Statistics for geoscientists

Plotting data

I		II		III		IV	
x	y	x	y	x	y	x	y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

– the mean of x is 9

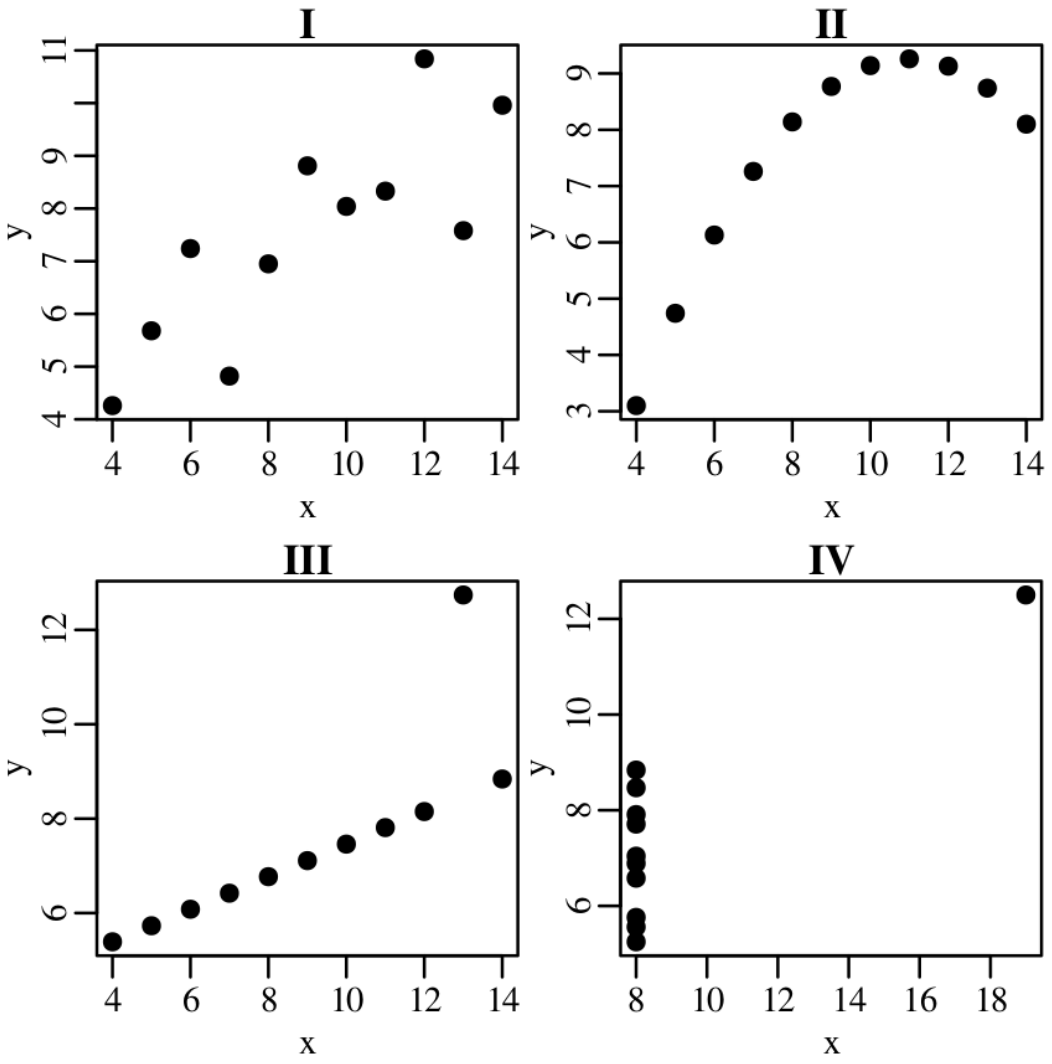
– the variance of x is 11

– the mean of y is 7.50

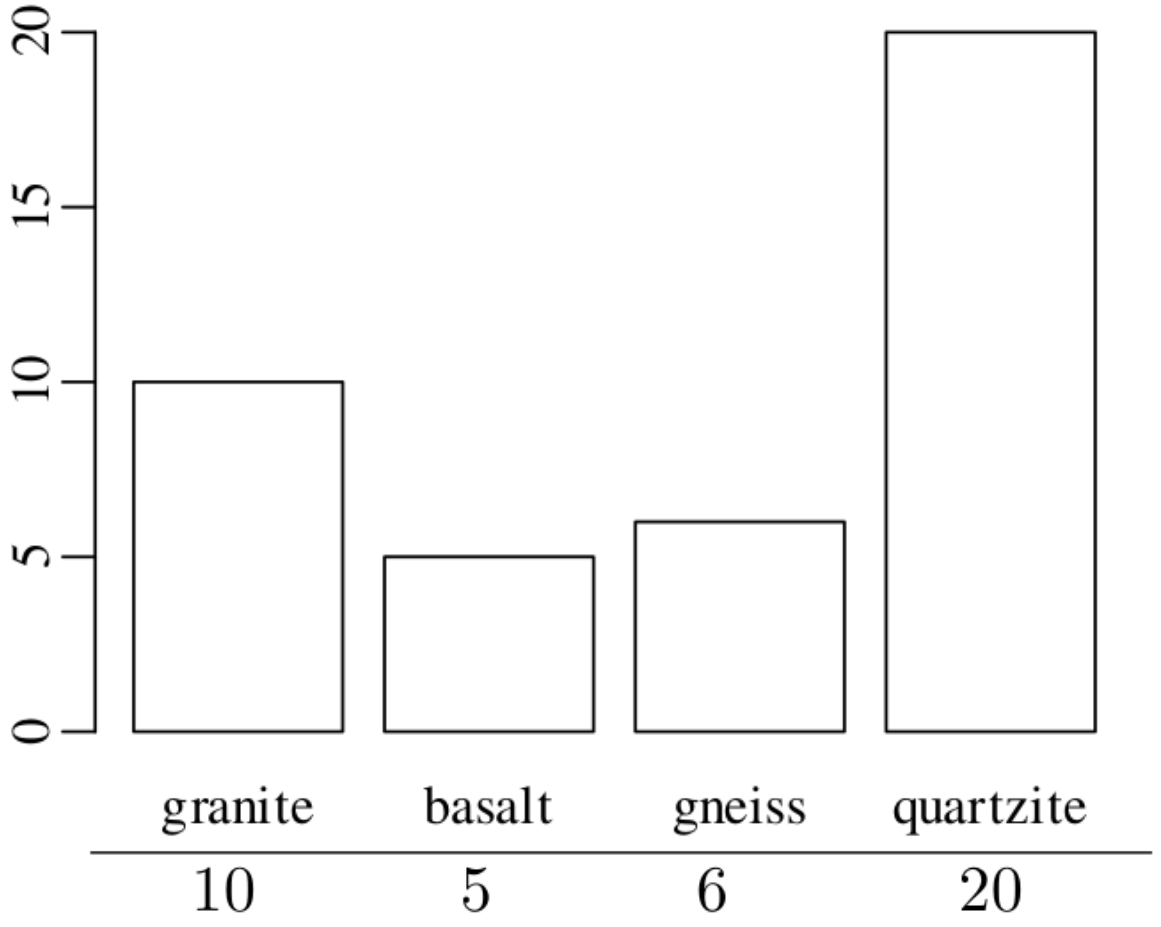
– the variance of y is 4.125

– the correlation coefficient of x and y is 0.816

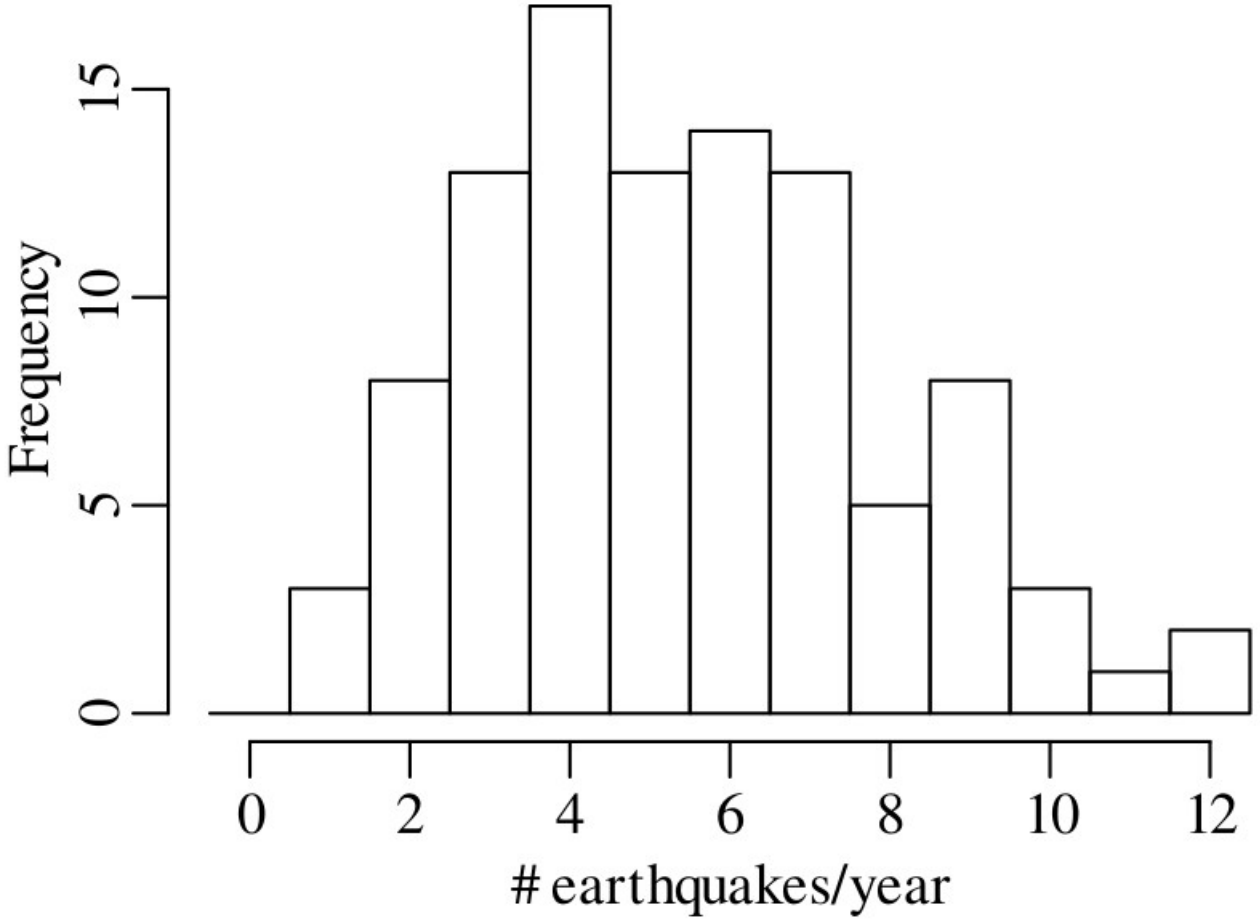
– the best fit line is given by $y = 3.00 + 0.500x$


