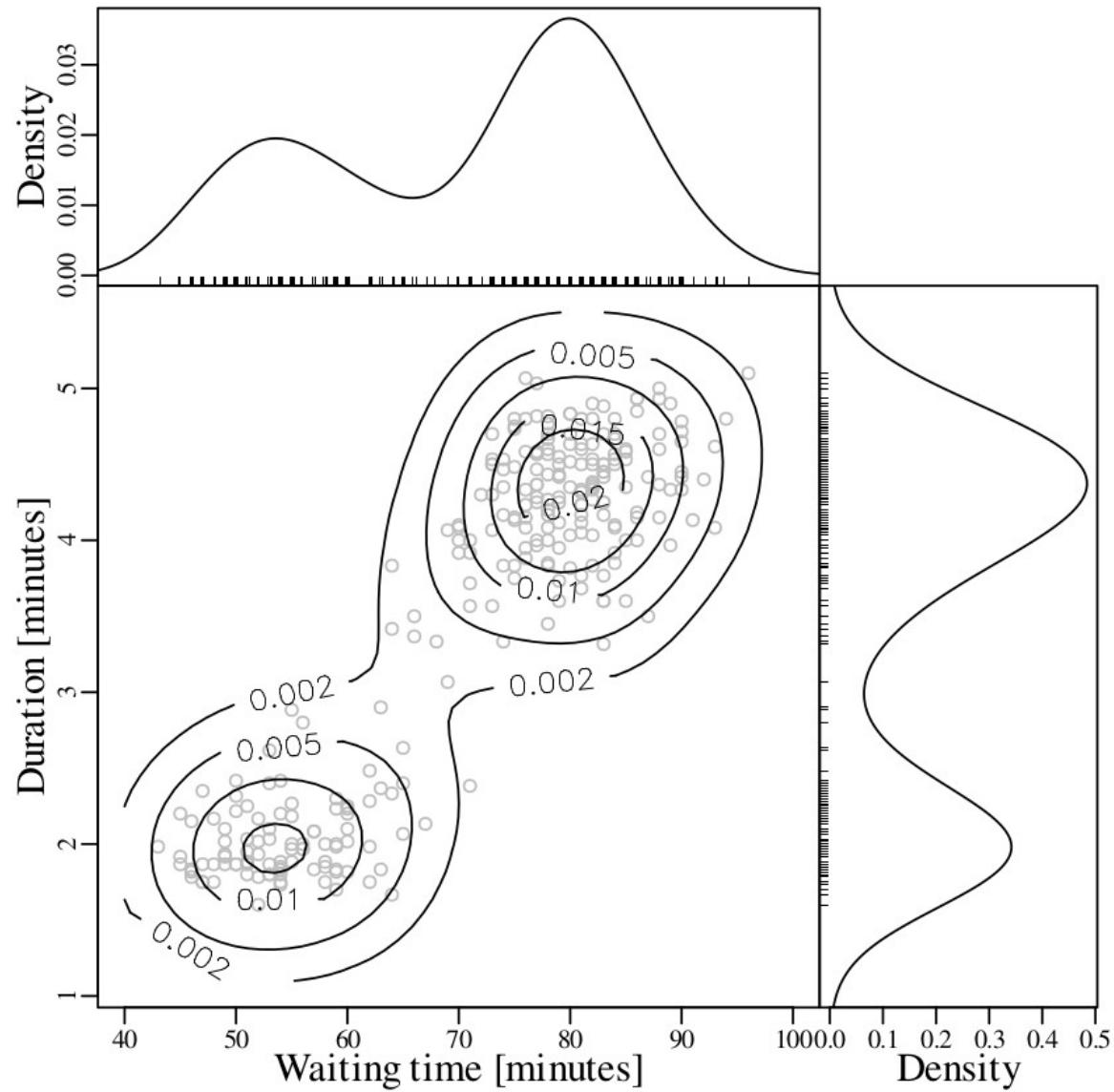
logistic transformation

$$u = \text{logit}(x) = \ln \left[ \frac{x}{1 - x} \right]$$

$$x = \text{logit}^{-1}(u) = \frac{\exp[u]}{\exp[u] + 1}$$

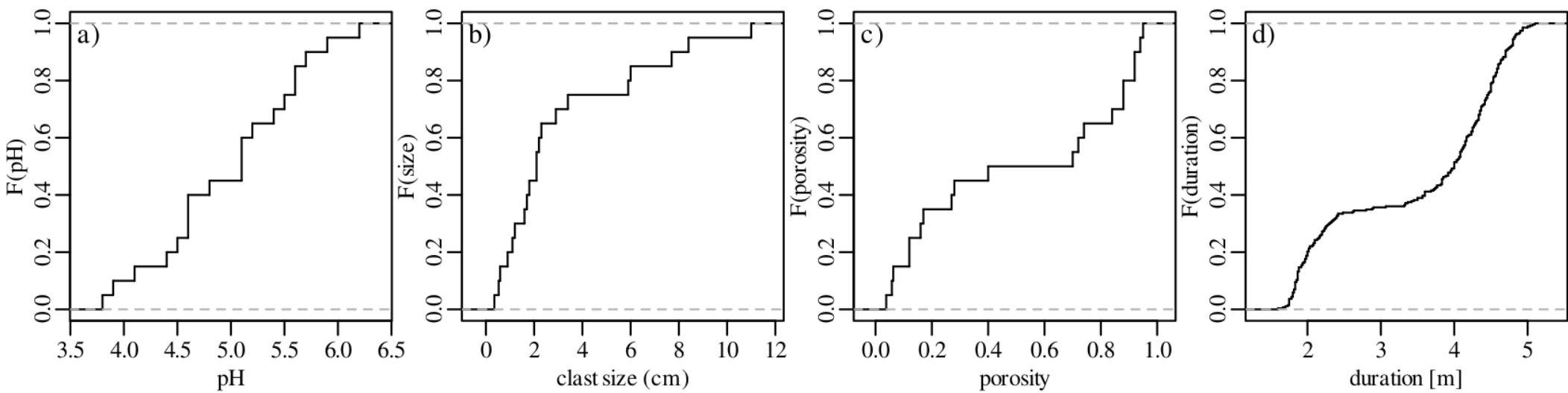


# Multivariate distributions



# Empirical cumulative distribution fuctions

$$F(x) = \sum_{i=1}^n 1(x_i < x)/n$$

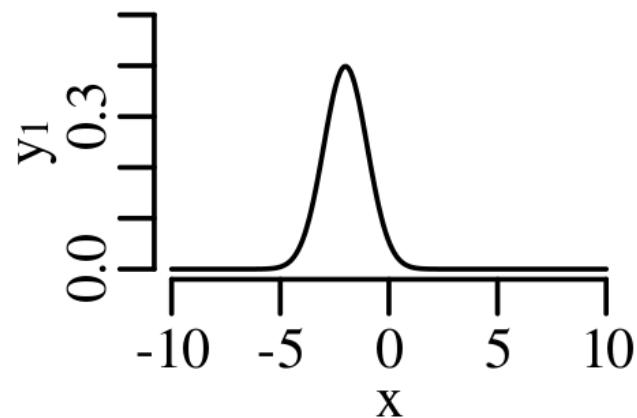


# Statistics for geoscientists

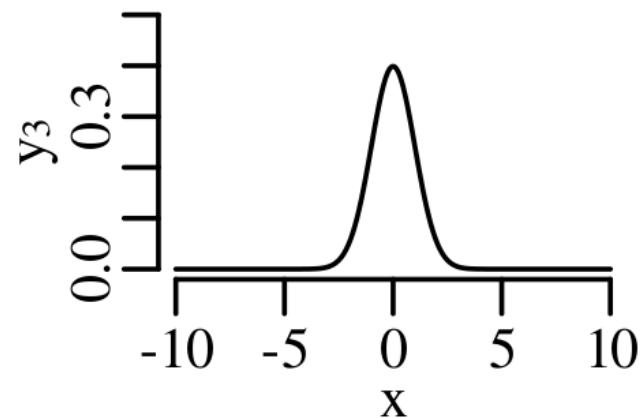
Summary statistics

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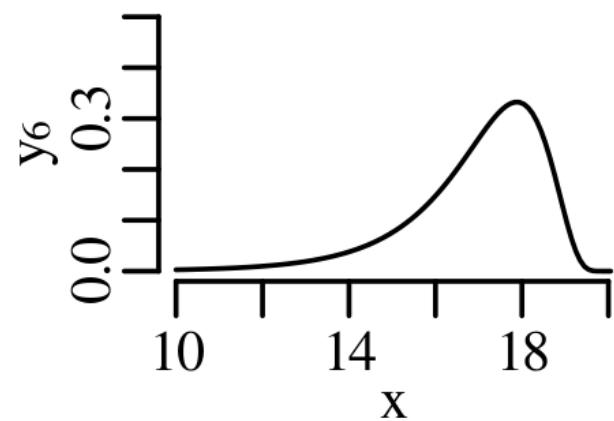
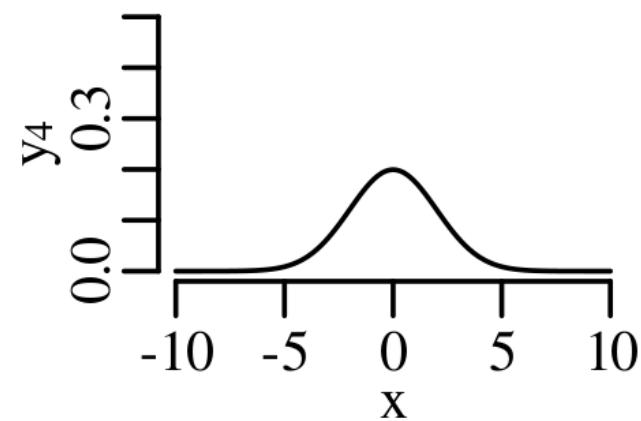
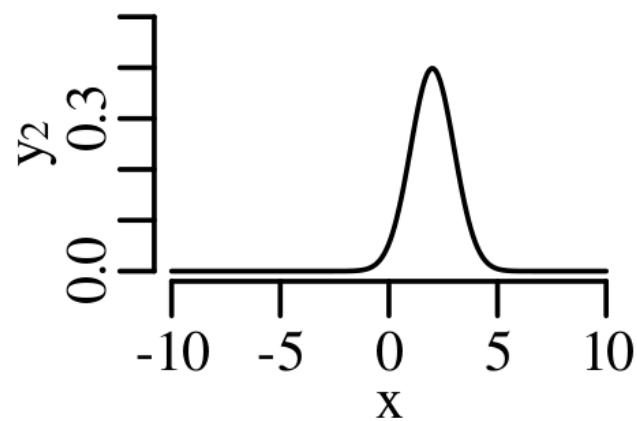
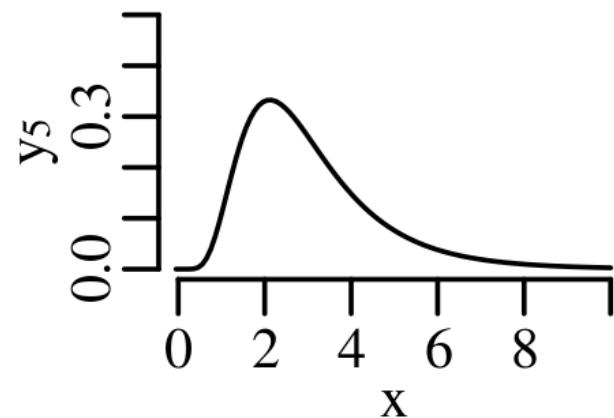
Location



Dispersion



Shape

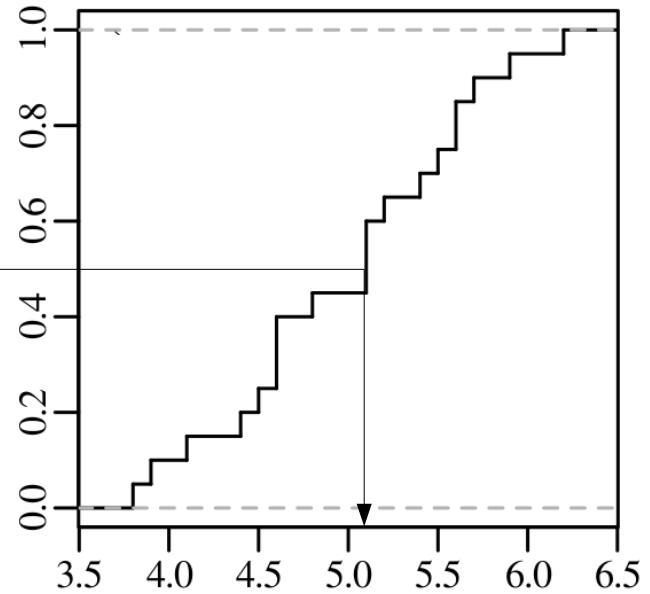
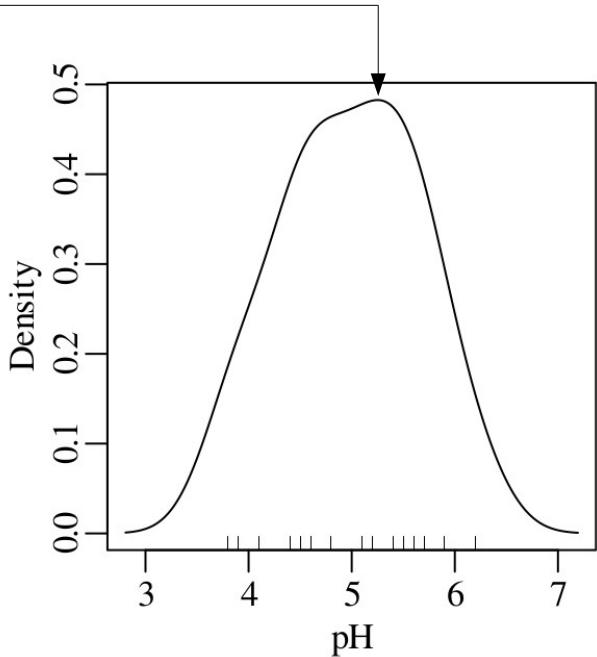


# Location

**Mean**  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$

**Median**  $\text{med}(x) = x_1 < \dots < x_{n/2} < \dots < x_n$

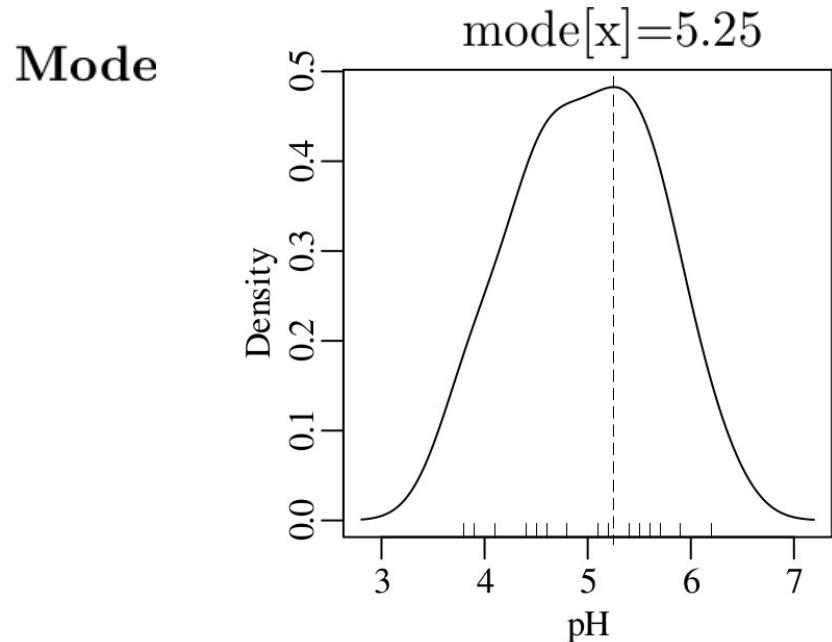
**Mode**

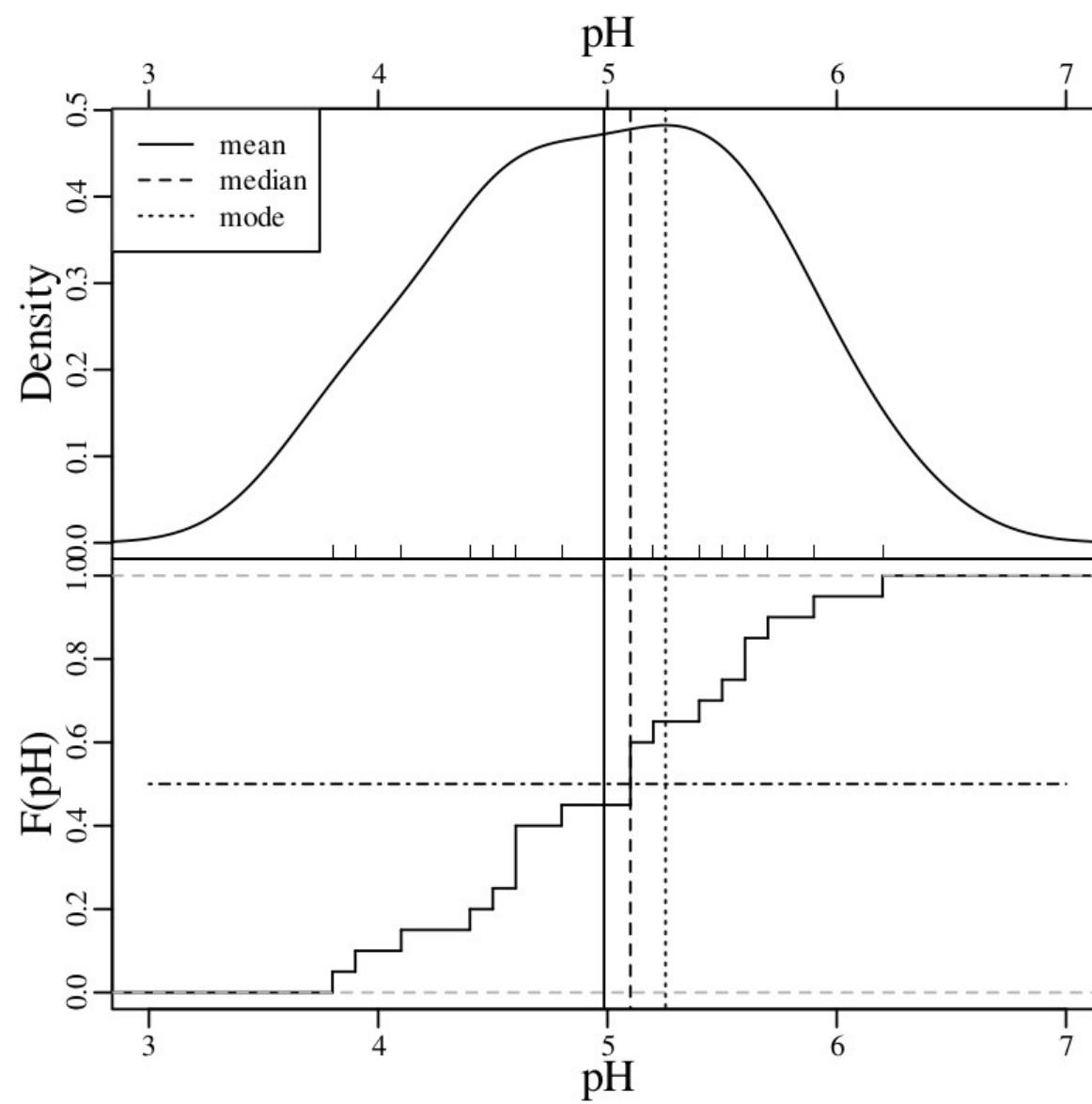


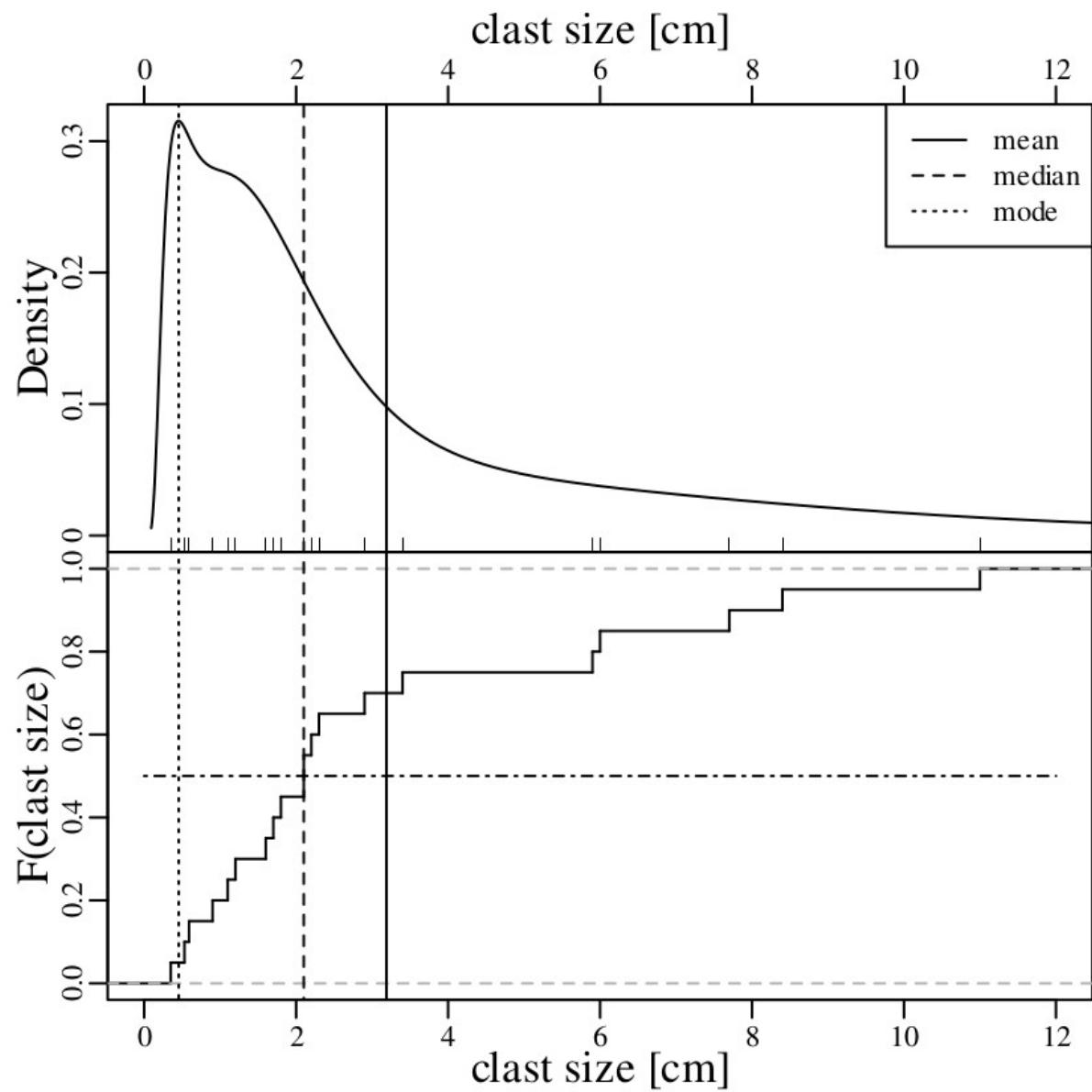
pH data    6.2, 4.4, 5.6, 5.2, 4.5, 5.4, 4.8, 5.9, 3.9, 3.8, 5.1, 4.1, 5.1, 5.5, 5.1, 4.6, 5.7, 4.6, 4.6, 5.6

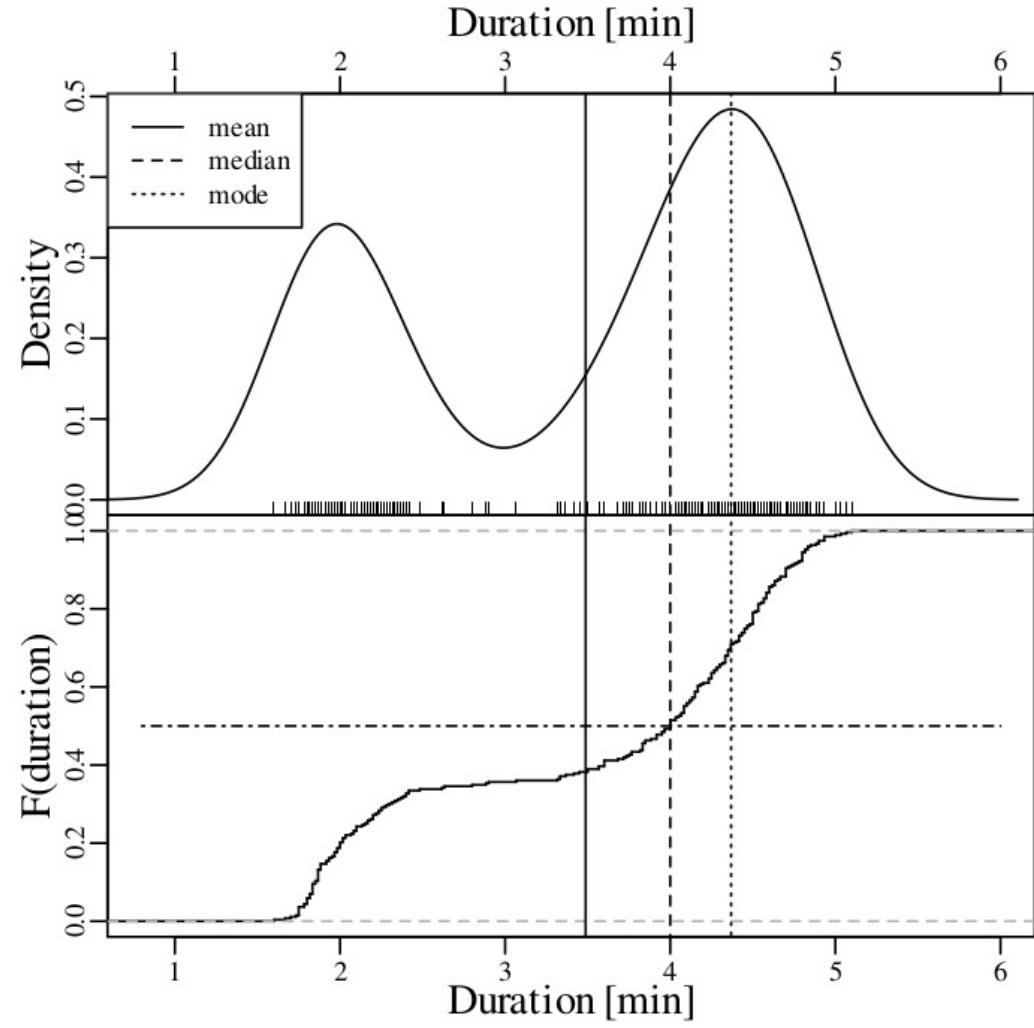
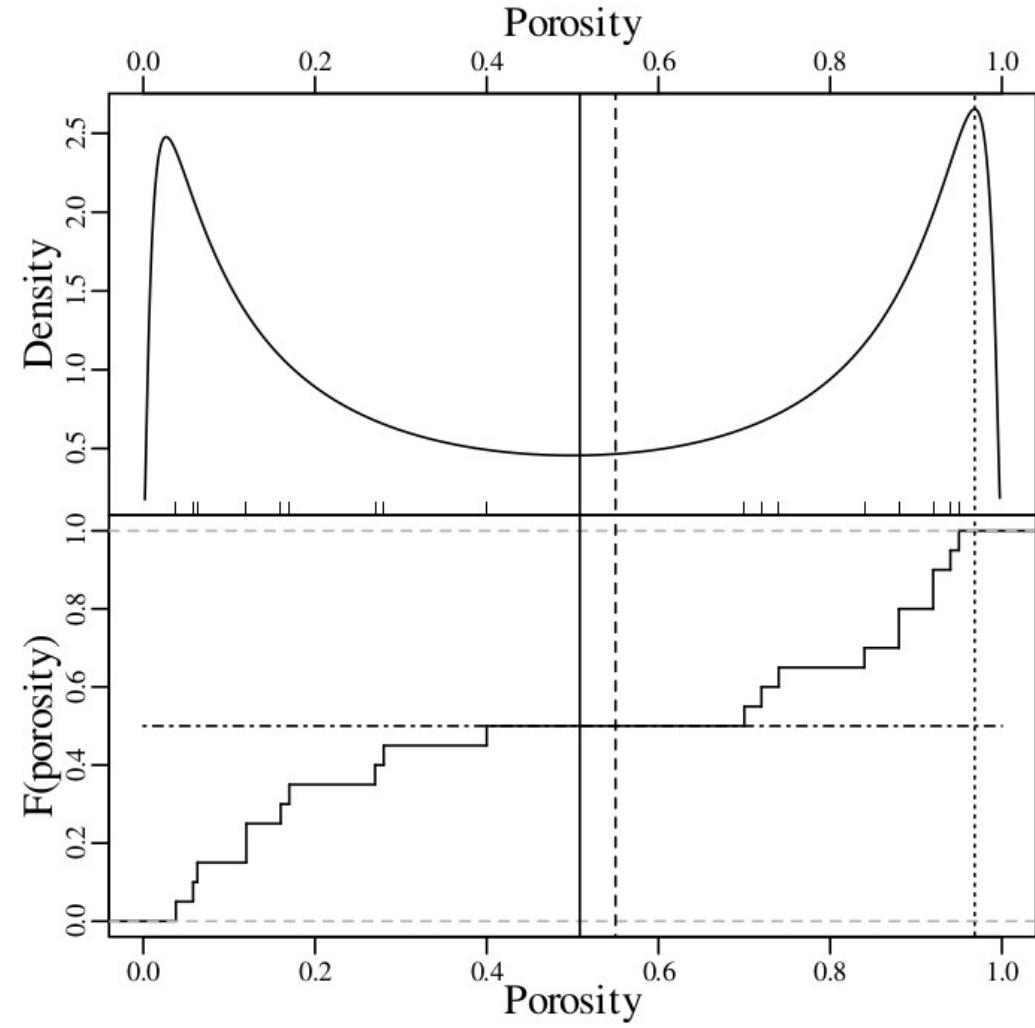
$$\text{Mean } \bar{x} = \frac{6.2 + 4.4 + 5.6 + 5.2 + 4.5 + 5.4 + 4.8 + 5.9 + 3.9 + 3.8 + 5.1 + 4.1 + 5.1 + 5.5 + 5.1 + 4.6 + 5.7 + 4.6 + 4.6 + 5.6}{20} = 5.00$$

Median    3.8, 3.9, 4.1, 4.4, 4.5, 4.6, 4.6, 4.6, 4.8, **5.1**, **5.1**, 5.1, 5.2, 5.4, 5.5, 5.6, 5.6, 5.7, 5.9, 6.2









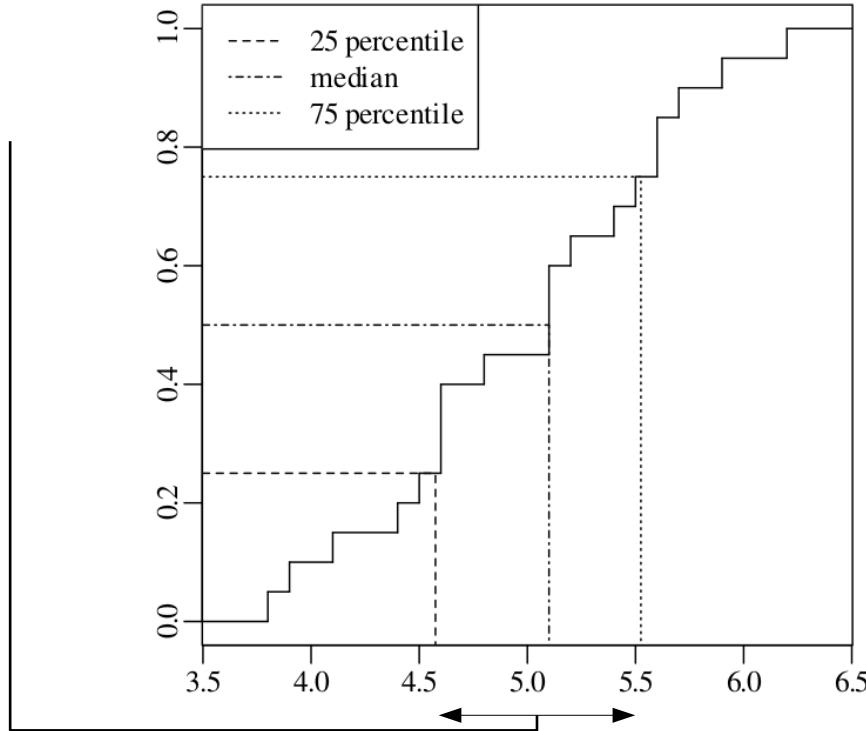
# Dispersion

## Standard deviation

$$s[x] = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

Median Absolute Deviation  $\text{MAD} = \text{median}|x_i - \text{median}(x)|$

## Interquartile range



## Standard deviation

$$s[x] = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

$i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$x_i$	6.2	4.4	5.6	5.2	4.5	5.4	4.8	5.9	3.9	3.8	5.1	4.1	5.1	5.5	5.1	4.6	5.7	4.6	4.6	5.6
$(x_i - \bar{x})$	1.20	-.58	.61	.21	-.49	.42	-.19	.92	-1.1	-1.2	.11	-.89	.11	.51	.11	-.39	.71	-.39	-.39	.61
$(x_i - \bar{x})^2$	1.5	.34	.38	.046	.24	.17	.034	.84	1.2	1.4	.013	.78	.013	.27	.013	.15	.51	.15	.15	.38
$\sum_{i=1}^{20} (x_i - \bar{x})^2 = 8.52$	$s[x] = \sqrt{8.52/19} = 0.70$																			

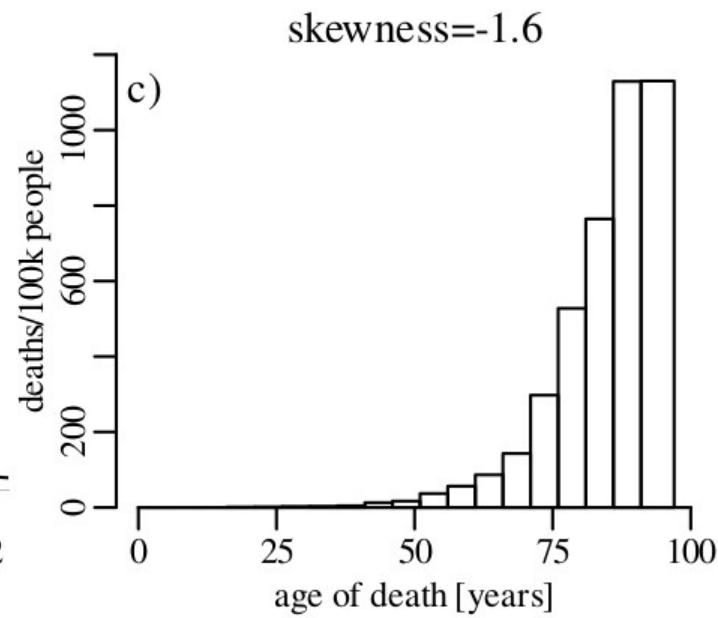
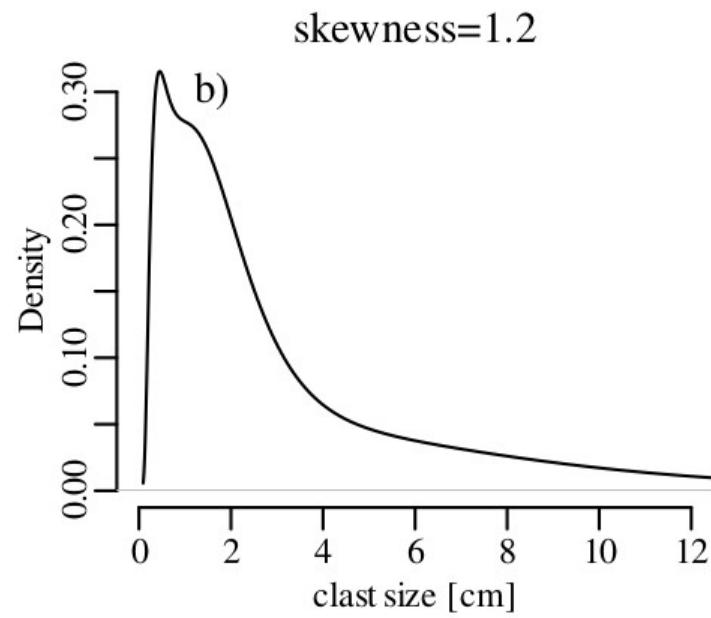
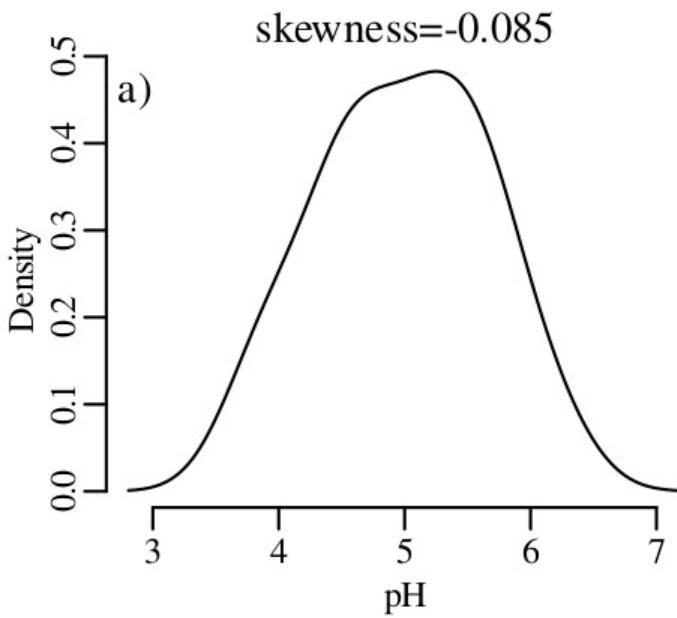
Interquartile range IQR =  $5.55 - 4.55 = 1.00$

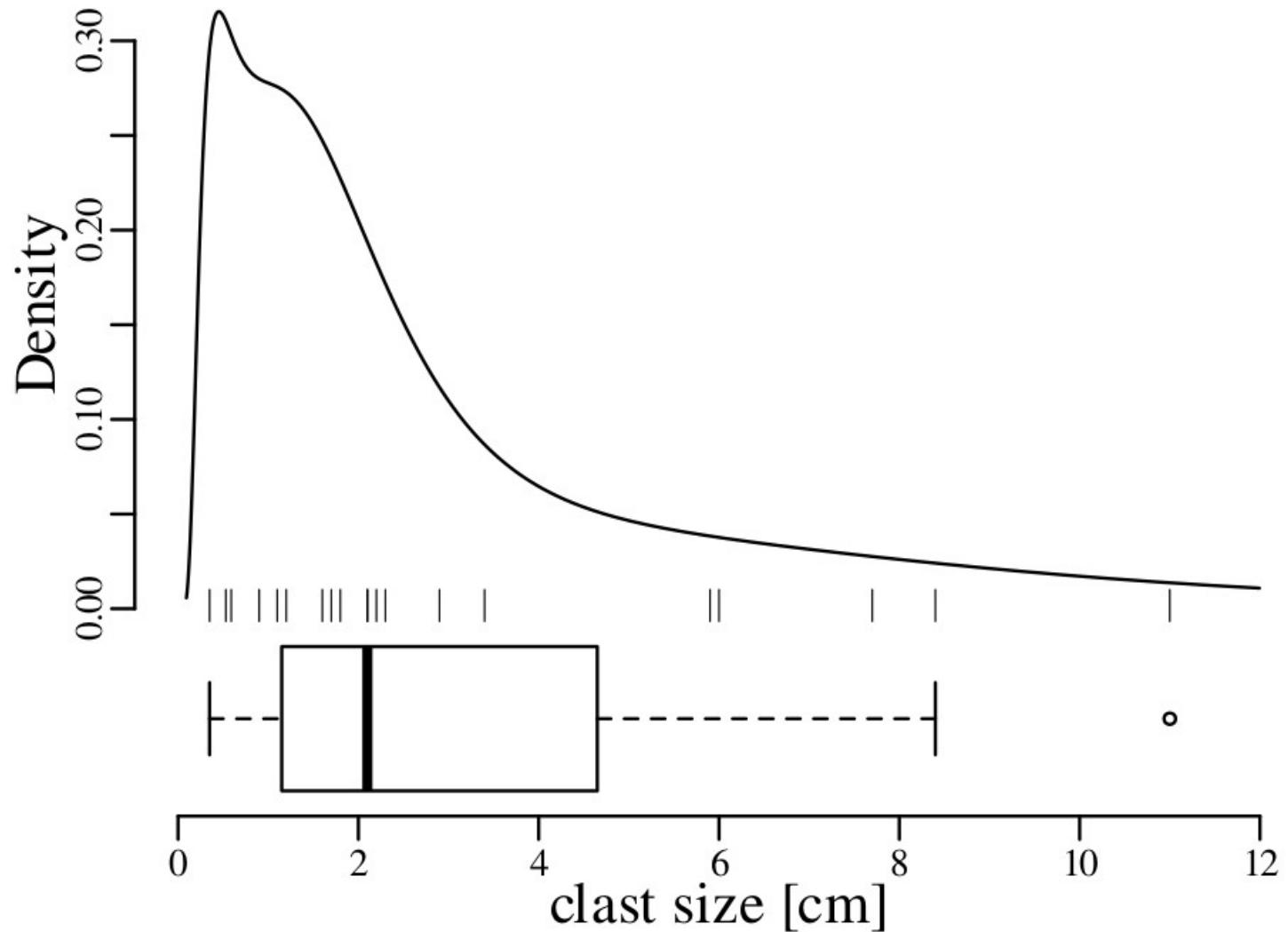
$i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$x_i$	3.8	3.9	4.1	4.4	4.5	4.6	4.6	4.6	4.8	5.1	5.1	5.1	5.2	5.4	5.5	5.6	5.6	5.7	5.9	6.2
$x_i - \text{med}(x)$	-1.3	-1.2	-1.0	-0.7	-0.6	-0.5	-0.5	-0.5	-0.3	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.6	0.8	1.1
$ x_i - \text{med}(x) $	1.3	1.2	1.0	0.7	0.6	0.5	0.5	0.5	0.3	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.6	0.8	1.1
sorted	0.0	0.0	0.0	0.1	0.3	0.3	0.4	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.8	1.0	1.1	1.2	1.3	

Median Absolute Deviation MAD =  $\text{median}|x_i - \text{median}(x)|$

**skewness**

$$\text{skew}(x) = \frac{1}{n \cdot s[x]^3} \sum_{i=1}^n (x_i - \bar{x})^3$$





# Statistics for geoscientists

## Probability

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

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$$\frac{\{H\}}{\{H\}\{T\}} = \frac{1}{2} = 0.5$$

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$P(2 \times H \cap 1 \times T) = \frac{\{THH\}\{HTH\}\{HHT\}}{\{HHH\}\{THH\}\{HTH\}\{HHT\}\{TTH\}\{THT\}\{HTT\}\{TTT\}} = \frac{3}{8}$$

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$P(1 \times \begin{array}{|c|}\hline \bullet \\ \hline \end{array} \cap 1 \times \begin{array}{|c|c|}\hline \bullet & \\ \hline & \bullet \\ \hline \end{array}) = \frac{\left\{ \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \\ \hline \end{array}, \begin{array}{|c|c|}\hline \bullet & \\ \hline \bullet & \bullet \\ \hline \end{array} \right\} \left\{ \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}, \begin{array}{|c|c|}\hline \bullet & \\ \hline \bullet & \\ \hline \end{array} \right\}}{\text{Total Number of Outcomes}} = \frac{2}{36} = \frac{1}{18}$$

**multiplicative rule**

$$P(\{2 \times H \cap 1 \times T\} \cap \{1 \times \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \cap 1 \times \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}\}) = \frac{3}{8} \frac{1}{18} = \frac{3}{144} = 0.021$$

**additive rule**

$$P(\{2 \times H \cap 1 \times T\} \cup \{3 \times H\}) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = 0.5$$

If the two outcomes are *not* mutually exclusive:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(\{2 \times H \cap 1 \times T\} \cup \{1 \times \begin{array}{|c|}\hline \bullet \\ \hline \end{array} \cap 1 \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \end{array}\}) = \frac{3}{8} + \frac{1}{18} - \frac{3}{144} = \frac{59}{144} = 0.410$$

$$P(\{2 \times H\} \cup \{3 \times H\}) = \frac{4}{8} + \frac{1}{8} - \frac{1}{8} = \frac{4}{8} = 0.5$$



$$P(\{2 \times H\} \cap \{1 \times T\}) + P(3 \times H)$$

# Permutations

1    2    3    4    5    6    7    8    9    10    11    12    13    14    15    16    17    18    19    20

1    2    3    4    5    6    7    8    9    10    11    12    13    14    15    16    17    18    19    20

sampling with replacement    5    2    12    19    10    5    3    19    11    4

$$\overbrace{n \times n \times \dots \times n}^{k \text{ times}} = n^k$$

1    2    3    4    5    6    7    8    9    10    11    12    13    14    15    16    17    18    19    20

sampling with replacement     $\boxed{5}$     2    12     $\boxed{19}$     10     $\boxed{5}$     3     $\boxed{19}$     11    4

$$\overbrace{n \times n \times \dots \times n}^{k \text{ times}} = n^k$$

1    2    3    4    5    6    7    8    9    10    11    12    13    14    15    16    17    18    19    20

**sampling without replacement**    15    8    13    5    4    19    7    1    20    10

$$n \times (n - 1) \times (n - 2) \times \dots \times (n - k + 1) = \frac{n!}{(n - k)!}$$

*what is the probability that two students in a classroom of  $k$  celebrate their birthdays on the same day?*

*what is the probability that two students in a classroom of  $k$  celebrate their birthdays on the same day?*

$365^k$  possible combinations of birthdays (sampling with replacement)

$365!/(365-k)!$  of these combinations do not overlap (sampling without replacement)

*what is the probability that two students in a classroom of  $k$  celebrate their birthdays on the same day?*

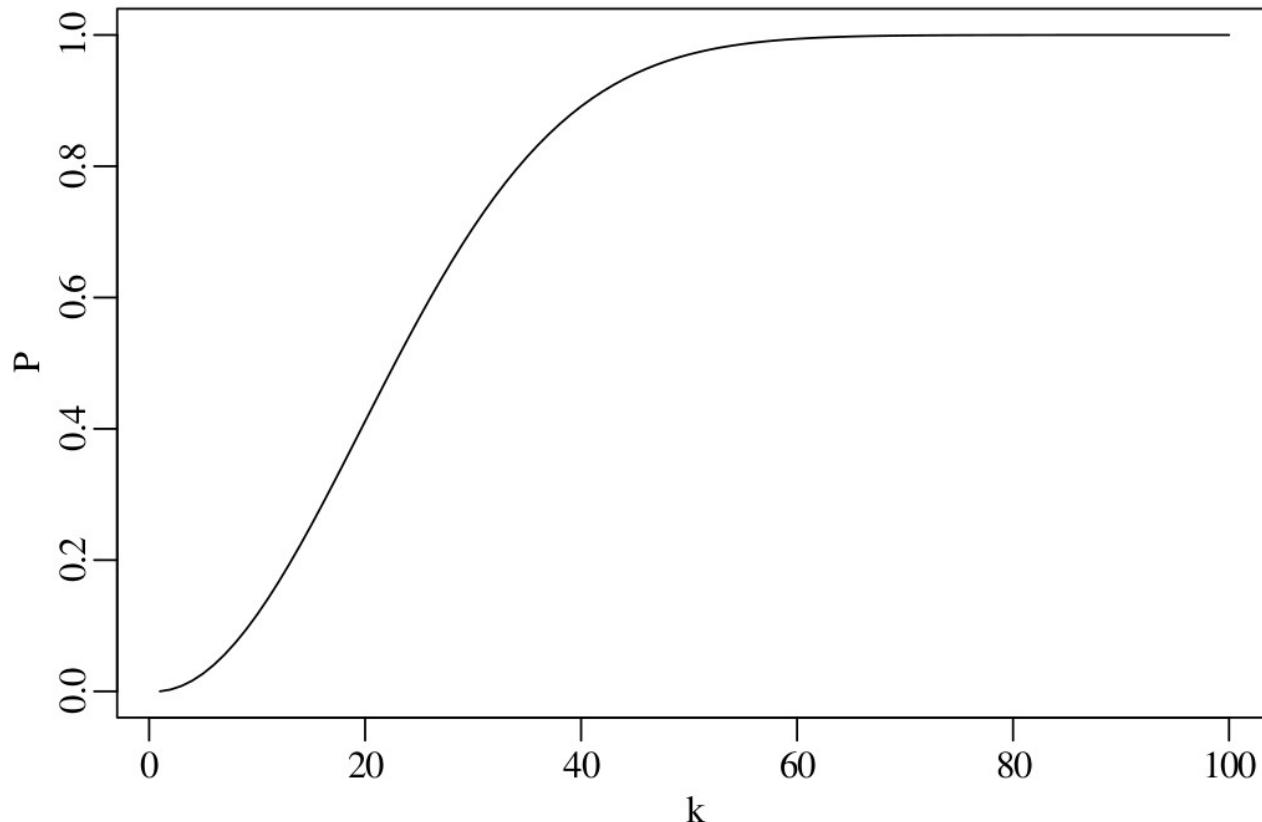
$365^k$  possible combinations of birthdays (sampling with replacement)

$365!/(365-k)!$  of these combinations do not overlap (sampling without replacement)

$$P(\text{no overlapping birthdays}) = \frac{365!}{(365 - k)!365^k}$$

$$P(> 1 \text{ overlapping birthdays}) = 1 - \frac{365!}{(365 - k)!365^k}$$

$$P(> 1 \text{ overlapping birthdays}) = 1 - \frac{365!}{(365 - k)!365^k}$$

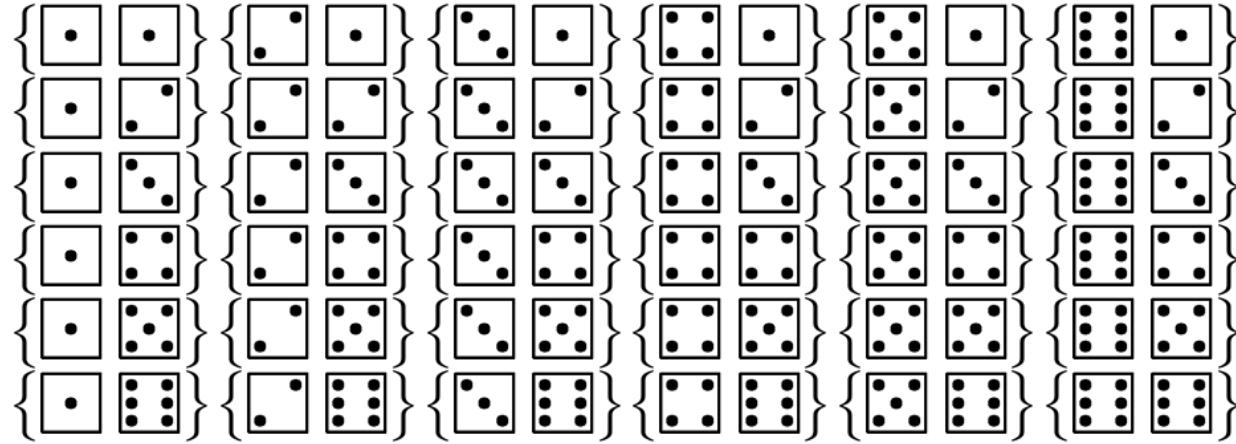


If  $k = 23$ , then  $P(> 1 \text{ overlapping birthdays}) = 0.507$

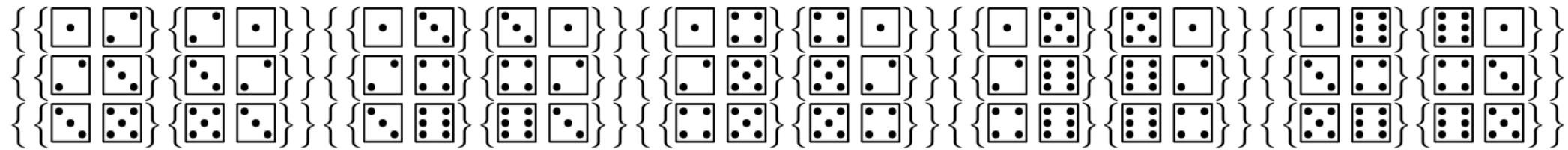
# Combinations

$\{HHH\}\{THH\}\{HTH\}\{HHT\}\{TTH\}\{THT\}\{HTT\}\{TTT\}$

duplicates:  $\{\{HTT\}\{THT\}\{HHT\}\}$  and  $\{\{THH\}\{HTH\}\{TTH\}\}$



duplicates:



$$(\# \text{ ordered samples}) = (\# \text{ unordered samples}) \times (\# \text{ ways to order the samples})$$

$$(\# \text{ unordered samples}) = \frac{(\# \text{ ordered samples})}{(\# \text{ ways to order the samples})}$$

$$= \frac{n!/(n-k)! \text{ ways to select } k \text{ objects from a collection of } n}{k! \text{ ways to order these } k \text{ objects}} = \frac{n!}{(n-k)!k!}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

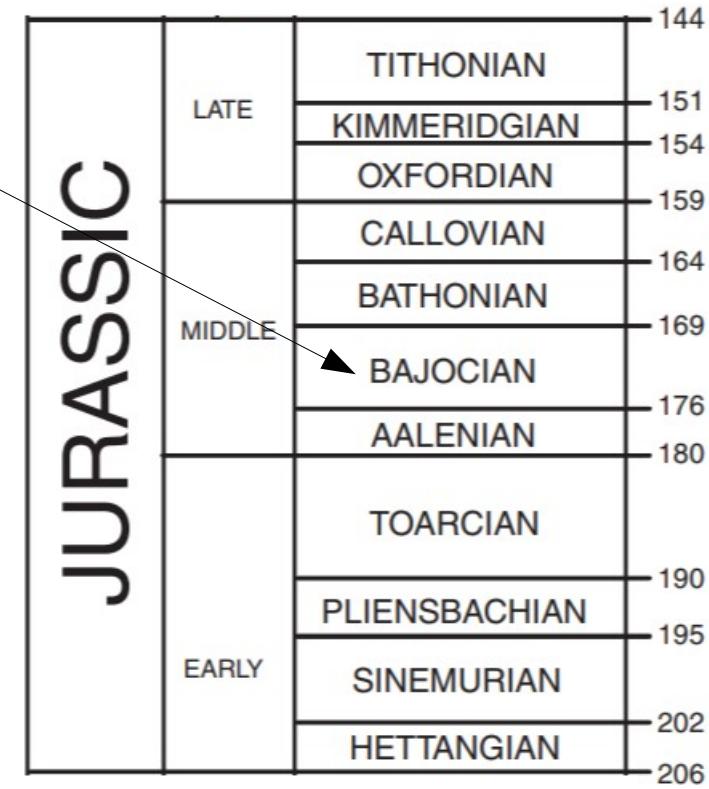
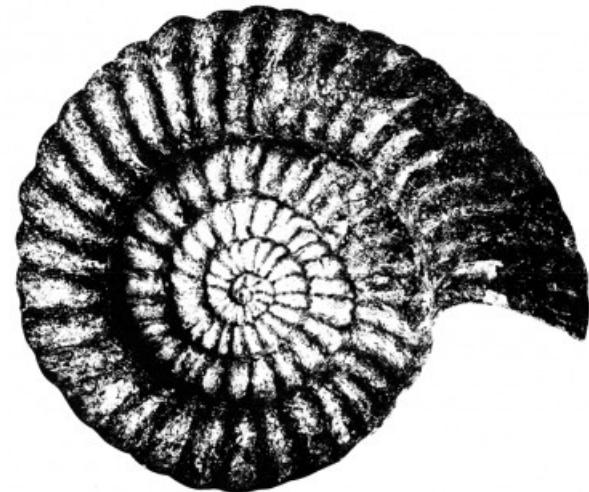
$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\{THH\}\{HTH\}\{HHT\} \hspace{10cm} \left\{\begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \end{array}\right\} \left\{\begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}\right\} \left\{\begin{array}{|c|}\hline \bullet \\ \hline \end{array}\right\}$$

$$\binom{3}{2}=\frac{3!}{1!2!}=\frac{6}{2}=3 \hspace{10cm} \binom{2}{1}=\frac{2!}{1!1!}=2$$

# Conditional probability

$P(A|B) = \text{“The conditional probability of } A \text{ given } B\text{”}$



## multiplication law

$P(A|B)$  = “The conditional probability of  $A$  given  $B$ ”

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$$

$$\left. \begin{array}{l} P(B) = 0.7 \\ P(A|B) = 0.2 \end{array} \right\} 0.7 \times 0.2 = 14\%$$

## law of total probability

$P(A|B)$  = “The conditional probability of  $A$  given  $B$ ”

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

## law of total probability

$P(A|B)$  = “The conditional probability of  $A$  given  $B$ ”

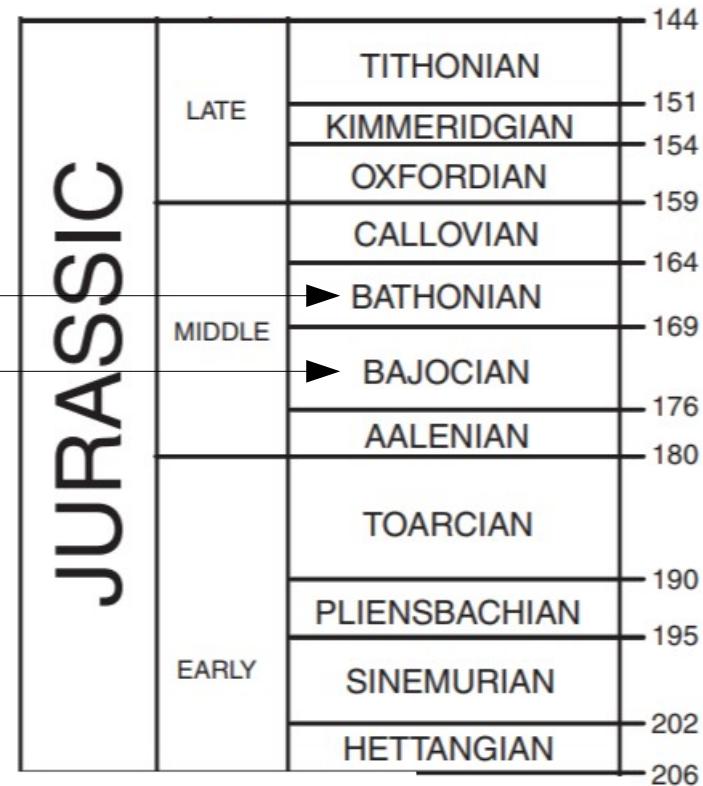
$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

$$P(B_2) = 0.3$$

$$P(B_1) = 0.7$$

$$P(A|B_2) = 0.5$$

$$P(A|B_1) = 0.2$$



$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = 0.2 \times 0.7 + 0.5 \times 0.3 = 0.29$$

**multiplication law**  $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$

**Bayes' Rule** 
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

**multiplication law**  $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$

**law of total probability**  $P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$

**Bayes' Rule**  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)}$

**Bayes' Rule**

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)}$$

Suppose that we have found an ammonite fossil in the river bed. What is its likely age?

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.5 \times 0.3} = 0.48$$

# Statistics for geoscientists

The binomial distribution

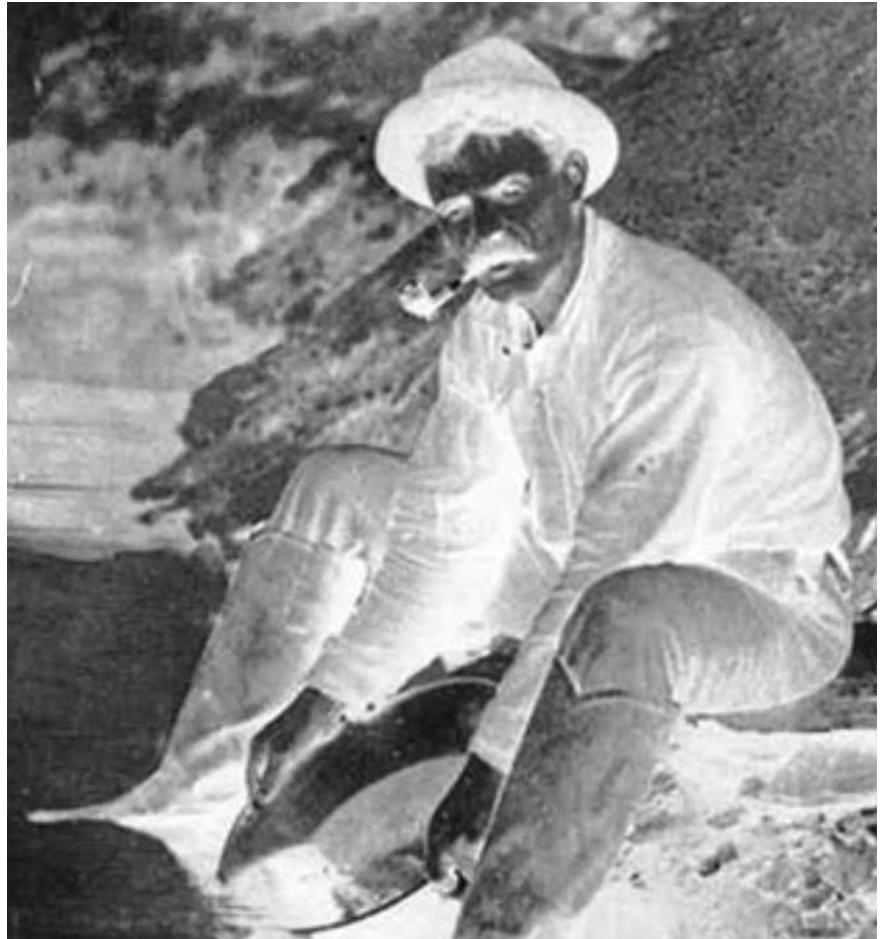
## Bernoulli variable

1. a coin may land on its head (1) or tail (0);
2. a die may land on  (1) or not (0);
3. a ‘wildcat’ exploration well may find petroleum (1) or be dry (0).

five gold claims

$$p = 2/3$$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$



five gold claims               $p = 2/3$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$

$$P(1 \times \text{gold}) = P(10000) + P(01000) + P(00100) + P(00010) + P(00001)$$

$$P(10000) = (2/3)(1/3)^4 = 0.0082$$

$$P(01000) = (1/3)(2/3)(1/3)^3 = 0.0082$$

$$P(00100) = (1/3)^2(2/3)(1/3)^2 = 0.0082$$

$$P(00010) = (1/3)^3(2/3)(1/3) = 0.0082$$

$$P(00001) = (1/3)^4(2/3) = 0.0082$$

$$P(1 \times \text{gold}) = \binom{5}{1} (2/3)(1/3)^4 = 5 \times 0.0082 = 0.041$$

$$\text{five gold claims} \quad p = 2/3$$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$

$$P(1 \times \text{gold}) = \binom{5}{1} (2/3)(1/3)^4 = 5 \times 0.0082 = 0.041$$

$$P(2 \times \text{gold}) = \binom{5}{2} (2/3)^2 (1/3)^3 = 10 \times 0.016 = 0.16$$

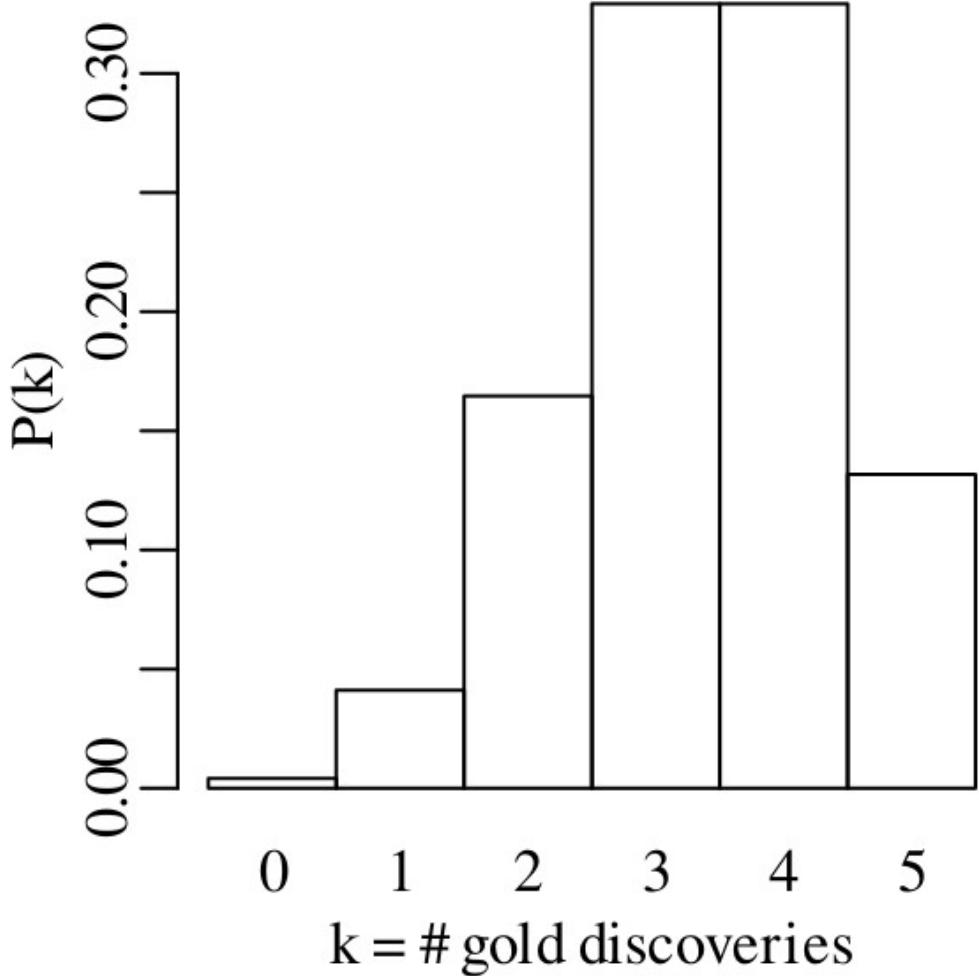
$$P(3 \times \text{gold}) = \binom{5}{3} (2/3)^3 (1/3)^2 = 10 \times 0.033 = 0.33$$

$$P(4 \times \text{gold}) = \binom{5}{4} (2/3)^4 (1/3) = 5 \times 0.066 = 0.33$$

$$P(5 \times \text{gold}) = (2/3)^5 = 0.13$$

## probability mass function (PMF)

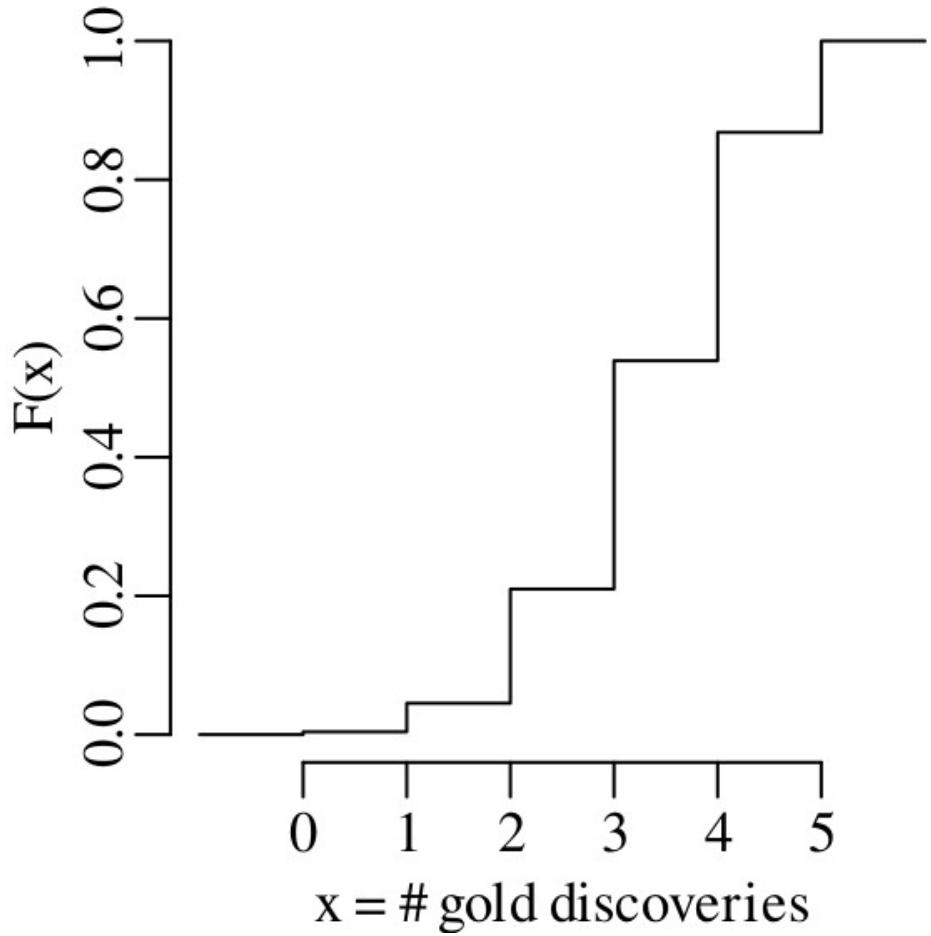
$$P(k|n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$



## cumulative distribution function (CDF)

$$P(k|n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$F(x) = P(X \leq x)$$



# method of maximum likelihood

$$P(k|n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathcal{L}(p|n, k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\hat{p} = \frac{k}{n} \quad \text{e.g.} \quad \hat{p} = \frac{2}{5} = 0.4$$

# Hypothesis tests

$$\hat{p} = 2/5 \quad \longleftrightarrow \quad p = 2/3$$

1. Formulate two hypotheses:

$$H_0 \text{ (null hypothesis)} \qquad \qquad p = 2/3$$

$$H_a \text{ (alternative hypothesis):} \qquad \qquad p < 2/3$$

2. Calculate the **test statistic**  $T$  ( $k$ )
3. Determine the **null distribution** of  $T$  under  $H_0$ .

$k$	0	1	2	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	0.0041	0.0453	0.2099	0.5391	0.8683	1.0000


**p-value**

4. Choose a **significance level** ( $\alpha = 0.05$ )

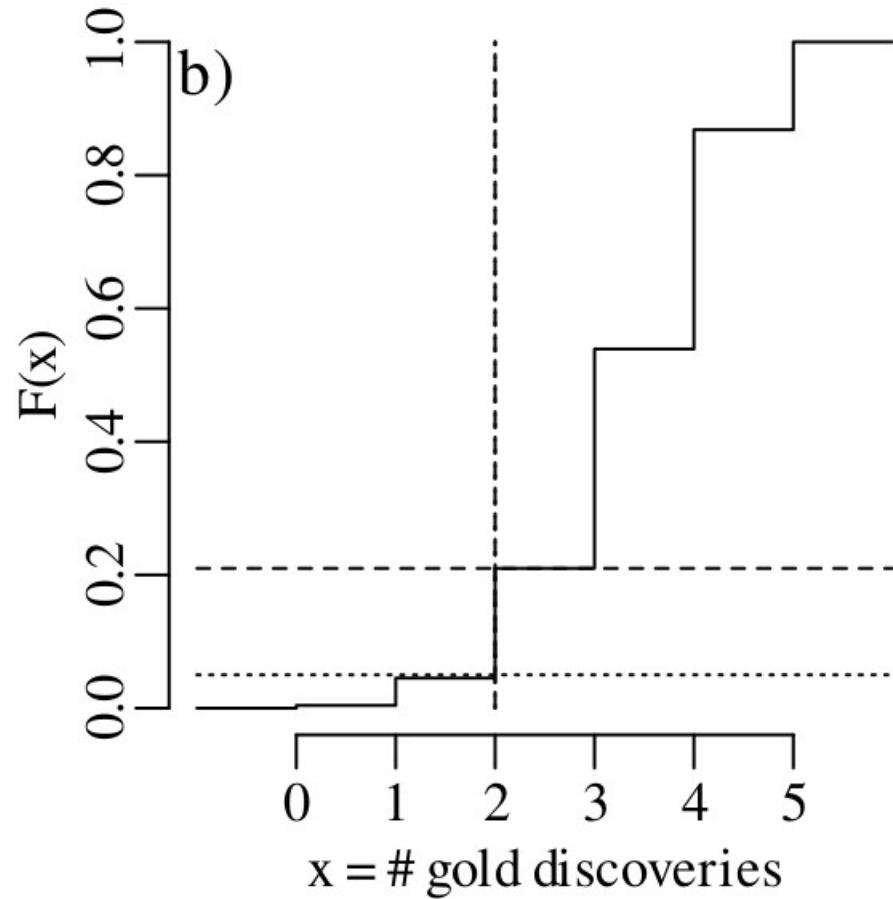
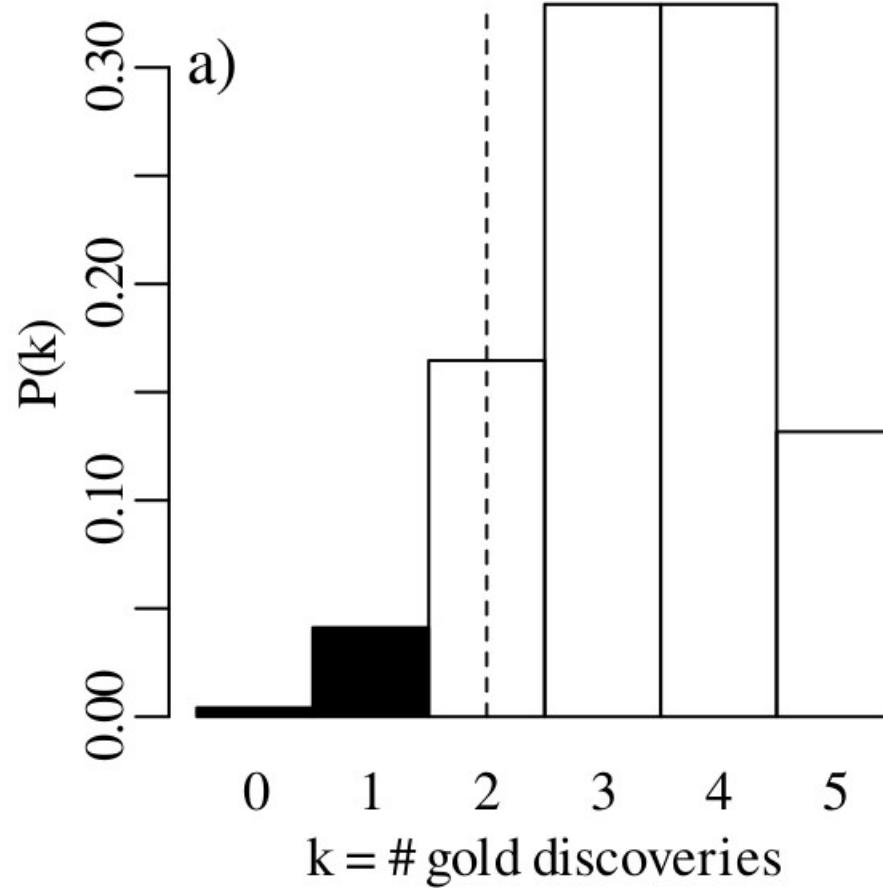
5. Mark the **rejection region**

k	0	1	2	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	<b>0.0041</b>	<b>0.0453</b>	<i>0.2099</i>	0.5391	0.8683	1.0000

rejection region     $R = \{0, 1\}$

6. Reach a **decision**  $k = 2 \notin R$

7. Alternatively,               $0.2099 > \alpha$



$$\hat{p} = 2/5 \quad \longleftrightarrow \quad p = 2/3$$

**one-sided** hypothesis test

$$H_0 \text{ (null hypothesis)} \qquad \qquad \qquad p = 2/3$$

$$H_a \text{ (alternative hypothesis):} \qquad \qquad \qquad p < 2/3$$

**two-sided** hypothesis test

$$H_0 \text{ (null hypothesis)} \qquad \qquad \qquad p = 2/3$$

$$H_a \text{ (alternative hypothesis):} \qquad \qquad \qquad p \neq 2/3$$

2. Calculate the **test statistic**  $T^-(k)$
3. Determine the **null distribution** of  $T$  under  $H_0$ .

$k$	0	1	2	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	0.0041	0.0453	0.2099	0.5391	0.8683	1.0000
$P(T \geq k)$	1.000	0.9959	0.9547	0.7901	0.4609	0.1317

4. Choose a **significance level** ( $\alpha = 0.05$ )

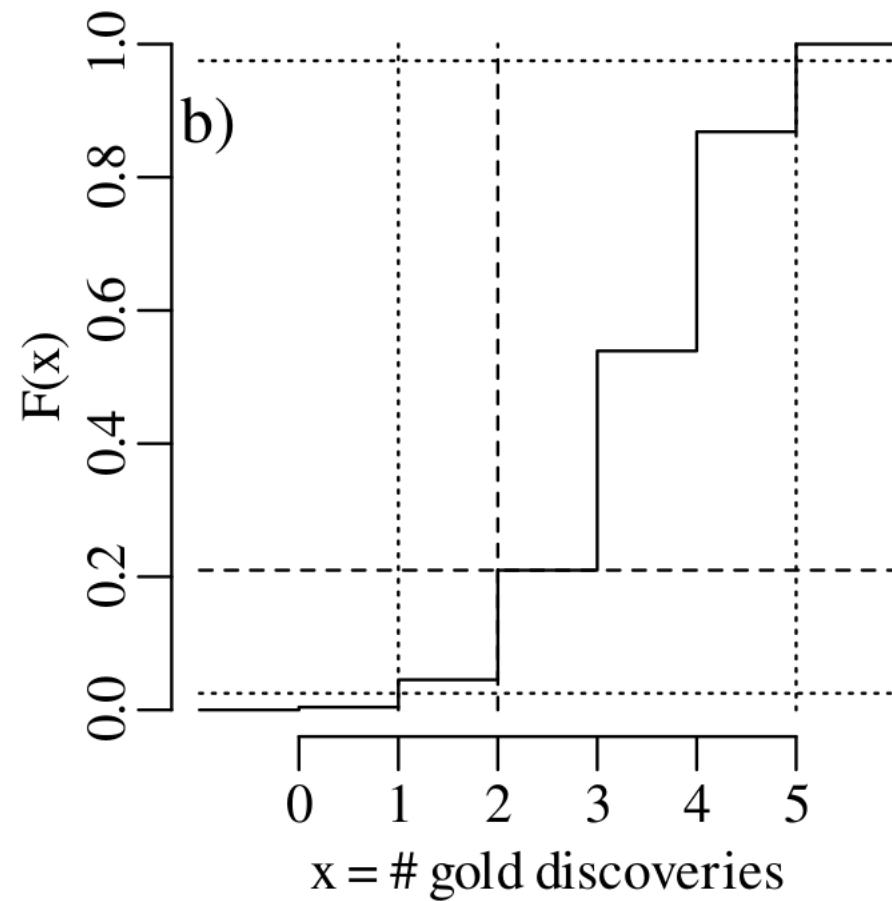
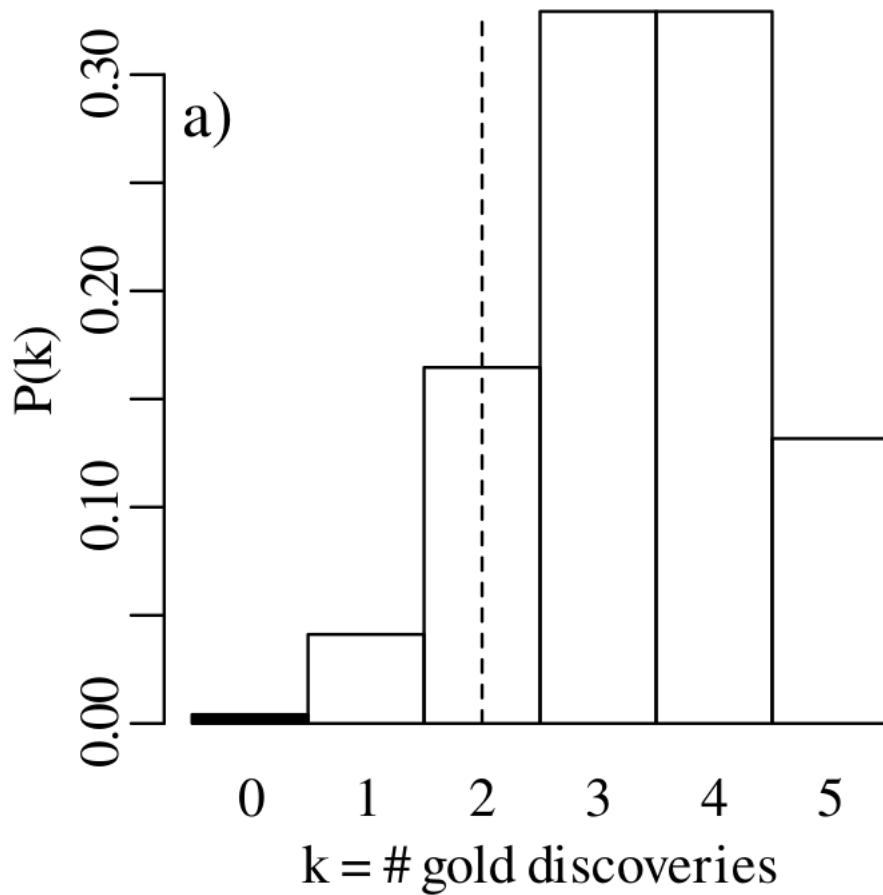
but evaluated twice at  $\alpha/2$

5. Mark the **rejection region**

k	0	1	2	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	<b>0.0041</b>	0.0453	<i>0.2099</i>	0.5391	0.8683	1.0000
$P(T \geq k)$	1.000	0.9959	<i>0.9547</i>	0.7901	0.4609	0.1317

$$R = \{0\}$$

6. Reach a **decision**  $k = 2 \notin R$



# Statistical power

We failed to reject  $H_0$  for  $\hat{p} = \frac{2}{5} = 0.4$ . How about  $\hat{p} = \frac{6}{15} = 0.4$ ?

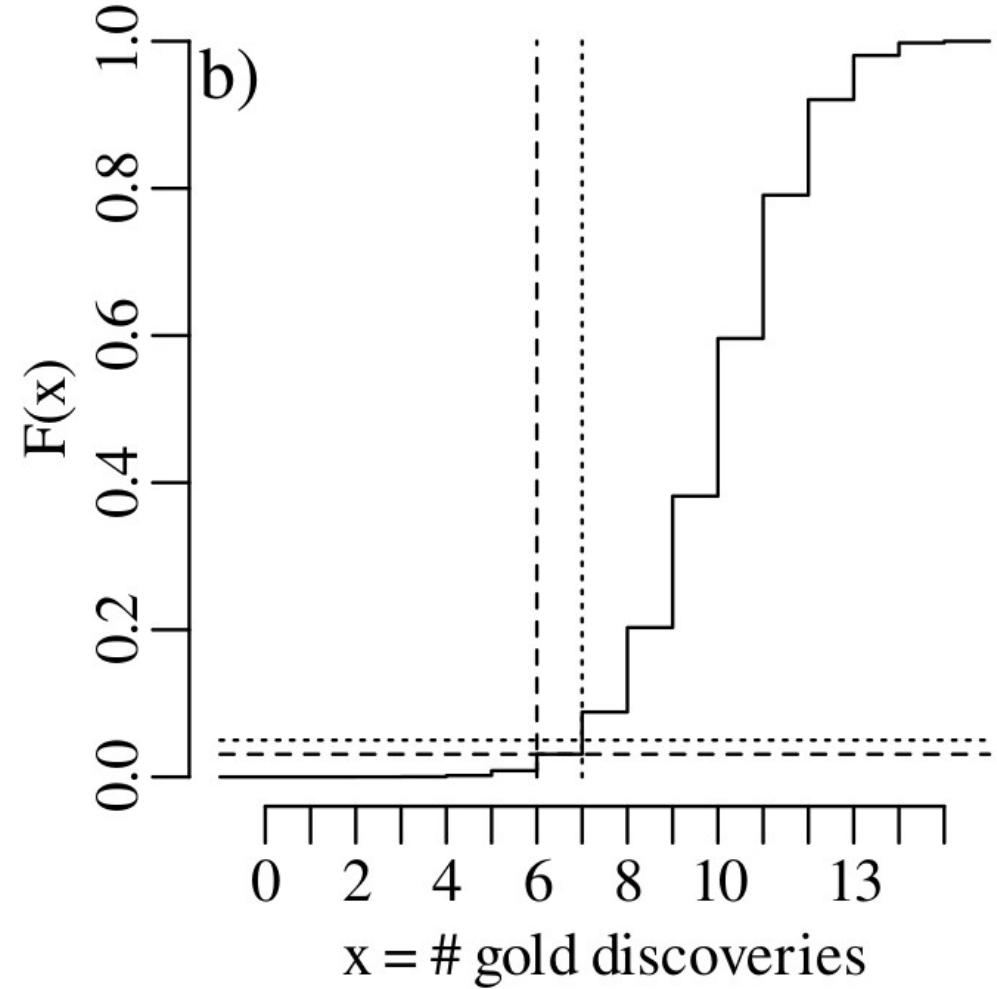
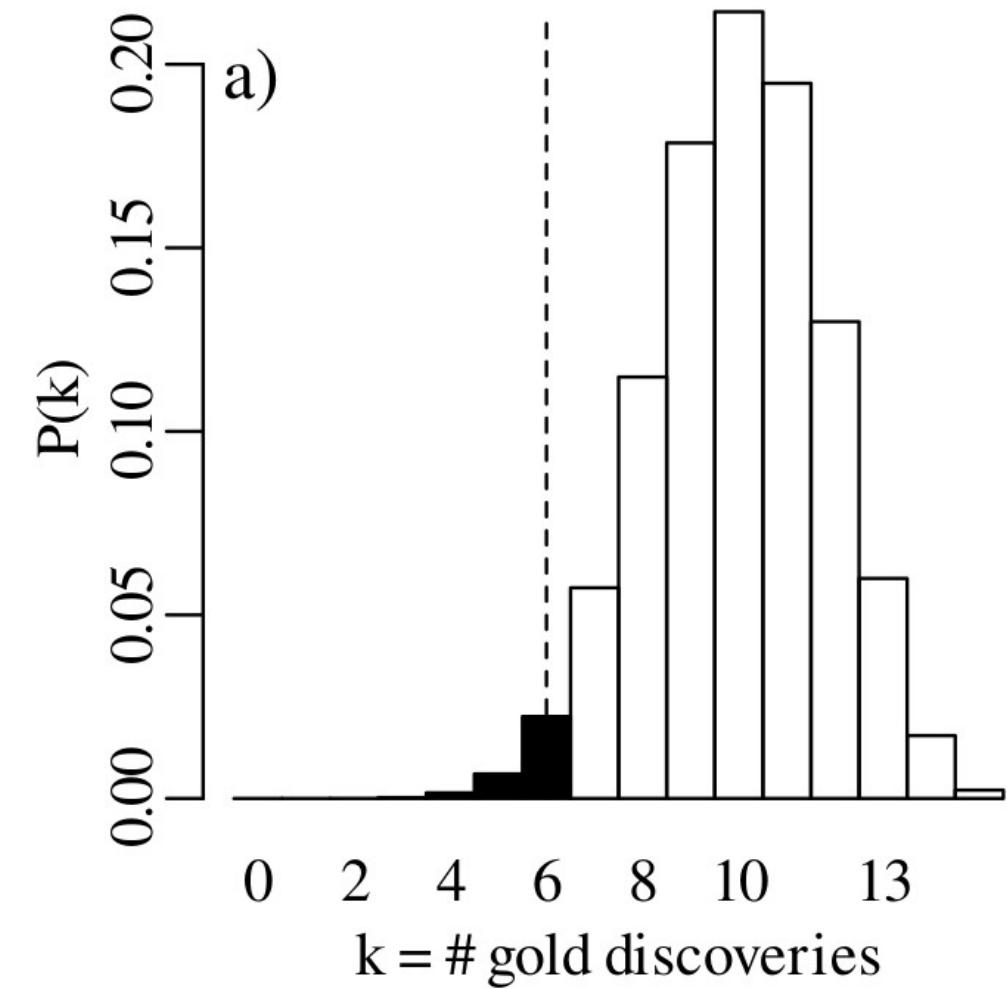
one-sided hypothesis test ( $H_0 : p = 2/3$  vs.  $H_a : p < 2/3$ )

k	0	1	2	3	4	5	6	7
$P(T = k)$	$7.0 \times 10^{-8}$	$2.1 \times 10^{-6}$	$2.9 \times 10^{-5}$	$2.5 \times 10^{-4}$	0.0015	0.0067	0.0223	0.0574
$P(T \leq k)$	<b><math>7.0 \times 10^{-8}</math></b>	<b><math>2.2 \times 10^{-6}</math></b>	<b><math>3.1 \times 10^{-5}</math></b>	<b><math>2.8 \times 10^{-4}</math></b>	<b>0.0018</b>	<b>0.0085</b>	<b>0.0308</b>	0.0882
k	8	9	10	11	12	13	14	15
$P(T = k)$	0.1148	0.1786	0.2143	0.1948	0.1299	0.0599	0.0171	0.0023
$P(T \leq k)$	0.2030	0.3816	0.5959	0.7908	0.9206	0.9806	0.9977	1.0000

$$R = \{0, 1, 2, 3, 4, 5, 6\}$$

$$k \in R$$

$$\text{p-value} = 0.0308 < \alpha$$



two-sided hypothesis test ( $H_0 : p = 2/3$  vs.  $H_a : p \neq 2/3$ ):

k	0	1	2	3	4	5	6	7
$P(T = k)$	$7.0 \times 10^{-8}$	$2.1 \times 10^{-6}$	$2.9 \times 10^{-5}$	$2.5 \times 10^{-4}$	0.0015	0.0067	0.0223	0.0574
$P(T \leq k)$	<b><math>7.0 \times 10^{-8}</math></b>	<b><math>2.2 \times 10^{-6}</math></b>	<b><math>3.1 \times 10^{-5}</math></b>	<b><math>2.8 \times 10^{-4}</math></b>	<b>0.0018</b>	<b>0.0085</b>	<b>0.0308</b>	0.0882
$P(T \geq k)$	1.0000	$1 - 7.0 \times 10^{-8}$	$1 - 2.2 \times 10^{-6}$	$1 - 3.1 \times 10^{-5}$	$1 - 2.8 \times 10^{-4}$	0.9982	0.9915	0.9692
k	8	9	10	11	12	13	14	15
$P(T = k)$	0.1148	0.1786	0.2143	0.1948	0.1299	0.0599	0.0171	0.0023
$P(T \leq k)$	0.2030	0.3816	0.5959	0.7908	0.9206	0.9806	0.9977	1.0000
$P(T > k)$	0.9118	0.7970	0.6184	0.4041	0.2092	0.0794	<b>0.0194</b>	<b>0.0023</b>

$$R = \{0, 1, 2, 3, 4, 5, 14, 15\}$$

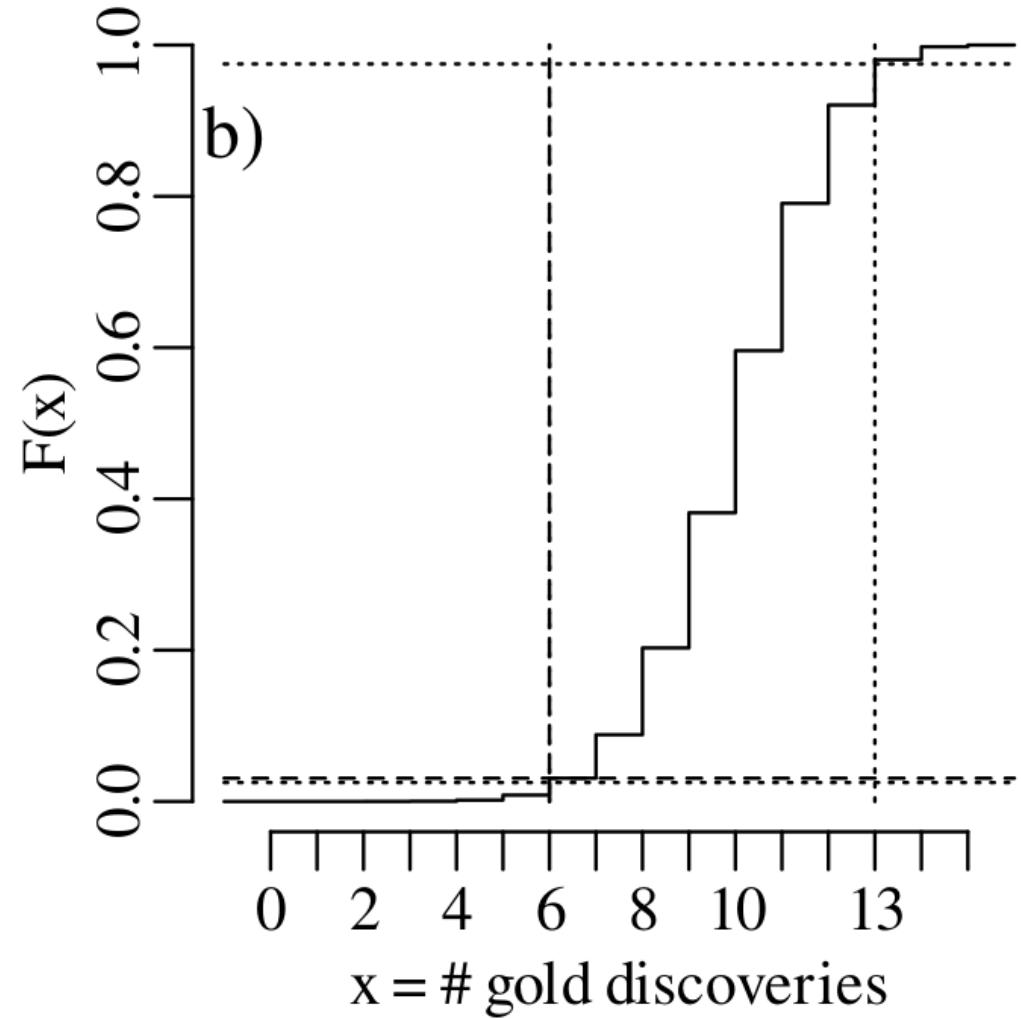
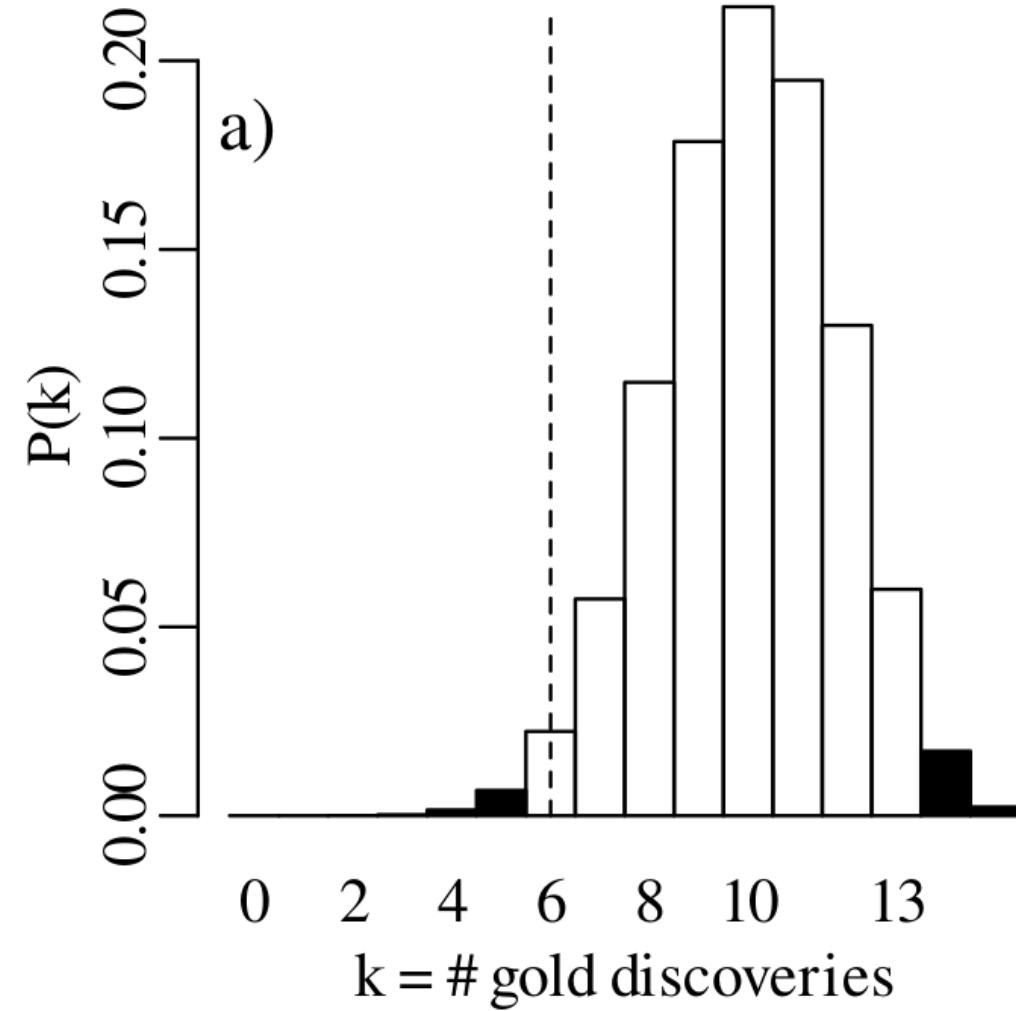
$$k \notin R$$

two-sided hypothesis test ( $H_0 : p = 2/3$  vs.  $H_a : p \neq 2/3$ ):

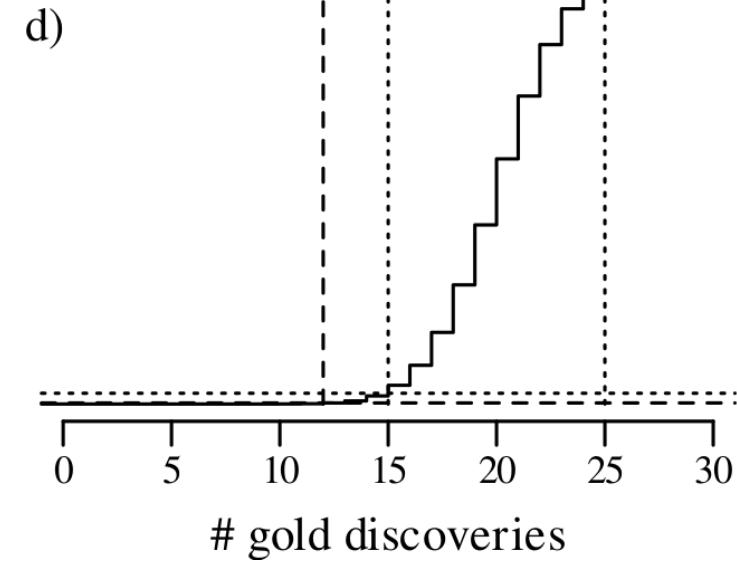
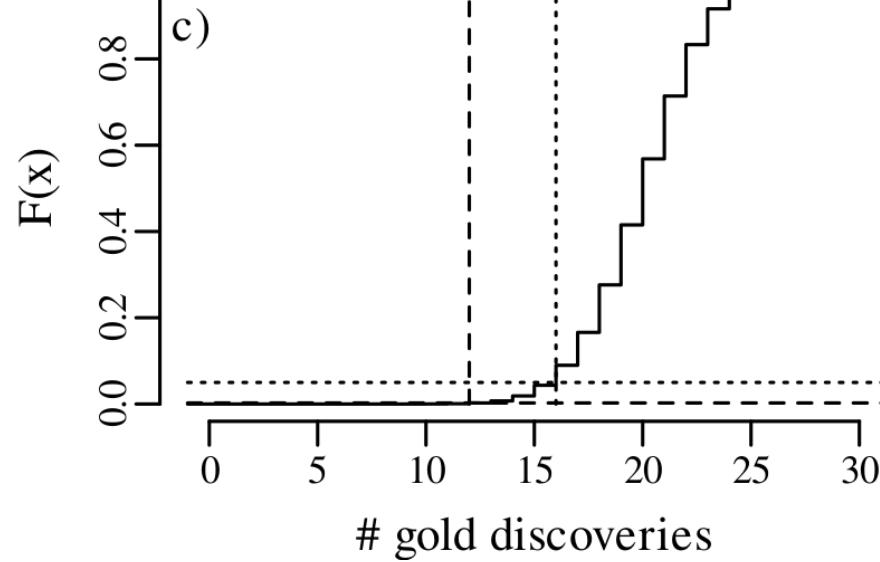
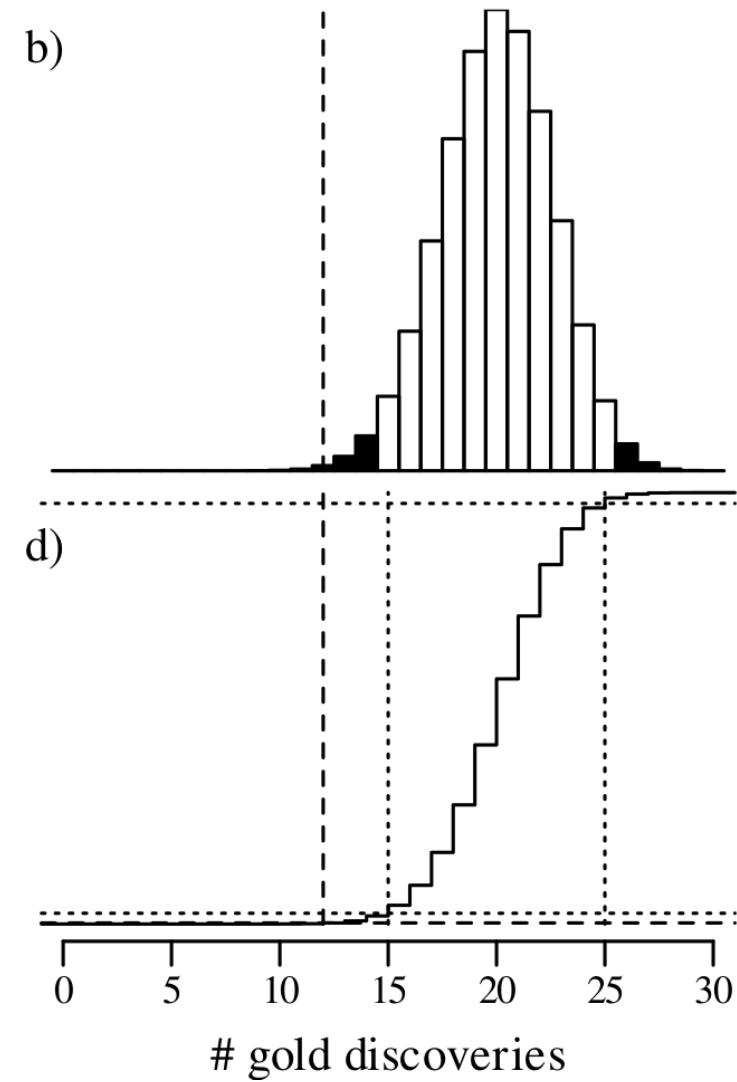
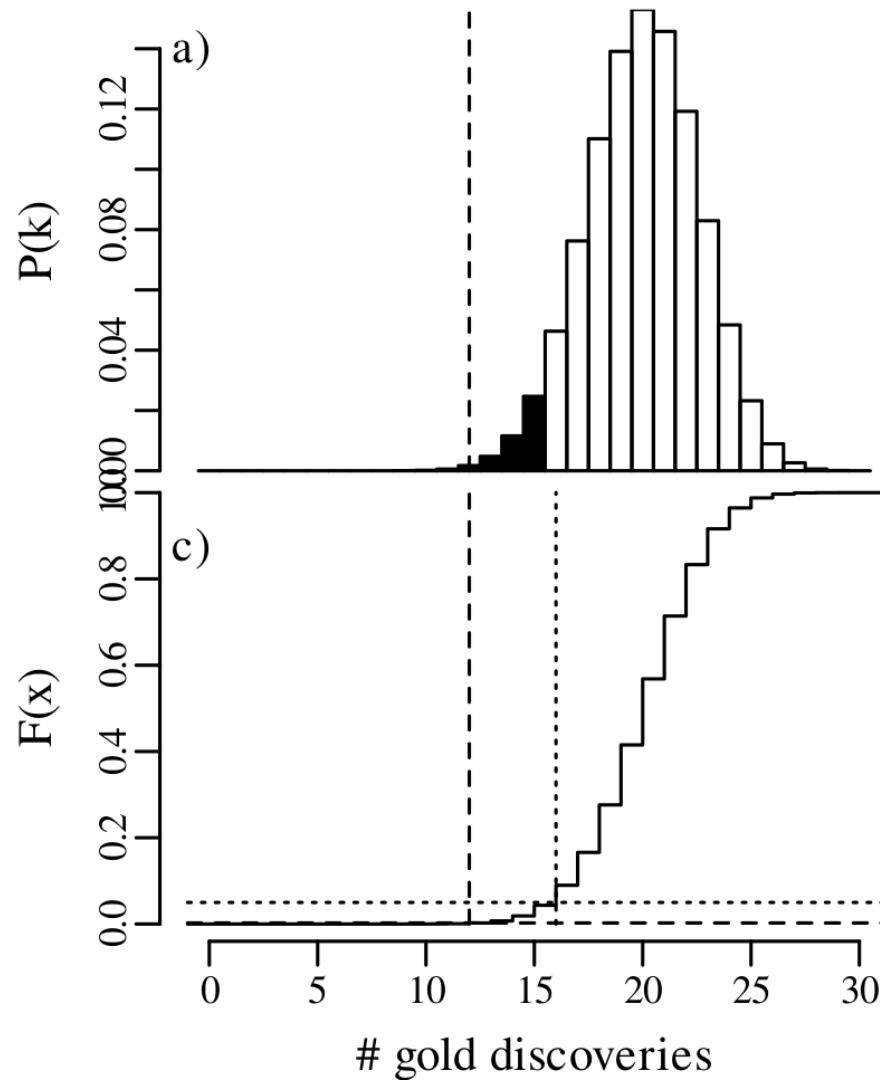
k	0	1	2	3	4	5	6	7
$P(T = k)$	$7.0 \times 10^{-8}$	$2.1 \times 10^{-6}$	$2.9 \times 10^{-5}$	$2.5 \times 10^{-4}$	0.0015	0.0067	0.0223	0.0574
$P(T \leq k)$	<b><math>7.0 \times 10^{-8}</math></b>	<b><math>2.2 \times 10^{-6}</math></b>	<b><math>3.1 \times 10^{-5}</math></b>	<b><math>2.8 \times 10^{-4}</math></b>	<b>0.0018</b>	<b>0.0085</b>	<b>0.0308</b>	0.0882
$P(T \geq k)$	1.0000	$1 - 7.0 \times 10^{-8}$	$1 - 2.2 \times 10^{-6}$	$1 - 3.1 \times 10^{-5}$	$1 - 2.8 \times 10^{-4}$	0.9982	0.9915	0.9692
k	8	9	10	11	12	13	14	15
$P(T = k)$	0.1148	0.1786	0.2143	0.1948	0.1299	0.0599	0.0171	0.0023
$P(T \leq k)$	0.2030	0.3816	0.5959	0.7908	0.9206	0.9806	0.9977	1.0000
$P(T > k)$	0.9118	0.7970	0.6184	0.4041	0.2092	0.0794	<b>0.0194</b>	<b>0.0023</b>

$$R = \{0, 1, 2, 3, 4, 5, 14, 15\}$$

$$k \notin R$$



Next, how about  $\hat{p} = \frac{12}{30} = 0.4$ ?



$H_0$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

the accused is ...	guilty	innocent
sentenced	correct decision	Type-I error
acquitted	Type-II error	correct decision

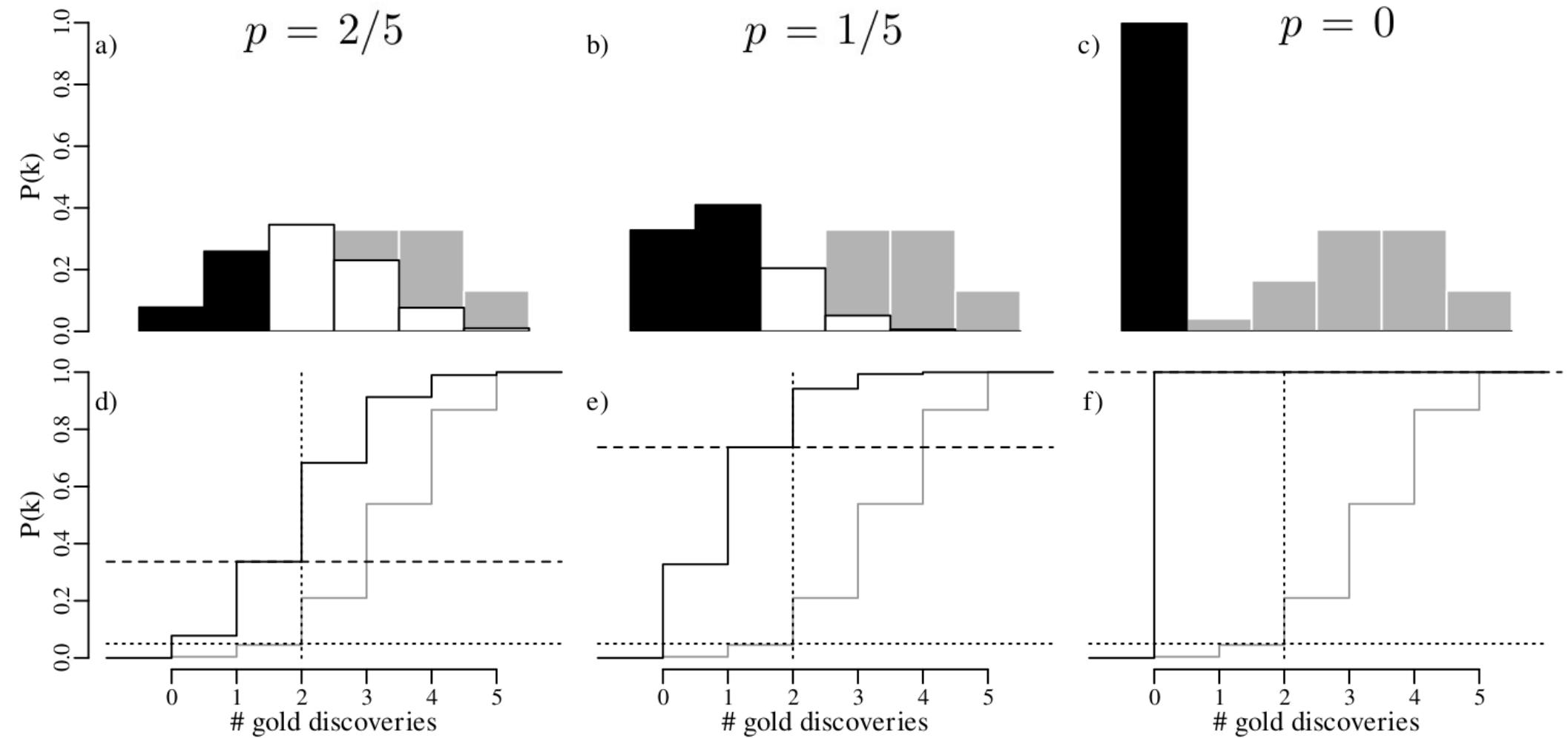
depends on the **confidence level**  $\alpha$

$H_0$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$\beta = 1 - \text{power}$

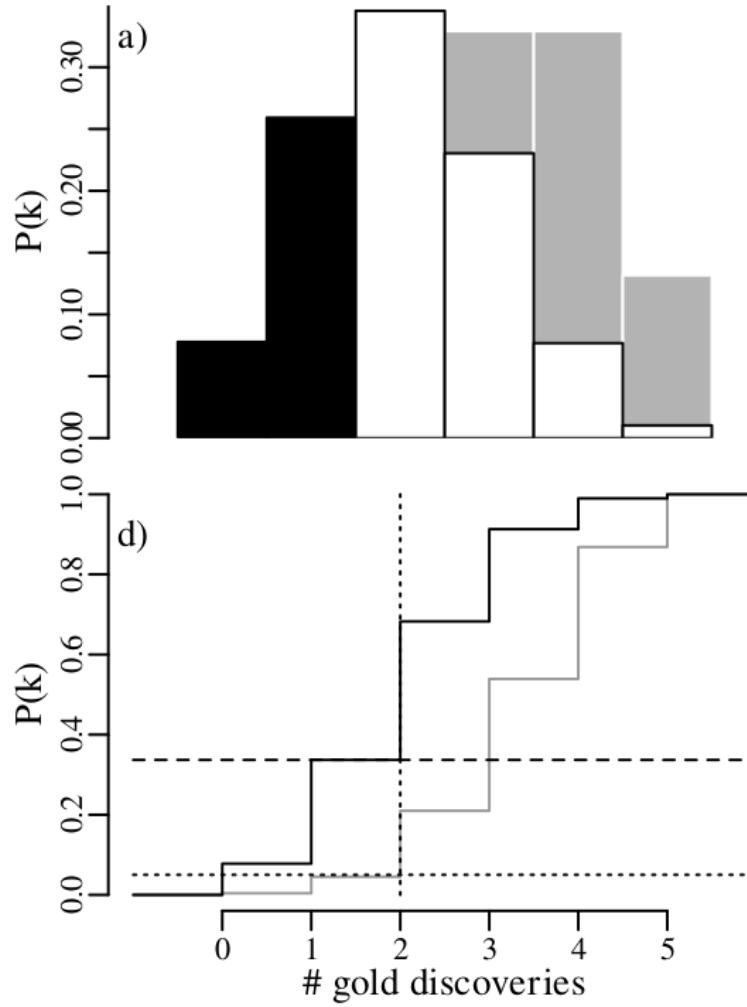
depends on 1. The **degree to which  $H_0$  is false**  
2. **Sample size**

# 1. The degree to which $H_0$ is false

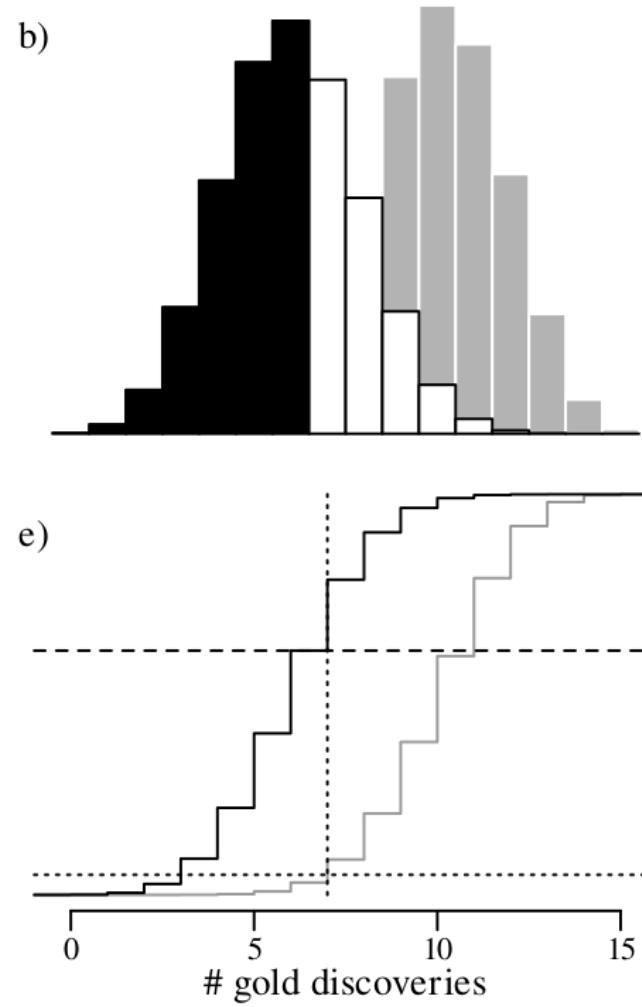


## 2. Sample size

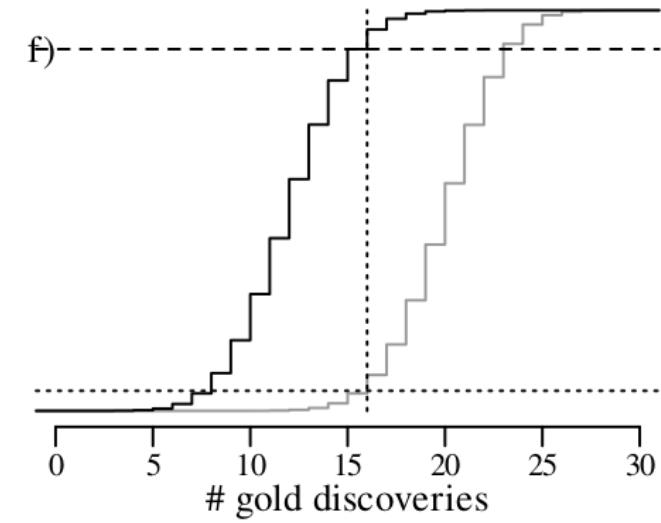
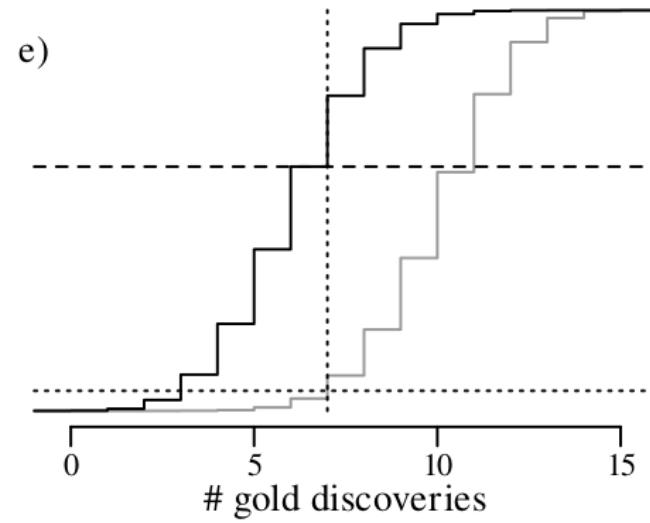
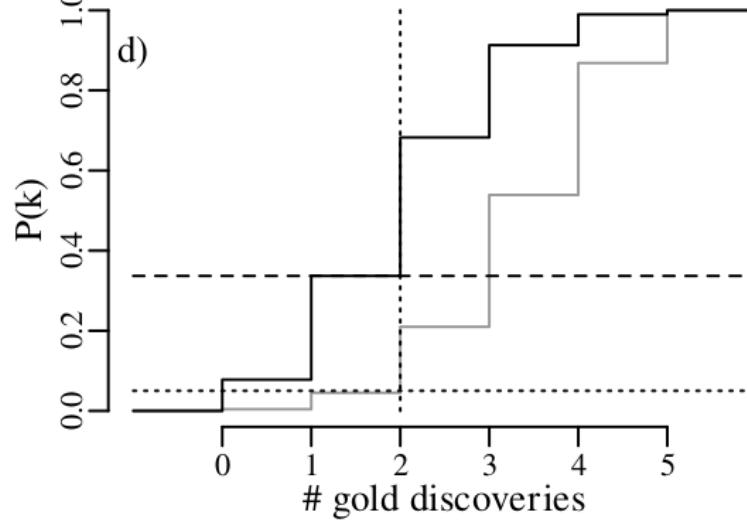
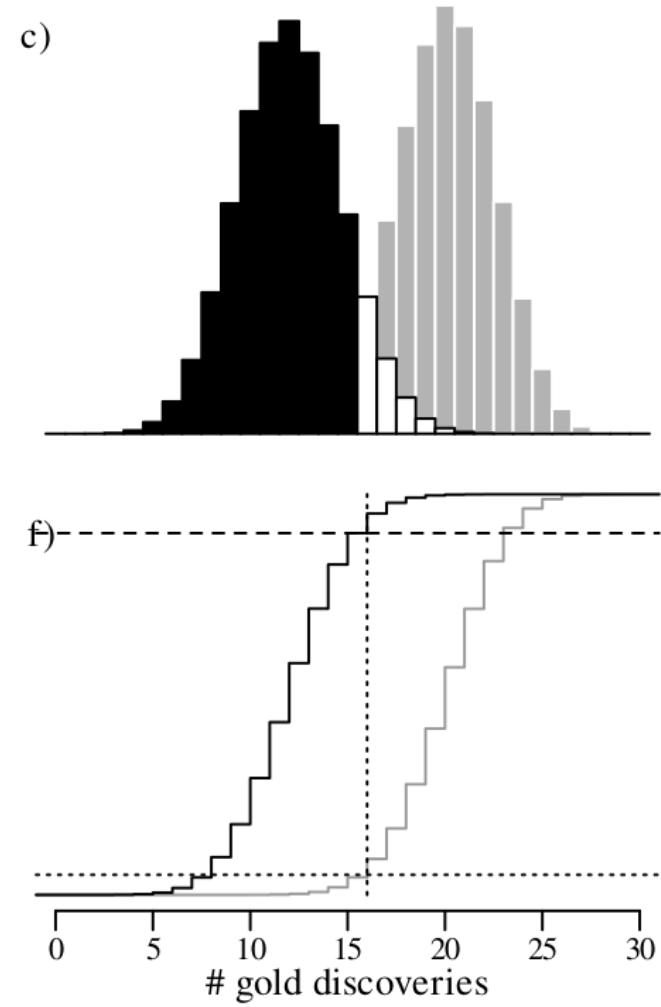
$n = 5$



$n = 15$



$n = 30$



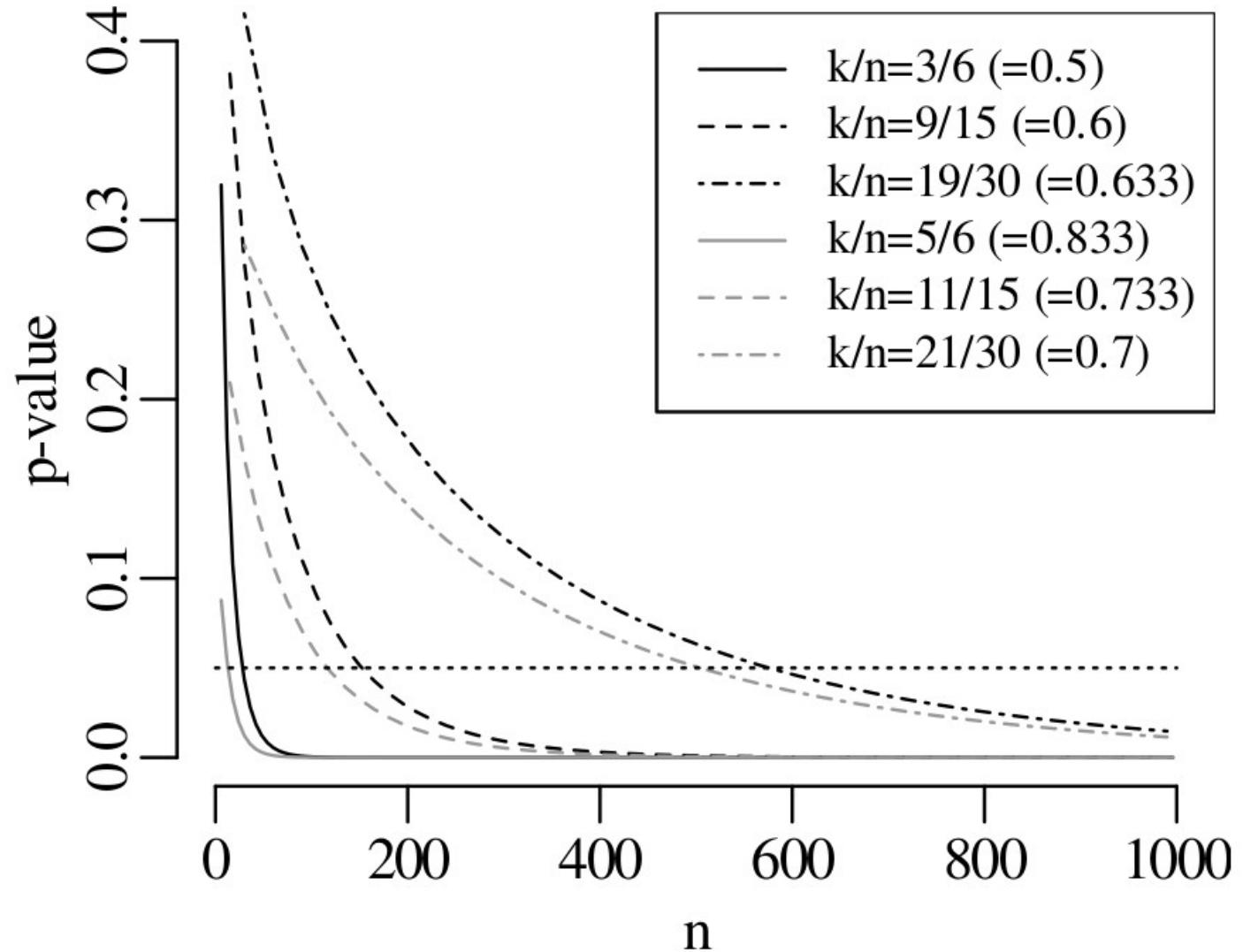
# The scientific method

1. Formulate a hypothesis.
2. Design an experiment to test said hypothesis.
3. Carry out the experiment and check to see if it matches the prediction.

1. Hypothesis: Earth's lower mantle is made of olivine.
2. Test: Study the stability of olivine at lower mantle pressures (24-136 GPa).
3. Result: Olivine is not stable at lower mantle pressures.

1. Hypothesis: Earth's lower mantle is made of perovskite.
2. Test: Study the stability of perovskite at lower mantle pressures.
3. Result: Perovskite is stable at lower mantle pressures.

$p = 2/3$  ?



# Pitfalls

*All hypotheses are wrong ... in some decimal place*

– John Tukey (paraphrased)

*All models are wrong, but some are useful*

– George Box

# Confidence intervals

Which values of  $p$  are compatible with the observation of  $k = 2$  successes out of  $n = 5$  claims?

1.  ~~$p=0?$~~

2.  $p=0.1?$

$$P(k \geq 2 | p = 0.1, n = 5) = \sum_{i=2}^5 \binom{5}{i} (0.1)^i (0.9)^{n-i} = 0.081 \geq \alpha/2$$

3.  $p=2/5? = \hat{p}$

4.  $p=2/3?$

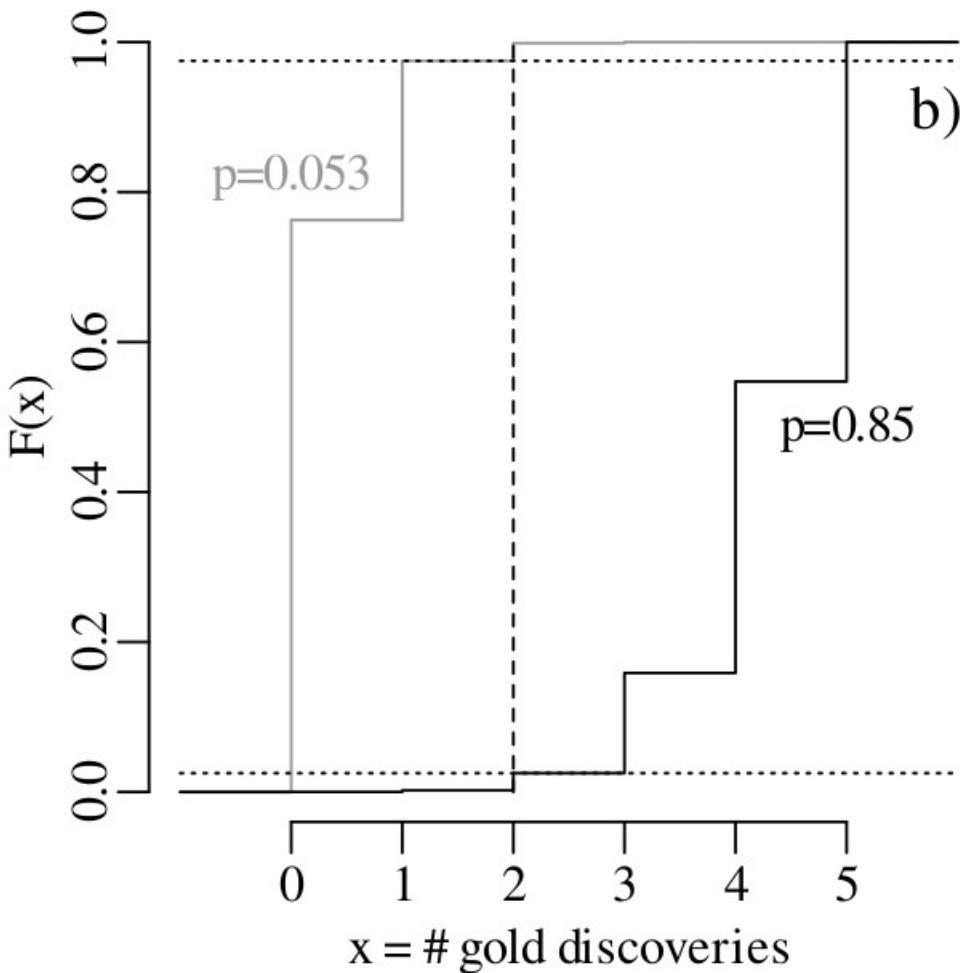
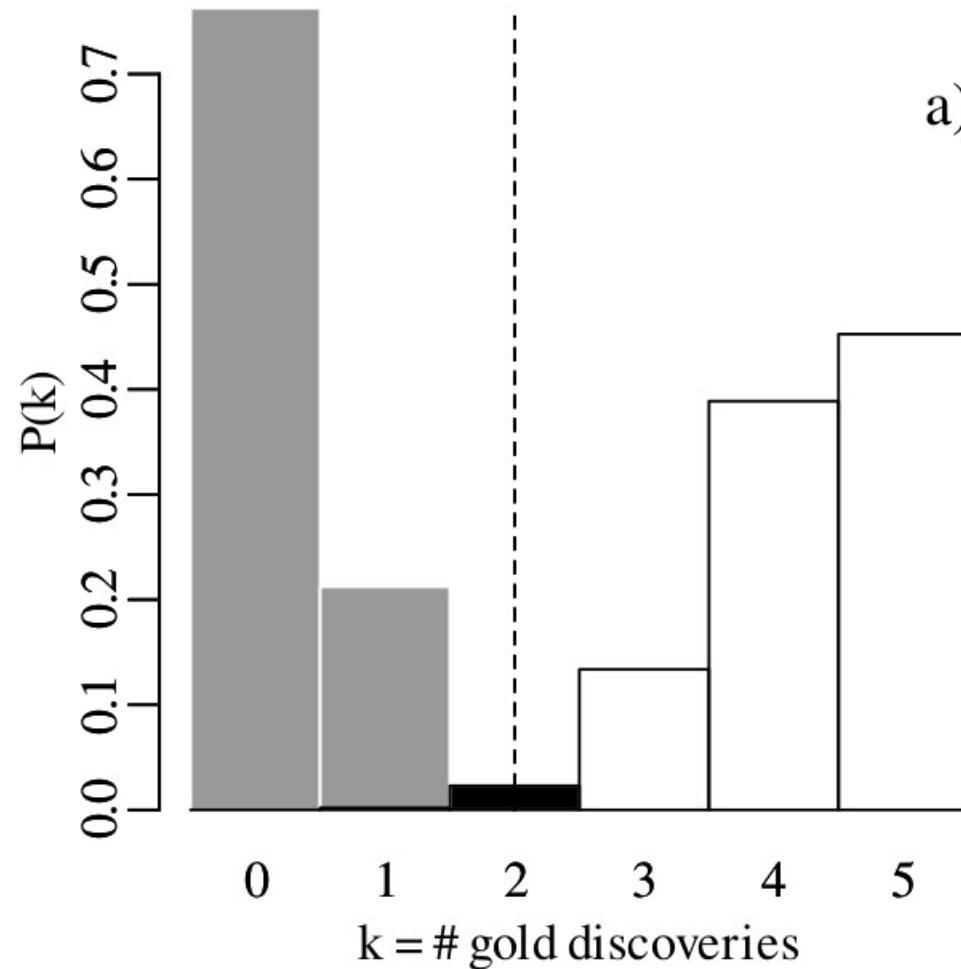
$$P(k \leq 2 | p = 2/3, n = 5) = \sum_{i=0}^2 \binom{5}{i} (2/3)^i (1/3)^{n-i} = 0.21 \geq \alpha/2$$

5.  ~~$p=0.9?$~~

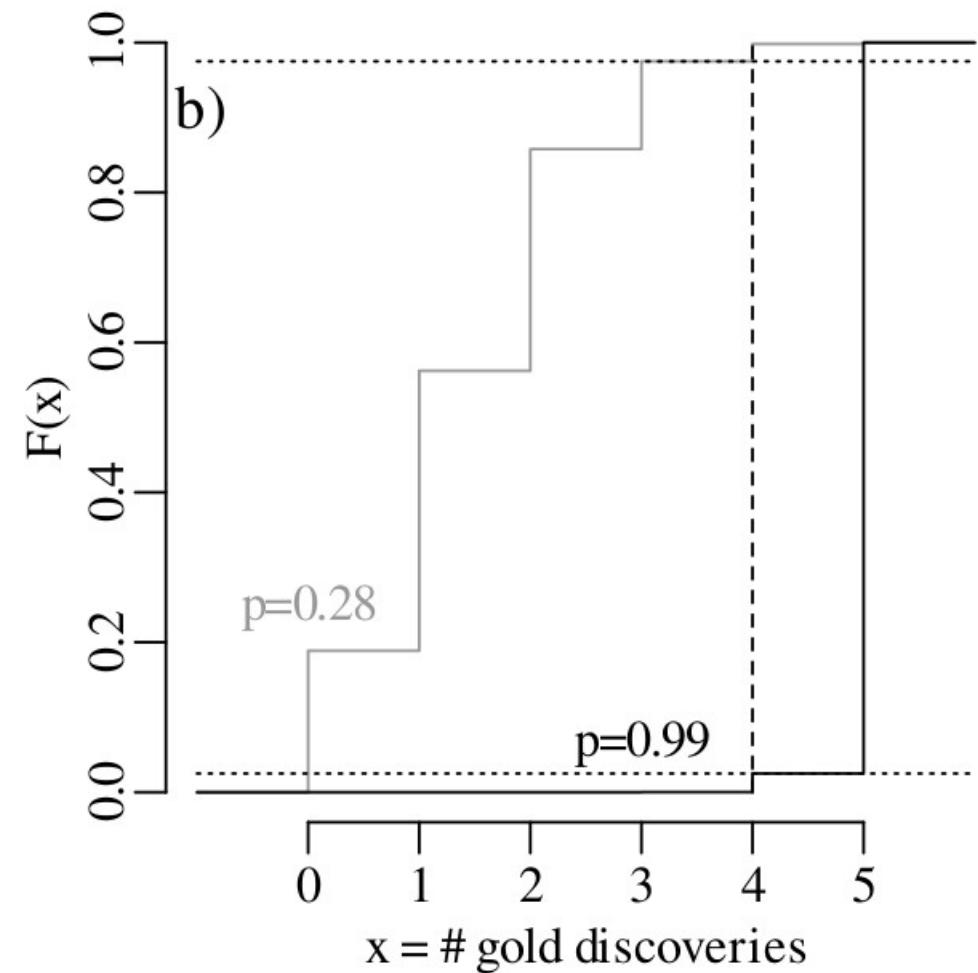
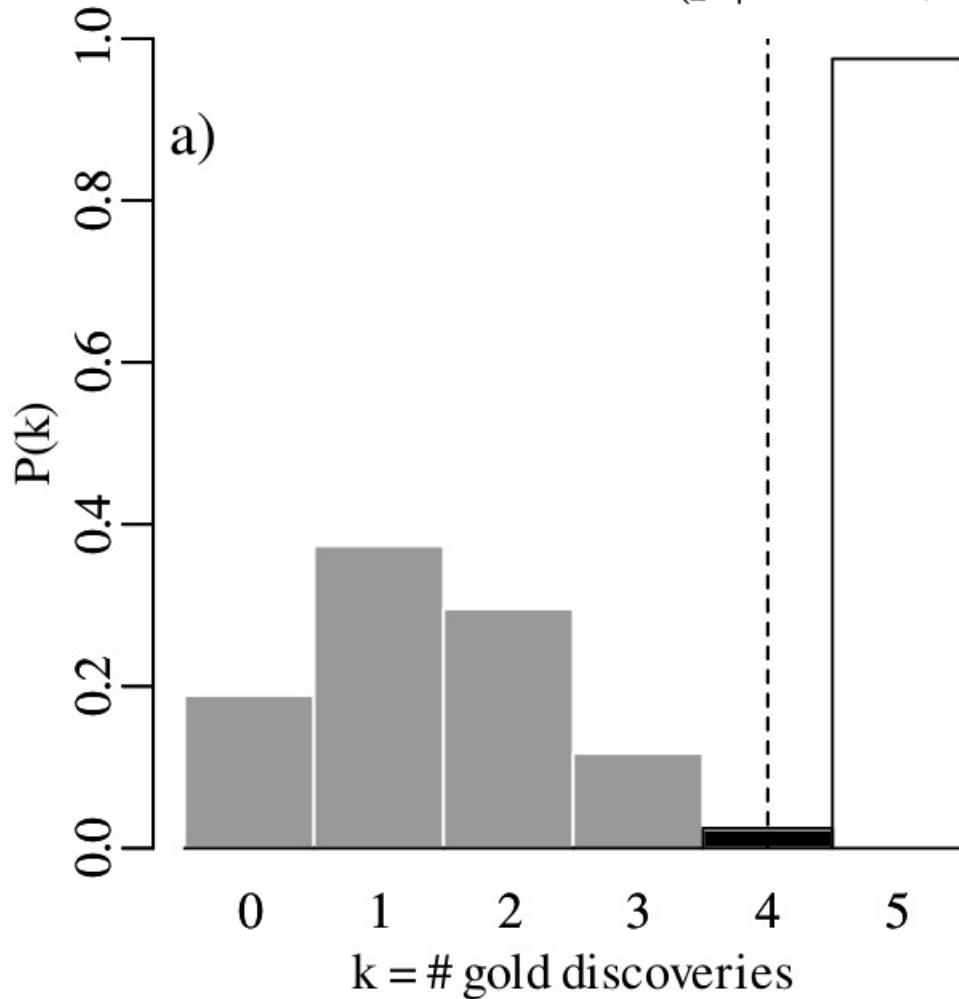
$$P(k \leq 2 | p = 0.9, n = 5) = \sum_{i=0}^2 \binom{5}{i} (0.9)^i (0.1)^{n-i} = 0.0086 \leq \alpha/2$$

6.  ~~$p=1?$~~

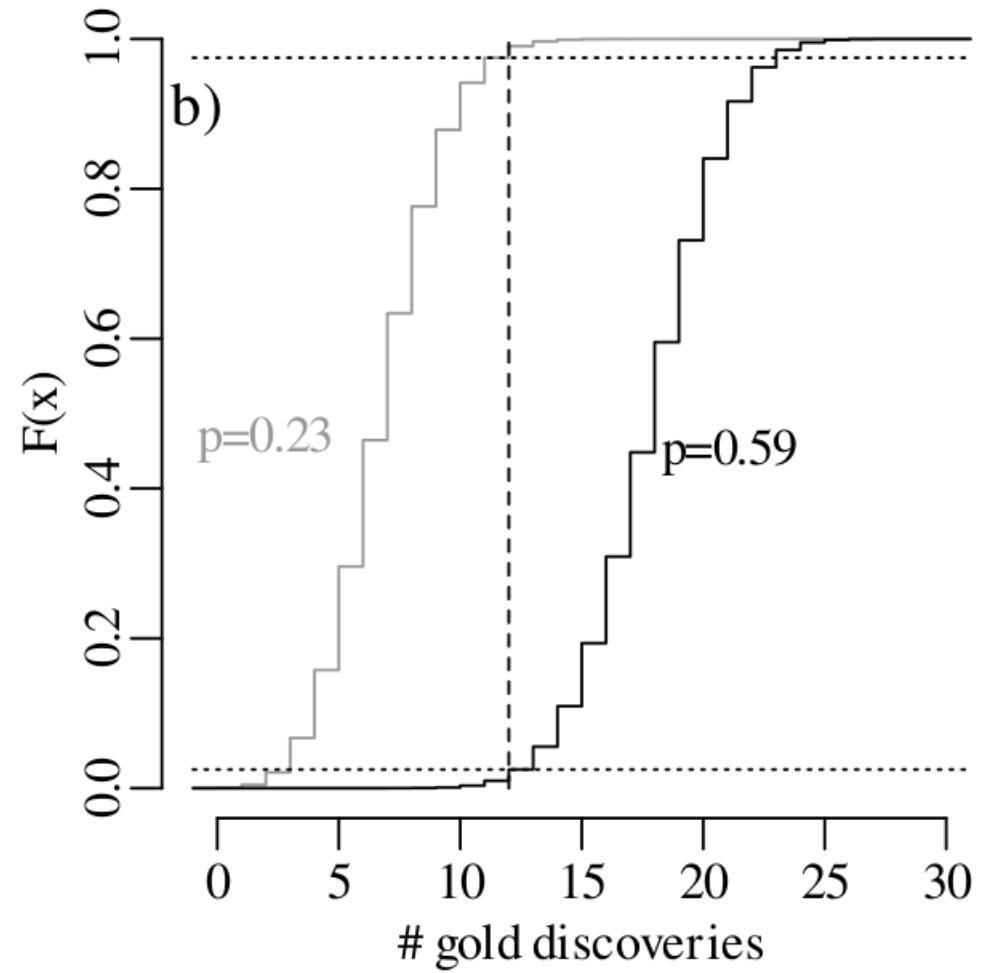
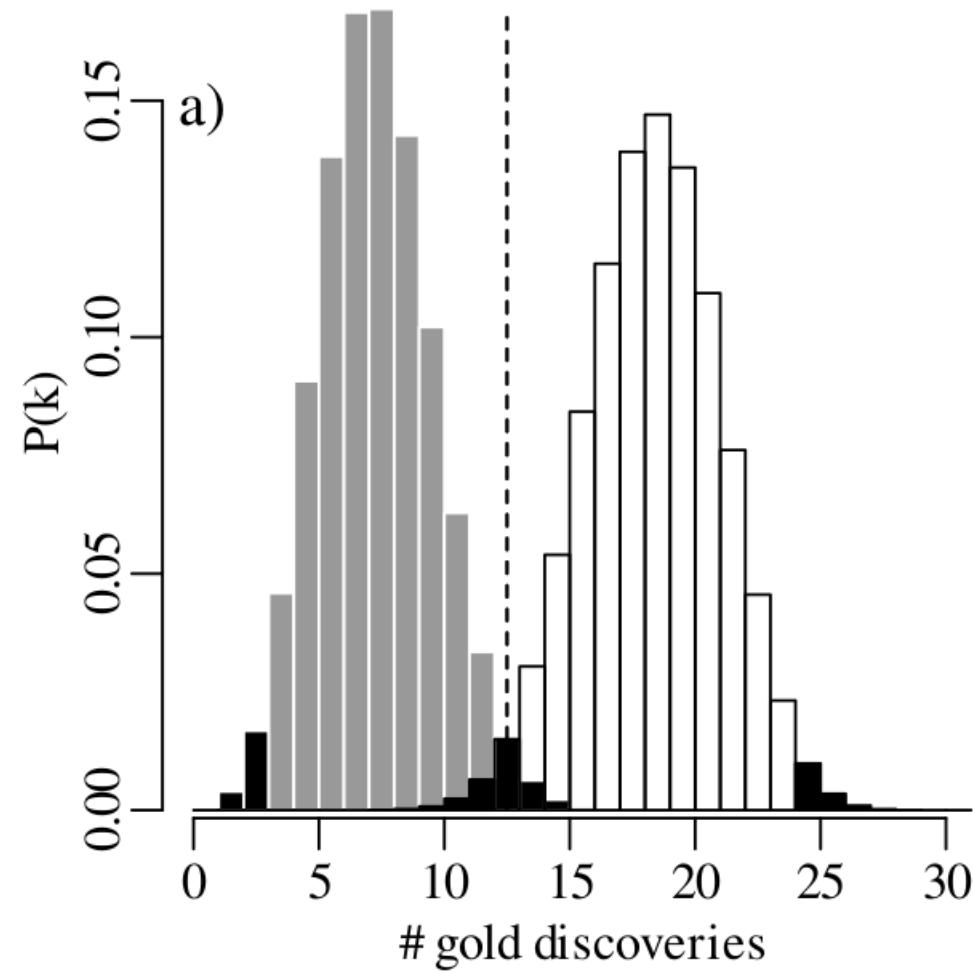
$$\text{C.I.}(p|k=2, n=5) = [0.053, 0.85]$$

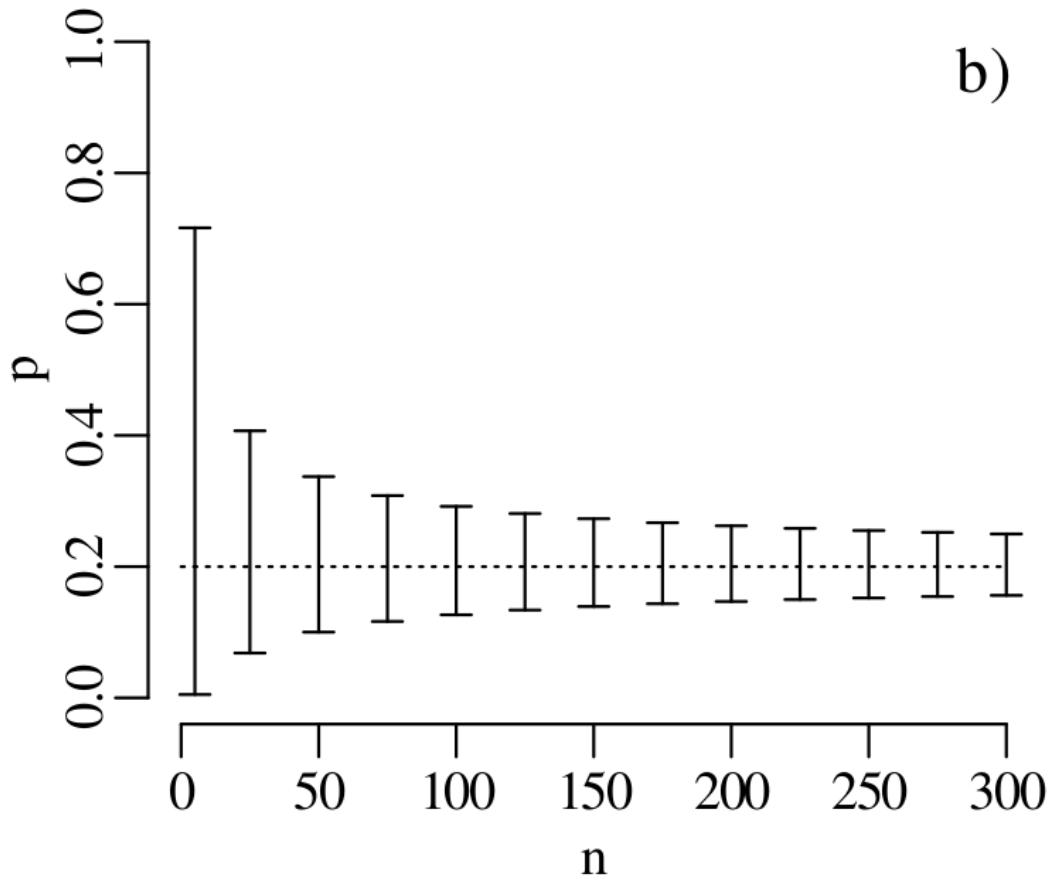
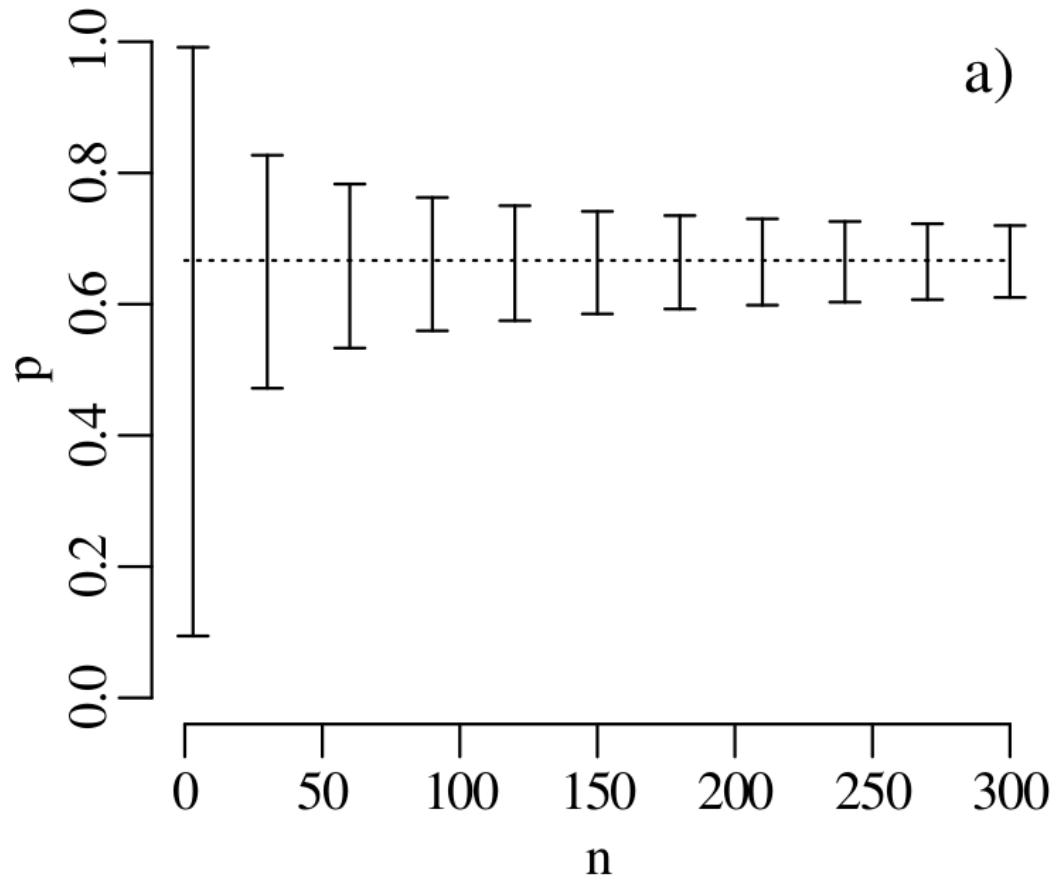


$$\text{C.I.}(p|k=4, n=5) = [0.284, 0.995]$$



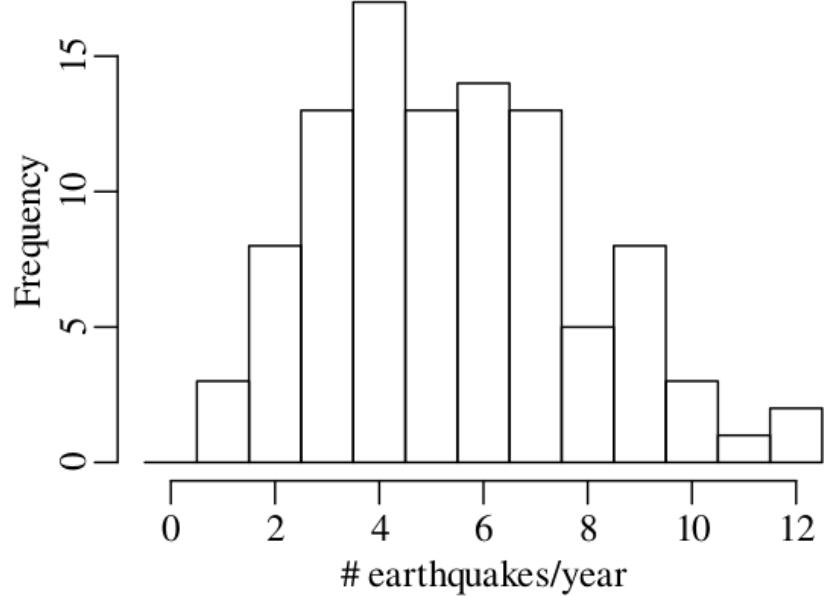
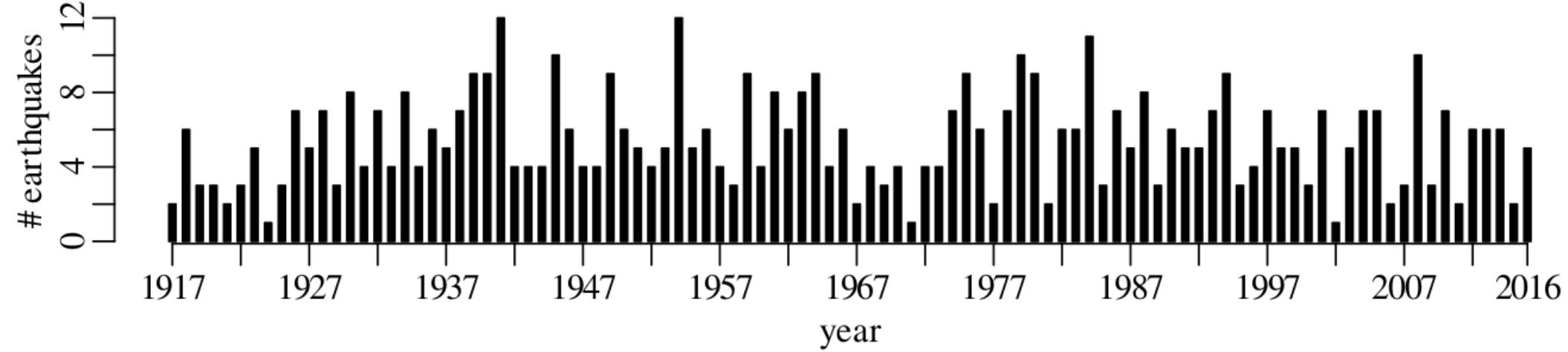
$$\text{C.I.}(p|k = 12, n = 30) = [0.23, 0.59]$$





# Statistics for geoscientists

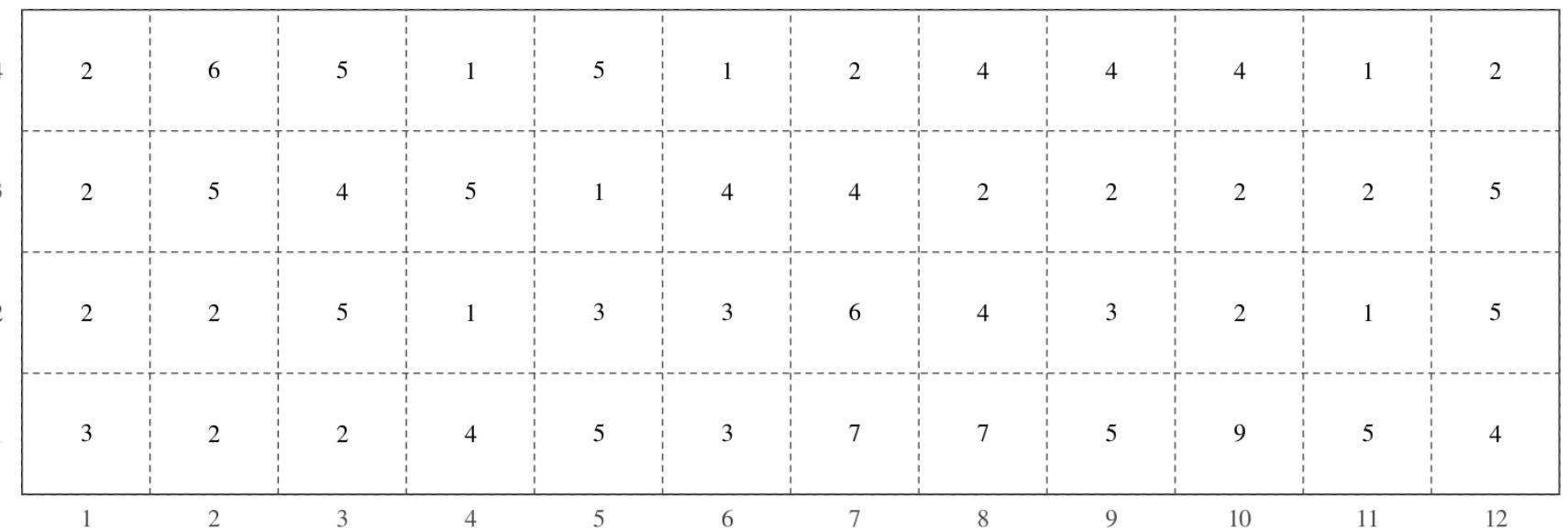
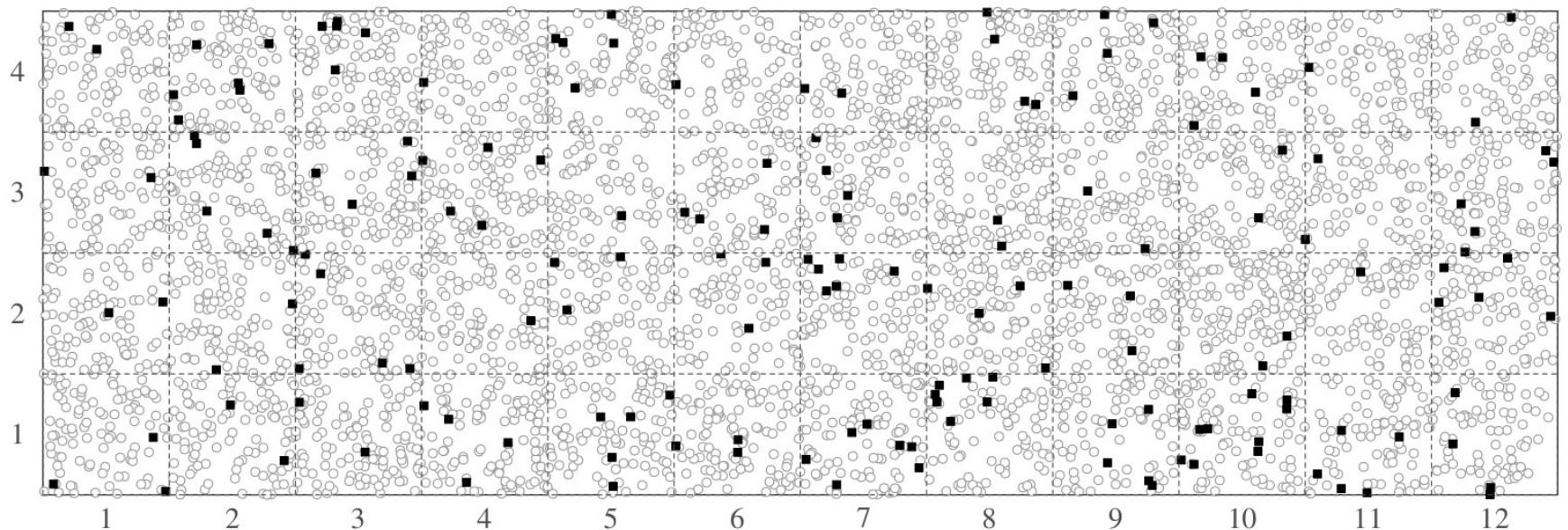
## The Poisson distribution

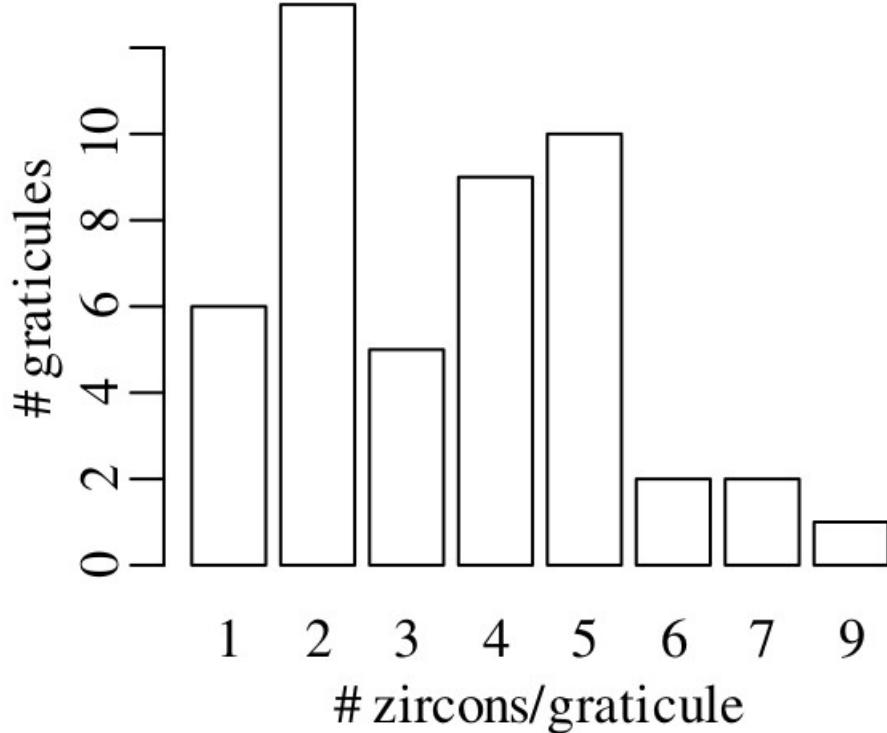


**mean: 5.43**

standard deviation: 2.50

**variance: 6.24**





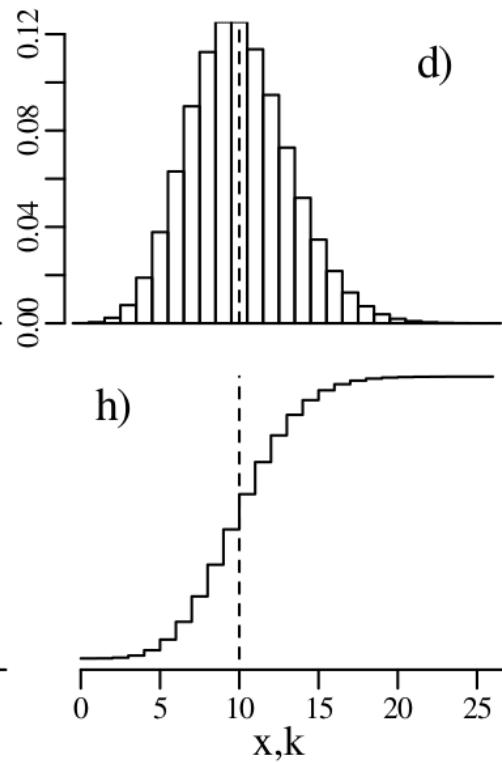
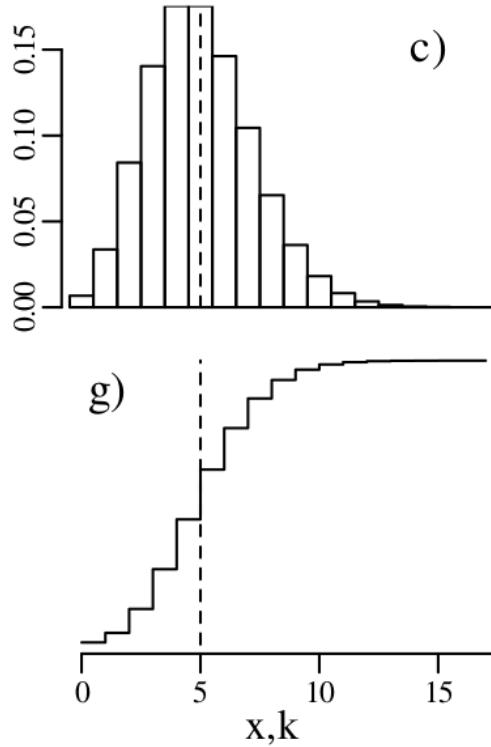
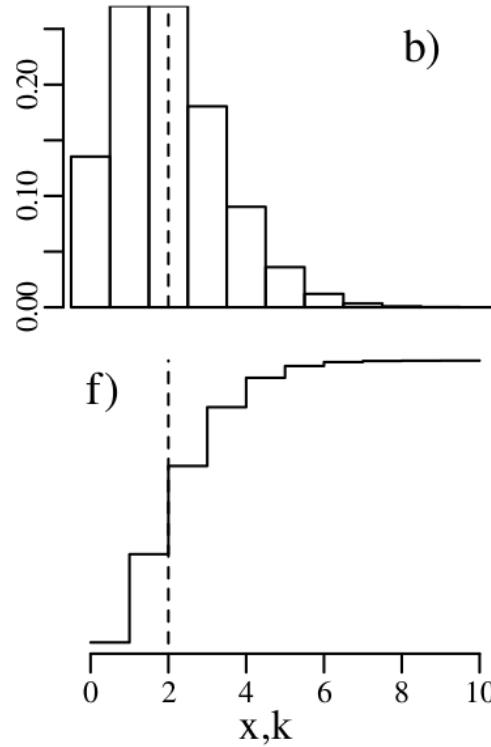
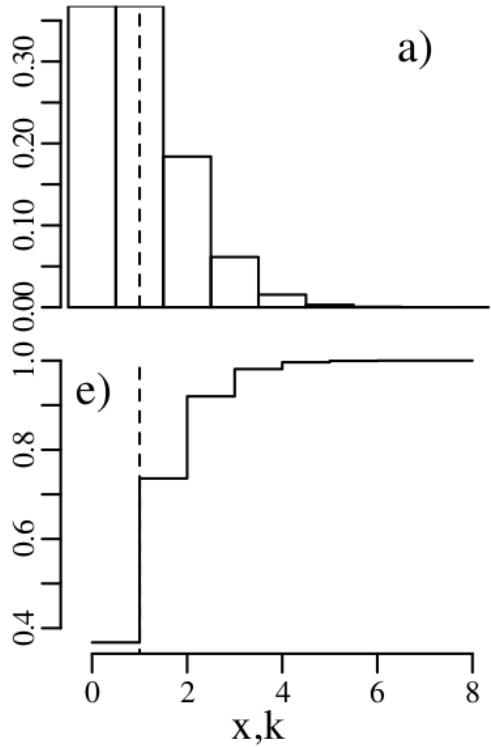
**mean: 3.50**

standard deviation: 1.85

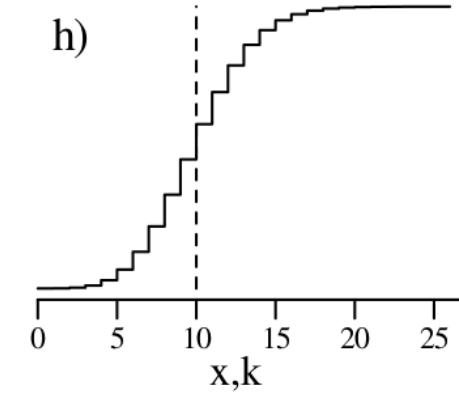
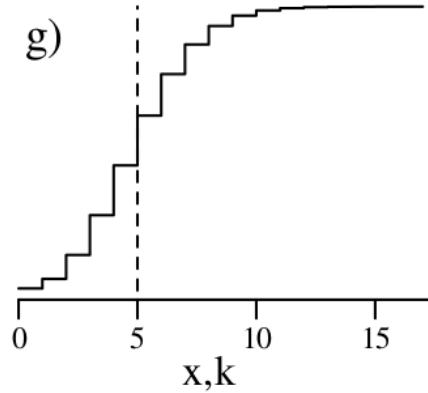
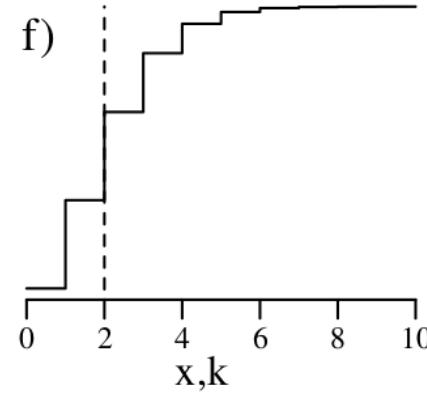
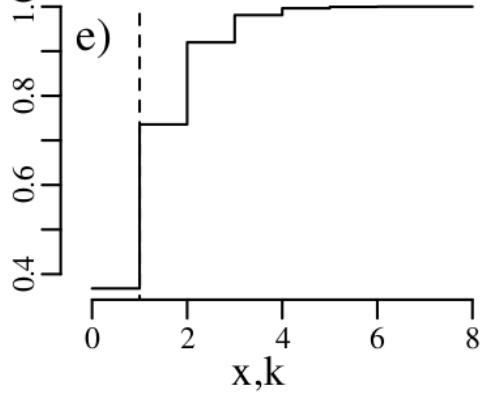
**variance: 3.40**

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$P(k)$



$F(x)$



$$P(k|n, p) = \binom{n}{k} p^k (1-p)^{n-k} \quad \xrightarrow{\begin{array}{l} n \rightarrow \infty \\ p \rightarrow 0 \end{array}} \quad P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{where} \quad \lambda = np$$

$n$	10	20	50	100	200	500	1000	2000	5000	10000
$p$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
$P(k \leq 2)$	0.0547	0.0913	0.112	0.118	0.121	0.1234	0.124	0.1243	0.1245	0.1246

## Examples of Poisson variables

1. people killed by lightning per year;
2. mutations in DNA per generation;
3. radioactive disintegrations per unit time;
4. mass extinctions per 100 million years.

~~—earthquakes including aftershocks—~~

~~—floods per year~~

# Parameter estimation

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathcal{L}(\lambda|k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda = k$$

## Hypothesis tests

$k = 9$  zircons given  $\lambda = 3.5$

1.  $H_0$  (null hypothesis)  $\lambda = 3.5$

$H_a$  (alternative hypothesis):  $\lambda > 3.5$

2. test statistic :  $k = 9$  successes

3. The null distribution

k	0	1	2	3	4	5	6	7	8	9	10
$P(T = k)$	0.030	0.106	0.185	0.216	0.189	0.132	0.077	0.038	0.017	0.007	0.002
$P(T \geq k)$	1.000	0.970	0.864	0.679	0.463	0.275	0.142	0.065	0.027	0.010	0.003

4. significance level  $\alpha = 0.05$

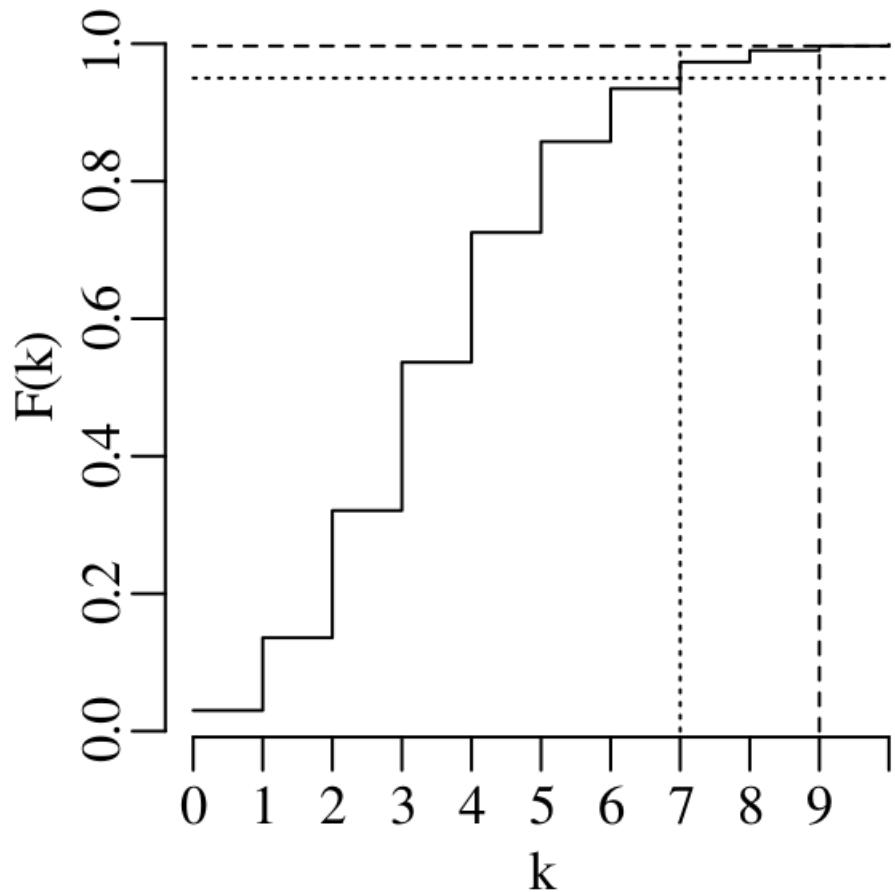
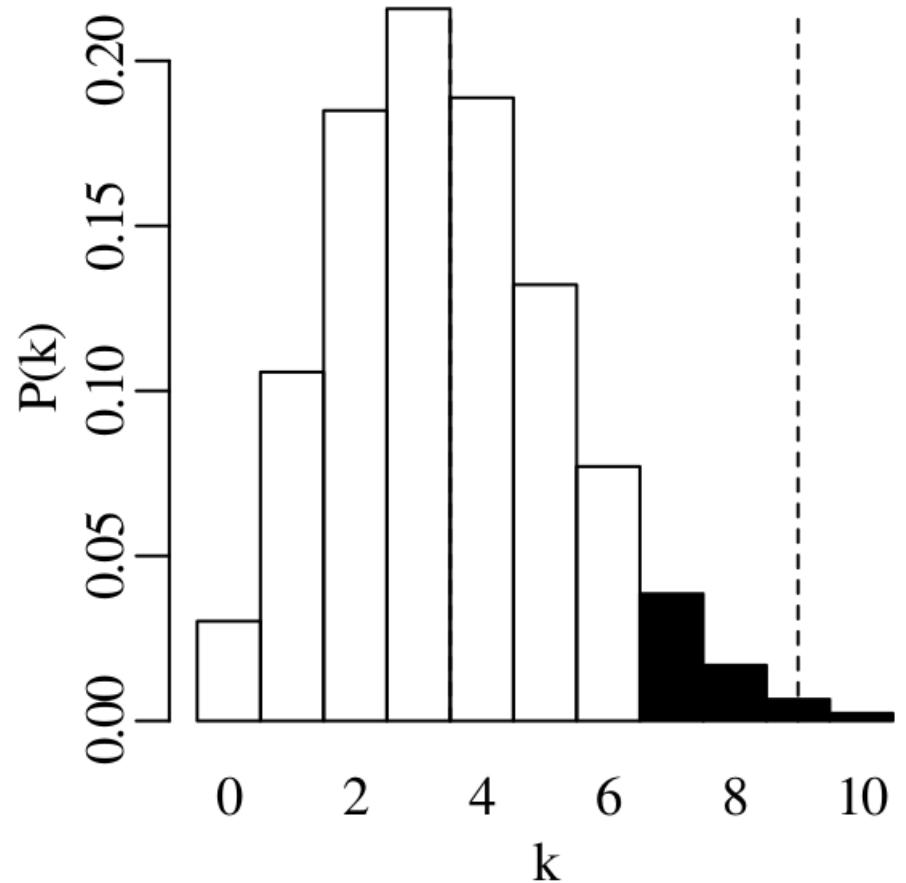
5. rejection region

k	0	1	2	3	4	5	6	7	8	9	10
$P(T = k)$	0.030	0.106	0.185	0.216	0.189	0.132	0.077	0.038	0.017	0.007	0.002
$P(T \geq k)$	1.000	0.970	0.864	0.679	0.463	0.275	0.142	0.065	<b>0.027</b>	<b>0.010</b>	<b>0.003</b>

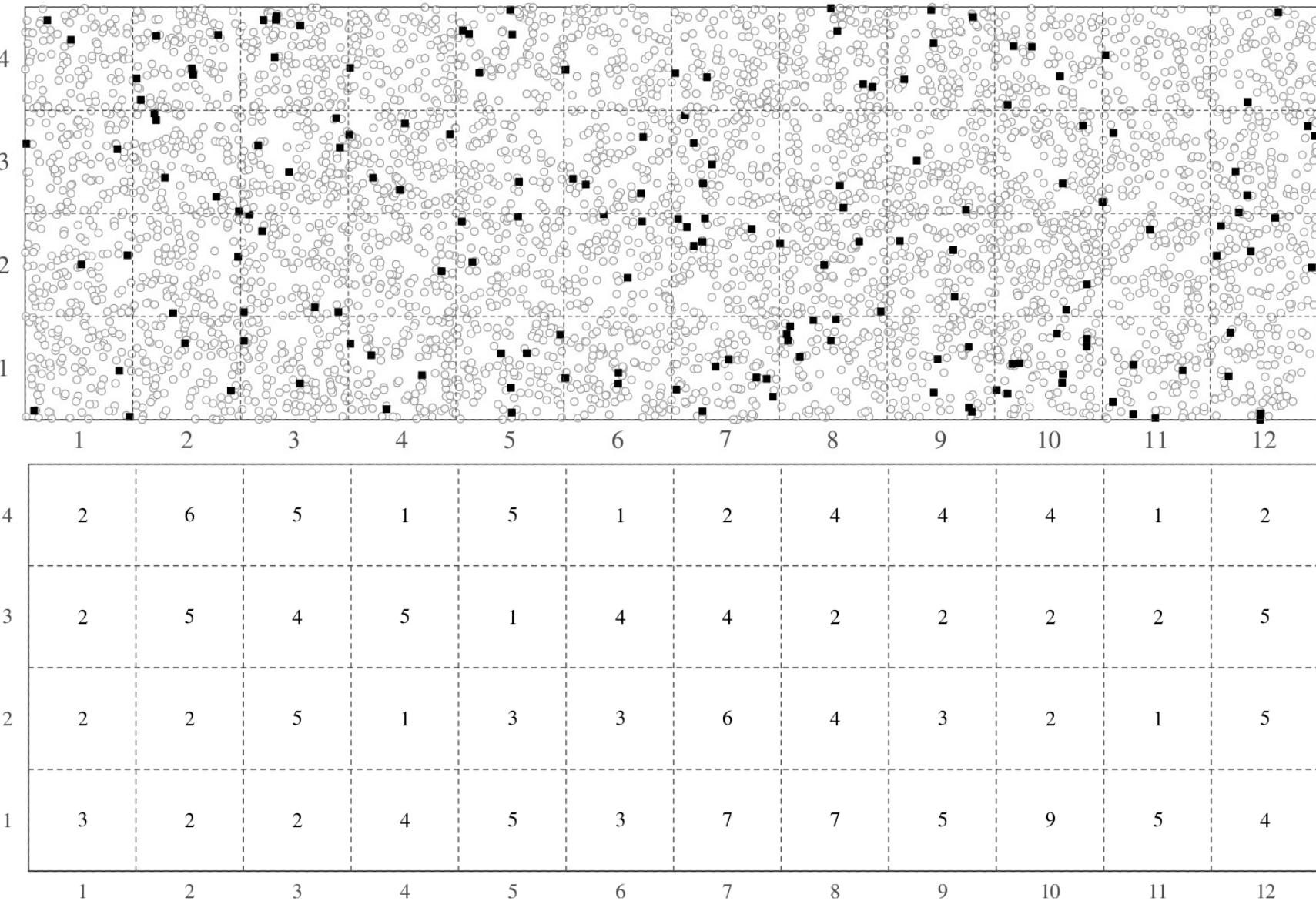
$$R = \{8, 9, 10, \dots, \infty\}$$

6.  $k \in R$

7. p-value =  $0.010 < \alpha$



# Multiple testing



Assuming that  $H_0$  is true, the probability of incurring a type-I error is:

$$1 \text{ experiment: } \alpha = 5\%$$

$$2 \text{ experiments: } 1 - (1 - \alpha)^2 = 9.75\%$$

$$3 \text{ experiments: } 1 - (1 - \alpha)^3 = 14\%$$

$$48 \text{ experiments: } 1 - (1 - \alpha)^{48} = 91.5\%$$

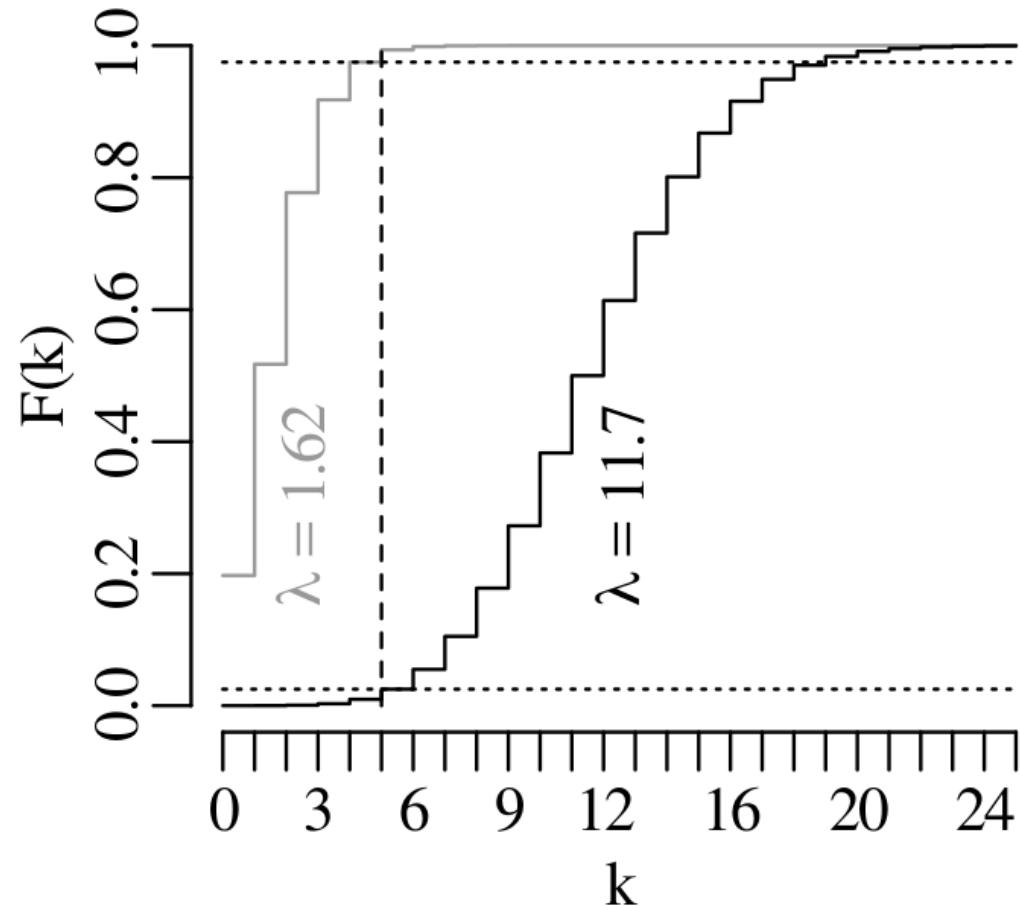
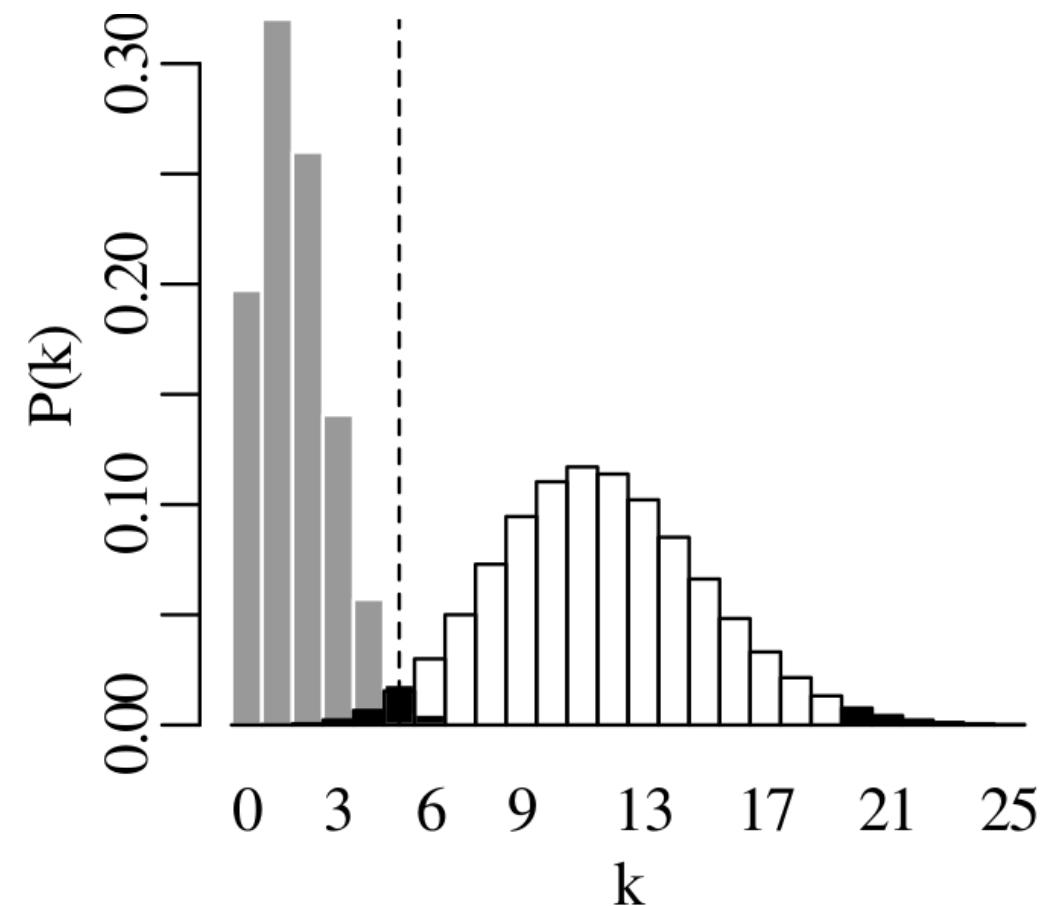
$$N \text{ experiments: } 1 - (1 - \alpha)^N\%$$

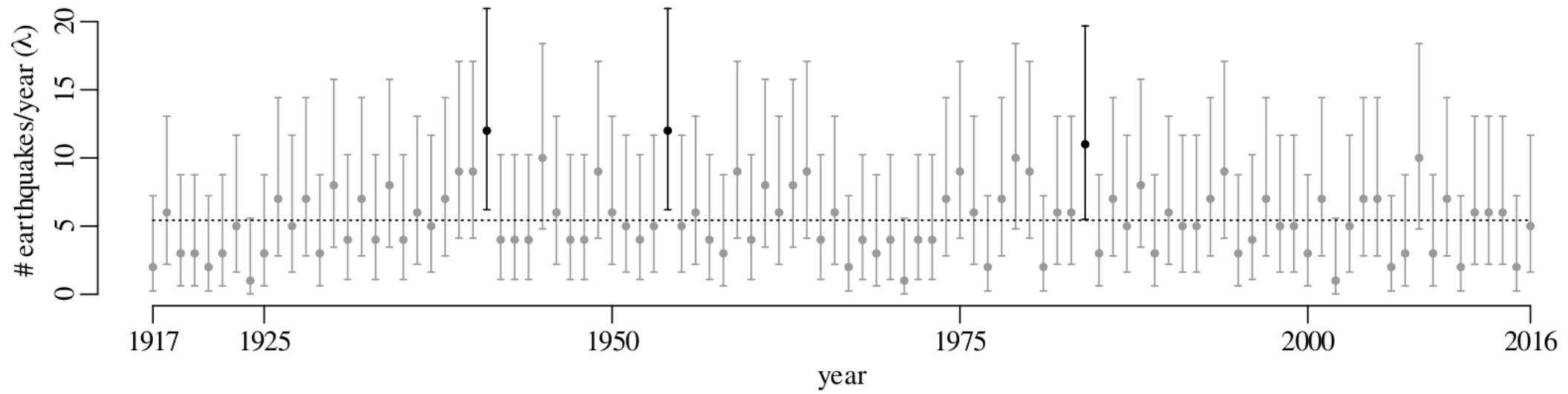
Bonferroni correction:  $\alpha/N$

# Confidence intervals

5 magnitude 5.0 or greater earthquakes occurred in the US in 2016

What is  $\lambda$ ?





# Statistics for geoscientists

The normal distribution

**the binomial distribution**

$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

**the Poisson distribution**

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

**the negative binomial distribution**

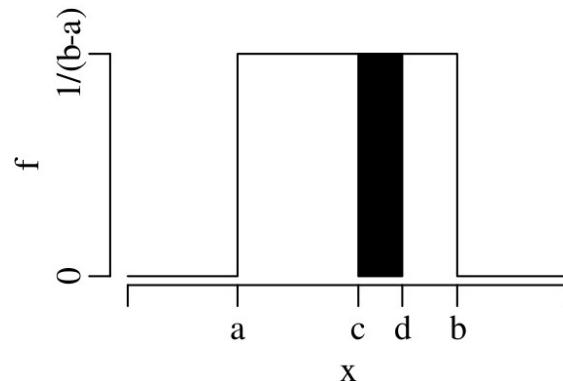
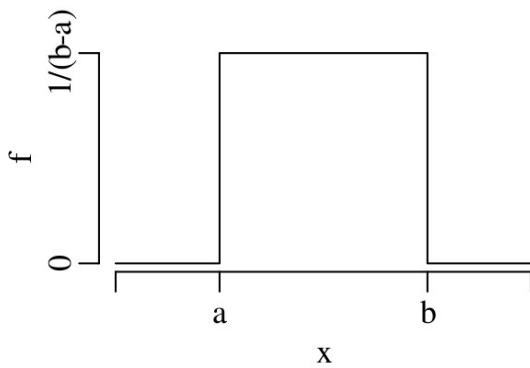
$$P(x|r,p) = \binom{r+x-1}{x} (1-p)^x p^r$$

**the multinomial distribution**

$$P(k_1, k_2, \dots, k_m | p_1, p_2, \dots, p_m) = \frac{n!}{\prod_{i=1}^m k_i!} \prod_{i=1}^m p_i^{k_i}$$

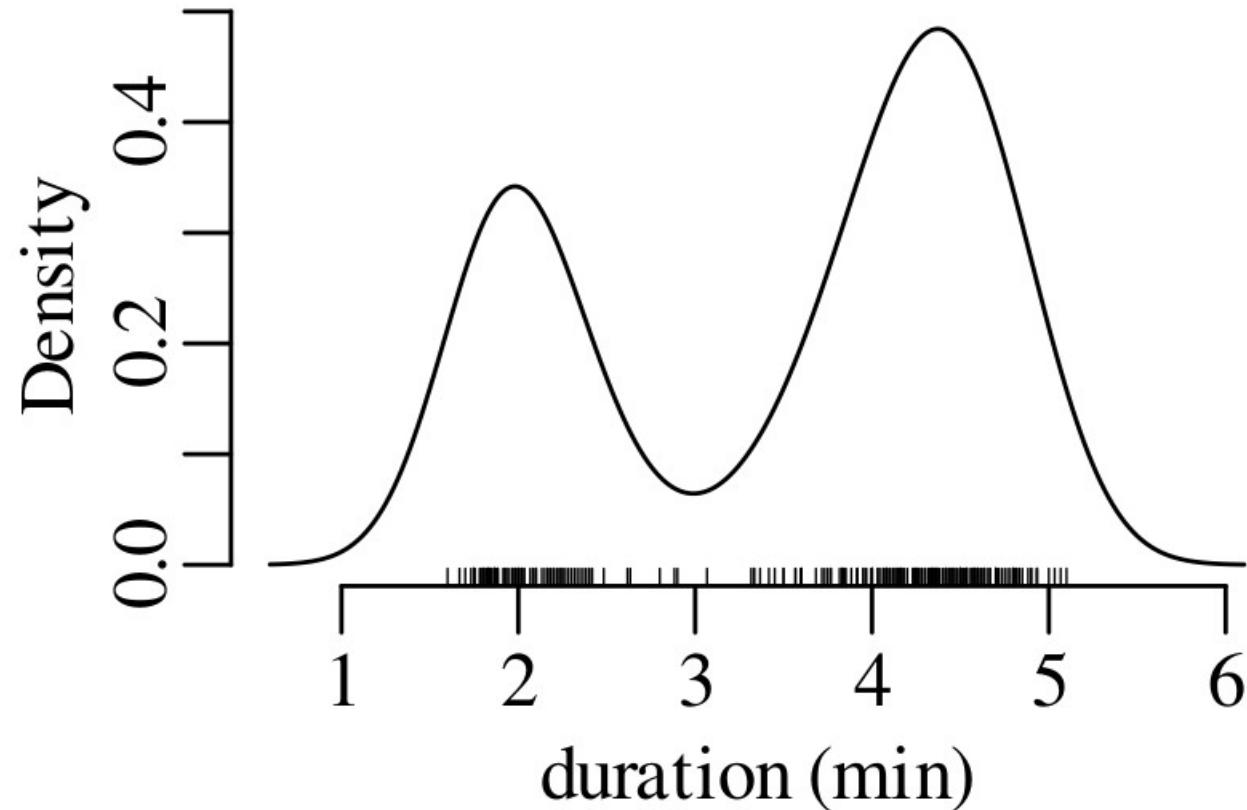
**the uniform distribution**

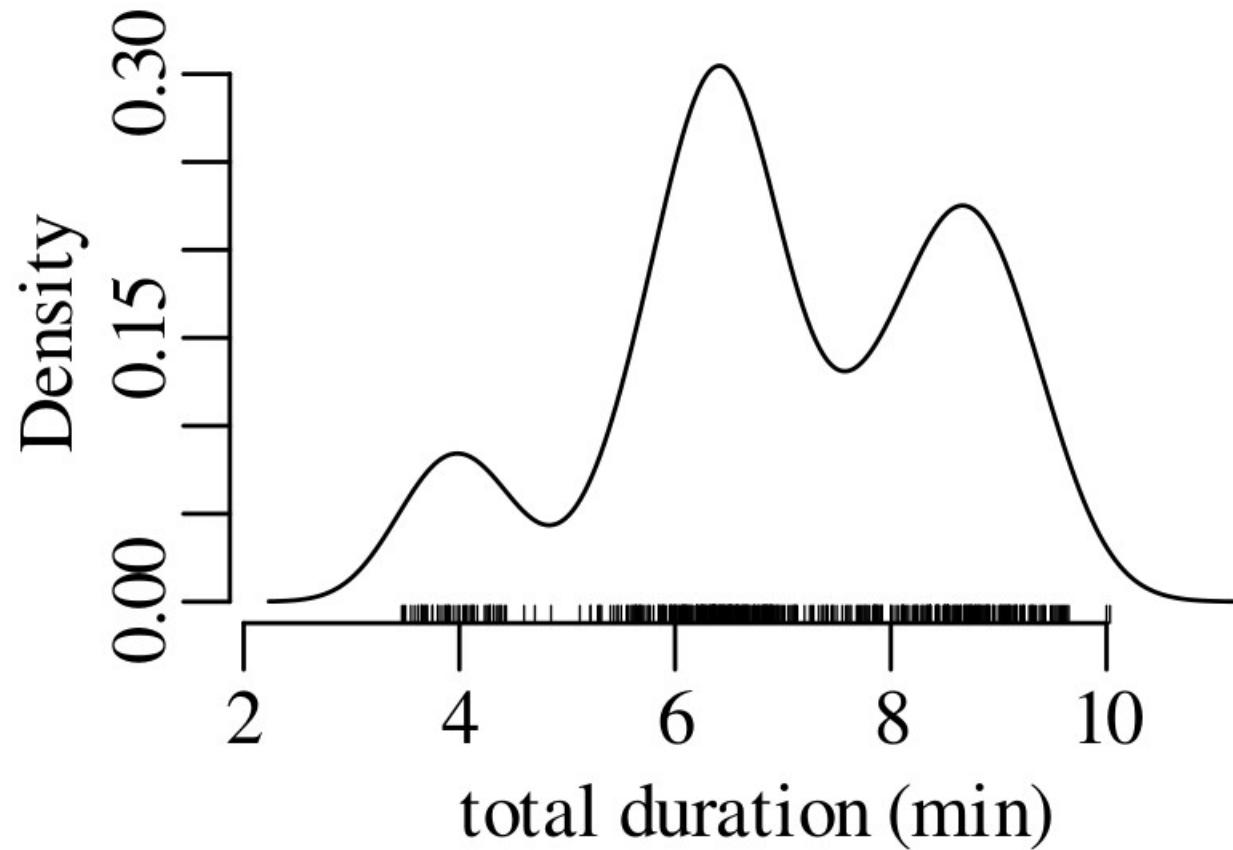
$$f(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

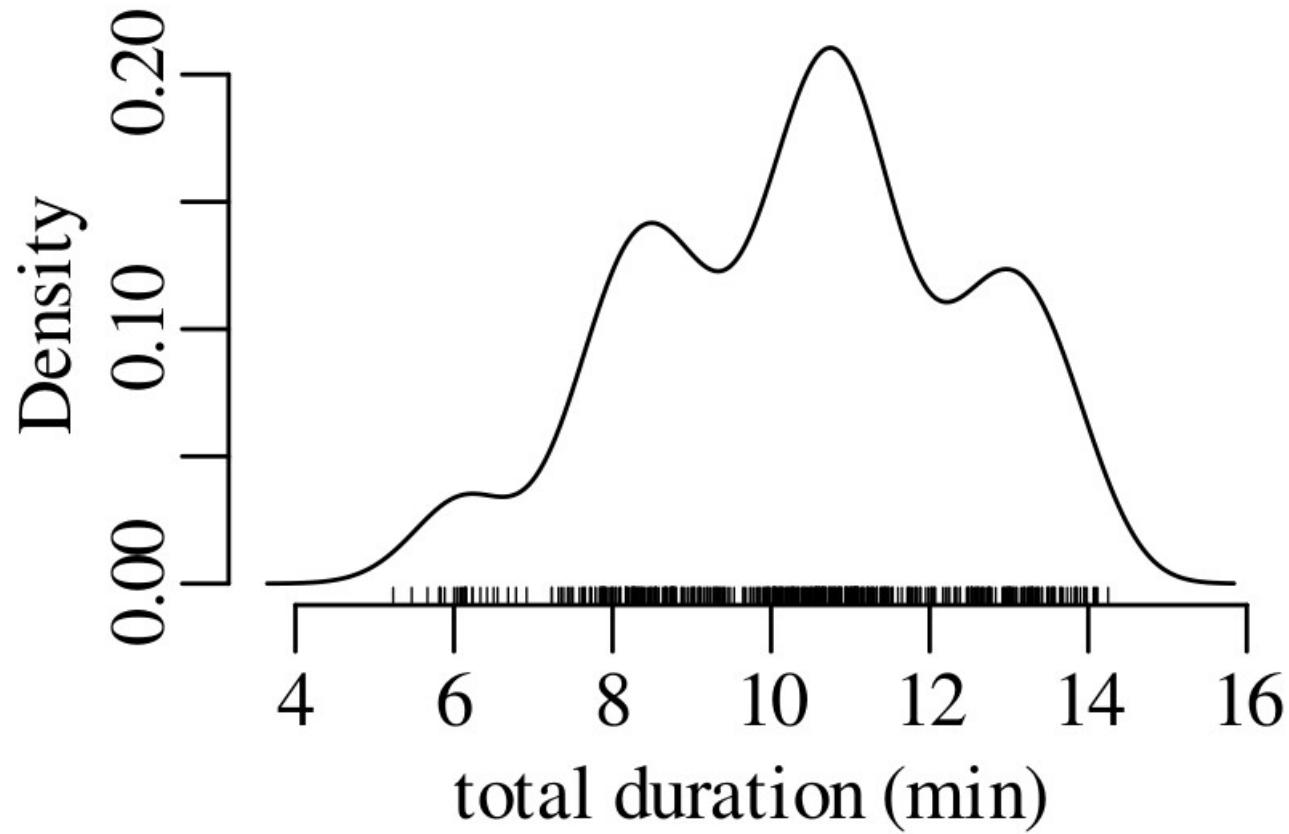


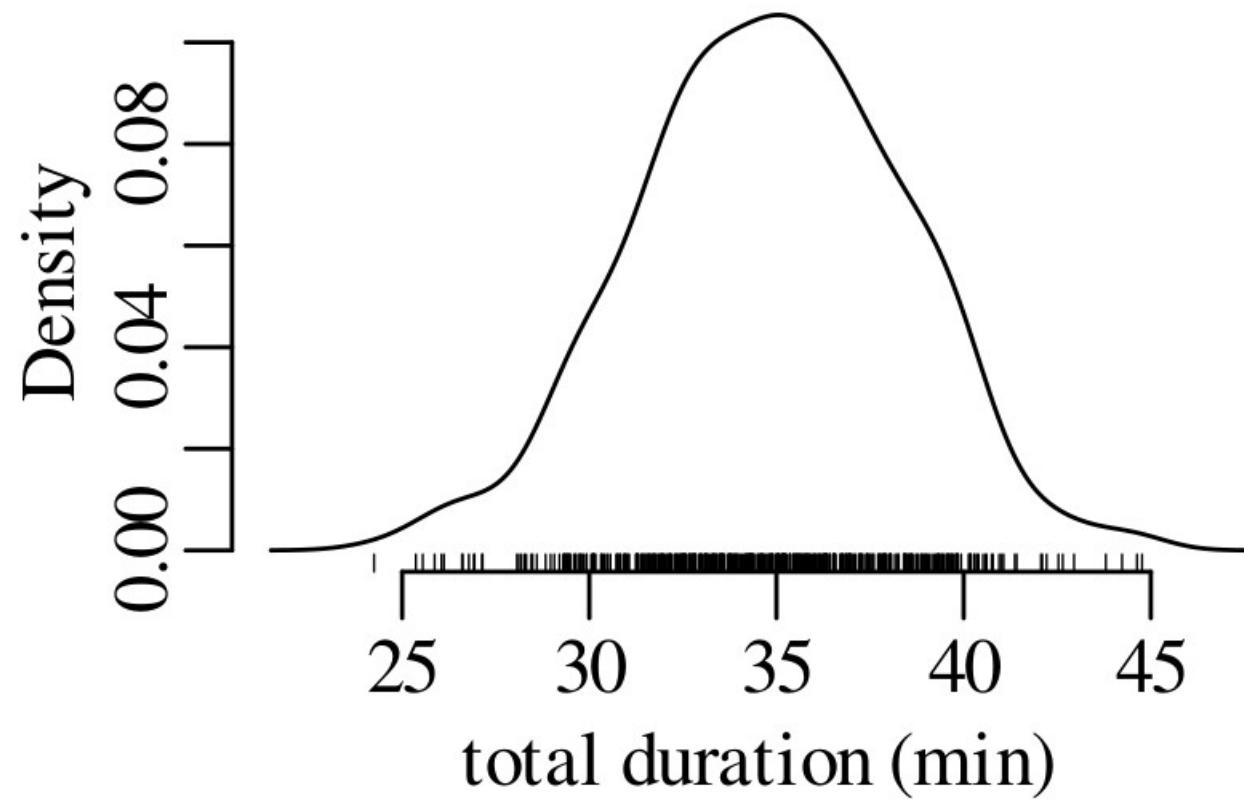
$$P(c \leq x \leq d) = \int_c^d f(x|a,b) dx$$

What is so normal about the Gaussian distribution?