I
II
III
IV
x
y
4
6
8
10
12
14
16
18
3
5
7
9
11

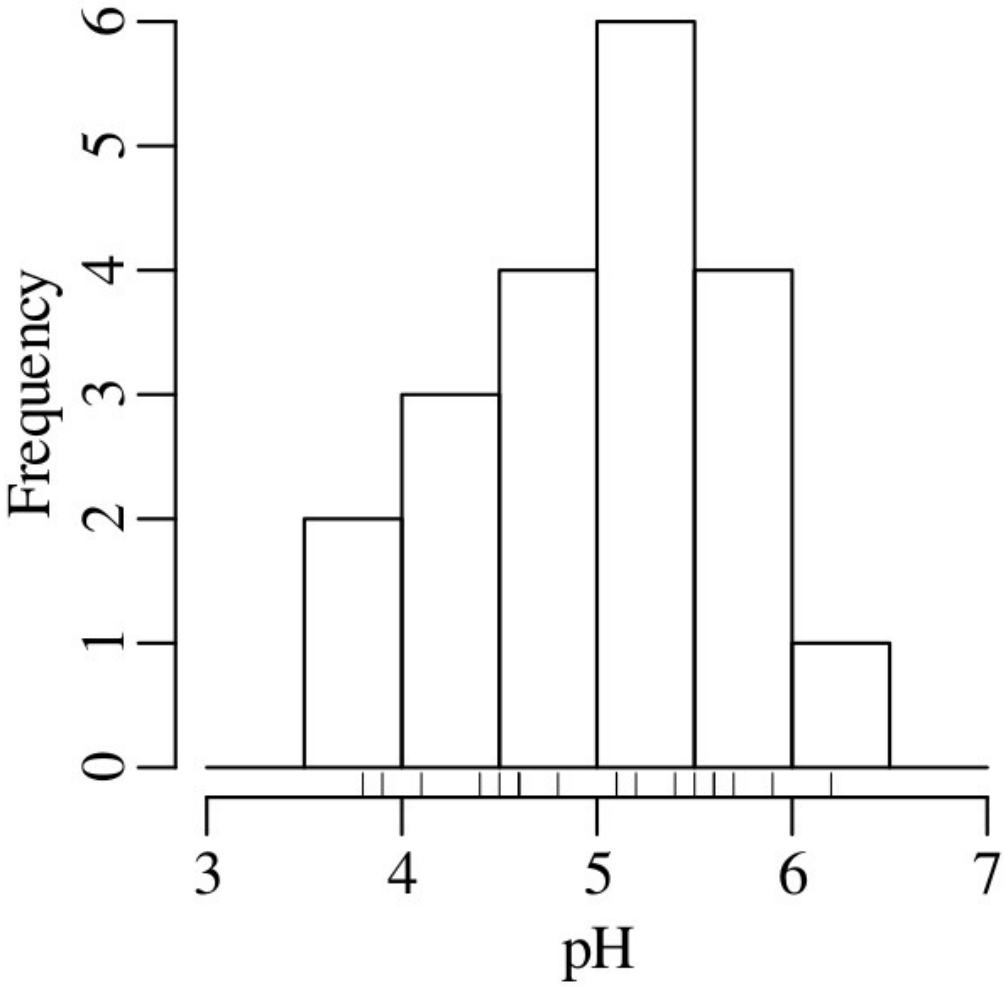
Categorical data



Count data

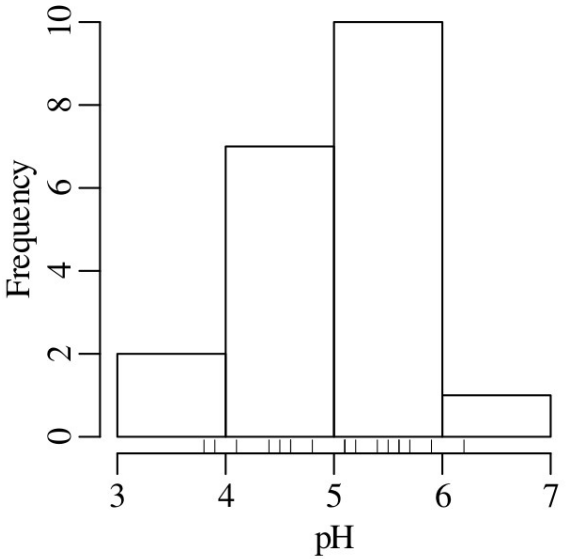


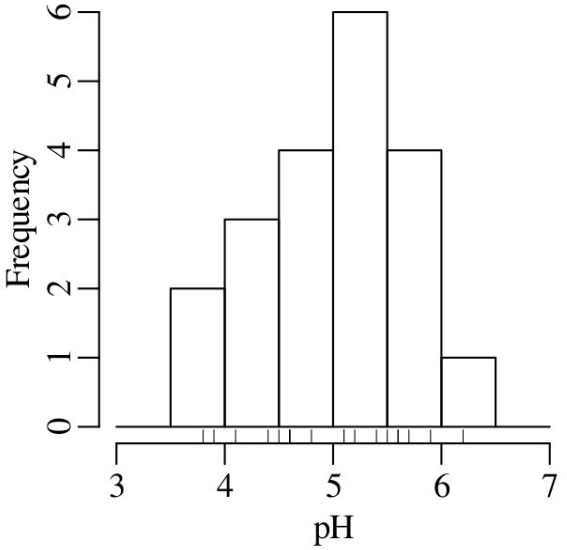
Continuous data

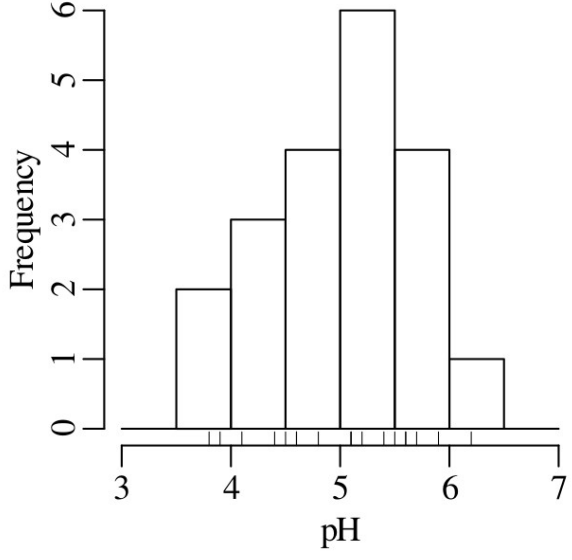


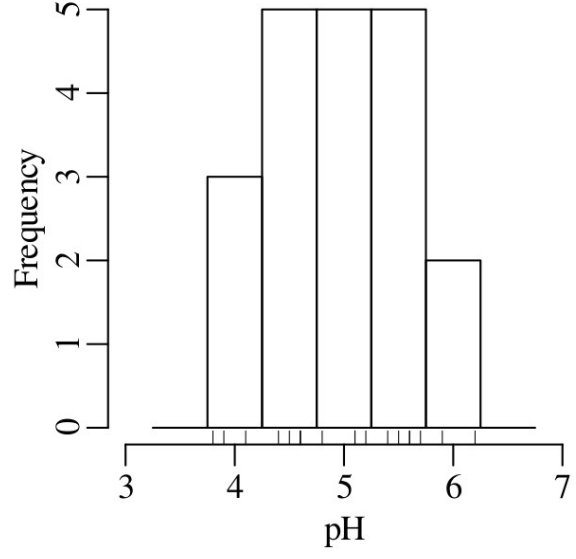
i. How many bins?

ii. Where to place the bins?