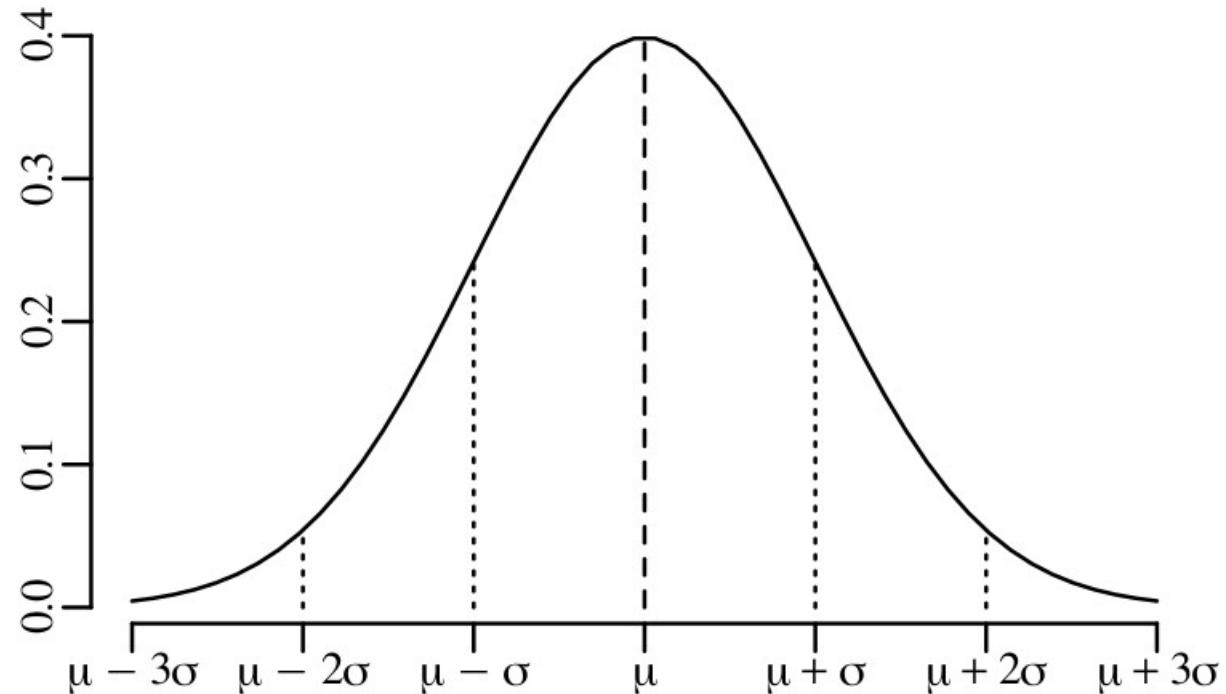


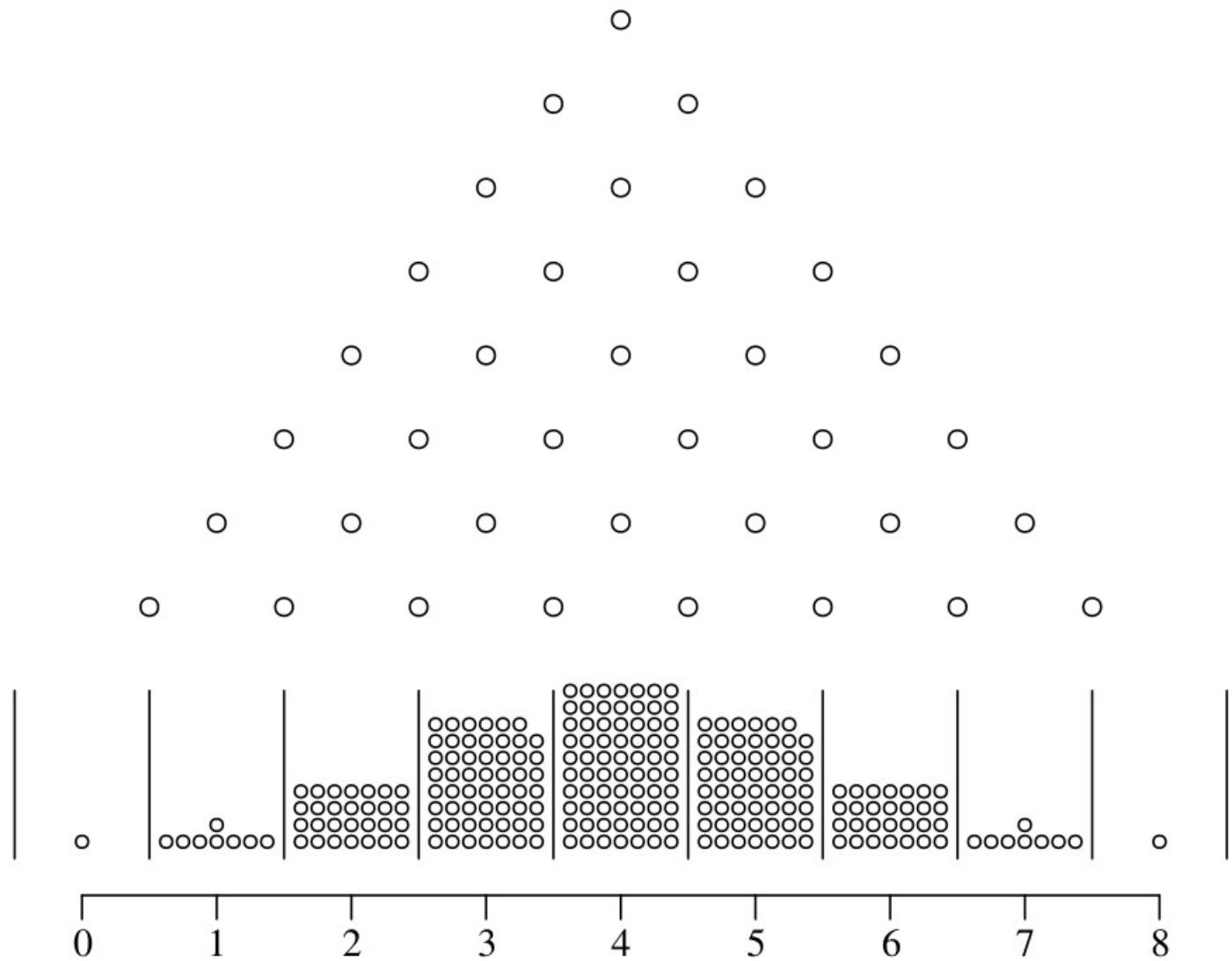


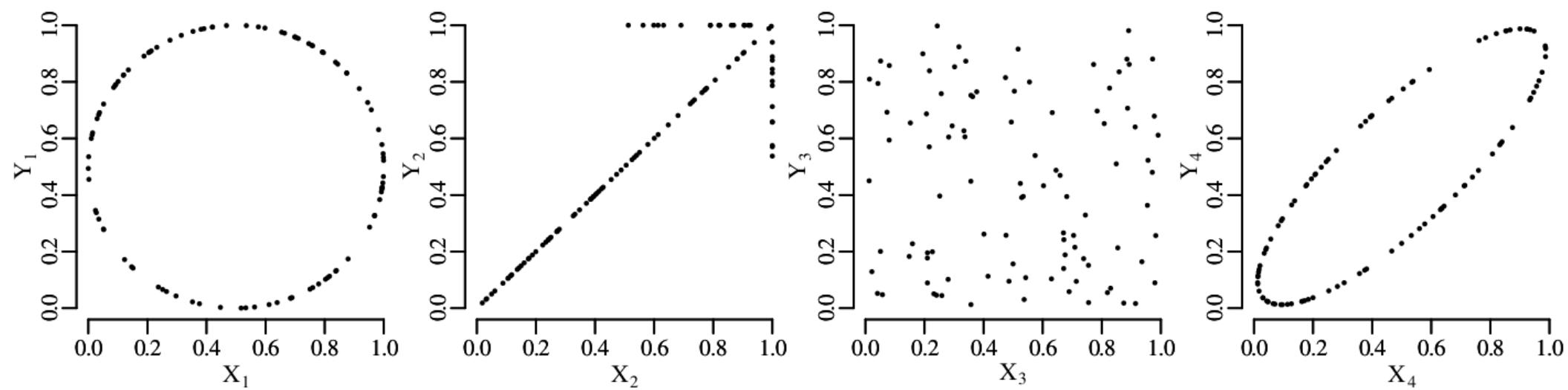
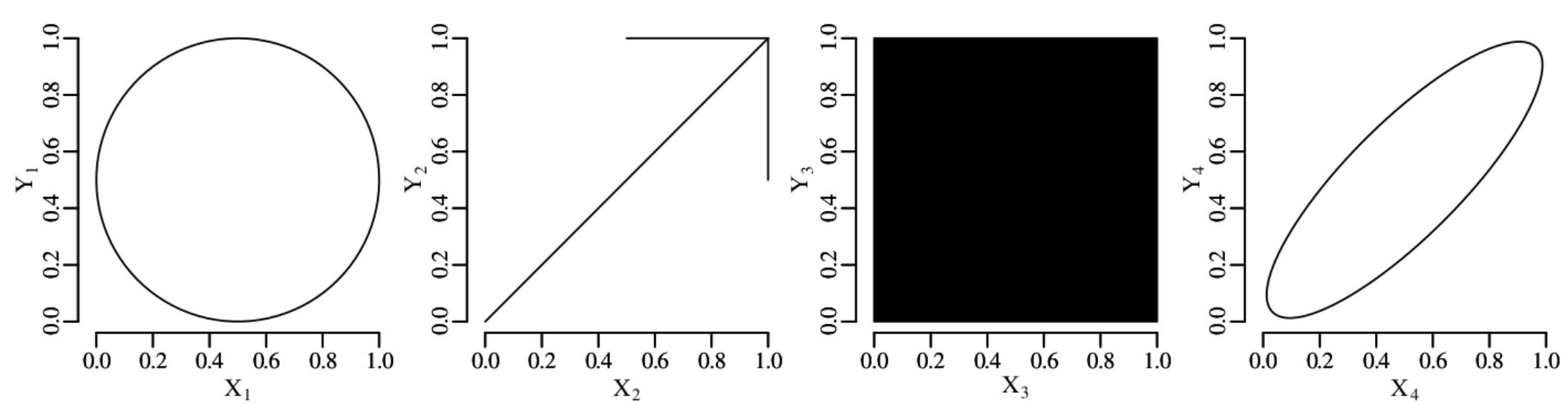


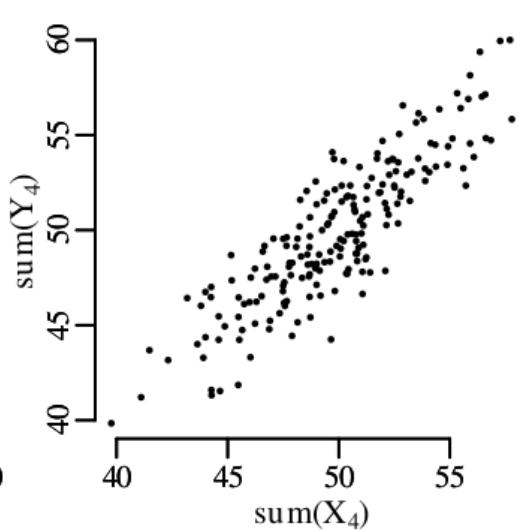
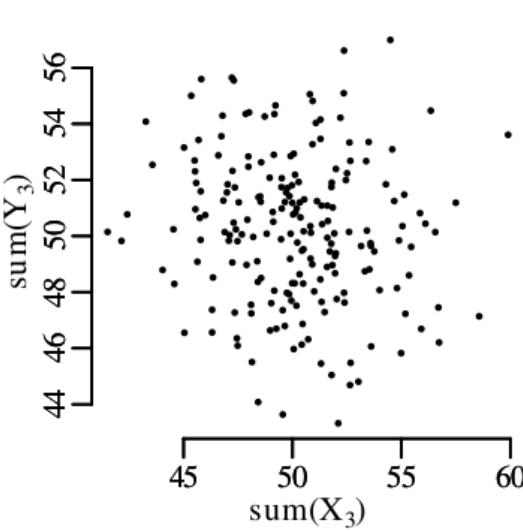
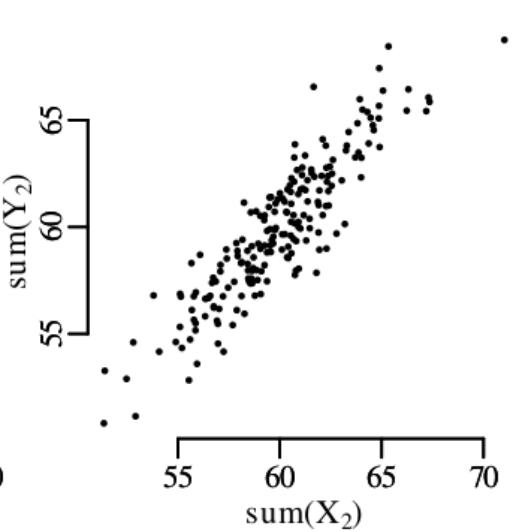
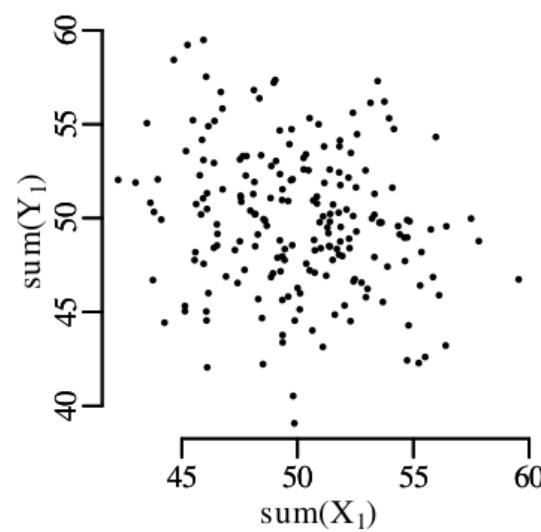
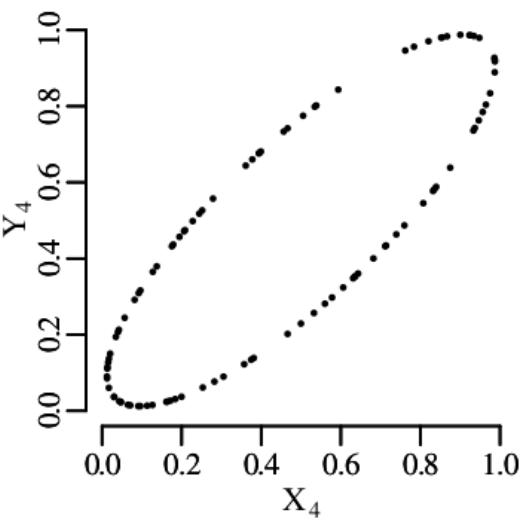
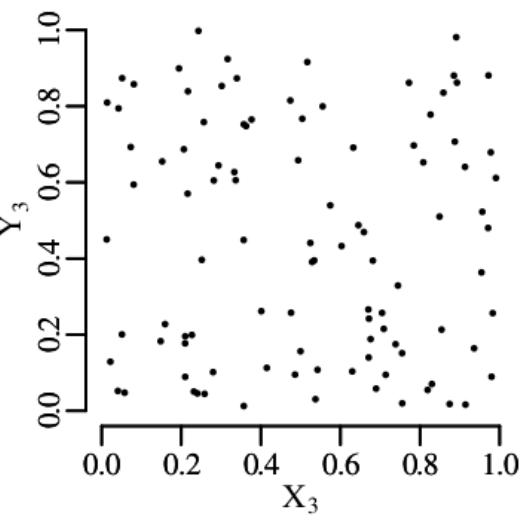
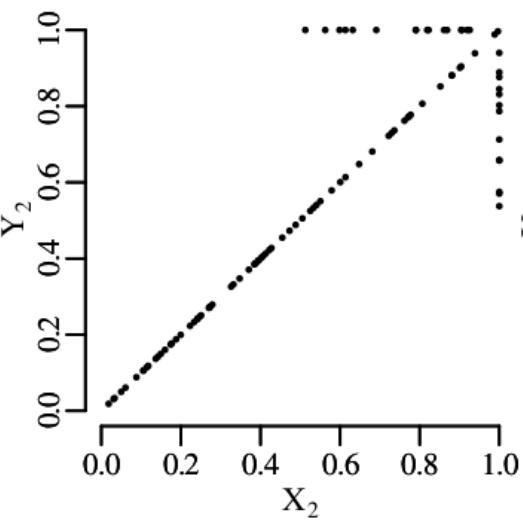
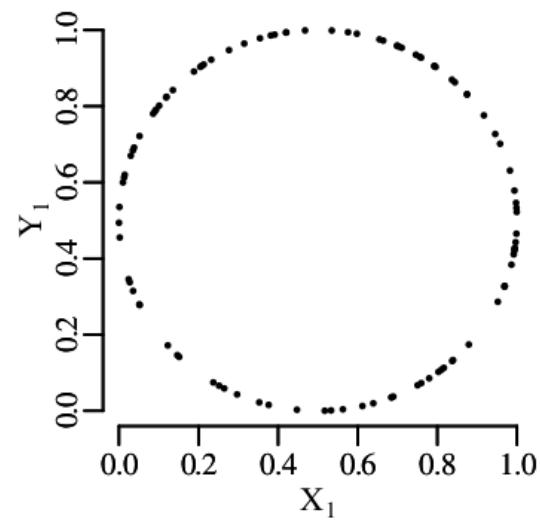


$$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

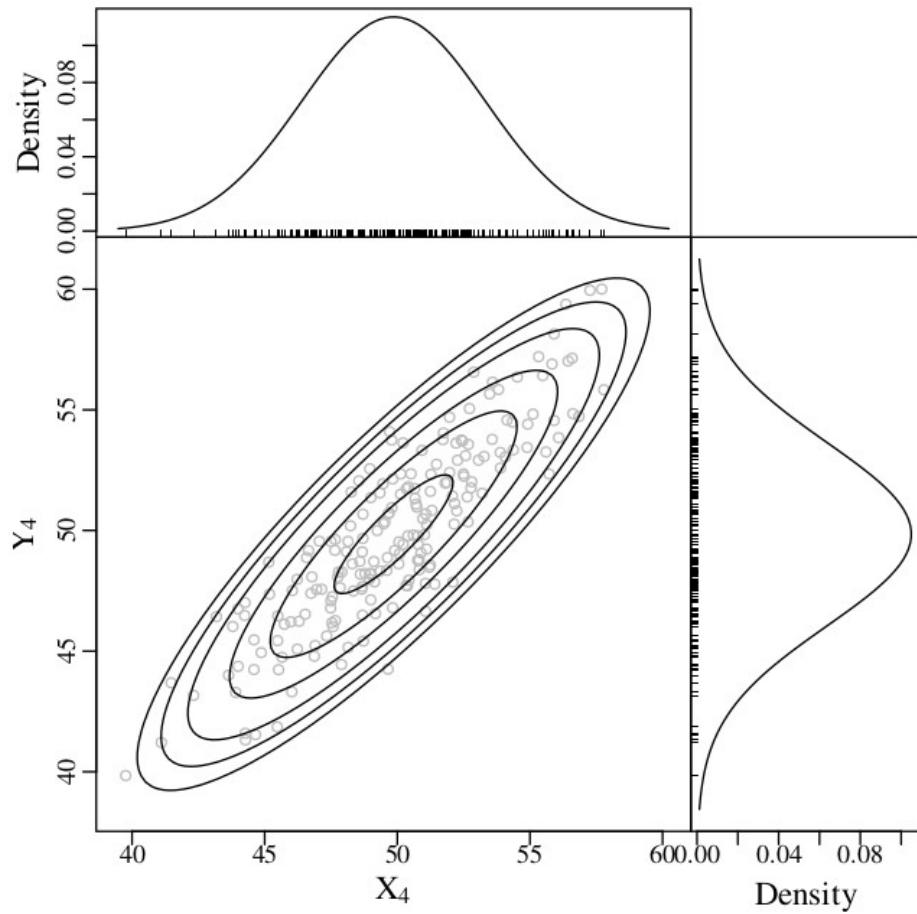




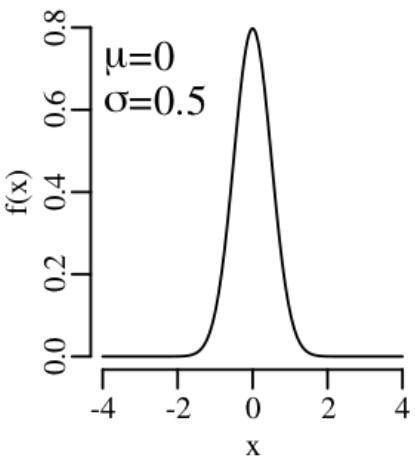
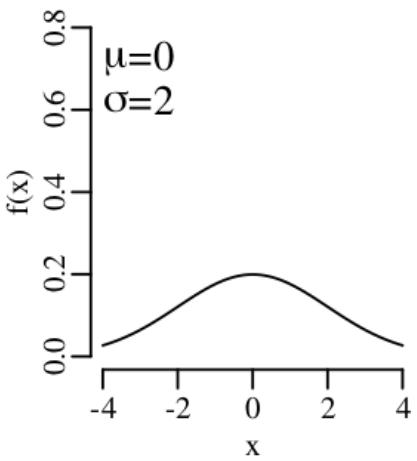
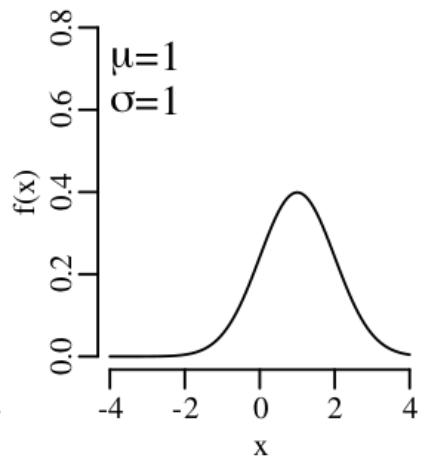
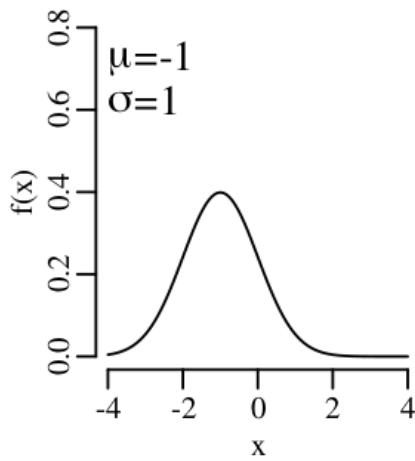
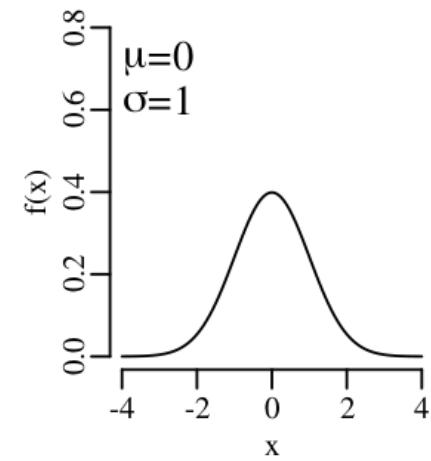




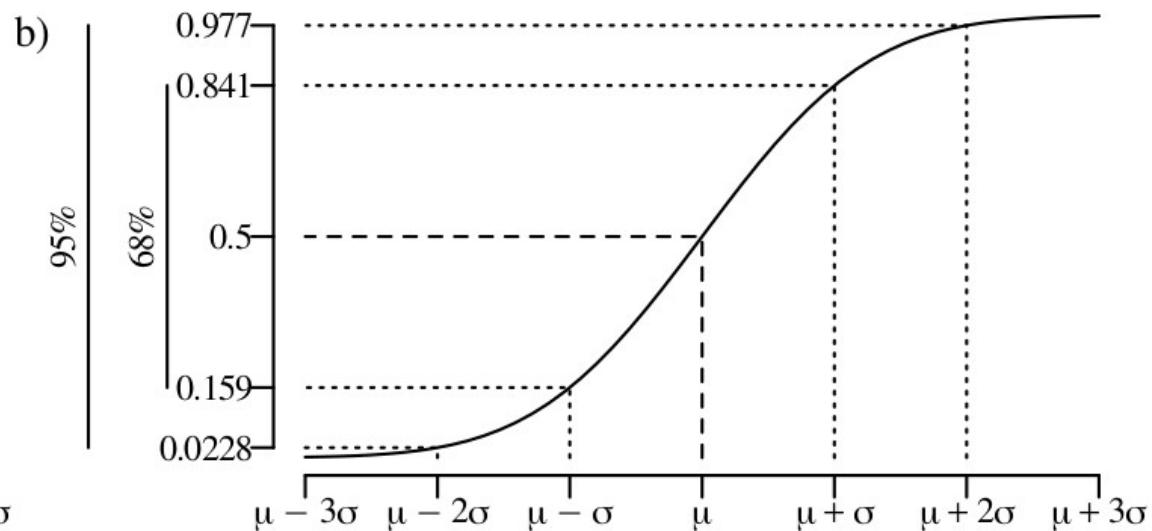
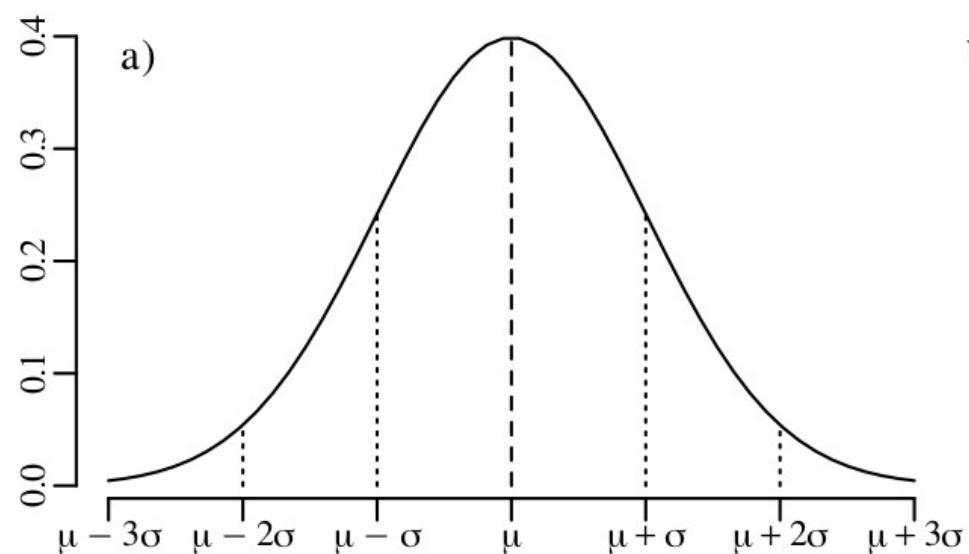
$$f(x, y | \mu_x, \mu_y, \sigma_x, \sigma_y, \sigma_{x,y}) = \frac{\exp\left(-\left[(x - \mu_x) \quad (y - \mu_y)\right] \begin{bmatrix} \sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix} / 2\right)}{2\pi \sqrt{\left|\begin{bmatrix} \sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2 \end{bmatrix}\right|}}$$



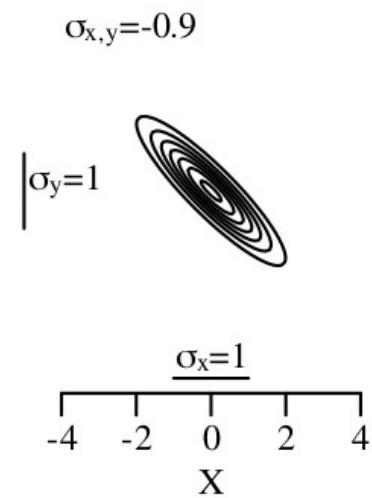
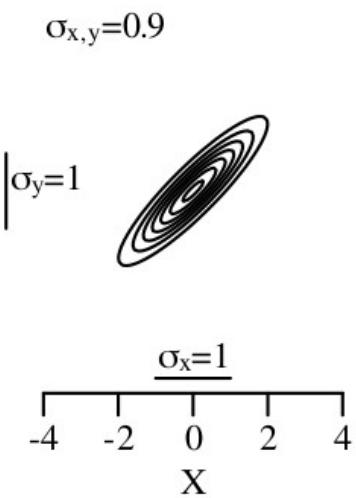
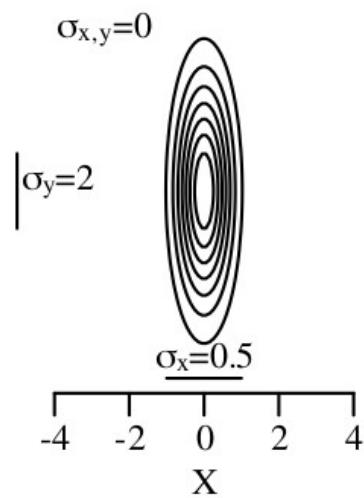
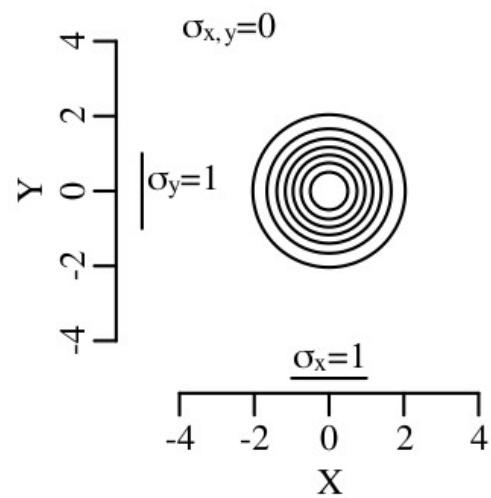
$$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$



$$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$



$$f(x, y | \mu_x, \mu_y, \sigma_x, \sigma_y, \sigma_{x,y}) = \frac{\exp\left(-\begin{bmatrix}(x - \mu_x) & (y - \mu_y)\end{bmatrix} \begin{bmatrix}\sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2\end{bmatrix}^{-1} \begin{bmatrix}x - \mu_x \\ y - \mu_y\end{bmatrix} / 2\right)}{2\pi \sqrt{\begin{vmatrix}\sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2\end{vmatrix}}}$$



# Parameter estimation

$$\mathcal{L}(\mu, \sigma | x_1, x_2, \dots, x_n) = \prod_{i=1}^n f(x_i | \mu, \sigma) \quad \text{with} \quad f(x | \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i \qquad \longleftrightarrow \qquad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

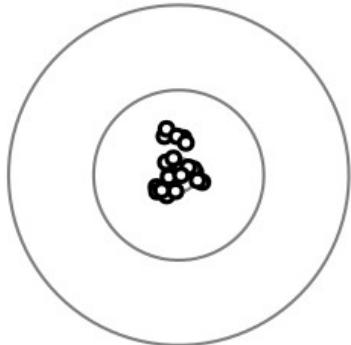
$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2} \qquad \longleftrightarrow \qquad s[x] = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\sigma}_{x,y} = \sum_{i=1}^n \frac{1}{n} (x_i - \mu_x)(y_i - \mu_y) \qquad \longleftrightarrow \qquad s[x, y] = \sum_{i=1}^n \frac{1}{n-1} (x_i - \bar{x})(y_i - \bar{y})$$

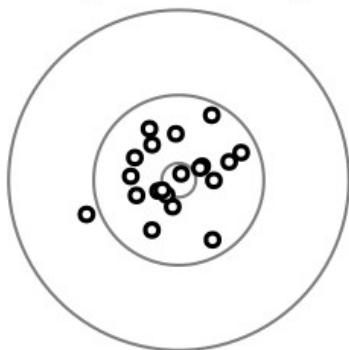
# Statistics for geoscientists

Error propagation

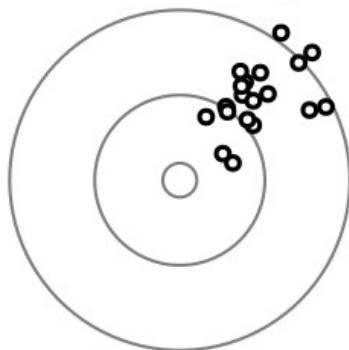
high precision  
high accuracy



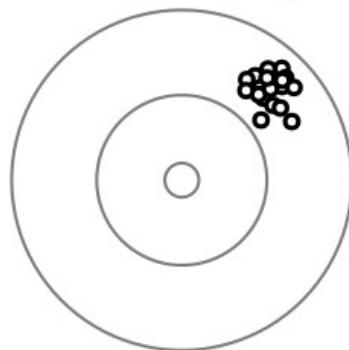
low precision  
high accuracy



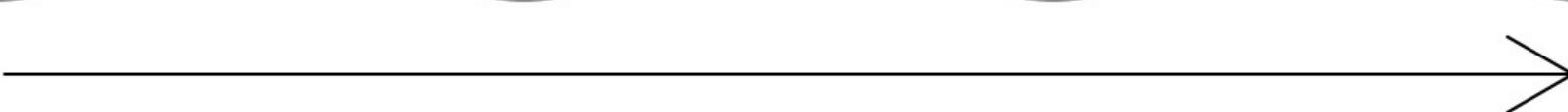
low precision  
low accuracy



high precision  
low accuracy



good



bad

$z = g(x) \rightarrow$  given  $x_i$ , for  $i = 1...n$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$s[x] = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

$\bar{z} = g(\bar{x}) \longrightarrow$  What is  $s[z]$ ?

# Linear approximation

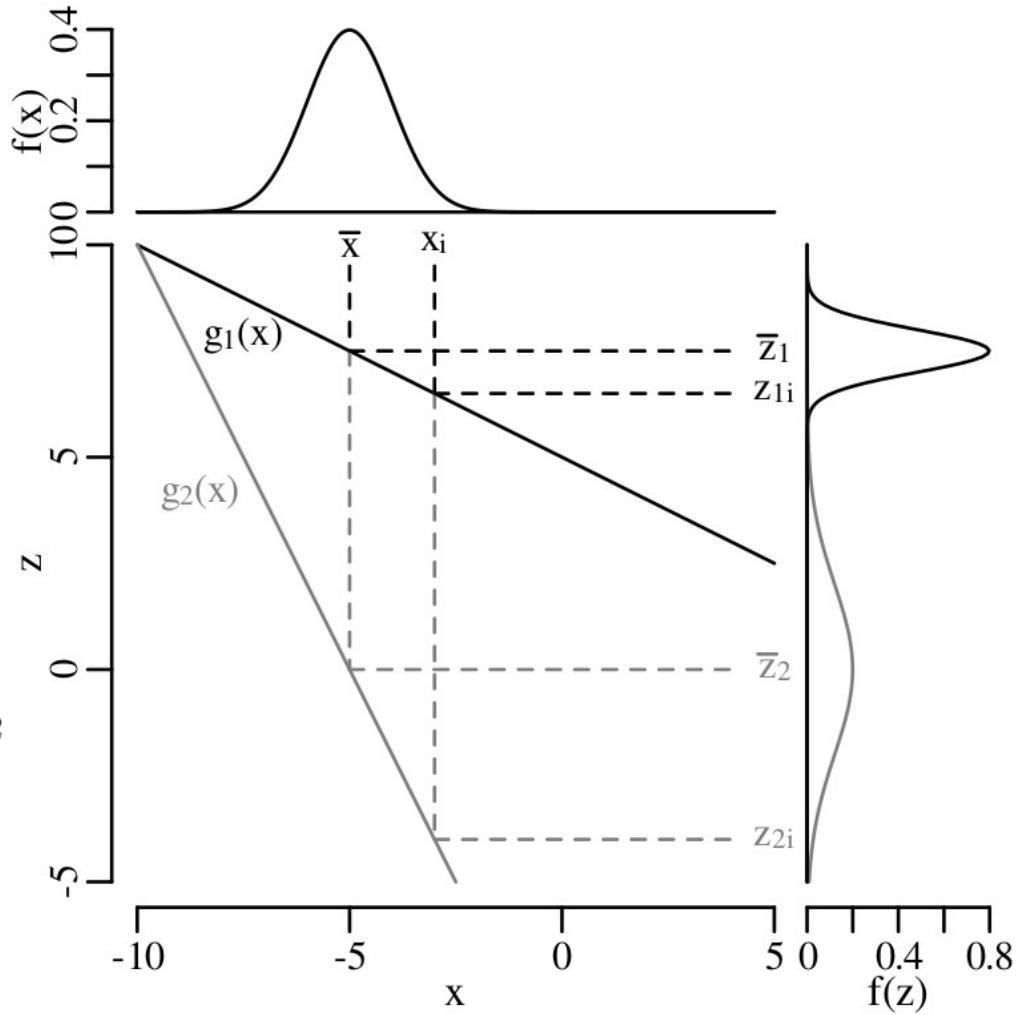
$$z = g(x)$$

$$(z_i - \bar{z}) \approx \frac{\partial z}{\partial x} (x_i - \bar{x})$$

$$s[z]^2 \equiv \frac{1}{n-1} \sum_{i=1}^n (z_i - \bar{z})^2$$

$$\begin{aligned} s[z]^2 &\approx \frac{1}{n-1} \sum_{i=1}^n \left[ (x_i - \bar{x}) \frac{\partial z}{\partial x} \right]^2 \\ &= \left[ \frac{\partial z}{\partial x} \right]^2 \left[ \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \right] = \left[ \frac{\partial z}{\partial x} \right]^2 s[x]^2 \end{aligned}$$

$$\Rightarrow s[z] = \left| \frac{\partial z}{\partial x} \right| s[x]$$

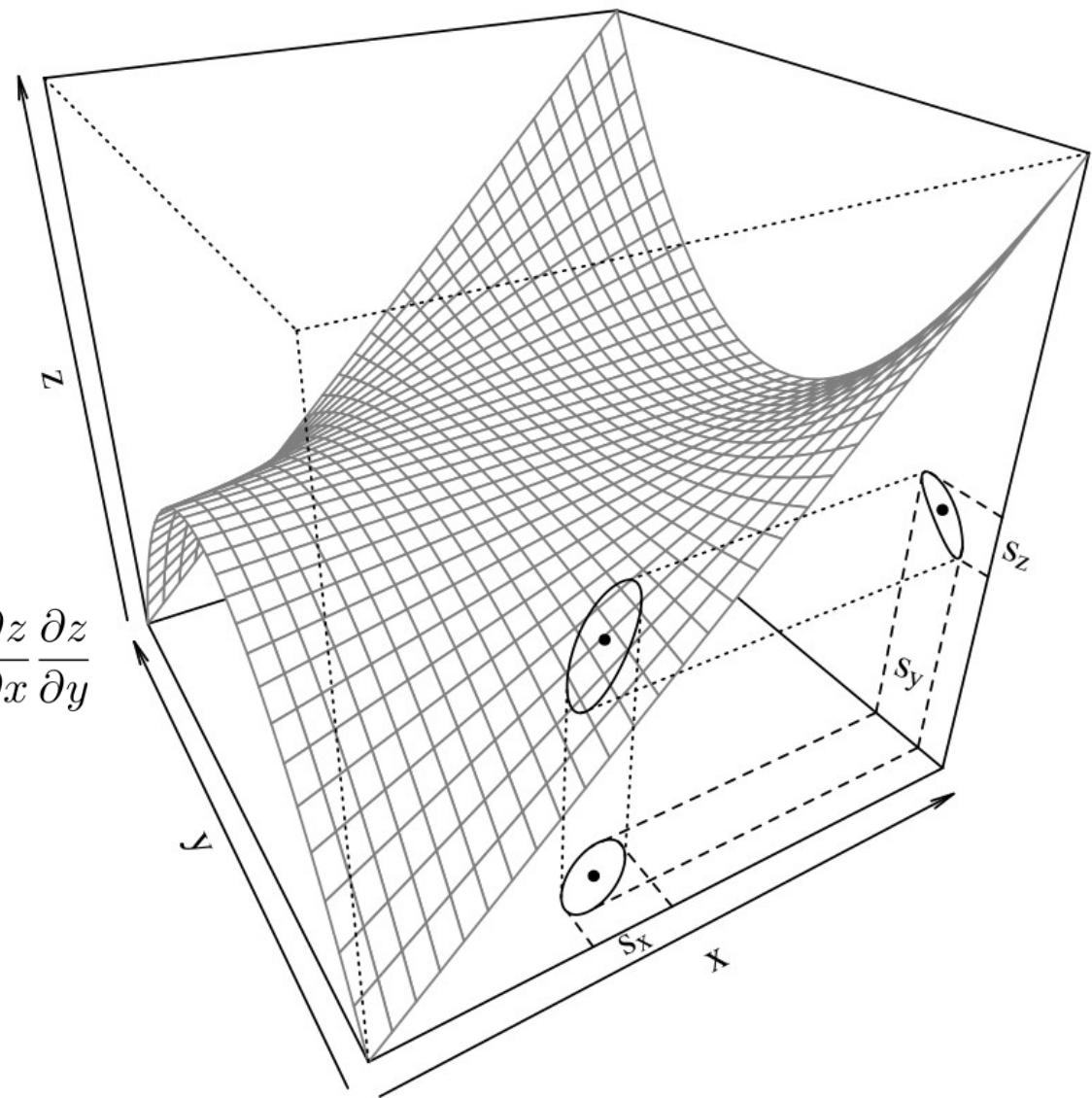


$$z = g(x, y)$$

$$s[z]^2 \approx \frac{1}{n-1} \sum_{i=1}^n \left[ (x_i - \bar{x}) \frac{\partial z}{\partial x} + (y_i - \bar{y}) \frac{\partial z}{\partial y} \right]^2$$

$$s[z]^2 \approx s[x]^2 \left( \frac{\partial z}{\partial x} \right)^2 + s[y]^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2 s[x, y] \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

$$s[z]^2 \approx \begin{bmatrix} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} s[x]^2 & s[x, y] \\ s[x, y] & s[y]^2 \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix}$$



$$s[z]^2 \approx s[x]^2 \left( \frac{\partial z}{\partial x} \right)^2 + s[y]^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2 s[x, y] \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

**addition**

$$z = a + bx + cy$$

$$\frac{\partial z}{\partial x} = b \text{ and } \frac{\partial z}{\partial y} = c$$

$$s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2 + 2bc s[x, y]$$

$$z = x + y \longrightarrow s[z]^2 = s[x]^2 + s[y]^2$$

$$s[z]^2 \approx s[x]^2 \left( \frac{\partial z}{\partial x} \right)^2 + s[y]^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2 s[x, y] \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

**subtraction**

$$z = ax - by$$

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = -b$$

$$s[z]^2 = a^2 s[x]^2 + b^2 s[y]^2 - 2ab s[x, y]$$

$$z = x - y \longrightarrow s[z]^2 = s[x]^2 + s[y]^2$$

$$s[z]^2 \approx s[x]^2 \left( \frac{\partial z}{\partial x} \right)^2 + s[y]^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2 s[x, y] \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

**multiplication**

$$z = axy$$

$$\frac{\partial z}{\partial x} = ay \text{ and } \frac{\partial z}{\partial y} = ax$$

$$s[z]^2 = (ay)^2 s[x]^2 + (ax)^2 s[y]^2 + 2(ay)(ax) s[x, y]$$

$$\left( \frac{s[z]}{z} \right)^2 = \left( \frac{ay}{axy} s[x] \right)^2 + \left( \frac{ax}{axy} s[y] \right)^2 + 2 \left( \frac{ay}{axy} \right) \left( \frac{ax}{axy} \right) s[x, y]$$

$$\left( \frac{s[z]}{z} \right)^2 = \left( \frac{s[x]}{x} \right)^2 + \left( \frac{s[y]}{y} \right)^2 + 2 \frac{s[x, y]}{xy}$$

**addition**       $z = a + bx + cy$        $s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2 + 2bc s[x, y]$

**subtraction**       $z = ax - by$        $s[z]^2 = a^2 s[x]^2 + b^2 s[y]^2 - 2ab s[x, y]$

**multiplication**       $z = axy$        $\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 + 2\frac{s[x, y]}{xy}$

**division**       $z = a\frac{x}{y}$        $\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 - 2\frac{s[x, y]}{xy}$

**exponentiation**       $z = ae^{bx}$        $\left(\frac{s[z]}{z}\right)^2 = b^2 s[x]^2$

**logarithms**       $z = a \ln[bx]$        $s[z]^2 = a^2 \left(\frac{s[x]}{x}\right)^2$

**power**       $z = ax^b$        $\left(\frac{s[z]}{z}\right)^2 = b^2 \left(\frac{s[x]}{x}\right)^2$

# the chain rule

$$d = d_o + v_o t + g t^2$$

can be written as

$$d = x + y \quad \text{where} \quad x \equiv d_o + v_o t \quad \text{and} \quad y \equiv g t^2$$
$$s[d] = \sqrt{s[x]^2 + s[y]^2}$$
$$s[x]^2 = (v_o s[t])^2$$
$$s[y]^2 = g^2 \left( \frac{s[t]}{t} \right)^2$$
$$s[d] = \sqrt{(v_o s[t])^2 + g^2 \left( \frac{s[t]}{t} \right)^2}$$

```
graph TD; A["d = x + y  
x ≡ d_o + v_o t  
y ≡ g t^2"] --> B["s[d] = sqrt(s[x]^2 + s[y]^2)"]; B --> C["s[x]^2 = (v_o s[t])^2"]; B --> D["s[y]^2 = g^2 (s[t]/t)^2"]; C --> E["s[d] = sqrt((v_o s[t])^2 + g^2 (s[t]/t)^2)"]; D --> E;
```

$$z = g(x)$$

$$\bar{x} = \sum_{i=1}^n \frac{x_i}{n} = \frac{1}{n} \sum_{i=1}^n x_i$$

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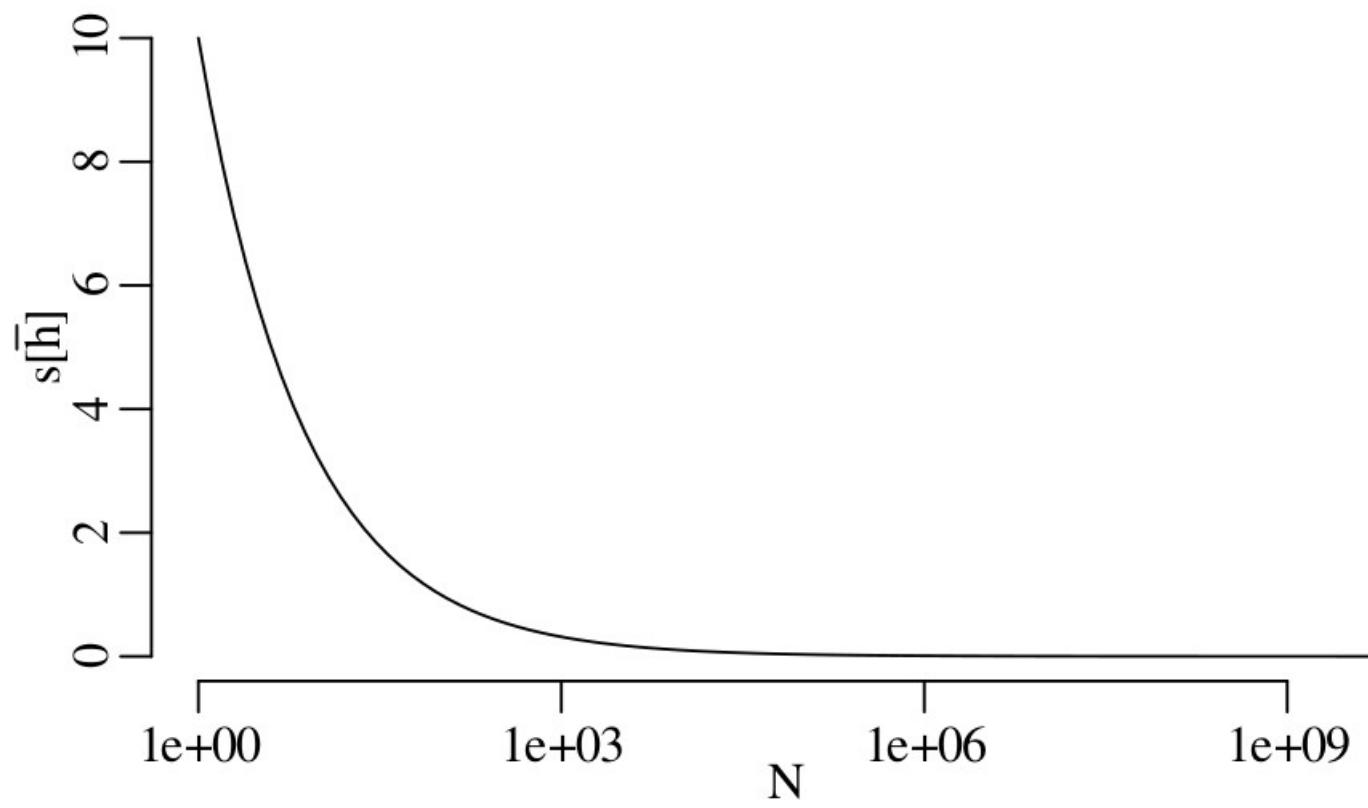
**addition**       $z = a + bx + cy$        $s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2$

---

$$s[\bar{x}]^2 = \sum_{i=1}^n \left( \frac{s[x_i]}{n} \right)^2 = \left( \frac{1}{n} \right)^2 \sum_{i=1}^n s[x_i]^2$$

$$s[\bar{x}]^2 = \left( \frac{1}{n} \right)^2 \sum_{i=1}^n s[x]^2 = n \left( \frac{1}{n} \right)^2 s[x]^2$$

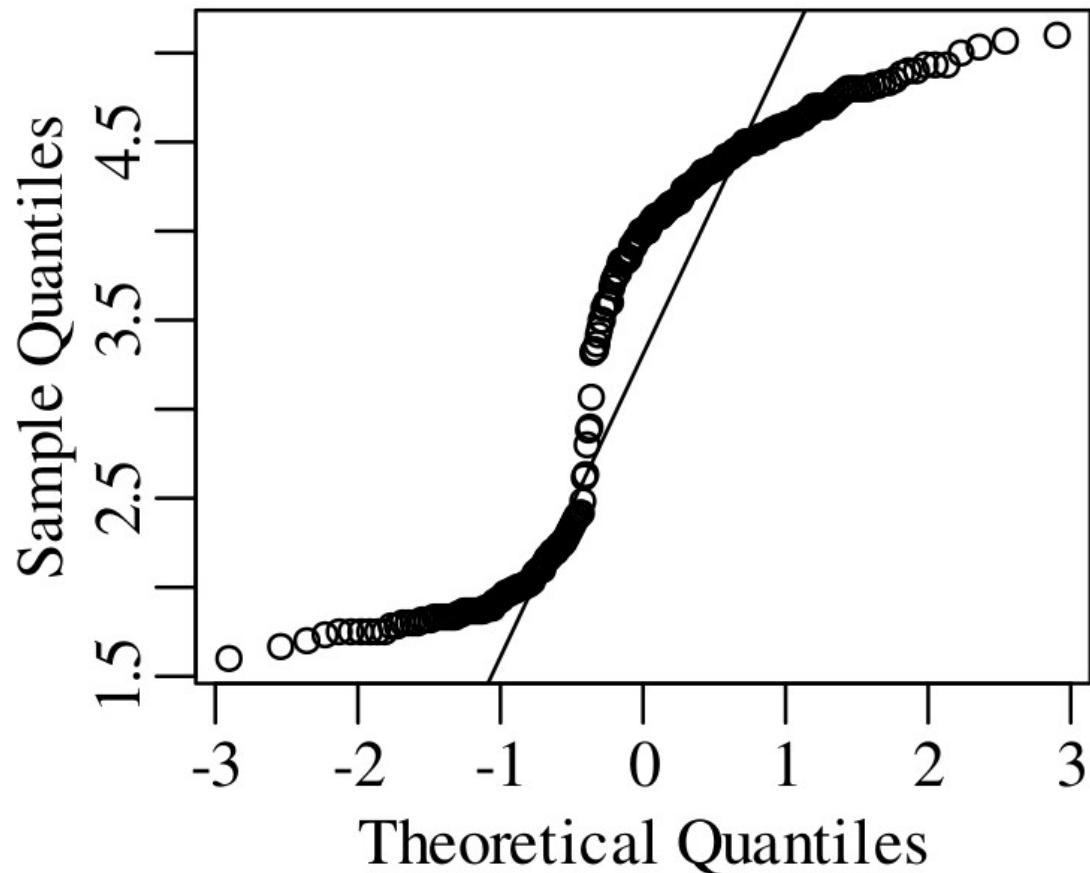
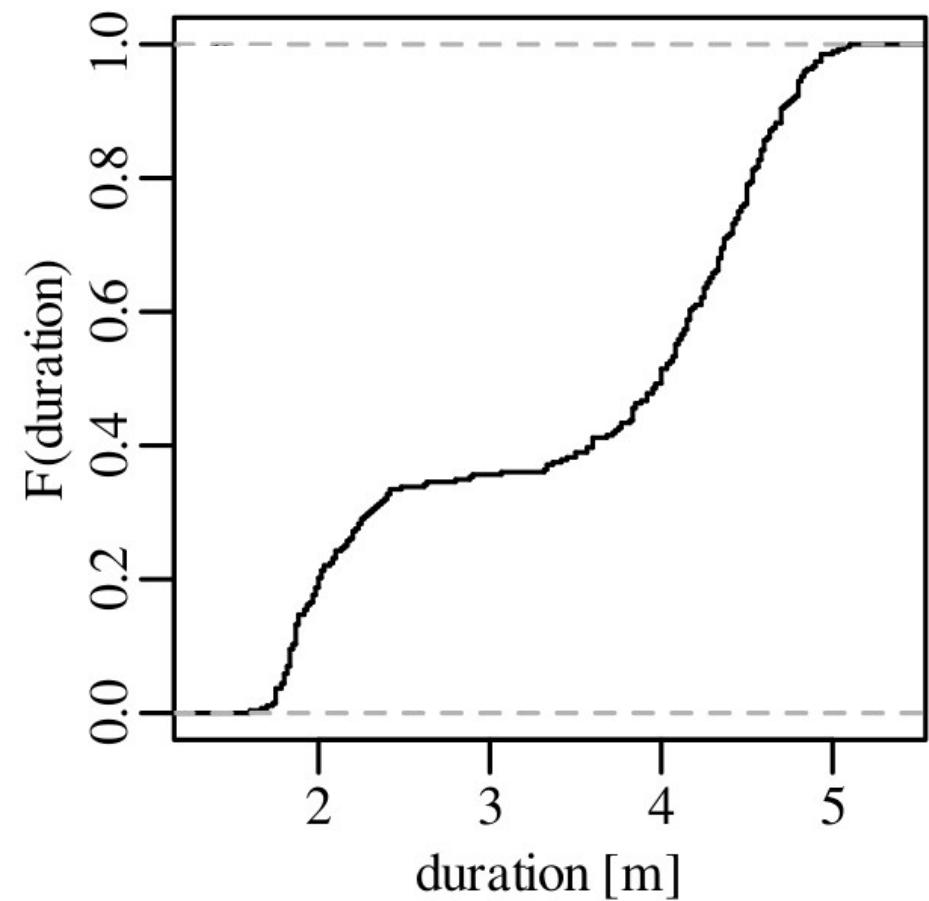
$$s[\bar{x}] = \frac{s[x]}{\sqrt{n}}$$

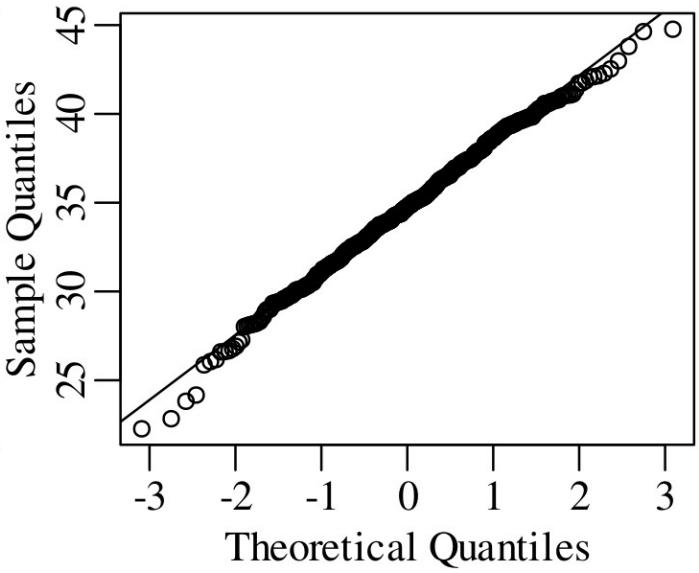
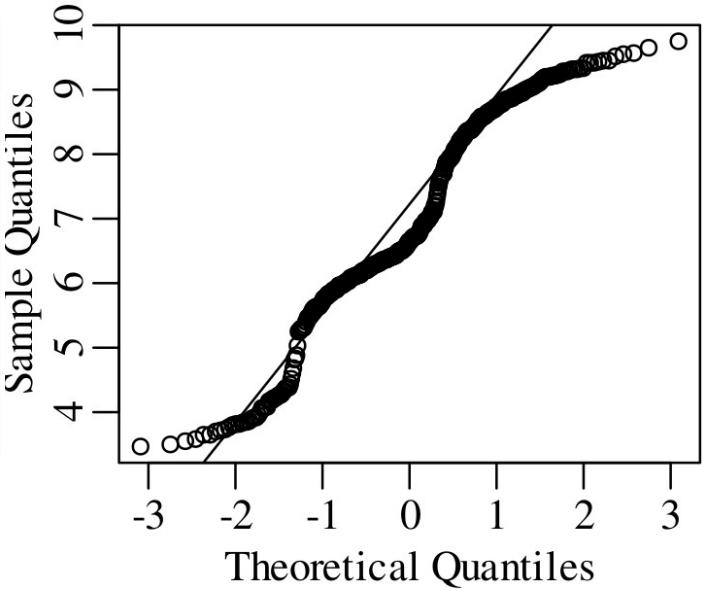
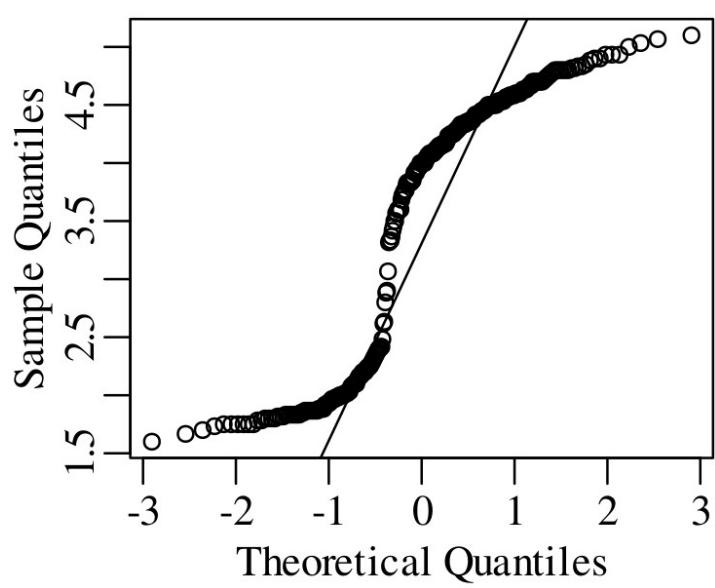


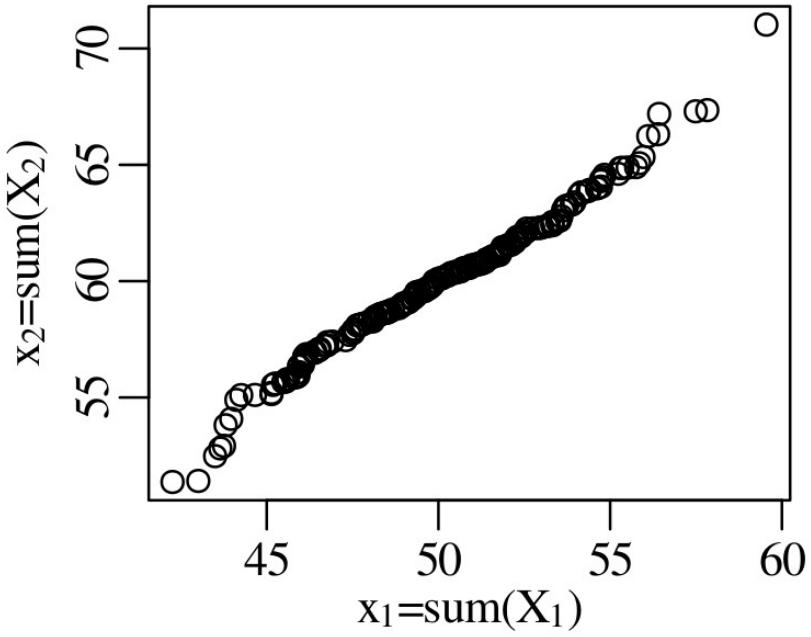
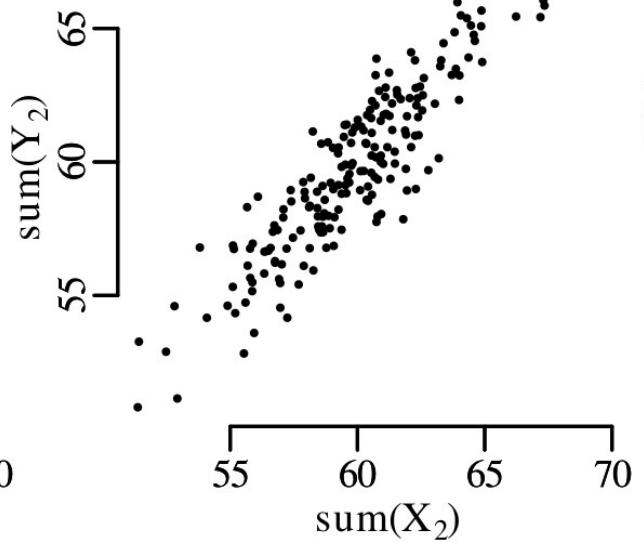
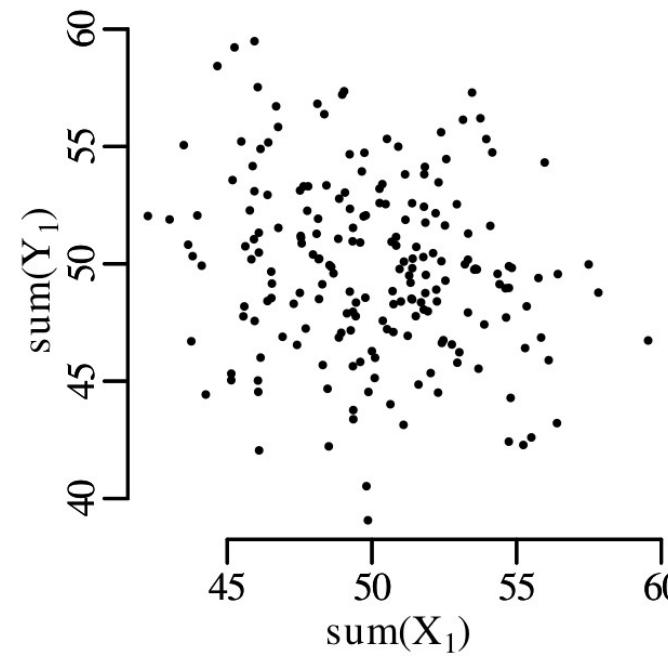
# Statistics for geoscientists

Comparing distributions

# Q-Q plots







# The t-test

coin #	1	2	3	4	5
density (g/cm <sup>3</sup> )	19.07	19.09	19.17	19.18	19.31

$$\bar{x} = 19.164$$

density of pure gold = 19.30 g/cm<sup>3</sup>

coin #	1	2	3	4	5
density (g/cm <sup>3</sup> )	19.07	19.09	19.17	19.18	19.31

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$  follows a **standard normal distribution**

$t = \frac{\bar{x} - \mu}{s[\bar{x}]/\sqrt{n}}$  follows a **Student t-distribution**  
with  $(n - 1)$  **degrees of freedom**

1.  $H_0$  (null hypothesis)  $\mu = 19.30$

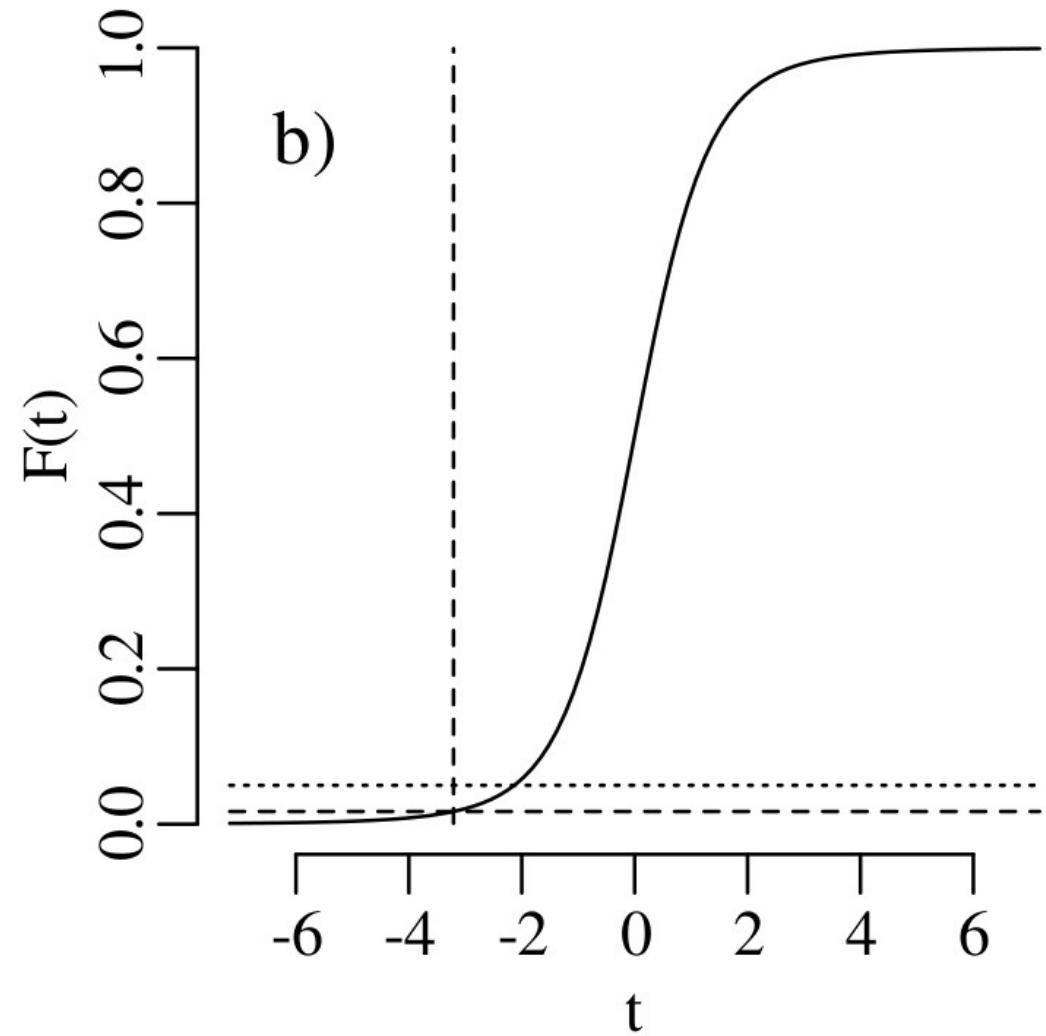
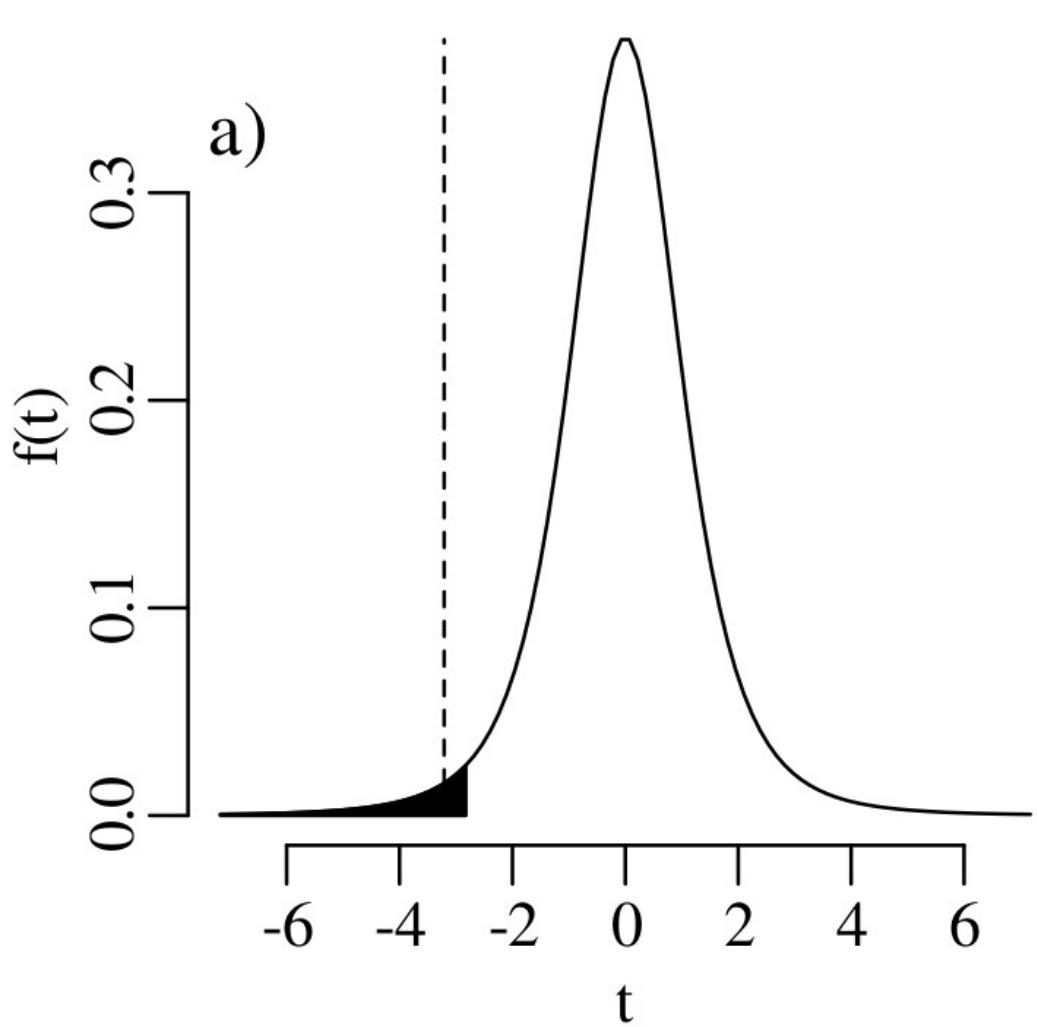
$H_a$  (alternative hypothesis):  $\mu < 19.30$

2.  $t = \frac{\bar{x} - \mu_0}{s[x]/\sqrt{n}}$  where  $\mu_0 = 19.30$   $\bar{x} = 19.164$   
 $s[x] = 0.0948$  so that  $t = -3.2091$   
 $n = 5$

3.	$t$	-3.70 -3.2091 -2.80 -2.10 -1.50 -0.74 0.00 0.74 1.50 2.10 2.80 3.70
	$P(T \leq t)$	0.01 0.0163 0.025 0.05 0.1 0.25 0.5 0.75 0.9 0.95 0.975 0.99

4.  $\alpha = 0.05$

5.	$t$	-3.70 -3.2091 -2.80 -2.10 -1.50 -0.74 0.00 0.74 1.50 2.10 2.80 3.70
	$P(T \leq t)$	0.01 0.0163 0.025 0.05 0.1 0.25 0.5 0.75 0.9 0.95 0.975 0.99



## two sample t-test

coin #	1	2	3	4	5
density (1 <sup>st</sup> collection)	19.07	19.09	19.17	19.18	19.31
density (2 <sup>nd</sup> collection)	19.17	19.30	19.31	19.32	

$$\bar{x}_1 = 19.164 \longleftrightarrow \bar{x}_2 = 19.275$$

1.  $H_0$  (**null hypothesis**)  $\mu_1 = \mu_2$

$H_a$  (**alternative hypothesis**):  $\mu_1 \neq \mu_2$

2.  $t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$  where  $s_p = \sqrt{\frac{(n_1 - 1)s[x_1]^2 + (n_2 - 1)s[x_2]^2}{n_1 + n_2 - 2}}$   
 $= -2.014$

3. Student t-distribution with  $(n_1 + n_2 - 2)$  degrees of freedom

$t$	-3.00	-2.40	-2.014	-1.90	-1.40	-0.71	0.00	0.71	1.40	1.90	2.40	3.00
$P(T \leq t)$	0.01	0.025	0.042	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(T \geq t)$	0.99	0.975	0.958	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.01

4.  $\alpha = 0.05$

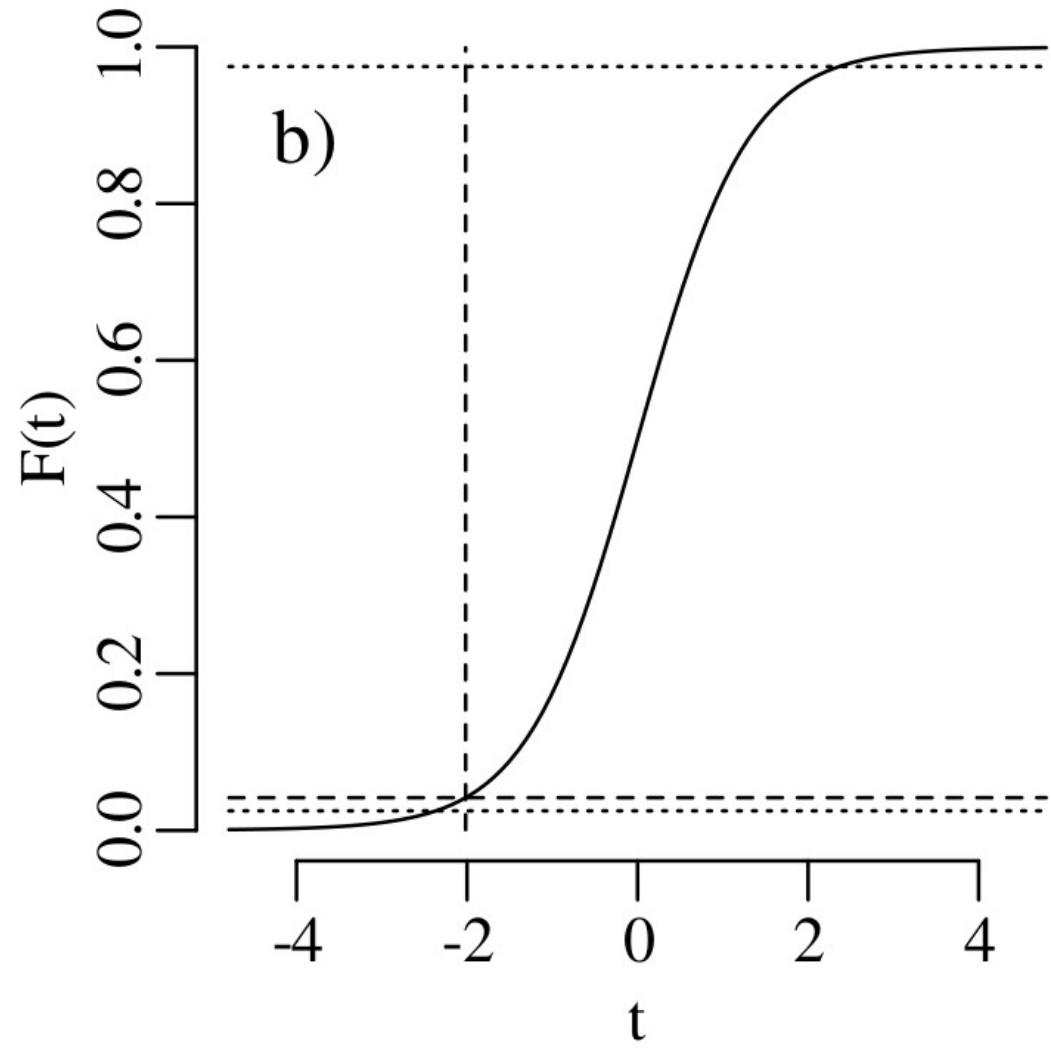
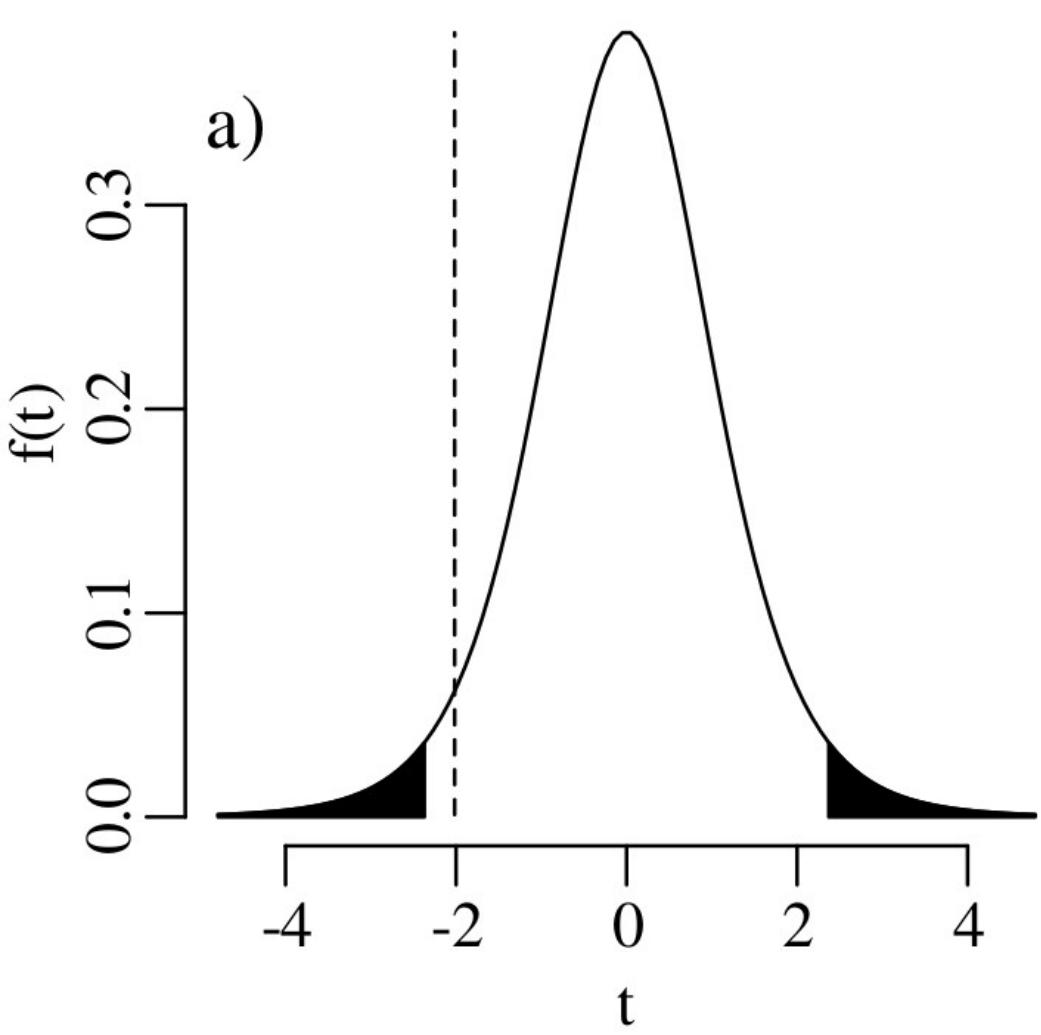
5.

$t$	<b>-3.00</b>	<b>-2.40</b>	<b>-2.014</b>	-1.90	-1.40	-0.71	0.00	0.71	1.40	1.90	<b>2.40</b>	<b>3.00</b>
$P(T \leq t)$	<b>0.01</b>	<b>0.025</b>	<b>0.042</b>	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(T \geq t)$	0.99	0.975	0.958	0.95	0.9	0.75	0.5	0.25	0.1	0.05	<b>0.025</b>	<b>0.01</b>

$$R = \{t < -2.40 \text{ and } t > 2.40\}$$

6.  $t = -2.014 \notin R$

7. p-value = 0.084 ( $= 2 \times 0.042$ )  $> 0.05$



# Confidence intervals

$$t = \frac{\bar{x} - \mu}{s[x]/\sqrt{n}}$$

$$t_{df,\alpha/2} \leq t \leq t_{df,1-\alpha/2}$$

$$t_{df,\alpha/2} \leq \frac{\bar{x} - \mu}{s[x]/\sqrt{n}} \leq t_{df,1-\alpha/2}$$

$$\bar{x} - t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \geq \mu \geq \bar{x} - t_{df,1-\alpha/2} \frac{s[x]}{\sqrt{n}}$$

with  $t_{df,1-\alpha/2} = -t_{df,\alpha/2}$

$$\bar{x} + t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \leq \mu \leq \bar{x} - t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}}$$

$\rightarrow \mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\}$

$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\}$$

coin #	1	2	3	4	5
density (g/cm <sup>3</sup> )	19.07	19.09	19.17	19.18	19.31

$$\bar{x} = 19.164$$

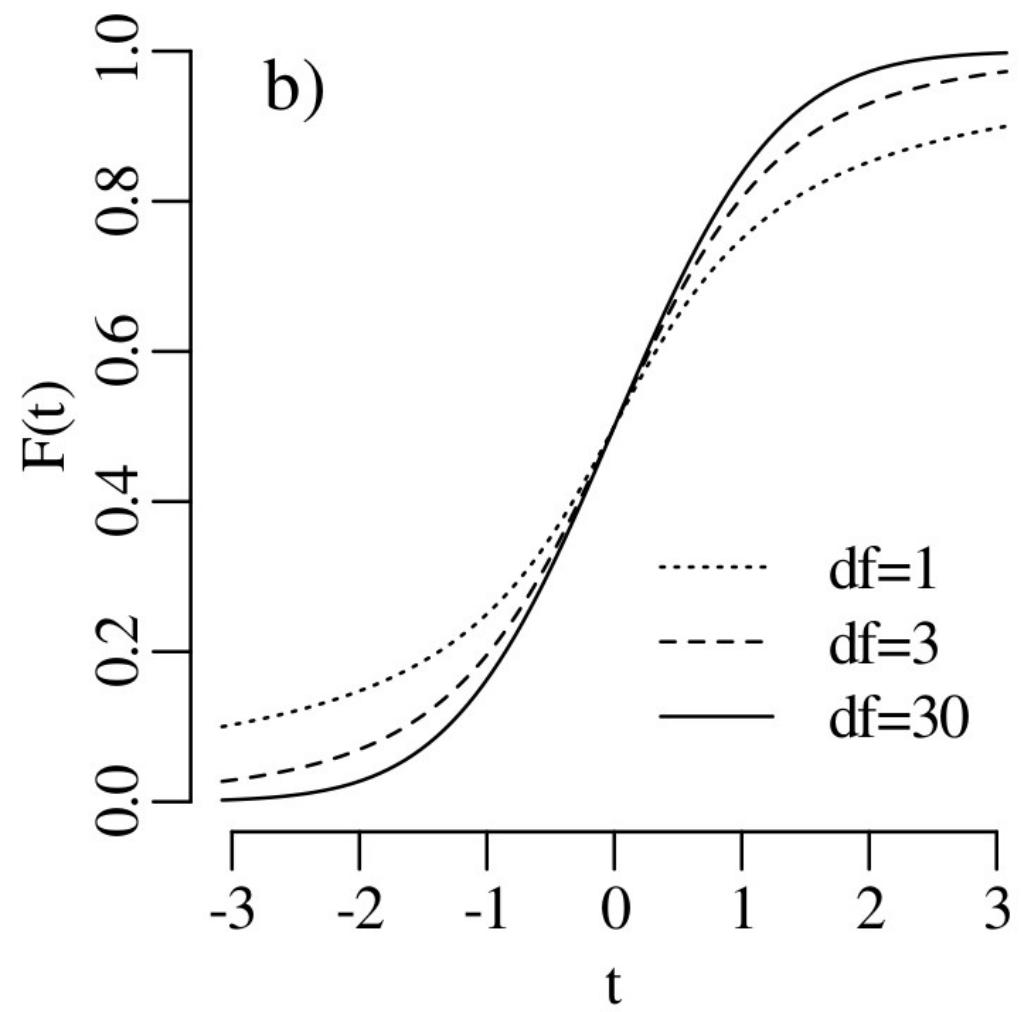
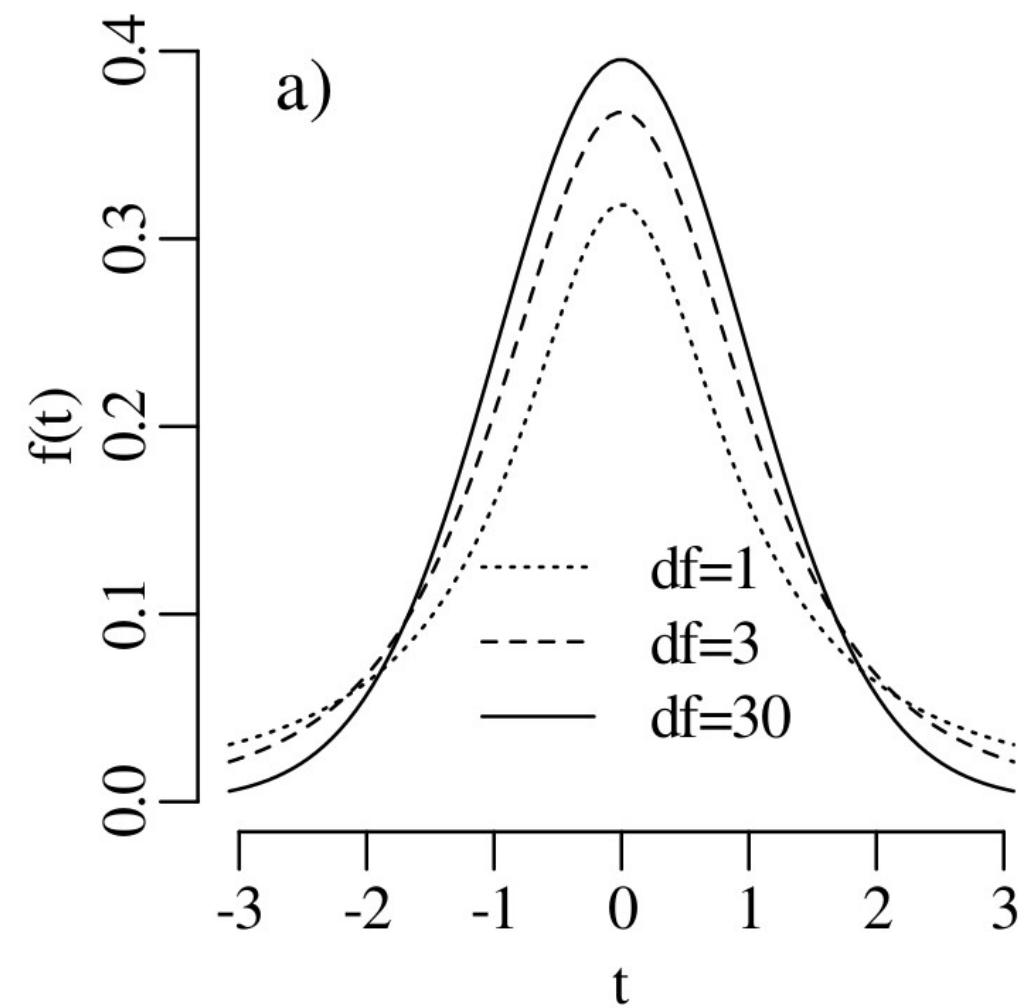
$$s[x] = 0.0948 \qquad \qquad \mu = 19.16 \pm 0.12 \text{ g/cm}^3$$

$$df = 4$$

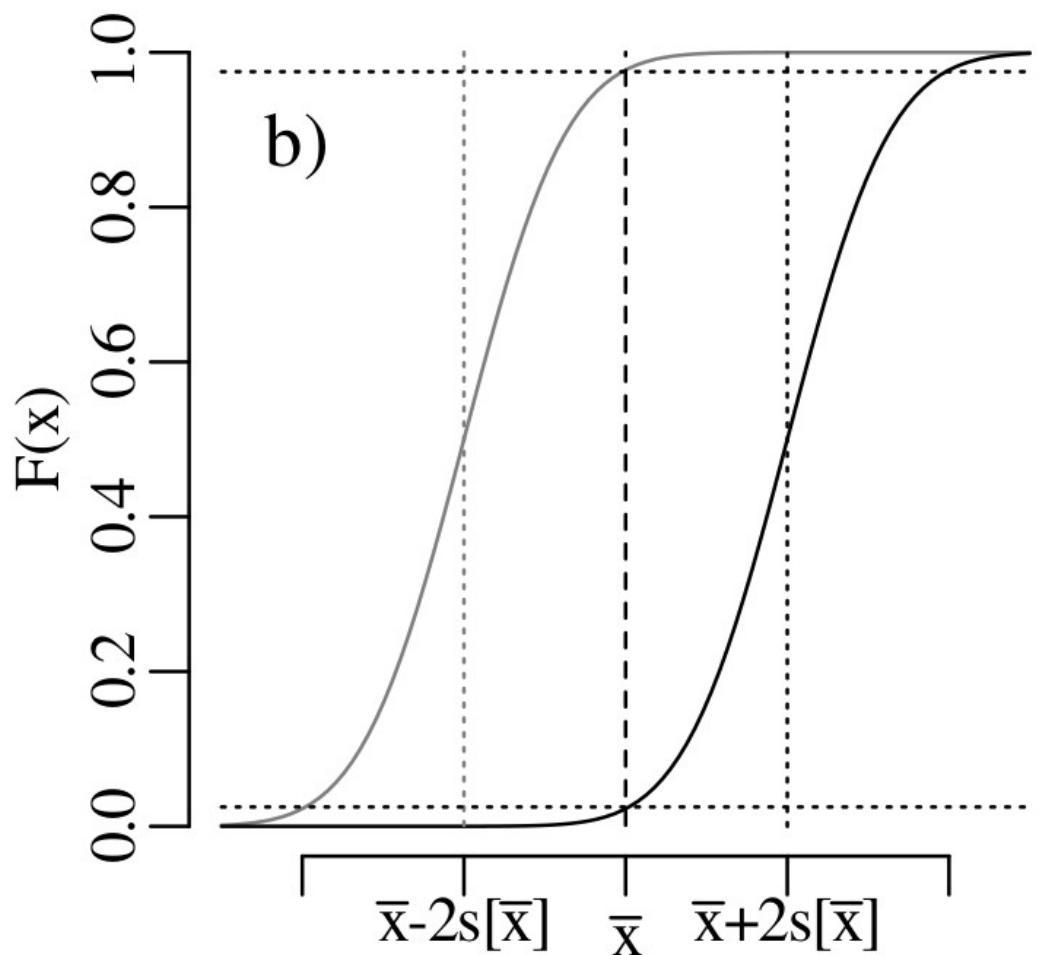
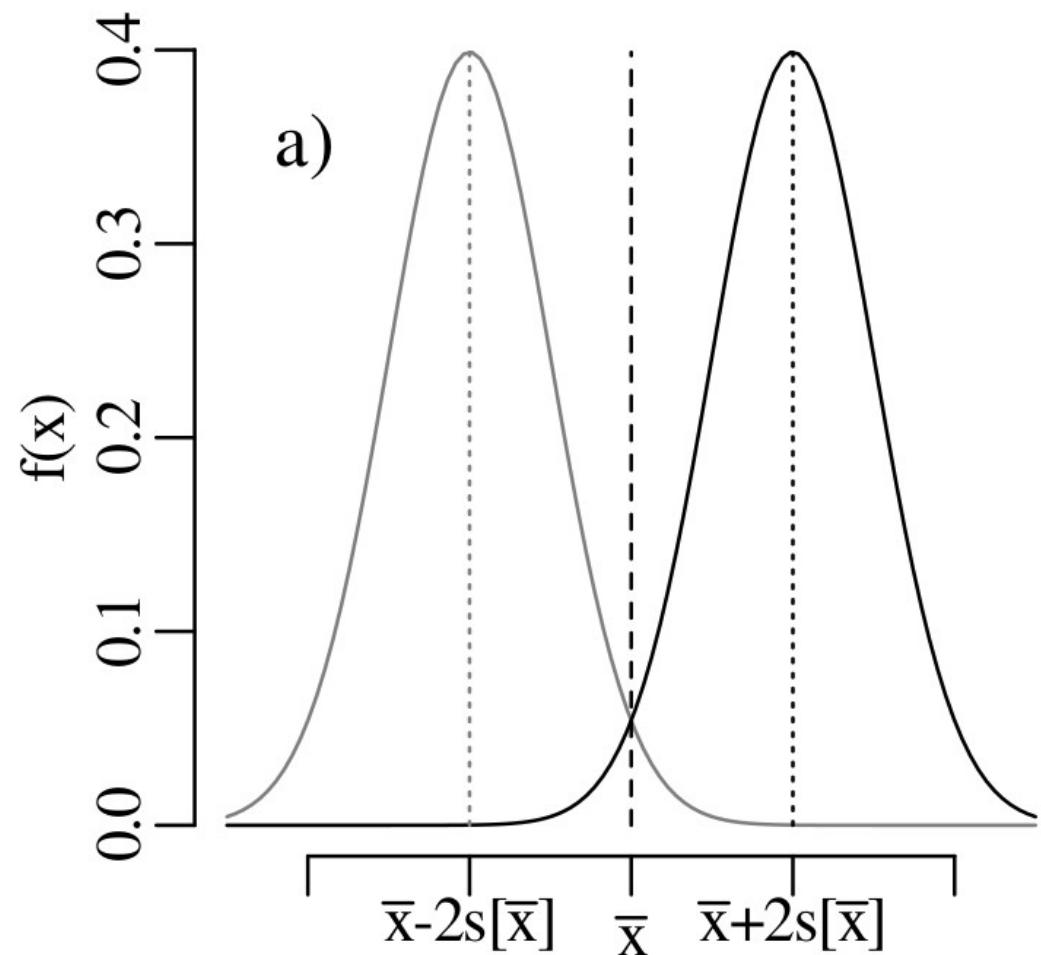
$$t_{4,0.025} = -2.776$$

$df$	1	2	3	4	5	6	7	8	9	10	30	100	1000
$t_{df,0.975}$	12.710	4.303	3.182	2.776	2.571	2.447	2.365	2.306	2.262	2.228	2.042	1.984	<b>1.962</b>

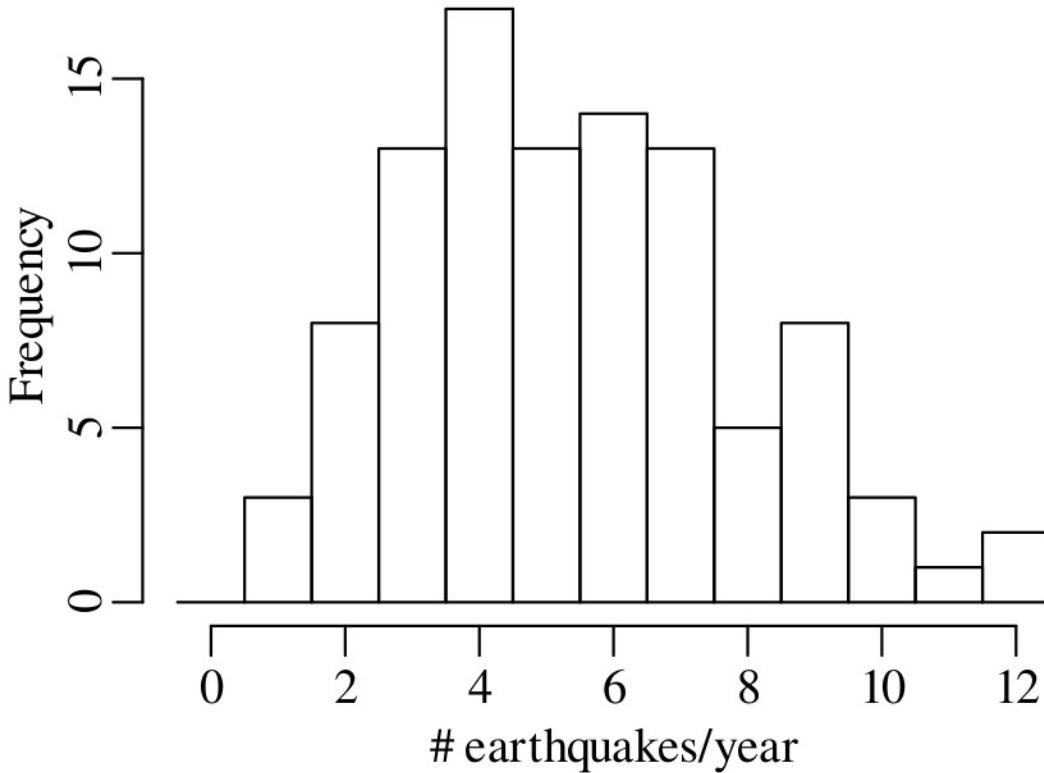
$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[\bar{x}]}{\sqrt{n}} \right\} \xrightarrow{} \mu \in \{ \bar{x} \pm 2s[\bar{x}] \}$$



$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\} \longrightarrow \mu \in \{ \bar{x} \pm 2s[\bar{x}] \}$$



# The $\chi^2$ -test



**mean: 5.43**

standard deviation: 2.50

**variance: 6.24**

number of earthquakes per year	0 1 2 3 4 5 6 7 8 9 10 11 12
number of years	0 3 8 13 17 13 14 13 5 8 3 1 2

observed

number of earthquakes per year	0	1	2	3	4	5	6	7	8	9	10	11	12
number of years	0	3	8	13	17	13	14	13	5	8	3	1	2

number of earthquakes per year	$\leq 2$	3	4	5	6	7	8	9	$\geq 10$
number of years	11	13	17	13	14	13	5	8	6

expected

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

number of earthquakes per year ( $k$ )	$\leq 2$	3	4	5	6	7	8	9	$\geq 10$
$N \times P(k \lambda = 5.43)$	9.28	11.7	15.9	17.2	15.6	12.1	8.22	4.96	5.02

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

1.  $H_0$  (**null hypothesis**): the earthquake data follow a Poisson distribution

$H_a$  (**alternative hypothesis**): the earthquake data do not follow a Poisson distribution

$$2. \chi^2 = \frac{(11 - 9.28)^2}{9.28} + \frac{(13 - 11.7)^2}{11.7} + \frac{(17 - 15.9)^2}{15.9} + \frac{(13 - 17.2)^2}{17.2} + \\ \frac{(14 - 15.6)^2}{15.6} + \frac{(13 - 12.1)^2}{12.1} + \frac{(5 - 8.22)^2}{8.22} + \frac{(8 - 4.96)^2}{4.96} + \frac{(6 - 5.02)^2}{5.02} = 5.14$$

3.  $\chi^2$  distribution with  $n - 2$  degrees of freedom

$\chi^2$	1.24	1.69	2.17	2.83	4.25	5.14	6.35	9.04	12.0	14.1	16.0	18.5
$P(X \leq \chi^2)$	0.01	0.025	0.05	0.1	0.25	0.36	0.5	0.75	0.9	0.95	0.975	0.99
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	0.743	0.5	0.25	0.1	0.05	0.025	0.01

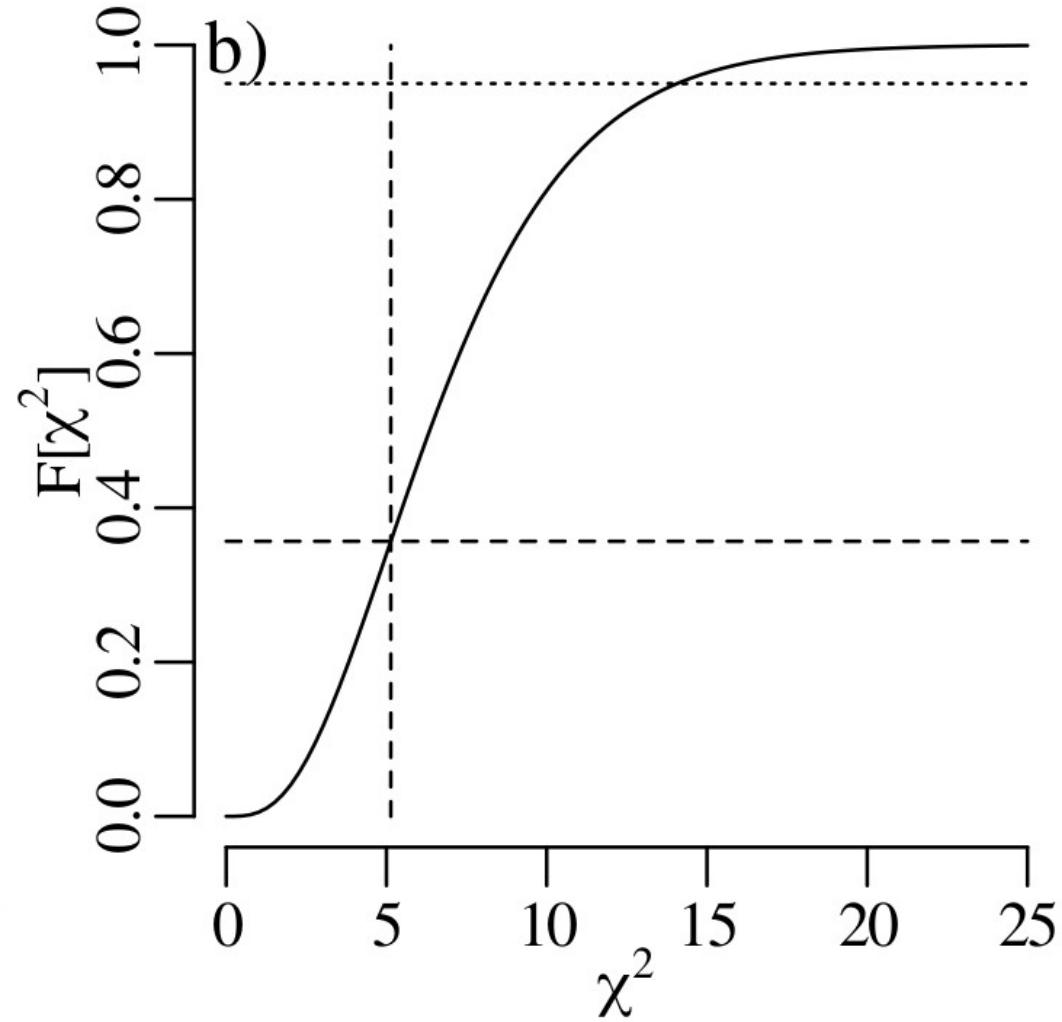
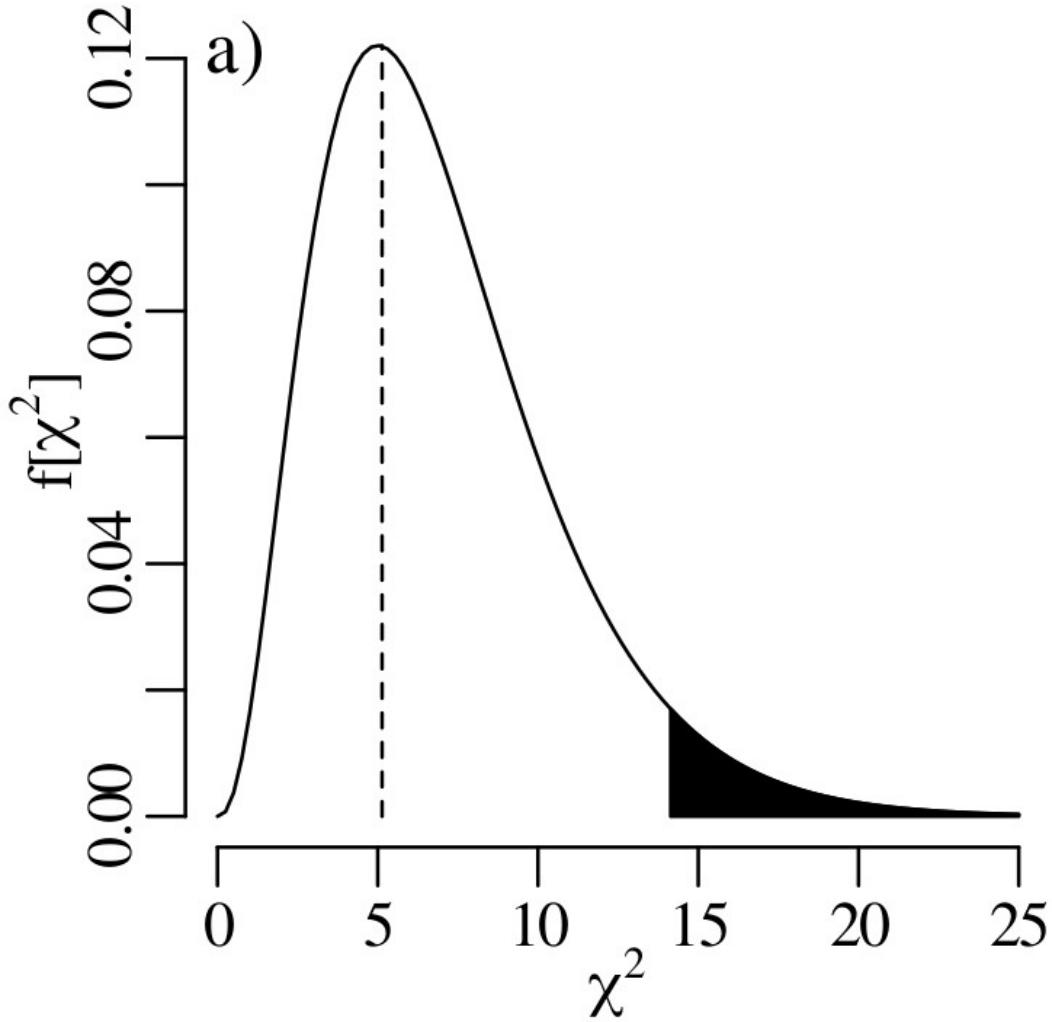
4.  $\alpha = 0.05$

5.	$\chi^2$	1.24 1.69 2.17 2.83 4.25 5.14 6.35 9.04 12.0	<b>14.1</b> 16.0 18.5
	$P(X \geq \chi^2)$	0.99 0.975 0.95 0.9 0.75 0.743 0.5 0.25 0.1	<b>0.05</b> <b>0.025</b> <b>0.01</b>

$$R = \{\chi^2 \geq 14.1\}$$

6.  $\chi^2 = 5.14 \notin R$

7. p-value = 0.743 > 0.05



# Comparing two or more samples

lithology	granite	basalt	gneiss	quartzite
sample A	10	5	6	20
sample B	25	12	10	35

lithology	granite	basalt	gneiss	quartzite	row sum
sample A	10	5	6	20	41
sample B	25	12	10	35	82
column sum	35	17	16	55	123

expected counts of bin  $(i, j) = \frac{(\text{sum of row } i) \times (\text{sum of column } j)}{\text{(sum of all the cells)}}$

lithology	granite	basalt	gneiss	quartzite
sample A	11.7	5.67	5.33	18.3
sample B	23.3	11.30	10.70	36.7

1.  $H_0$  (**null hypothesis**): samples A and B have the same composition

$H_a$  (**alternative hypothesis**): samples A and B do not have the same composition

$$2. \chi^2 = \frac{(10 - 11.7)^2}{11.7} + \frac{(5 - 5.67)^2}{5.67} + \frac{(6 - 5.33)^2}{5.33} + \frac{(20 - 18.3)^2}{18.3} + \\ \frac{(25 - 23.3)^2}{23.3} + \frac{(12 - 11.30)^2}{11.30} + \frac{(10 - 10.70)^2}{10.70} + \frac{(35 - 36.7)^2}{36.7} = 0.86$$

3.  $\chi^2$ -distribution with  $(2-1) \times (4-1) = 3$  degrees of freedom

$\chi^2$	0.115	0.216	0.352	0.584	0.86	1.21	2.37	4.11	6.25	7.81	9.35	11.3
$P(X \leq \chi^2)$	0.01	0.025	0.05	0.1	0.157	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.843	0.75	0.5	0.25	0.1	0.05	0.025	0.01

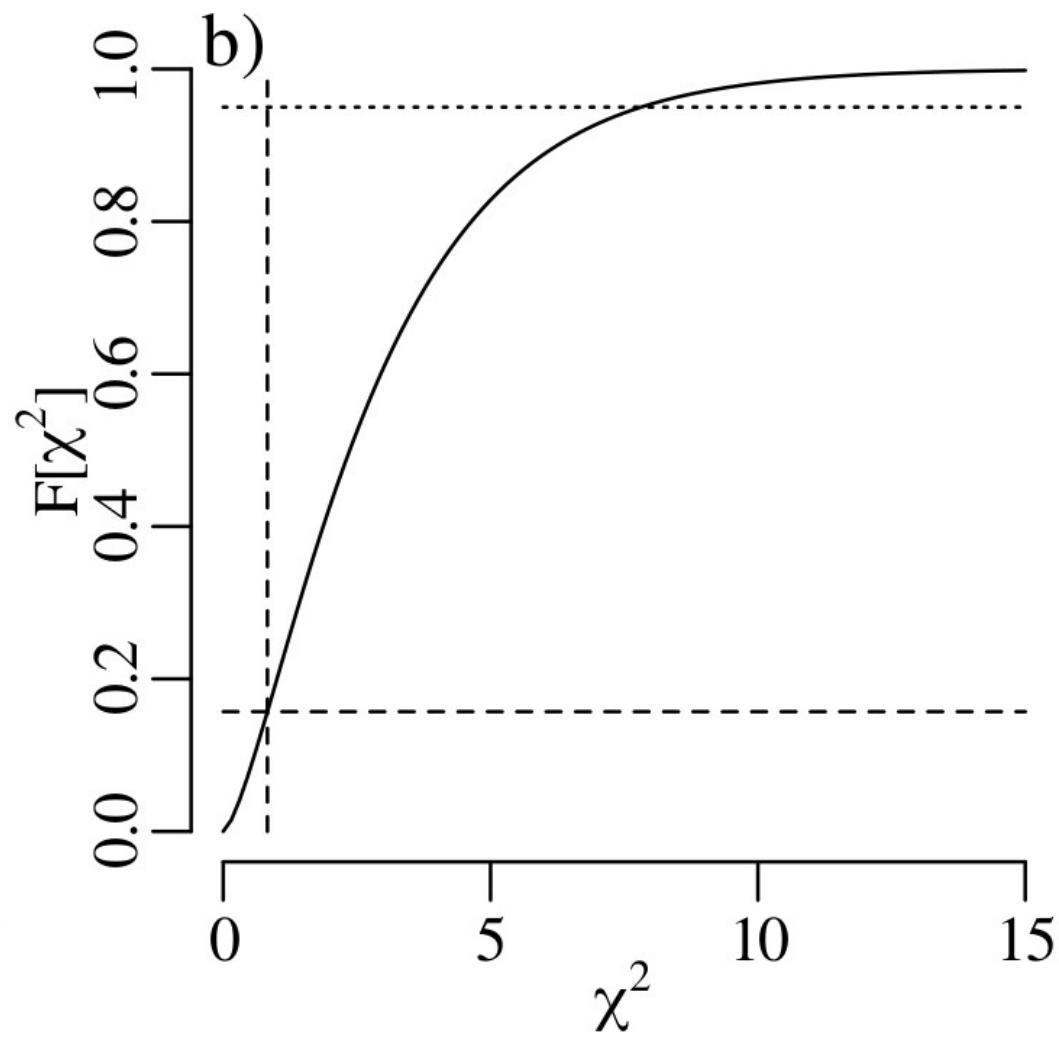
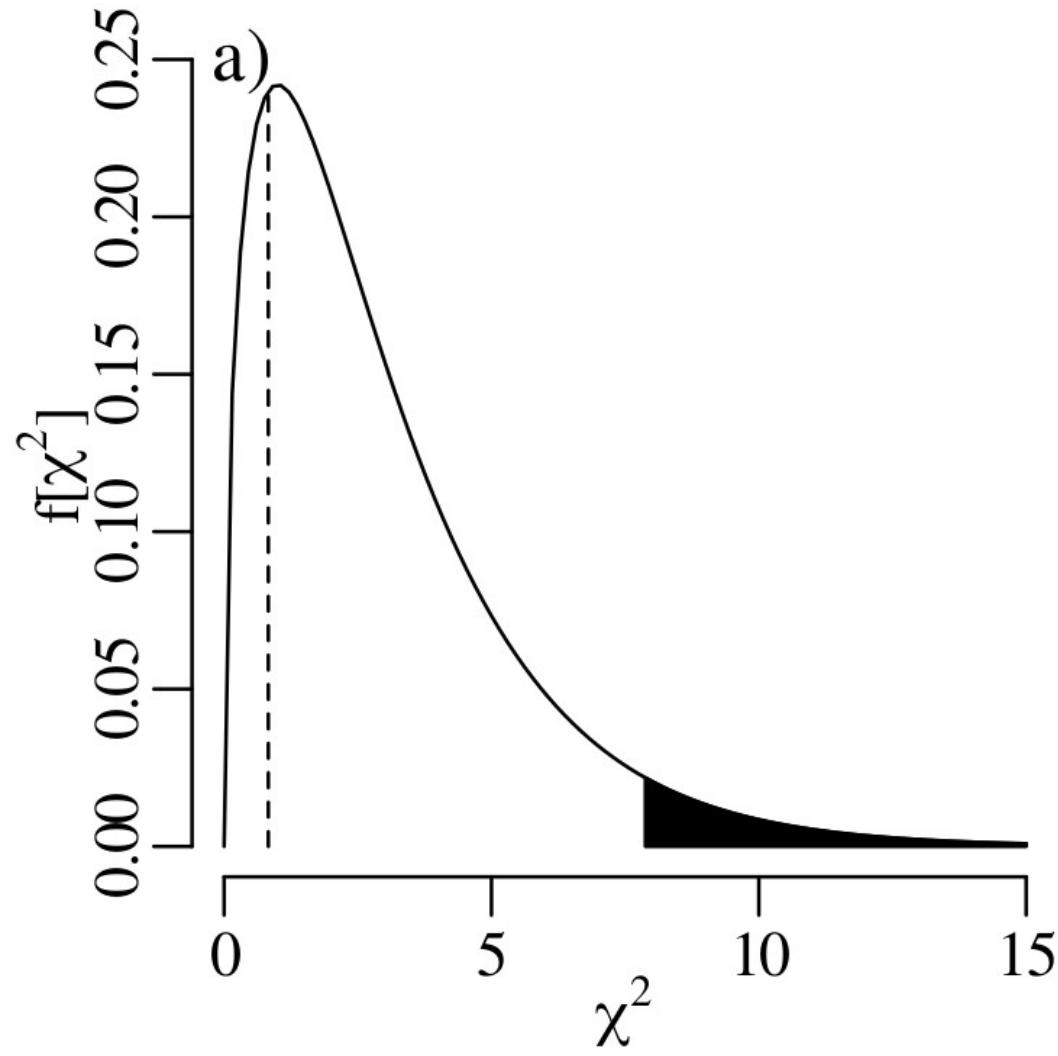
4.  $\alpha = 0.05$

5.	$\chi^2$	0.115 0.216 0.352 0.584 0.86 1.21 2.37 4.11 6.25 <b>7.81</b> <b>9.35</b> 11.3
	$P(X \geq \chi^2)$	0.99 0.975 0.95 0.9 0.843 0.75 0.5 0.25 0.1 <b>0.05</b> <b>0.025</b> <b>0.01</b>

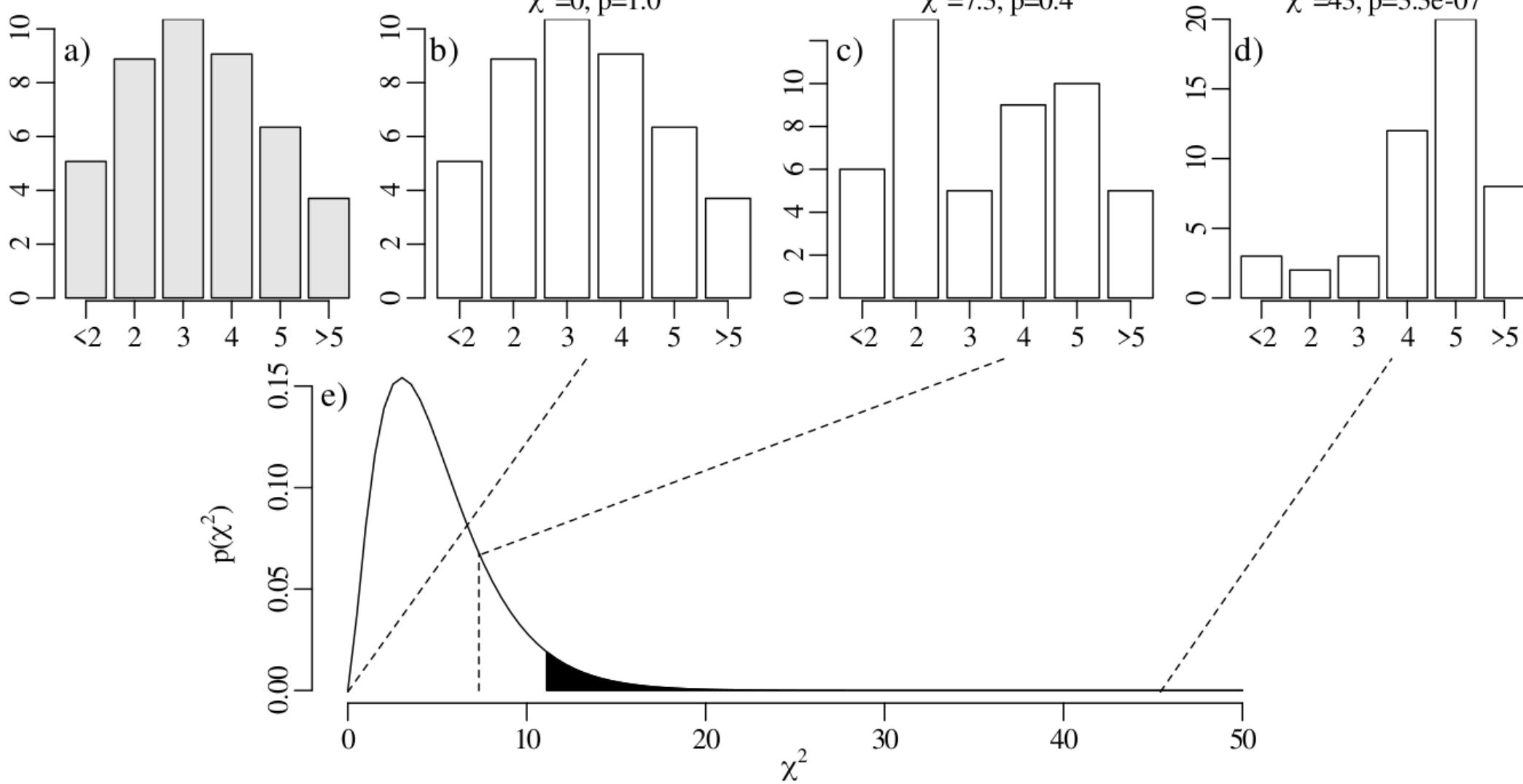
$$R = \{\chi^2 > 7.81\}$$

6.  $\chi^2 = 0.86 \notin R$

7. p-value = 0.843 > 0.05



# Cherry picking (Type-I errors revisited)



## Effect size (Type-II errors revisited)

lithology	quartz	plagioclase	alkali feldspar	lithics
sample A	29544	14424	13706	47864
sample B	29454	14788	13948	47311

lithology	quartz	plagioclase	alkali feldspar	lithics	row sum
sample A	29544	14424	13706	47864	105538
sample B	29454	14788	13948	47311	105501
column sum	58998	29212	27654	95175	211039

1.  $H_0$  (**null hypothesis**): samples A and B have identical compositions

$H_a$  (**alternative hypothesis**): samples A and B have different compositions

lithology	quartz	plagioclase	alkali feldspar	lithics	
sample A	29504	14609	13829	47596	$\chi^2 = 10.0$
sample B	29494	14603	13825	47579	

3.  $\chi^2$ -distribution 3 degrees of freedom

$\chi^2$	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.81	9.35	10.0	11.3
$P(X \leq \chi^2)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.9814	0.99
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.0186	0.010

4.  $\alpha = 0.05$

5.

$\chi^2$	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	<b>7.81</b>	<b>9.35</b>	<b>10.0</b>	11.3
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	<b>0.05</b>	<b>0.025</b>	<b>0.0186</b>	0.010

$$R = \{\chi^2 > 7.81\}$$

6.  $\chi^2 = 10.0 \in R$

7. p-value = 0.0186 <  $\alpha$

depends on the **confidence level**  $\alpha$

$H_0$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$$\beta = 1 - \text{power}$$

depends on 1. The degree to which  $H_0$  is false

2. Sample size

depends on the **confidence level**  $\alpha$

$H_0$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$\beta = 1 - \text{power}$

depends on

1. The degree to which  $H_0$  is false
2. Sample size

# effect size

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$w = \sqrt{\sum_{i=1}^n \frac{(o_i - e_i)^2}{e_i}}$$

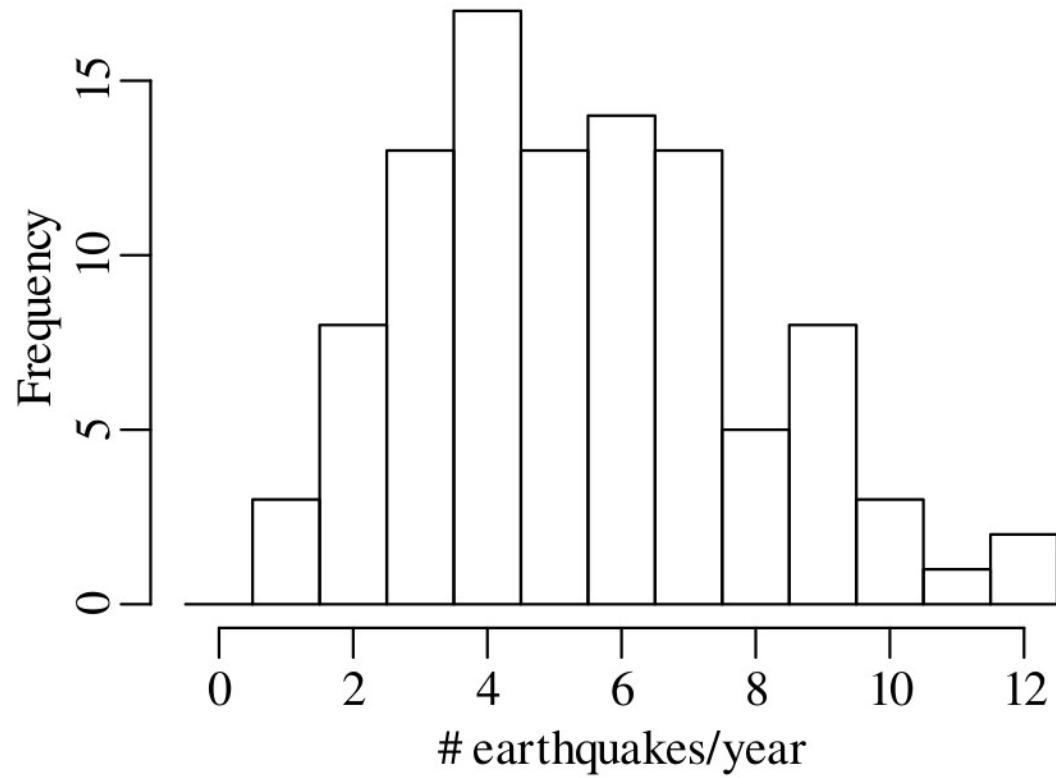
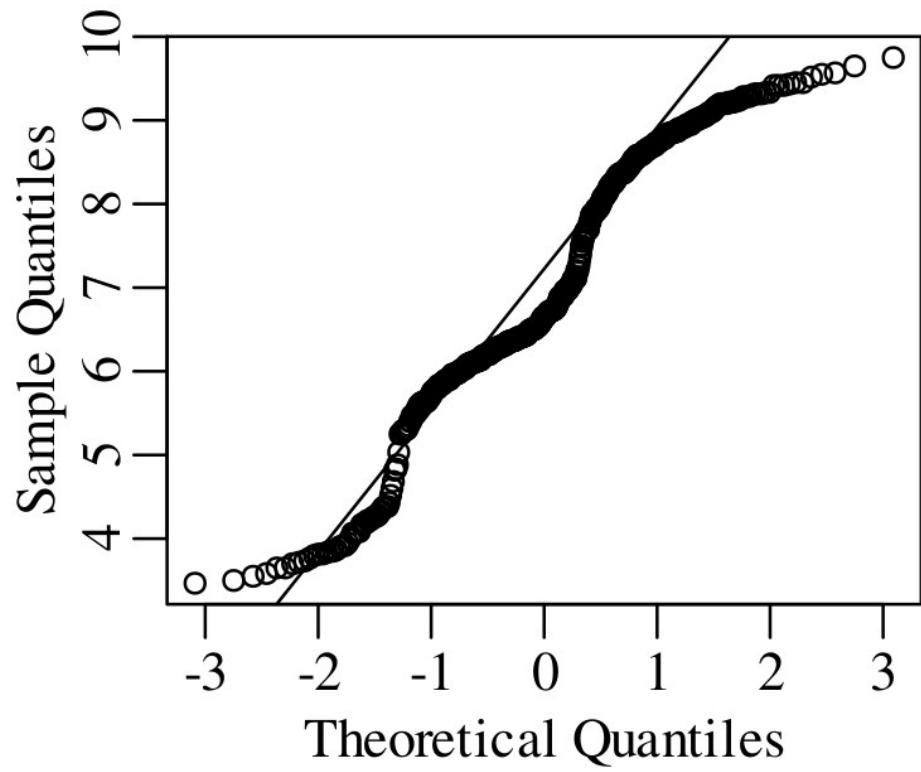
where  $o_i = O_i/N$  and  $e_i = E_i/N$

with  $N = \sum_i^n O_i = \sum_i^n E_i$

effect size	small	medium	large
w	0.1	0.3	0.5

effect size		small	medium	large	
	w	0.1	0.3	0.5	
$o =$	lithology	quartz	plagioclase	alkali feldspar	lithics
	sample A	0.1400	0.06835	0.06495	0.2268
	sample B	0.1396	0.07007	0.06609	0.2242
$e =$	lithology	quartz	plagioclase	alkali feldspar	lithics
	sample A	0.1398	0.06922	0.06553	0.2255
	sample B	0.1398	0.06920	0.06551	0.2255

$$w = 0.00688$$



# Non-parametric tests

## 1. Wilcoxon test

coin #	1	2	3	4	5
density (1 <sup>st</sup> collection)	19.07	19.09	19.17	19.18	19.31
density (2 <sup>nd</sup> collection)	19.17	19.30	19.31	19.32	

sample	1	2	3	4	5
$A$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
$B$	$B_1$	$B_2$	$B_3$	$B_4$	

sample	1	2	3	4	5
A	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
B	$B_1$	$B_2$	$B_3$	$B_4$	

suppose that

$$A_4 < A_1 < A_2 < B_3 < A_5 < B_1 < B_4 < A_3 < B_2$$

then

rank	1	2	3	4	5	6	7	8	9
value	$A_4$	$A_1$	$A_2$	$B_3$	$A_5$	$B_1$	$B_4$	$A_3$	$B_2$

$$W = 4 + 6 + 7 + 9 = 26$$

e.g.  $W = 10$

arrangement 1:	$B_1$	$B_2$	$B_3$	$B_4$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
arrangement 2:	$B_2$	$B_1$	$B_3$	$B_4$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
arrangement 2:	$B_1$	$B_2$	$B_3$	$B_4$	$A_2$	$A_1$	$A_3$	$A_4$	$A_5$
etc.									

$$P(W = 10 | n_A = 5, n_B = 4) = \frac{4!5!}{9!} = 0.00794$$

coin #	1	2	3	4	5
density (1 <sup>st</sup> collection)	19.07	19.09	19.17	19.18	19.31
<b>density (2<sup>nd</sup> collection)</b>	<b>19.17</b>	<b>19.30</b>	<b>19.31</b>	<b>19.32</b>	

1.  $H_0$  (null hypothesis)  $\text{median}(\text{sample 1}) = \text{median}(\text{sample 2})$

$H_a$  (alternative hypothesis):  $\text{median}(\text{sample 1}) \neq \text{median}(\text{sample 2})$

rank	1	2	3.5	<b>3.5</b>	5	<b>6</b>	7.5	<b>7.5</b>	<b>9</b>
density	19.07	19.09	19.17	<b>19.17</b>	19.18	<b>19.30</b>	<b>19.31</b>	19.31	<b>19.32</b>

$$W = 3.5 + 6 + 7.5 + 9 = 26$$

3.

$W$	10	11	12	14	16	19	22	24	26	27	28
$P(w \leq W)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	<i>0.95</i>	0.975	0.99
$P(w \geq W)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	<i>0.05</i>	0.025	0.010

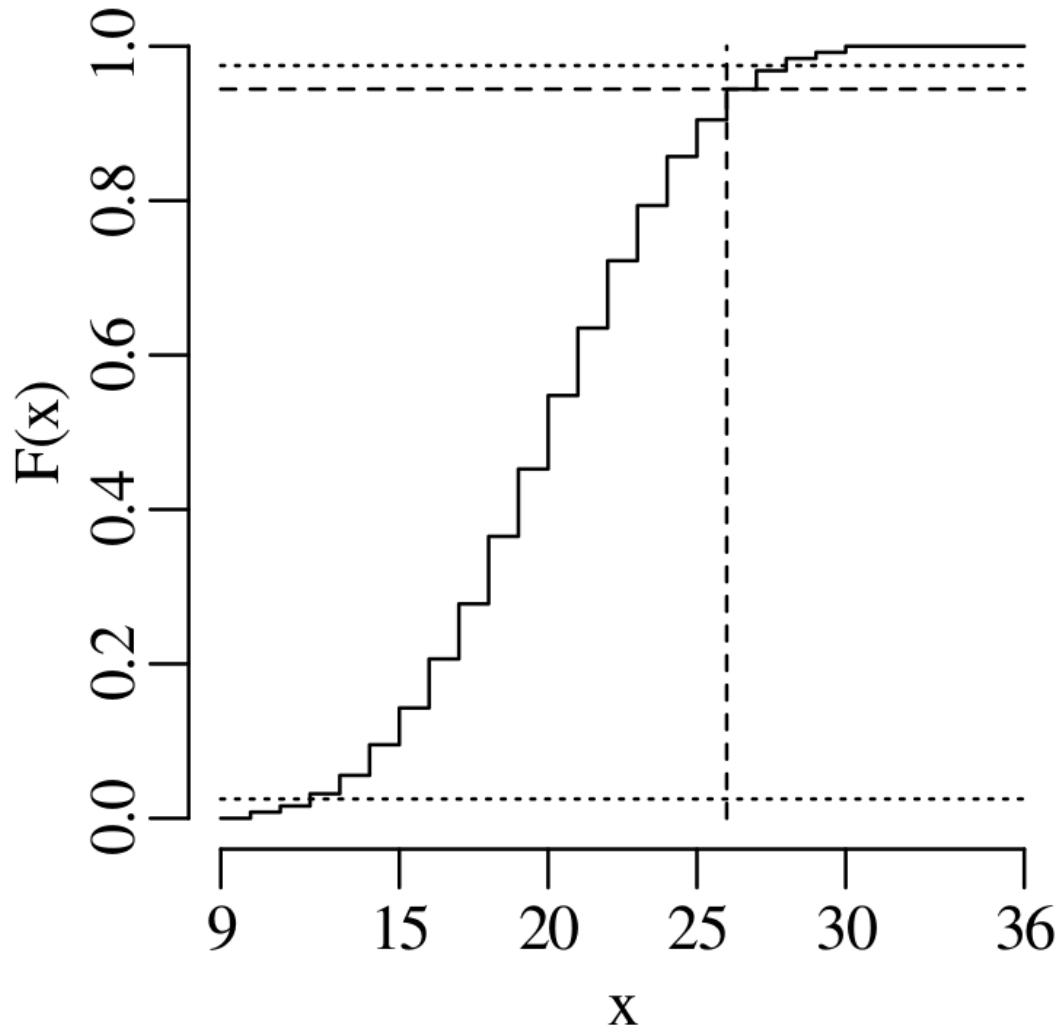
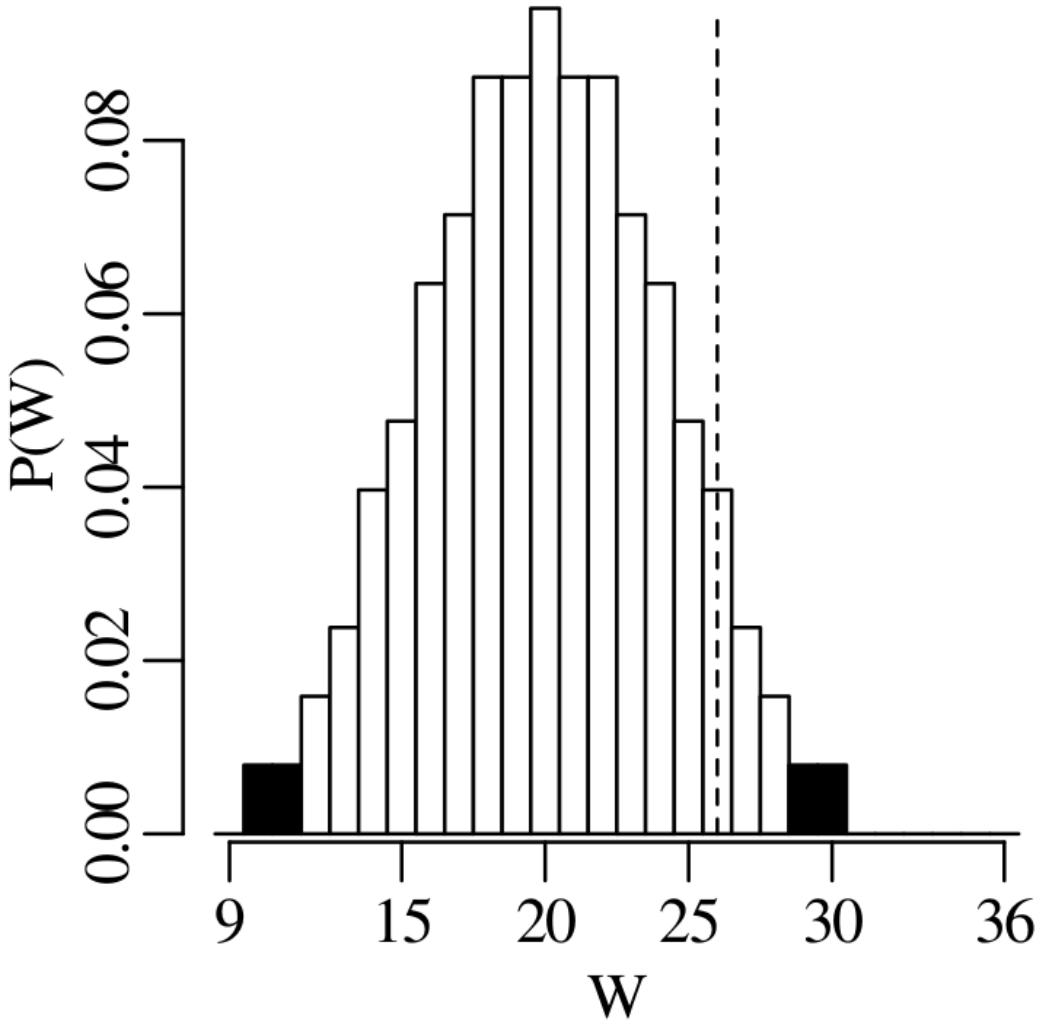
4.  $\alpha = 0.05$

5.	$W$	10	11	12	14	16	19	22	24	26	27	<b>28</b>
	$P(w \leq W)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
	$P(w \geq W)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	<b>0.01</b>

$$R = \{ W = 10 \text{ and } W > 27 \}$$

6.  $W = 26 \notin R$

7. p-value = 0.10 >  $\alpha$



## 2. Kolmogorov-Smirnov test

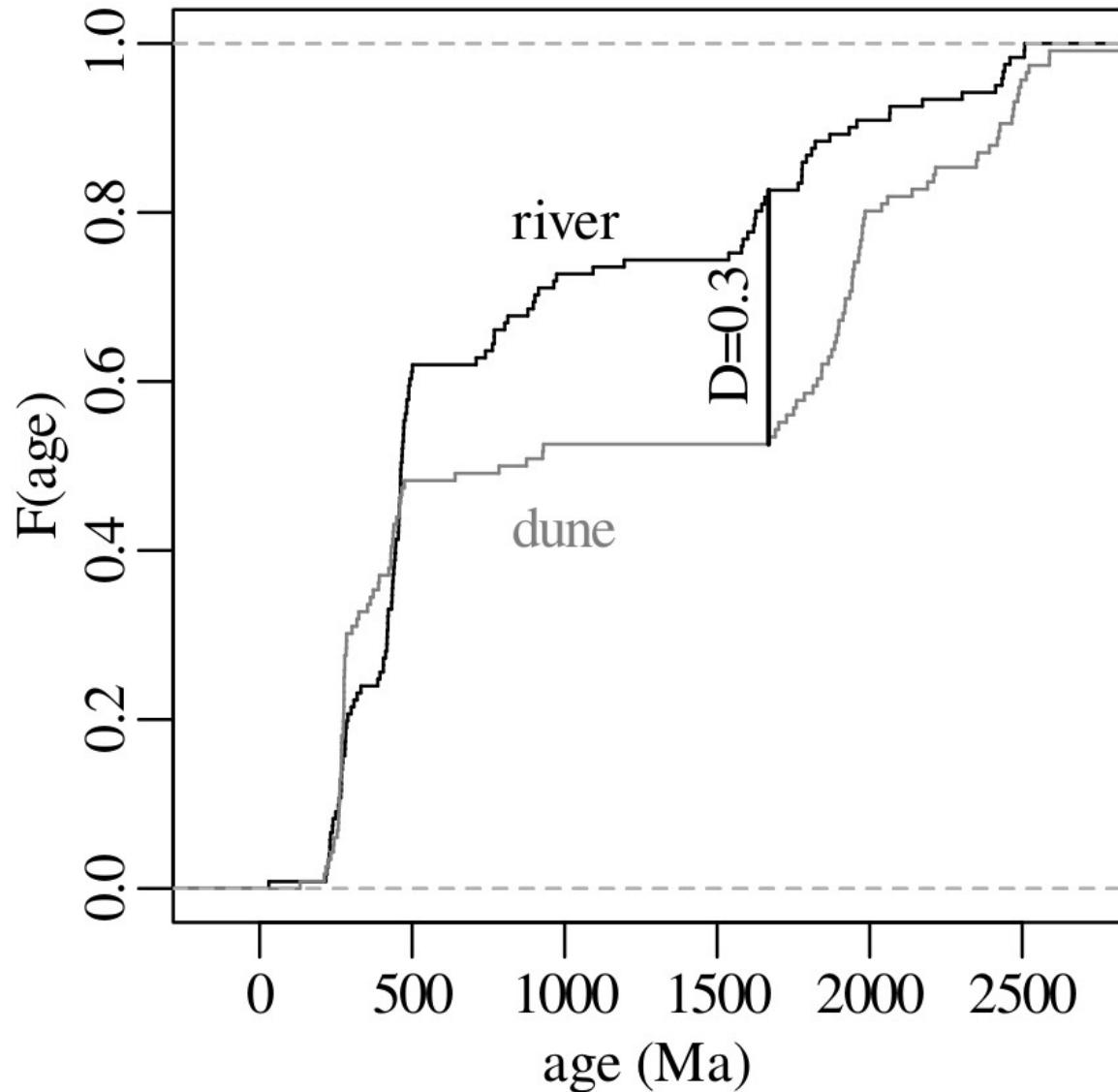
Given

$$x = \{x_1, x_2, \dots, x_n\}$$

and

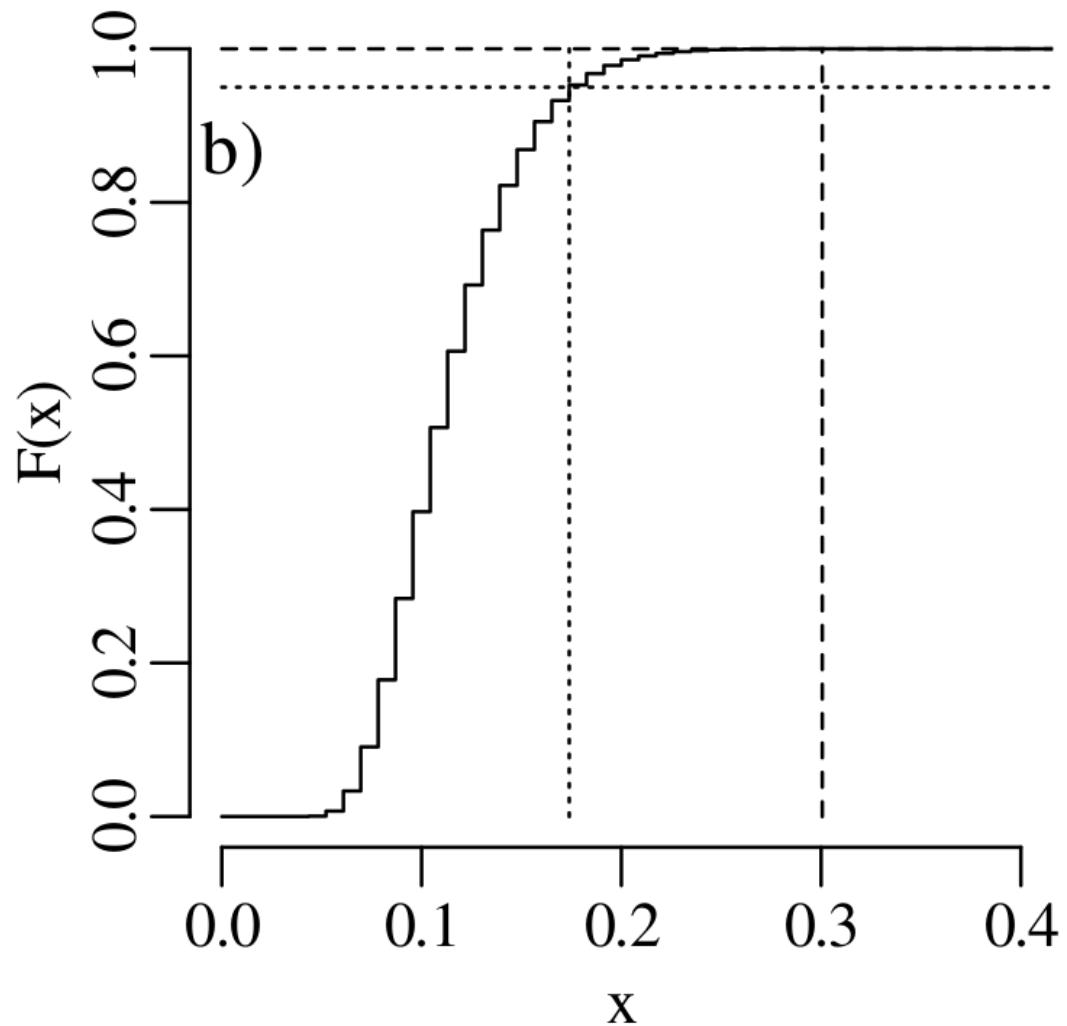
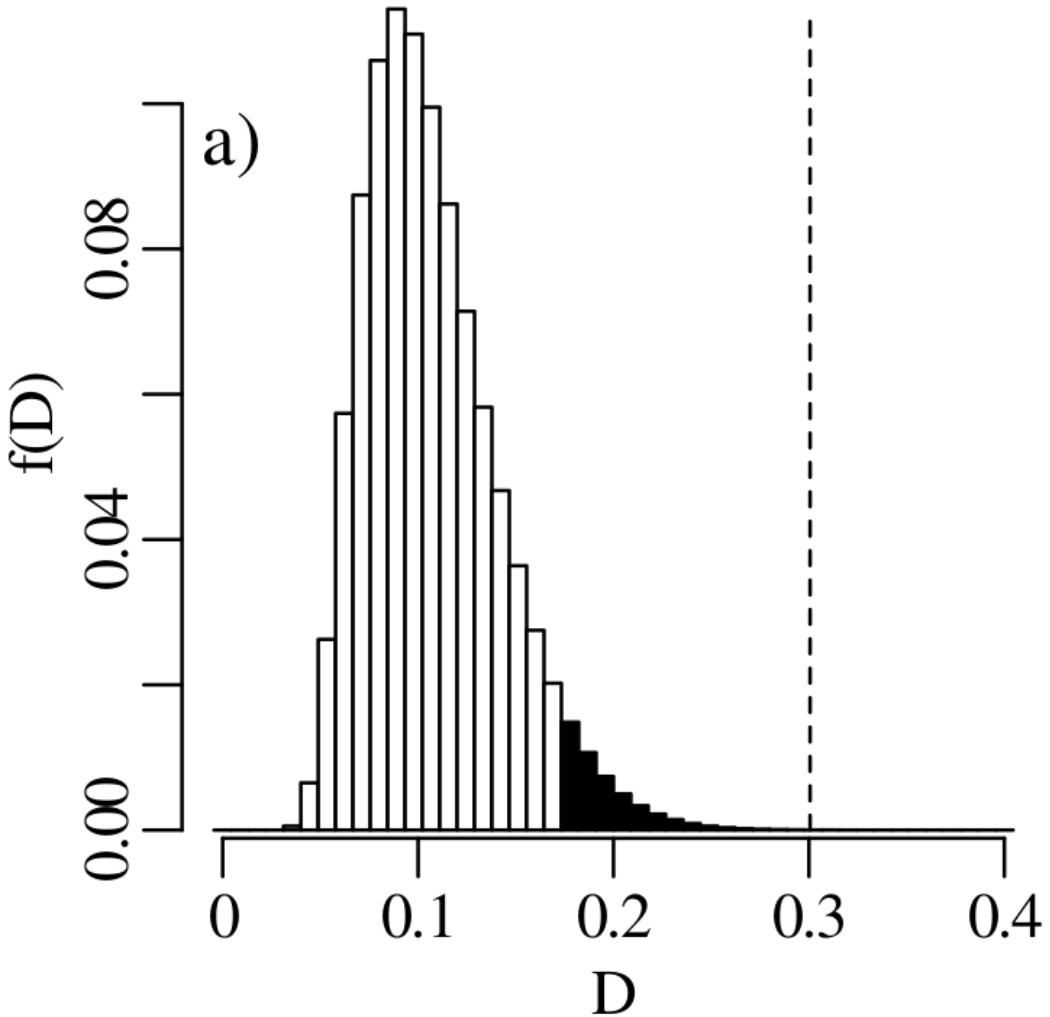
$$y = \{y_1, y_2, \dots, y_m\}$$

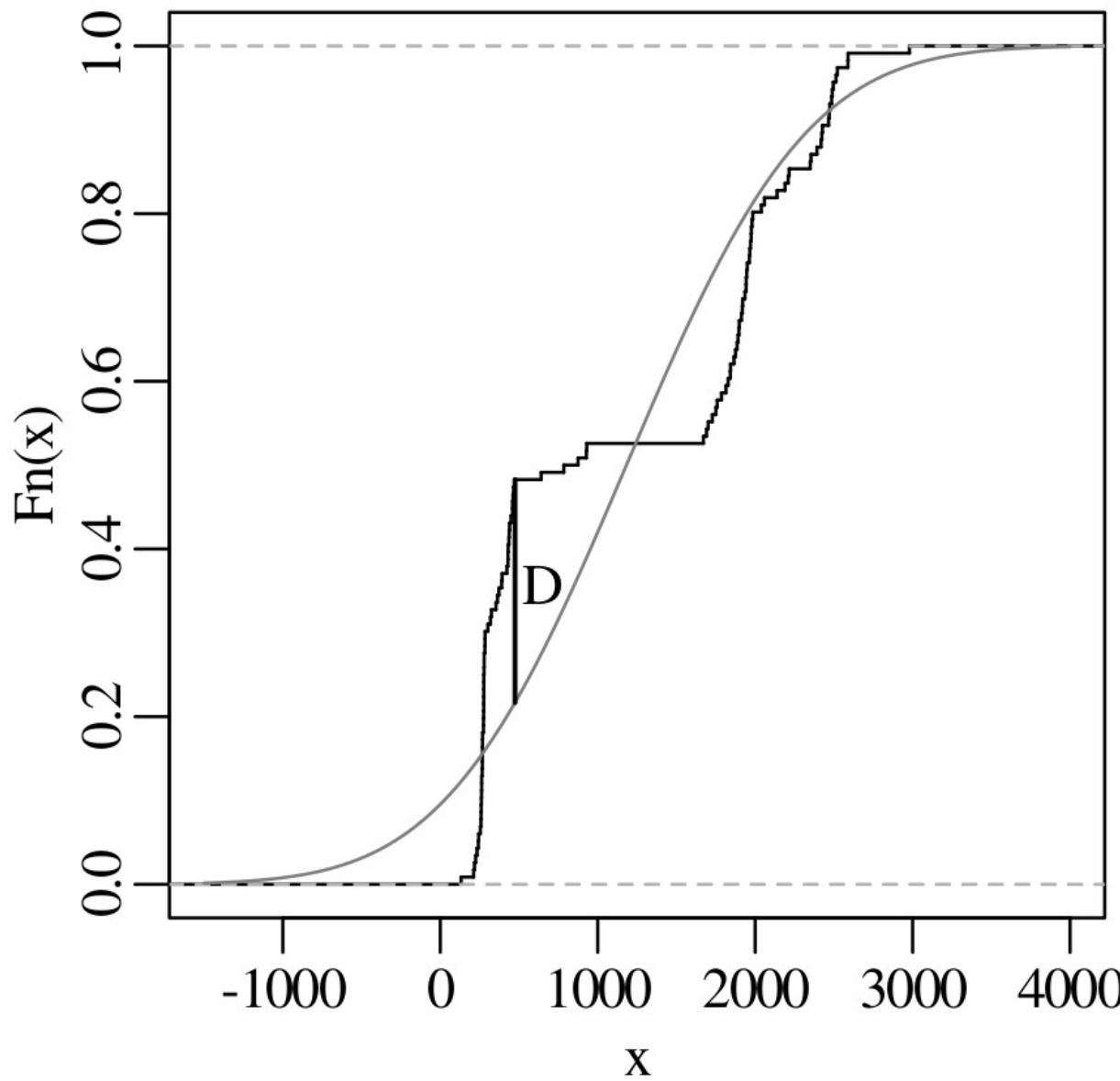
$$D = \max_z |F_x(z) - F_y(z)|$$



- $H_0$  (**null hypothesis**): samples 1 and 2 were drawn from the same distribution  
 $H_a$  (**alternative hypothesis**): samples 1 and 2 were drawn from different distributions
- $D = 0.30$
- | $D$           | 0.061 | 0.061 | 0.070 | 0.078 | 0.087 | 0.104 | 0.130 | 0.157 | 0.174 | 0.191 | 0.209 | 0.301                |
|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------------------|
| $P(d \leq D)$ | 0.01  | 0.025 | 0.05  | 0.1   | 0.25  | 0.5   | 0.75  | 0.9   | 0.95  | 0.975 | 0.99  | 0.99996              |
| $P(d \geq D)$ | 0.99  | 0.975 | 0.95  | 0.9   | 0.75  | 0.5   | 0.25  | 0.1   | 0.05  | 0.025 | 0.010 | $4.5 \times 10^{-5}$ |
- $\alpha = 0.05$
- | $D$           | 0.061 | 0.061 | 0.070 | 0.078 | 0.087 | 0.104 | 0.130 | 0.157 | <b>0.174</b> | <b>0.191</b> | <b>0.209</b> | 0.301                |
|---------------|-------|-------|-------|-------|-------|-------|-------|-------|--------------|--------------|--------------|----------------------|
| $P(d \geq D)$ | 0.99  | 0.975 | 0.95  | 0.9   | 0.75  | 0.5   | 0.25  | 0.1   | <b>0.05</b>  | <b>0.025</b> | <b>0.010</b> | $4.5 \times 10^{-5}$ |

 $R = \{D > 0.174\}$
- $D = 0.301 \in R$
- p-value =  $4.5 \times 10^{-5} < \alpha$





# Statistics for geoscientists

## Regression

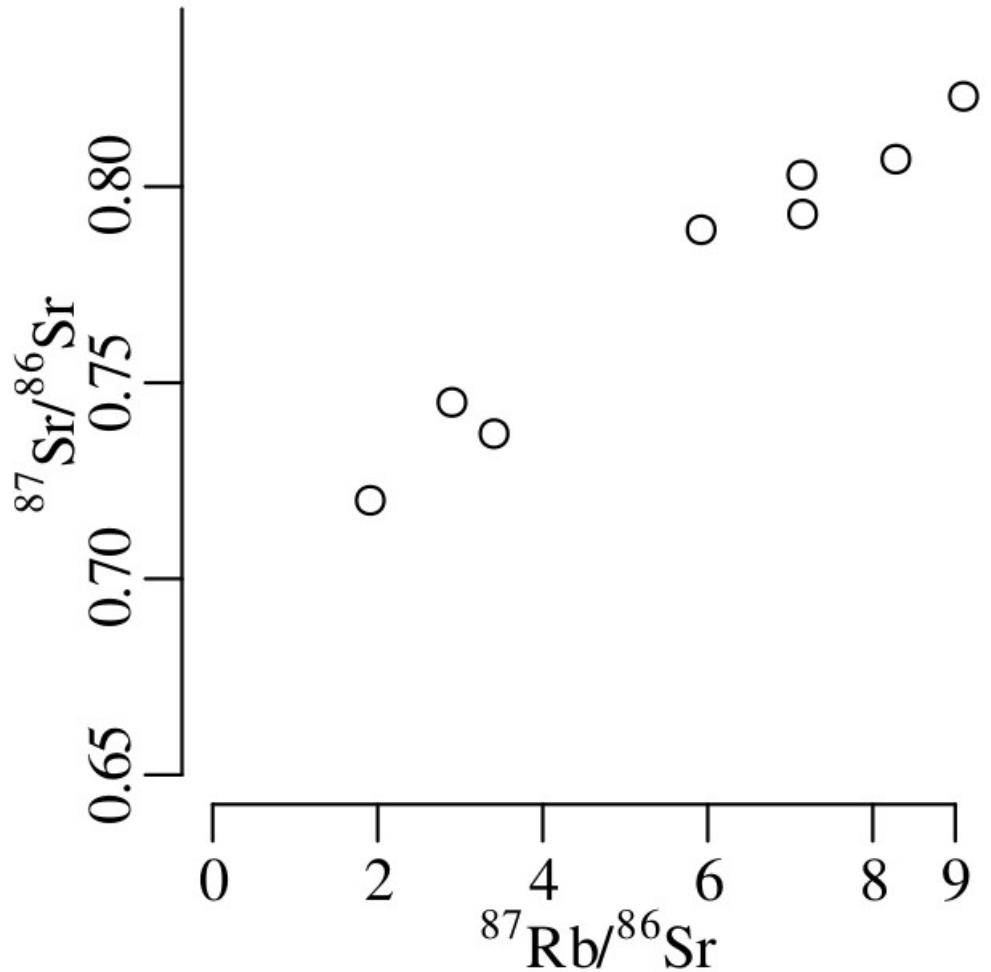
$$\left[ \frac{^{87}Sr}{^{86}Sr} \right] = \left[ \frac{^{87}Sr}{^{86}Sr} \right]_0 + \left[ \frac{^{87}Rb}{^{86}Sr} \right] (e^{\lambda t} - 1)$$

$$y_i = \beta_0 + \beta_1 x_i$$

$i$	1	2	3	4	5	6	7	8
$[^{87}Rb/^{86}Sr] = x_i$	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28
$[^{87}Sr/^{86}Sr] = y_i$	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807

$$\left[ \frac{^{87}Sr}{^{86}Sr} \right] = \left[ \frac{^{87}Sr}{^{86}Sr} \right]_o + \left[ \frac{^{87}Rb}{^{86}Sr} \right] (e^{\lambda t} - 1)$$

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$



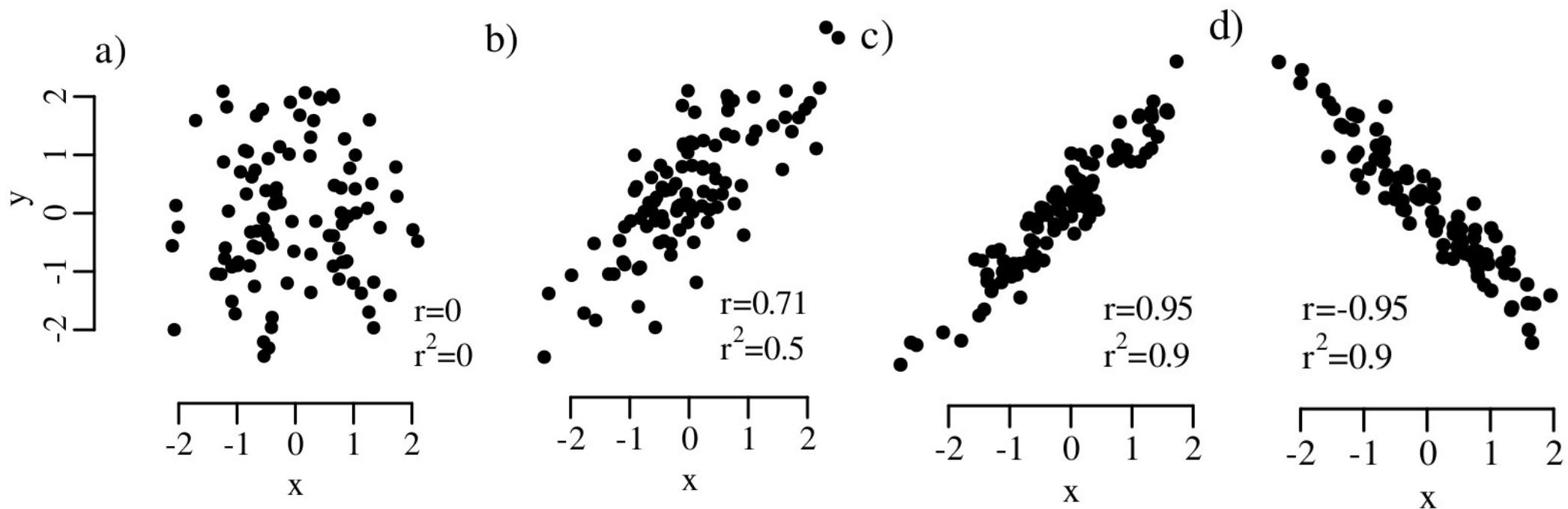
$$\rho = \frac{\sigma_{x,y}}{\sigma_x\sigma_y}$$

$$s[x] = \sqrt{\sum_{i=1}^n \frac{1}{n-1}(x_i - \bar{x})^2}$$

$$s[x,y] = \sum_{i=1}^n \frac{1}{n-1}(x_i - \bar{x})(y_i - \bar{y})$$

$$r=\frac{s[x,y]}{s[x]s[y]}$$

$$r = \frac{s[x, y]}{s[x]s[y]}$$



$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

1.  $H_0$  (null hypothesis)  $\rho = 0$

$H_a$  (alternative hypothesis):  $\rho \neq 0$

2.  $t = \frac{0.985\sqrt{8-2}}{\sqrt{1-0.985^2}} = 13.98$

3.

$t$	-3.10 -2.40 -1.90 -1.40 -0.72 0 0.72 1.40 1.90 2.40 3.10 13.98
$P(t \leq T)$	0.01 0.025 0.05 0.1 0.25 0.5 0.75 0.9 0.95 0.975 0.99 0.9999958
$P(t \geq T)$	0.99 0.975 0.95 0.9 0.75 0.5 0.25 0.1 0.05 0.025 0.010 0.0000042

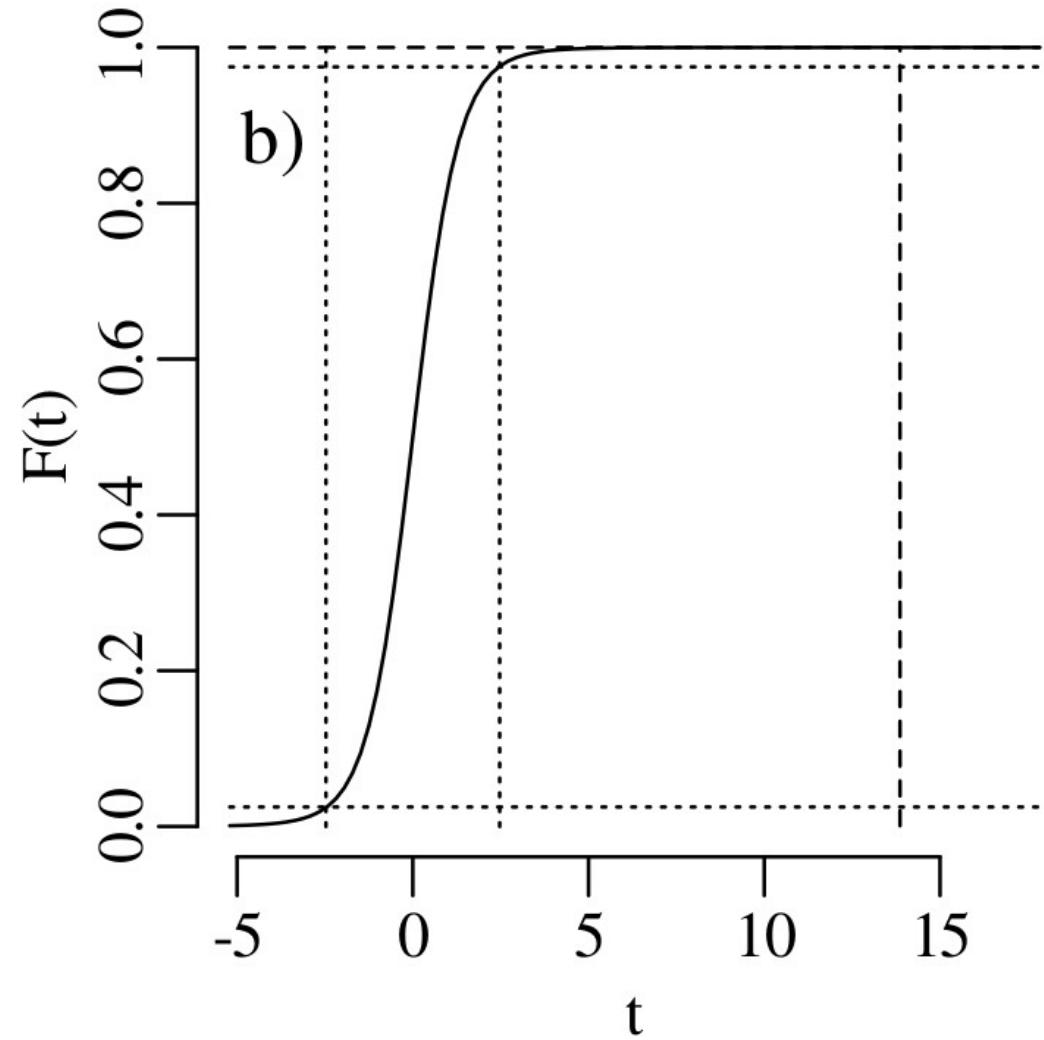
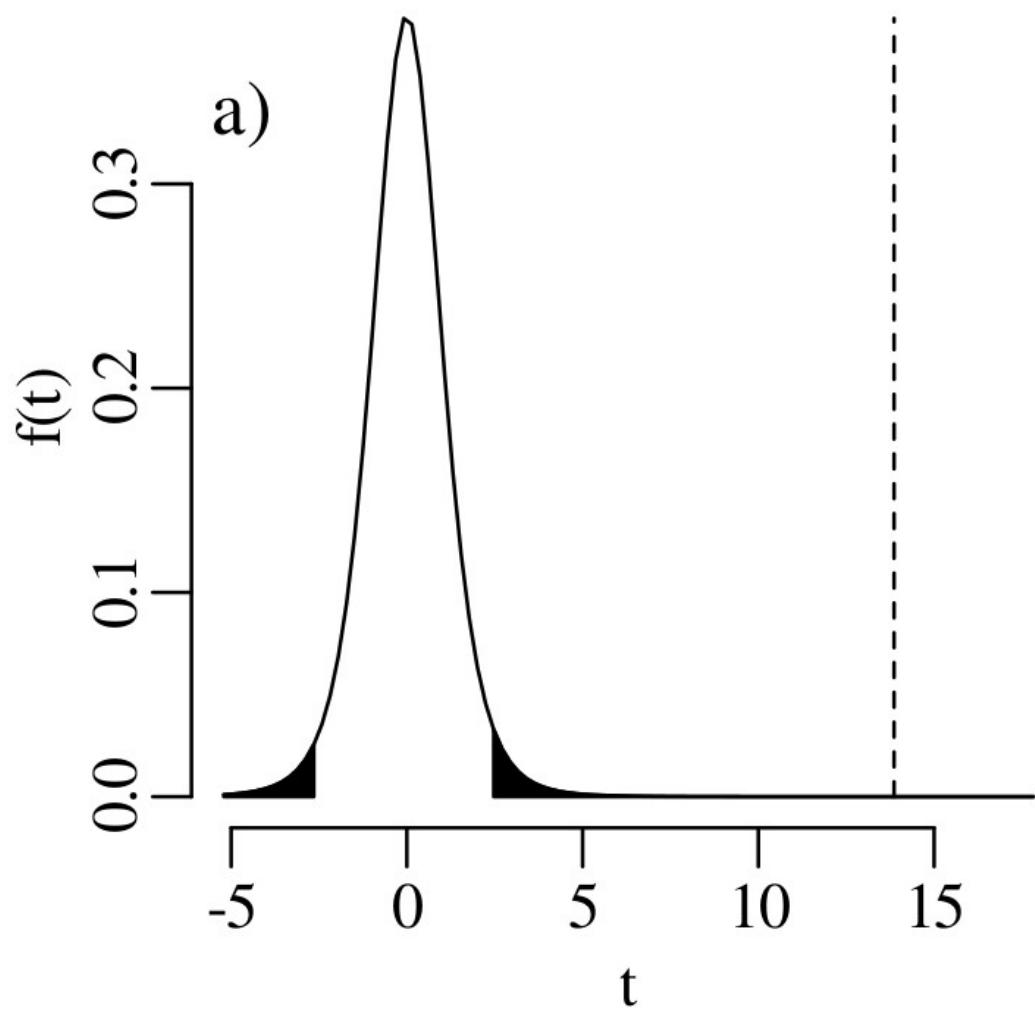
4.  $\alpha = 0.05$

5.	$t$	-3.10	-2.40	-1.90	-1.40	-0.72	0	0.72	1.40	1.90	2.40	3.10	13.98
	$P(t \leq T)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99	0.9999958
	$P(t \geq T)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.010	0.0000042

$$R = \{|t| > 2.40\}$$

6.  $t = 13.98 \in R$

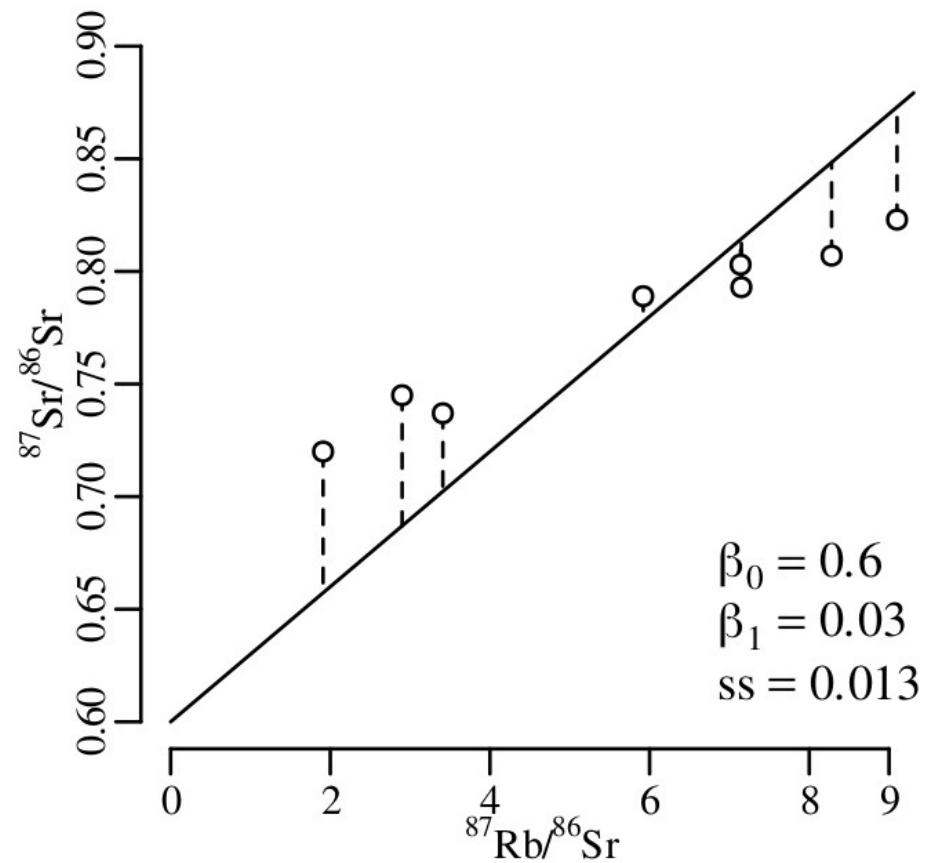
7. p-value =  $2 \times 0.0000042 = 0.0000084 < \alpha$



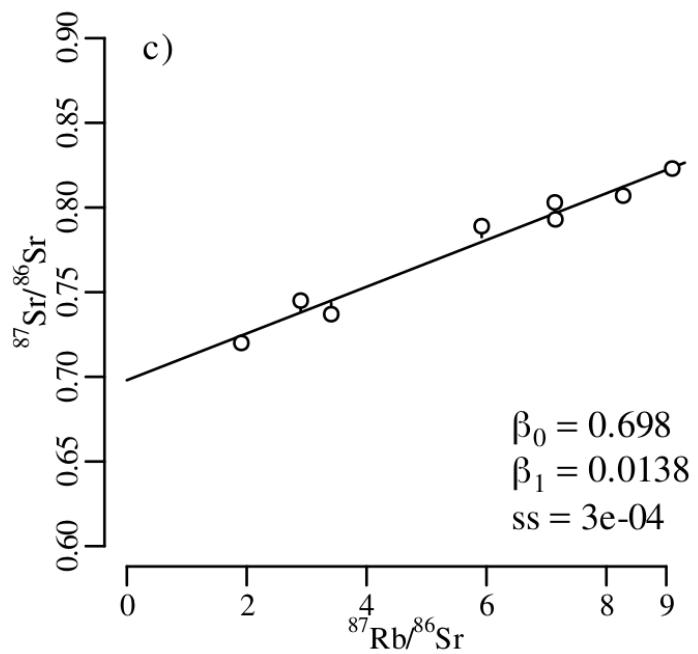
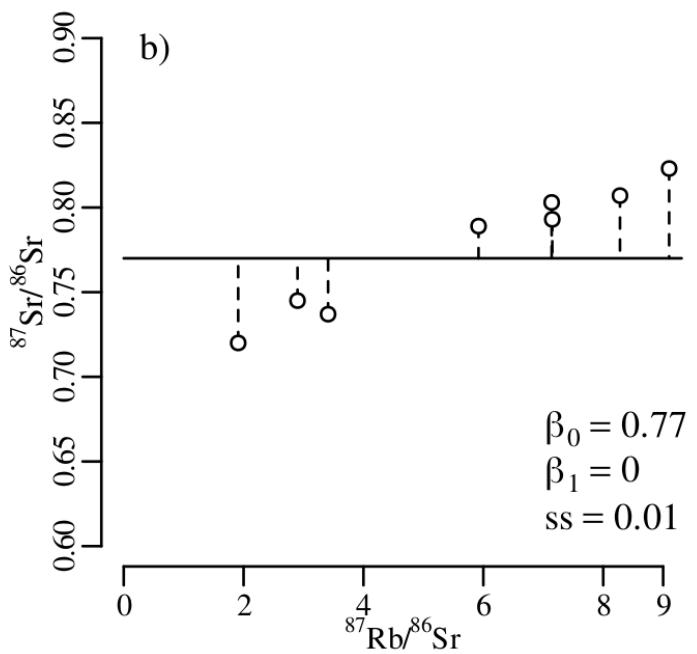
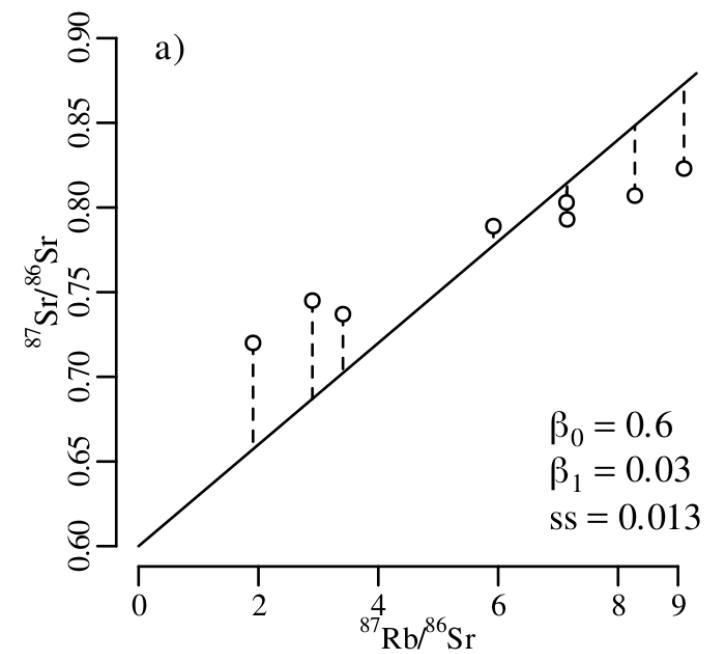
# Least Squares

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$ss \equiv \sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n (\beta_0 + \beta_1 x_i - y_i)^2$$



$i$	1	2	3	4	5	6	7	8
$x_i$	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28
$y_i$	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807
$\beta_0 + \beta_1 x_i$	0.687	0.8142	0.873	0.7023	0.6573	0.8145	0.7776	0.8484
$\epsilon_i$	-0.058	0.0112	0.05	-0.0347	-0.0627	0.0215	-0.0114	0.0414

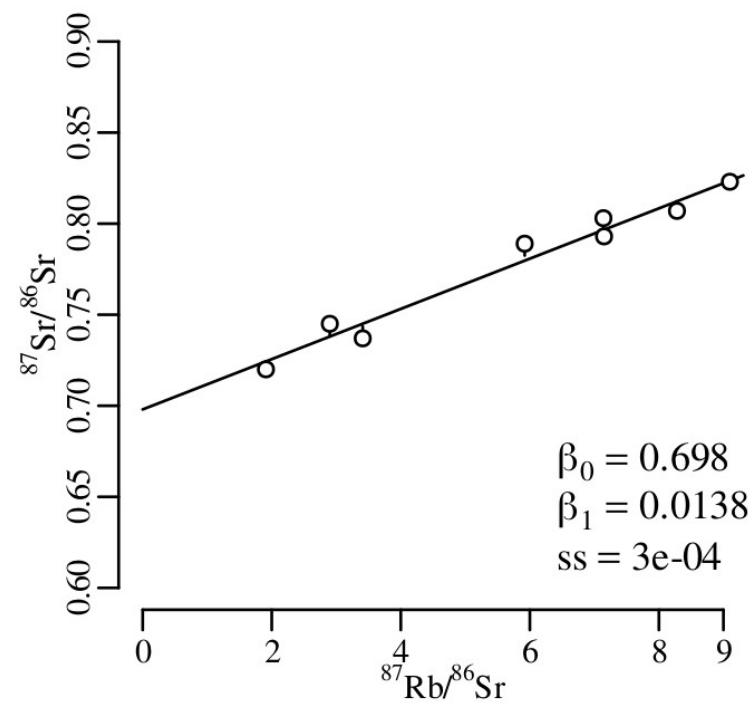


$$\begin{cases} \frac{\partial ss}{\partial \beta_0} = 2 \sum (\beta_0 + \beta_1 x_i - y_i) = 0 \\ \frac{\partial ss}{\partial \beta_1} = 2 \sum (\beta_0 + \beta_1 x_i - y_i) x_i = 0 \end{cases}$$

$$\begin{cases} \hat{\beta}_0 = (\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i) / \left( n \sum x_i^2 - (\sum x_i)^2 \right) \\ \hat{\beta}_1 = (n \sum x_i y_i - \sum x_i \sum y_i) / \left( n \sum x_i^2 - (\sum x_i)^2 \right) \end{cases}$$

$i$	1	2	3	4	5	6	7	8	$\sum_{i=1}^8$
$x_i$	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28	45.81
$y_i$	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807	6.217
$x_i^2$	8.41	50.98	82.81	11.63	3.648	51.12	35.05	68.56	312.2
$x_i y_i$	2.160	5.733	7.489	2.513	1.375	5.670	4.671	6.682	36.29

$$\begin{cases} \hat{\beta}_0 = (312.2 \times 6.217 - 45.81 \times 36.29) / (8 \times 312.2 - 45.81^2) = 0.698 \\ \hat{\beta}_1 = (8 \times 36.29 - 45.81 \times 6.217) / (8 \times 312.2 - 45.81^2) = 0.0138 \end{cases}$$



# Maximum Likelihood

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$f(\epsilon_i | \beta_0, \beta_1, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{\epsilon_i^2}{2\sigma^2}\right], \text{ where } \epsilon_i = \beta_0 + \beta_1 x_i - y_i$$

$$\mathcal{L}(\beta_0, \beta_1, \sigma | \{(x_1, y_1), \dots, (x_n, y_n)\}) = \prod_{i=1}^n \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{\epsilon_i^2}{2\sigma^2}\right]$$

$$\begin{aligned}
\max_{\beta_0, \beta_1} \left[ \prod_{i=1}^n \mathcal{L} \right] &= \max_{\beta_0, \beta_1} \left[ \sum_{i=1}^n \ln \mathcal{L} \right] = \max_{\beta_0, \beta_1} \left[ \ln \left( \frac{1}{\sigma \sqrt{2\pi}} \right) - \sum_{i=1}^n \left( \frac{\epsilon_i^2}{2\sigma^2} \right) \right] \\
&= \max_{\beta_0, \beta_1} \left[ - \sum_{i=1}^n \epsilon_i^2 \right] = \min_{\beta_0, \beta_1} \left[ \sum_{i=1}^n \epsilon_i^2 \right] = \min_{\beta_0, \beta_1} (ss)
\end{aligned}$$

$$\Sigma_{\hat{\beta}} = \begin{bmatrix} s[\hat{\beta}_0]^2 & s[\hat{\beta}_0, \hat{\beta}_1] \\ s[\hat{\beta}_0, \hat{\beta}_1] & s[\hat{\beta}_1]^2 \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2 \sum_{i=1}^n x_i^2}{n \sum_{i=1}^n (x_i^2 - \bar{x}^2)} & -\frac{\sigma^2 \bar{x}}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} \\ -\frac{\sigma^2 \bar{x}}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} & \frac{\sigma^2}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} \end{bmatrix}$$

$$\beta_0 \in \left\{ \hat{\beta}_0 \pm t_{df,\alpha/2} \ s[\hat{\beta}_0] \right\}$$

$$\beta_1 \in \left\{ \hat{\beta}_1 \pm t_{df,\alpha/2} \ s[\hat{\beta}_1] \right\}$$

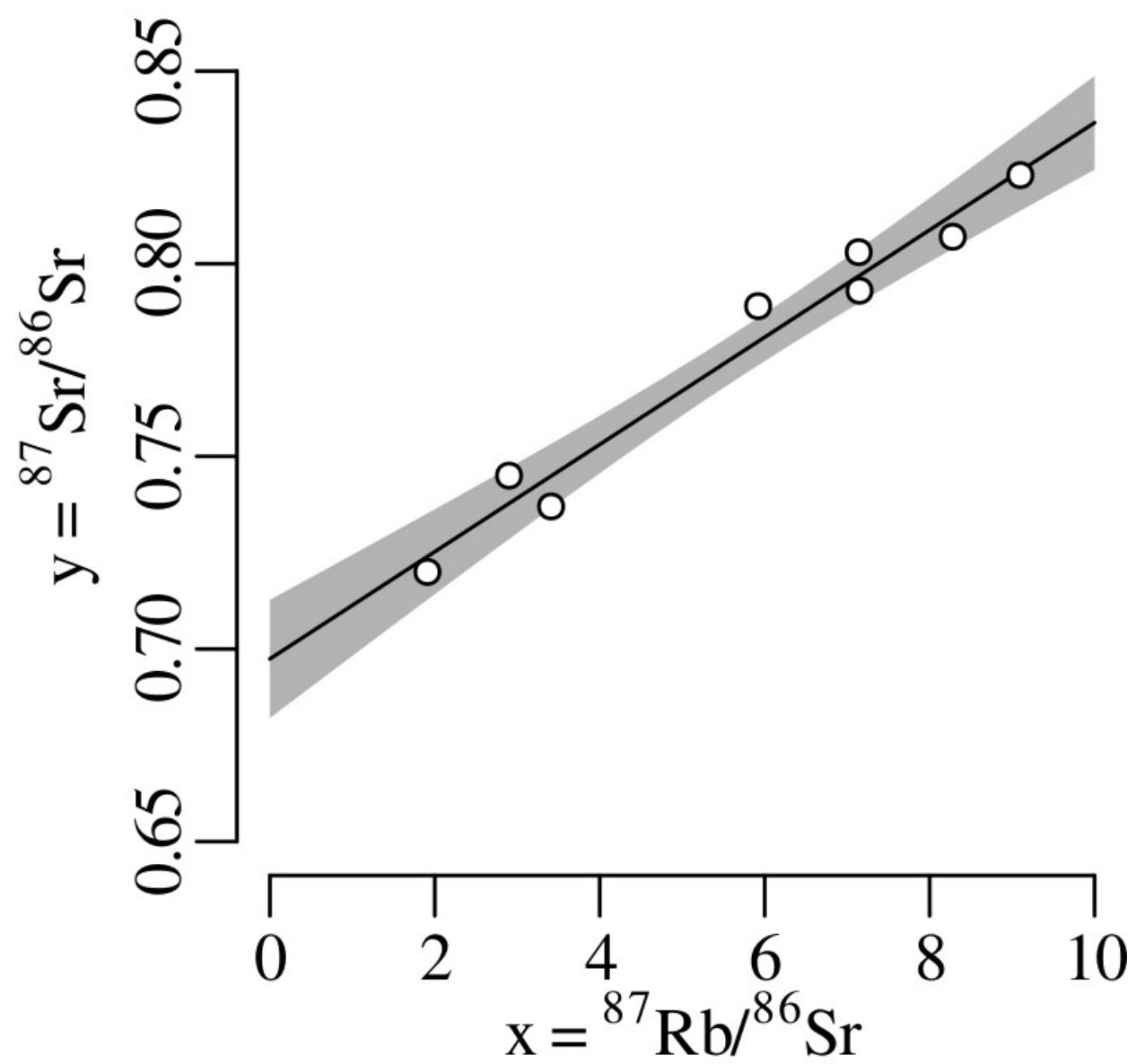
where  $df = n - 2$

$$y = \beta_0 + \beta_1 x$$



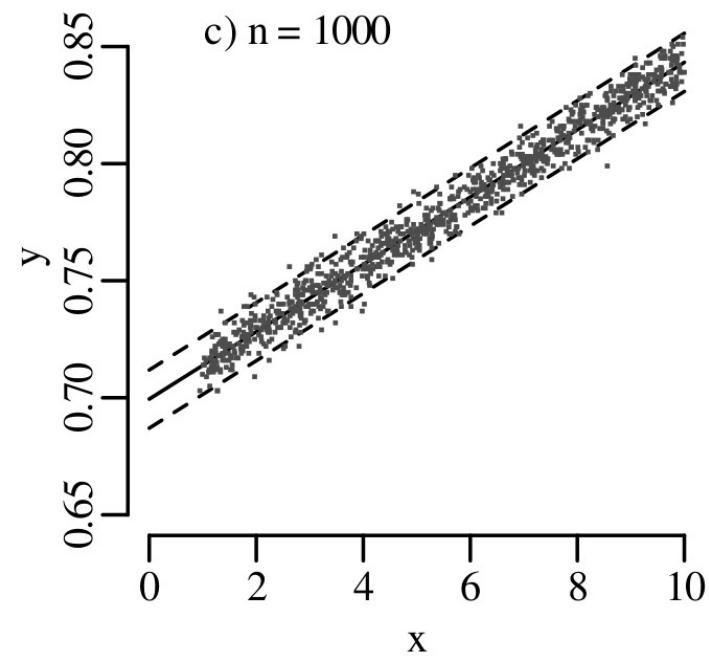
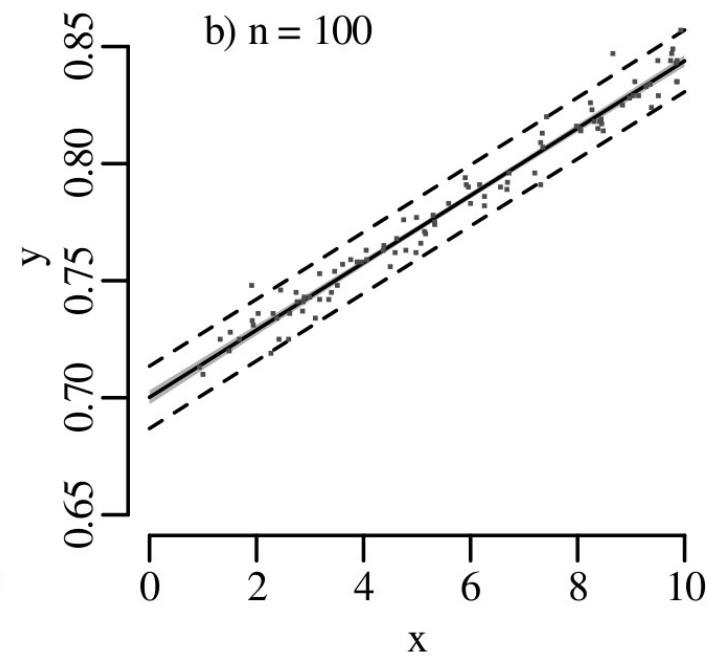
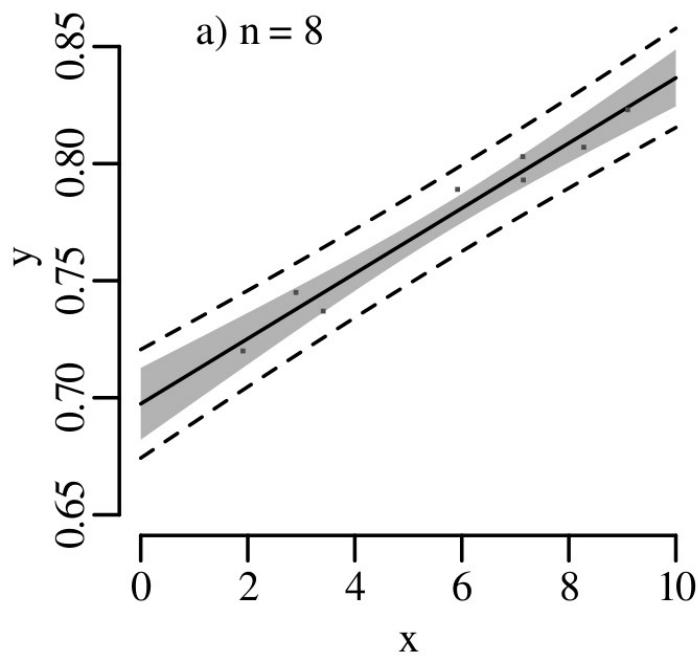
$$\Sigma_{\hat{\beta}} = \begin{bmatrix} s[\hat{\beta}_0]^2 & s[\hat{\beta}_0, \hat{\beta}_1] \\ s[\hat{\beta}_0, \hat{\beta}_1] & s[\hat{\beta}_1]^2 \end{bmatrix}$$

$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df, \alpha/2} \sqrt{s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1]x} \right\}$$