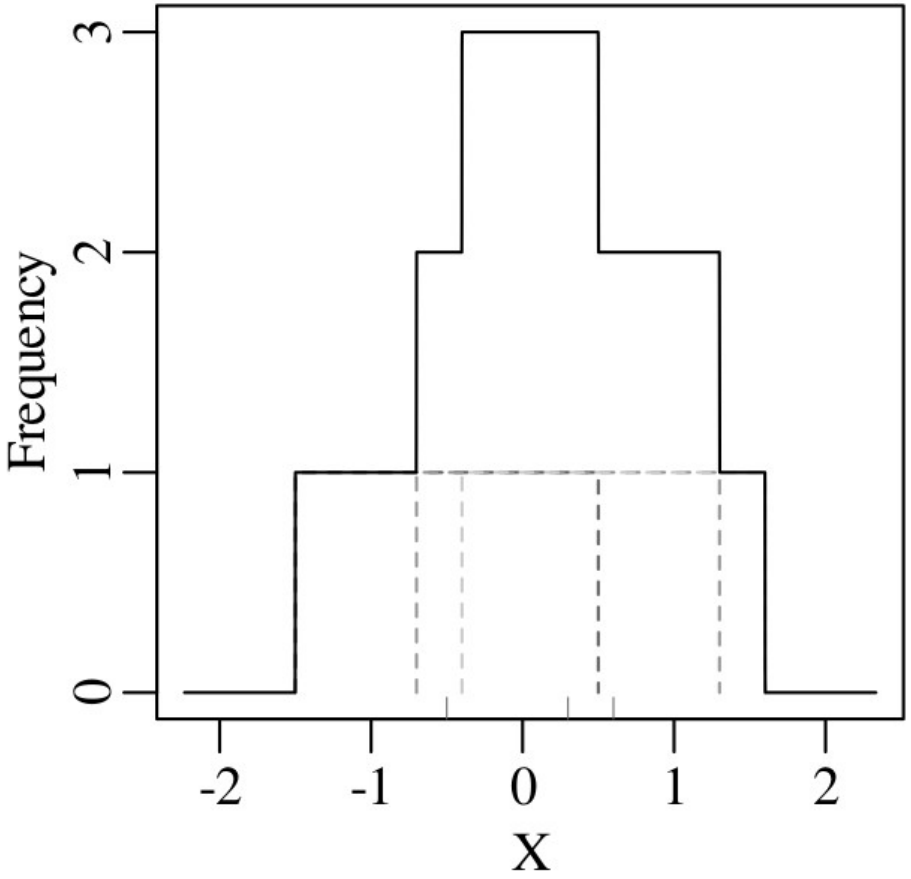


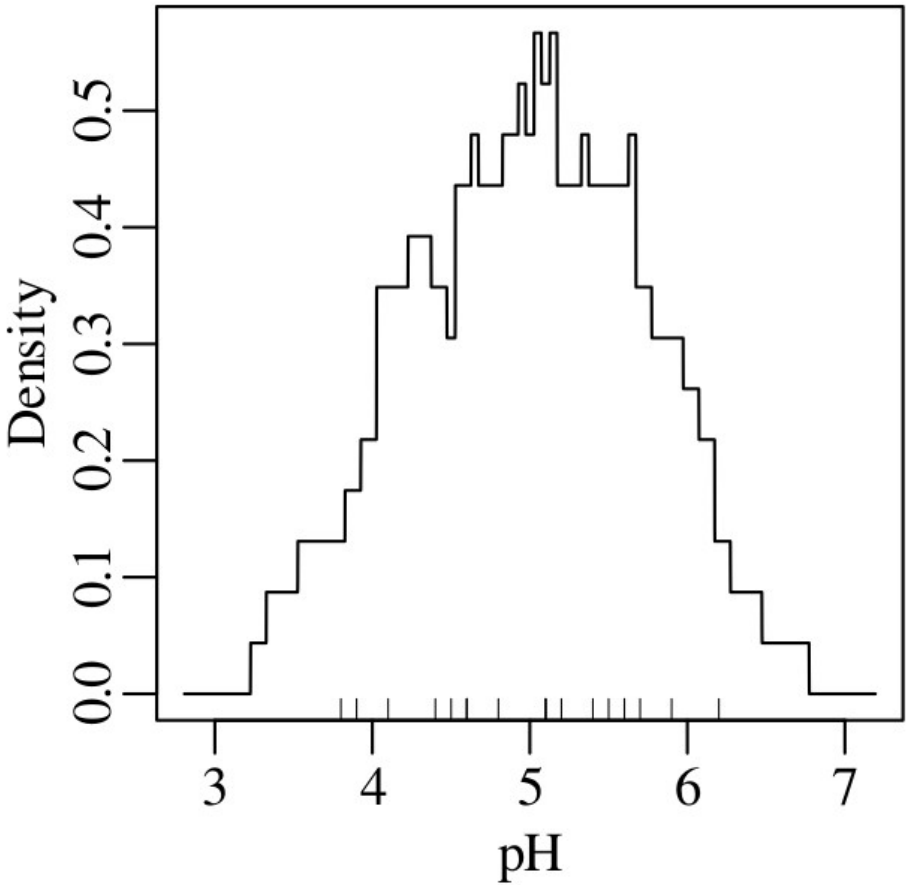




Kernel Density Estimate

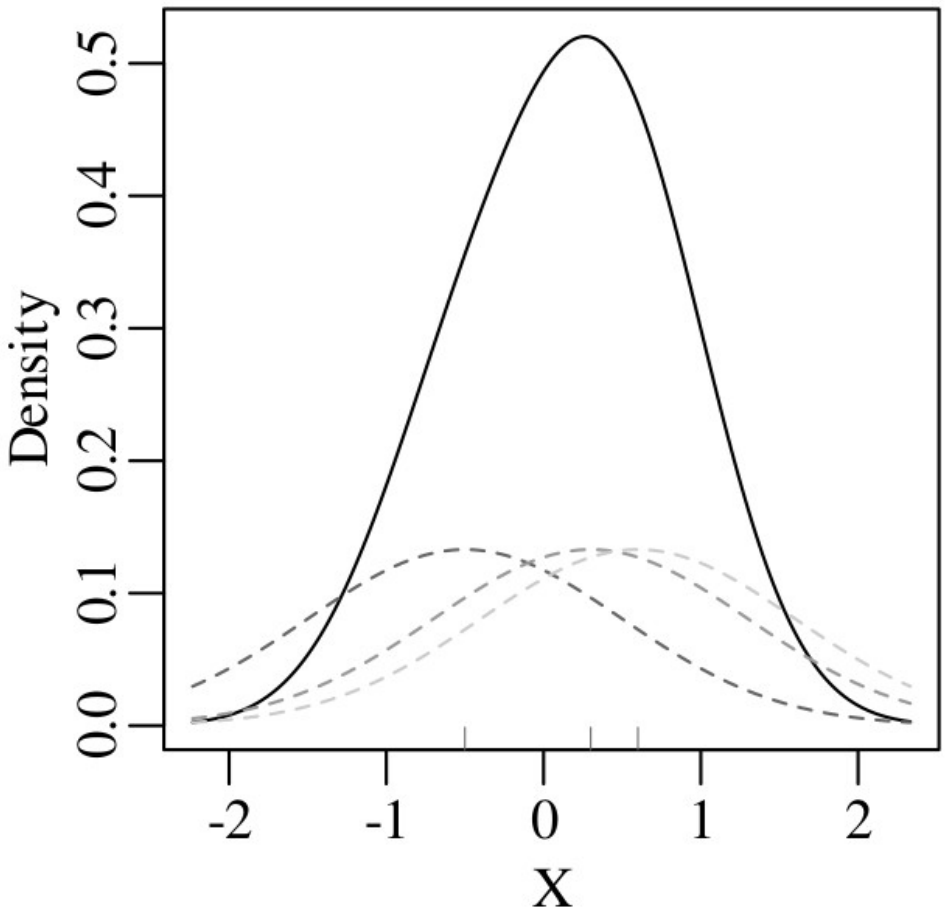
$$KDE(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

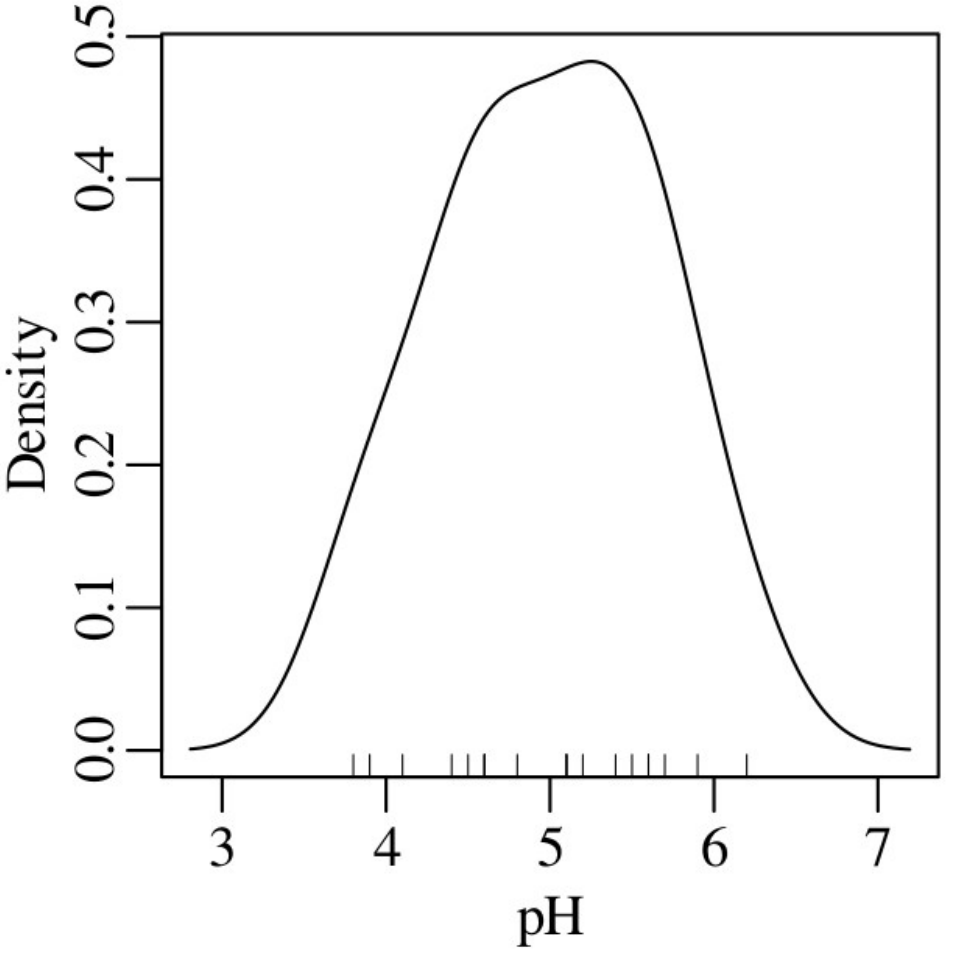




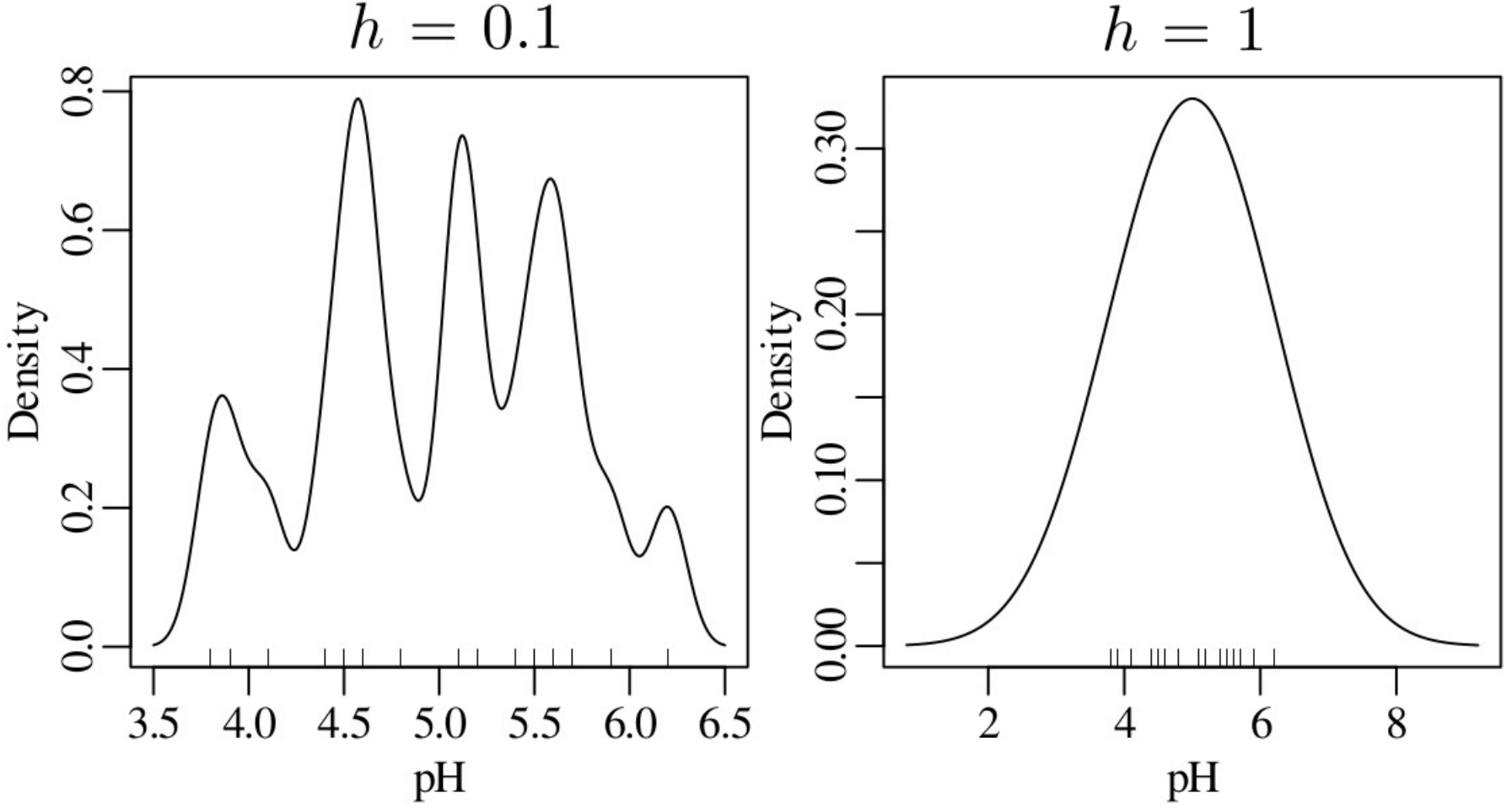
Gaussian kernel:

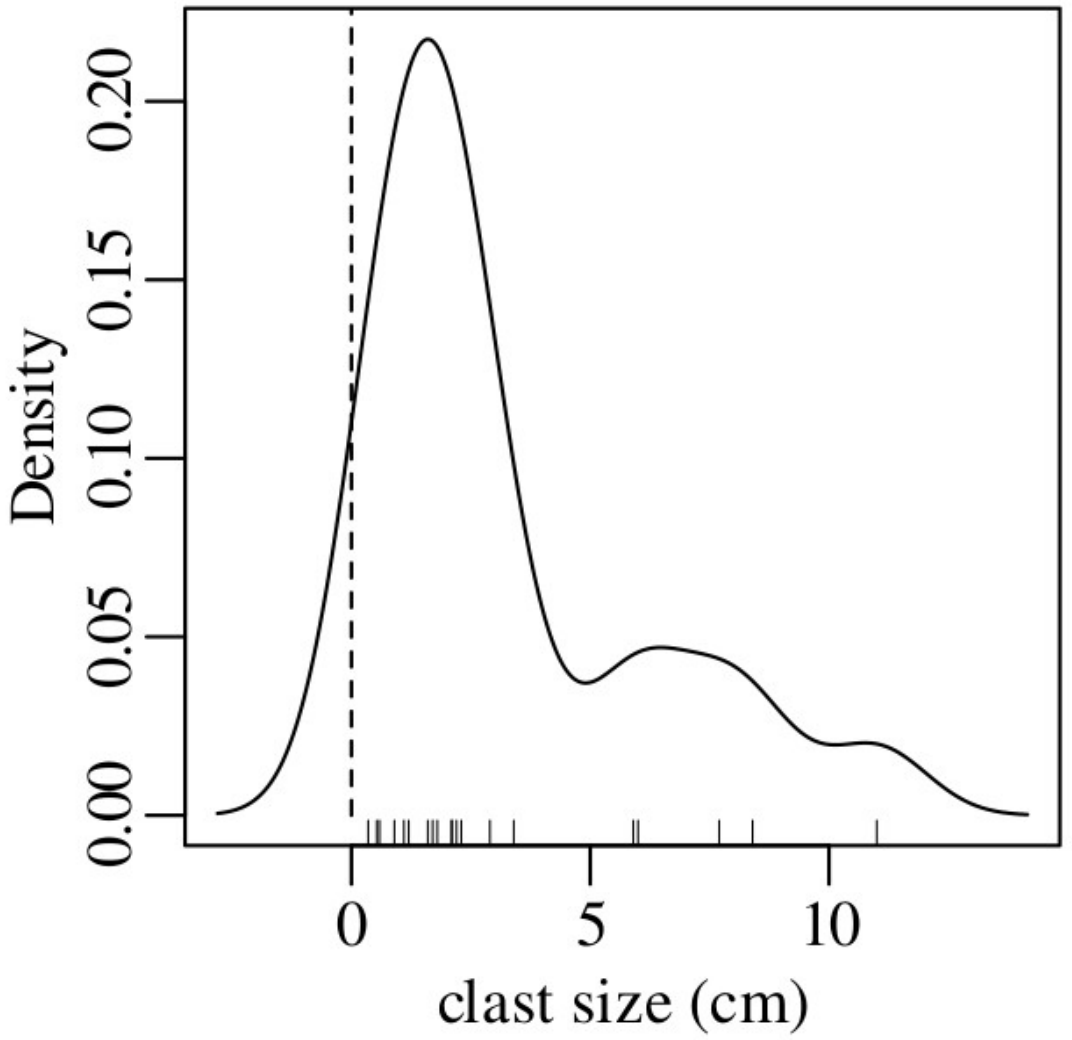
$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right]$$



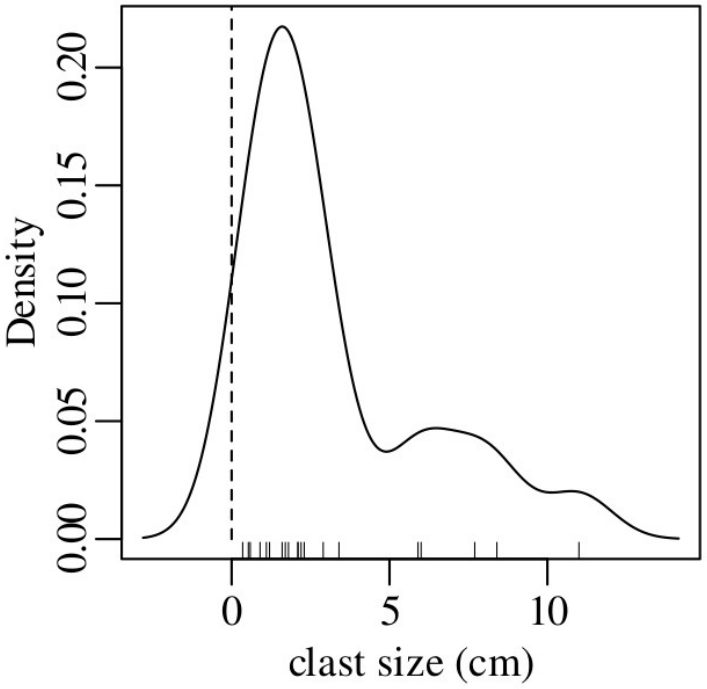


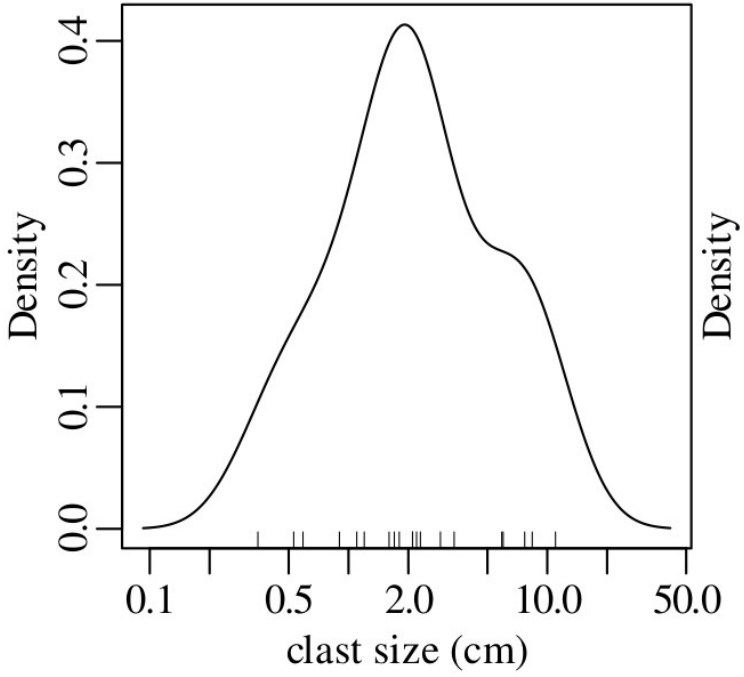
bandwidth

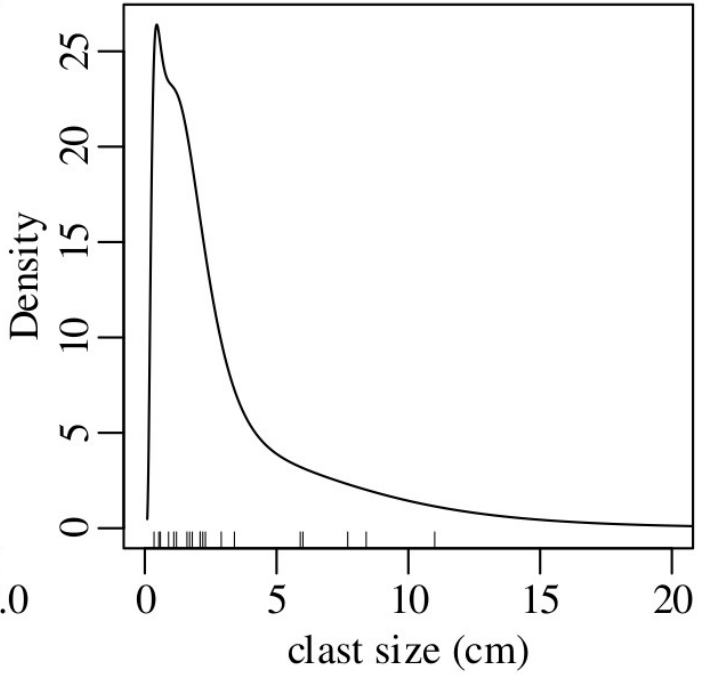

$h = 0.1$
Density
0.0
0.2
0.4
0.6
0.8
3.5
4.0
4.5
5.0
5.5
6.0
6.5
pH
$h = 1$
Density
0.00
0.10
0.20
0.30
2
4
6
8
pH

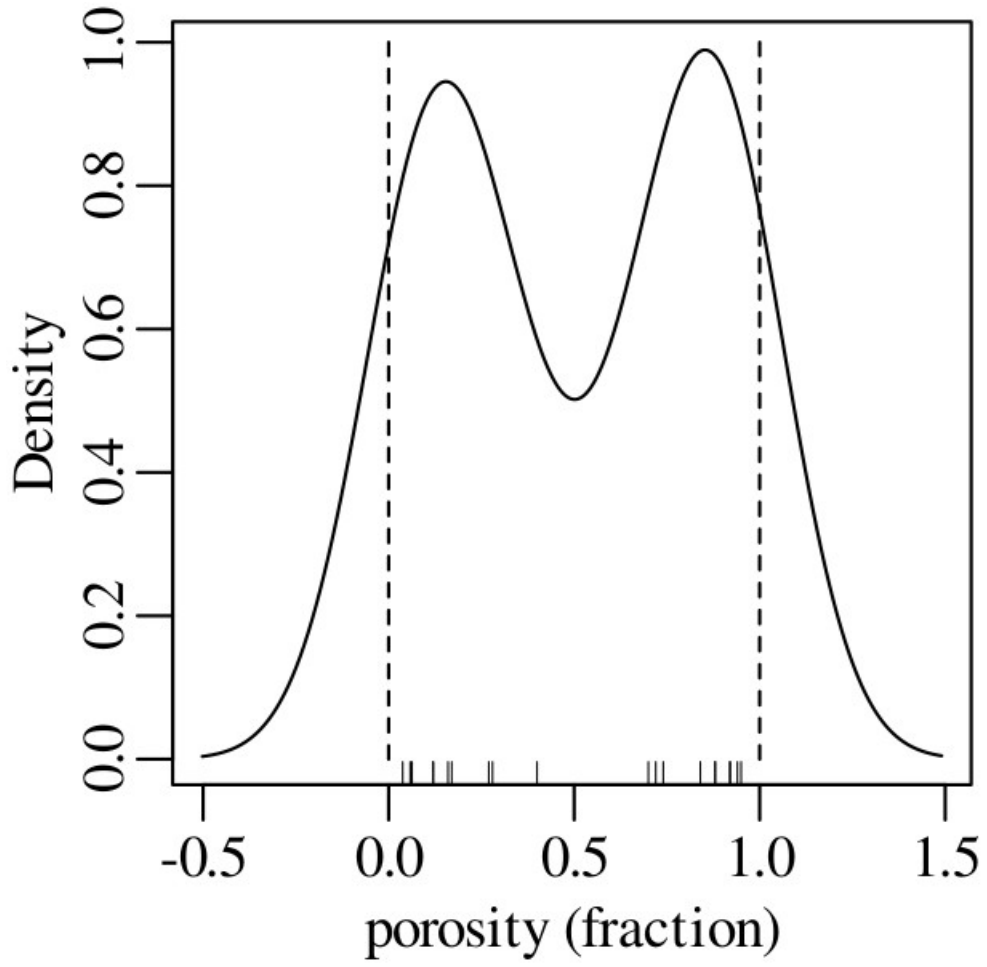

Density
0.00
0.05
0.10
0.15
0.20
0
5
10
clast size (cm)