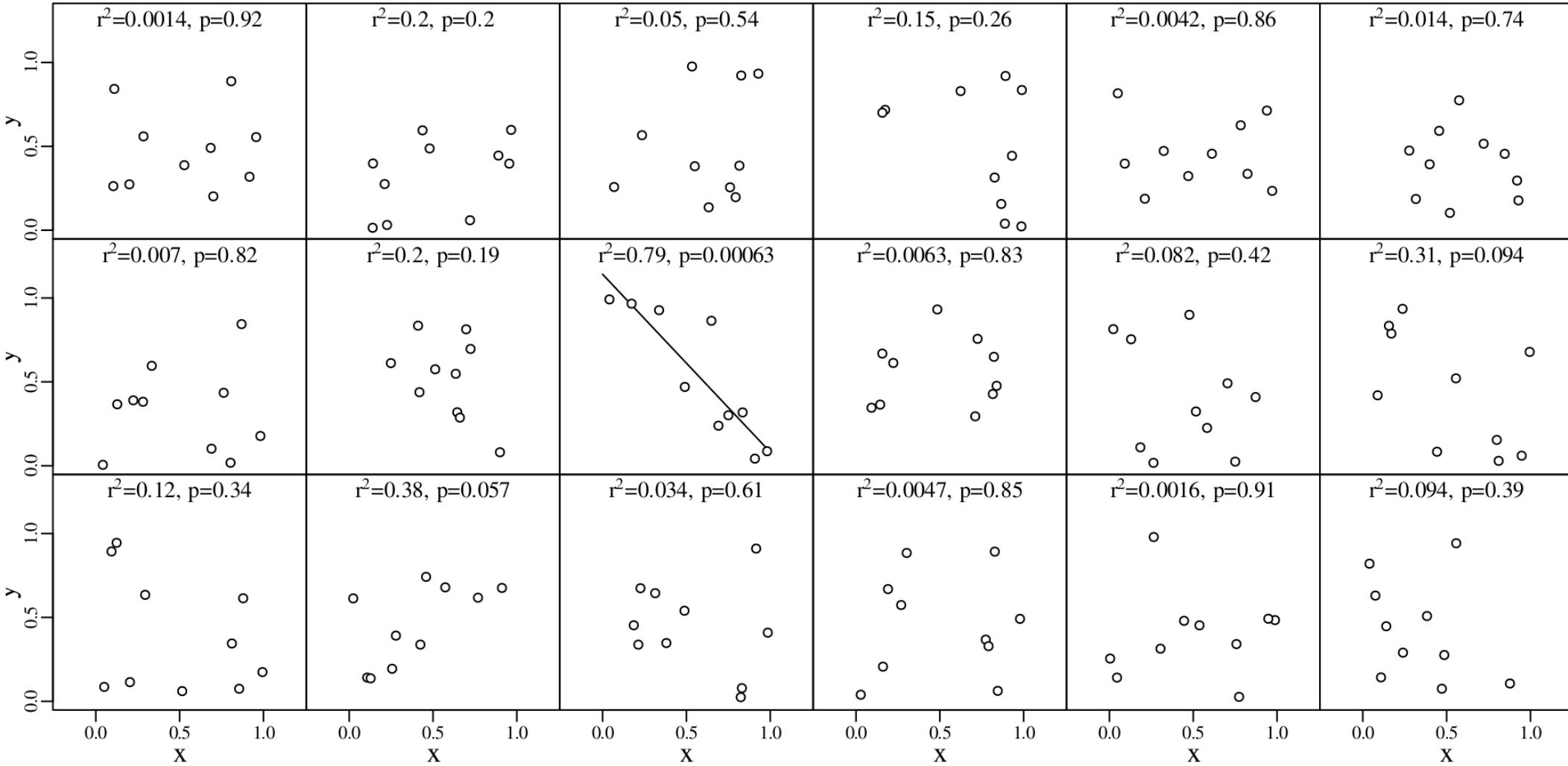


$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df,\alpha/2} \sqrt{s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1]x} \right\}$$

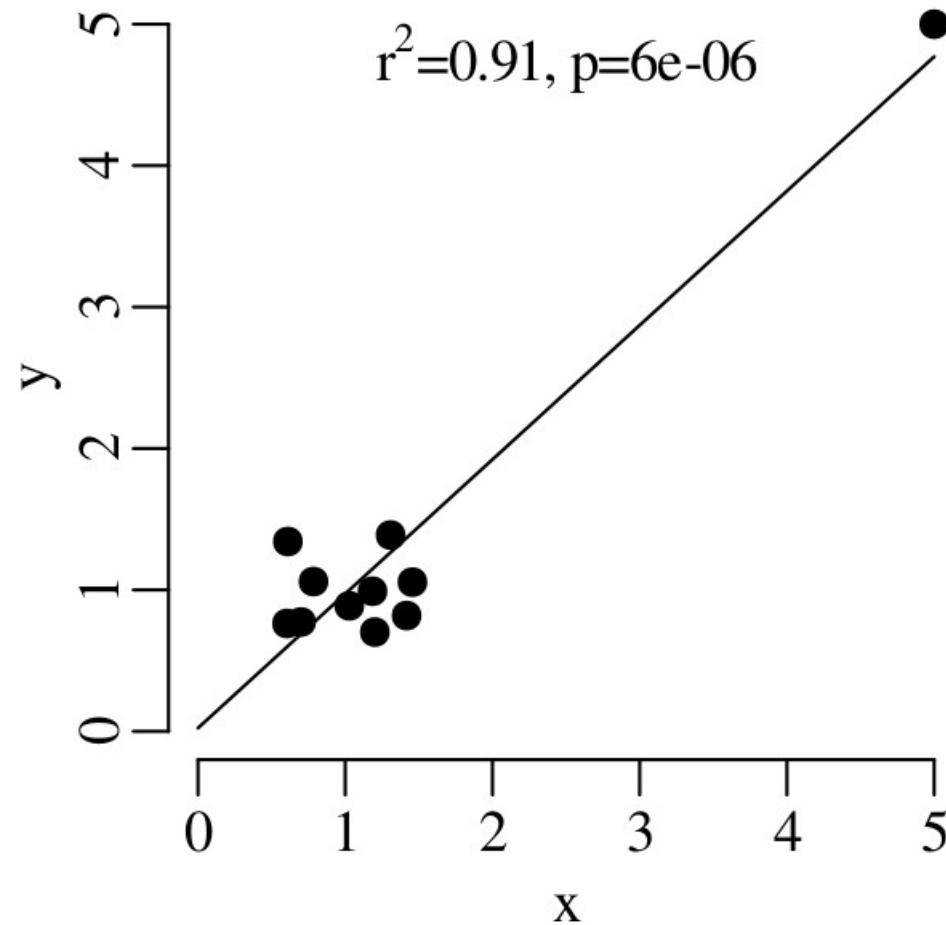
$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df,\alpha/2} \sqrt{\hat{\sigma}^2 + s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1]x} \right\}$$



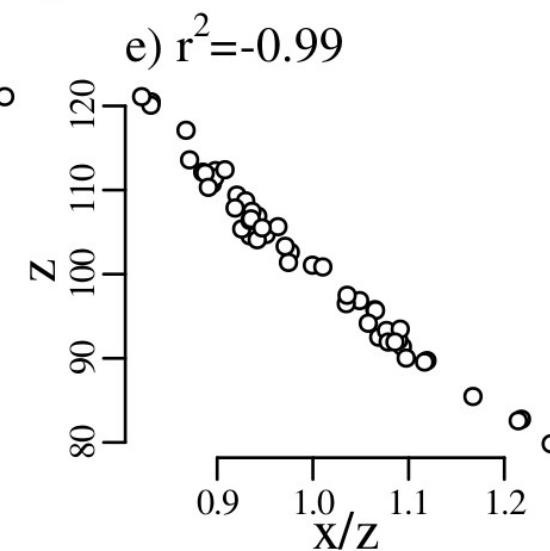
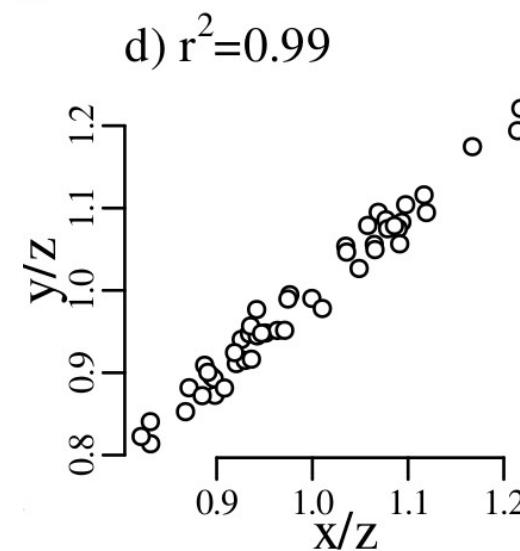
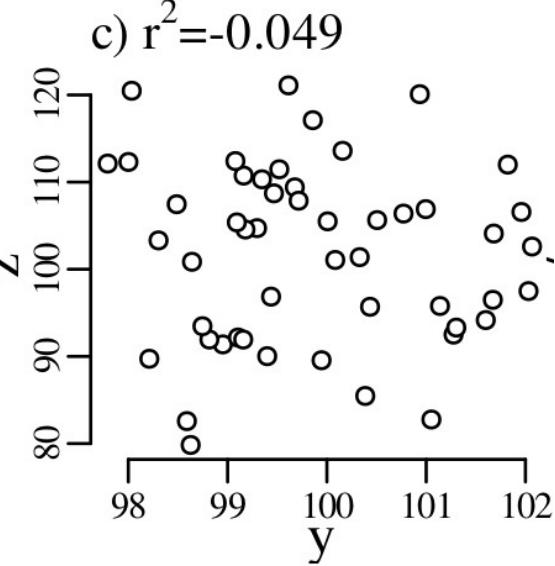
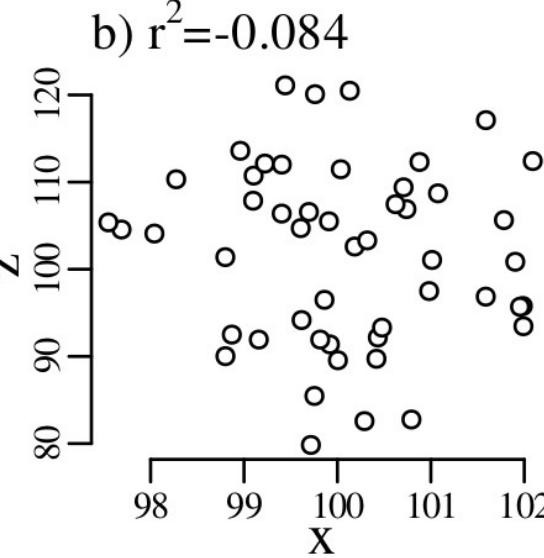
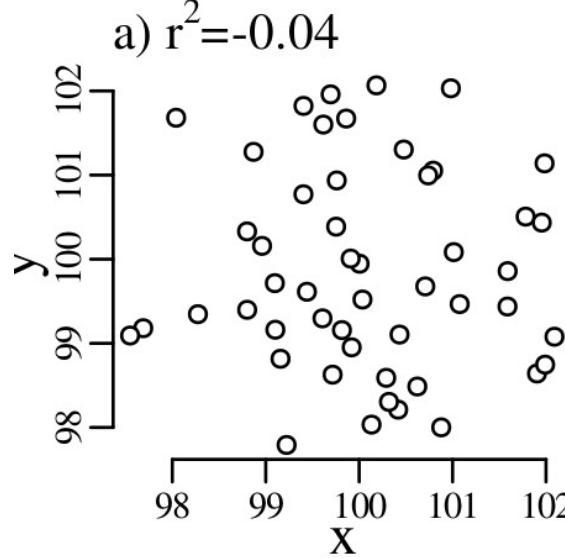
# p-hacking

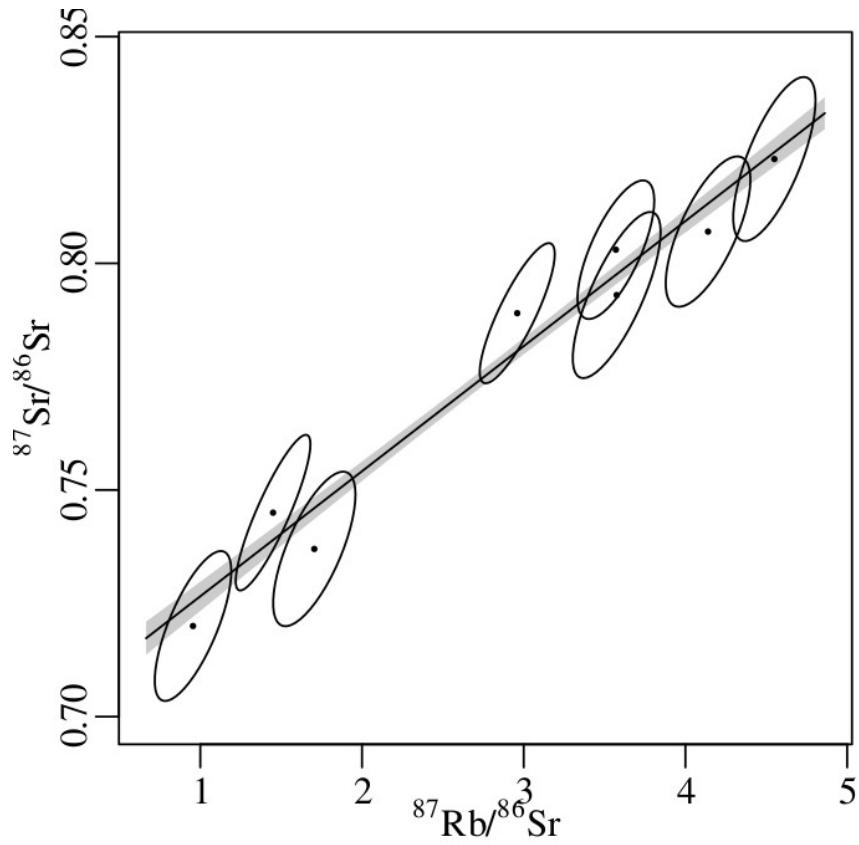
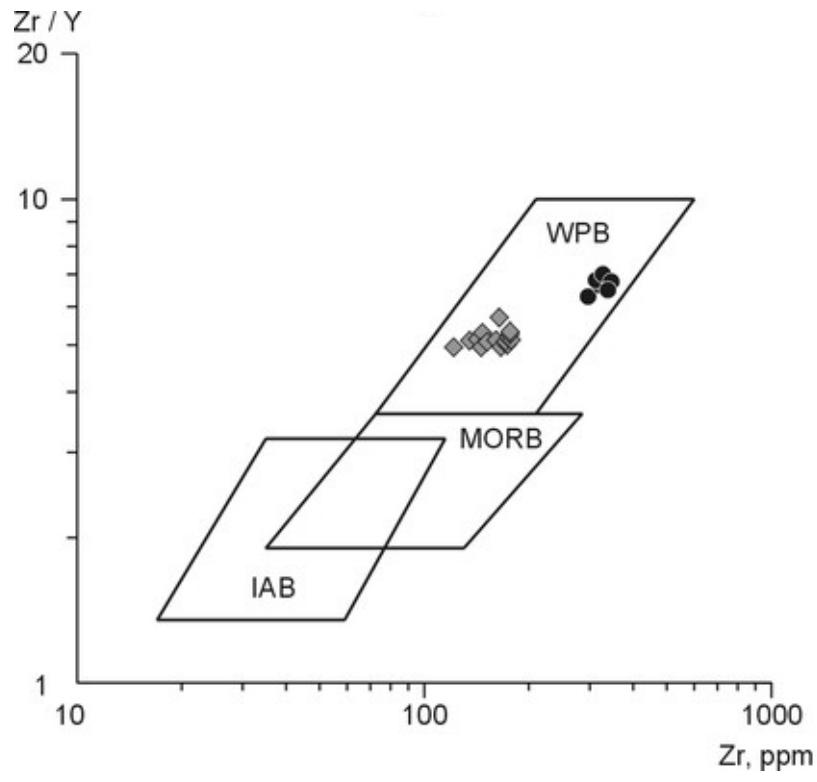


# outliers



# Spurious correlation



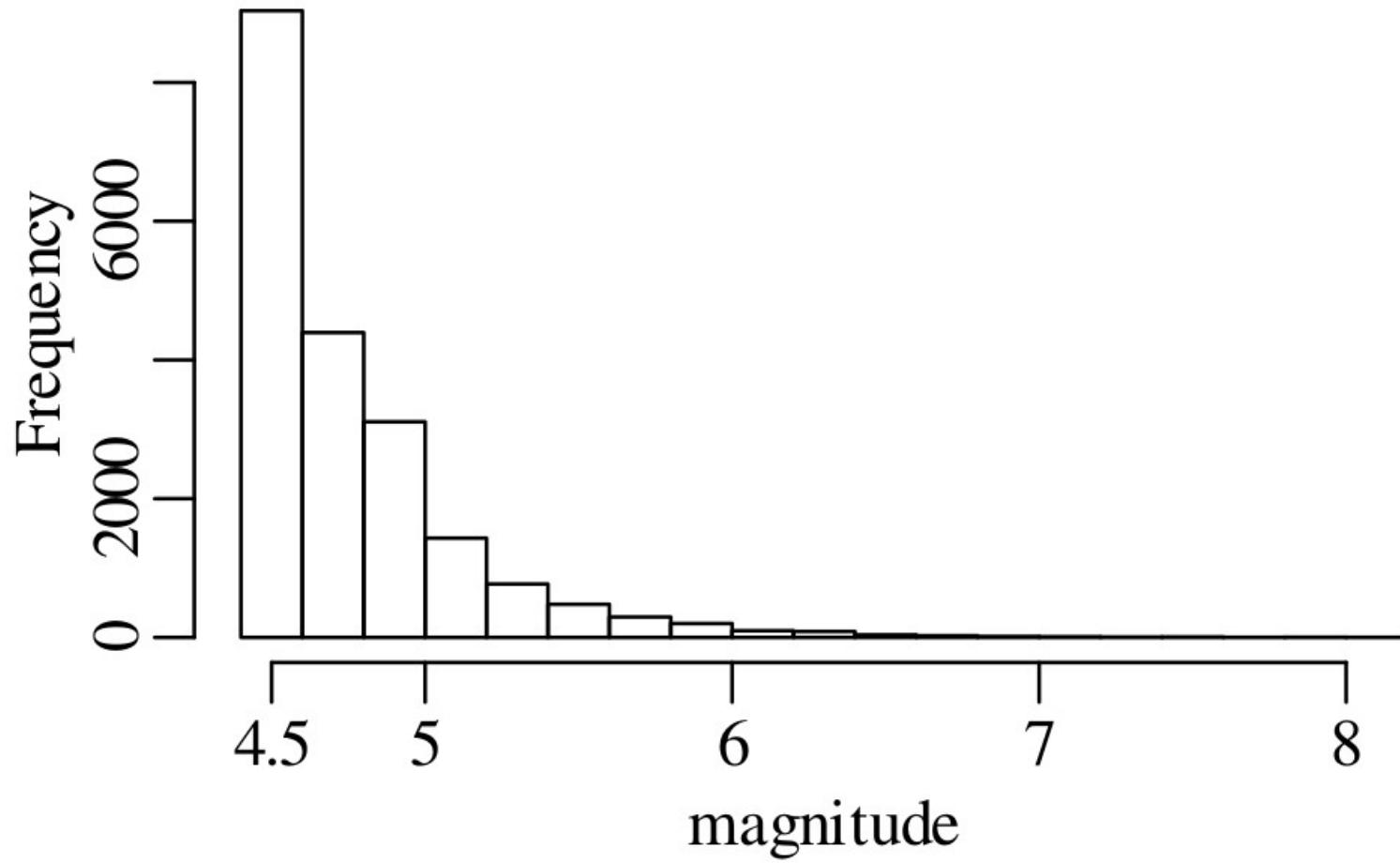


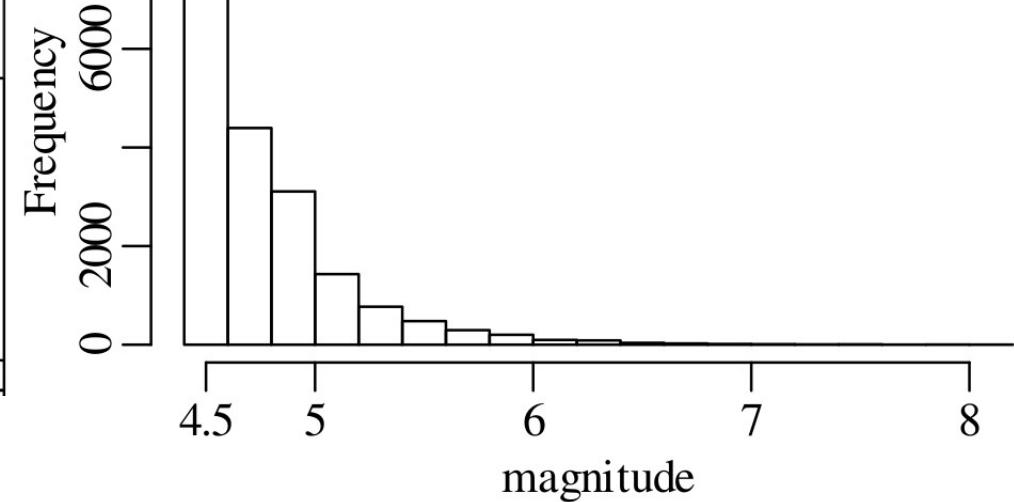
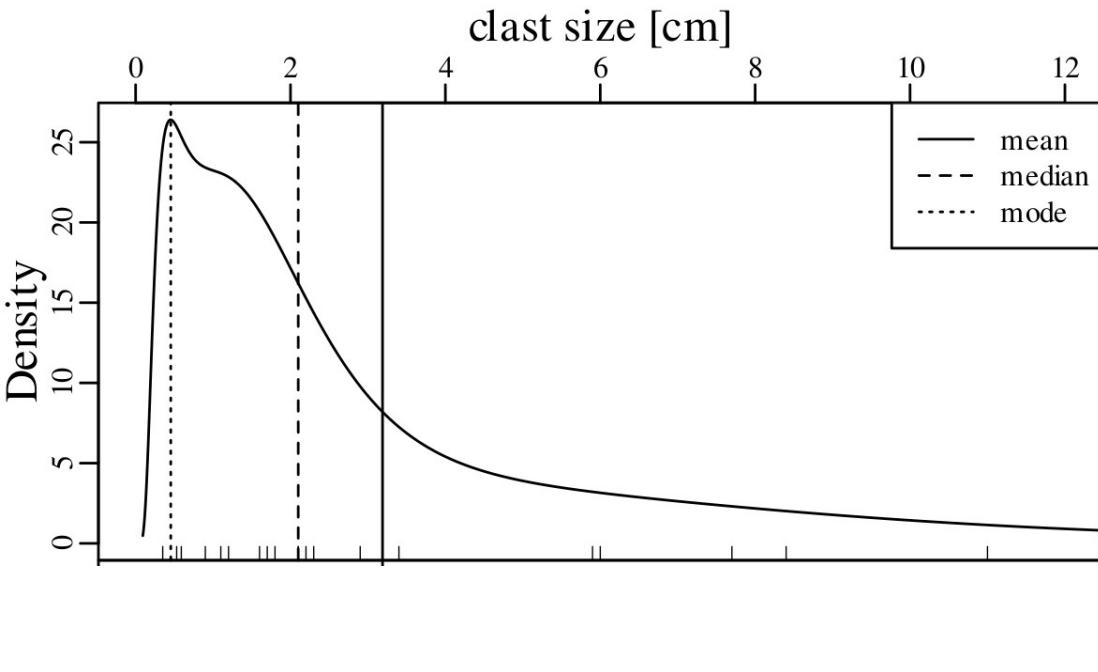
$$\rho_{\frac{w}{y}, \frac{x}{z}} \approx \frac{\rho_{w,x} \left[ \frac{\sigma_w}{\mu_w} \right] \left[ \frac{\sigma_x}{\mu_x} \right] - \rho_{w,z} \left[ \frac{\sigma_w}{\mu_w} \right] \left[ \frac{\sigma_z}{\mu_z} \right] - \rho_{x,y} \left[ \frac{\sigma_x}{\mu_x} \right] \left[ \frac{\sigma_y}{\mu_y} \right] + \rho_{y,z} \left[ \frac{\sigma_y}{\mu_y} \right] \left[ \frac{\sigma_z}{\mu_z} \right]}{\sqrt{\left[ \frac{\sigma_w}{\mu_w} \right]^2 + \left[ \frac{\sigma_y}{\mu_y} \right]^2 - 2\rho_{w,y} \left[ \frac{\sigma_w}{\mu_w} \right] \left[ \frac{\sigma_y}{\mu_y} \right]} \sqrt{\left[ \frac{\sigma_x}{\mu_x} \right]^2 + \left[ \frac{\sigma_z}{\mu_z} \right]^2 - 2\rho_{x,z} \left[ \frac{\sigma_x}{\mu_x} \right] \left[ \frac{\sigma_z}{\mu_z} \right]}}$$

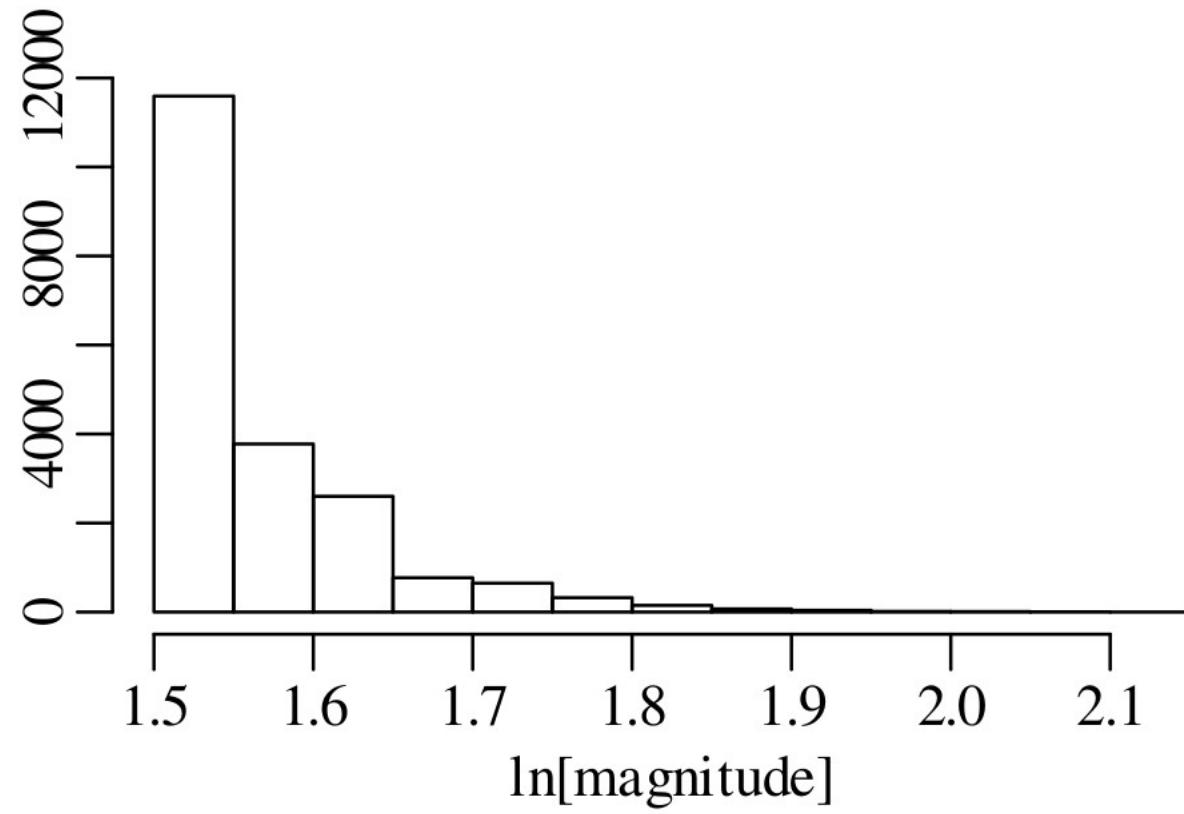
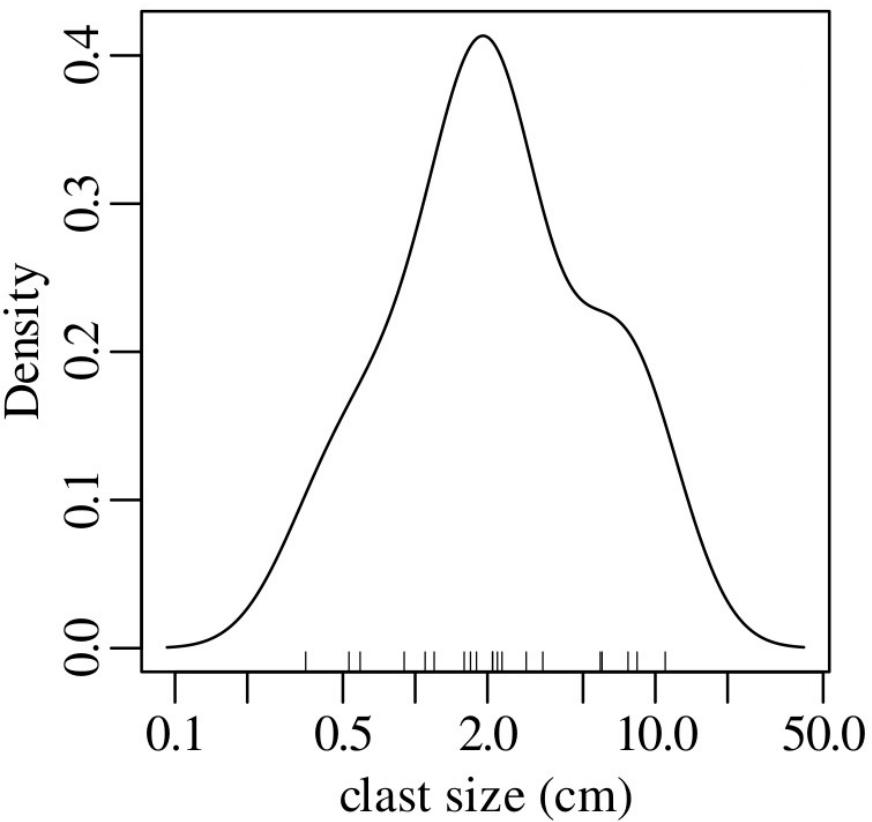
$$\rho_{\frac{y}{z}, \frac{x}{z}} \approx \frac{\left[ \frac{\sigma_z}{\mu_z} \right]^2}{\sqrt{\left[ \frac{\sigma_y}{\mu_y} \right]^2 + \left[ \frac{\sigma_z}{\mu_z} \right]^2} \sqrt{\left[ \frac{\sigma_x}{\mu_x} \right]^2 + \left[ \frac{\sigma_z}{\mu_z} \right]^2}}$$

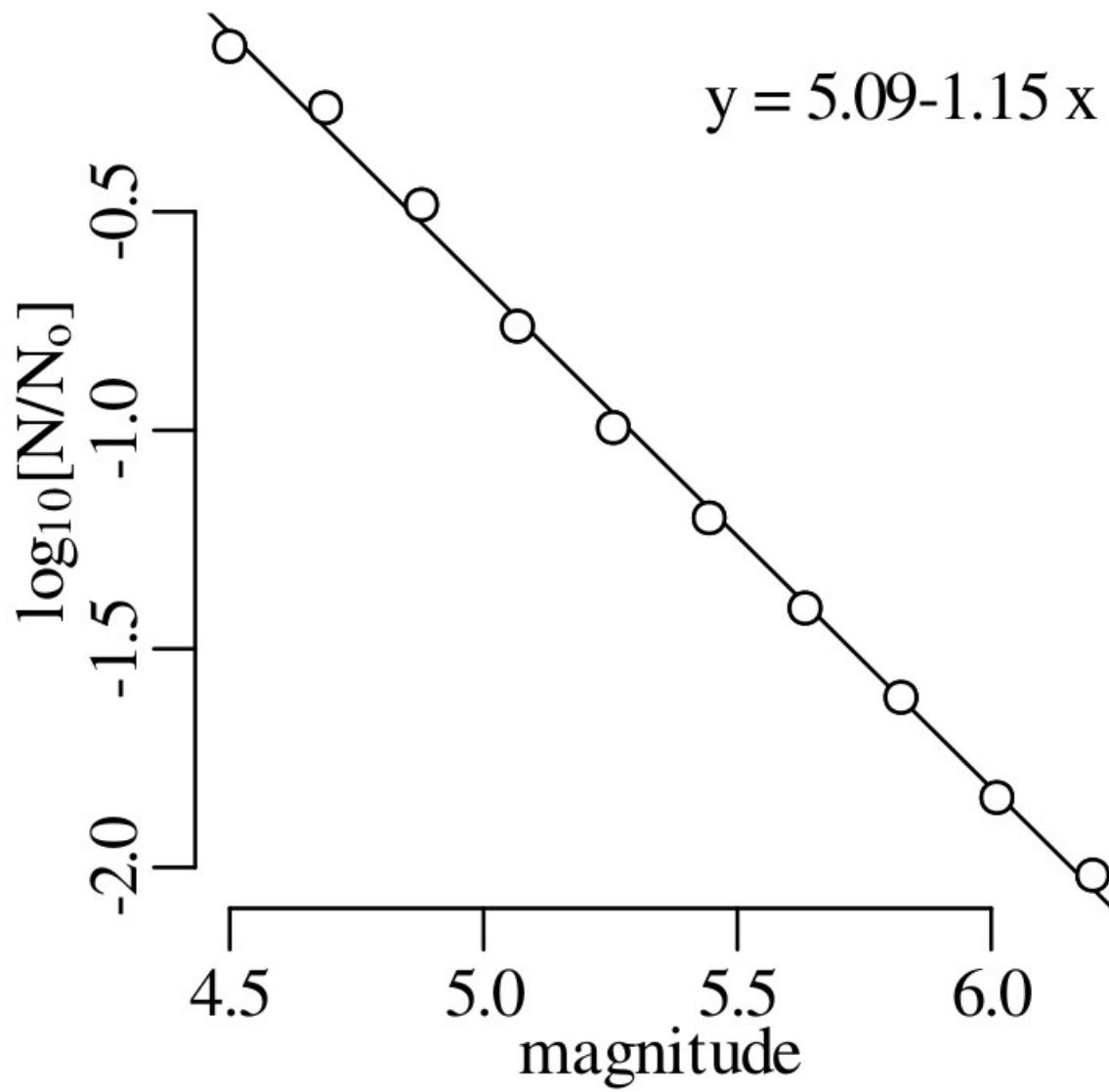
# Statistics for geoscientists

Fractals and chaos

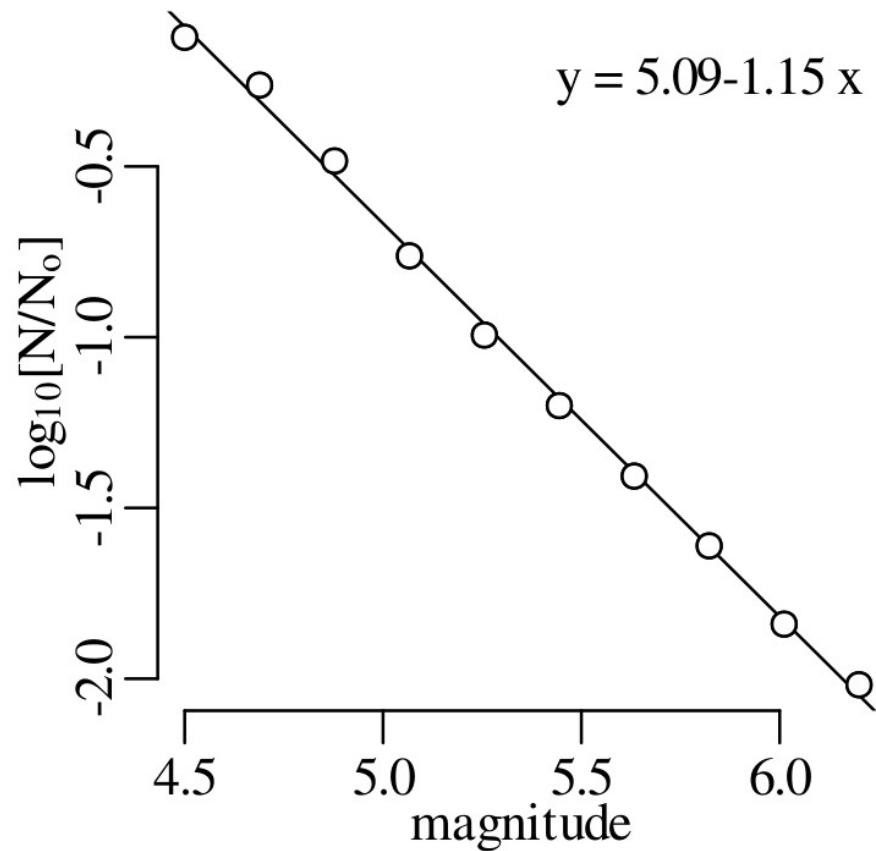








$P(\geq 9.0 \text{ earthquake})?$

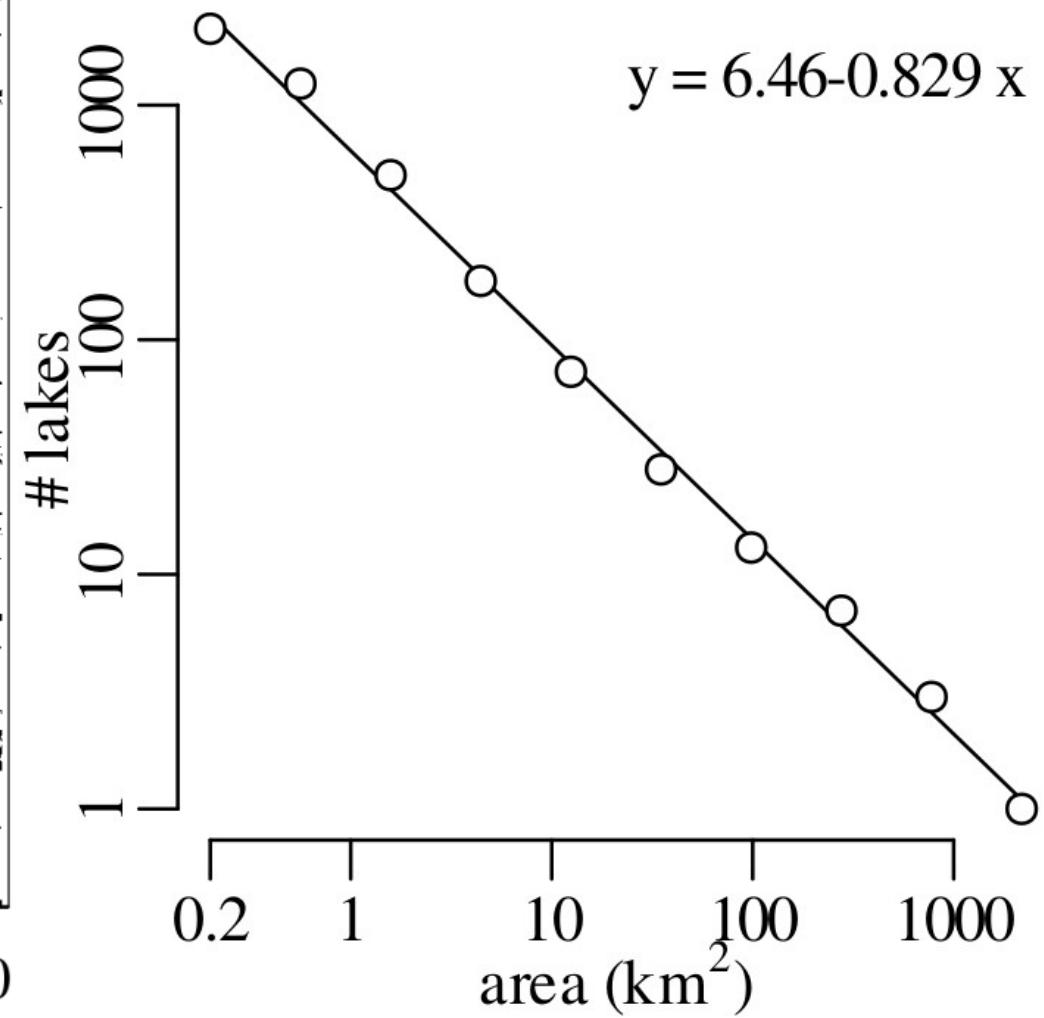
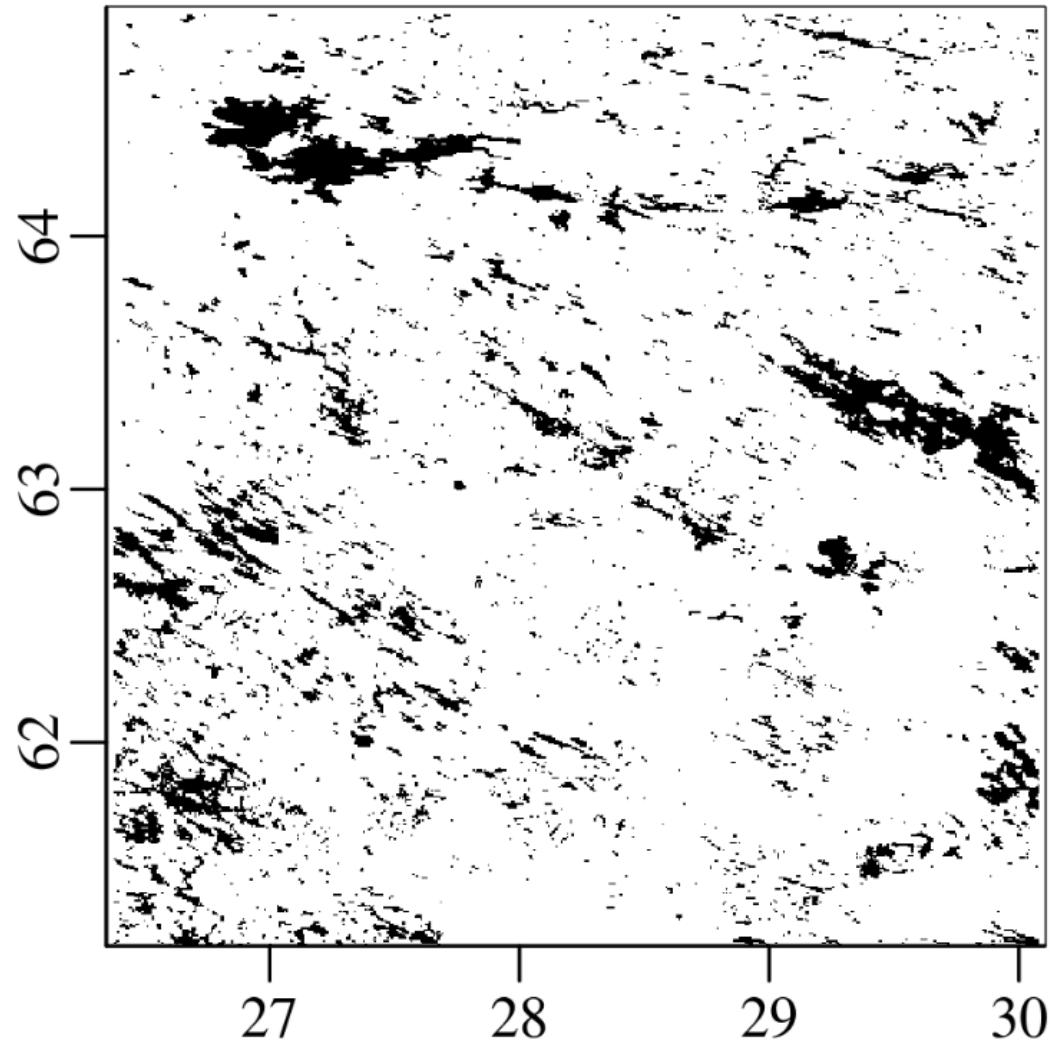


$$\log_{10}[N/N_o] = a + b \text{ magnitude}$$

$$N_o = \frac{20000 \text{ earthquakes}}{1031 \text{ days}} \times 365 \text{ days} \\ = 7080 \text{ earthquakes}$$

$$\log_{10}[N/N_o] = 5.09 - 1.15 \times 9.0 = -5.26$$

$$N = 7080 \times 10^{-5.26} = 0.039 \\ = 3.9\%$$

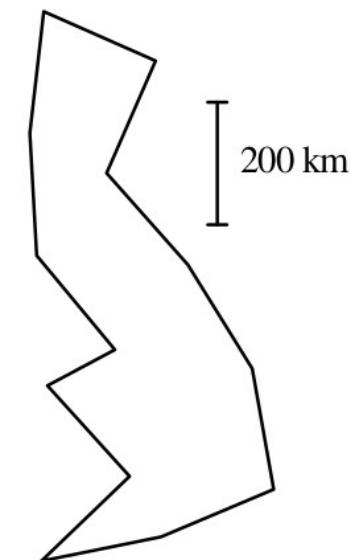


# Reports

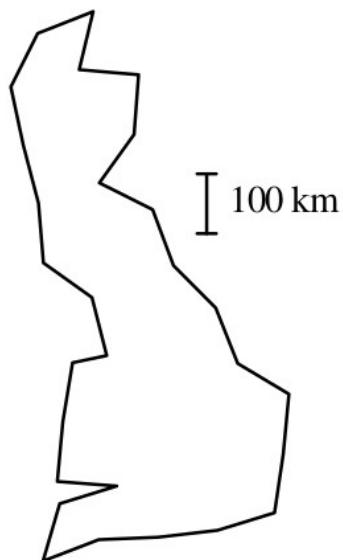
## How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

BENOIT MANDELBROT

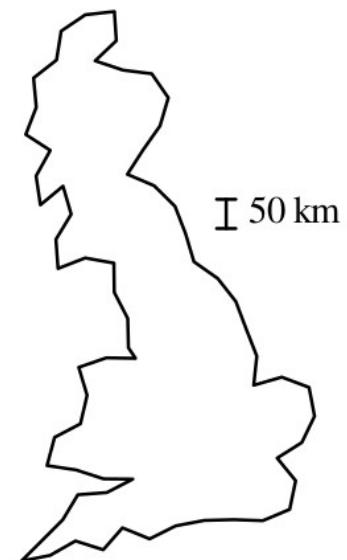
*International Business Machines,  
Thomas J. Watson Research Center,  
Yorktown Heights, New York 10598*



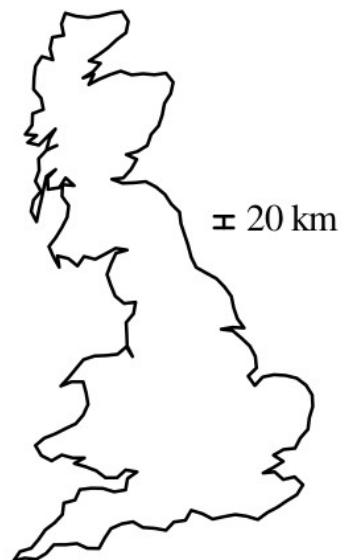
length = 2800km



length = 2900km



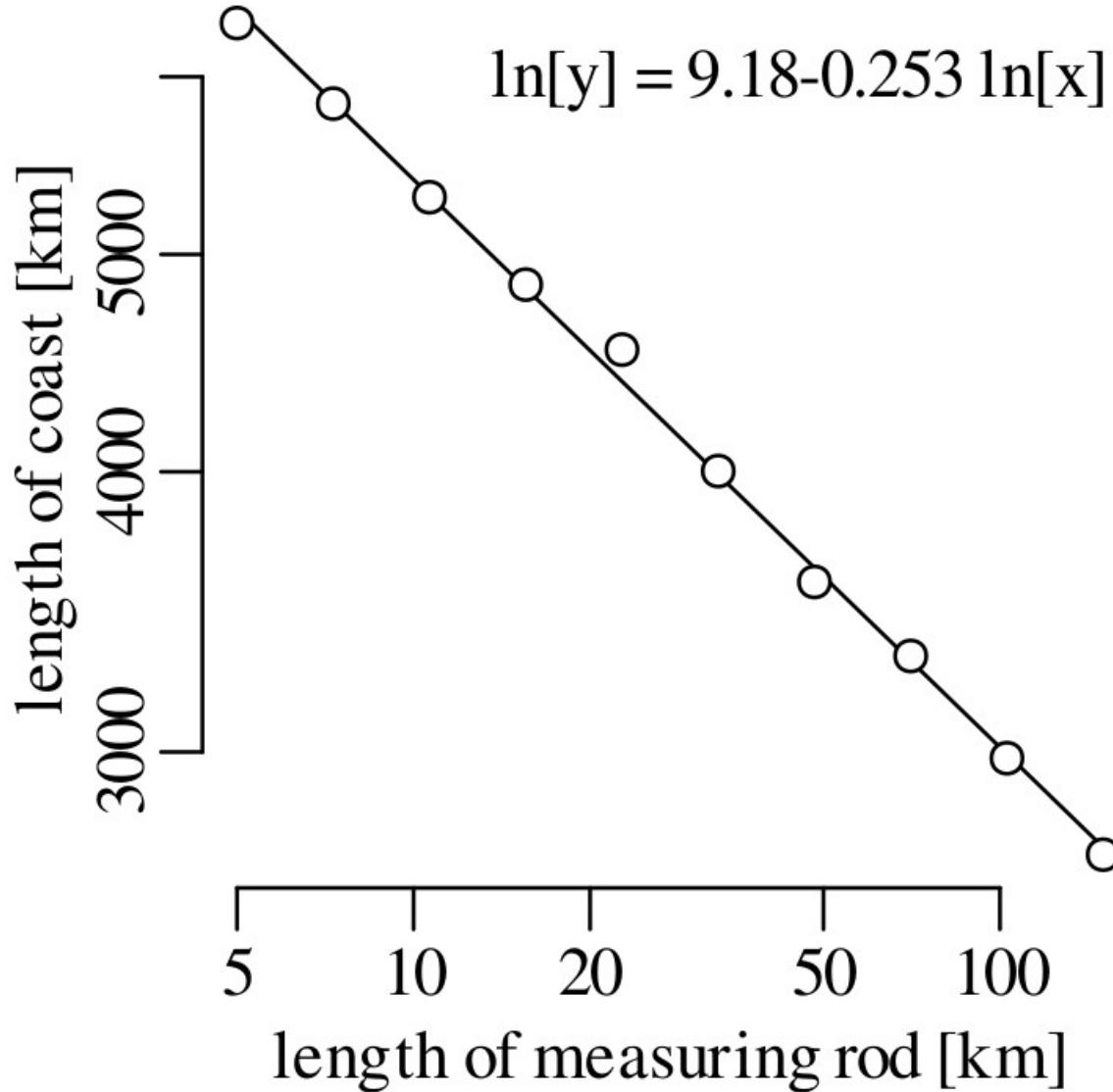
length = 3400km

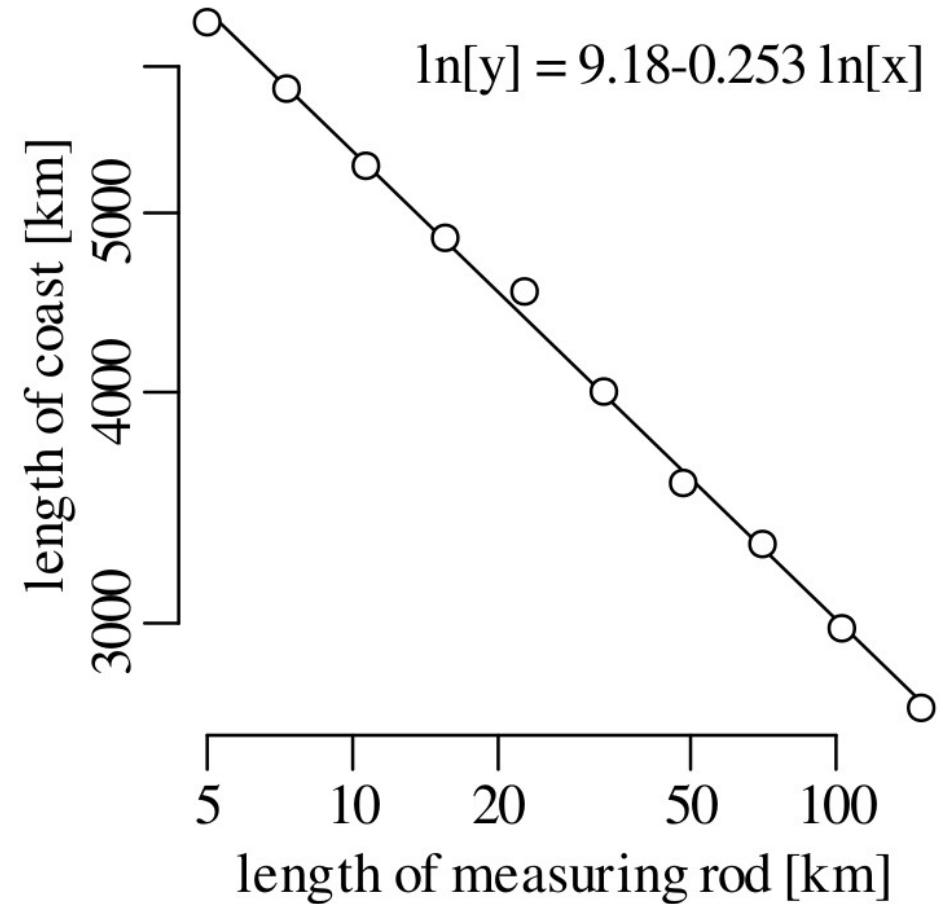


length = 4580km



length = 5490km

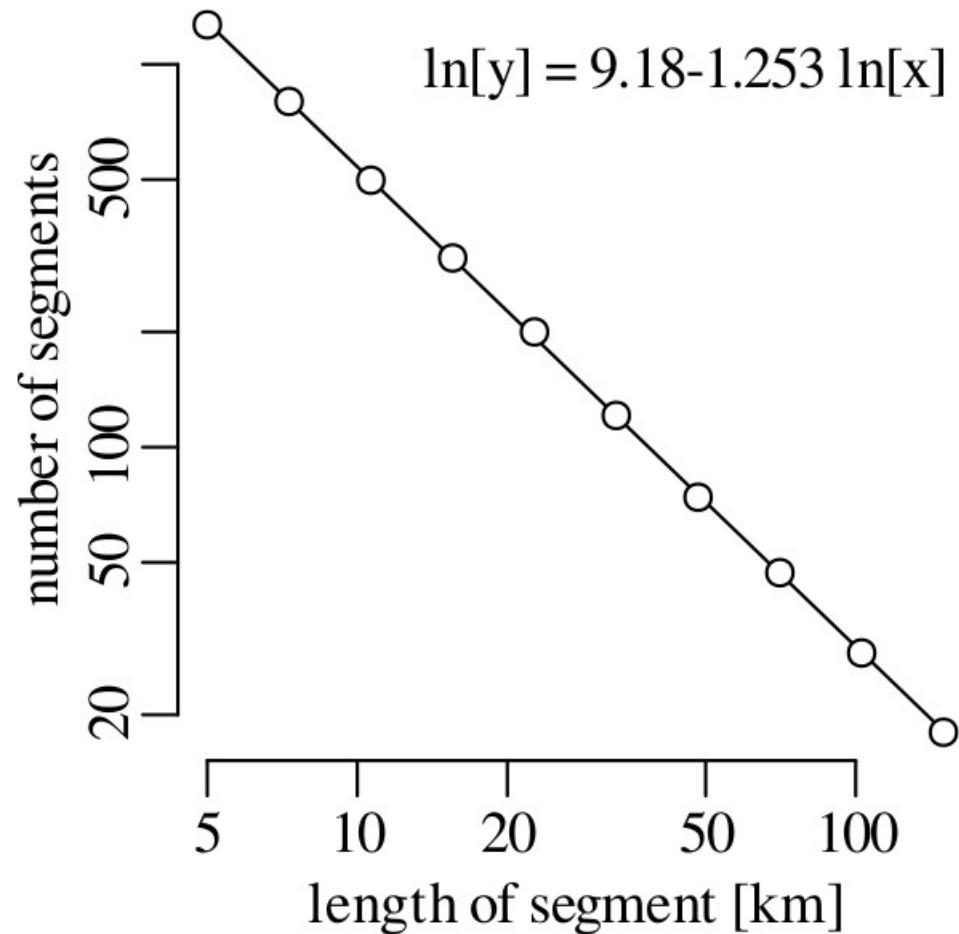
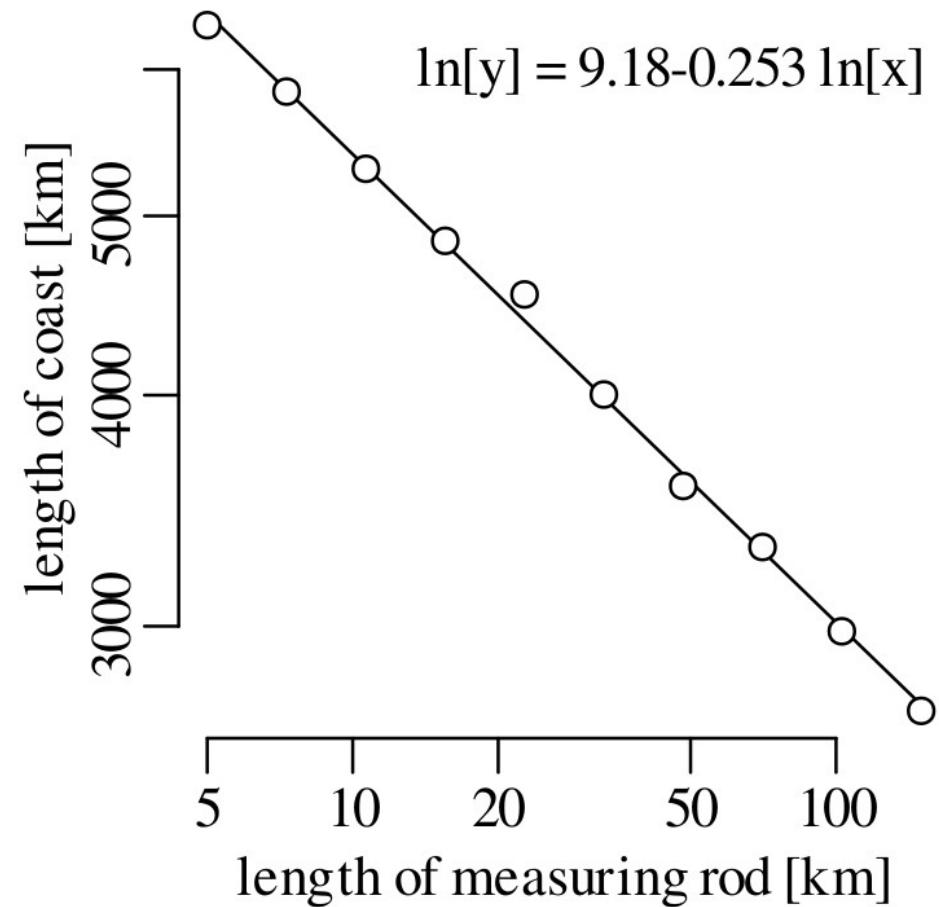


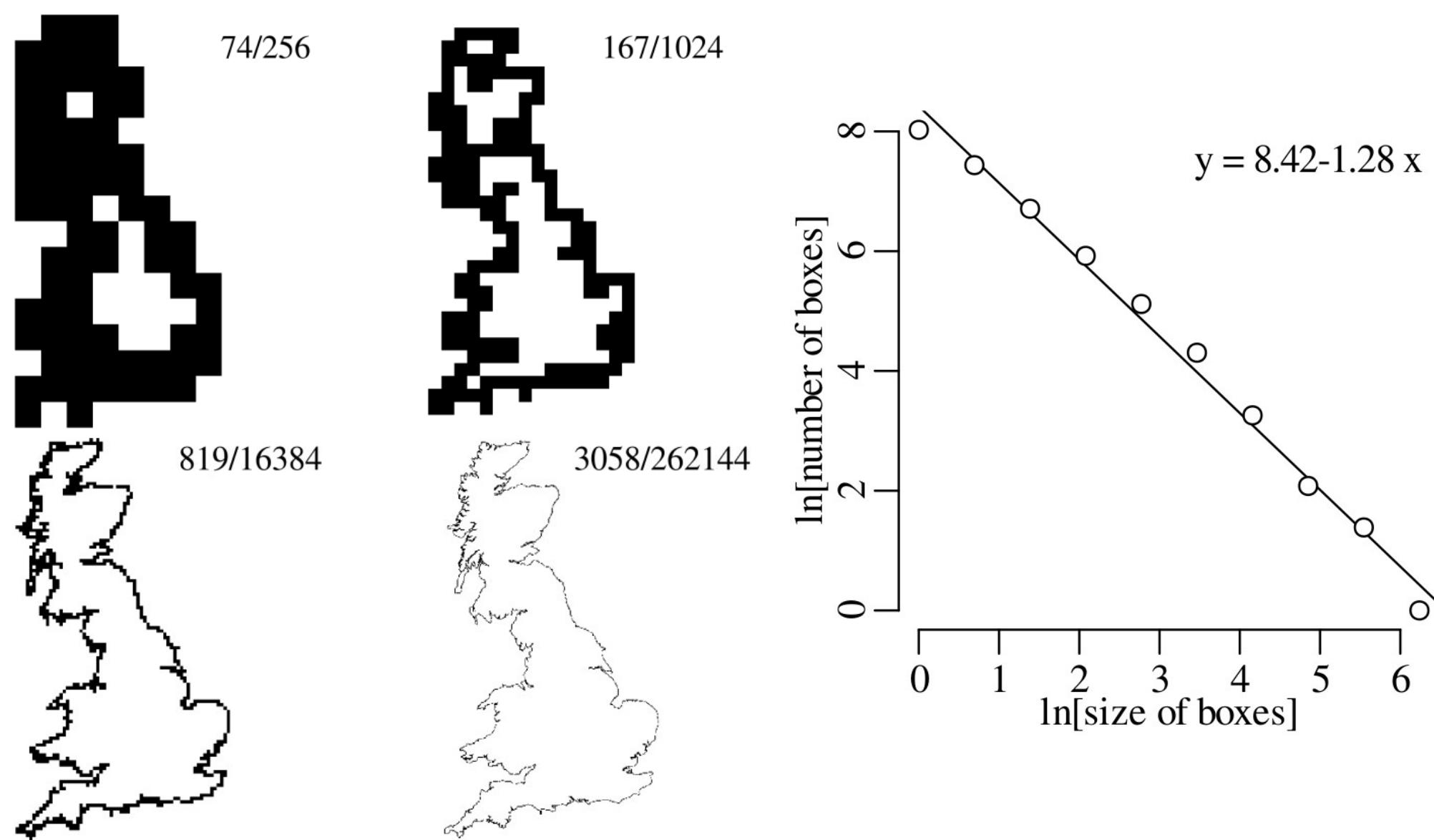


30 cm long ruler



$$\begin{aligned} & \exp(9.18 - 0.253 \ln[3 \times 10^{-4}]) \\ &= 42,060 \text{ km!} \end{aligned}$$

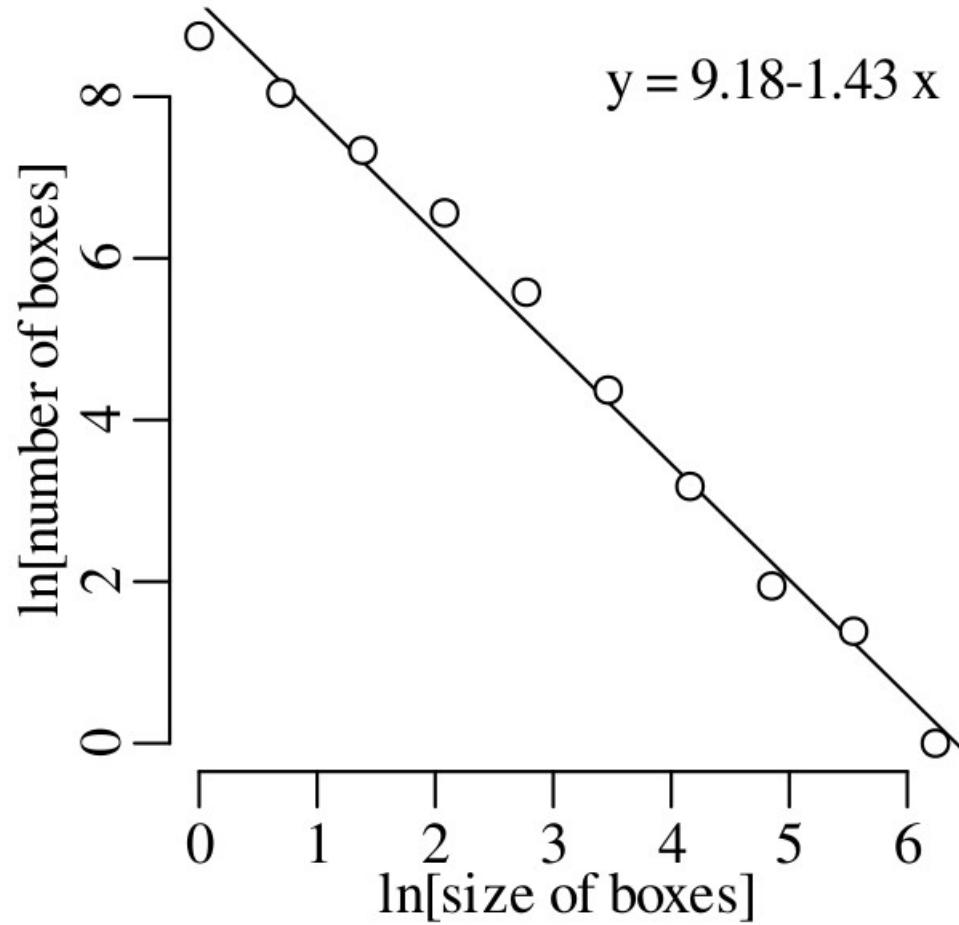




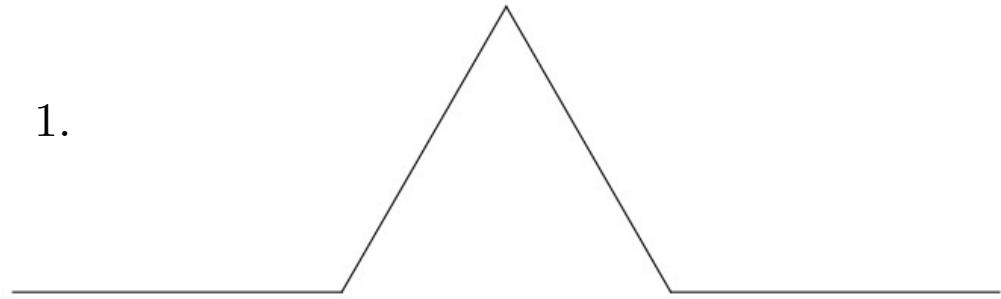
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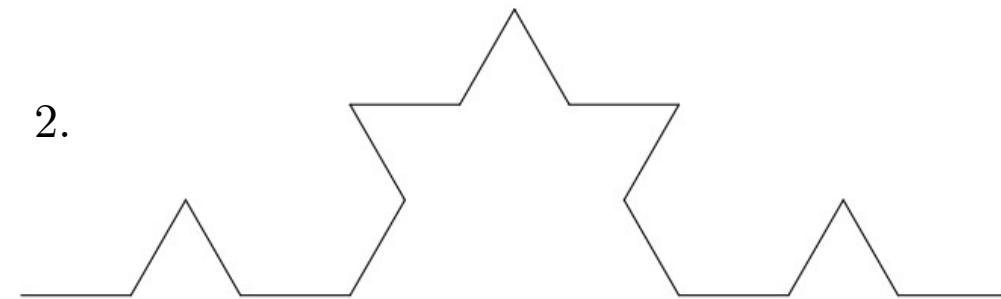
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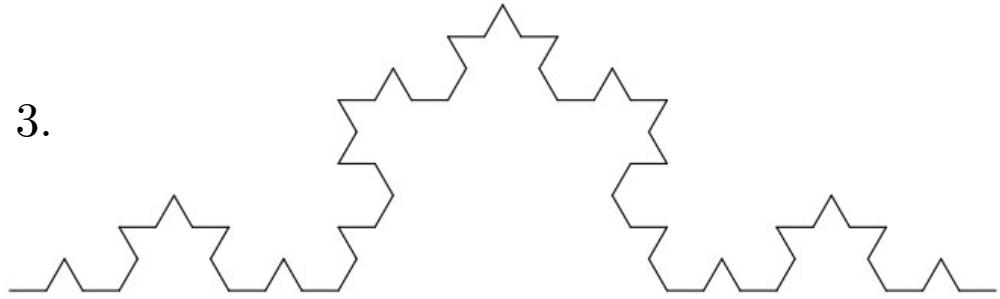
1.



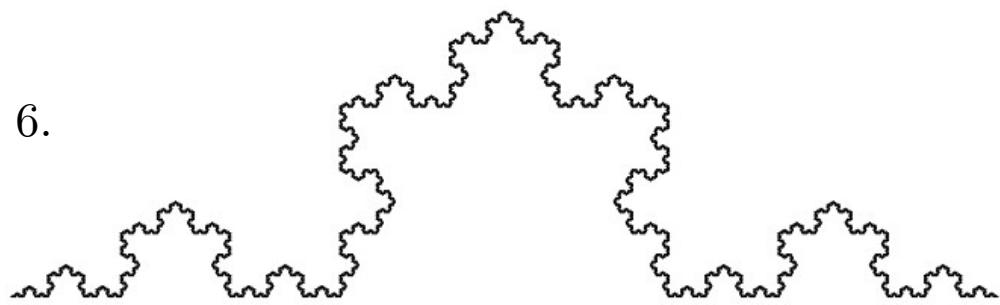
2.

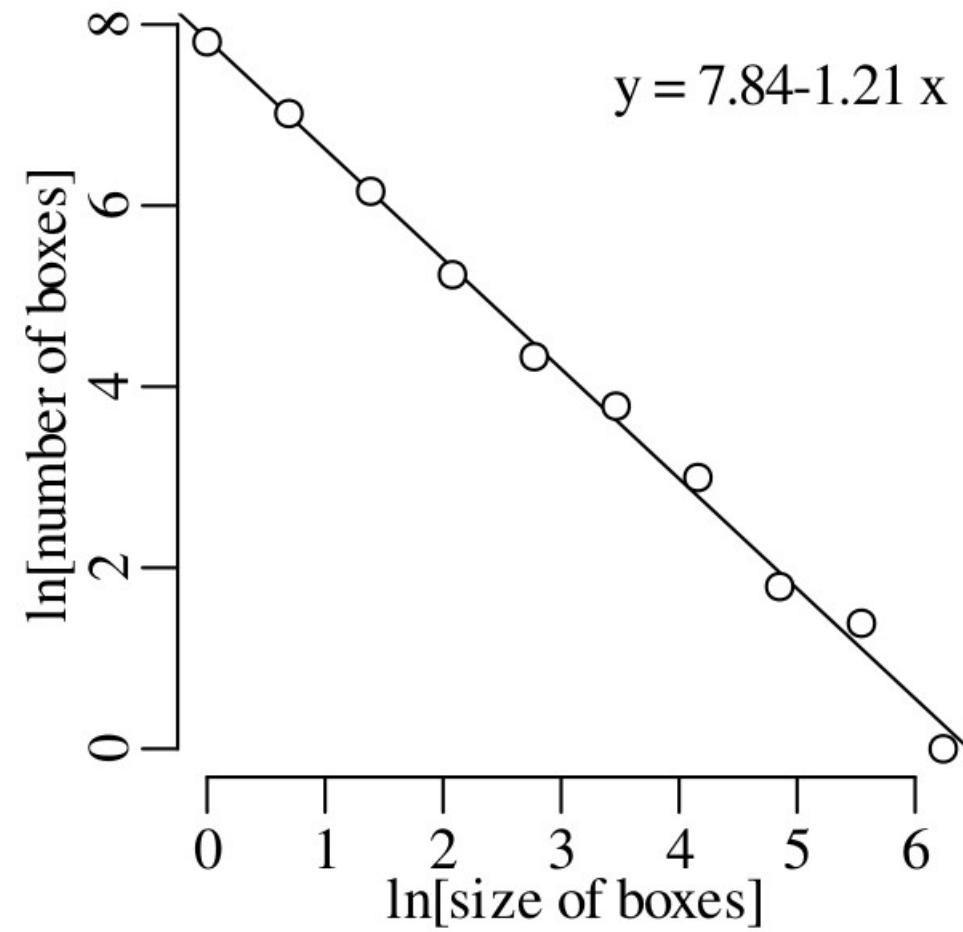
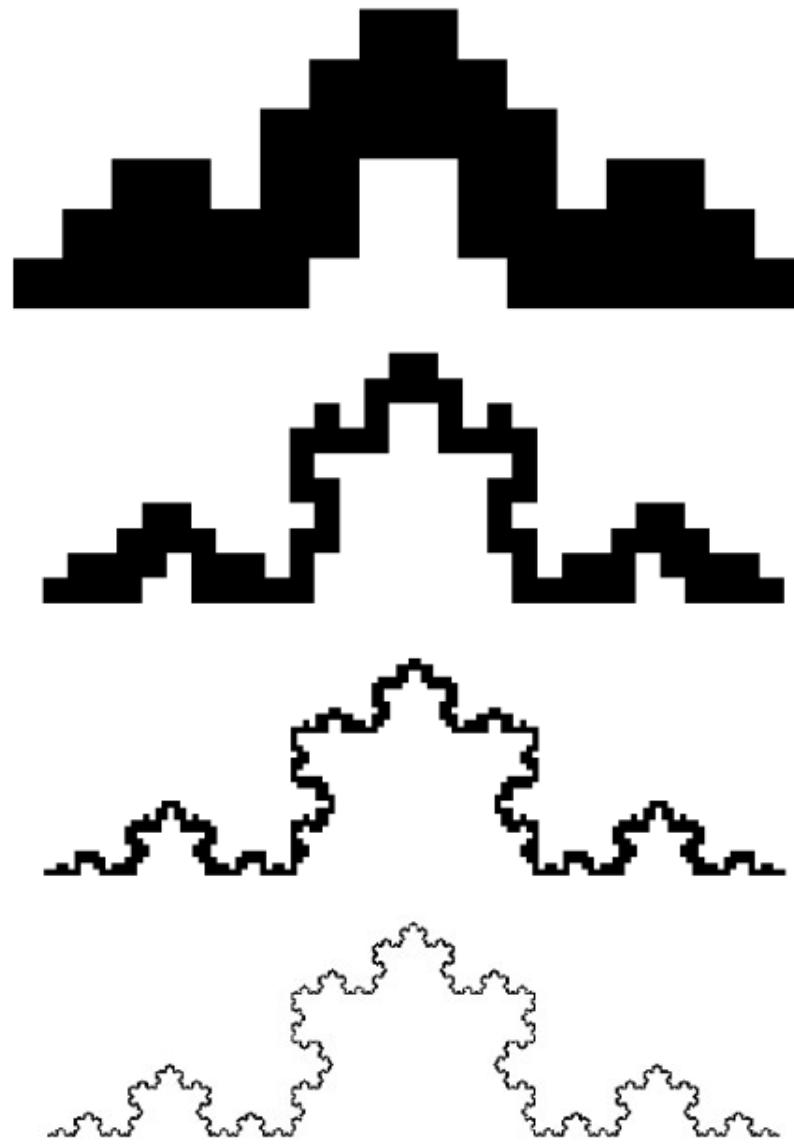


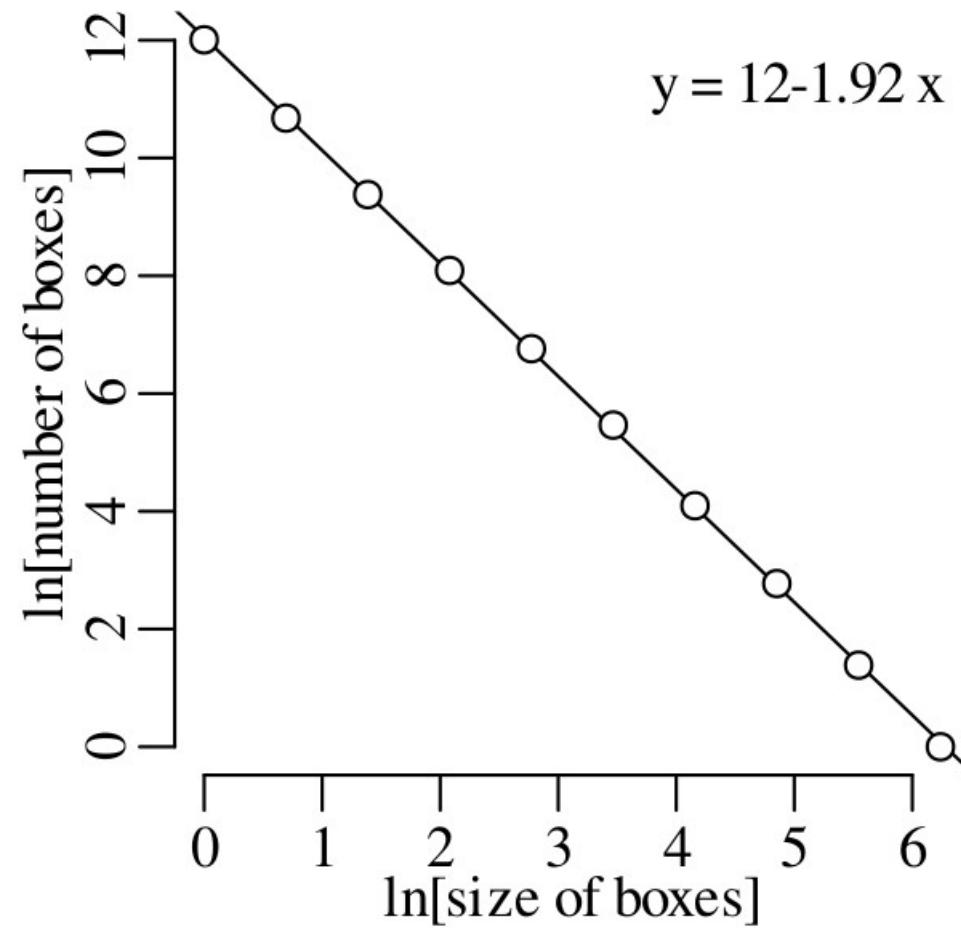
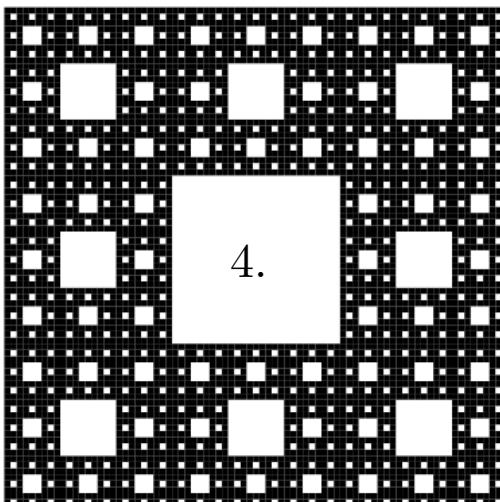
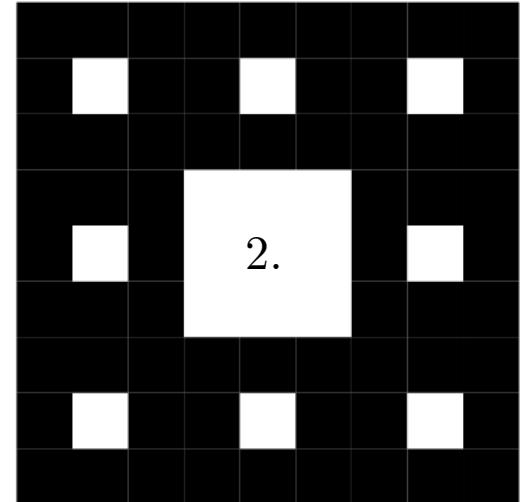
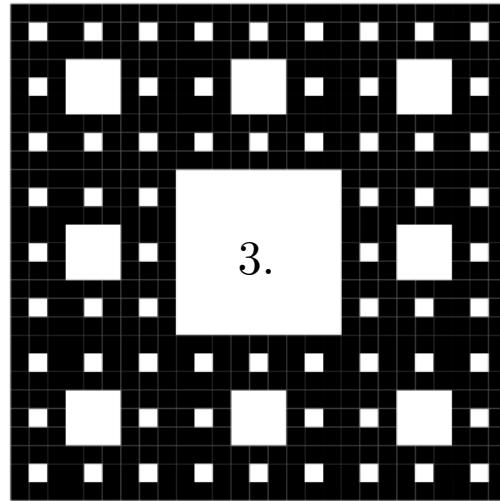
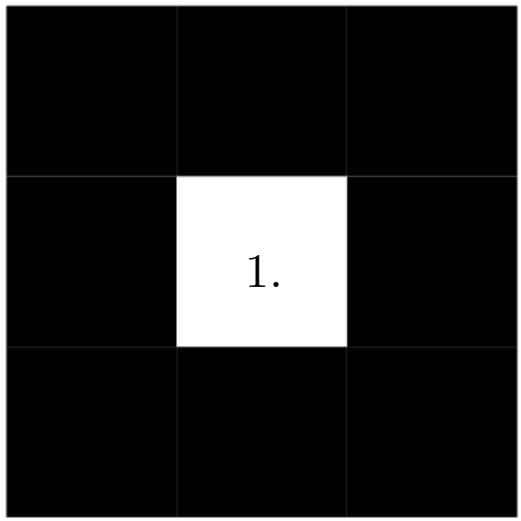
3.



6.



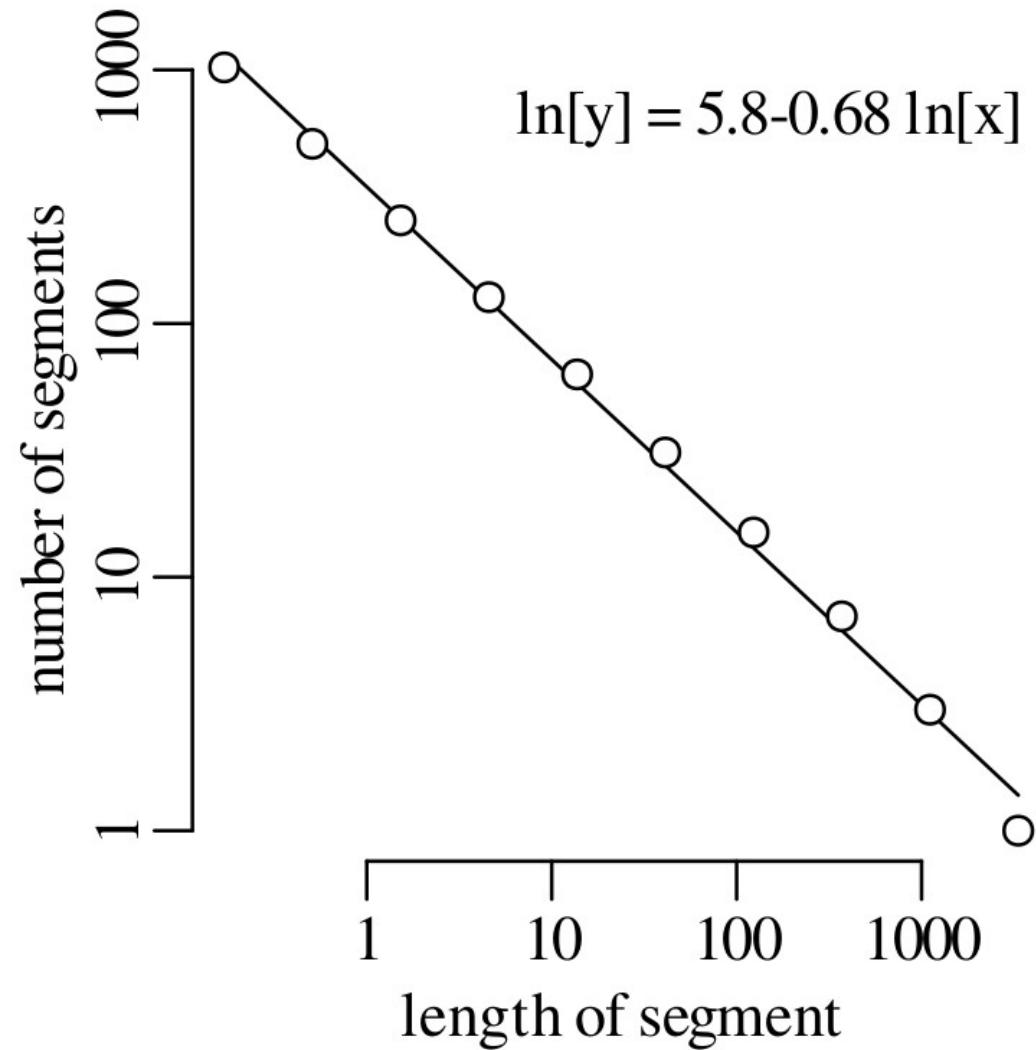




$$\ln[\text{number of boxes}] = C + D \times \ln[\text{size of boxes}]$$



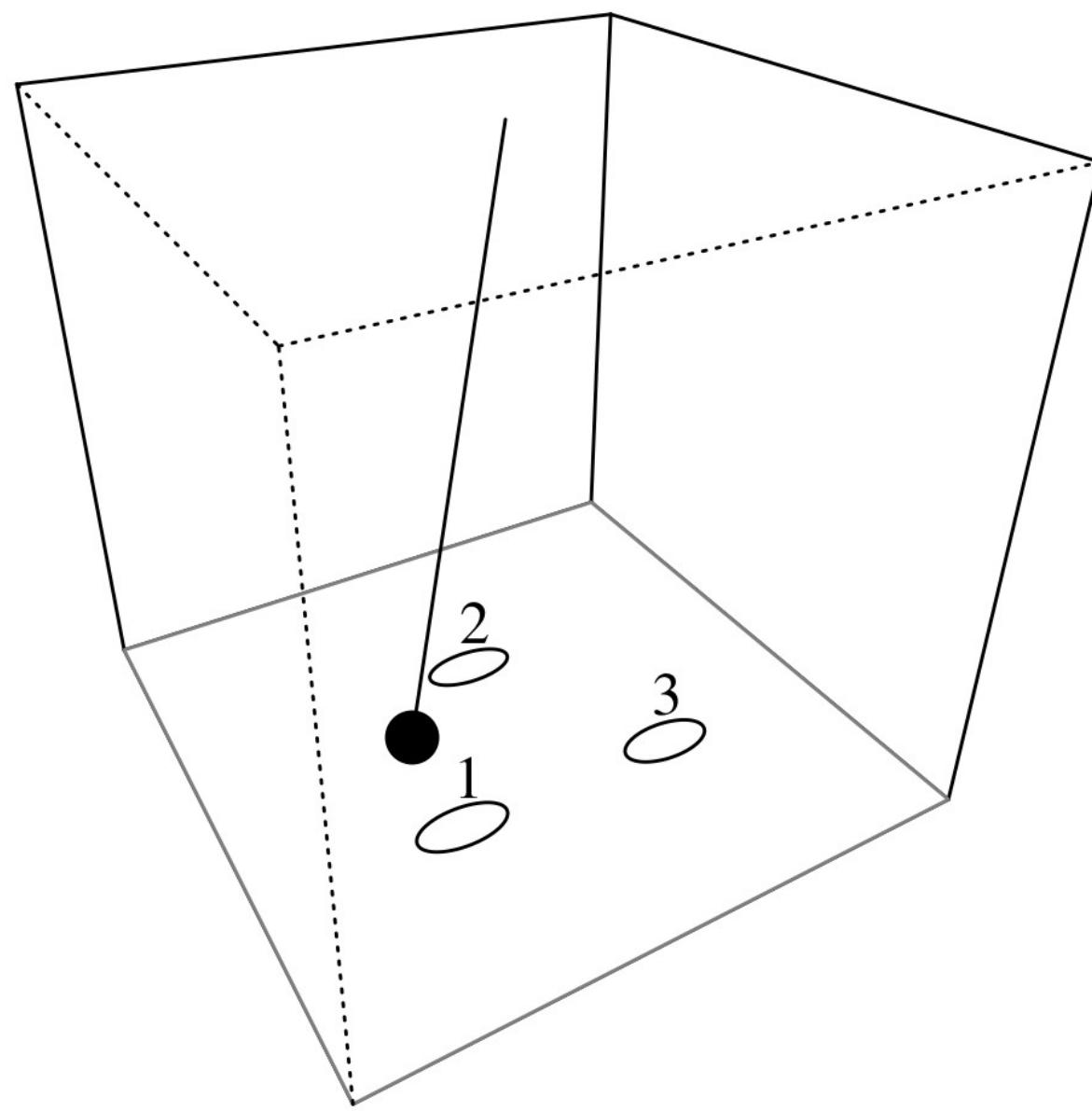
**fractal dimension**

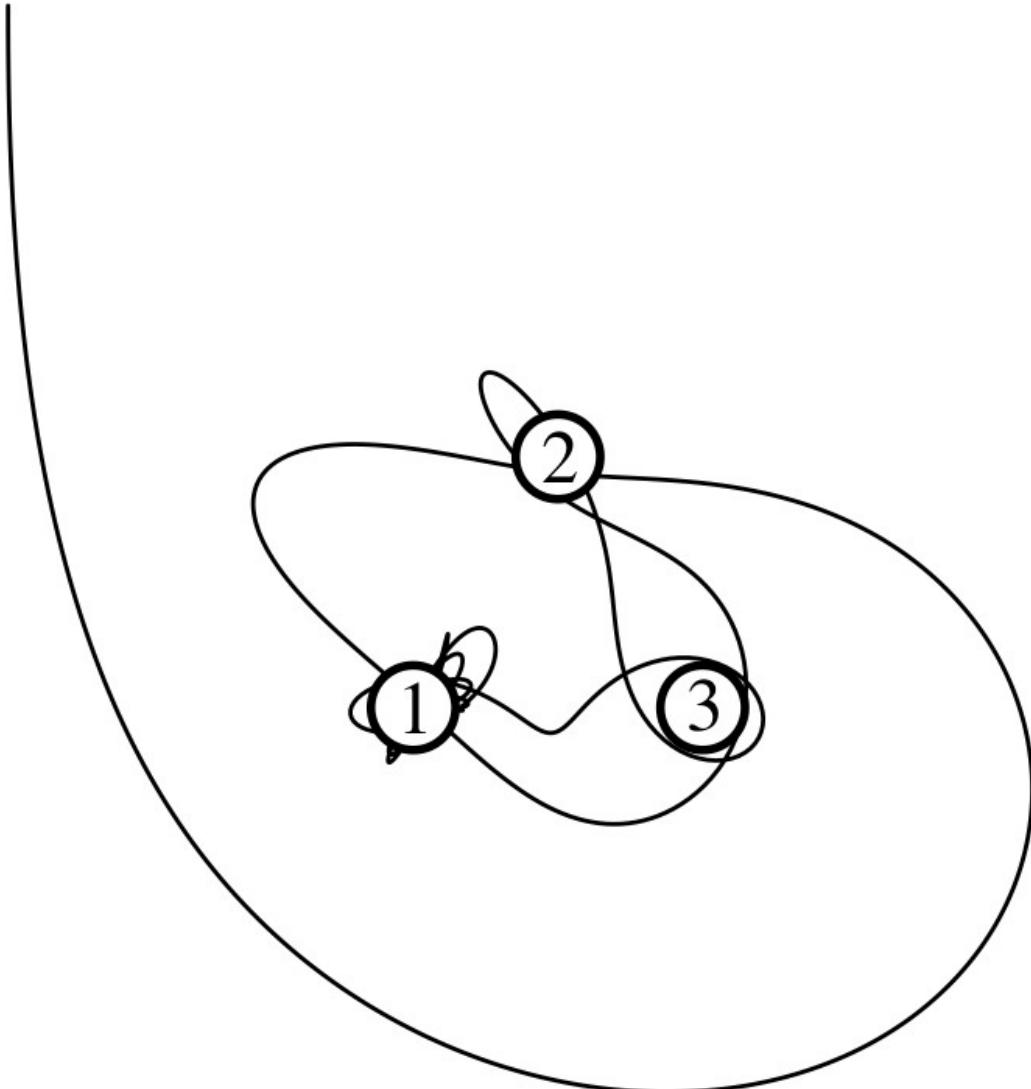
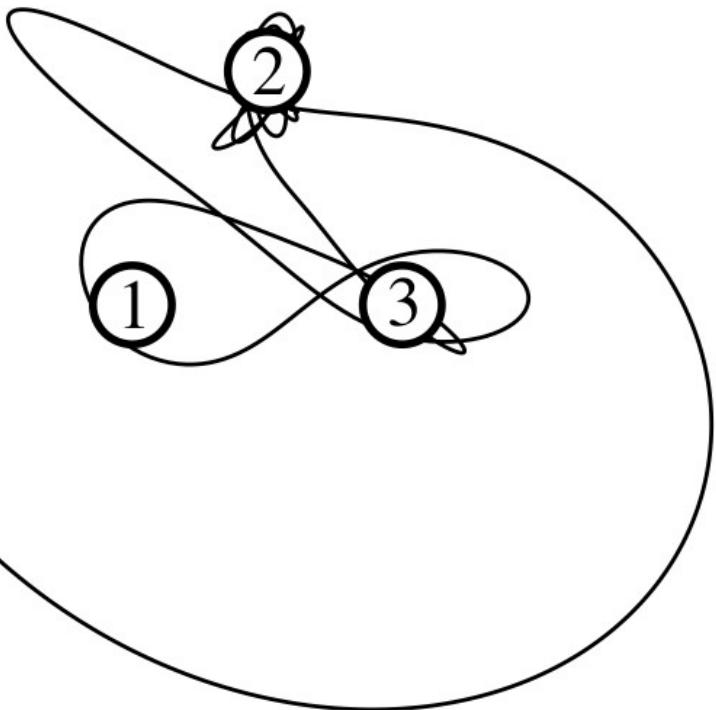


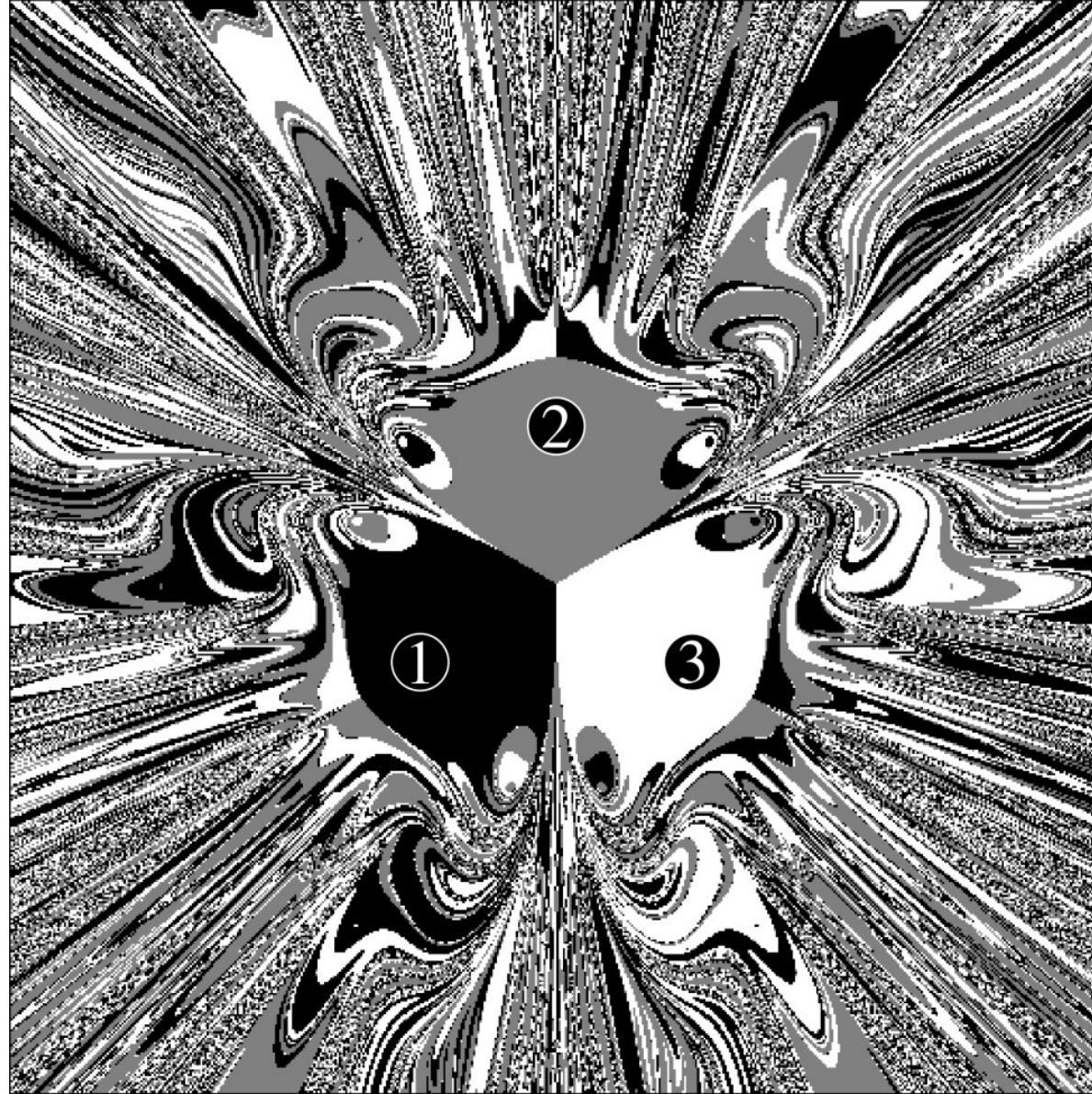
# Chaos

$$F_m(i) \propto 1/|d(i)|^2$$

$$F_f(i) \propto v$$





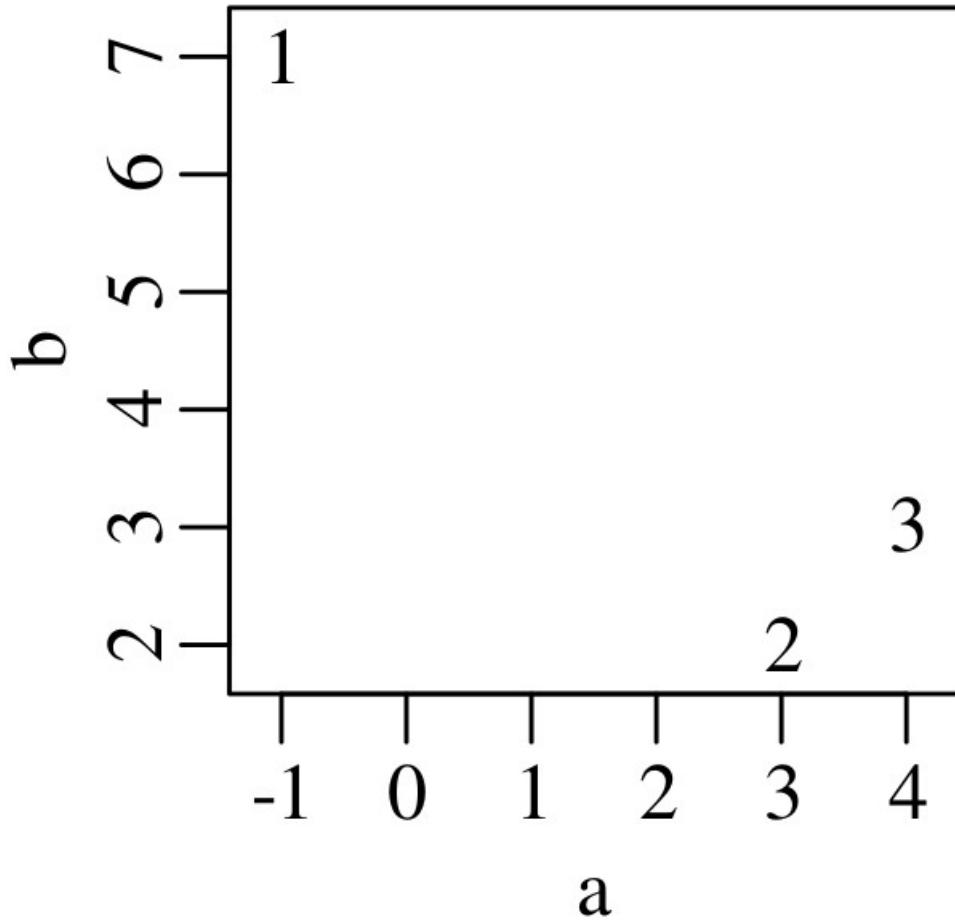


# Statistics for geoscientists

Unsupervised learning

# Principal Component Analysis

$$X = \begin{matrix} & a & b \\ 1 & \begin{bmatrix} -1 & 7 \\ 3 & 2 \end{bmatrix} \\ 2 & \\ 3 & \end{matrix}$$



$$X = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \end{matrix}$$

$$X = 1_{3,1} \ C + S \ V \ D$$

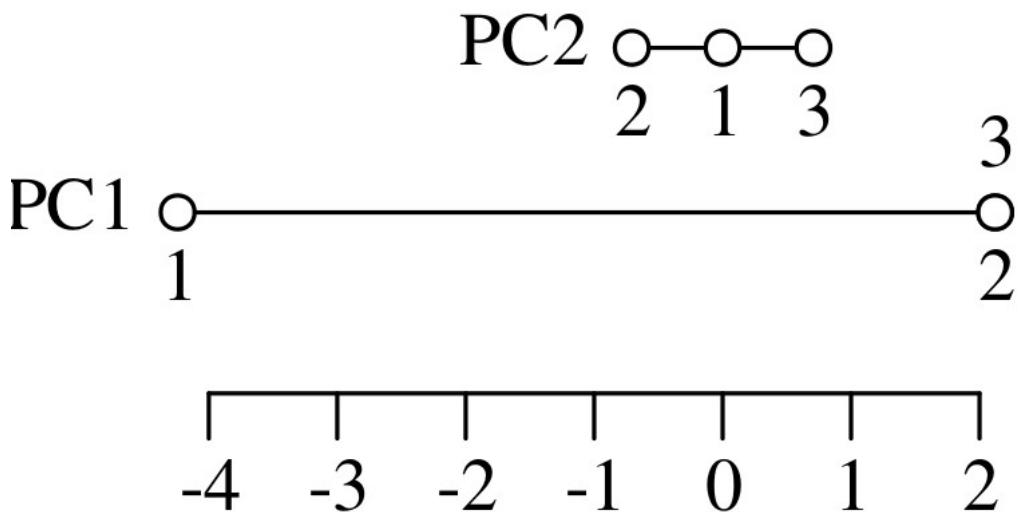
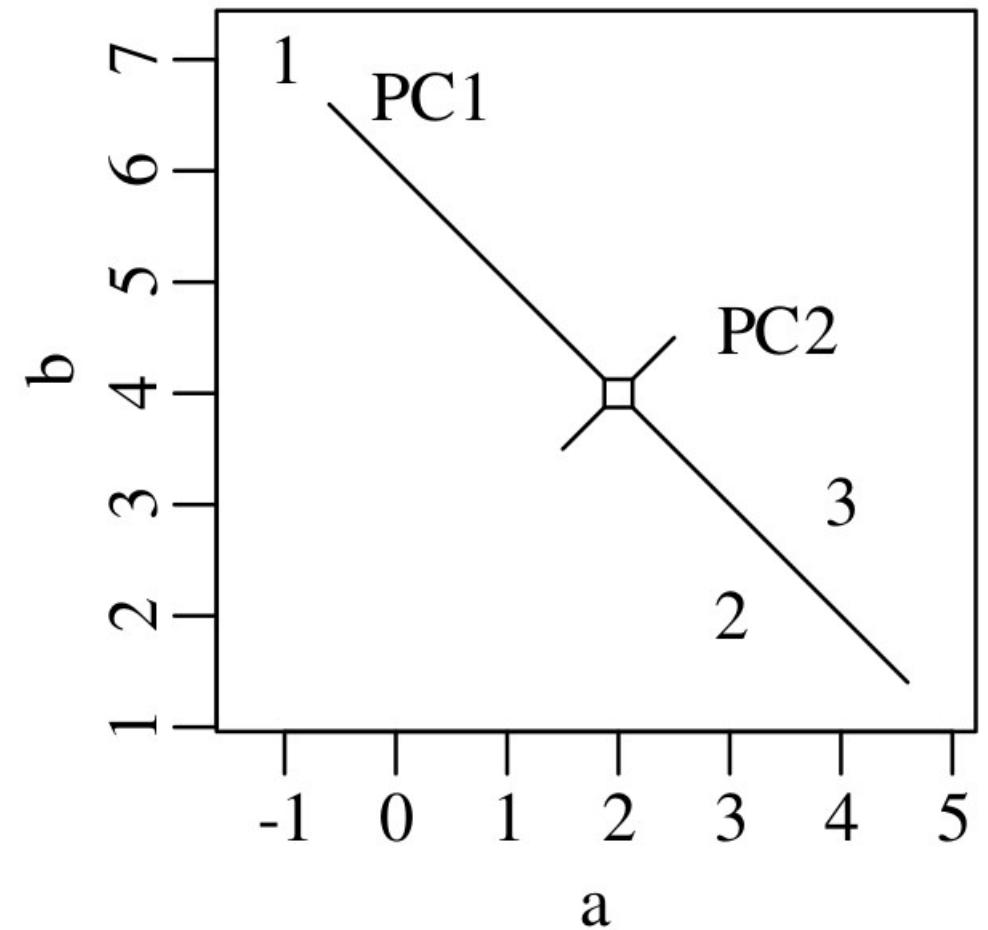
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} -1.15 & 0 \\ 0.58 & -1 \\ 0.58 & 1 \end{bmatrix} \begin{bmatrix} 3.67 & 0 \\ 0 & 0.71 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix}$$

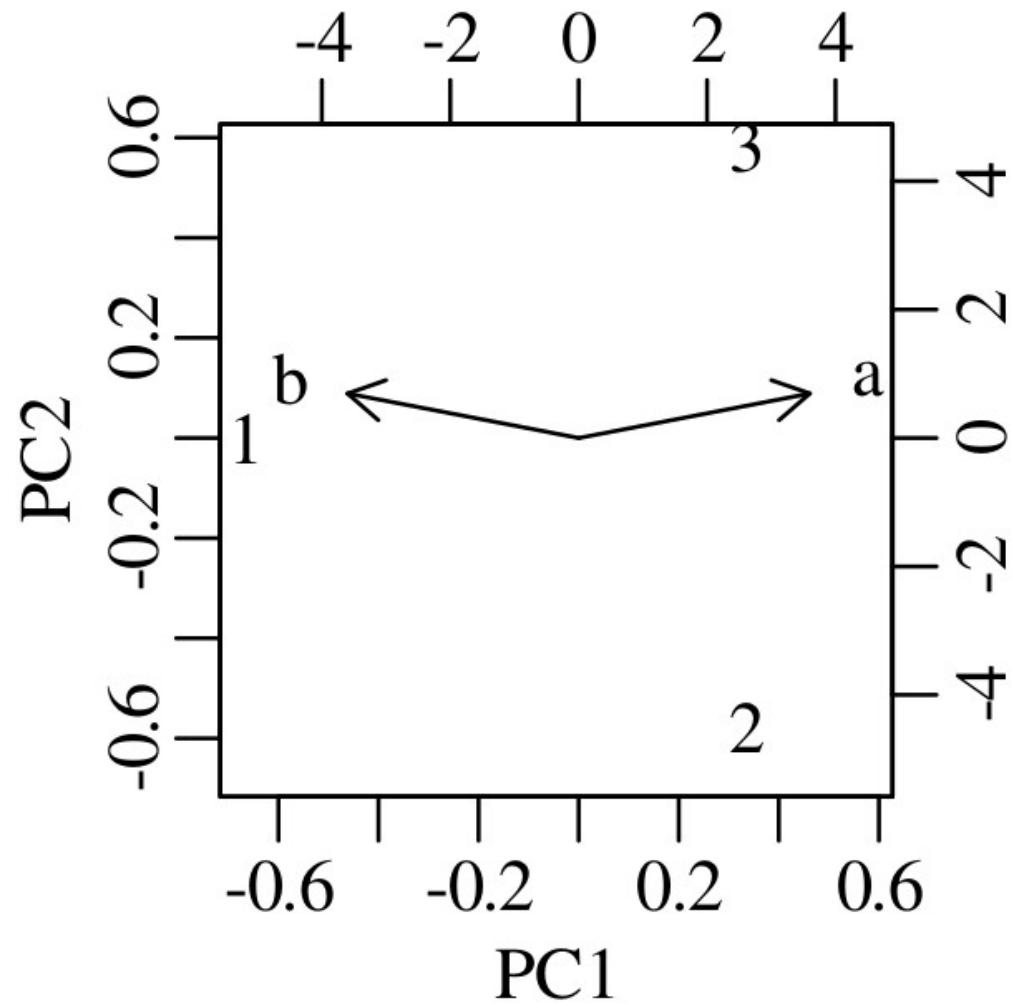
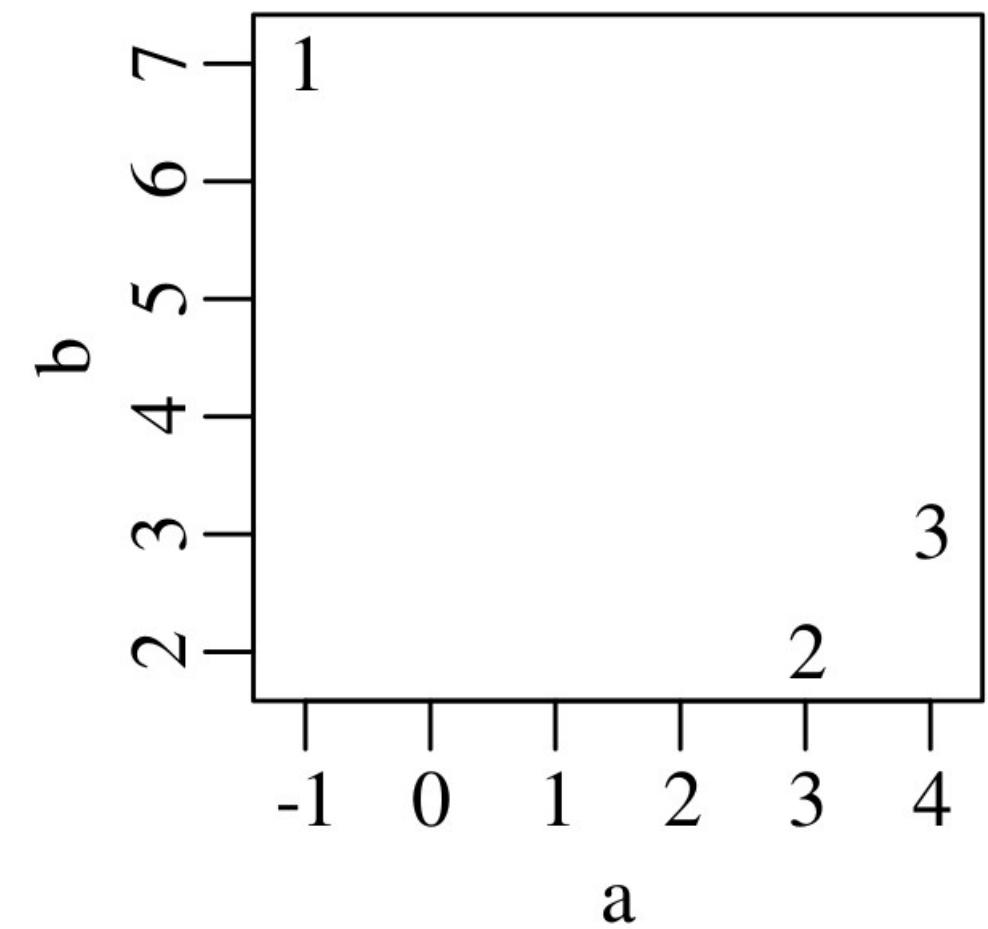
$$X = 1_{3,1} \ C + S \ V \ D$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} -1.15 & 0 \\ 0.58 & -1 \\ 0.58 & 1 \end{bmatrix} \begin{bmatrix} 3.67 & 0 \\ 0 & 0.71 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix}$$

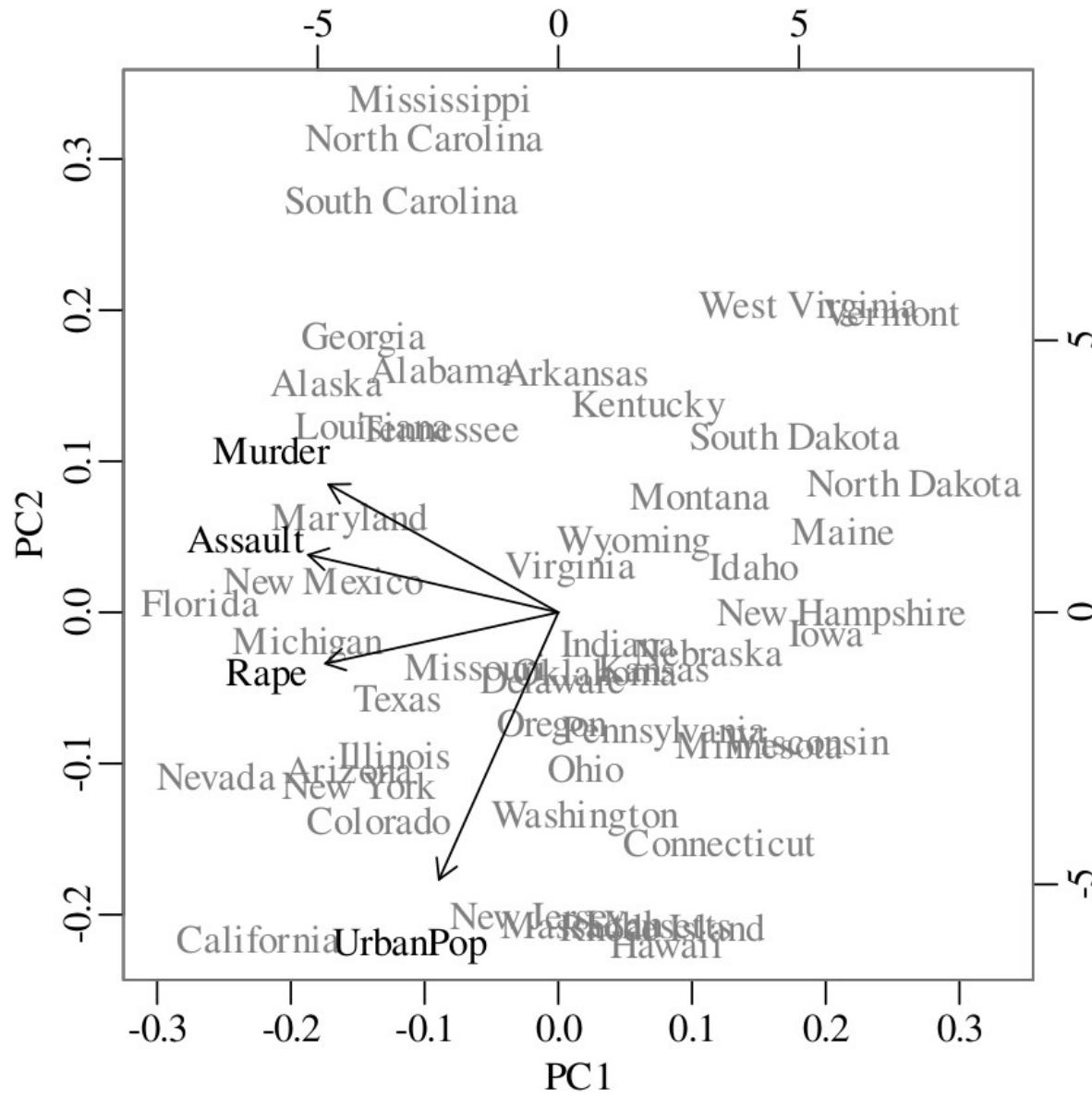
$$P = S \ V = \begin{bmatrix} -4.24 & 0 \\ 2.12 & -0.71 \\ 2.12 & 0.71 \end{bmatrix},$$

$$\text{and } L = V \ D = \begin{bmatrix} 2.6 & -2.6 \\ 0.5 & 0.5 \end{bmatrix}$$



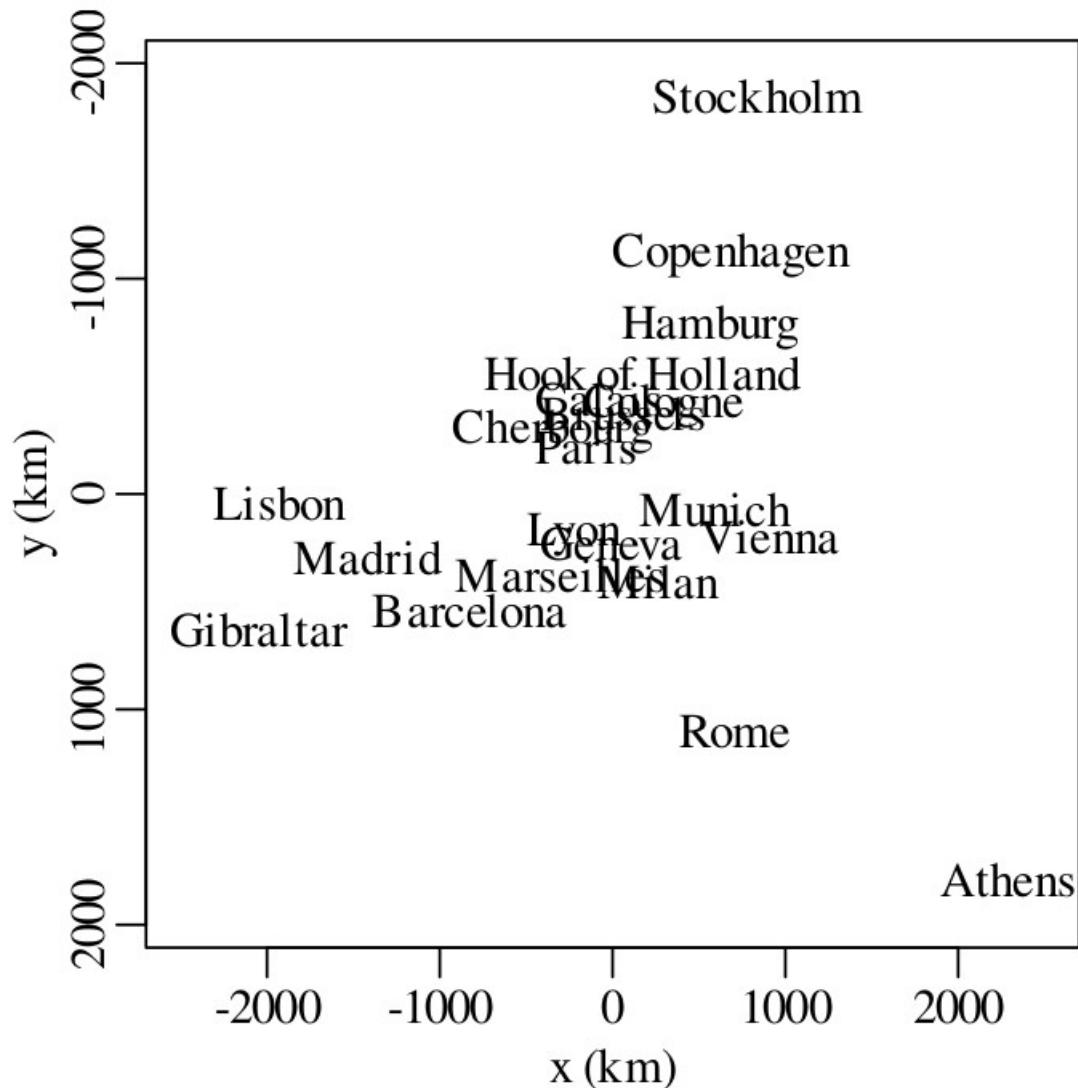


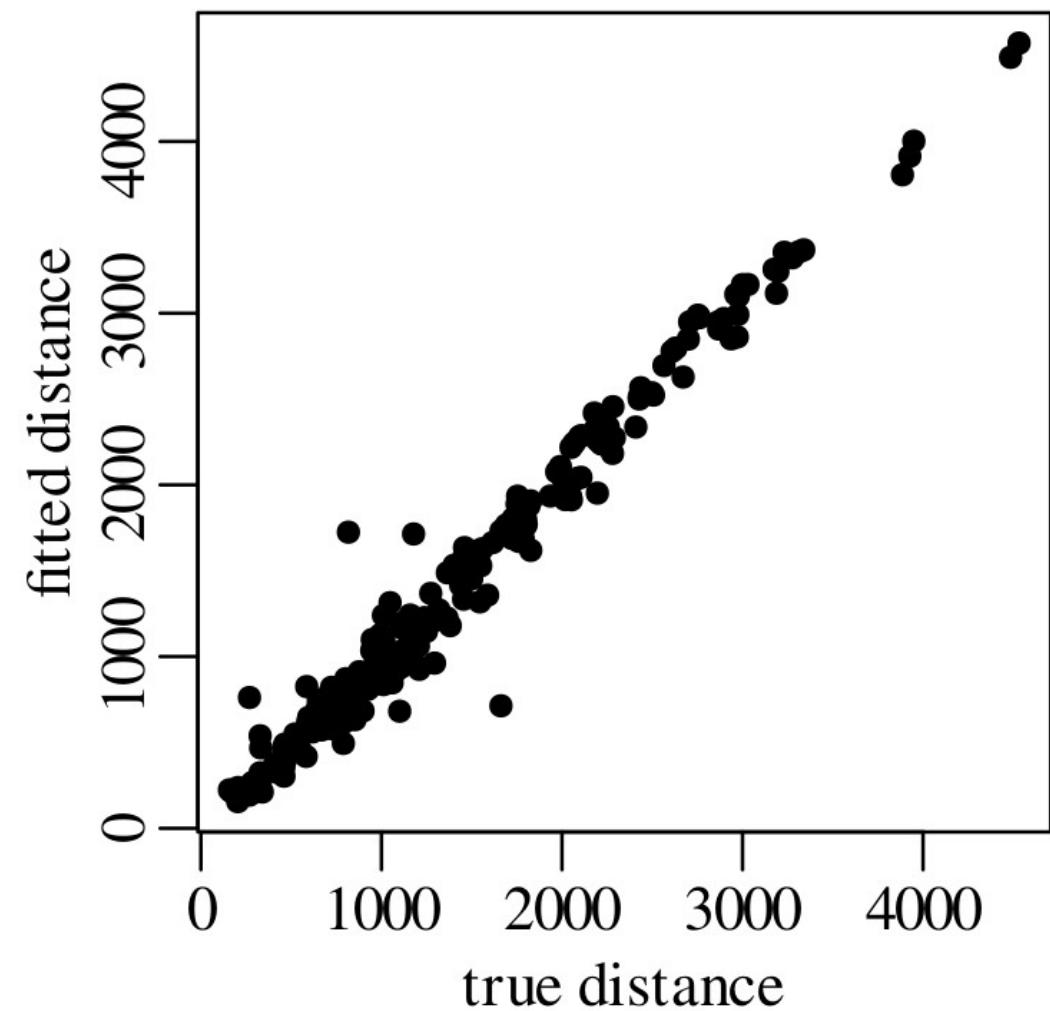
	Murder	Assault	Rape	UrbanPop
Alabama	13.2	236	21.2	58
Alaska	10.0	263	44.5	48
Arizona	8.1	294	31.0	80
Arkansas	8.8	190	19.5	50
California	9.0	276	40.6	91
Colorado	7.9	204	38.7	78
:	:	:	:	:
Wisconsin	2.6	53	10.8	66
Wyoming	6.8	161	15.6	60



# Multidimensional Scaling

	Athens	Barcelona	Brussels	...	Rome	Stockholm	Vienna
Athens	0	3313	2963	...	817	3927	1991
Barcelona	3313	0	1326	...	1460	2868	1802
Brussels	2963	1318	0	...	1511	1616	1175
:	:	:	:	..	:	:	:
Rome	817	1460	1511	...	0	2707	1209
Stockholm	3927	2868	1616	...	2707	0	2105
Vienna	1991	1802	1175	...	1209	2105	0





$$S = \sqrt{\frac{\sum_{i=1}^n \sum_{j=i+1}^n (f(d[i, j]) - \delta[i, j])^2}{\sum_{i=1}^n \sum_{j=i+1}^n \delta[i, j]^2}}$$

fit	poor	fair	good	excellent	perfect
S	0.2	0.1	0.05	0.025	0

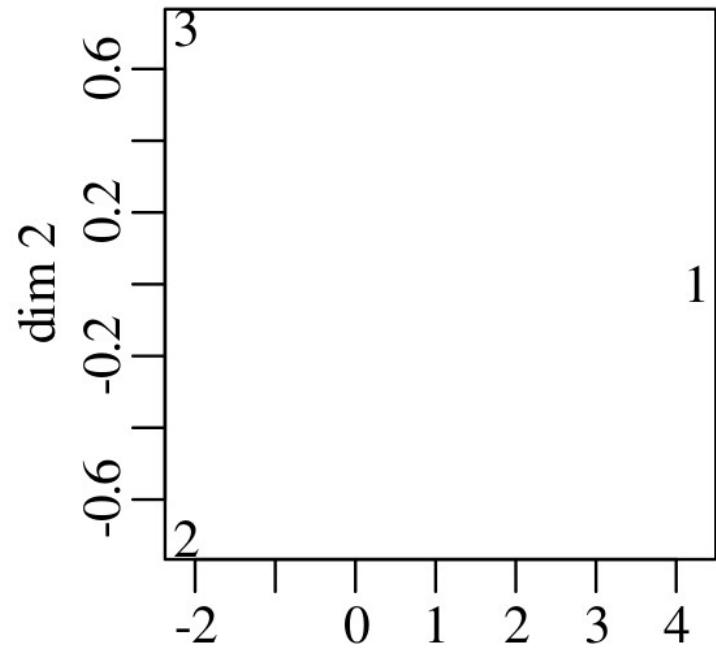
$$X = \begin{matrix} & a & b \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \left[ \begin{matrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{matrix} \right] \end{matrix}$$

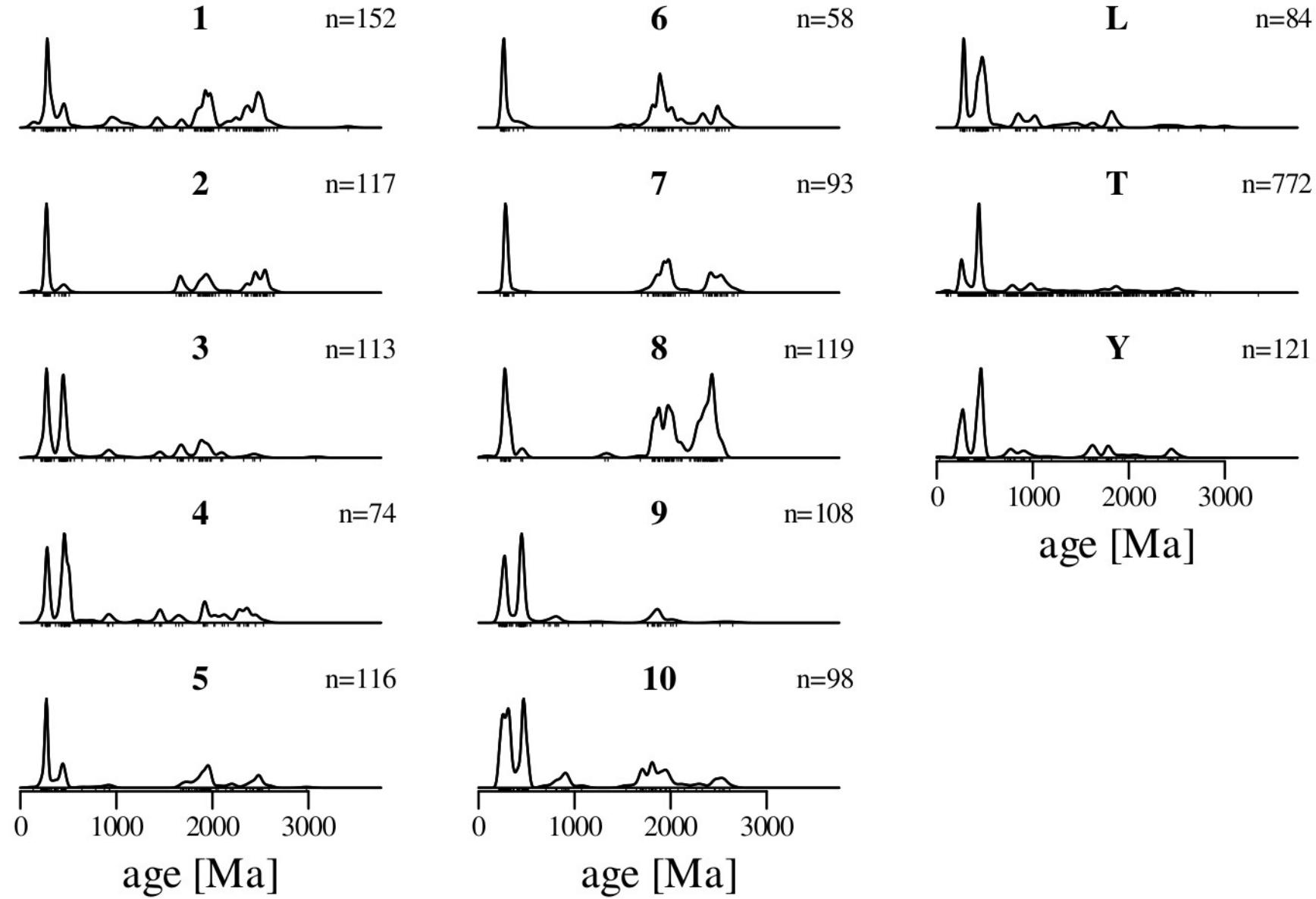
$$d[i, j] = \sqrt{(a[i] - a[j])^2 + (b[i] - b[j])^2}$$



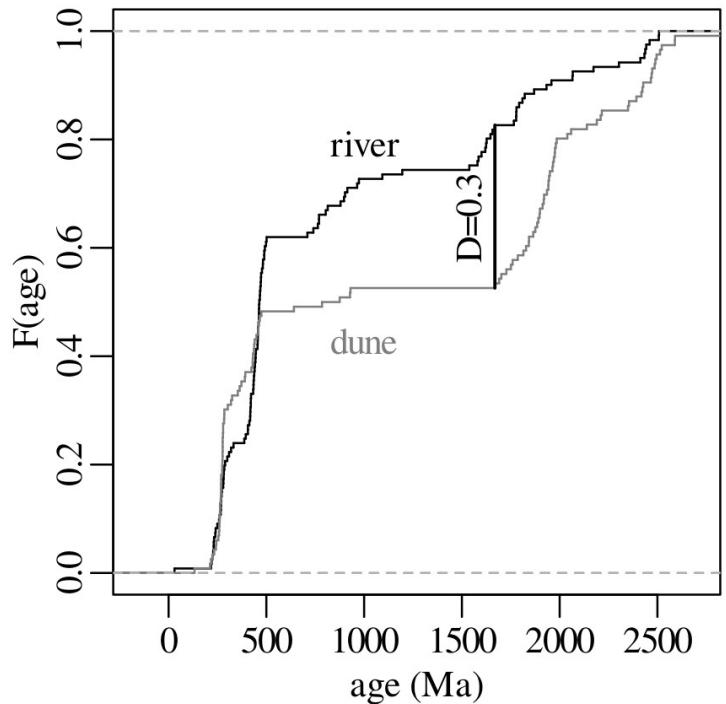
1      2      3

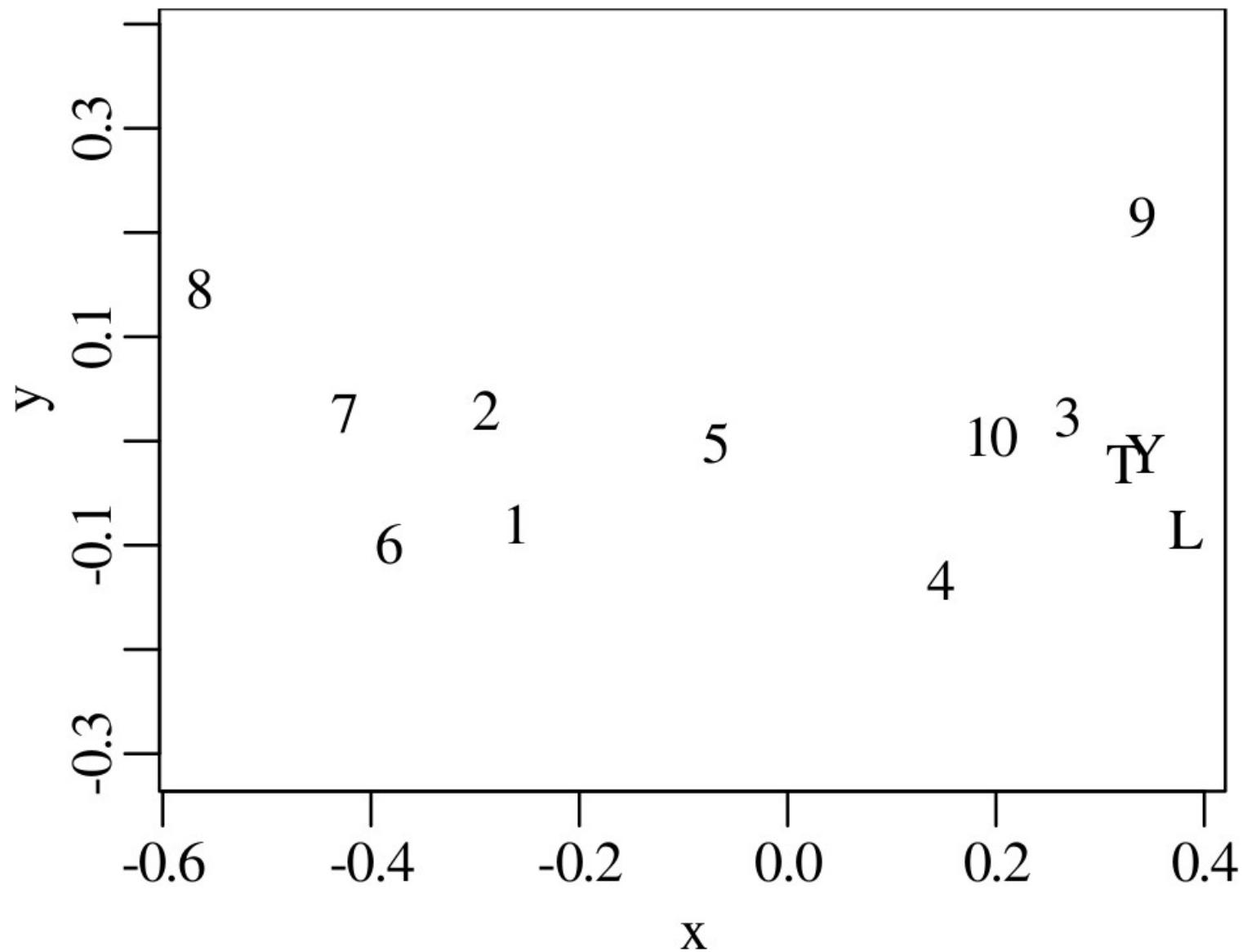
$$d = \begin{matrix} & x & y \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \left[ \begin{matrix} 0 & 6.4 & 6.4 \\ 6.4 & 0 & 1.4 \\ 6.4 & 1.4 & 0 \end{matrix} \right] \end{matrix} \longrightarrow m = \begin{matrix} & x & y \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \left[ \begin{matrix} 4.24 & 0 \\ -2.12 & -0.71 \\ -2.12 & 0.71 \end{matrix} \right] \end{matrix}$$



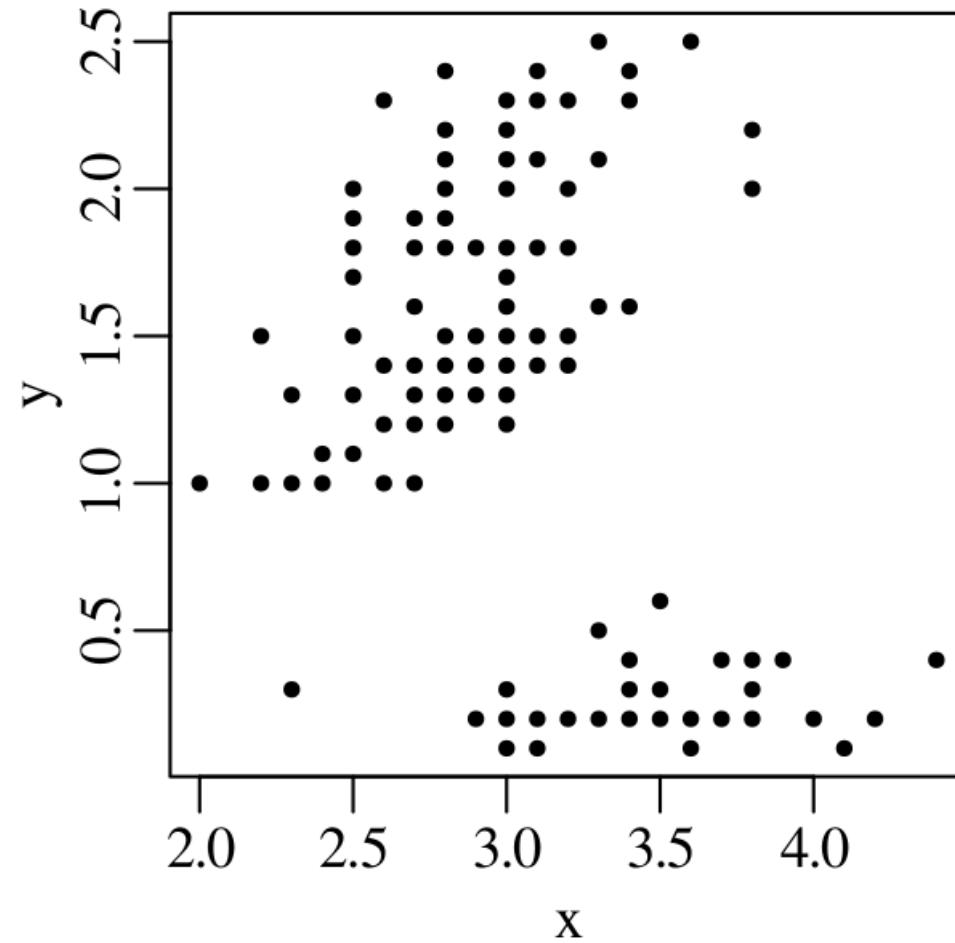


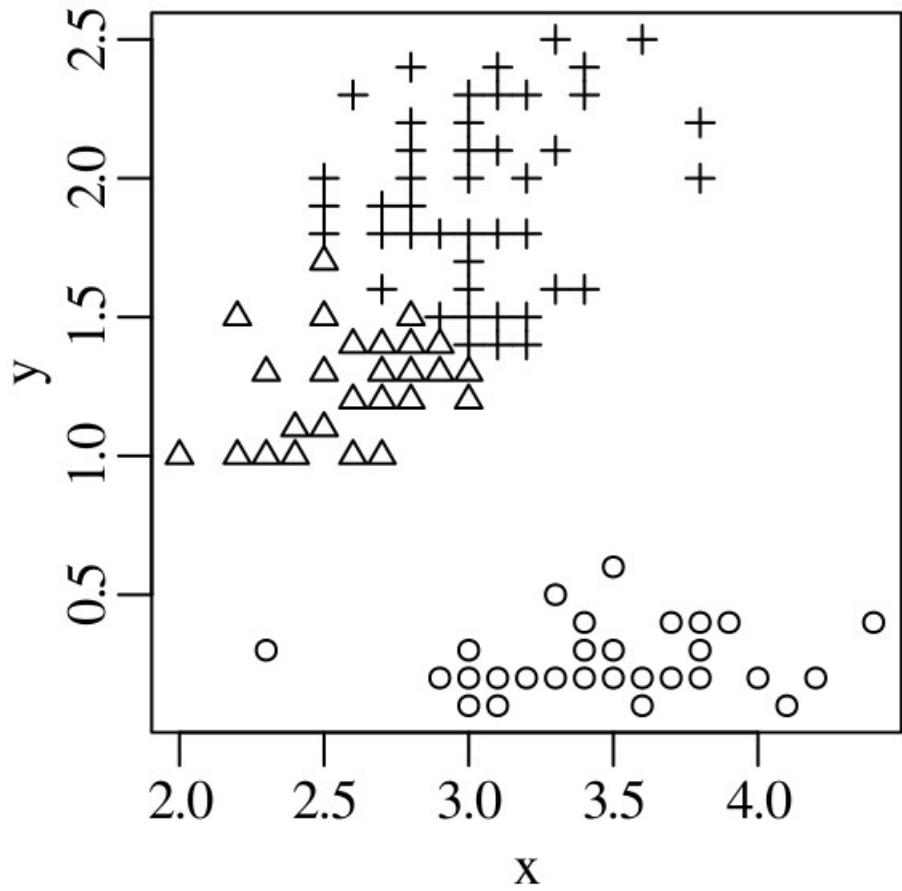
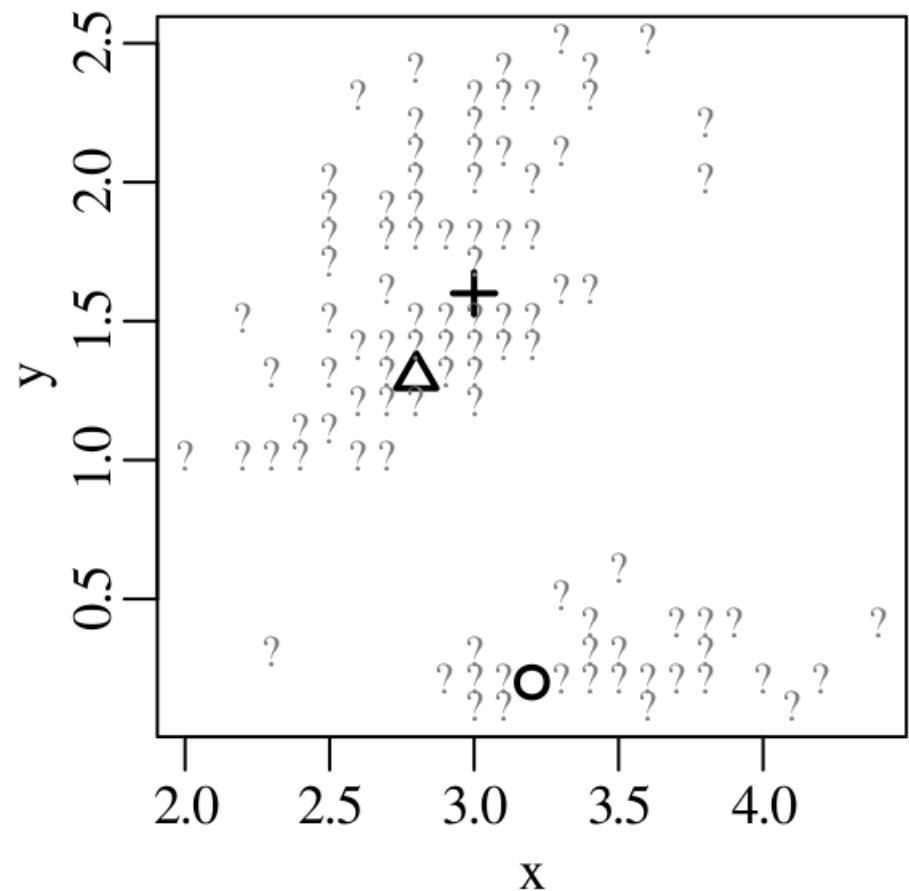
	1	2	3	4	5	6	7	8	9	10	L	T	Y
1	0	14	33	27	18	14	15	22	48	32	42	37	40
2	14	0	36	33	16	14	15	24	46	32	47	42	43
3	33	36	0	19	24	44	47	55	17	10	13	12	8
4	27	33	19	0	20	38	41	48	28	14	21	17	16
5	18	16	24	20	0	22	24	33	31	20	33	28	30
6	14	14	44	38	22	0	14	24	52	41	52	48	49
d = 7	15	15	47	41	24	14	0	16	51	43	54	49	52
8	22	24	55	48	33	24	16	0	61	53	63	59	62
9	48	46	17	28	31	52	51	61	0	20	22	18	16
10	32	32	10	14	20	41	43	53	20	0	17	15	13
L	42	47	13	21	33	52	54	63	22	17	0	10	11
T	37	42	12	17	28	48	49	59	18	15	10	0	7
Y	40	43	8	16	30	49	52	62	16	13	11	7	0

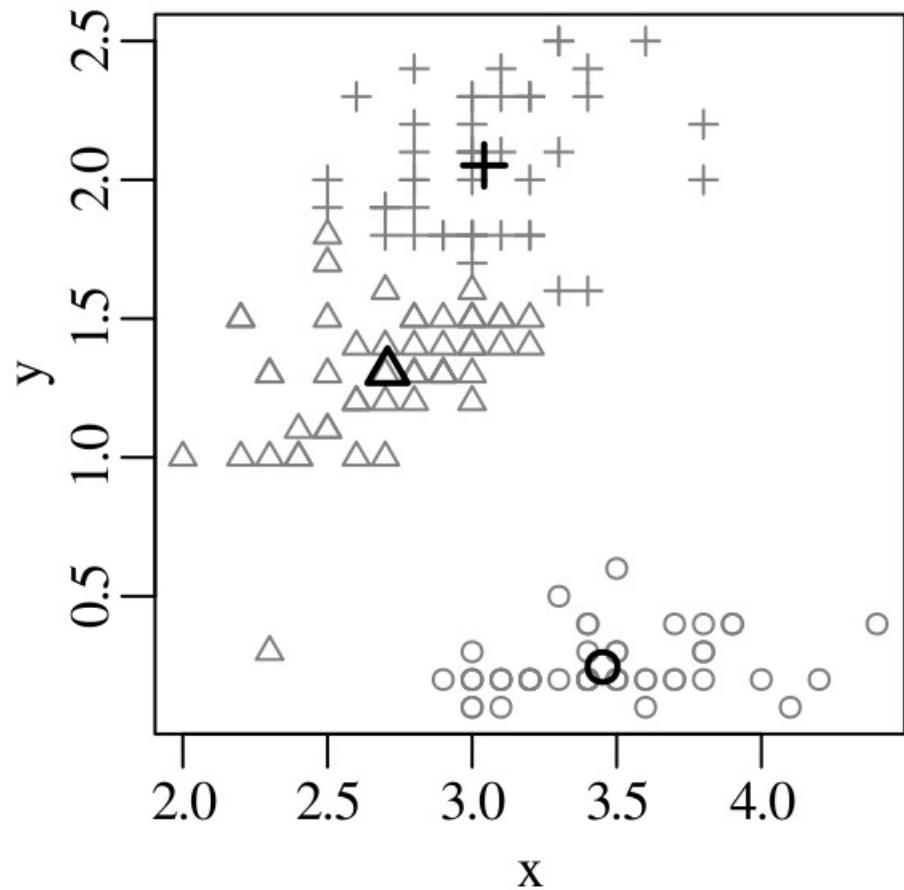
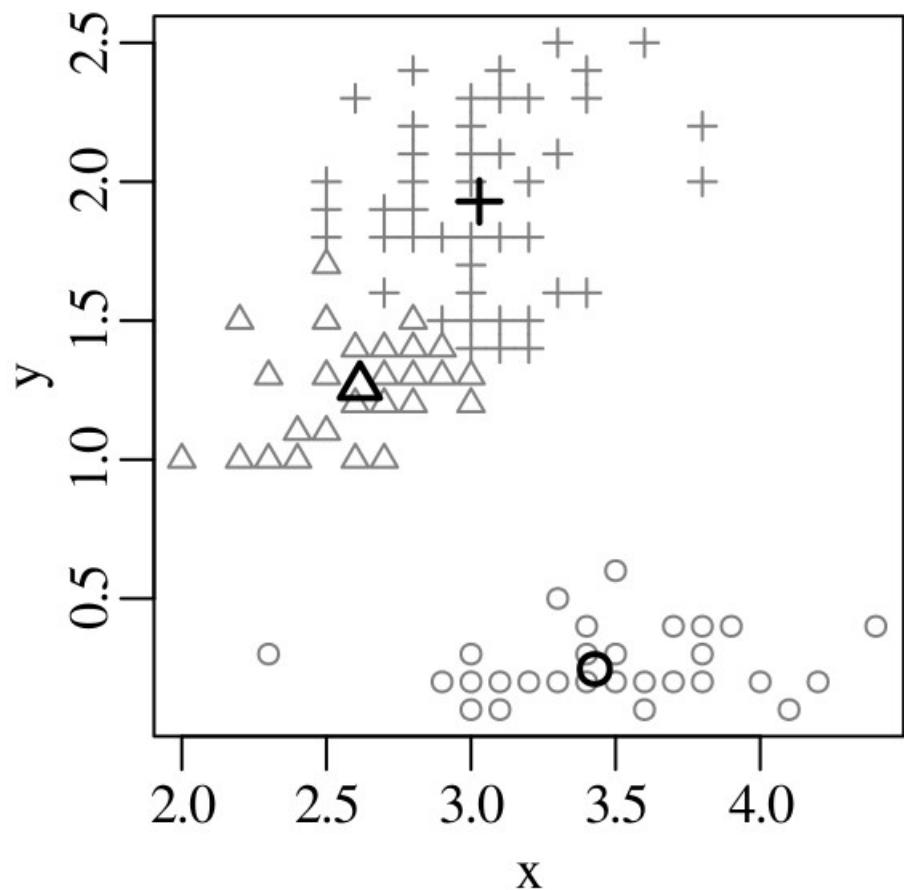


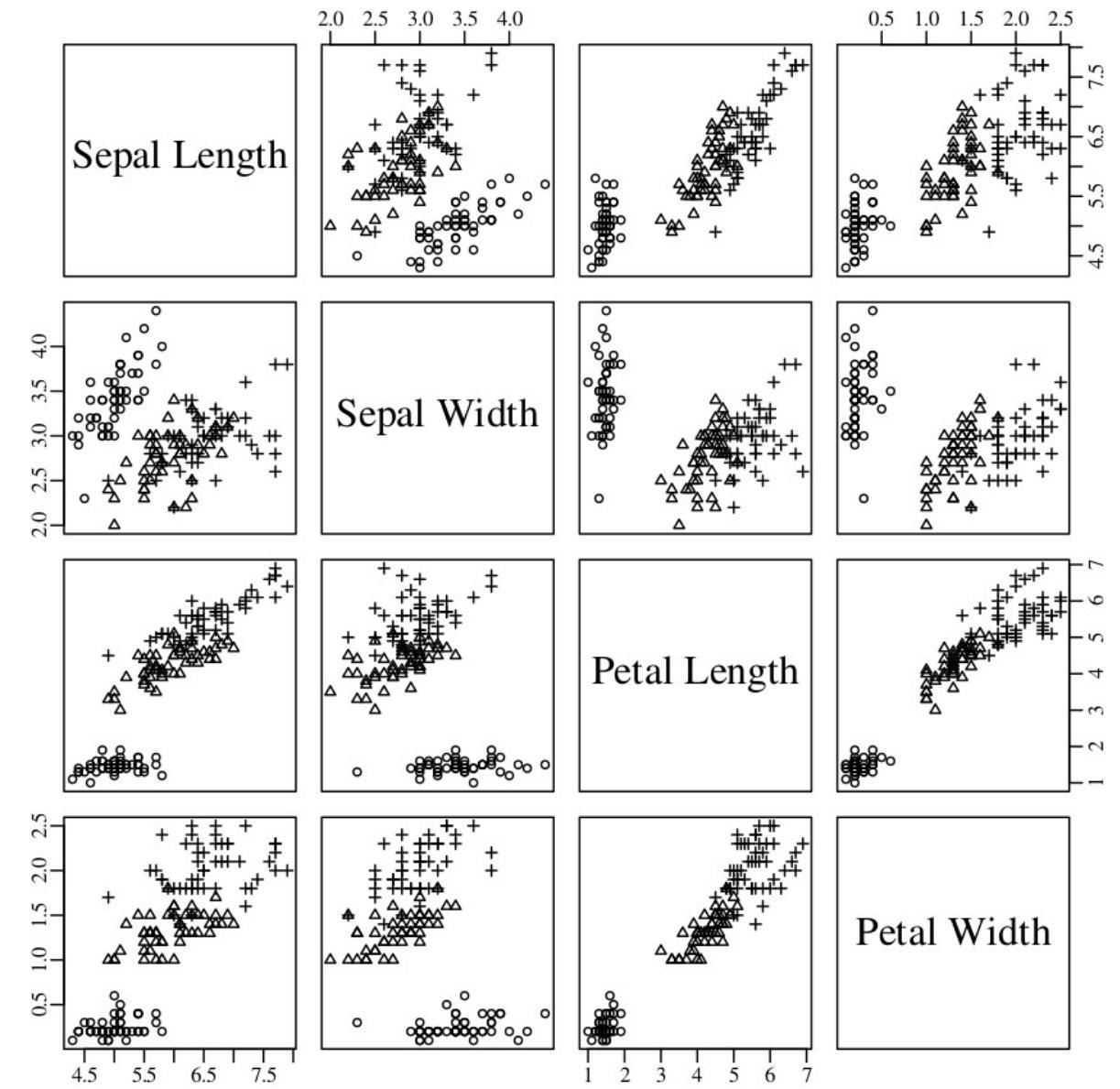


# K-means clustering





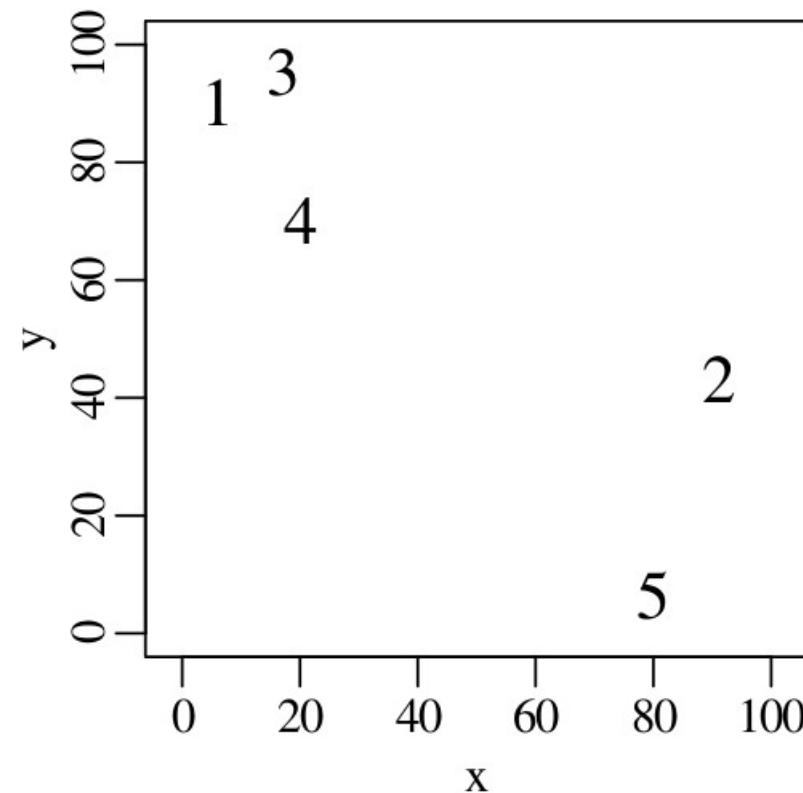




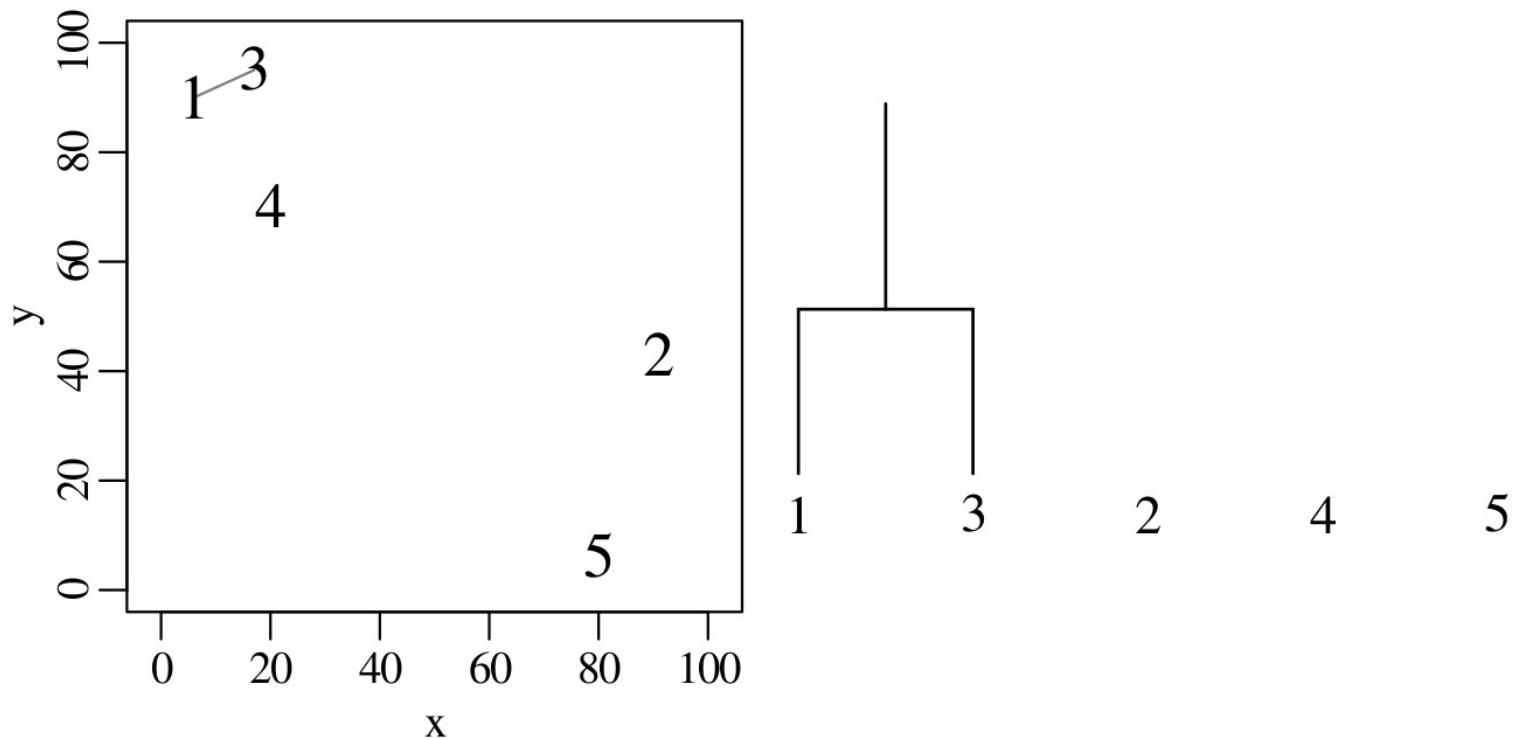
cluster	setosa	versicolor	virginica
1	50	0	0
2	0	48	14
3	0	2	36

# Hierarchical clustering

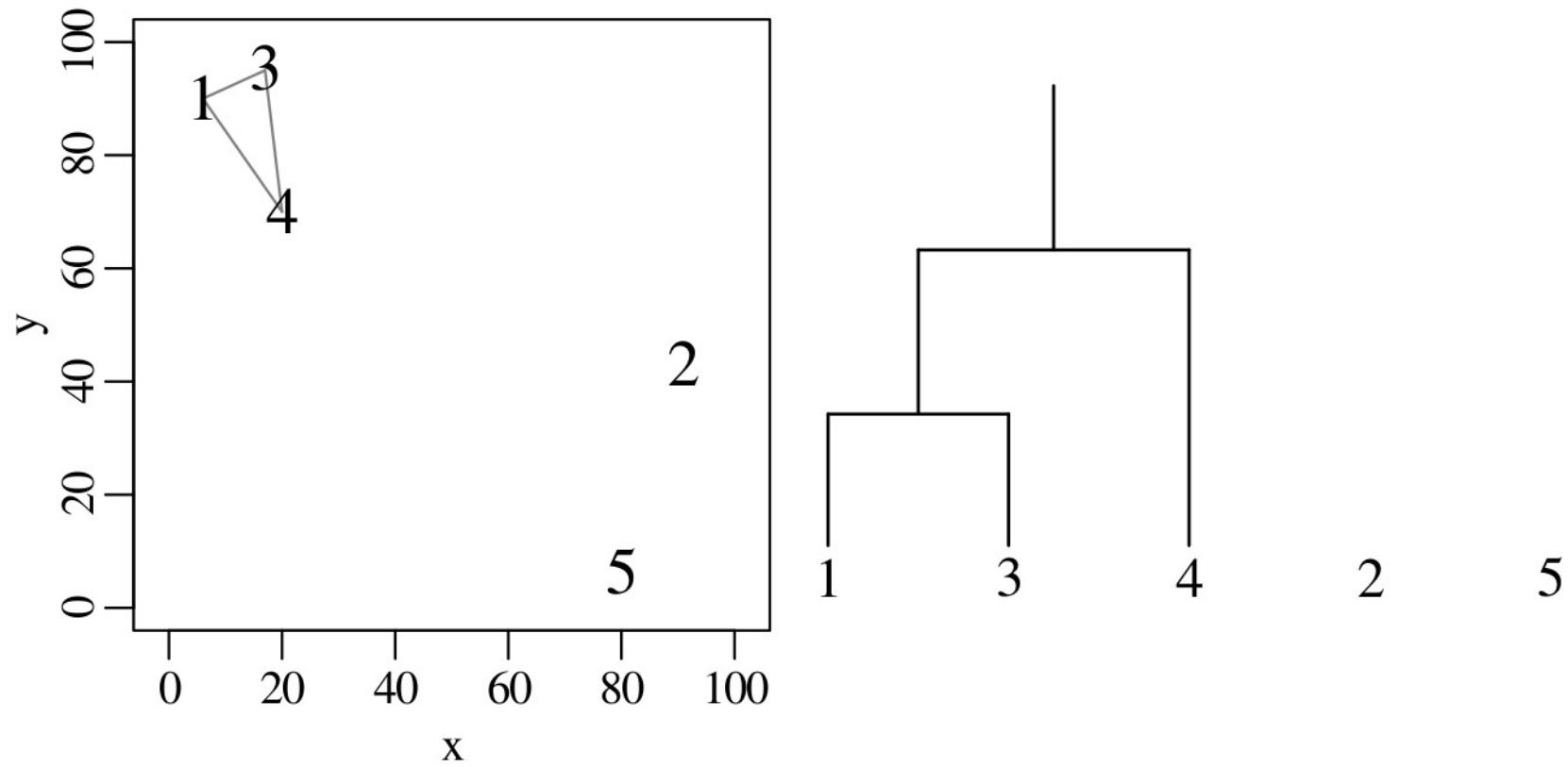
$$X = \begin{matrix} & x & y \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \left[ \begin{matrix} 6 & 90 \\ 91 & 43 \\ 17 & 95 \\ 20 & 70 \\ 80 & 6 \end{matrix} \right] \end{matrix}$$



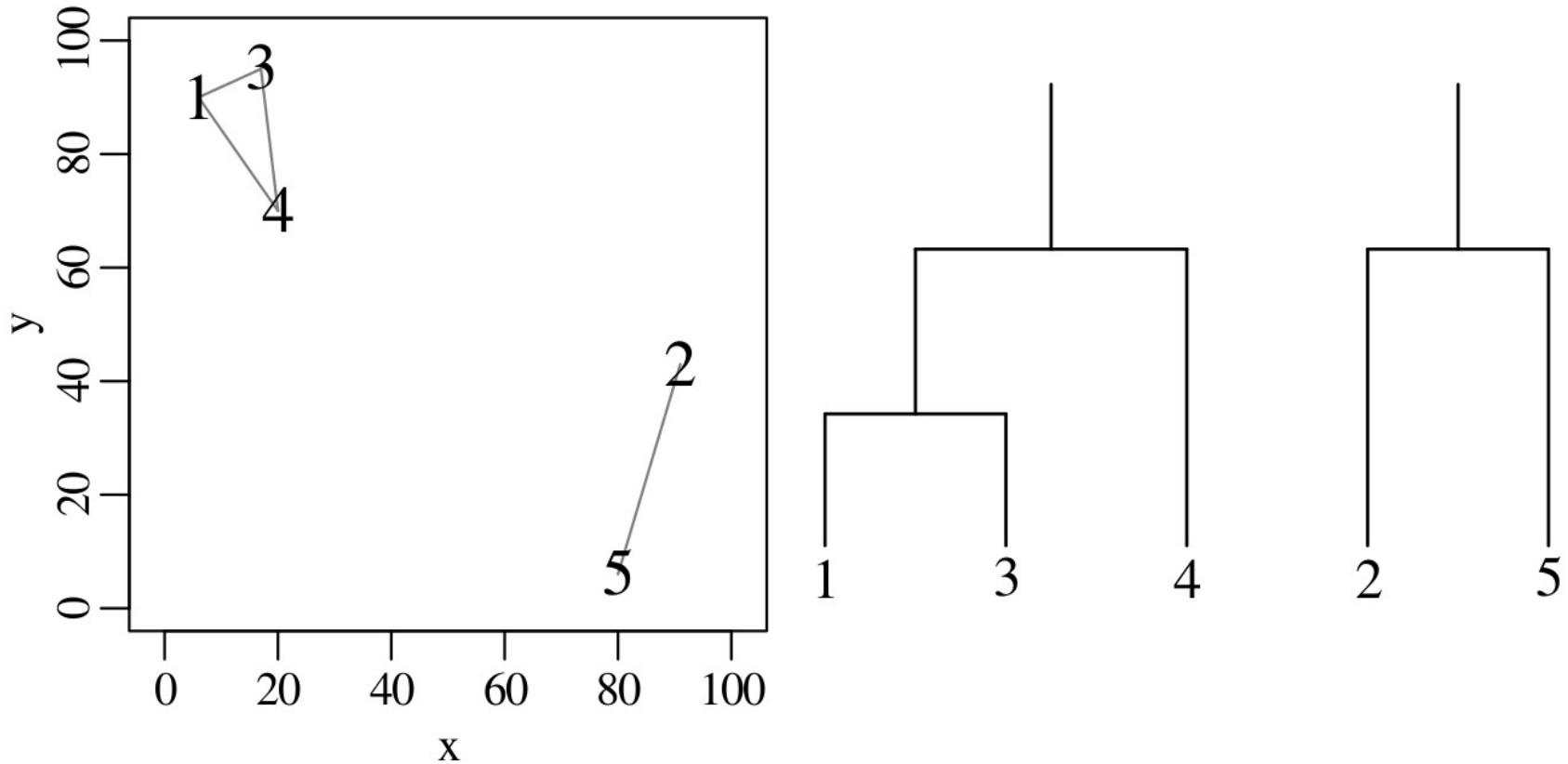
	1	2	3	4	5
1	0	97.1	12.1	24.4	112
2	97.1	0	90.4	76.0	38.6
3	12.1	90.4	0	25.2	109
4	24.4	76.0	25.2	0	87.7
5	112	38.6	109	87.7	0

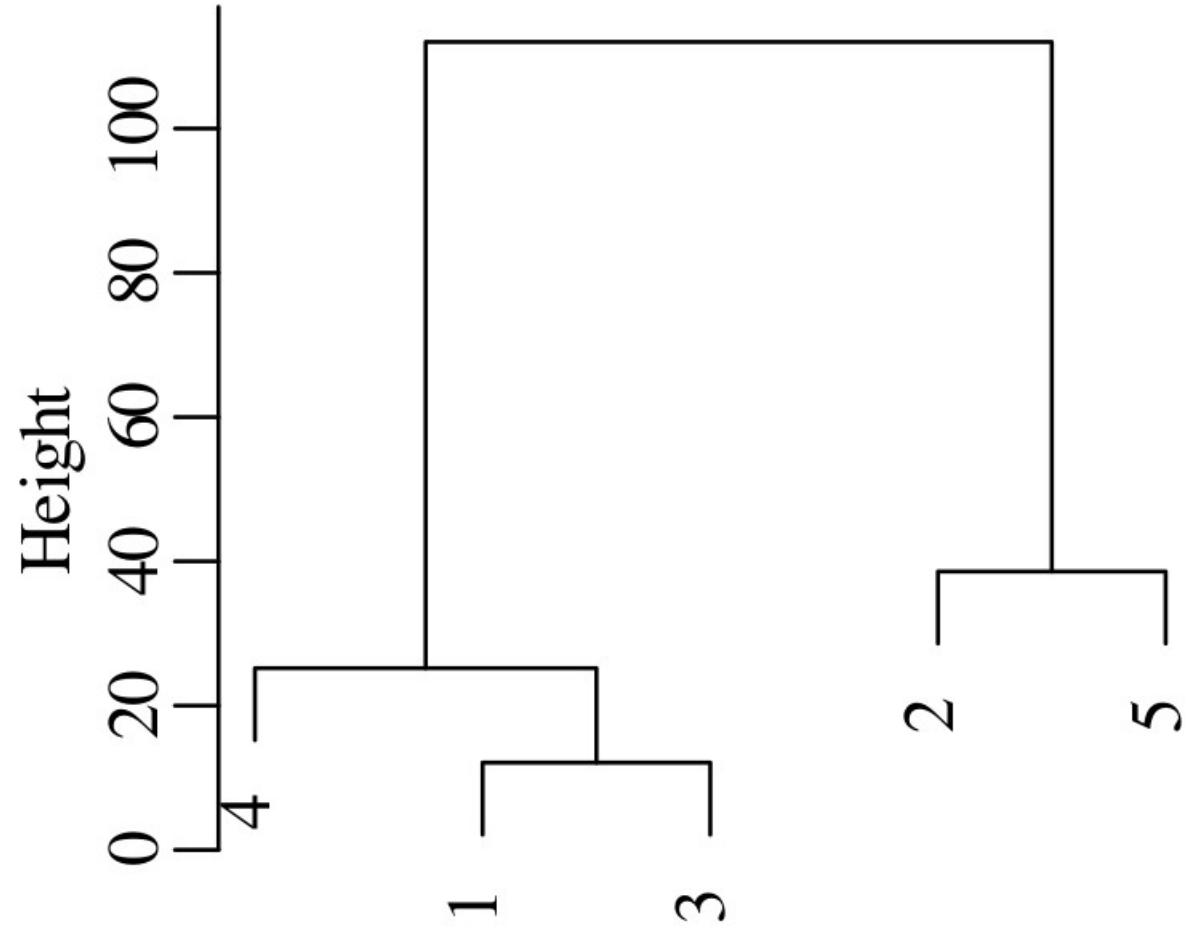


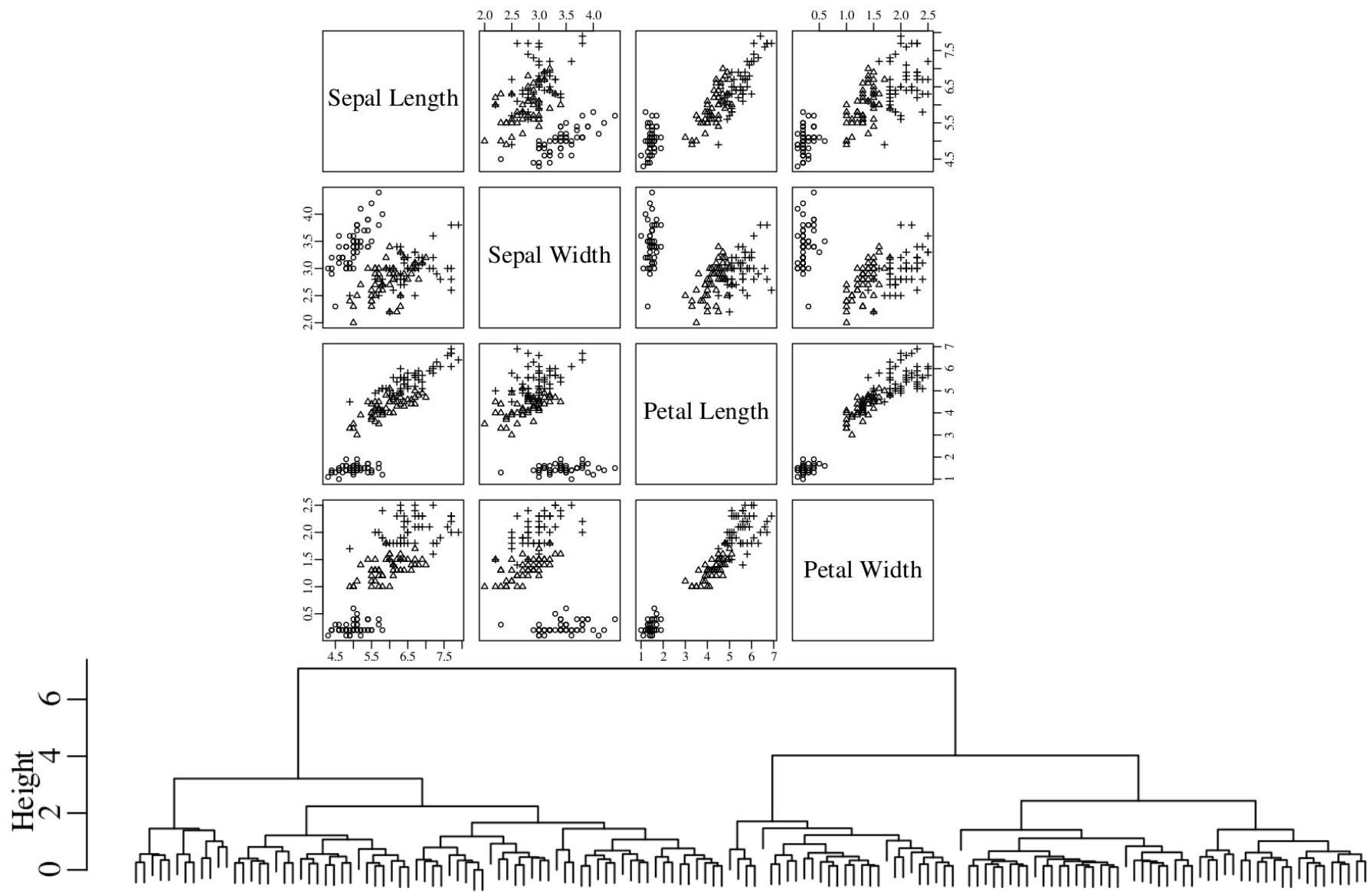
$$d = \begin{bmatrix} 13 & 2 & 4 & 5 \\ 13 & 0 & 97.1 & \boxed{25.2} & 112 \\ 2 & 97.1 & 0 & 76.0 & 38.6 \\ 4 & \boxed{25.2} & 76.0 & 0 & 87.7 \\ 5 & 112 & 38.6 & 87.7 & 0 \end{bmatrix} \text{ where } d[13, i] = \max(d[1, i], d[3, i]) \text{ for } i \in \{2, 4, 5\}$$

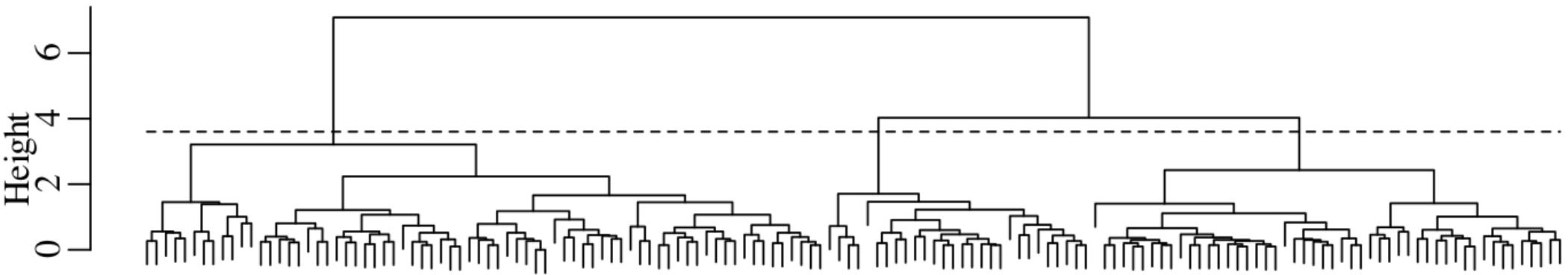


$$d = \begin{matrix} & 134 & 2 & 5 \\ & 134 & \left[ \begin{matrix} 0 & 97.1 & 112 \\ 97.1 & 0 & 76.0 \\ 112 & 76.0 & 0 \end{matrix} \right] \\ 2 & & & \\ 5 & & & \end{matrix}$$







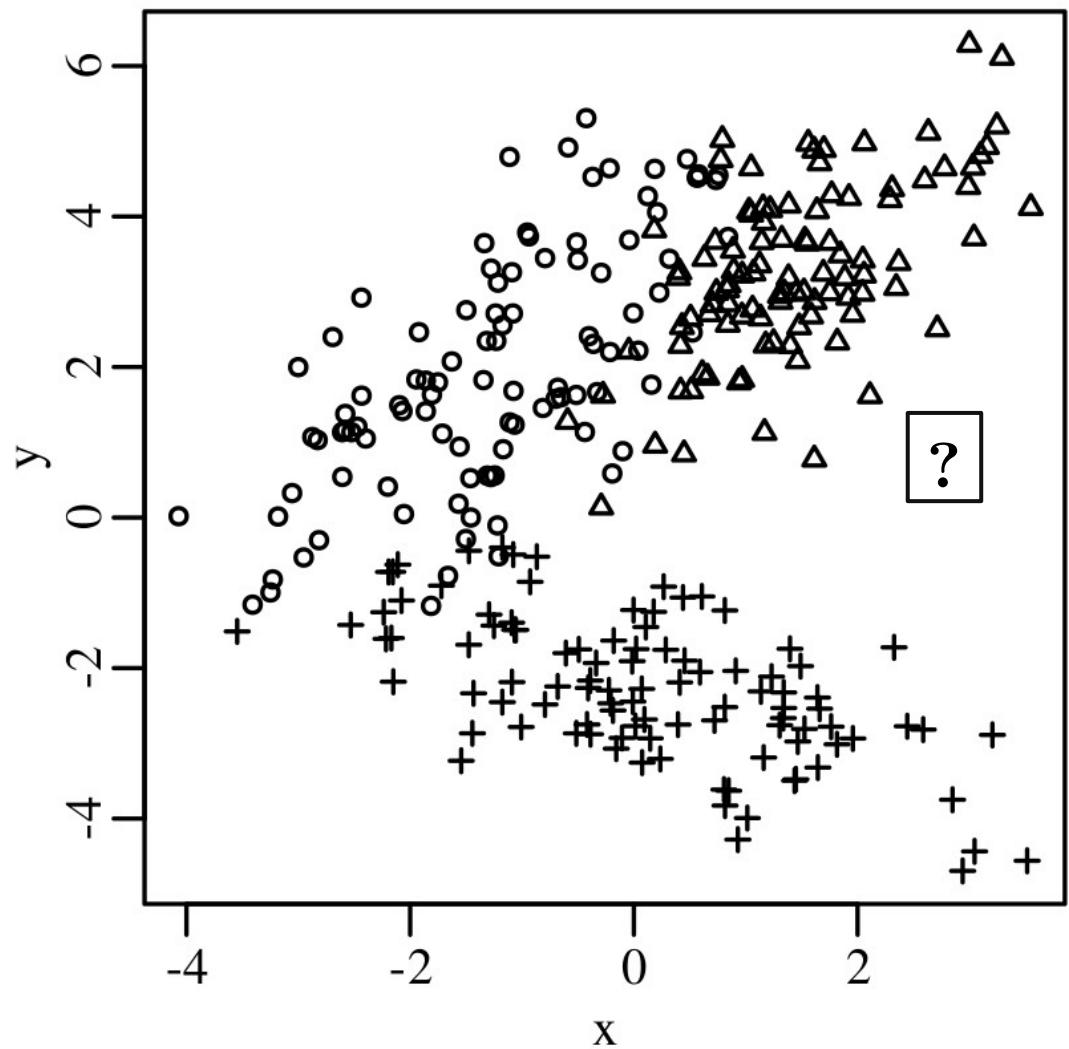


cluster	setosa	versicolor	virginica
1	50	0	0
2	0	23	49
3	0	27	1

# Statistics for geoscientists

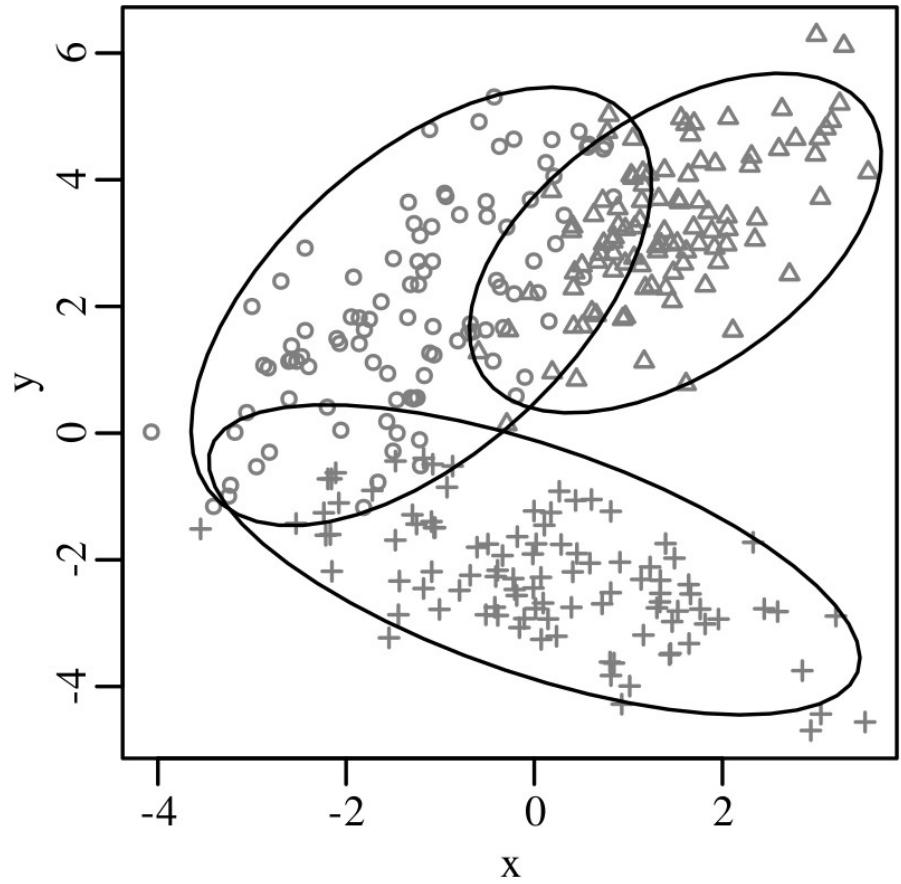
Supervised learning

# Discriminant Analysis



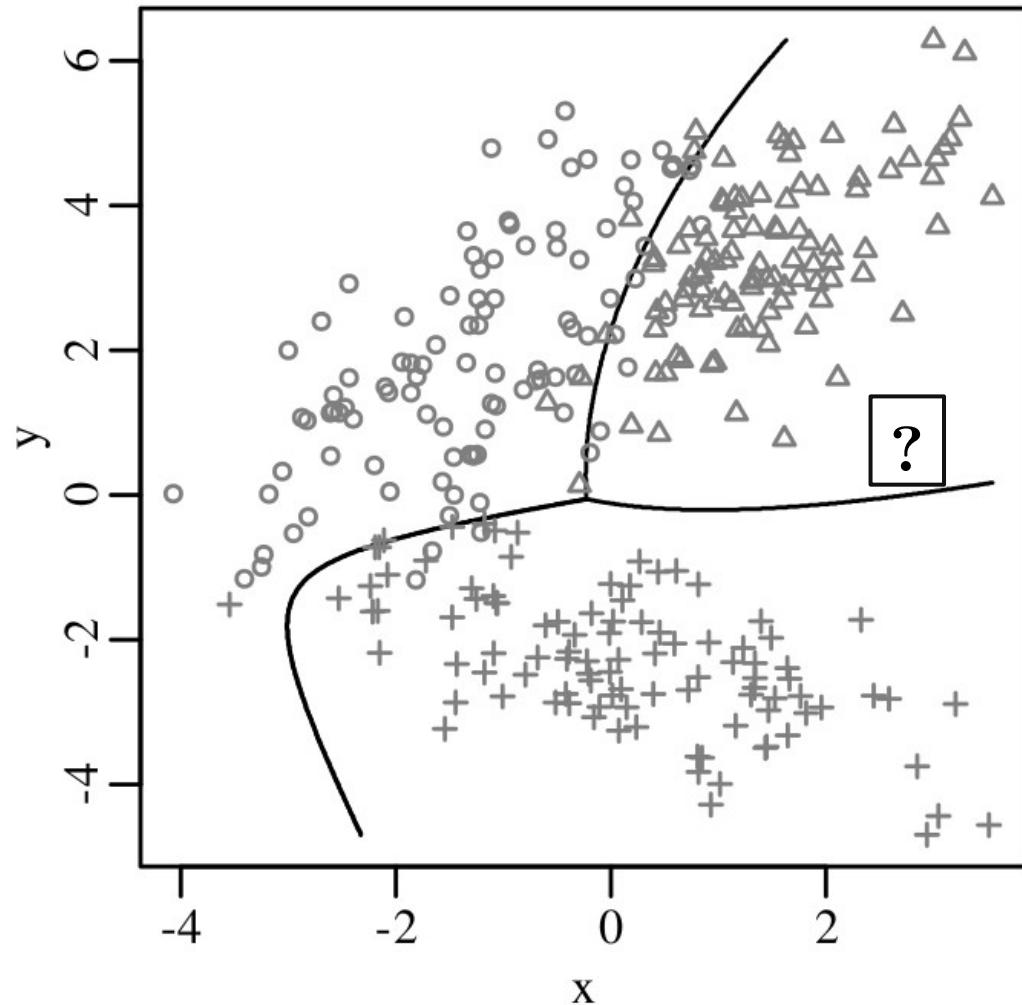
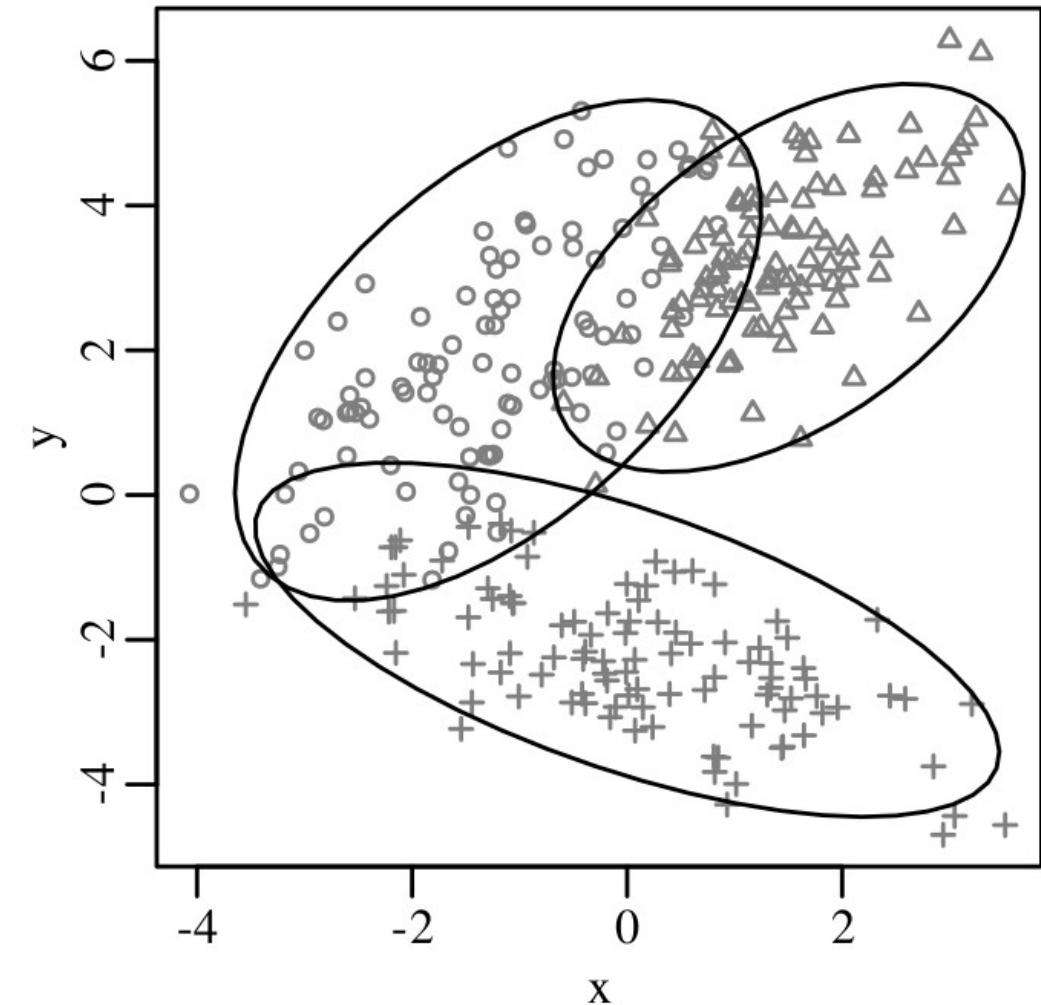
$$d = \max_{k=1,\dots,K} P(G = k | X = x)$$

$$P(G|X) \propto P(X|G)P(G)$$



$$P(X = x | G = k) = \frac{\exp\left(-\frac{1}{2}(x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k)\right)}{\sqrt{(2\pi)^N |\Sigma_k|}}$$

$$d = \max_{k=1,\dots,K} \left[ -\frac{1}{2} \ln |\Sigma_k| - \frac{1}{2}(x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k) \right]$$

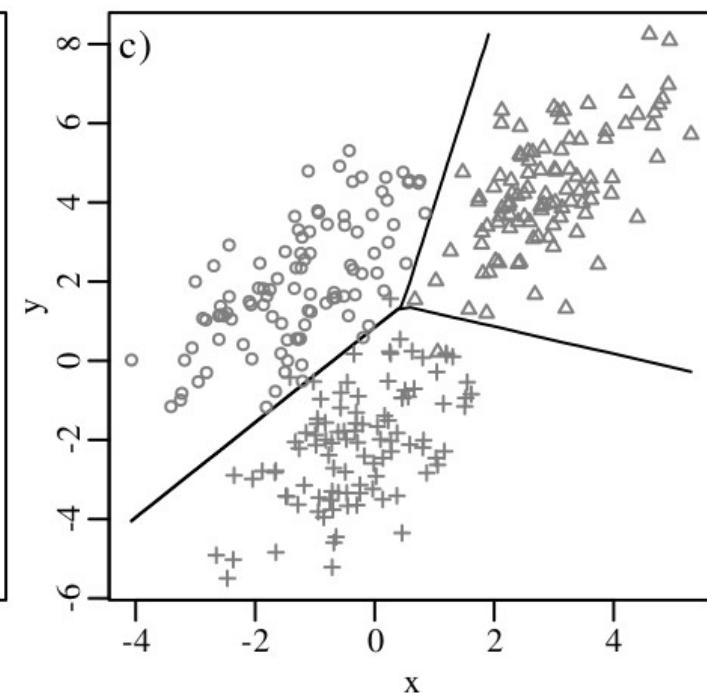
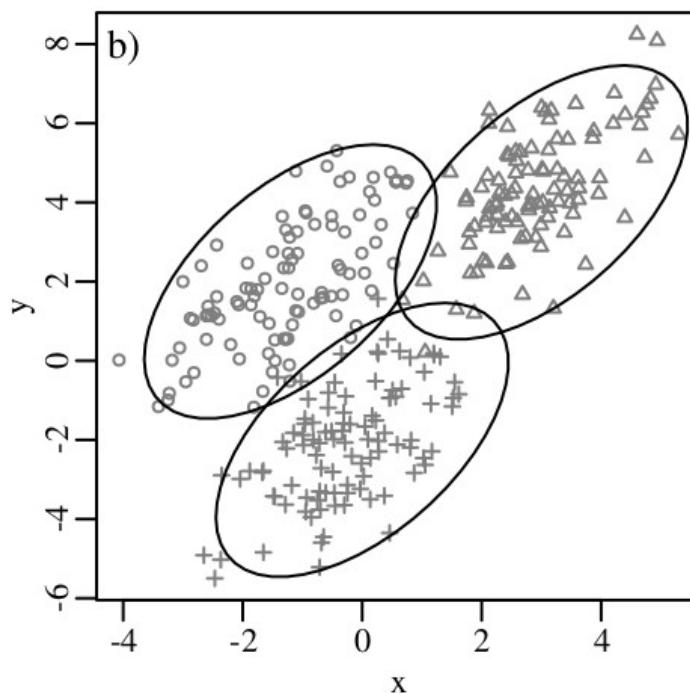
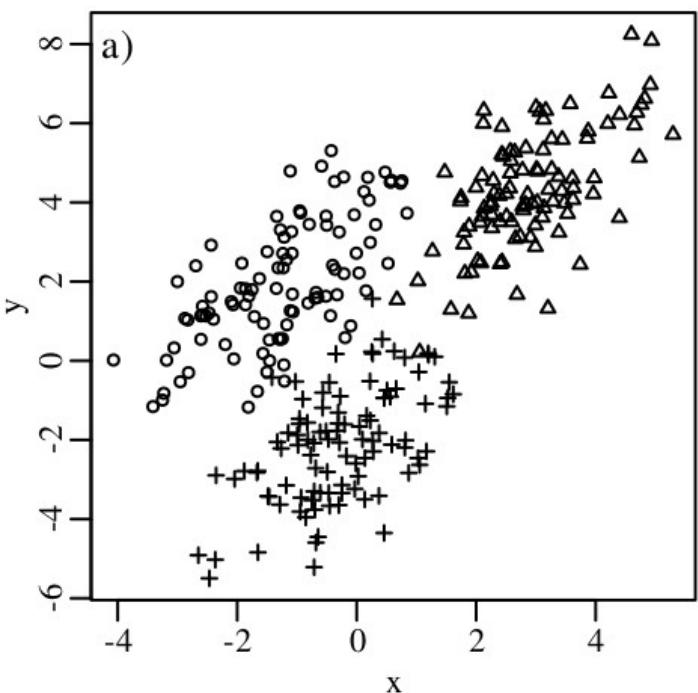