logarithmic transformation





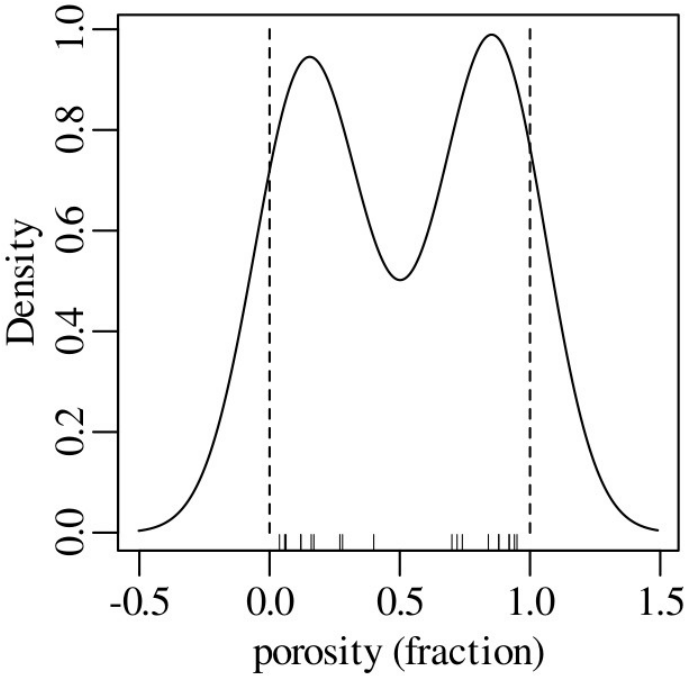


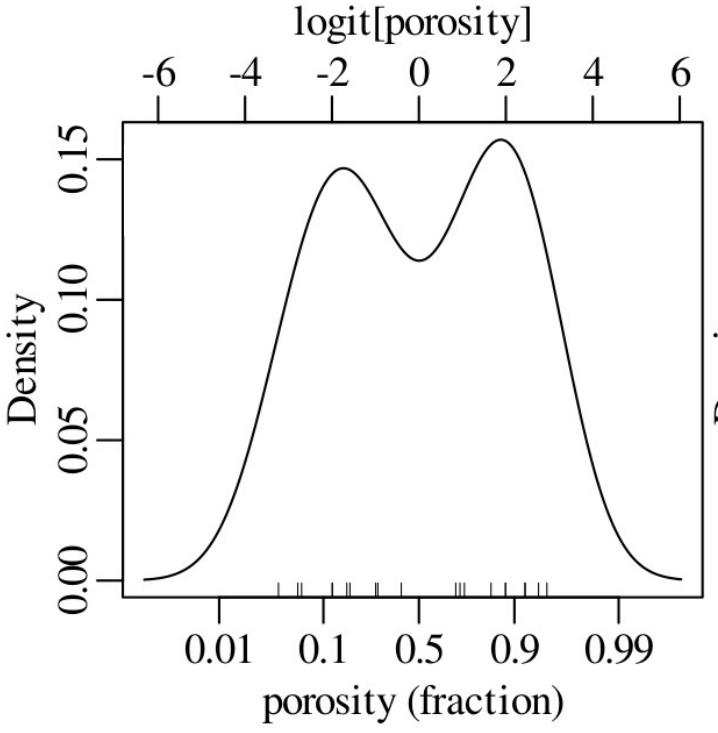

Density
0.0
0.2
0.4
0.6
0.8
1.0
-0.5
0.0
0.5
1.0
1.5
porosity (fraction)

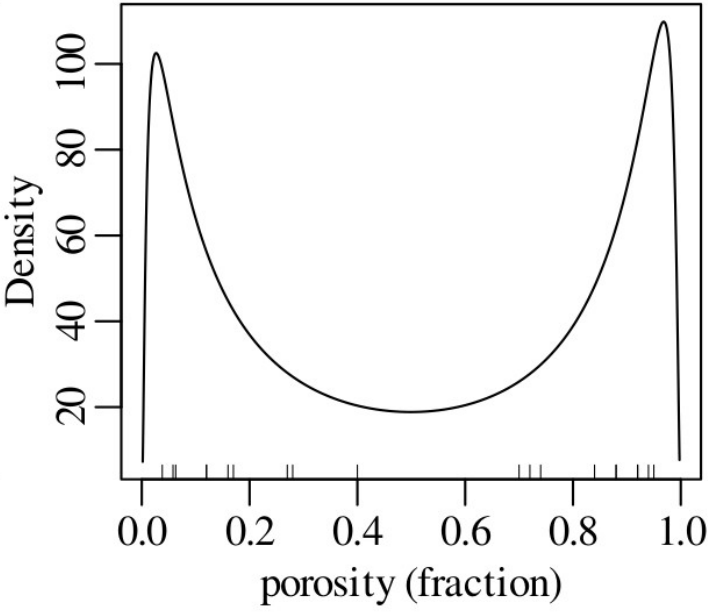
logistic transformation

$$u = \text{logit}(x) = \ln\left[\frac{x}{1-x}\right]$$

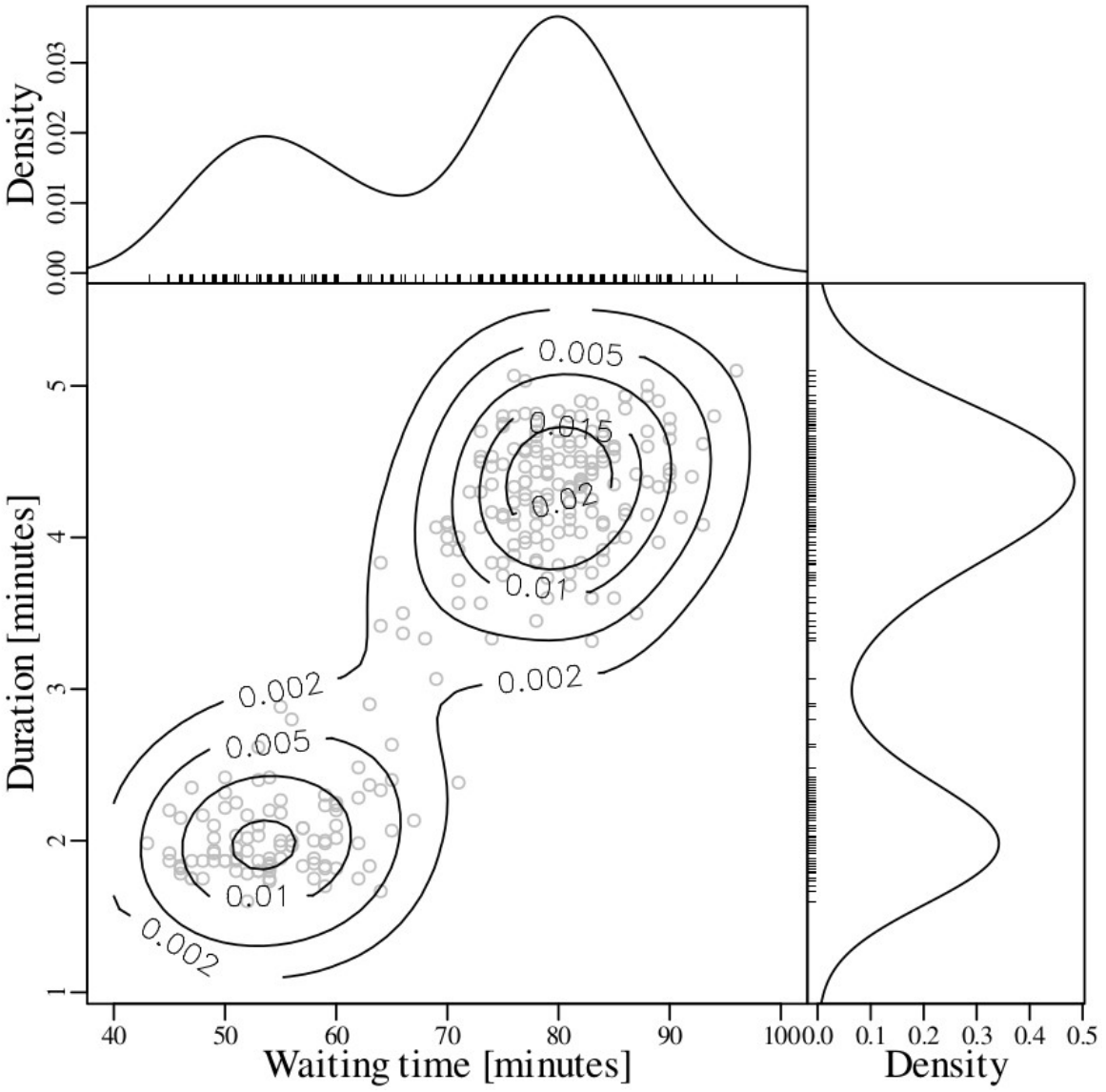
$$x = \text{logit}^{-1}(u) = \frac{\exp[u]}{\exp[u]+1}$$





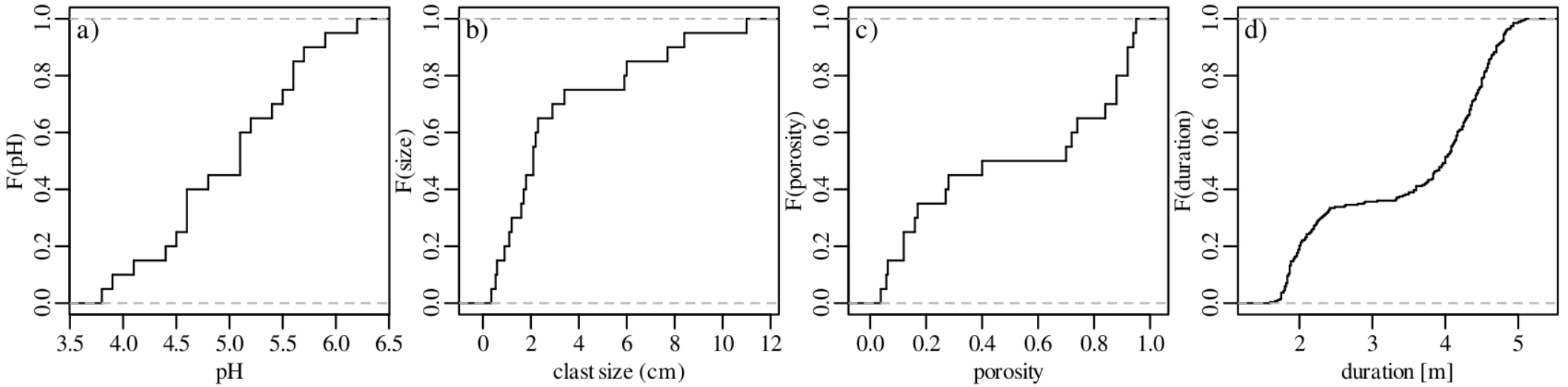


Multivariate distributions



Empirical cumulative distribution fuctions

$$F(x) = \sum_{i=1}^{n} 1(x_i < x)/n$$



Summary statistics

I		II		III		IV	
x	y	x	y	x	y	x	y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