## QDA

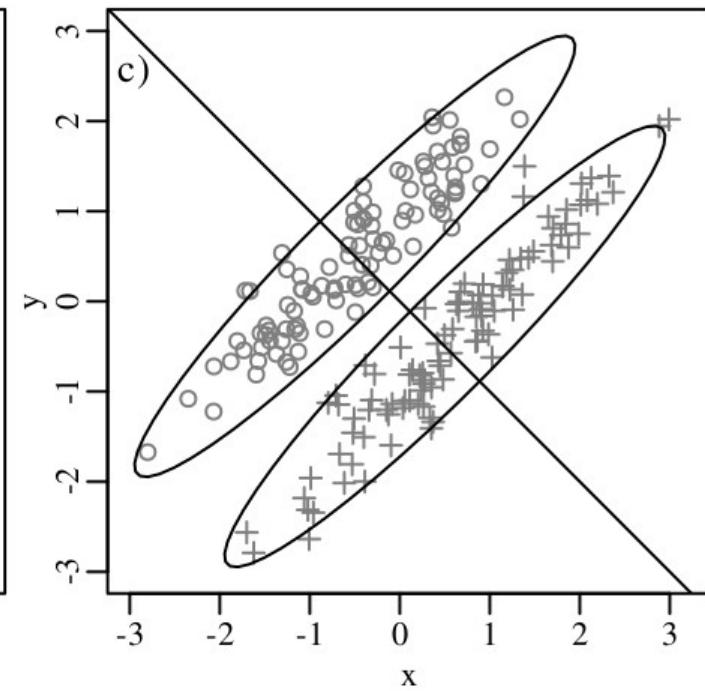
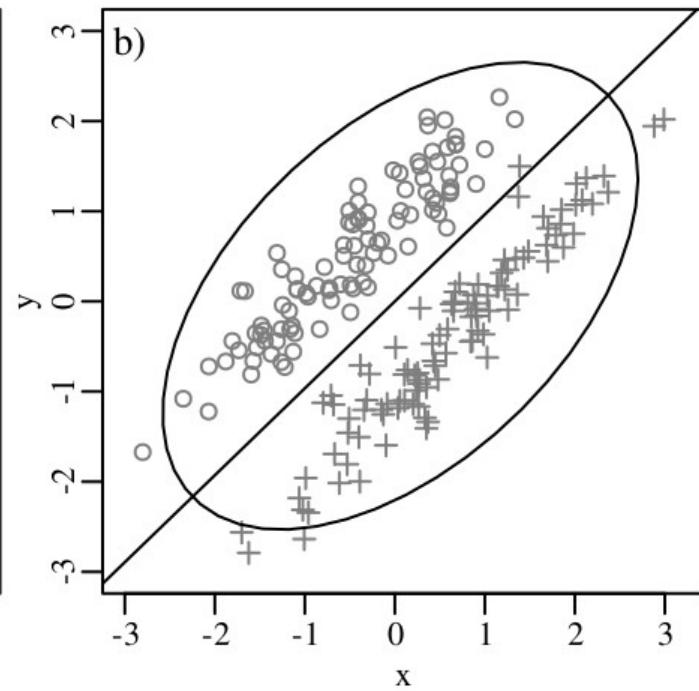
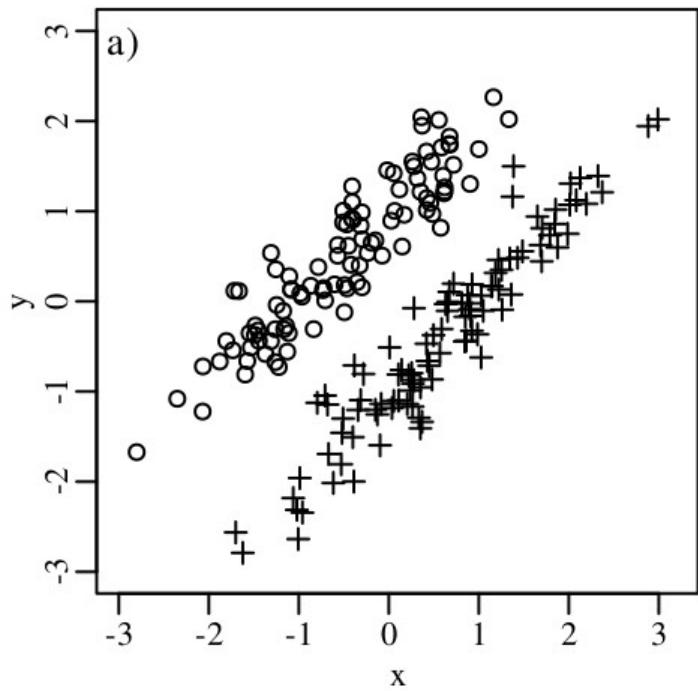
$$d = \max_{k=1,\dots,K} \left[ -\frac{1}{2} \ln |\Sigma_k| - \frac{1}{2} (x - \mu_k)^T \Sigma_k^{-1} (x - \mu_k) \right]$$

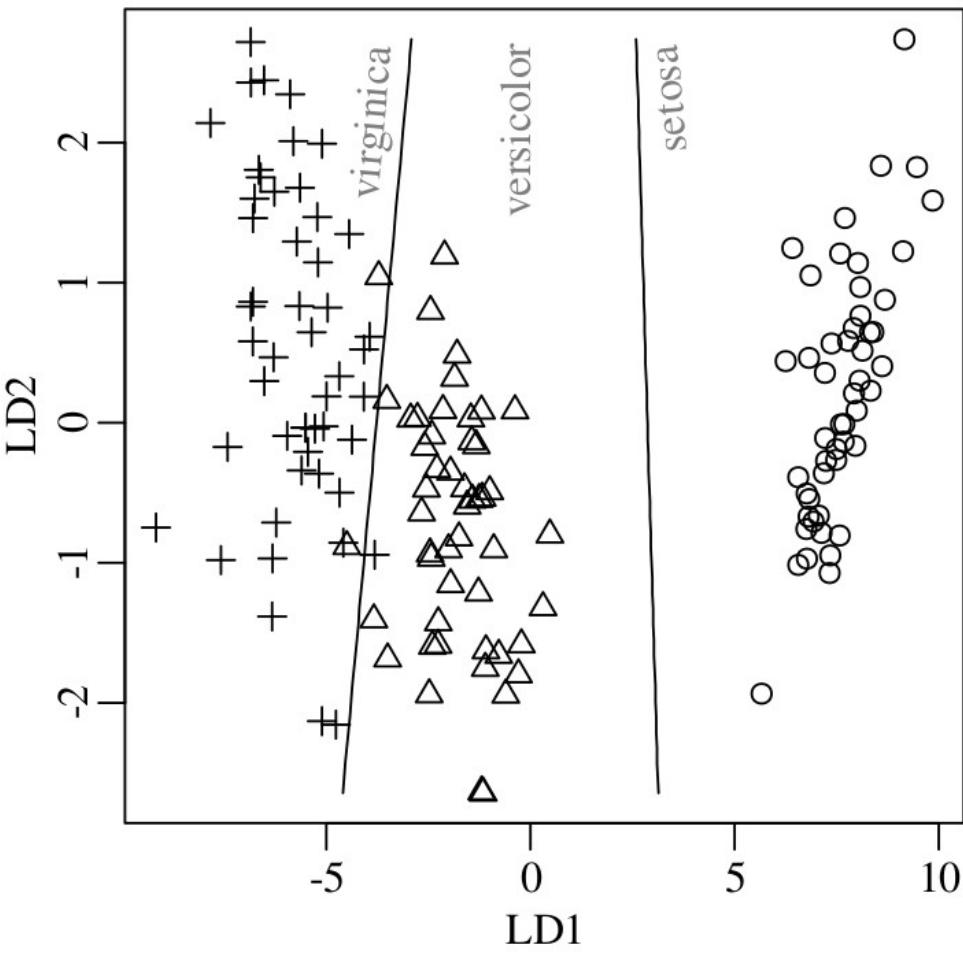
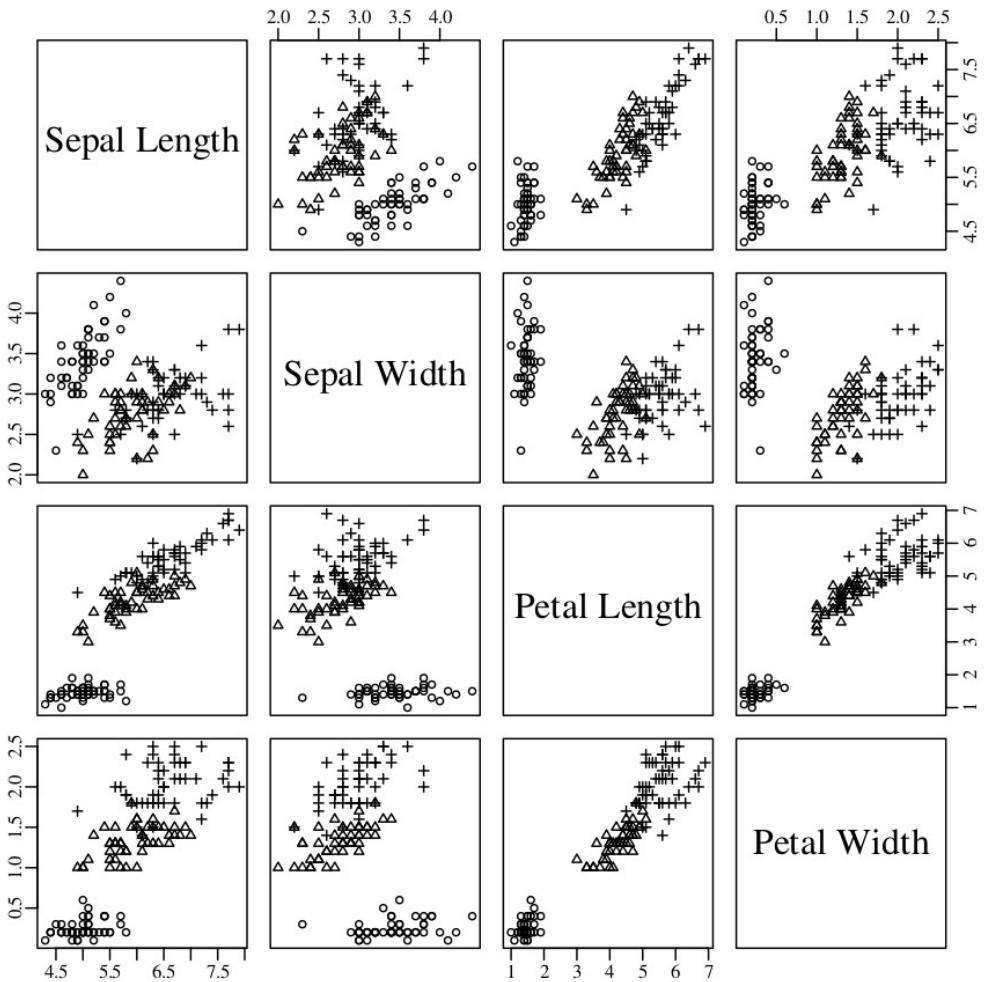
## LDA

$$d = \max_{k=1,\dots,K} \left[ x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k \right]$$



# PCA vs. LDA

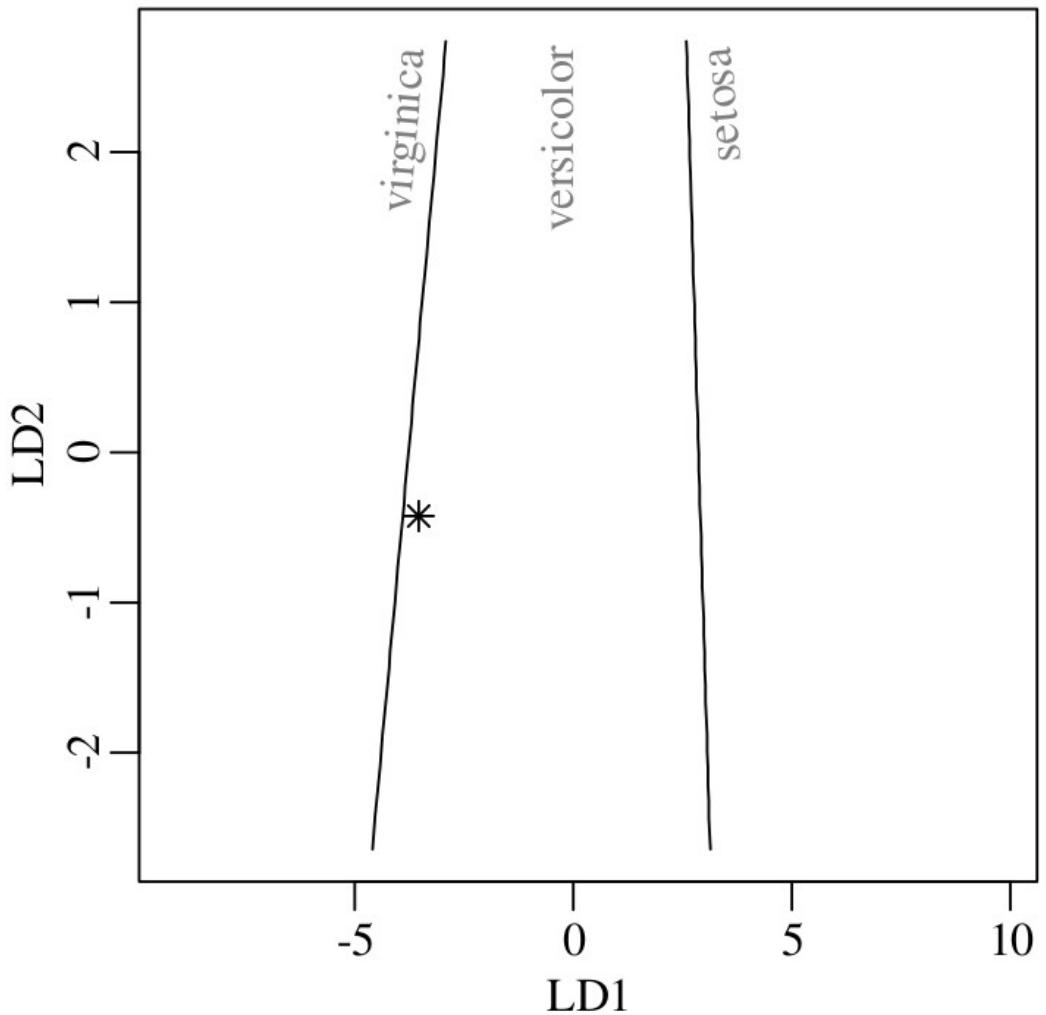




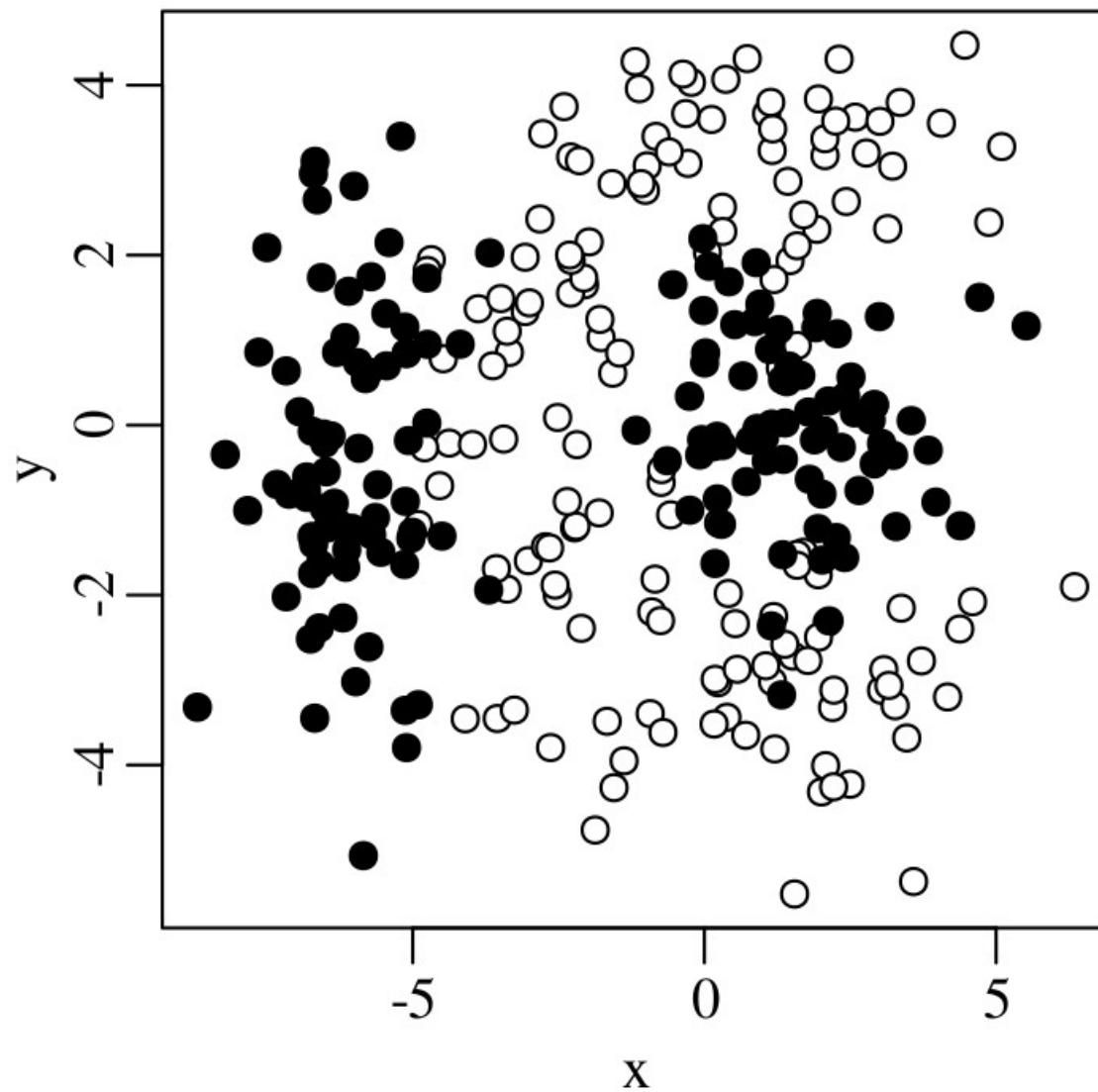
$$\begin{aligned} \text{LD1} = & 0.83 \times (\text{sepal length} - 5.84) \\ & + 1.53 \times (\text{sepal width} - 3.06) \\ & - 2.20 \times (\text{petal length} - 3.76) \\ & - 2.81 \times (\text{petal width} - 1.20) \end{aligned}$$

$$\begin{aligned} \text{LD2} = & 0.024 \times (\text{sepal length} - 5.85) \\ & + 2.16 \times (\text{sepal width} - 3.06) \\ & - 0.93 \times (\text{petal length} - 3.76) \\ & + 2.84 \times (\text{petal width} - 1.20) \end{aligned}$$

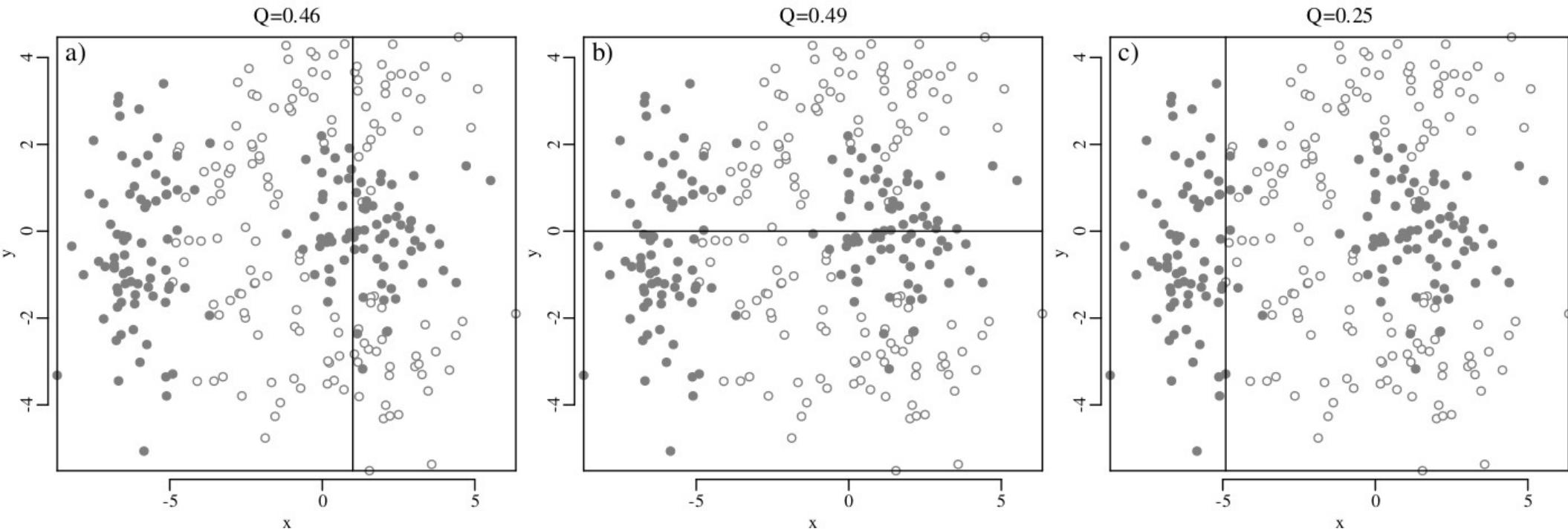
	<i>virginica</i>	<i>versicolor</i>	<i>setosa</i>
$P(G X)$	0.19	0.81	$3.2 \times 10^{-27}$

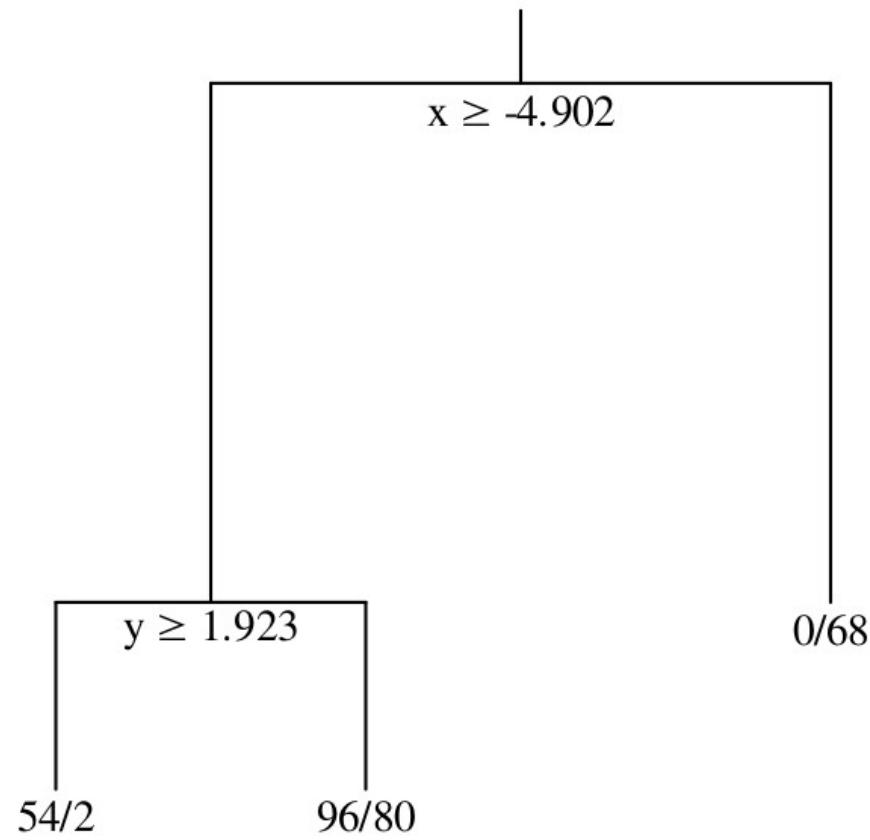
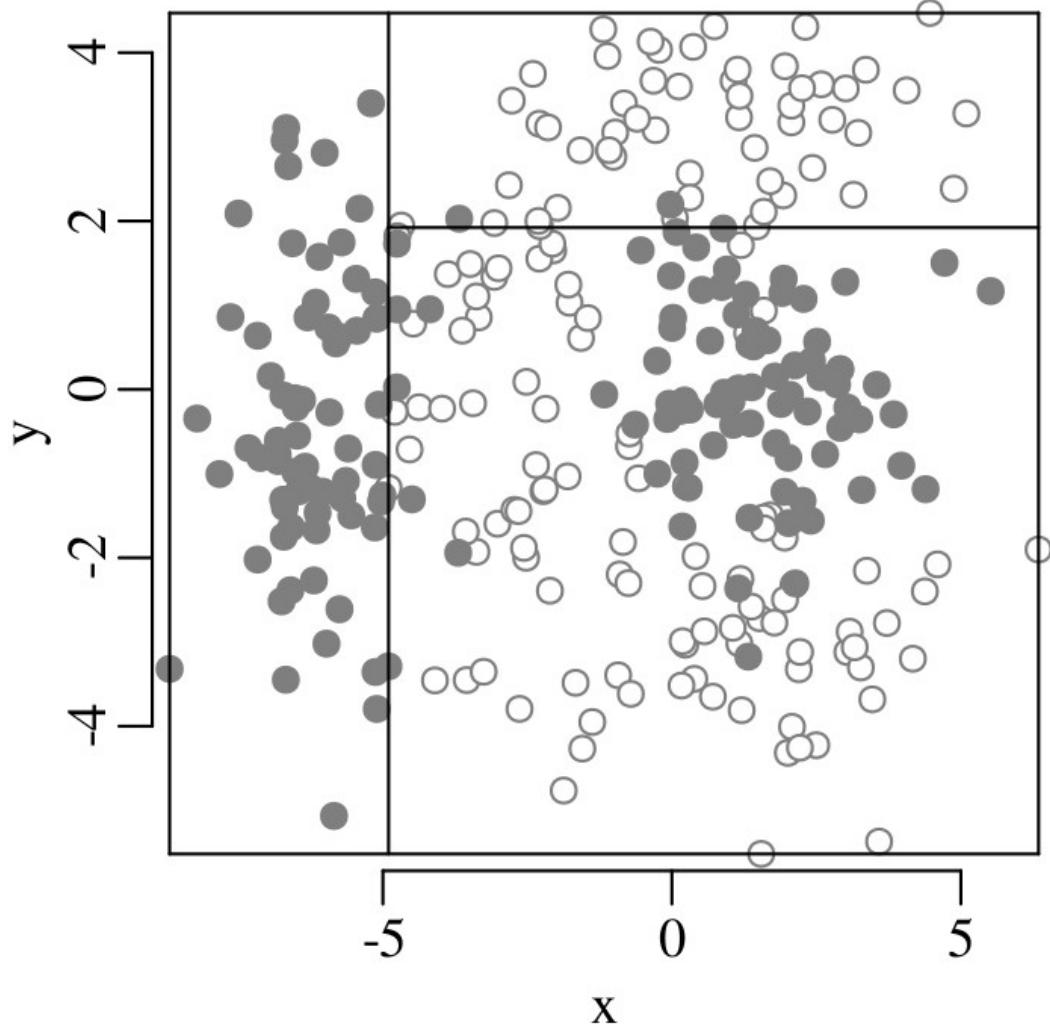


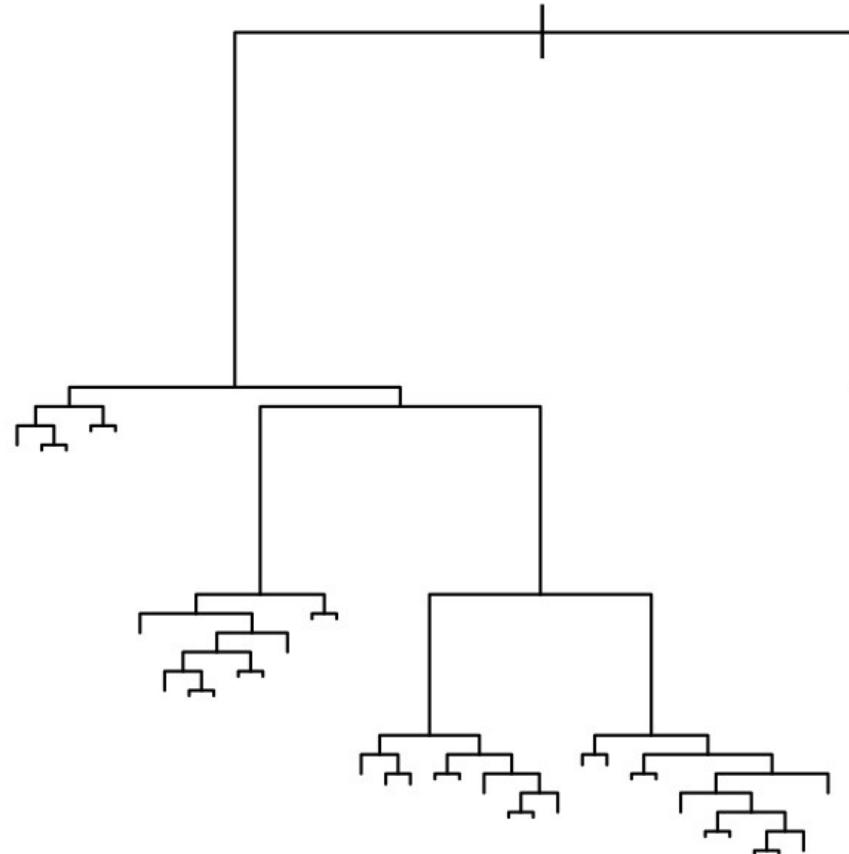
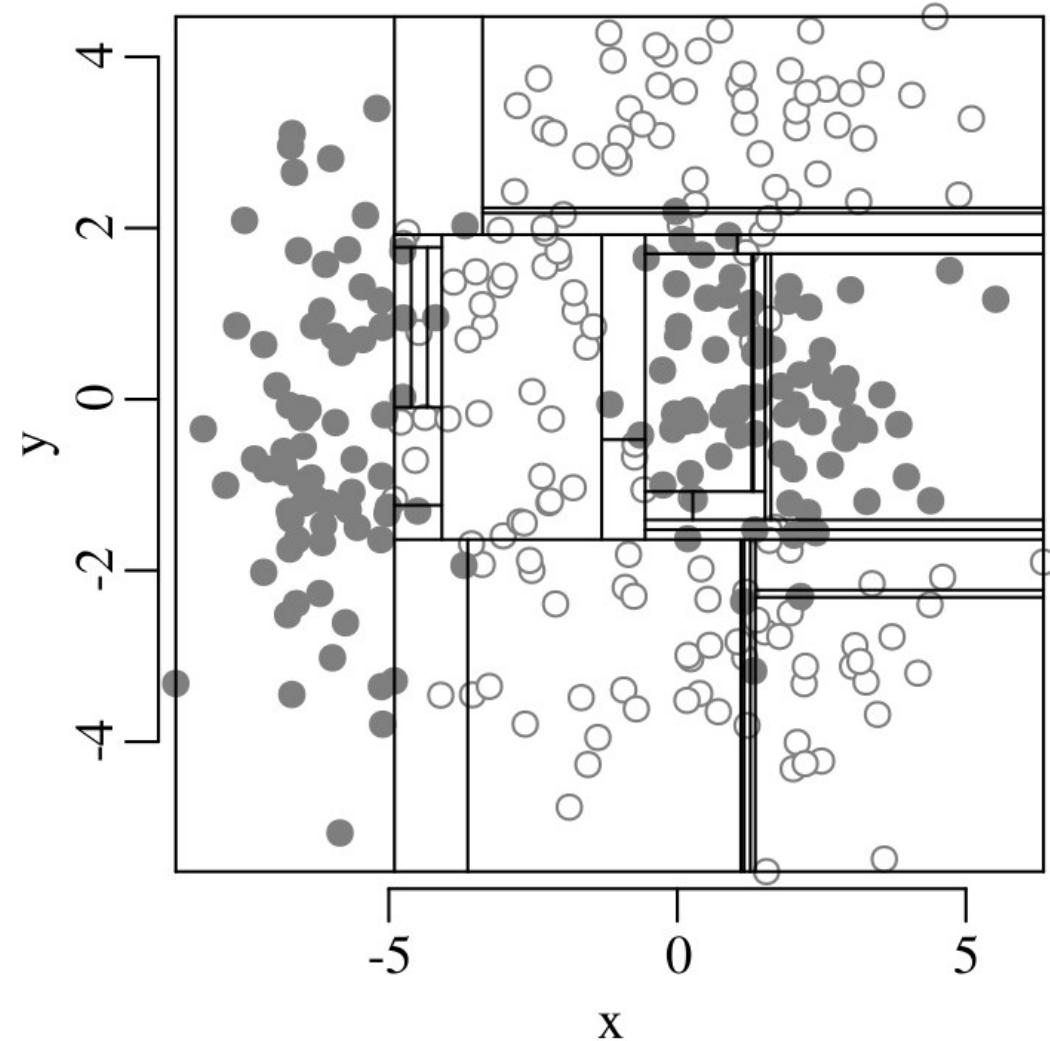
## Decision trees

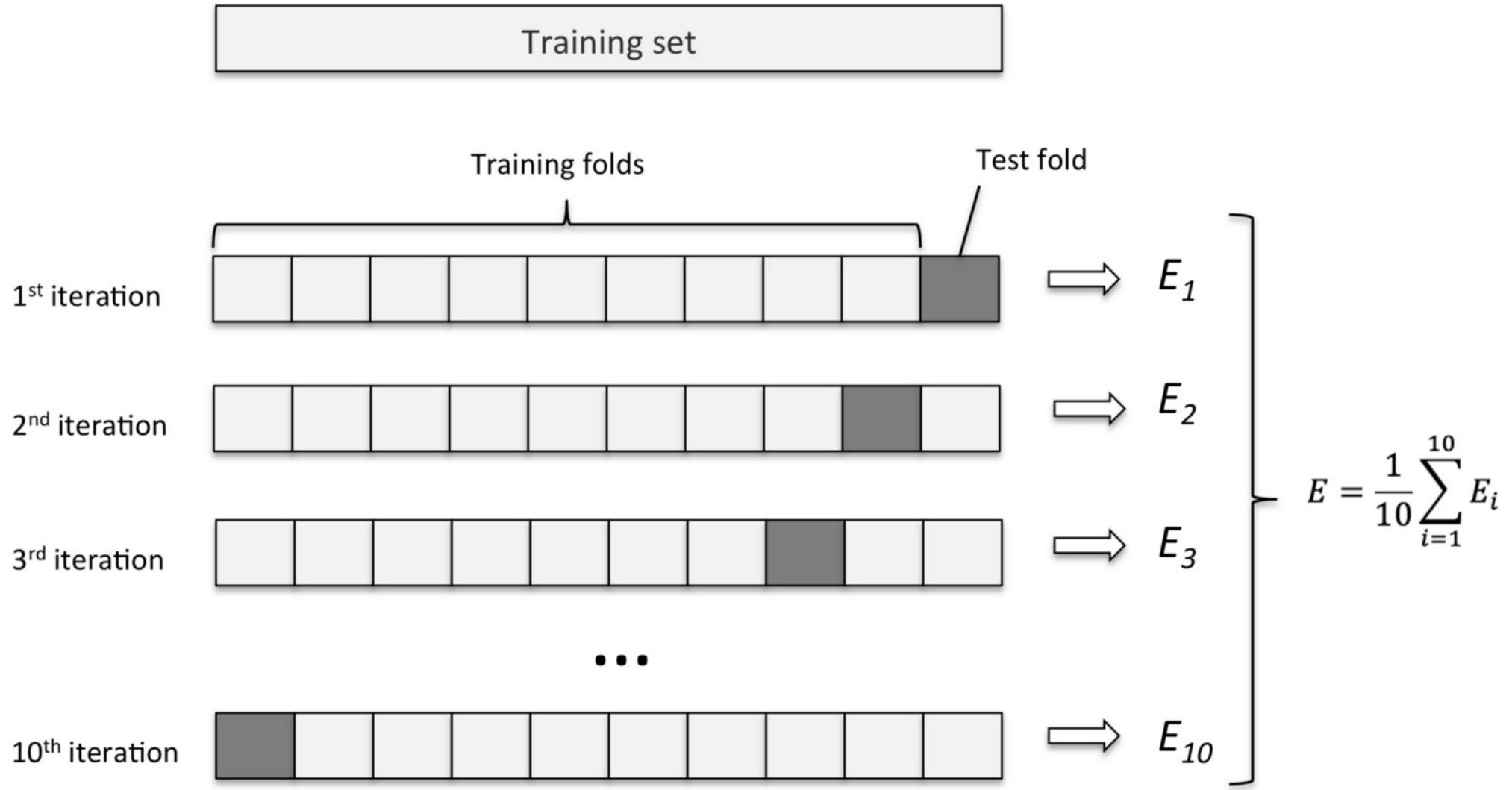


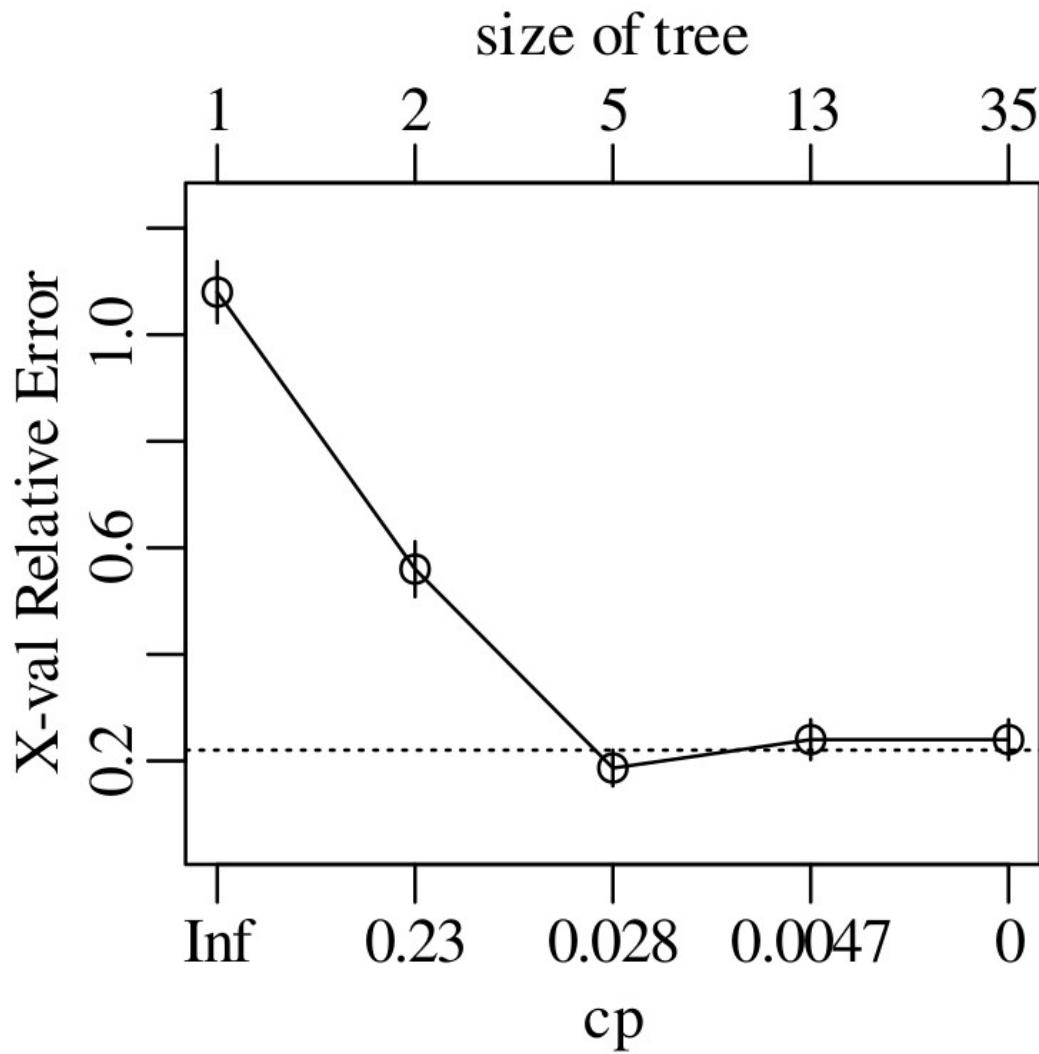
$$Q = p_{\circ}(1 - p_{\circ}) + p_{\bullet}(1 - p_{\bullet})$$



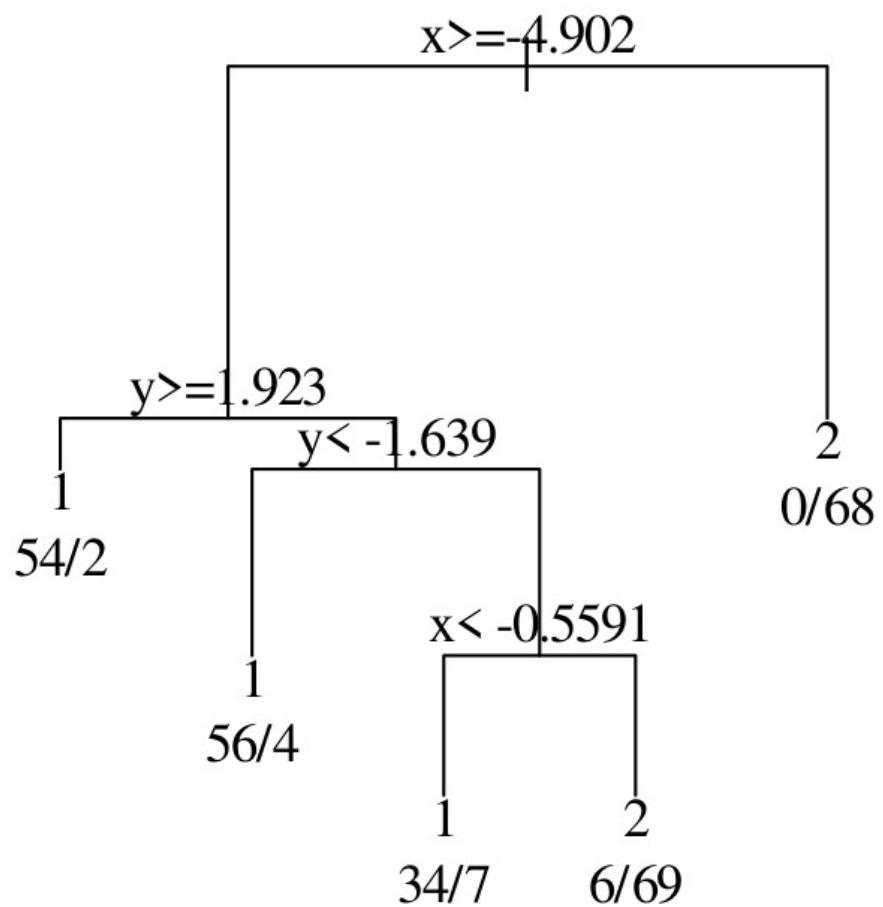
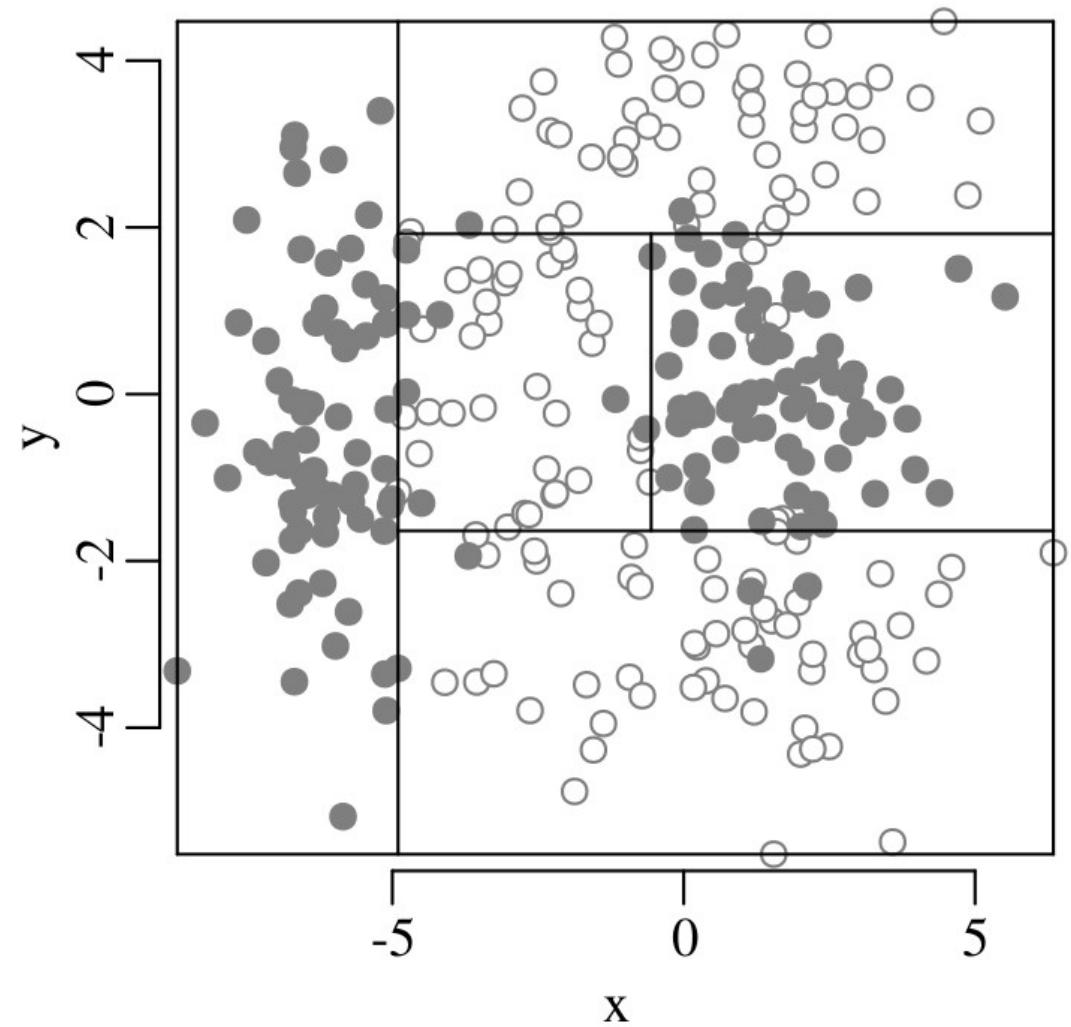


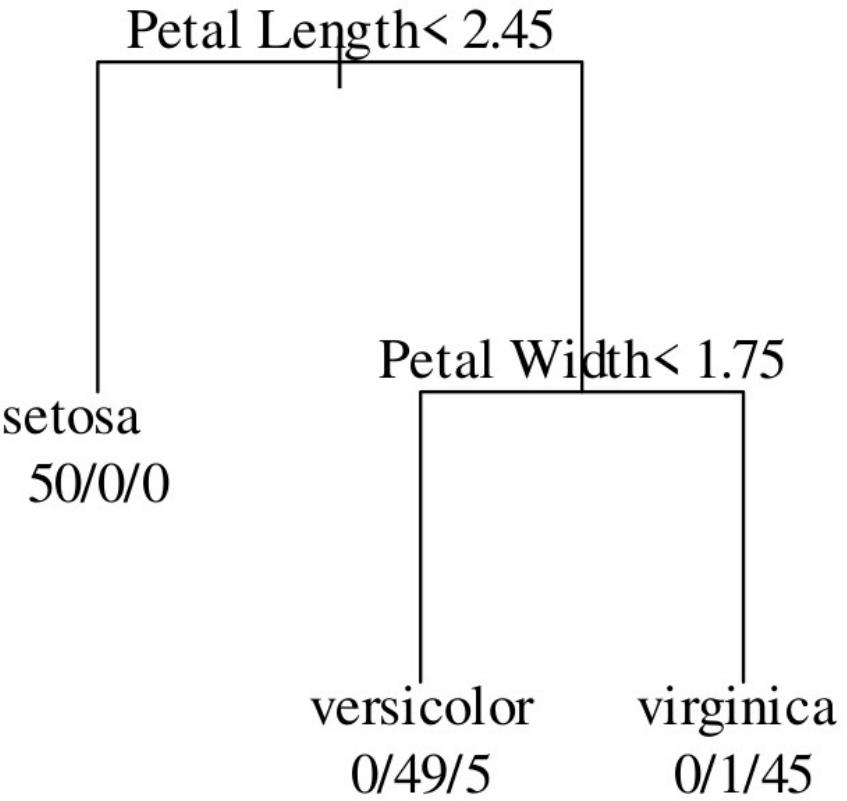
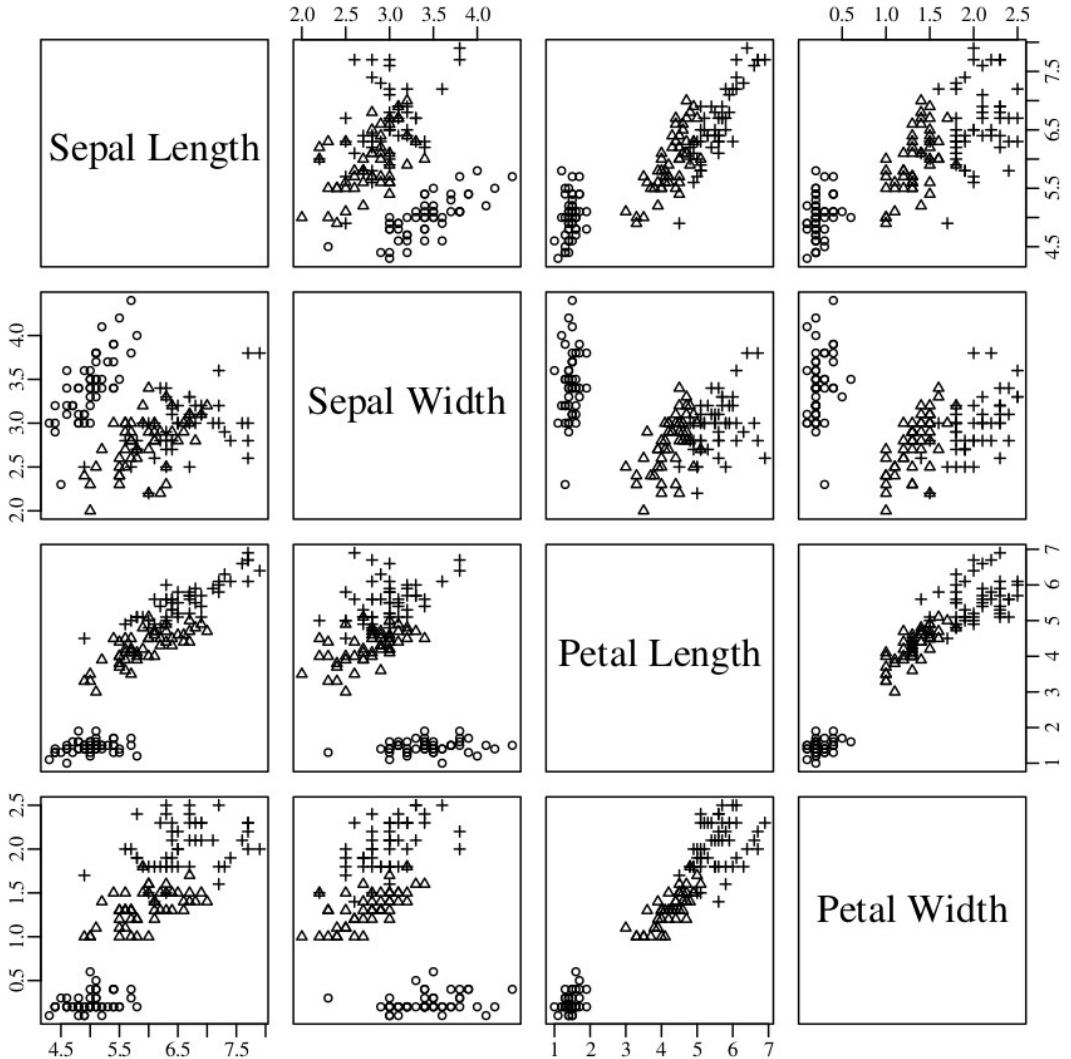






$$cp_\alpha(T) = \sum_{m=1}^{|T|} N_m Q_m + \alpha |T|$$
$$T_\alpha = \min_{T \subset T_0} cp_\alpha(T)$$





# Statistics for geoscientists

Compositional data

## Ratio data

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
$A$	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
$B$	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
$A/B$	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079
$B/A$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0

	1	2	3	4	5	6	7	8	9	10
$A$	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
$B$	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
$A/B$	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079
$B/A$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0
$1/(A/B)$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10	mean
$A/B$	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079	1.20
$B/A$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0	2.30
$1/(A/B)$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0	

$$\frac{1}{\overline{A/B}} = \frac{1}{1.20} = 0.81 \neq 2.30 = \overline{B/A}$$

and  $\frac{1}{\overline{B/A}} = \frac{1}{2.30} = 0.44 \neq 1.20 = \overline{A/B}$

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
$\ln[A/B]$	0.25	0.75	-0.18	0.86	-1.30	0.59	0.27	-0.41	0.50	-2.50
$\ln[B/A]$	-0.25	-0.75	0.18	-0.86	1.30	-0.59	-0.27	0.41	-0.50	2.50

	mean	$\exp[\text{mean}]$
$\ln[A/B]$	-0.12	0.88
$\ln[B/A]$	0.12	1.13

$$\frac{1}{g(A/B)} = \frac{1}{0.88} = 1.13 = g(B/A)$$

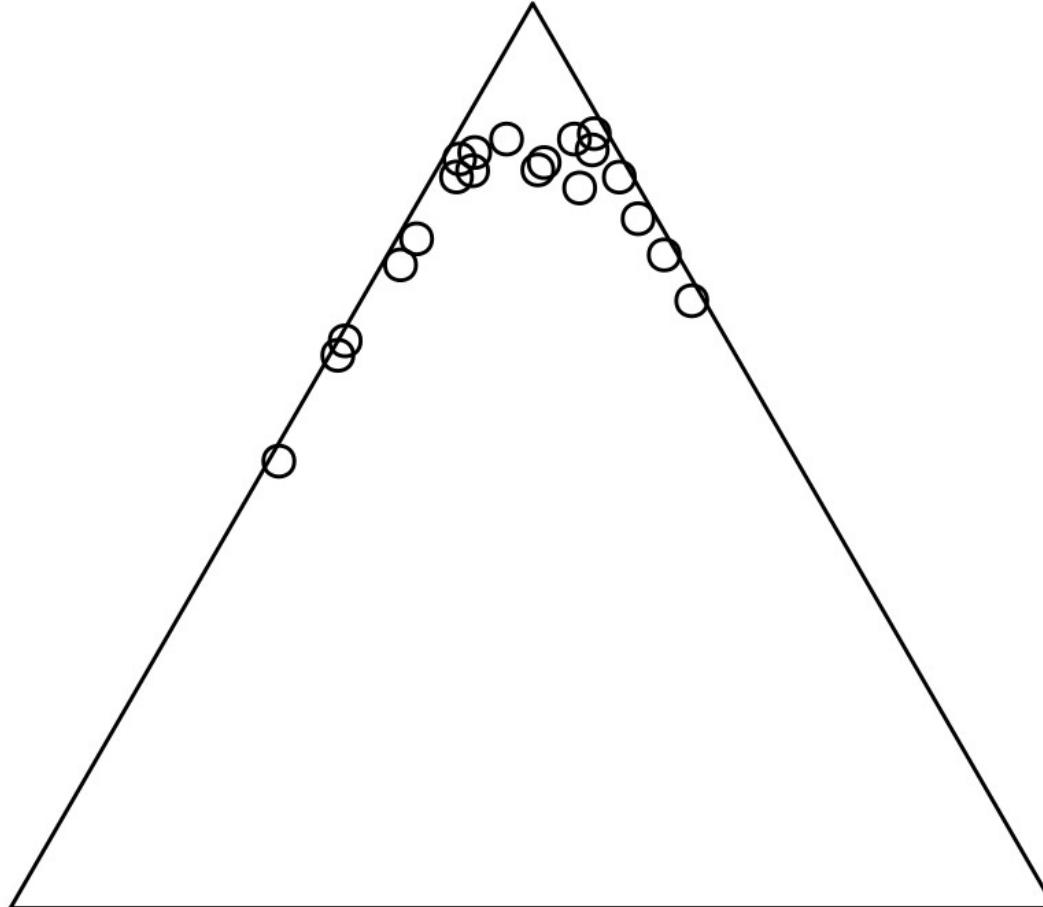
and  $\frac{1}{g(B/A)} = \frac{1}{1.13} = 0.88 = g(A/B)$

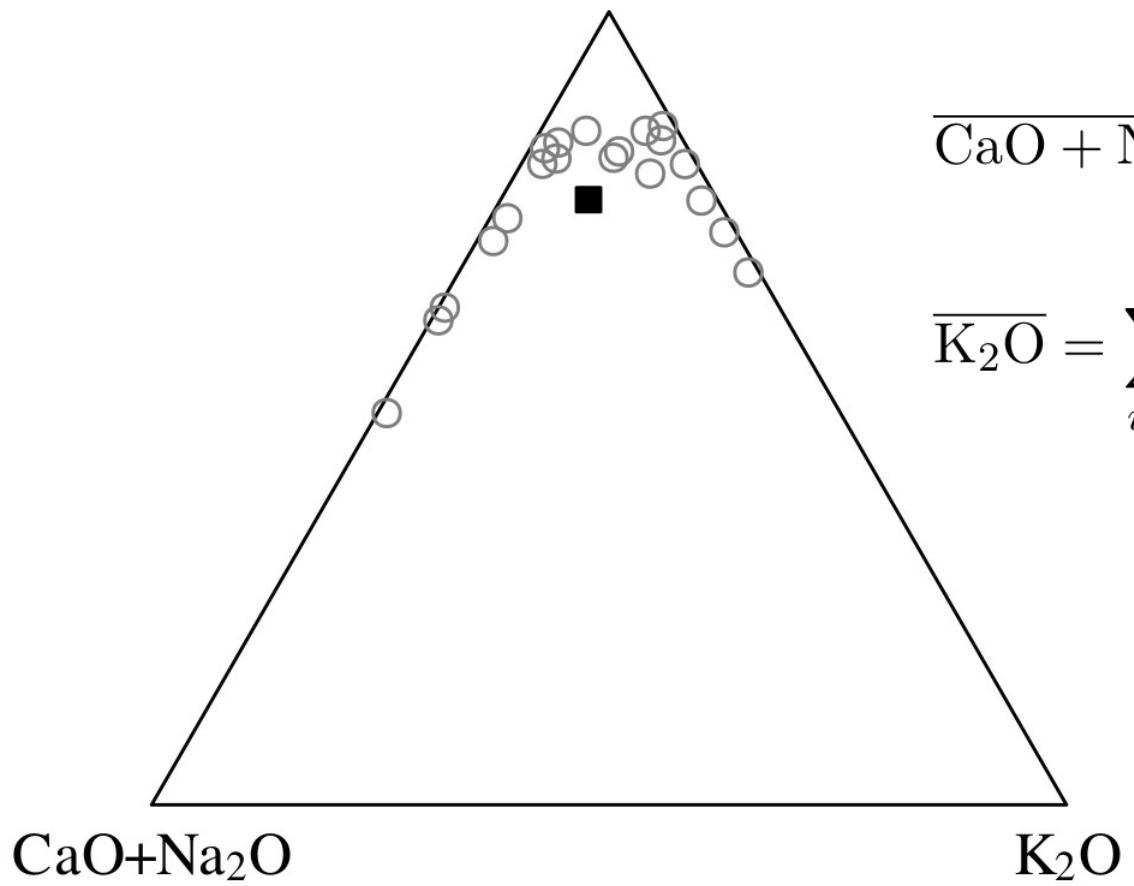
$$\sum_{i=1}^n C_i = 1$$

Al<sub>2</sub>O<sub>3</sub>

CaO+Na<sub>2</sub>O

K<sub>2</sub>O



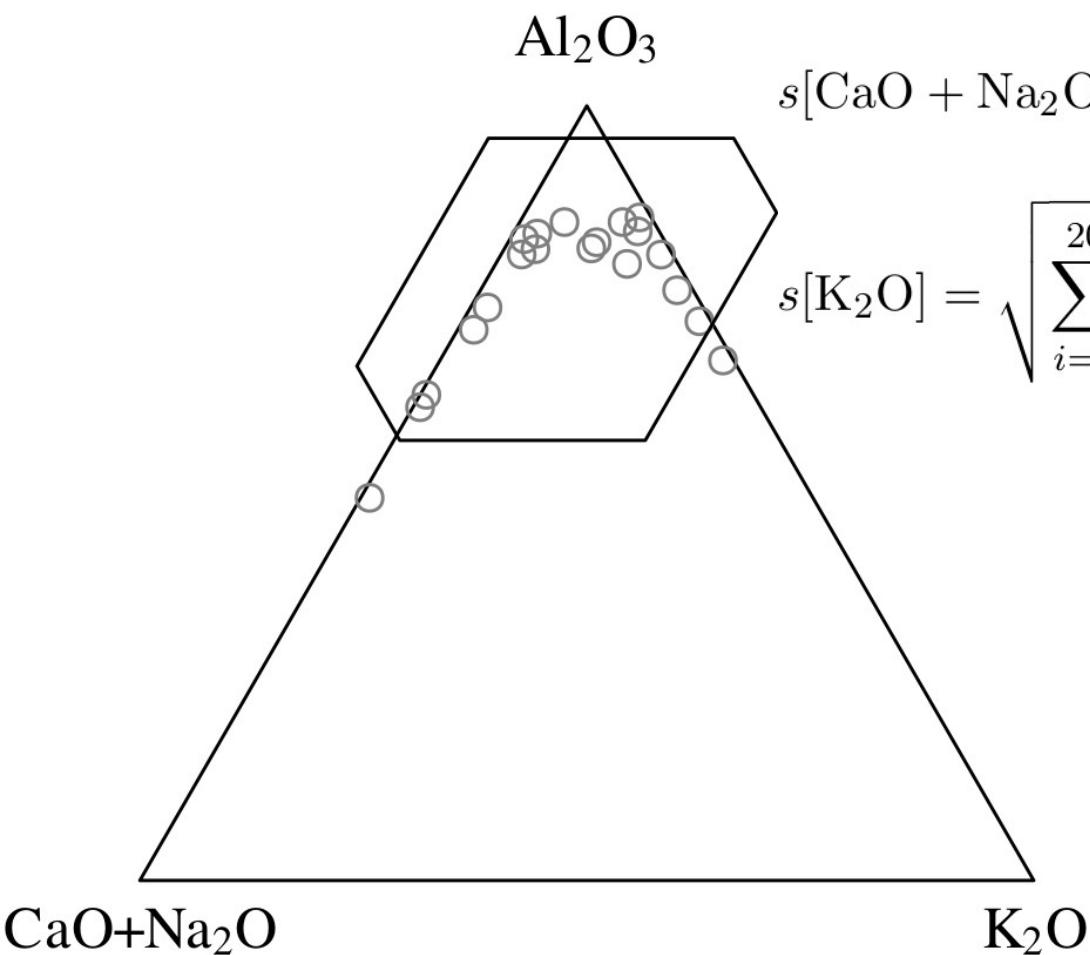


$$\overline{\text{Al}_2\text{O}_3} = \sum_{i=1}^{20} (\text{Al}_2\text{O}_3)_i / 20 = 0.763$$

$$\overline{\text{CaO} + \text{Na}_2\text{O}} = \sum_{i=1}^{20} (\text{CaO}_2 + \text{Na}_2\text{O})_i / 20 = 0.141$$

$$\overline{\text{K}_2\text{O}} = \sum_{i=1}^{20} (\text{K}_2\text{O})_i / 20 = 0.096$$

$$s[\text{Al}_2\text{O}_3] = \sqrt{\sum_{i=1}^{20} \frac{((\text{Al}_2\text{O}_3)_i - 0.763)^2}{19}} = 0.0975$$



$$s[\text{CaO} + \text{Na}_2\text{O}] = \sqrt{\sum_{i=1}^{20} \frac{((\text{CaO}_2 + \text{Na}_2\text{O})_i - 0.141)^2}{19}} = 0.142$$

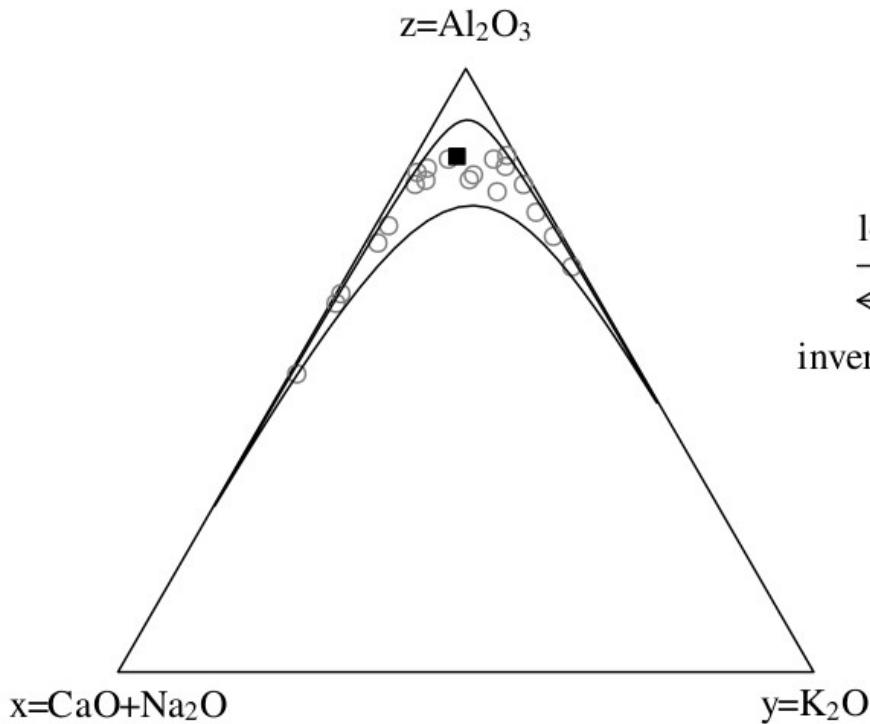
$$s[\text{K}_2\text{O}] = \sqrt{\sum_{i=1}^{20} \frac{((\text{K}_2\text{O})_i - 0.096)^2}{19}} = 0.0926$$

$\text{Al}_2\text{O}_3 : 0.763 \pm 0.195$

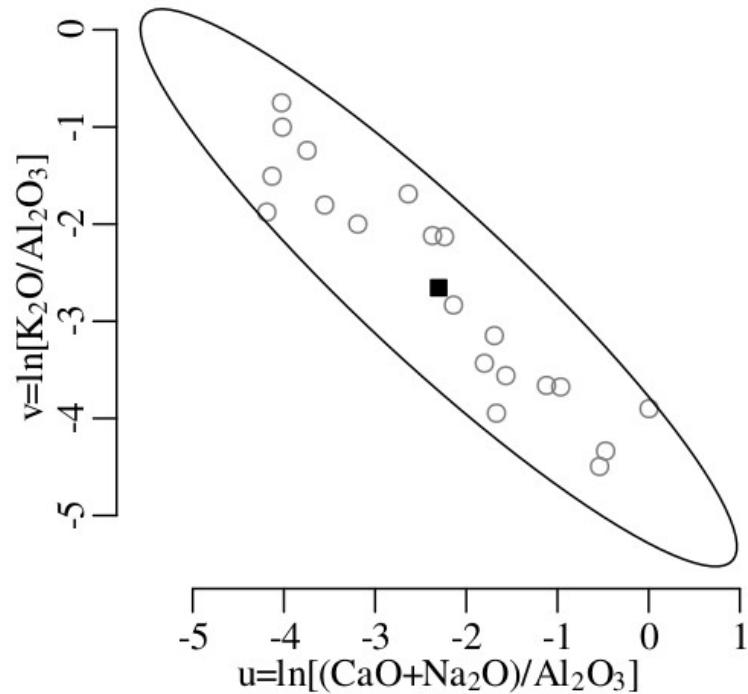
$\text{CaO} + \text{Na}_2\text{O} : 0.141 \pm 0.284$

$\text{K}_2\text{O} : 0.096 \pm 0.185$

$$v = \ln \left[ \frac{x}{z} \right], w = \ln \left[ \frac{y}{z} \right]$$



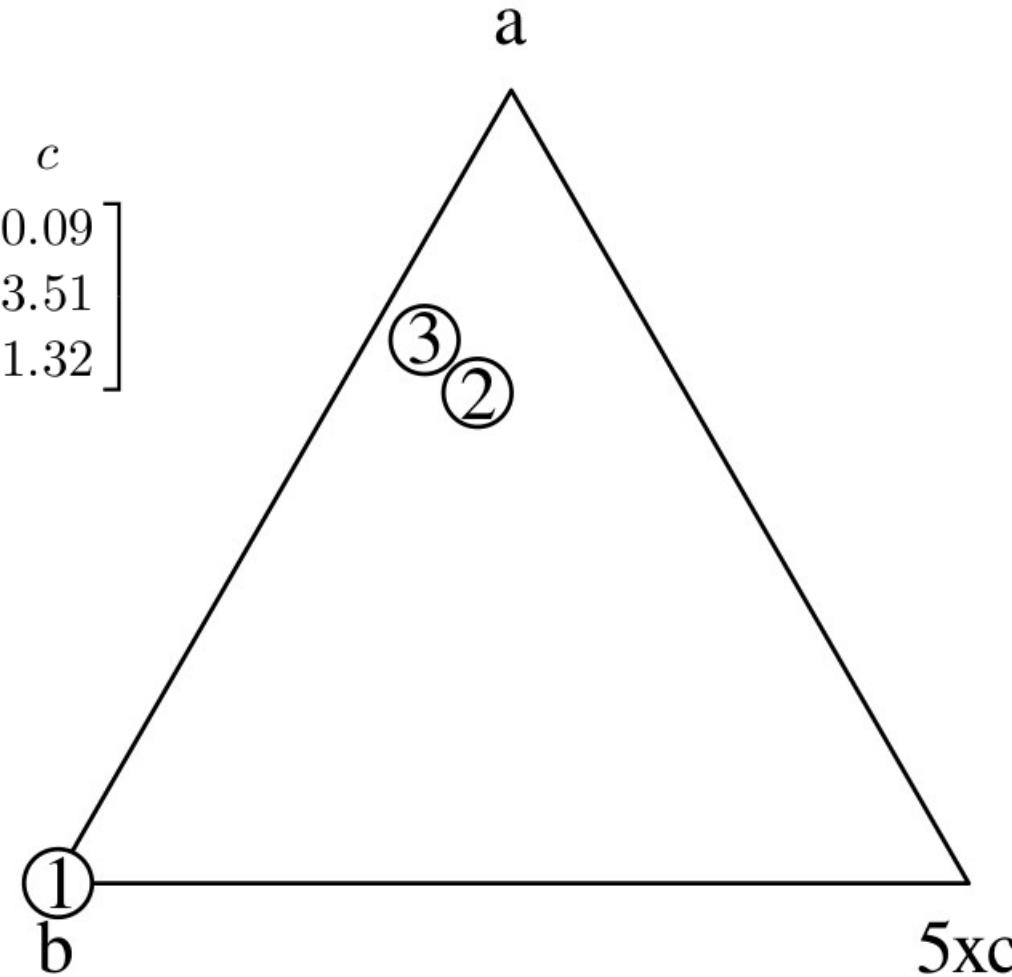
logratio transformation  
 ← →  
 inverse logratio transformation



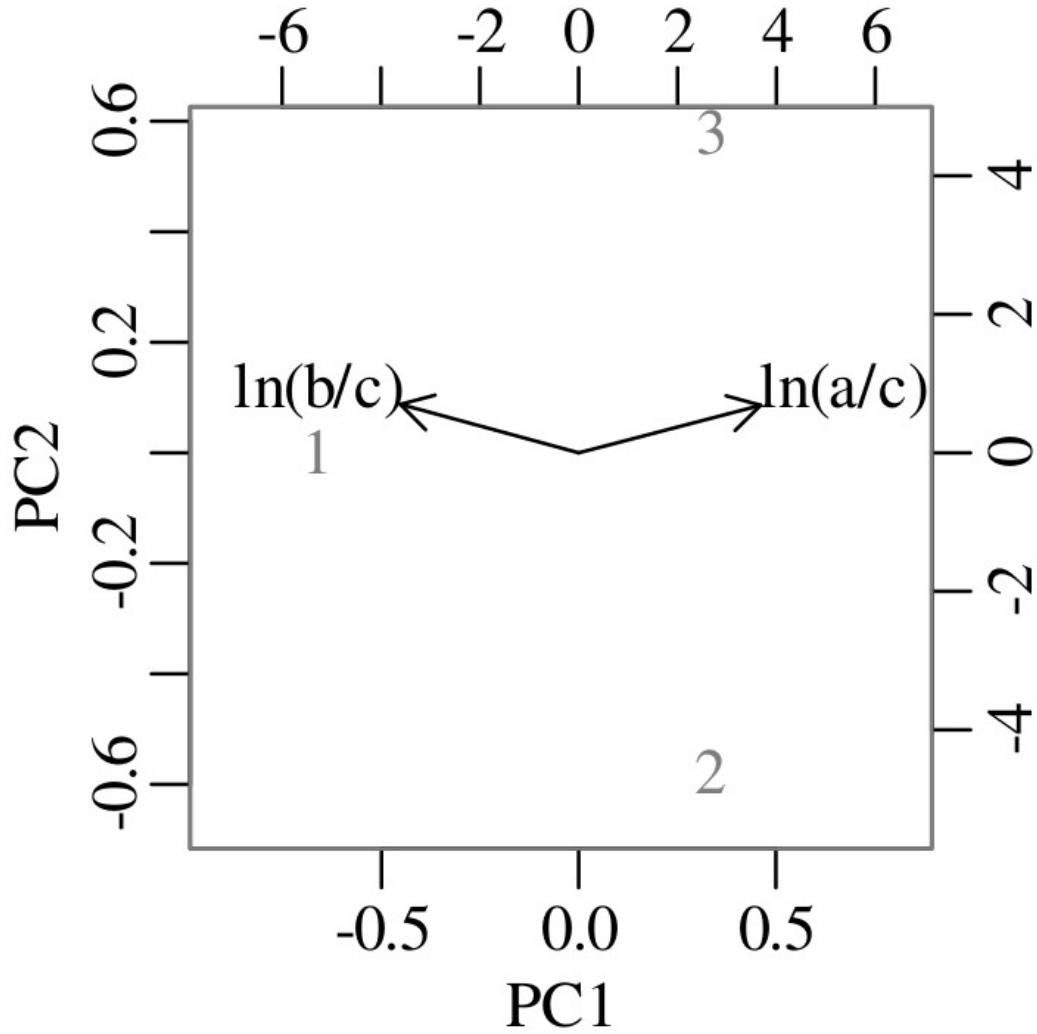
$$x = \frac{\exp[v]}{\exp[v] + \exp[w] + 1}, y = \frac{\exp[w]}{\exp[v] + \exp[w] + 1}, z = \frac{1}{\exp[v] + \exp[w] + 1}$$

# PCA of compositional data

$$X = \begin{bmatrix} 1 & 0.03 & 99.88 & 0.09 \\ 2 & 70.54 & 25.95 & 3.51 \\ 3 & 72.14 & 26.54 & 1.32 \end{bmatrix}$$



$$X_a = \begin{bmatrix} \ln(a/c) & \ln(b/c) \\ 1 & -1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$



## centred logratio transformation

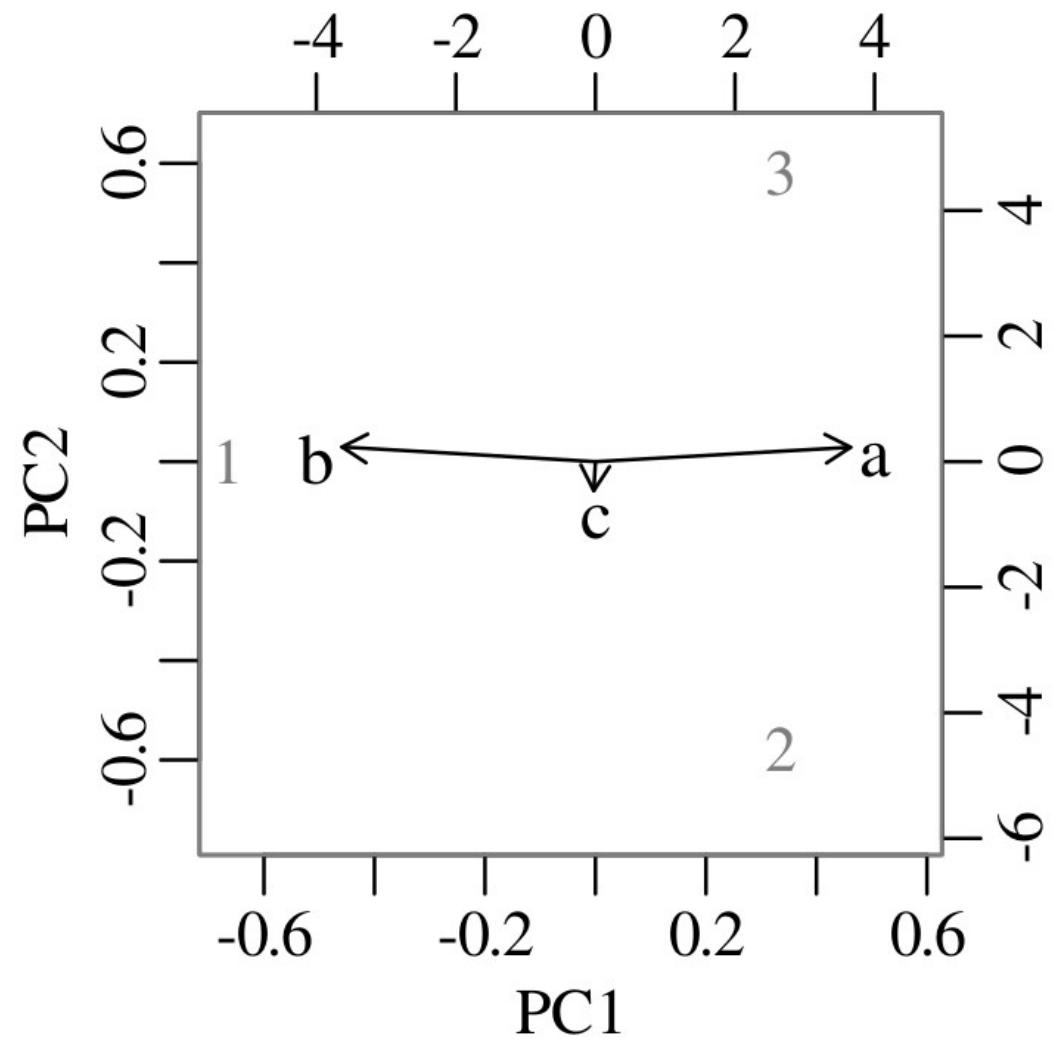
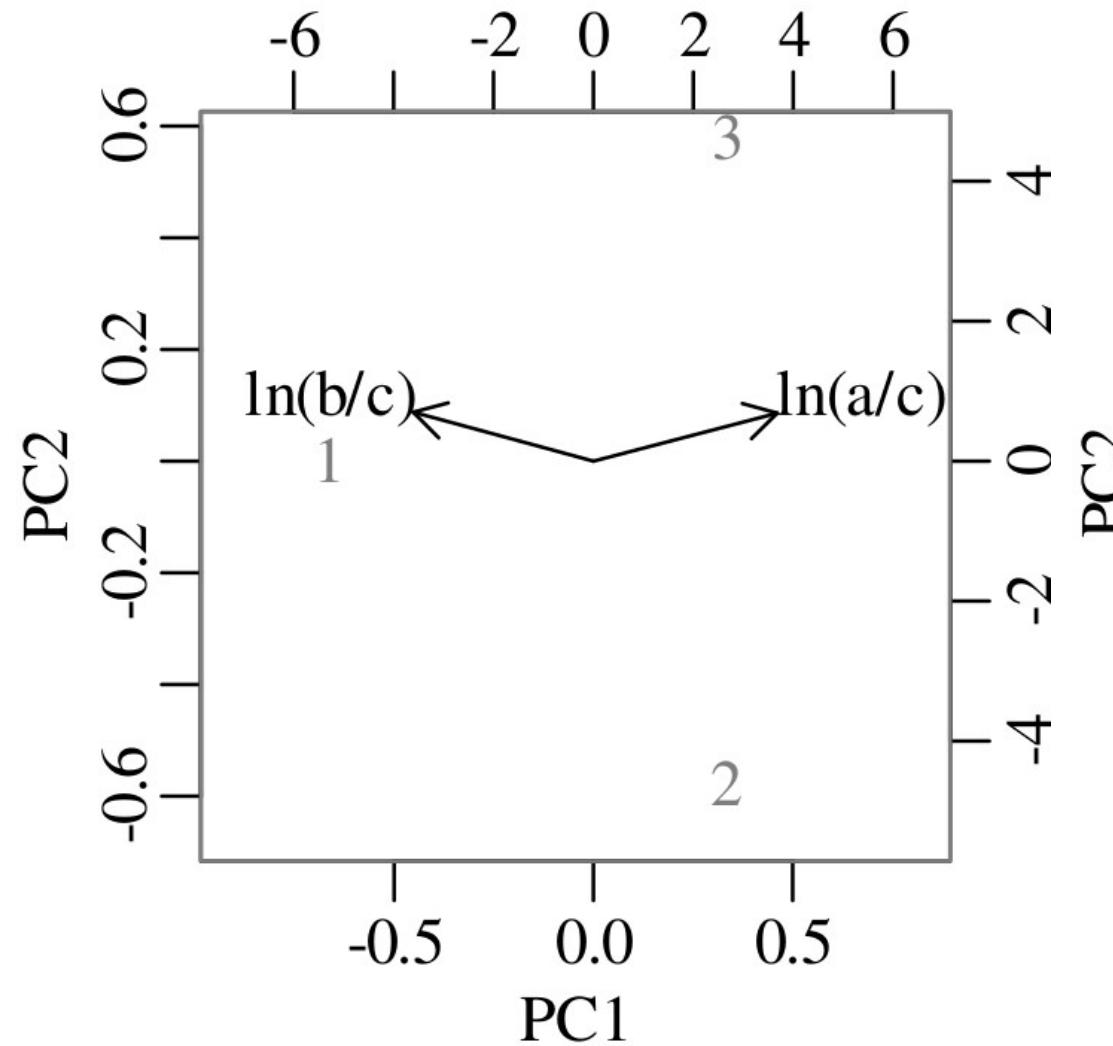
$$u_i = \ln \left[ \frac{x_i}{g_i} \right], v_i = \ln \left[ \frac{y_i}{g_i} \right], \text{ and } w_i = \ln \left[ \frac{z_i}{g_i} \right]$$

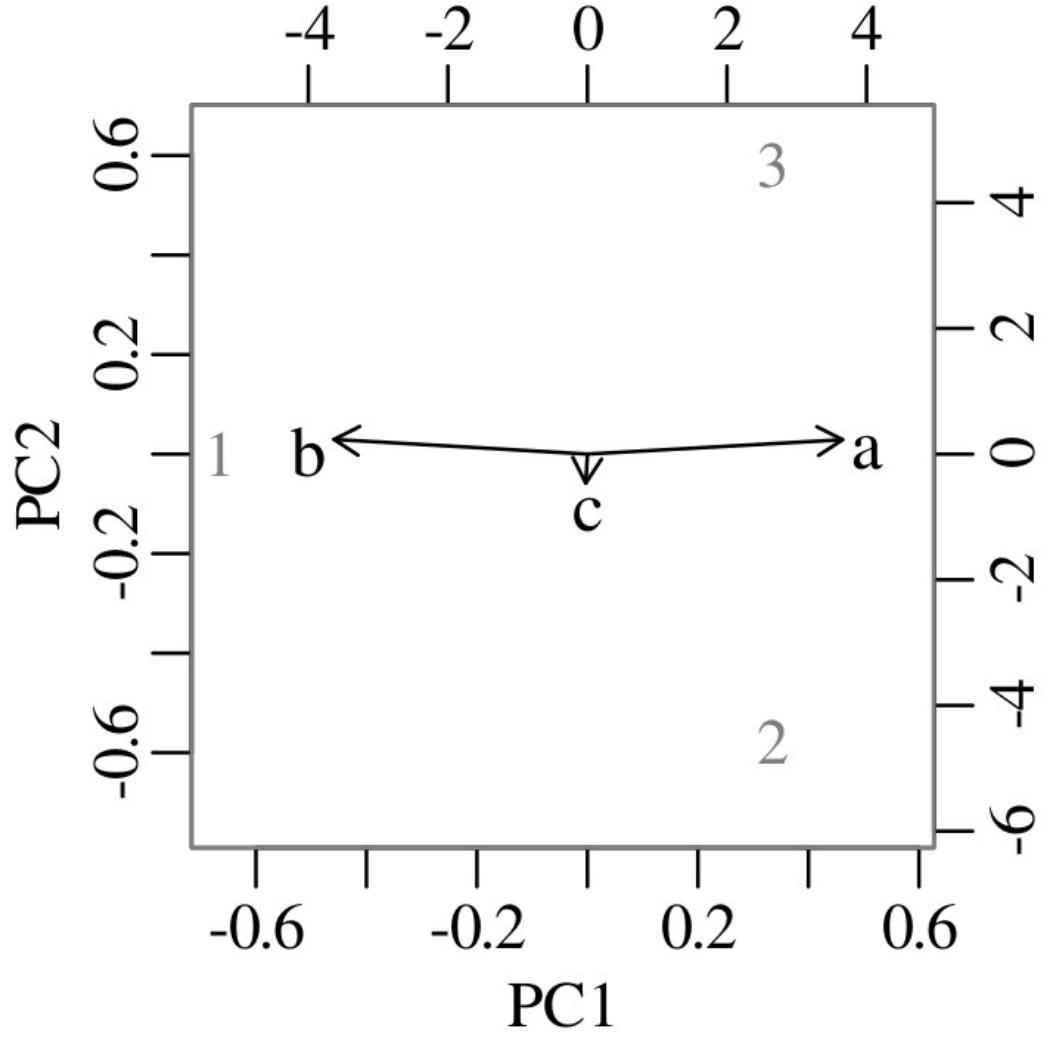
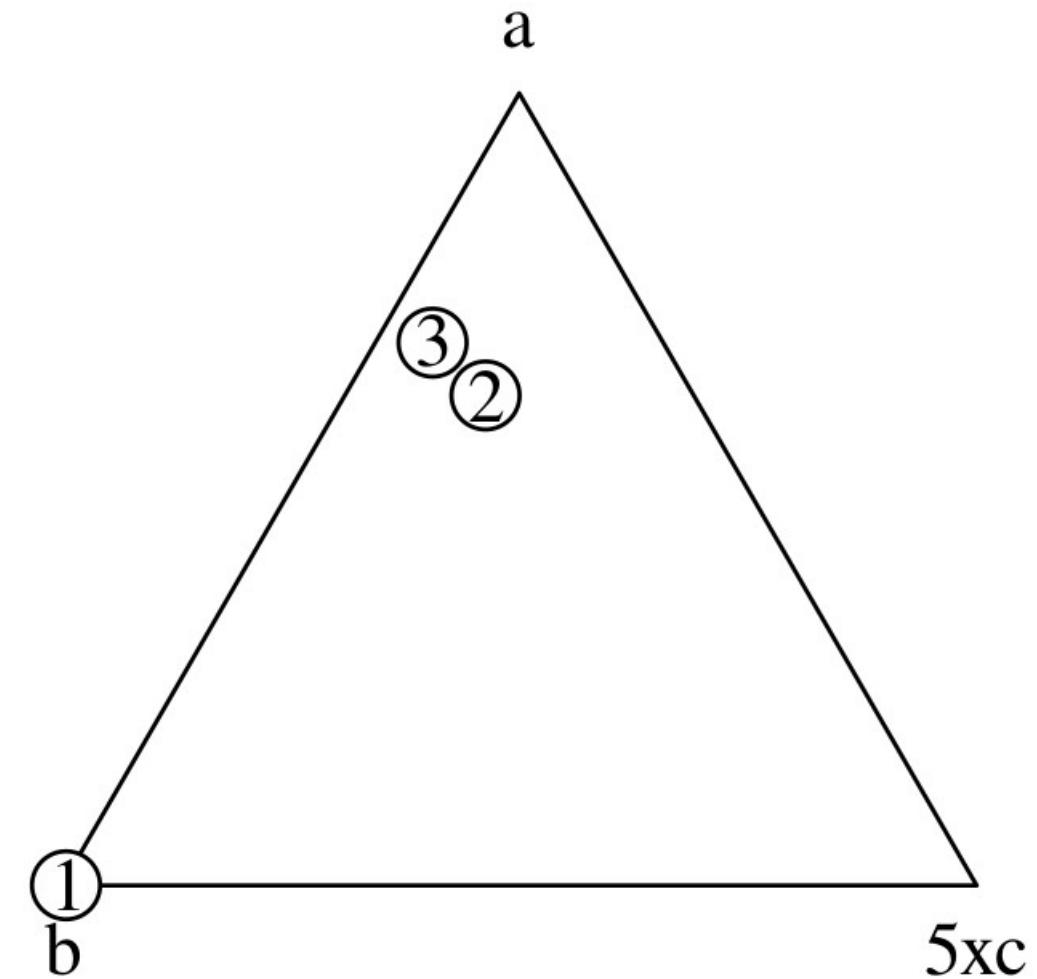
$$\text{where } g_i = \exp \left[ \frac{\ln[x_i] + \ln[y_i] + \ln[z_i]}{3} \right]$$

$$X_c = \begin{matrix} & \ln(a/g) & \ln(b/g) & \ln(c/g) \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} -3 & 5 & -2 \\ 1.33 & 0.33 & -1.67 \\ 1.67 & 0.67 & -2.33 \end{bmatrix} \end{matrix}$$

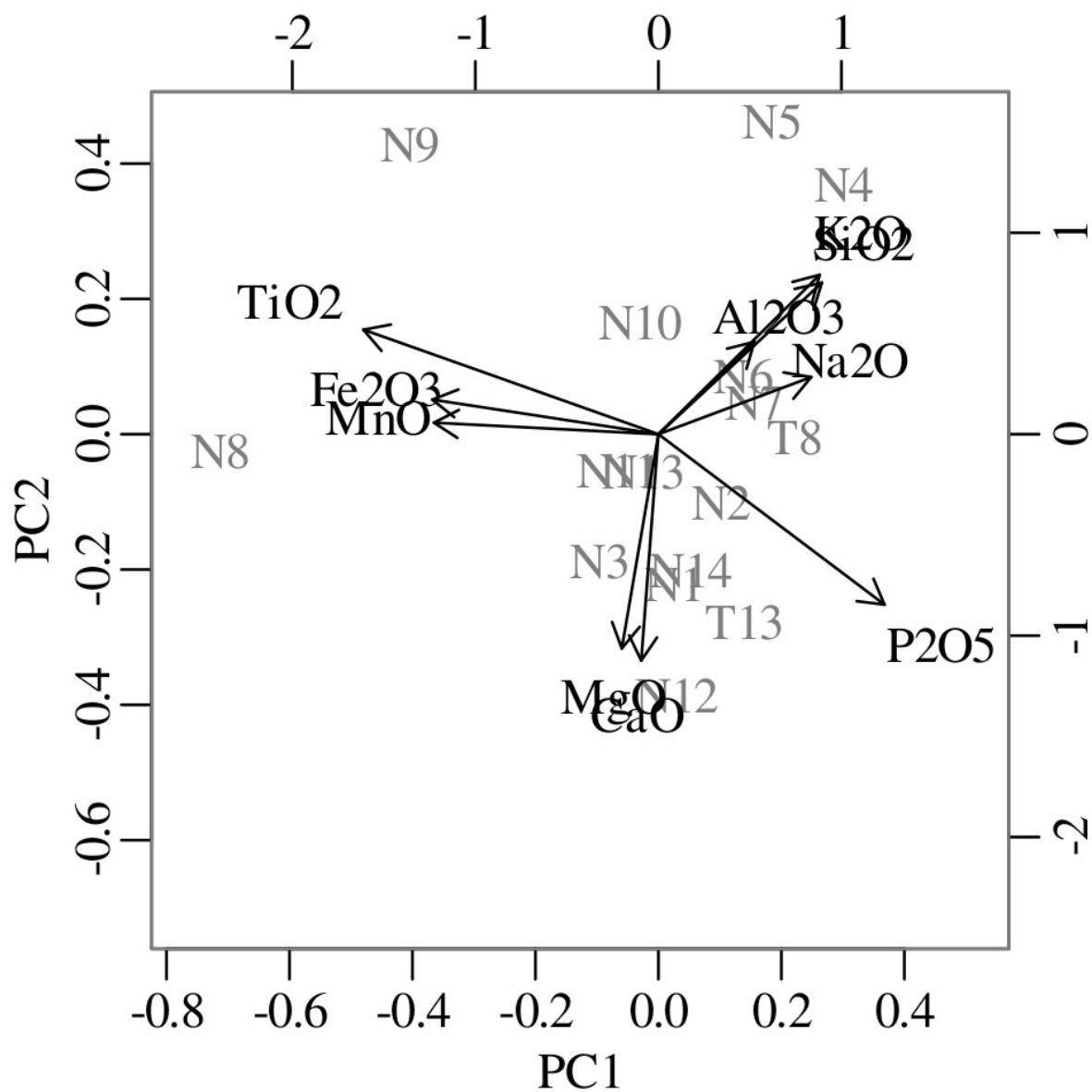
$$X_c = \begin{matrix} & \ln(a/g) & \ln(b/g) & \ln(c/g) \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} -3 & 5 & -2 \\ 1.33 & 0.33 & -1.67 \\ 1.67 & 0.67 & -2.33 \end{bmatrix} \end{matrix}$$

$$X_c = 1_{3,1} \ C + S \ V \ D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -2 \end{bmatrix} + \begin{bmatrix} -1.15 & 0 & 0.82 \\ 0.58 & -1 & 0.82 \\ 0.58 & 1 & 0.82 \end{bmatrix} \begin{bmatrix} 3.67 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 & 0 \\ 0.41 & 0.41 & -0.82 \\ 0.58 & 0.58 & 0.58 \end{bmatrix}$$