– the mean of x is 9

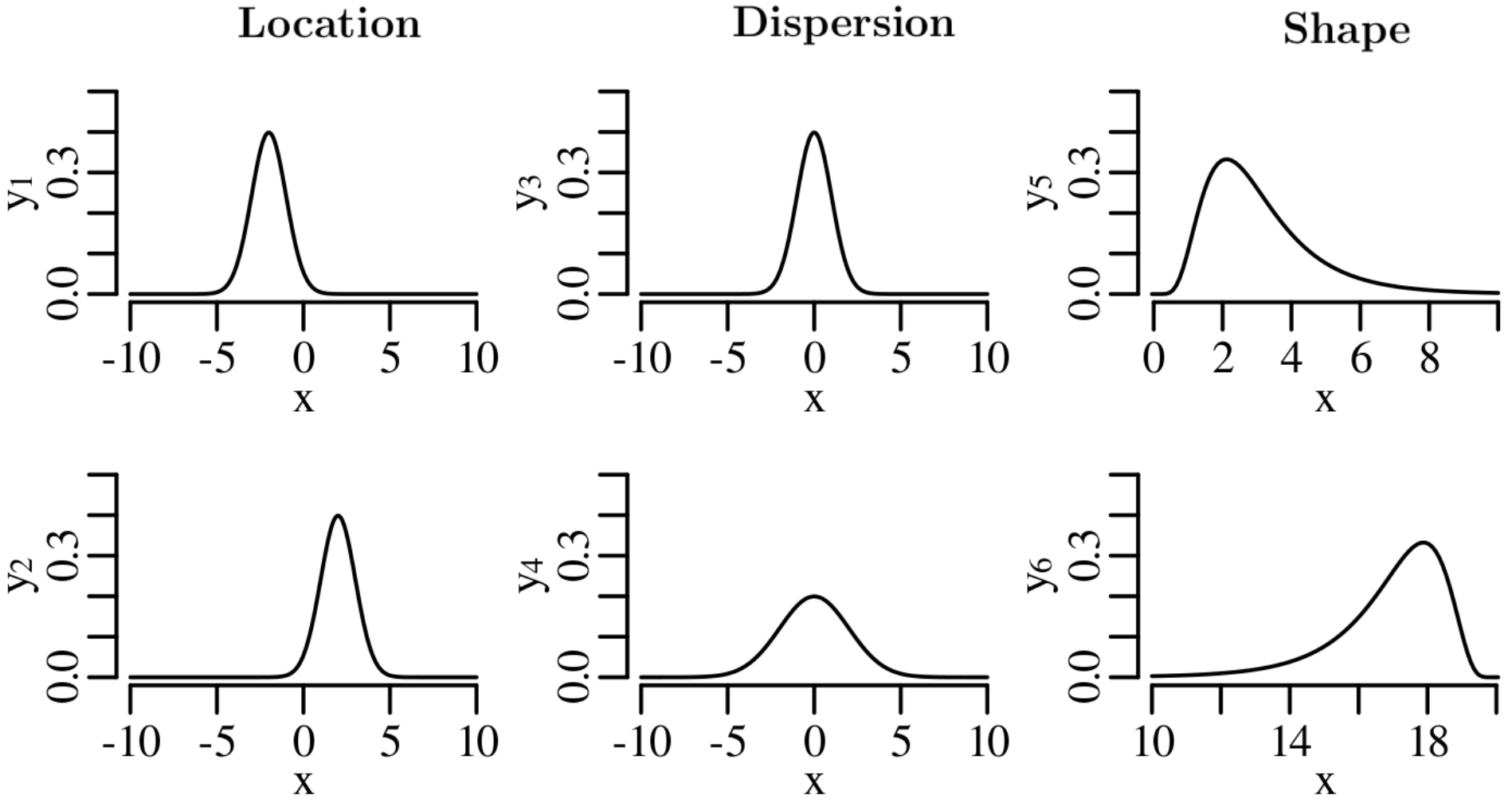
– the variance of x is 11

– the mean of y is 7.50

– the variance of y is 4.125

– the correlation coefficient of x and y is 0.816

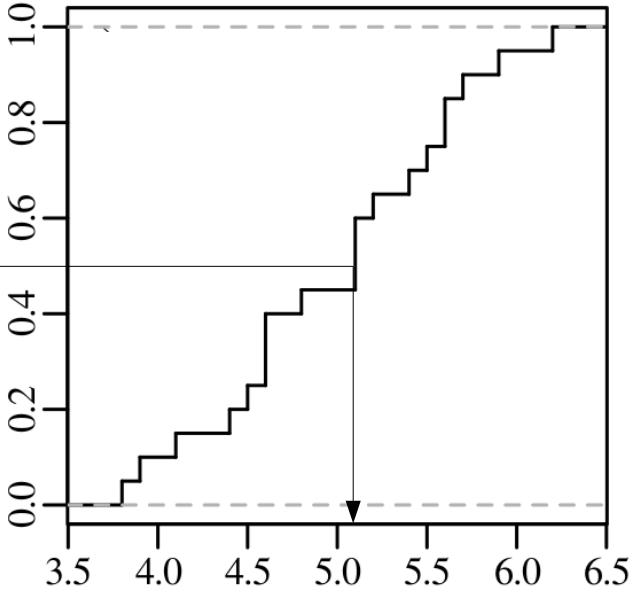
– the best fit line is given by $y = 3.00 + 0.500x$


Location
Dispersion
Shape
y1
y2
y3
y4
y5
y6
0.0
0.3
-10
-5
0
5
10
0
2
4
6
8
10
14
18
x

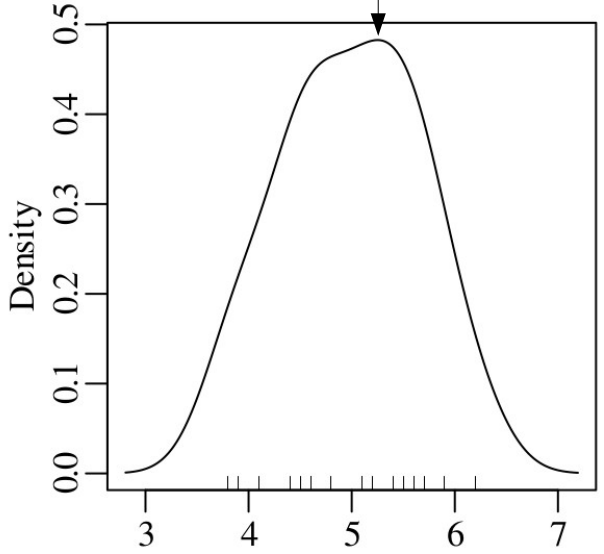
Location

Mean $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Median $\text{med}(x) = x_1 < \ldots < \mathbf{x_{n/2}} < \ldots < x_n$



Mode



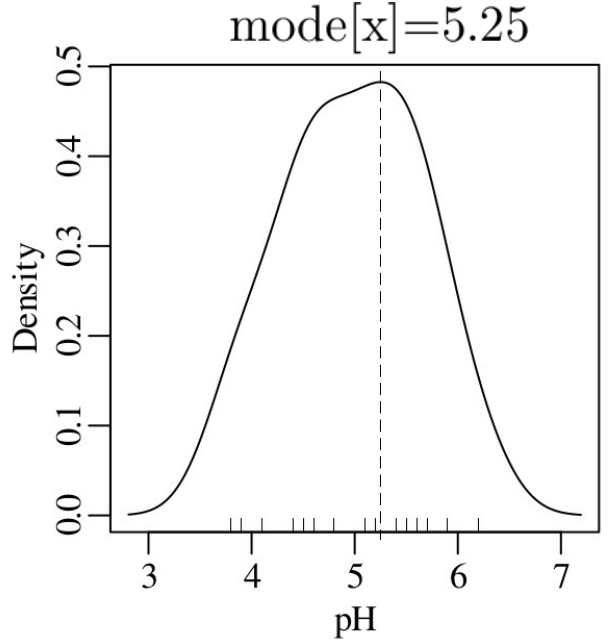
pH data 6.2, 4.4, 5.6, 5.2, 4.5, 5.4, 4.8, 5.9, 3.9, 3.8, 5.1, 4.1, 5.1, 5.5, 5.1, 4.6, 5.7, 4.6, 4.6, 5.6

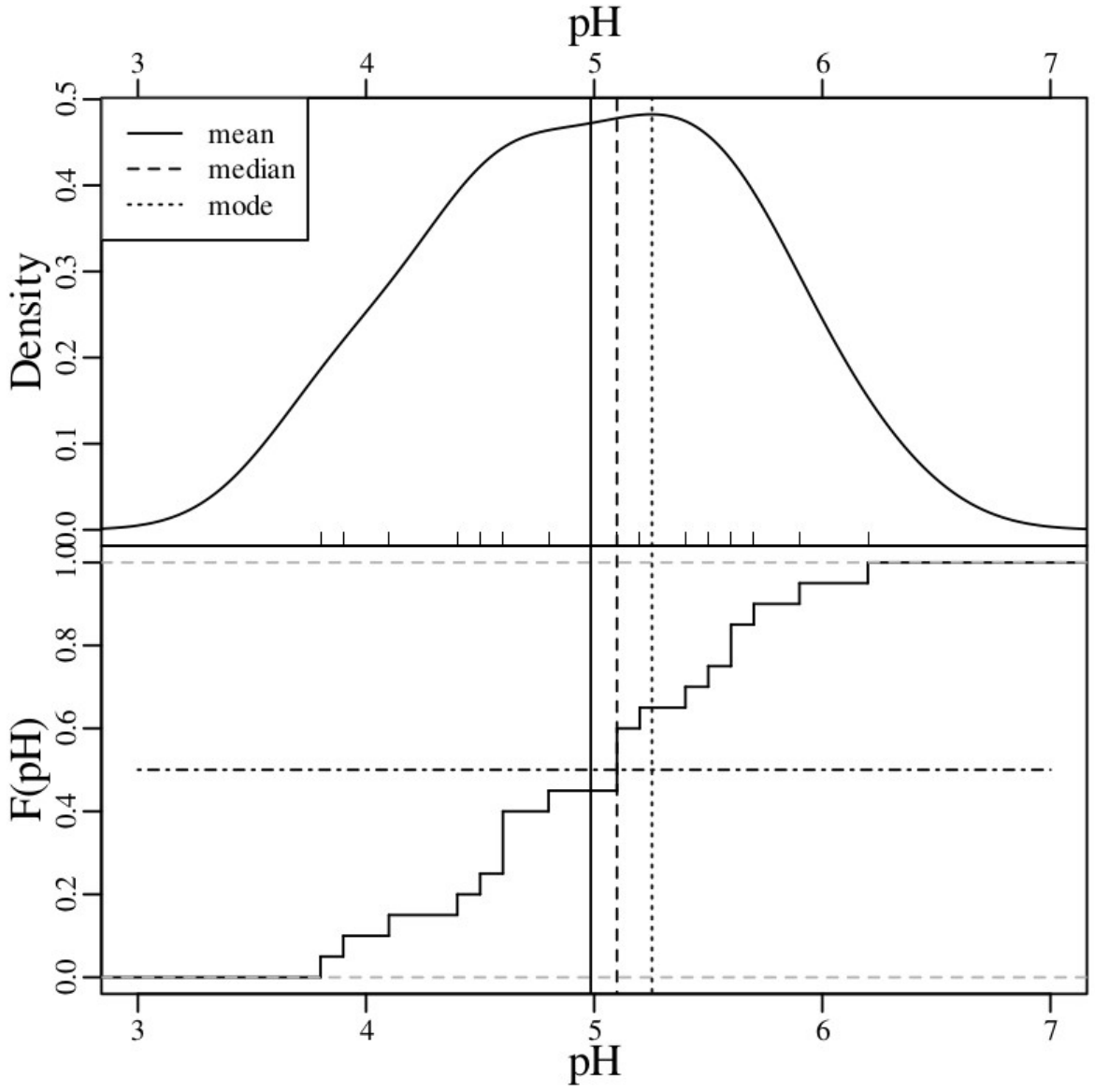
Mean

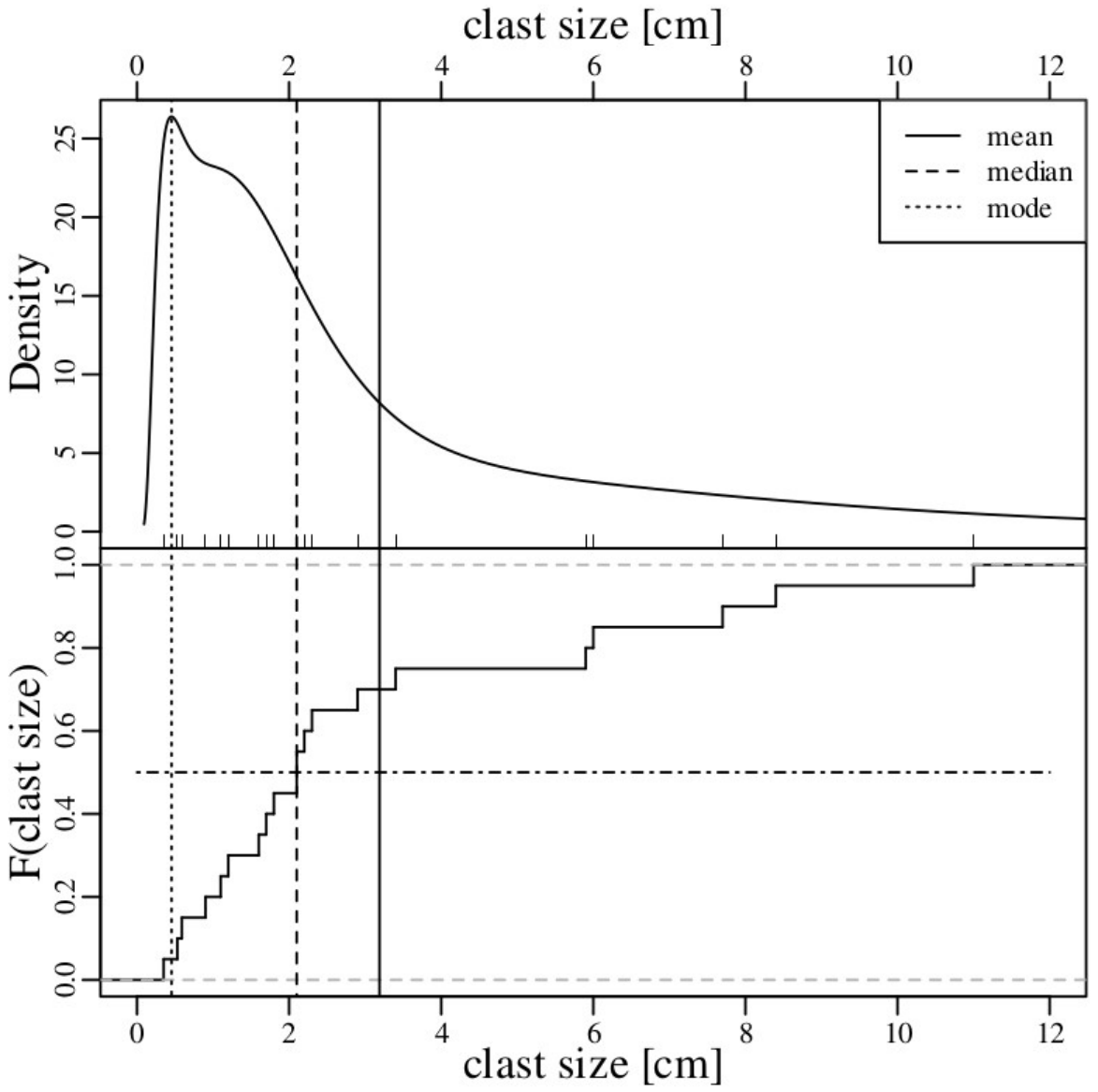
$$\bar{x} = \frac{\begin{array}{c}6.2 + 4.4 + 5.6 + 5.2 + 4.5 + 5.4 + 4.8 + 5.9 + 3.9 + 3.8 + \\ 5.1 + 4.1 + 5.1 + 5.5 + 5.1 + 4.6 + 5.7 + 4.6 + 4.6 + 5.6\end{array}}{20} = 5.00$$

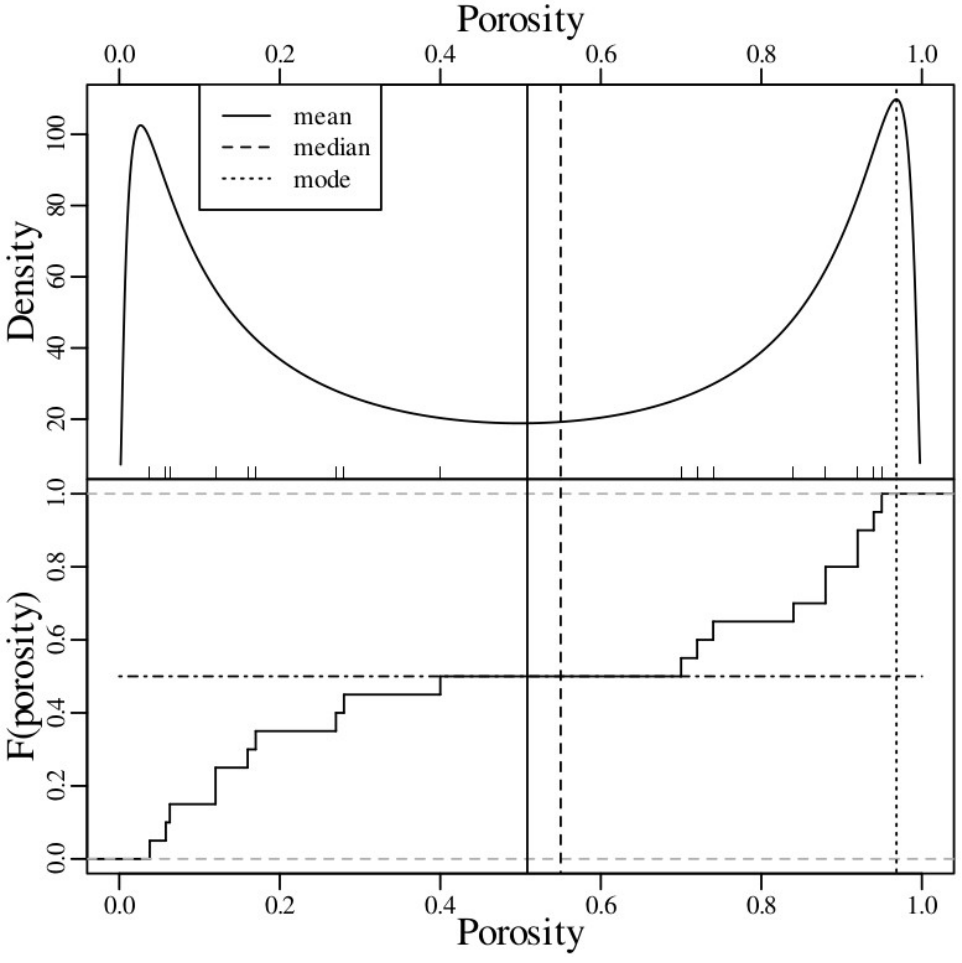
Median 3.8, 3.9, 4.1, 4.4, 4.5, 4.6, 4.6, 4.6, 4.8, **5.1**, **5.1**, 5.1, 5.2, 5.4, 5.5, 5.6, 5.6, 5.7, 5.9, 6.2

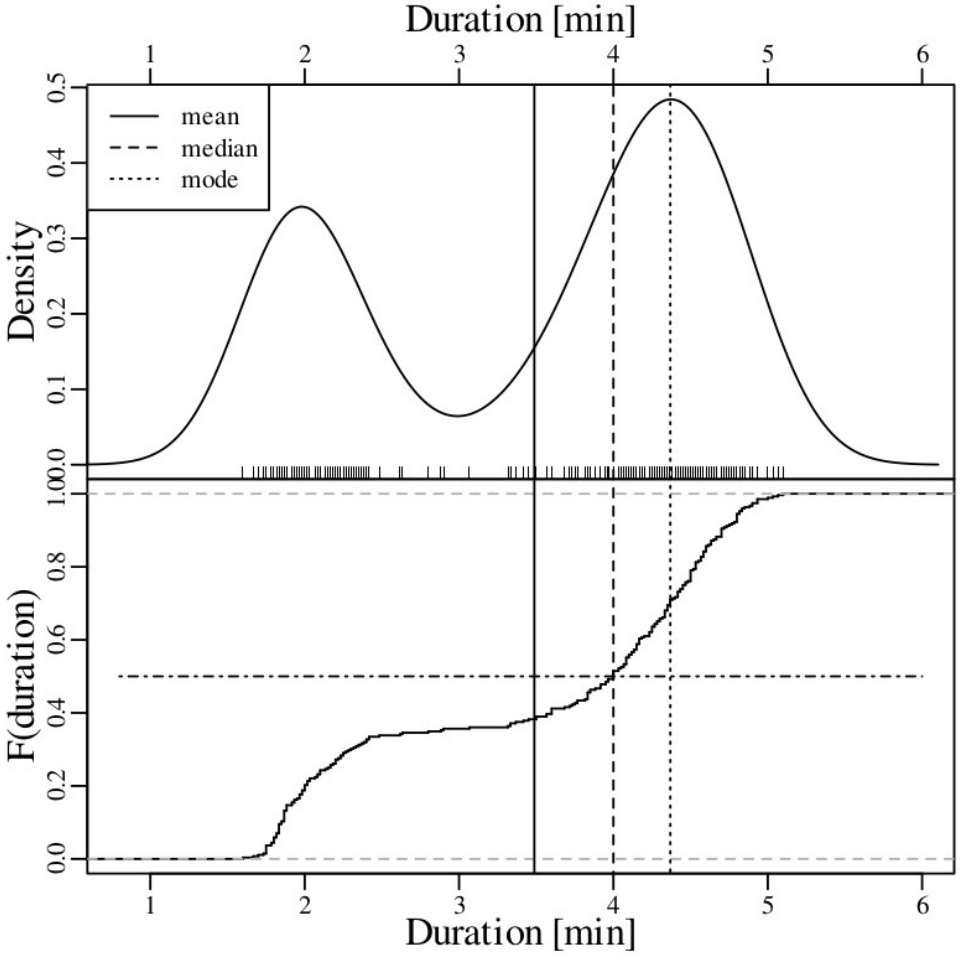
Mode




pH
3
4
5
6
7
mean
median
mode
Density
0.0
0.1
0.2
0.3
0.4
0.5
F(pH)
0.0
0.2
0.4
0.6
0.8
1.0
pH


clast size [cm]
0
2
4
6
8
10
12
Density
0
5
10
15
20
25
mean
median
mode
F(clast size)
0.0
0.2
0.4
0.6
0.8
1.0
clast size [cm]