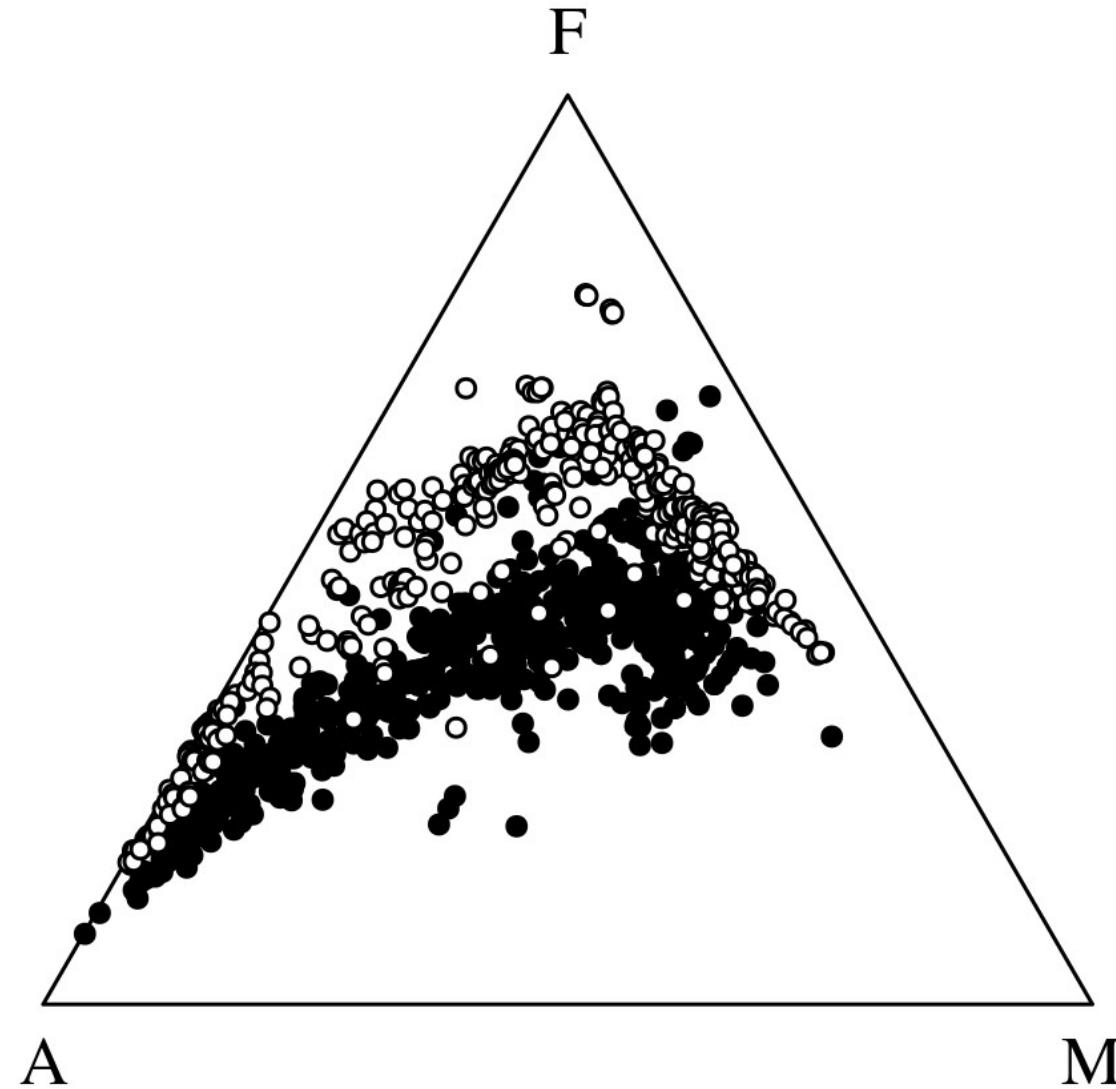


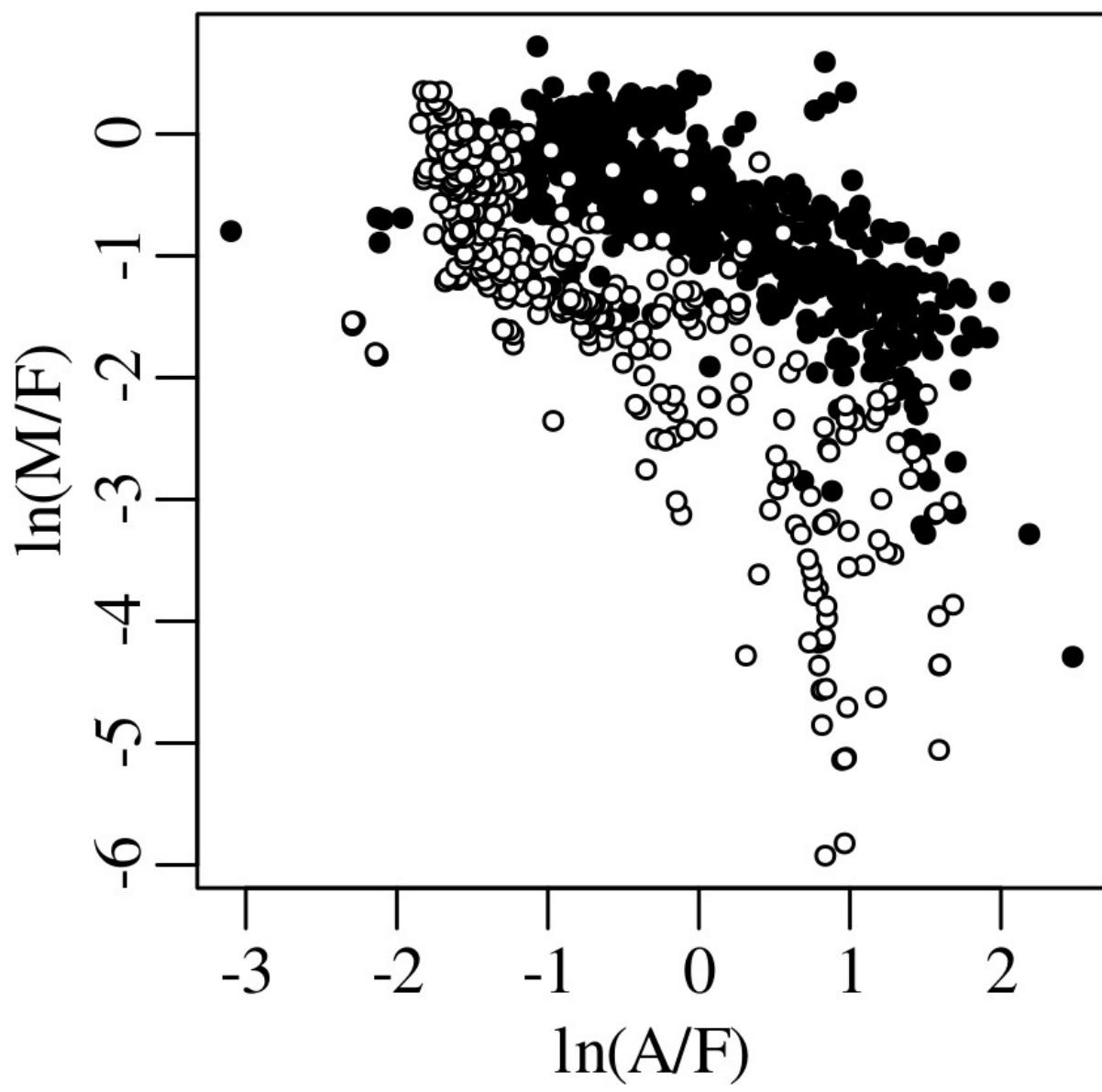


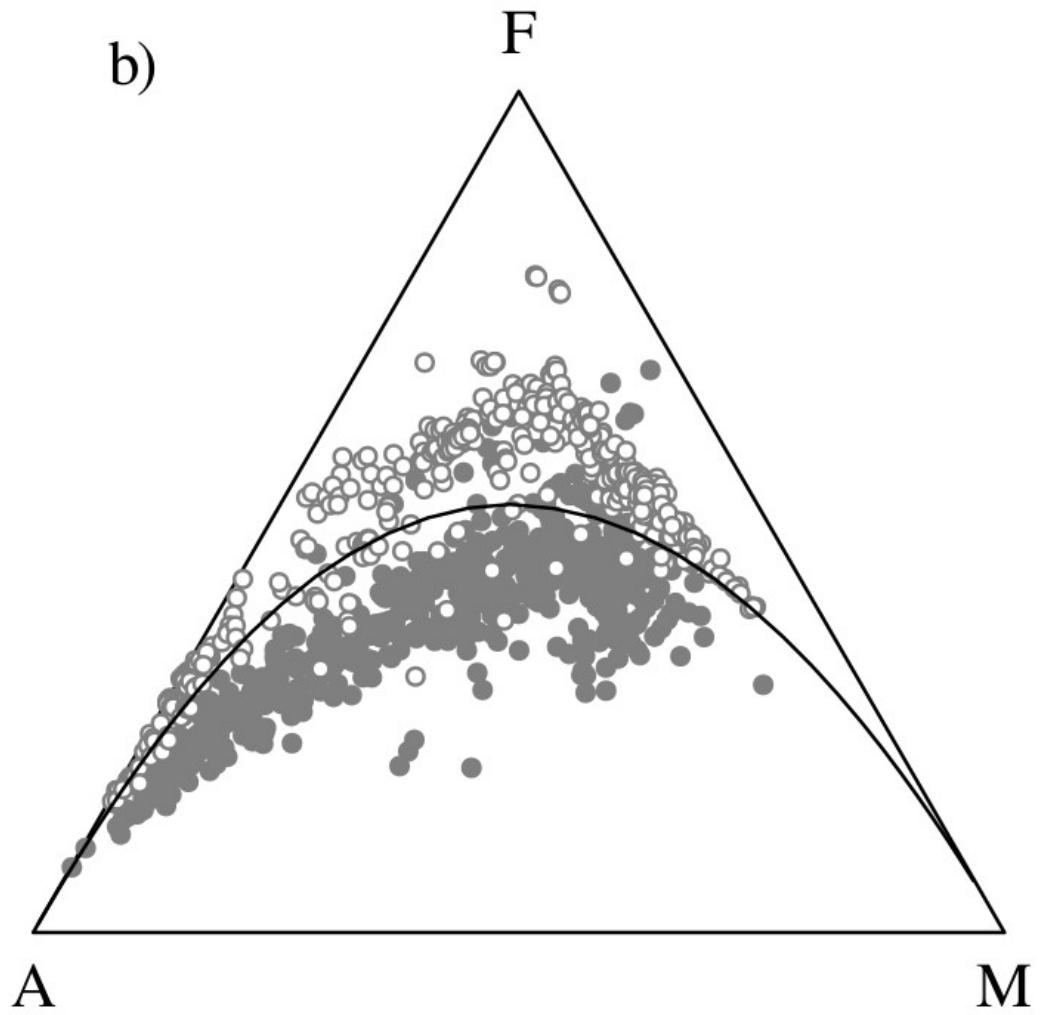
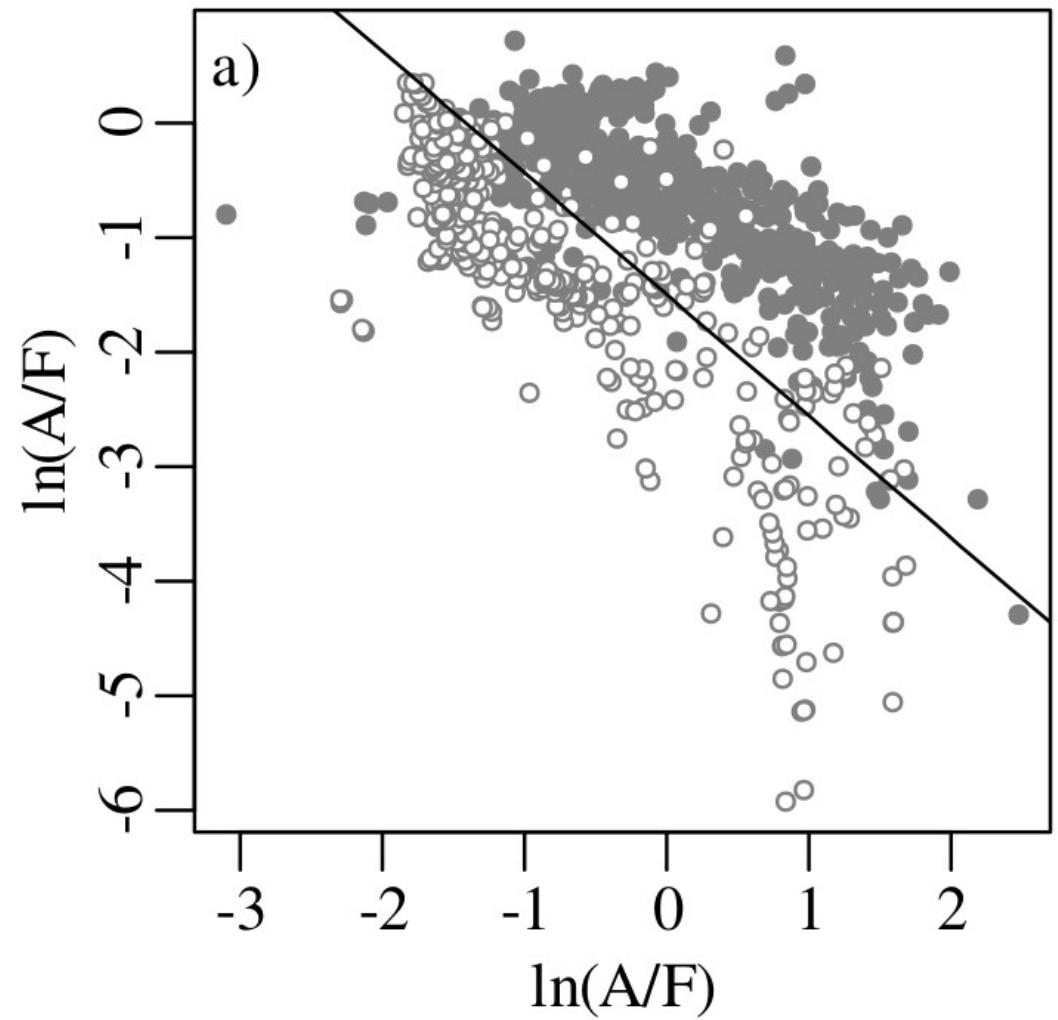
	SiO <sub>2</sub>	Al <sub>2</sub> O <sub>3</sub>	Fe <sub>2</sub> O <sub>3</sub>	MgO	CaO	Na <sub>2</sub> O	K <sub>2</sub> O	TiO <sub>2</sub>	P <sub>2</sub> O <sub>5</sub>	MnO
N1	82.54	6.14	3.18	1.65	2.24	1.16	1.35	0.44	0.11	0.06
N2	83.60	6.42	2.55	1.25	1.83	1.21	1.48	0.36	0.08	0.04
:	:	:	:	:	:	:	:	:	:	:
N13	79.96	6.41	3.19	1.31	2.10	1.09	1.62	0.51	0.06	0.05
N14	73.62	9.96	4.07	2.01	3.45	1.86	2.29	0.44	0.10	0.07
T8	85.70	5.89	1.82	0.81	1.44	1.12	1.67	0.23	0.07	0.03
T13	82.54	6.02	2.30	1.42	3.07	1.19	1.46	0.34	0.11	0.04



# LDA of compositional data



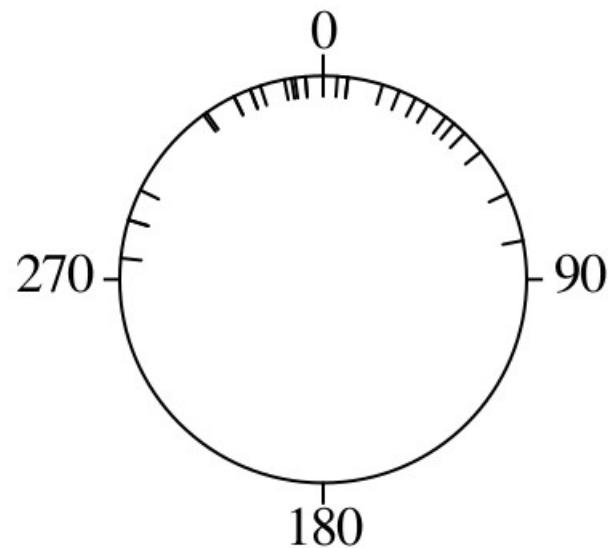




# Statistics for geoscientists

Directional data

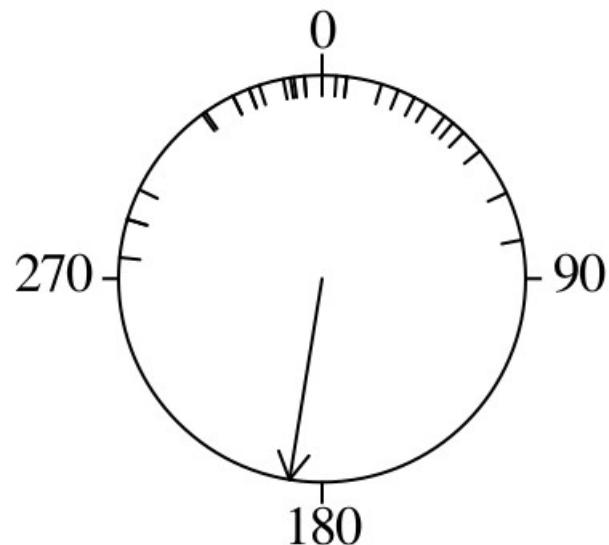
44	51	79	65	27	31	4	355	22	352	287	7	287	339	0
276	342	355	334	296	7	17	351	349	37	339	40	324	325	334



$$\begin{array}{r} 44 + 51 + 79 + 65 + 27 + 31 + 4 + 355 + 22 + 352 + 287 + 7 + 287 + 339 + 0 \\ 276 + 342 + 355 + 334 + 296 + 7 + 17 + 351 + 349 + 37 + 339 + 40 + 324 + 325 + 334 \end{array}$$

30

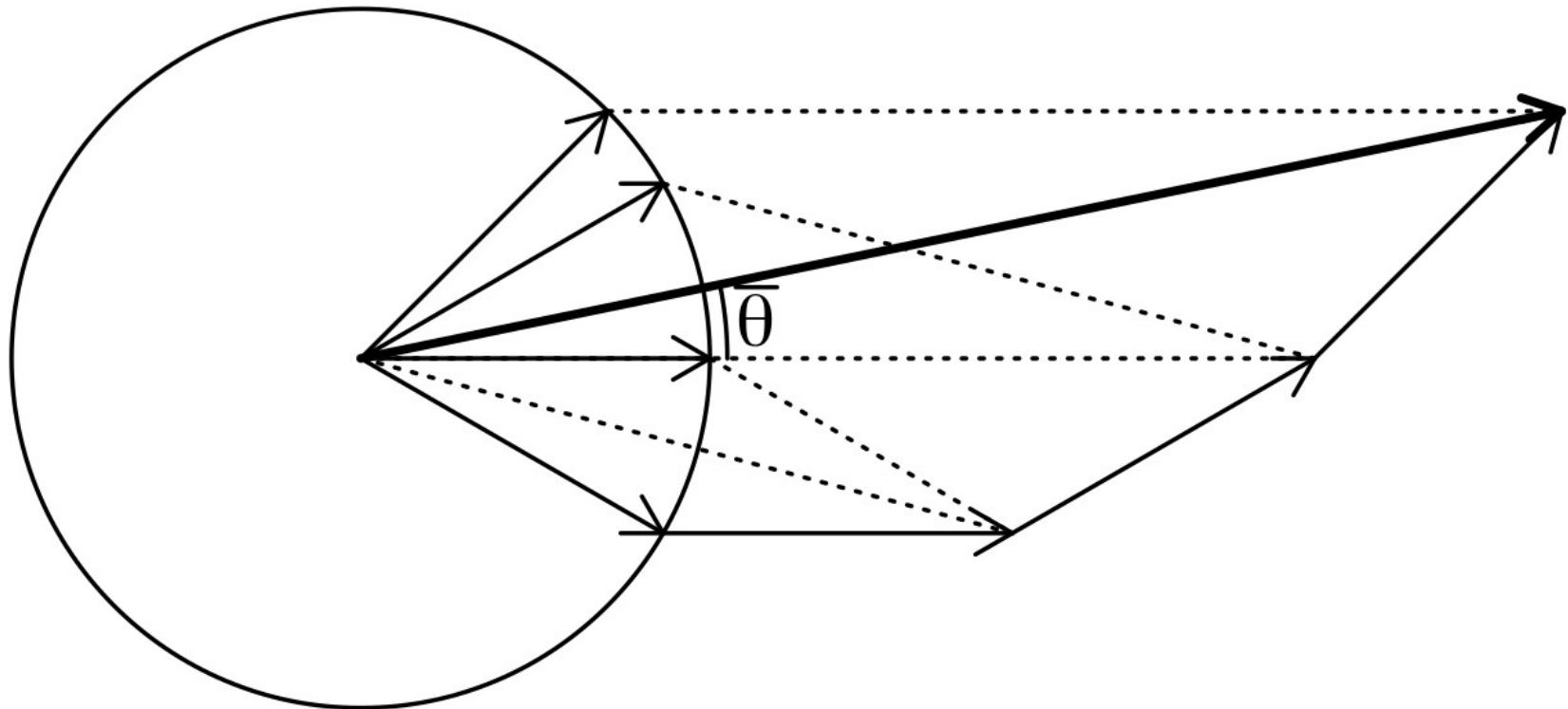
$$= 189.2^\circ$$



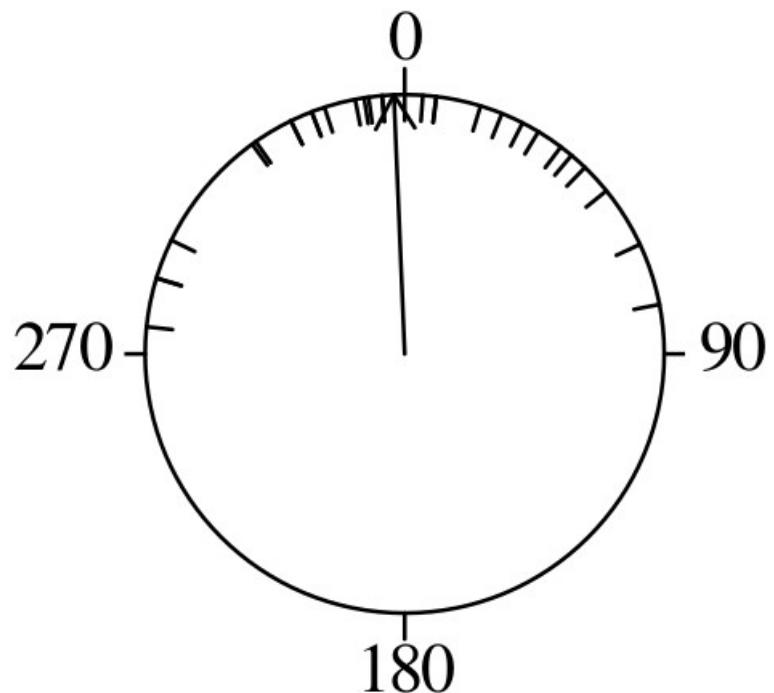
$$\sin[\theta \pm \phi] = \sin[\theta] \cos[\phi] \pm \cos[\theta] \sin[\phi]$$

$$\theta - \phi = \arcsin (\sin[\theta] \cos[\phi] - \cos[\theta] \sin[\phi])$$

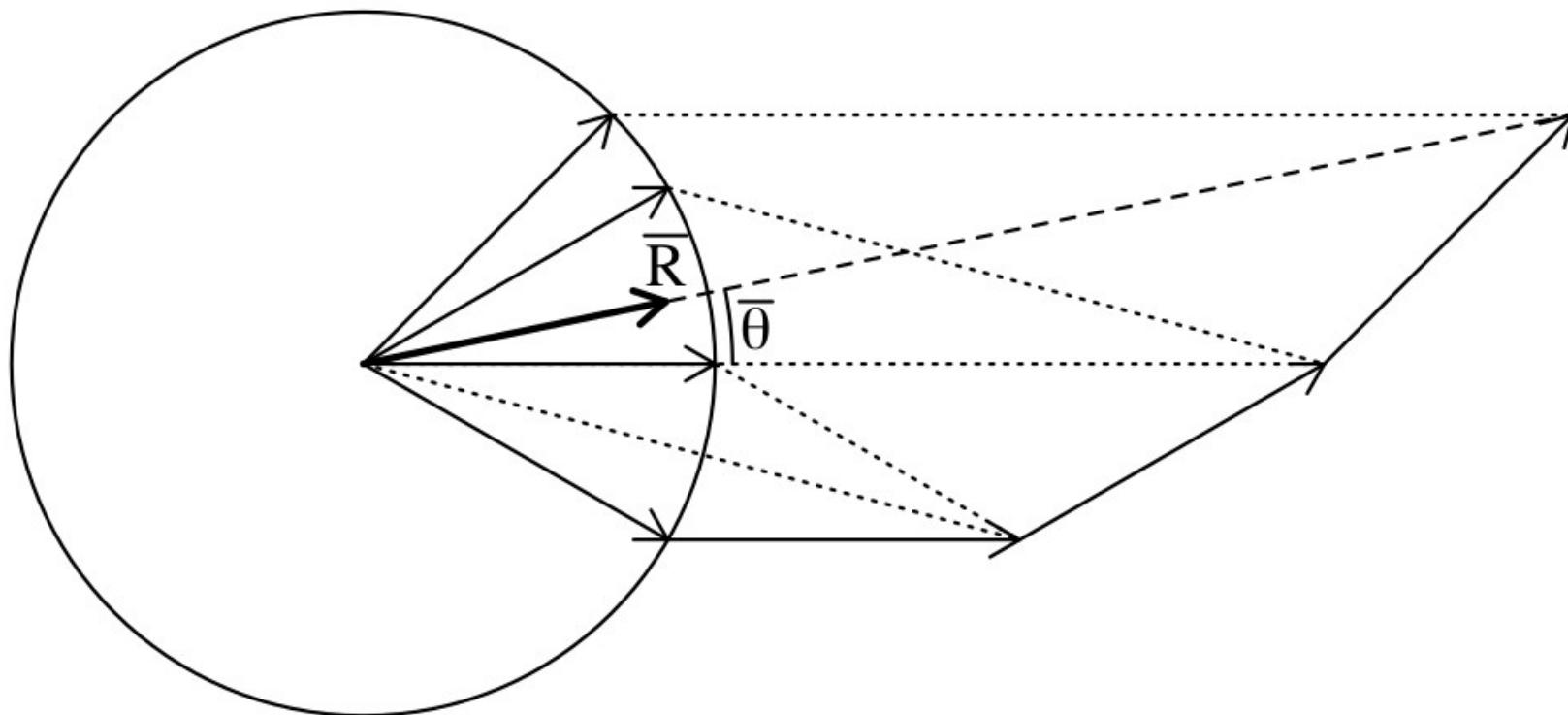
$$\bar{\theta} = \arctan \left( \frac{\sum_{i=1}^n \sin[\theta_i]}{\sum_{i=1}^n \cos[\theta_i]} \right)$$



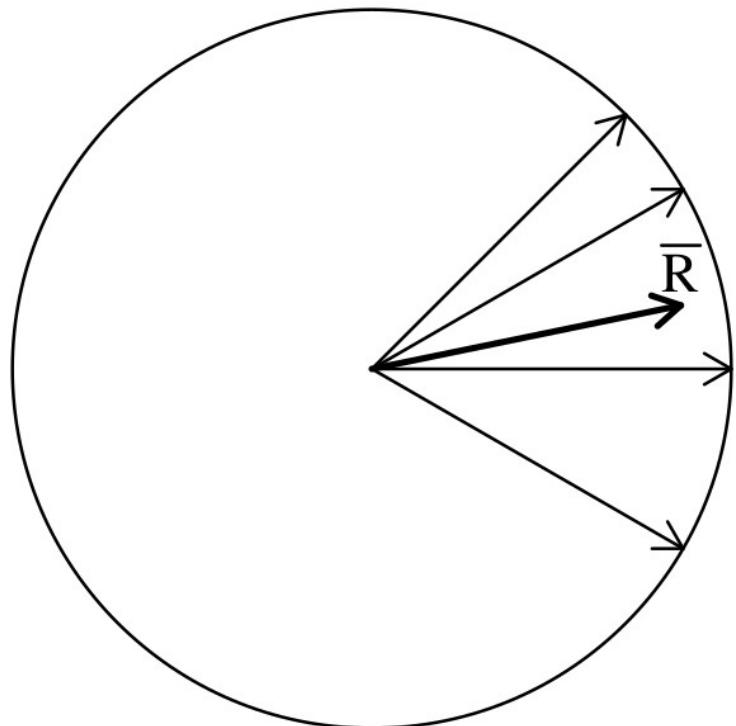
44	51	79	65	27	31	4	355	22	352	287	7	287	339	0
276	342	355	334	296	7	17	351	349	37	339	40	324	325	334



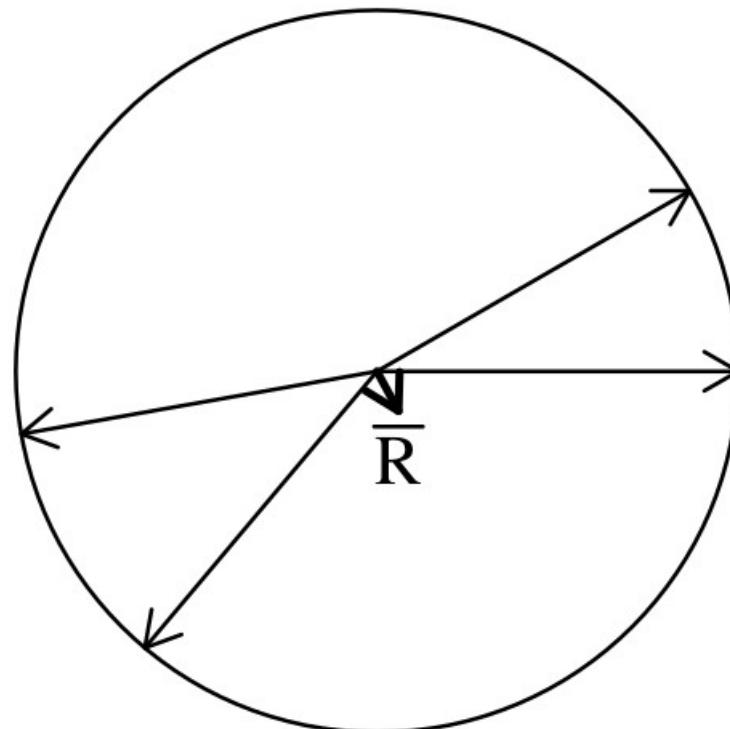
$$\bar{R} = \sqrt{\left( \sum_{i=1}^n \sin[\theta_i]/n \right)^2 + \left( \sum_{i=1}^n \cos[\theta_i]/n \right)^2}$$



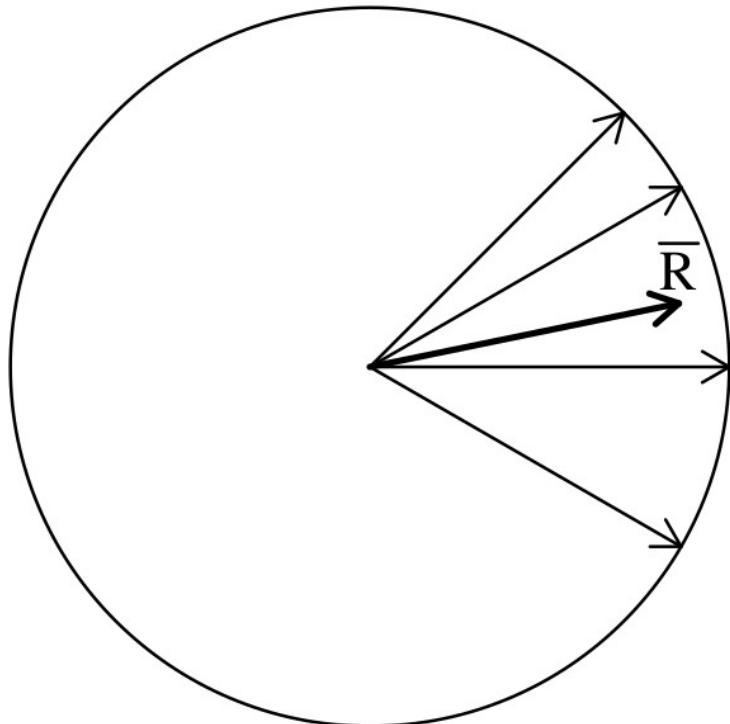
$$\bar{R} = 0.88$$



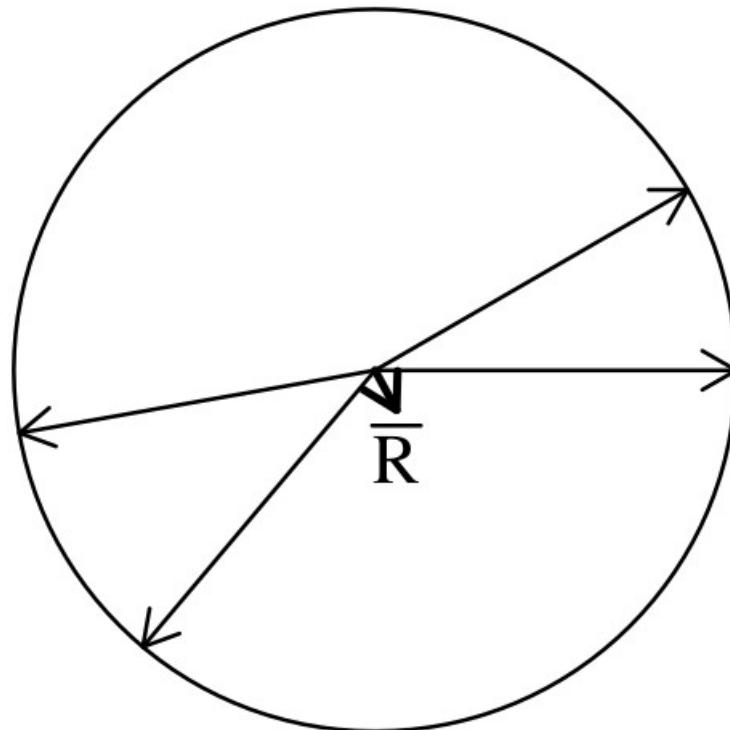
$$\bar{R} = 0.125$$



$$s_c = \sqrt{\ln(1/\bar{R}^2)}$$



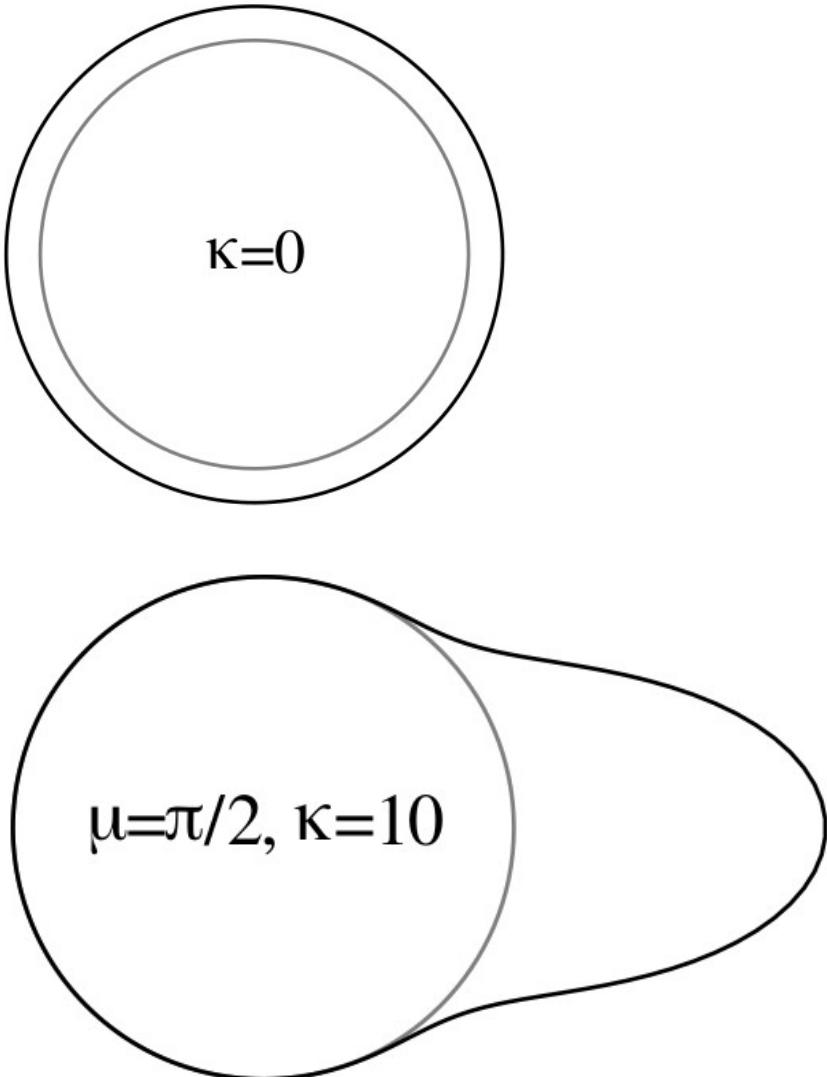
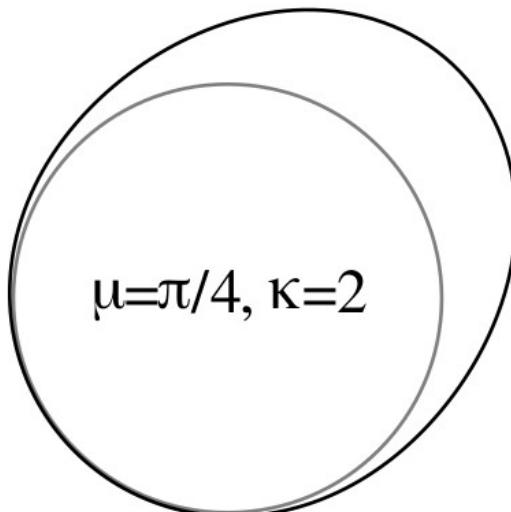
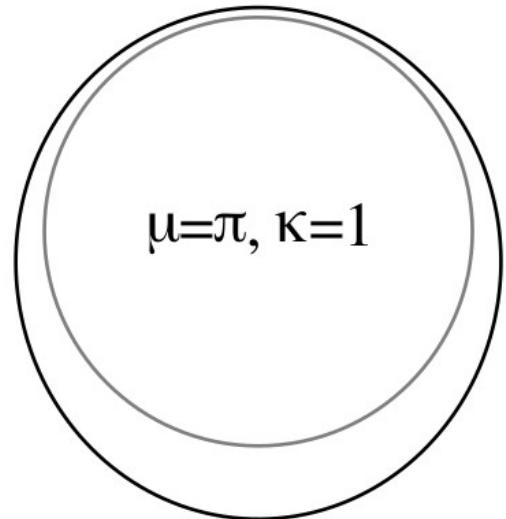
$$s_c = 0.51$$



$$s_c = 2.04$$

# The von Mises distribution

$$f(\theta|\mu, \kappa) \propto \exp[\kappa \cos(\theta - \mu)]$$

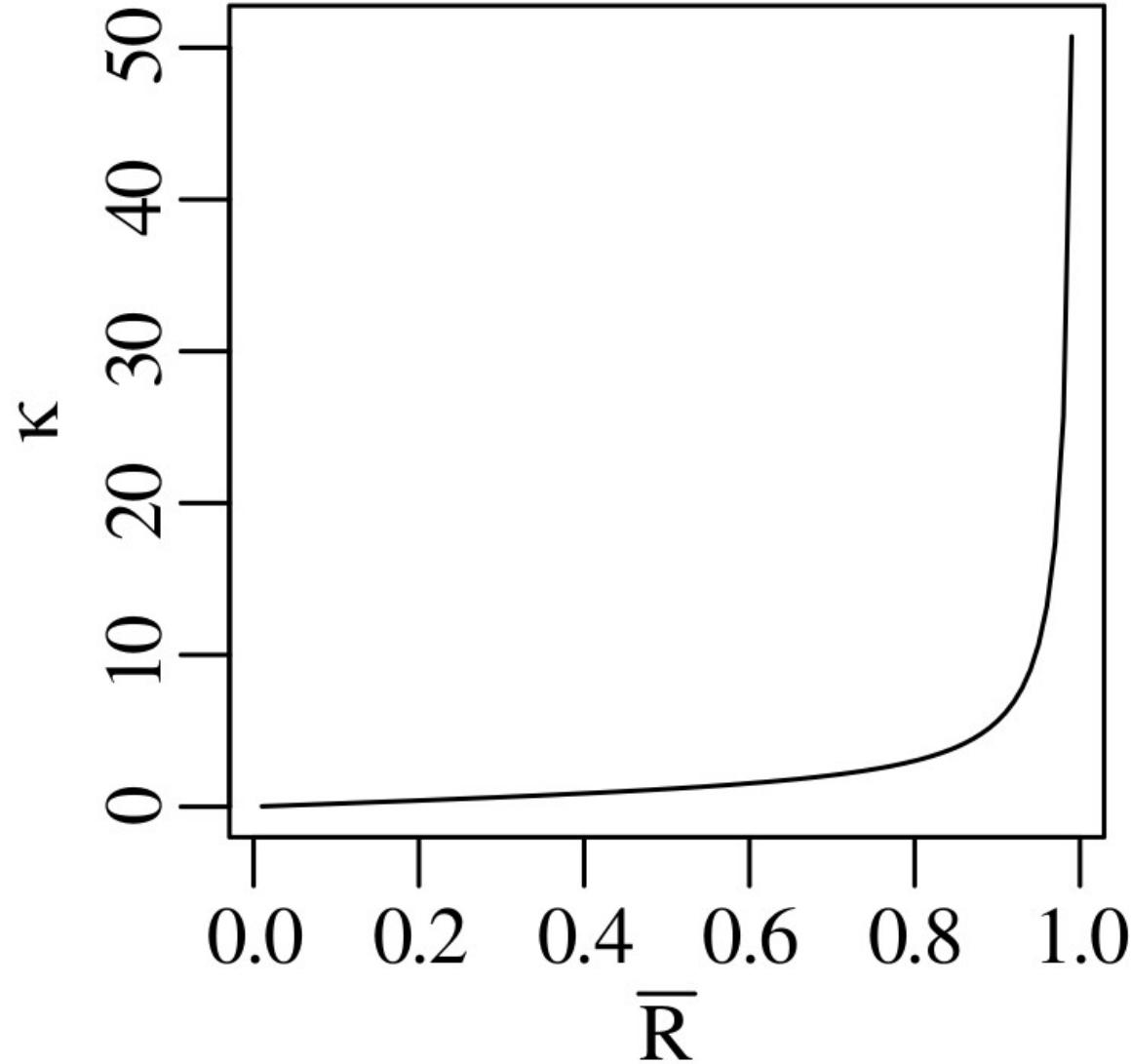


$\kappa=0$

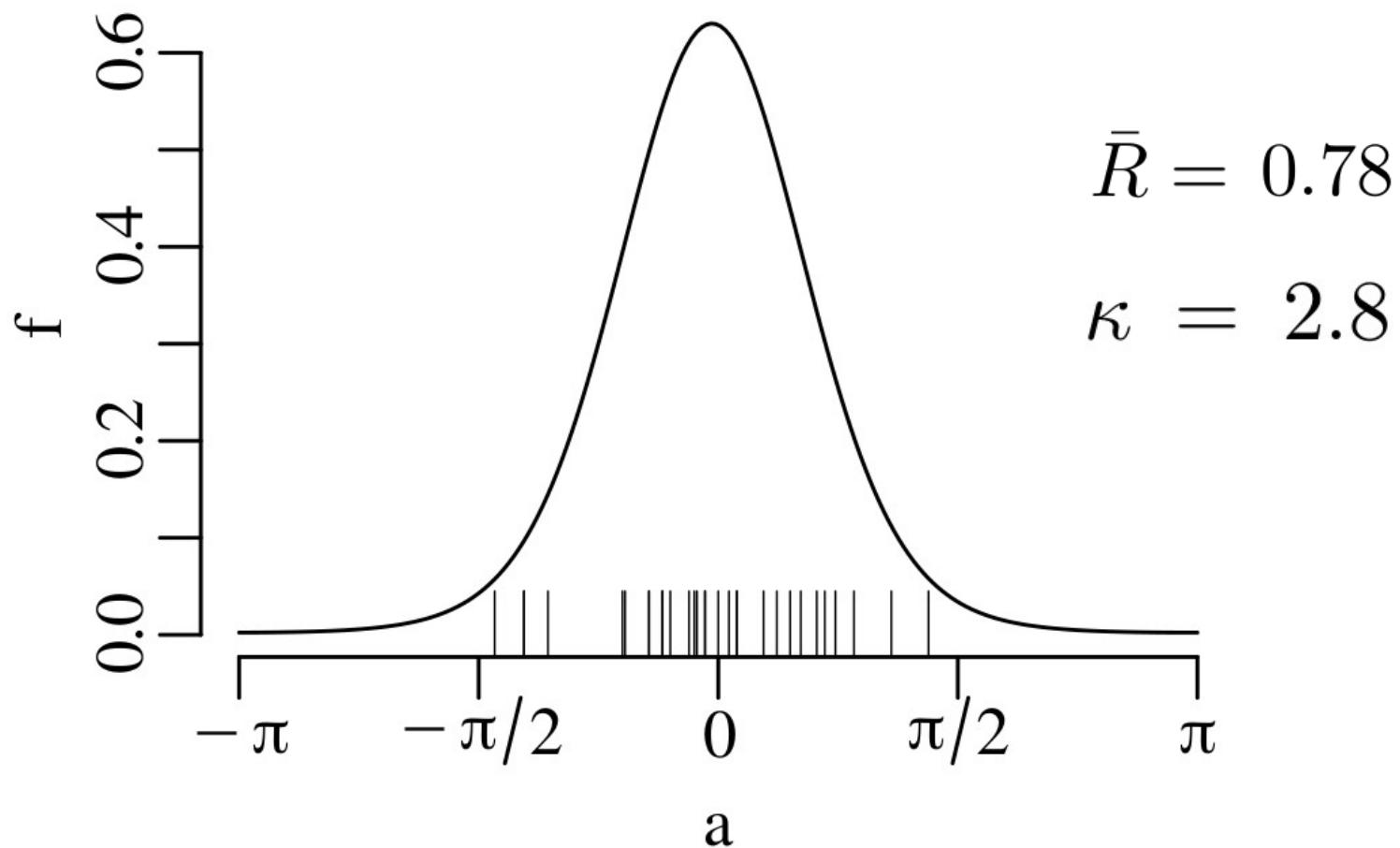
$\mu=\pi, \kappa=1$

$\mu=\pi/4, \kappa=2$

$\mu=\pi/2, \kappa=10$



44	51	79	65	27	31	4	355	22	352	287	7	287	339	0
276	342	355	334	296	7	17	351	349	37	339	40	324	325	334



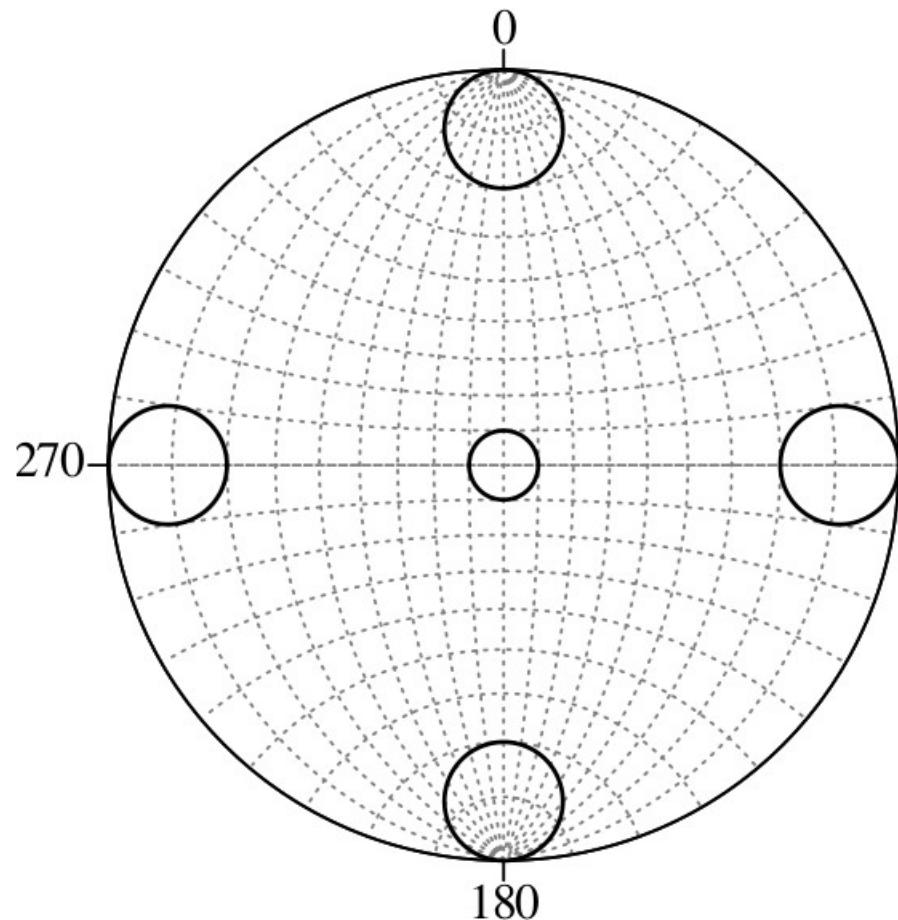
# Spherical data

$$\text{latitude } (l) \text{ and longitude } (L) \quad \left\{ \begin{array}{l} x = \cos[l] \sin[L] \\ y = \sin[l] \\ z = -\cos[l] \cos[L] \end{array} \right.$$

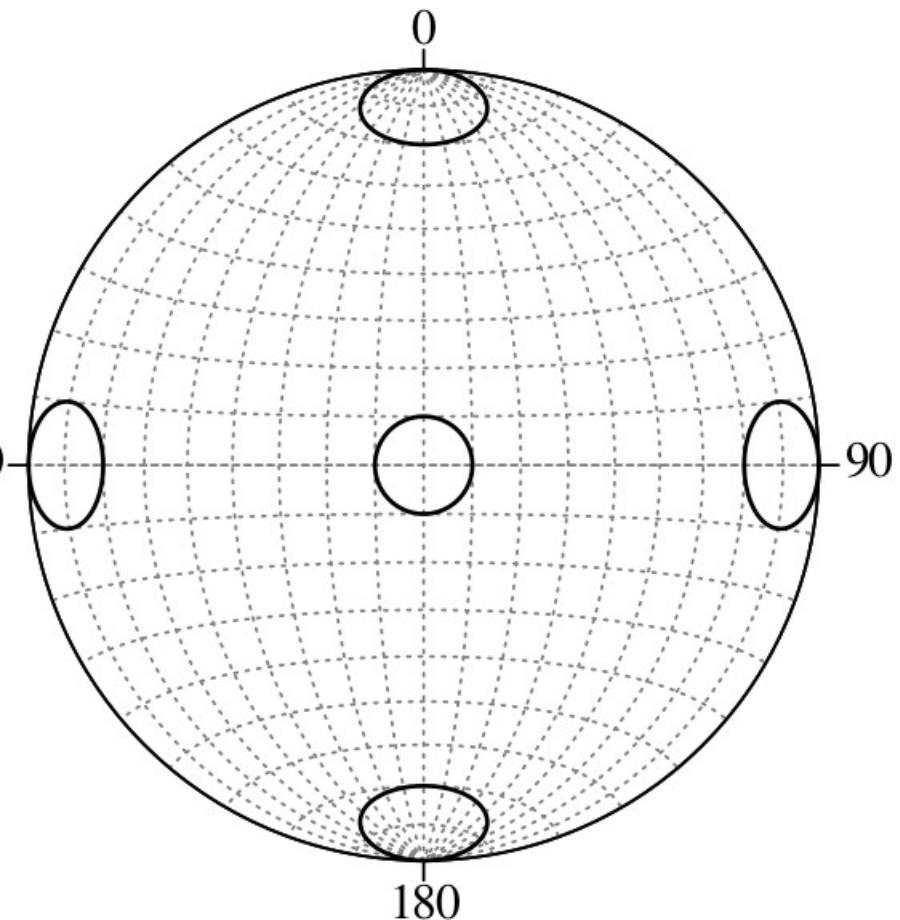
$$\text{strike } (S) \text{ and dip } (D) \quad \left\{ \begin{array}{l} x = -\cos[D] \sin[S] \\ y = \cos[D] \cos[S] \\ z = \sin[D] \end{array} \right.$$

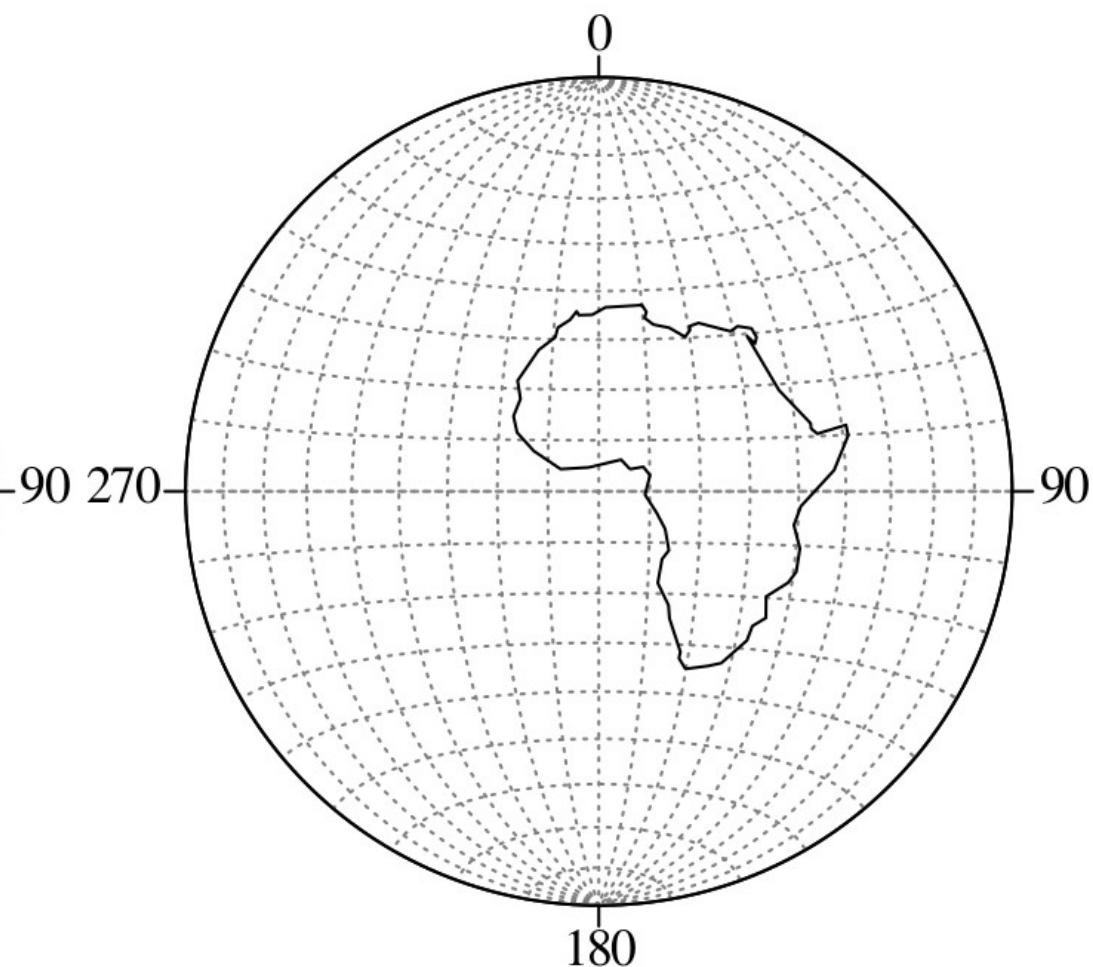
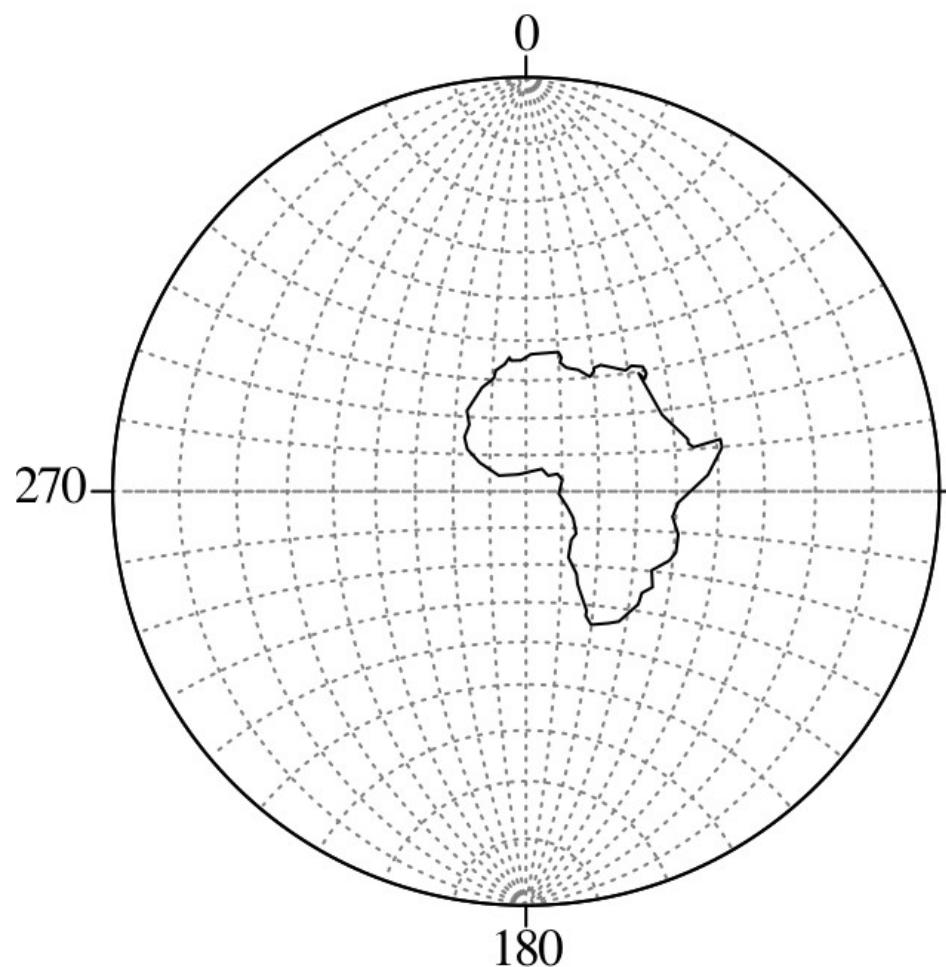
$$\text{dip } (D) \text{ and azimuth } (A) \quad \left\{ \begin{array}{l} x = \cos[D] \cos[A] \\ y = \cos[D] \sin[A] \\ z = \sin[D] \end{array} \right.$$

Wulff

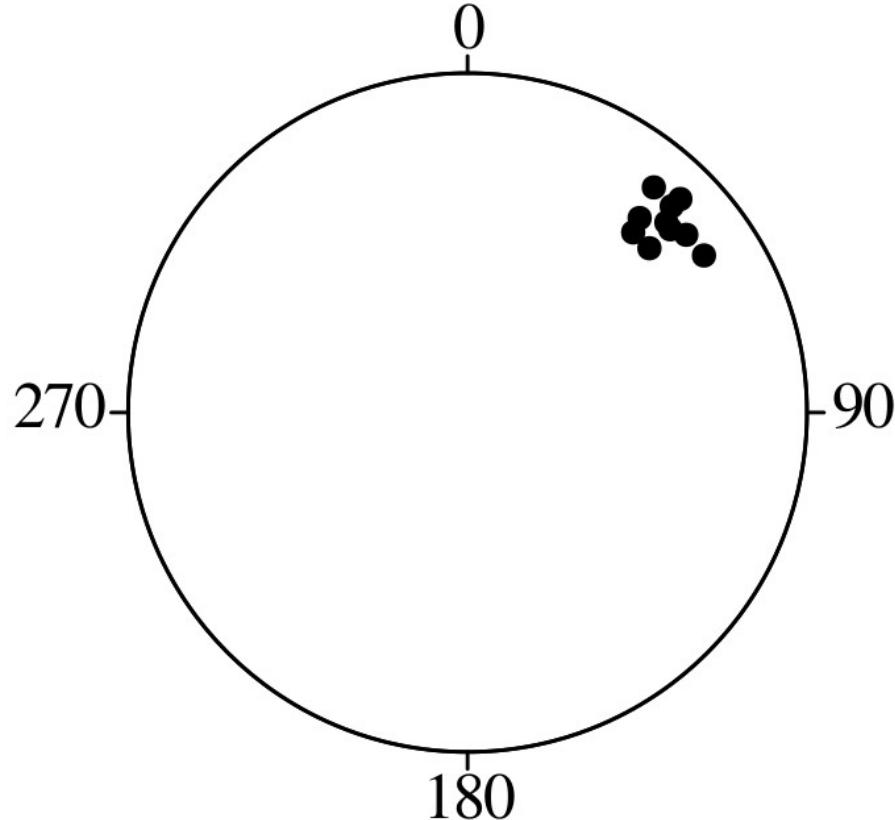
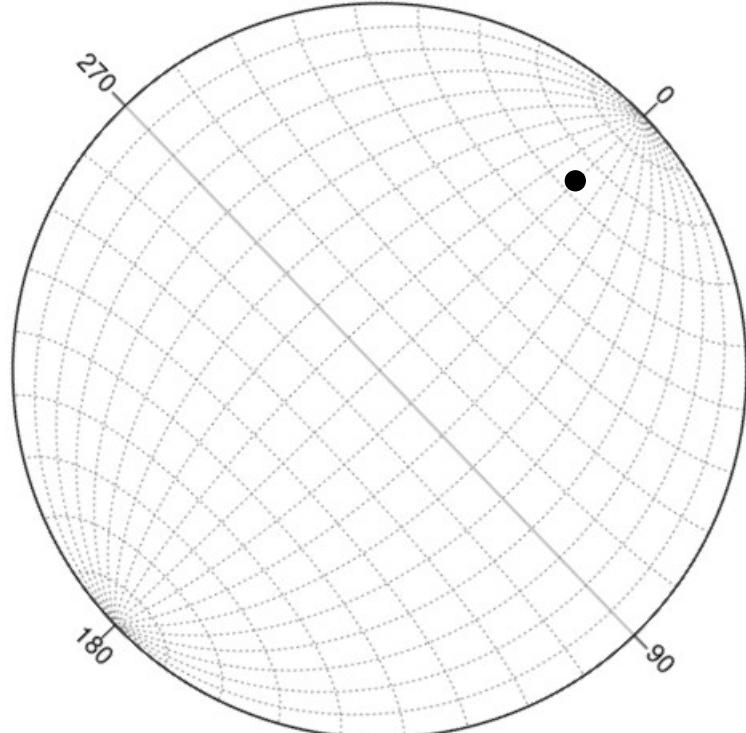


Schmidt

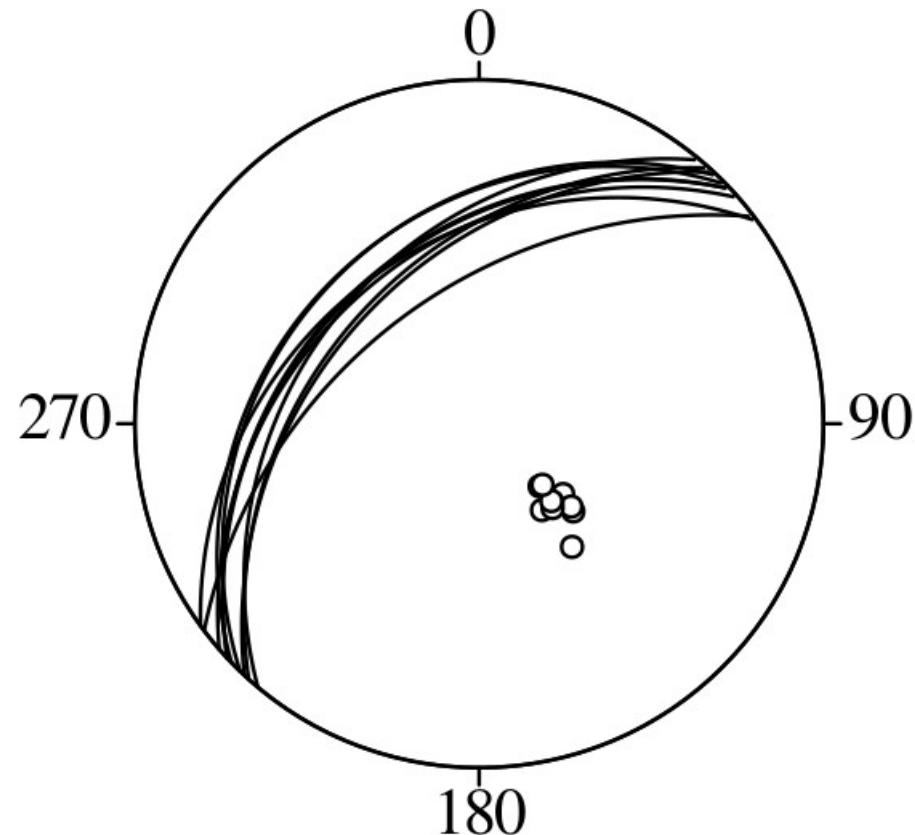
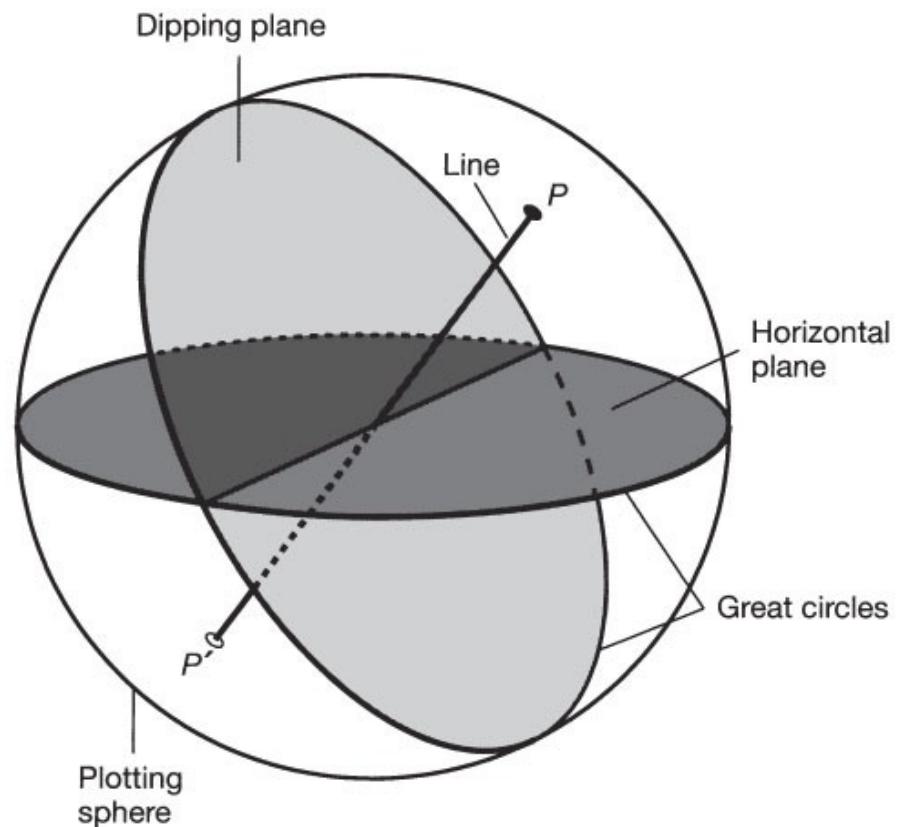




declination	47.9	46.3	44.7	50.9	56.4	42.6	44.9	41.5	47.9	39.6
inclination	28.6	20.1	15.6	18.1	17.5	28.7	12.2	24.5	20.6	15.0



strike	226	220	223	222	233	227	234	229	227	224
dip	28.4	35.3	41.0	39.6	48.3	34.7	34.5	36.0	34.2	28.7



$$\begin{cases} x = \cos[l] \sin[L] \\ y = \sin[l] \\ z = -\cos[l] \cos[L] \end{cases}$$

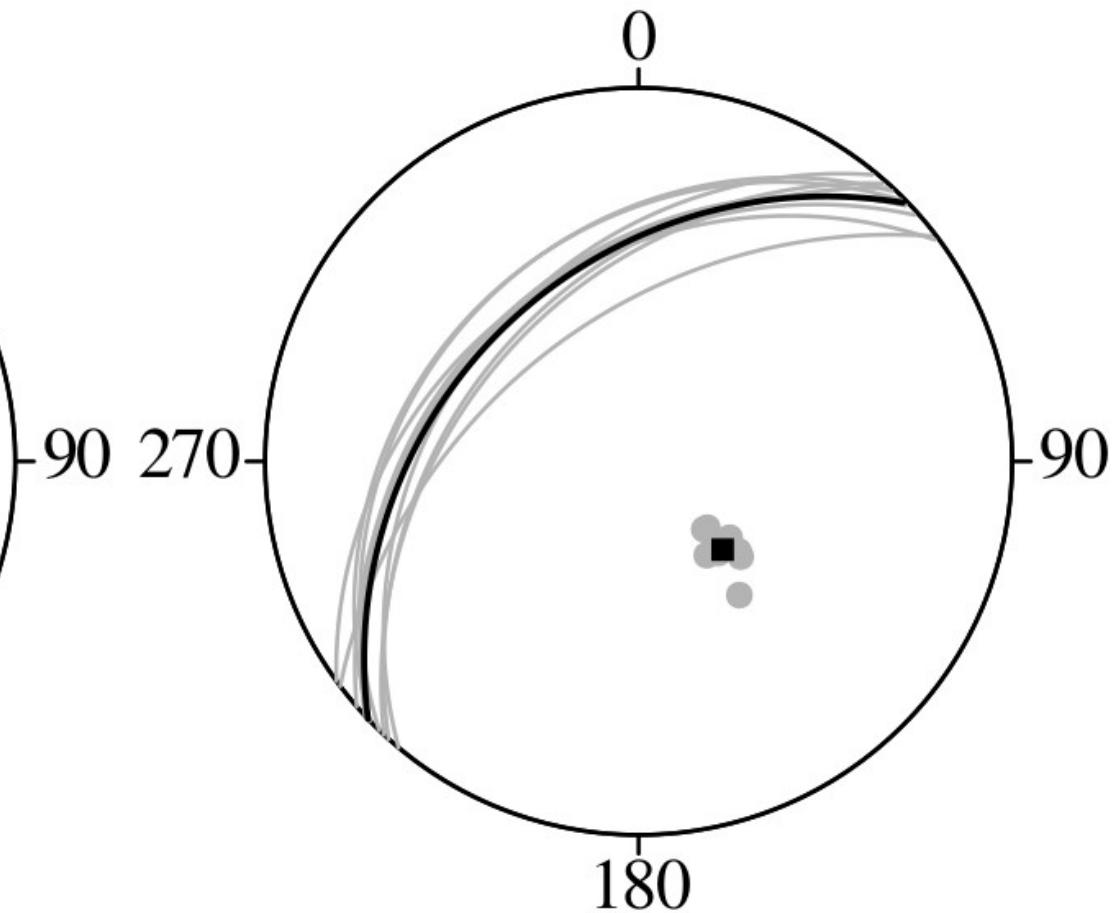
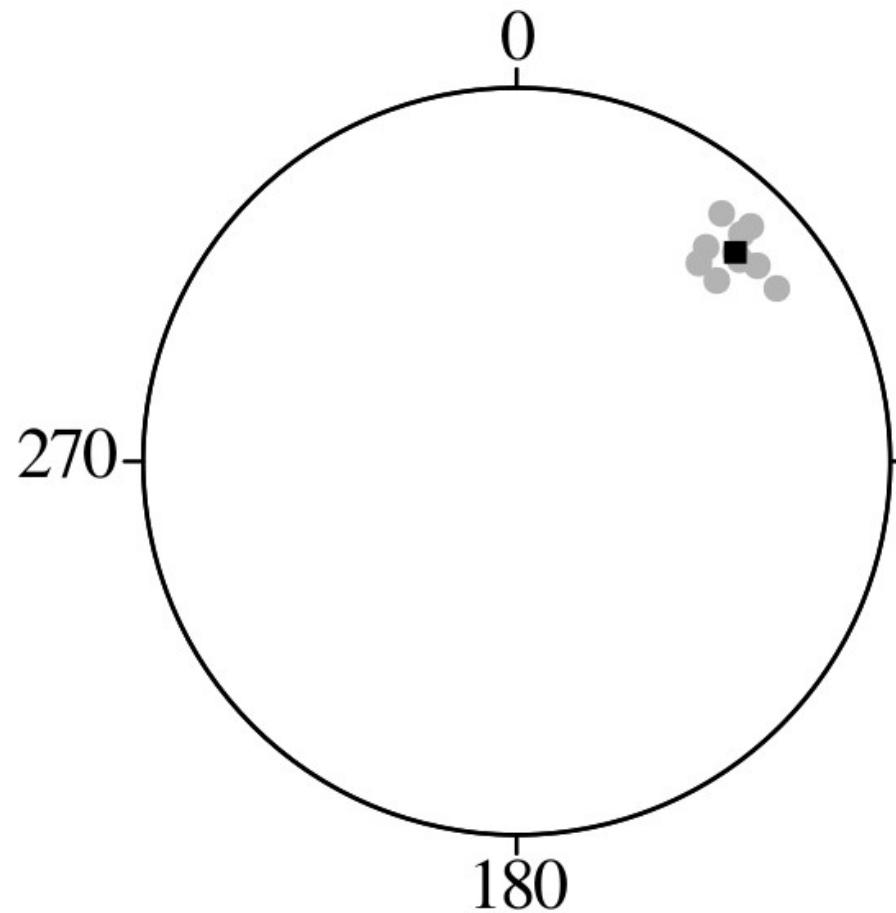
$$\begin{cases} \bar{L} = \arctan(-\bar{x}/\bar{z}) \\ \bar{l} = \arcsin(\bar{y}) \end{cases}$$

$$\begin{cases} x = -\cos[D] \sin[S] \\ y = \cos[D] \cos[S] \\ z = \sin[D] \end{cases}$$

$$\begin{cases} \bar{S} = \arctan(-\bar{x}/\bar{y}) \\ \bar{D} = \arcsin(\bar{z}) \end{cases}$$

$$\begin{cases} x = \cos[D] \cos[A] \\ y = \cos[D] \sin[A] \\ z = \sin[D] \end{cases}$$

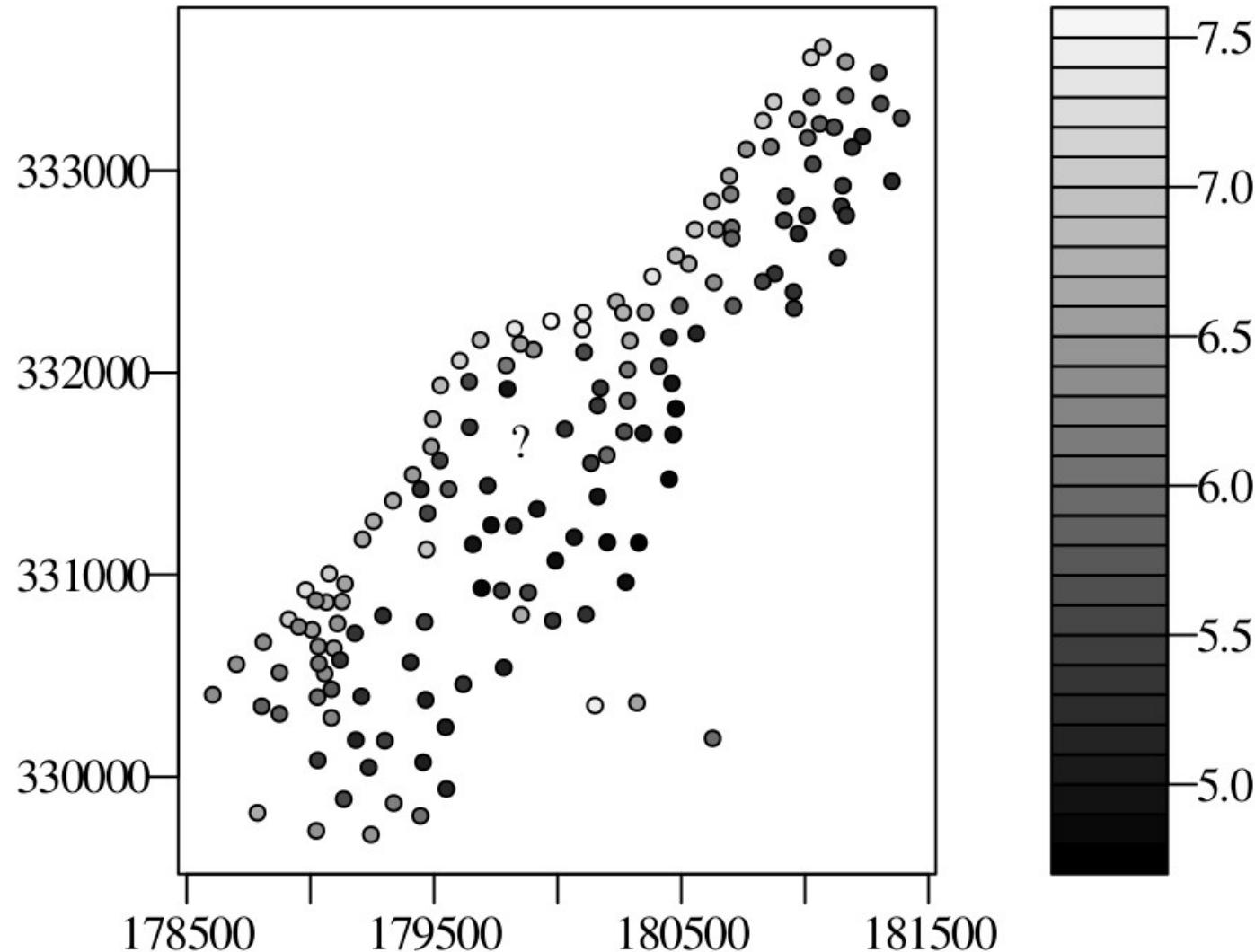
$$\begin{cases} \bar{D} = \arctan(\bar{y}/\bar{x}) \\ \bar{A} = \arcsin(\bar{z}) \end{cases}$$



# Statistics for geoscientists

Spatial data

#	$x$	$y$	Zn	$z = \ln[\text{Zn}]$
1	181072	333611	1022	6.93
2	181025	333558	1141	7.04
3	181165	333537	640	6.46
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
50	180199	331591	375	5.93
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
155	180627	330190	375	5.93

$\ln[\text{Zn}]$ 

“everything is related to everything else, but near things are more related than distant things.”

– Waldo R. Tobler (1969)



	1	2	3	4	5	6	7	8	9	10	...	155
1	0											
2	0.16	0										
3	0.66	0.82	0									
4	2.0	2.1	1.3	0								
5	1.9	2.0	1.2	0.065	0							
6	1.8	2.0	1.2	0.13	0.062	0						
7	1.5	1.7	0.87	0.42	0.36	0.29	0					
8	1.3	1.5	0.64	0.65	0.58	0.52	0.23	0				
9	1.5	1.7	0.87	0.42	0.36	0.30	0.0041	0.22	0			
10	2.4	2.6	1.8	0.48	0.54	0.61	0.90	1.1	0.90	0		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
155	1.4	1.6	0.76	0.53	0.47	0.41	0.11	0.11	0.11	1.0	...	0

	1	2	3	4	5	6	7	8	9	10	...	155
1	0	71	120	260	370	470	260	250	380	470	...	3400
2	0.16	0	140	280	360	470	230	200	330	440	...	3400
3	0.66	0.82	0	140	250	360	170	220	320	380	...	3400
4	2.0	2.1	1.3	0	150	240	180	300	350	320	...	3400
5	1.9	2.0	1.2	0.065	0	110	150	280	270	180	...	3200
6	1.8	2.0	1.2	0.13	0.062	0	250	380	330	180	...	3200
7	1.5	1.7	0.87	0.42	0.36	0.29	0	140	170	210	...	3200
8	1.3	1.5	0.64	0.65	0.58	0.52	0.23	0	140	280	...	3200
9	1.5	1.7	0.87	0.42	0.36	0.30	0.0041	0.22	0	180	...	3100
10	2.4	2.6	1.8	0.48	0.54	0.61	0.90	1.1	0.90	0	...	3000
:	:	:	:	:	:	:	:	:	:	:	..	:
155	1.4	1.6	0.76	0.53	0.47	0.41	0.11	0.11	0.11	1.0	...	0

# semivariance

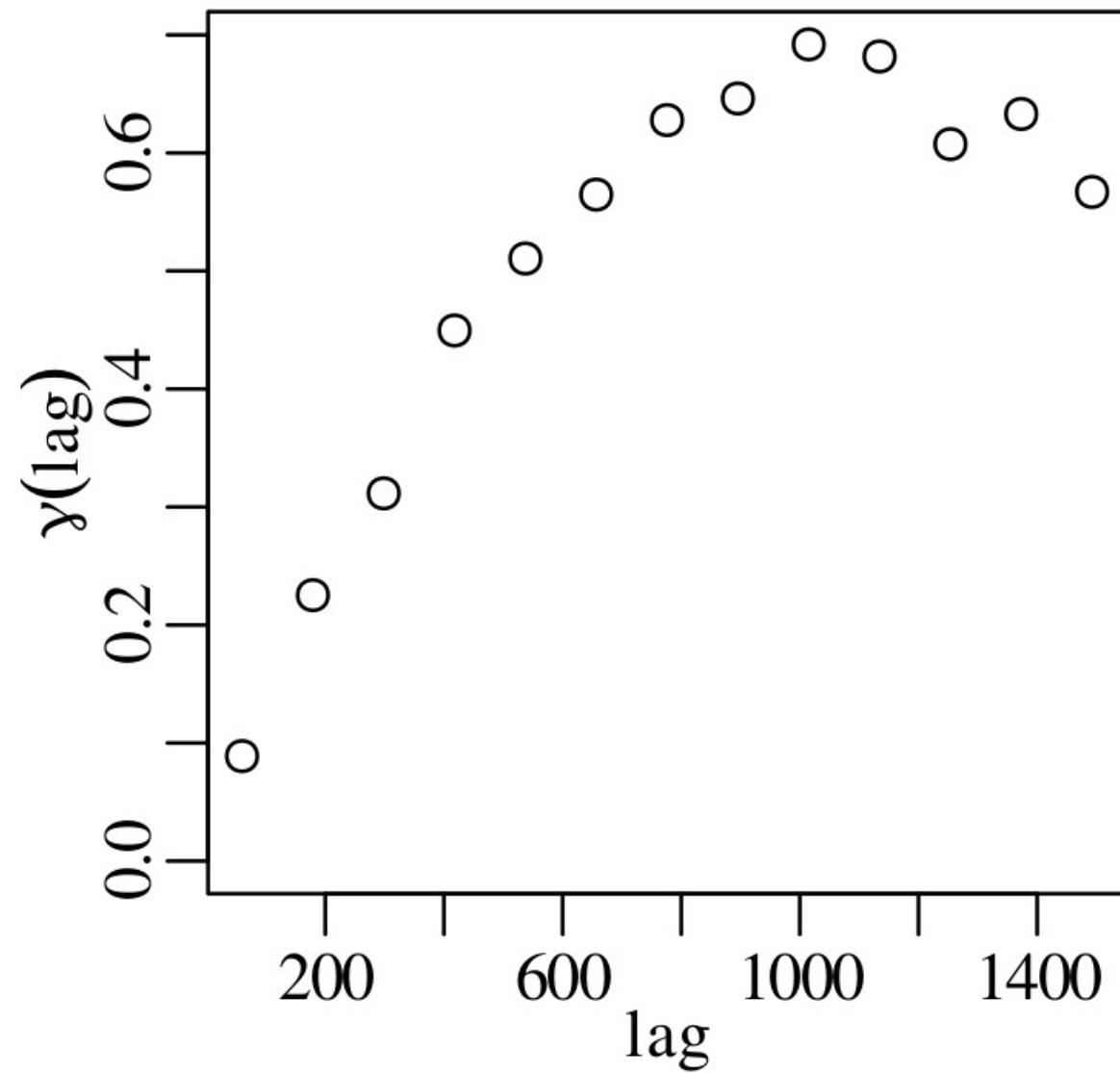
$$\gamma[h] = \sum_k^{n(h)} \frac{(z_{i[k]} - z_{j[k]})^2}{2n(h)}$$

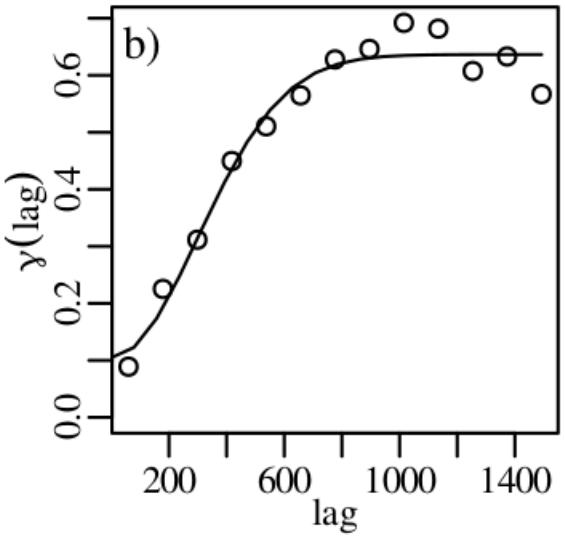
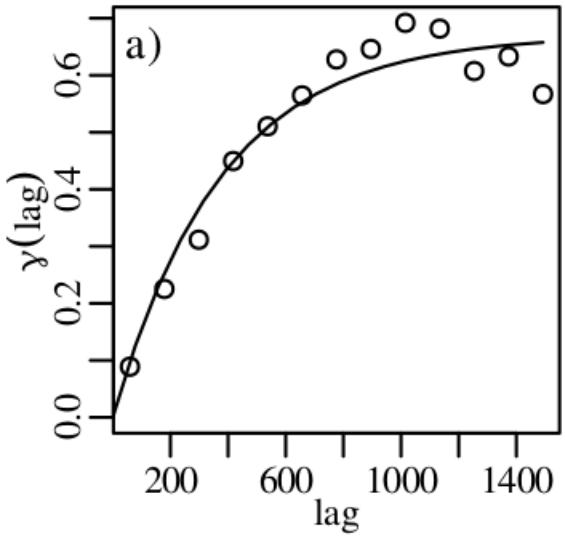
$$\gamma[0 < h \leq 150] = \frac{0.16^2 + 0.66^2 + 0.82^2 + \dots}{2 \times 166} = 0.10$$

	1	2	3	4	5	6	7	8	9	10	...	155
1	0	71	120	260	370	470	260	250	380	470	...	3400
2	0.16	0	140	280	360	470	230	200	330	440	...	3400
3	0.66	0.82	0	140	250	360	170	220	320	380	...	3400
4	2.0	2.1	1.3	0	150	240	180	300	350	320	...	3400
5	1.9	2.0	1.2	0.065	0	110	150	280	270	180	...	3200
6	1.8	2.0	1.2	0.13	0.062	0	250	380	330	180	...	3200
7	1.5	1.7	0.87	0.42	0.36	0.29	0	140	170	210	...	3200
8	1.3	1.5	0.64	0.65	0.58	0.52	0.23	0	140	280	...	3200
9	1.5	1.7	0.87	0.42	0.36	0.30	0.0041	0.22	0	180	...	3100
10	2.4	2.6	1.8	0.48	0.54	0.61	0.90	1.1	0.90	0	...	3000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
155	1.4	1.6	0.76	0.53	0.47	0.41	0.11	0.11	0.11	1.0	...	0

$$\gamma[150 < h \leq 300] = \frac{2.0^2 + 1.5^2 + 1.3^2 + 2.1^2 + \dots}{2 \times 530} = 0.28$$

	1	2	3	4	5	6	7	8	9	10	...	155
1	0	71	120	260	370	470	260	250	380	470	...	3400
2	0.16	0	140	280	360	470	230	200	330	440	...	3400
3	0.66	0.82	0	140	250	360	170	220	320	380	...	3400
4	2.0	2.1	1.3	0	150	240	180	300	350	320	...	3400
5	1.9	2.0	1.2	0.065	0	110	150	280	270	180	...	3200
6	1.8	2.0	1.2	0.13	0.062	0	250	380	330	180	...	3200
7	1.5	1.7	0.87	0.42	0.36	0.29	0	140	170	210	...	3200
8	1.3	1.5	0.64	0.65	0.58	0.52	0.23	0	140	280	...	3200
9	1.5	1.7	0.87	0.42	0.36	0.30	0.0041	0.22	0	180	...	3100
10	2.4	2.6	1.8	0.48	0.54	0.61	0.90	1.1	0.90	0	...	3000
:	:	:	:	:	:	:	:	:	:	:	..	:
155	1.4	1.6	0.76	0.53	0.47	0.41	0.11	0.11	0.11	1.0	...	0





a) exponential

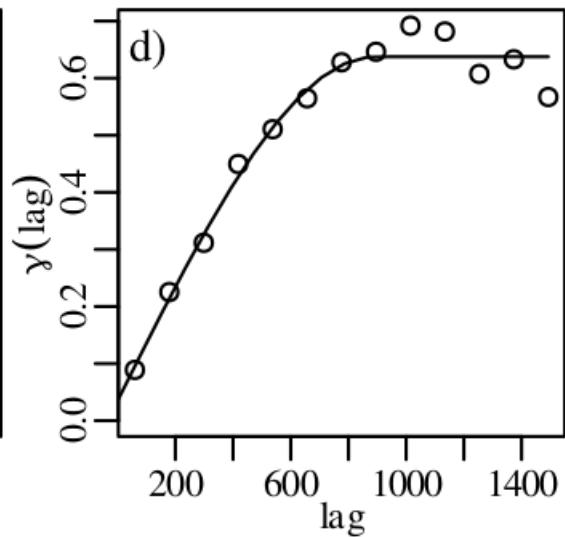
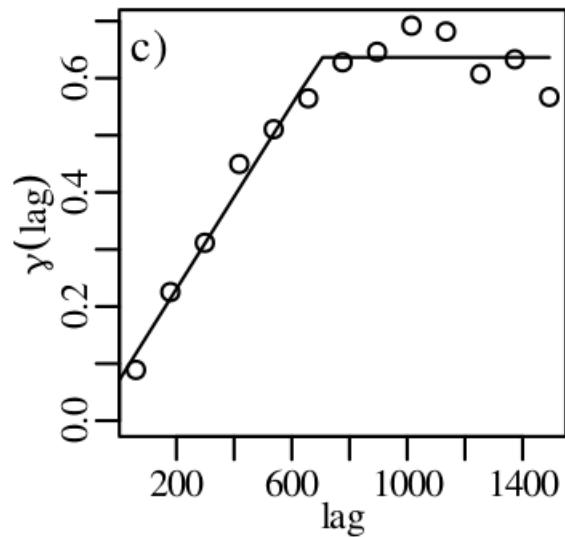
$$\gamma[h] = c_s + (c_n - c_s) \exp\left[-\frac{h}{c_r}\right]$$

b) Gaussian

$$\gamma[h] = c_s + (c_n - c_s) \exp\left[-\frac{h^2}{c_r^2}\right]$$

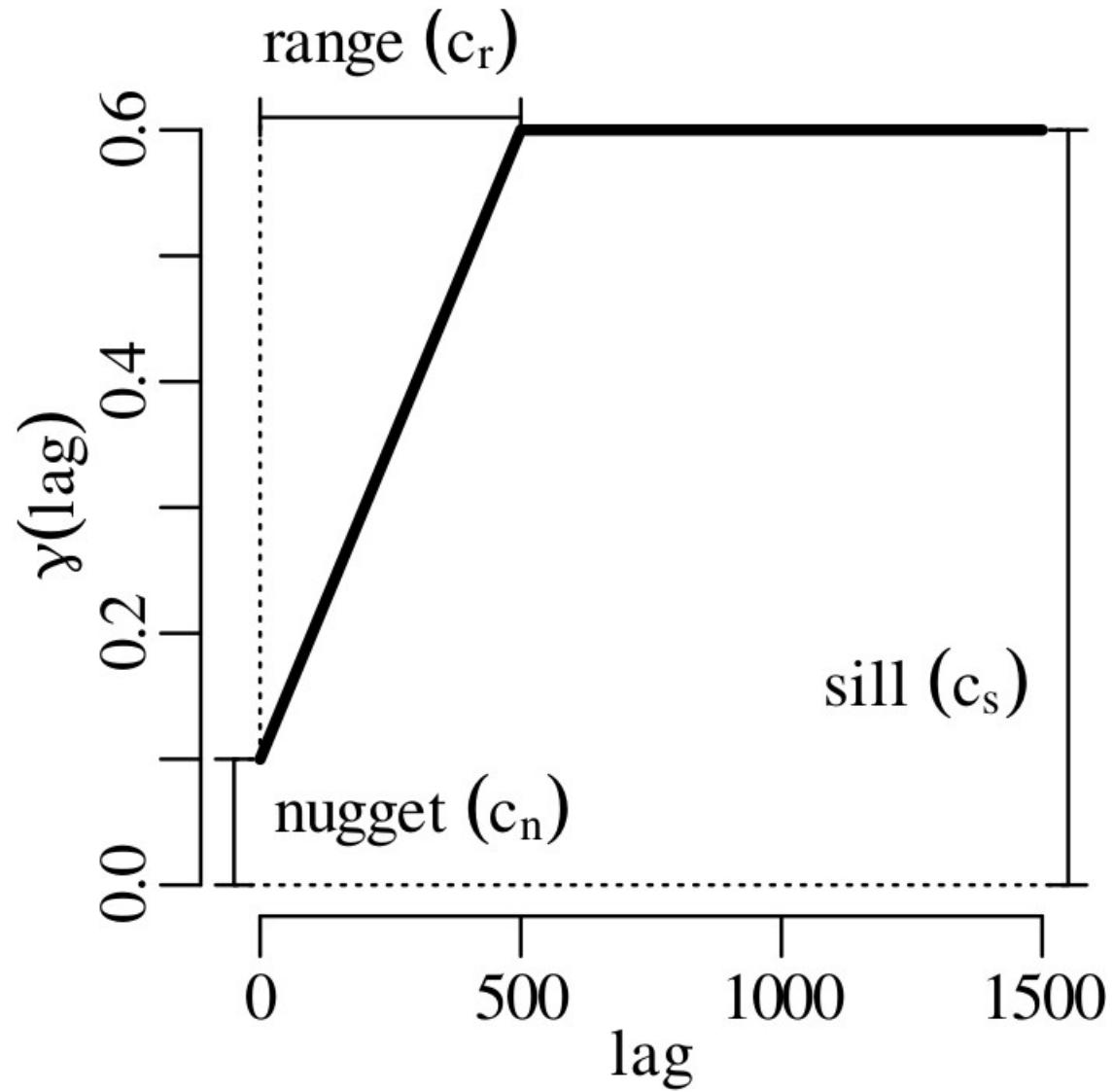
c) linear

$$\begin{cases} \gamma[h] = c_n + \frac{(c_s - c_n)h}{c_r} & \text{if } h \leq c_r \\ \gamma[h] = c_s & \text{if } h > c_r \end{cases}$$



d) spherical

$$\begin{cases} \gamma[h] = c_n + (c_s - c_n) \left( \frac{3}{2} \frac{h}{c_r} - \frac{1}{2} \left[ \frac{h}{c_r} \right]^3 \right) & \text{if } h \leq c_r \\ \gamma[h] = c_s & \text{if } h > c_r \end{cases}$$



# Kriging interpolation

$$z(x_o, y_o) = \sum_{i=1}^n w_i z(x_i, y_i)$$

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ \lambda \end{bmatrix} = \begin{bmatrix} c_n & \gamma[h_{1,2}] & \gamma[h_{1,3}] & \dots & \gamma[h_{1,n}] & 1 \\ \gamma[h_{2,1}] & c_n & \gamma[h_{2,3}] & \dots & \gamma[h_{2,n}] & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma[h_{n,1}] & \gamma[h_{n,2}] & \gamma[h_{n,3}] & \dots & c_n & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma[h_{1,o}] \\ \gamma[h_{2,o}] \\ \vdots \\ \gamma[h_{n,o}] \\ 1 \end{bmatrix}$$

$$\text{where } h_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

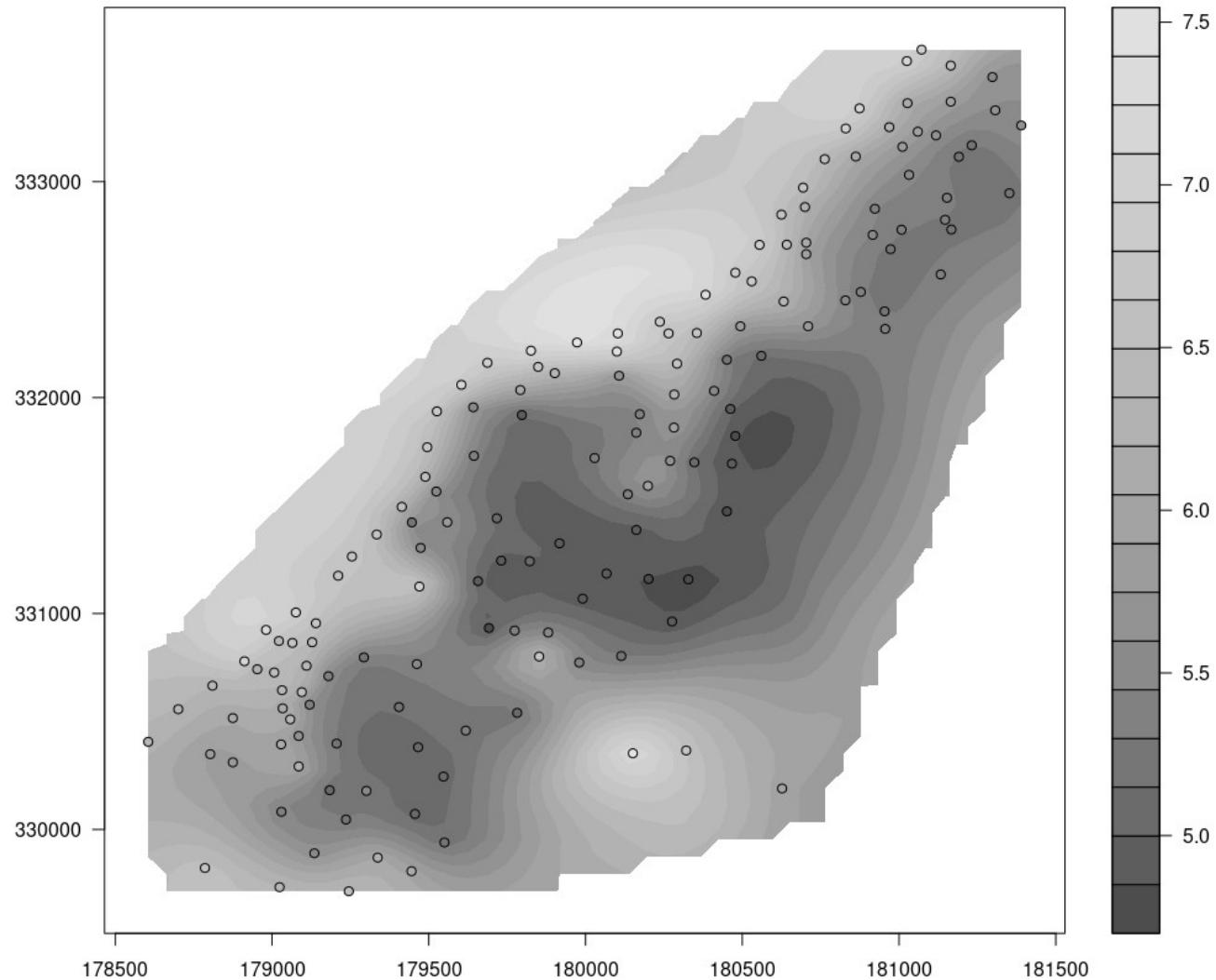
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_{50} \\ \vdots \\ w_{155} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0.0022 \\ -0.00030 \\ 0.00022 \\ \vdots \\ -0.028 \\ \vdots \\ 0.0019 \\ 0.0015 \end{bmatrix} = \begin{bmatrix} 0.035 & 0.11 & 0.15 & \dots & 0.64 & \dots & 0.64 & 1 \\ 0.11 & 0.035 & 0.18 & \dots & 0.64 & \dots & 0.64 & 1 \\ 0.15 & 0.18 & 0.035 & \dots & 0.64 & \dots & 0.64 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0.64 & 0.64 & 0.64 & \dots & 0.035 & \dots & 0.64 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0.64 & 0.64 & 0.64 & \dots & 0.64 & \dots & 0.035 & 1 \\ 1 & 1 & 1 & \dots & 1 & \dots & 1 & 0 \end{bmatrix}_n^{-1} \begin{bmatrix} 0.64 \\ 0.64 \\ 0.64 \\ \vdots \\ 0.37 \\ \vdots \\ 0.64 \\ 1 \end{bmatrix}$$

$$z(x_o, y_o) = \sum_{i=1}^n w_i z(x_i, y_i)$$

$$\begin{aligned} z_{x_o, y_o} &= 0.0022 \times 6.93 - 0.00030 \times 7.04 + 0.00022 \times 6.46 + \dots \\ &\quad - 0.028 \times 5.93 + \dots + 0.0019 \times 5.93 = 5.01 \end{aligned}$$

$$\exp[5.01] = 150 \text{ ppm}$$

$\ln[\text{Zn}]$



$$s^2(x_o, y_o) = \lambda + \sum_{i=1}^n \gamma[h_{i,o}]w_i$$

For location  $\{x_o = 179850, y_o = 331650\}$ :

$$s^2(x_o, y_o) = 0.00029 + 0.64 \times 0.0022 - 0.64 \times 0.00030 + \dots = 0.22$$

**addition**       $z = a + bx + cy$        $s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2 + 2bc s[x, y]$

**subtraction**       $z = ax - by$        $s[z]^2 = a^2 s[x]^2 + b^2 s[y]^2 - 2ab s[x, y]$

**multiplication**       $z = axy$        $\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 + 2\frac{s[x, y]}{xy}$

**division**       $z = a\frac{x}{y}$        $\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 - 2\frac{s[x, y]}{xy}$

**exponentiation**       $z = ae^{bx}$        $\left(\frac{s[z]}{z}\right)^2 = b^2 s[x]^2$

**logarithms**       $z = a \ln[bx]$        $s[z]^2 = a^2 \left(\frac{s[x]}{x}\right)^2$

**power**       $z = ax^b$        $\left(\frac{s[z]}{z}\right)^2 = b^2 \left(\frac{s[x]}{x}\right)^2$

$$s^2(x_{\circ}, y_{\circ}) = \lambda + \sum_{i=1}^n \gamma[h_{i,\circ}]w_i$$

For location  $\{x_{\circ} = 179850, y_{\circ} = 331650\}$ :

$$s^2(x_{\circ}, y_{\circ}) = 0.00029 + 0.64 \times 0.0022 - 0.64 \times 0.00030 + \dots = 0.22$$

$$s[z]/z = \sqrt{0.22} = 47\%$$

$s[\text{Zn}]/\text{Zn}$ 