Porosity
mean
median
mode
Density
F(porosity)
Porosity


Duration [min]
mean
median
mode
Density
F(duration)
Duration [min]

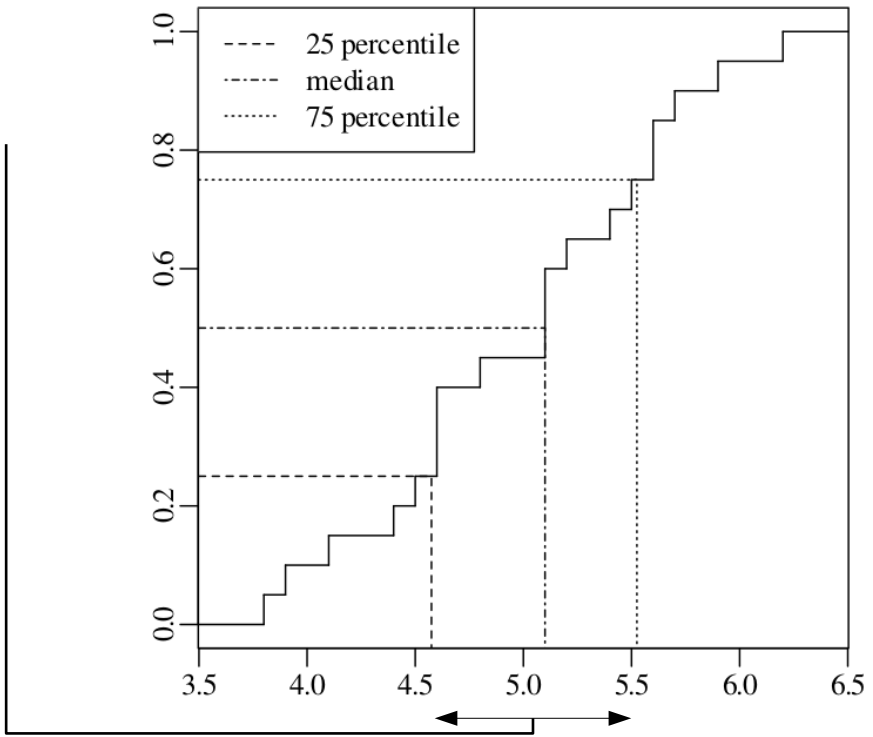
Dispersion

Standard deviation

$$s[x] = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Median Absolute Deviation $\mathrm{MAD} = \mathrm{median}|x_i - \mathrm{median}(x)|$

Interquartile range



Standard deviation

$$s[x] = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
x_i	6.2	4.4	5.6	5.2	4.5	5.4	4.8	5.9	3.9	3.8	5.1	4.1	5.1	5.5	5.1	4.6	5.7	4.6	4.6	5.6
$(x_i - \bar{x})$	1.20	-.58	.61	.21	-.49	.42	-.19	.92	-1.1	-1.2	.11	-.89	.11	.51	.11	-.39	.71	-.39	-.39	.61
$(x_i - \bar{x})^2$	1.5	.34	.38	.046	.24	.17	.034	.84	1.2	1.4	.013	.78	.013	.27	.013	.15	.51	.15	.15	.38

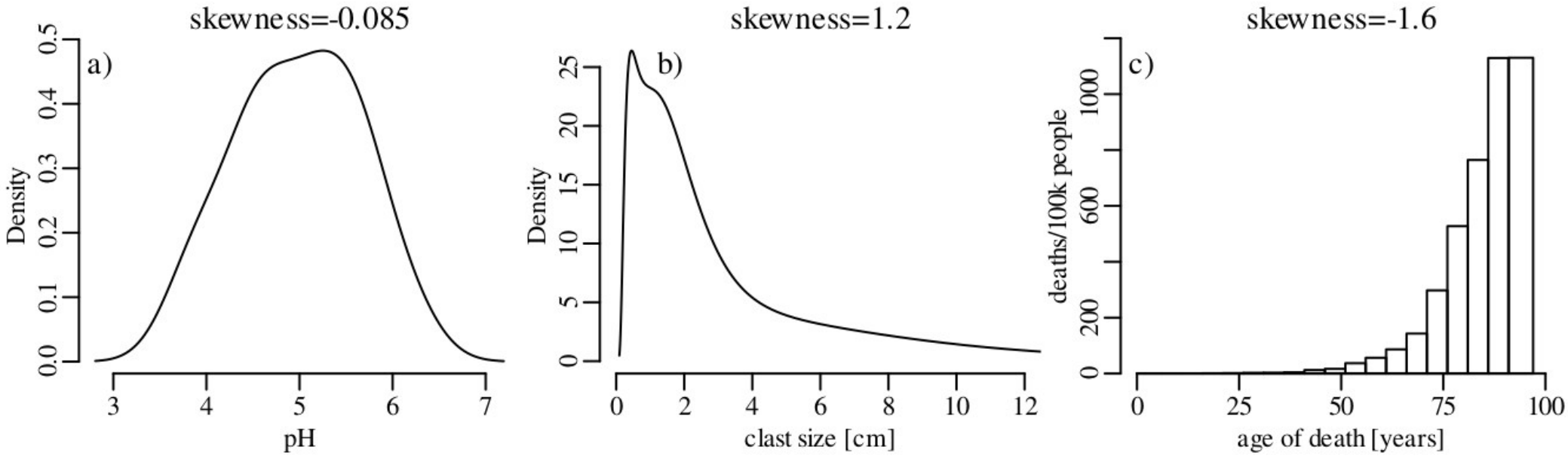
$$\sum_{i=1}^{20}(x_i - \bar{x})^2 = 8.52 \qquad s[x] = \sqrt{8.52/19} = 0.70$$

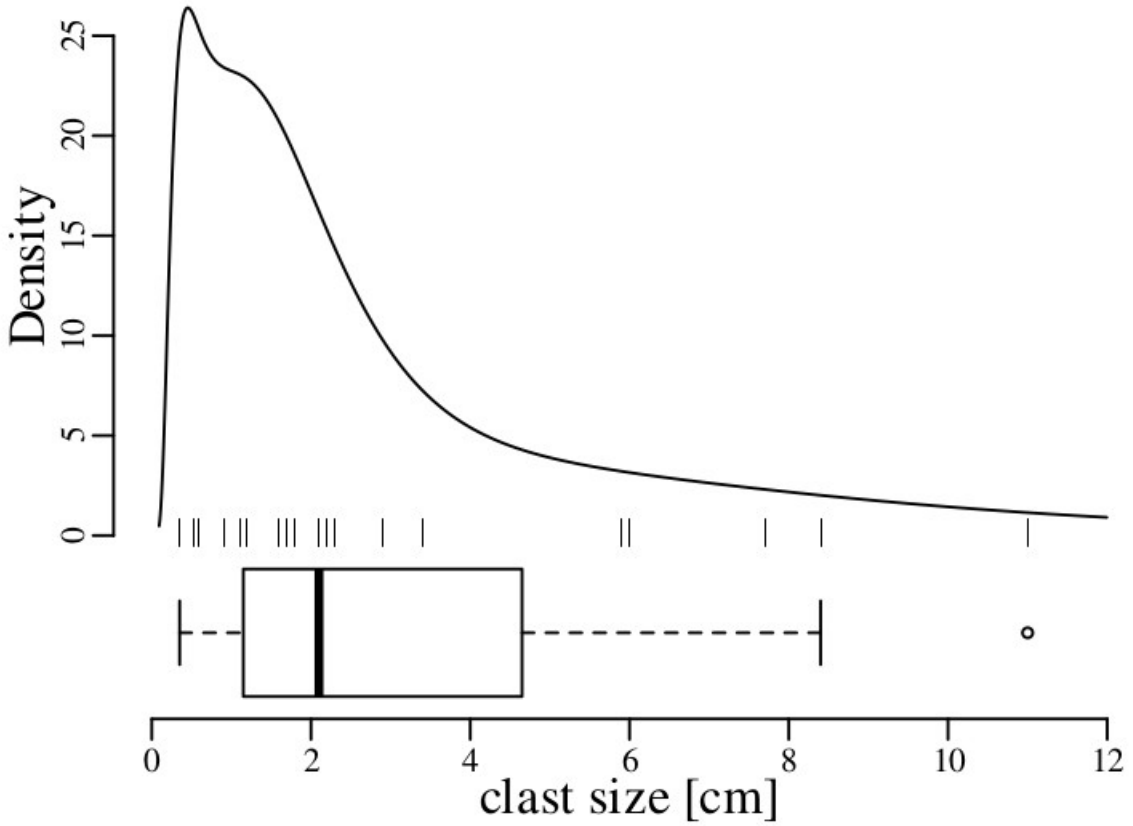
Interquartile range $\mathrm{IQR} = \boxed{5.55 - 4.55 = 1.00}$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
x_i	3.8	3.9	4.1	4.4	**4.5**	**4.6**	4.6	4.6	4.8	5.1	5.1	5.1	5.2	5.4	**5.5**	**5.6**	5.6	5.7	5.9	6.2
$x_i - \mathrm{med}(x)$	-1.3	-1.2	-1.0	-0.7	-0.6	-0.5	-0.5	-0.5	-0.3	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.6	0.8	1.1
$\lvert x_i - \mathrm{med}(x)\rvert$	1.3	1.2	1.0	0.7	0.6	0.5	0.5	0.5	0.3	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.6	0.8	1.1
sorted	0.0	0.0	0.0	0.1	0.3	0.3	0.4	0.5	0.5	**0.5**	**0.5**	0.5	0.6	0.6	0.7	0.8	1.0	1.1	1.2	1.3

Median Absolute Deviation $\mathrm{MAD} = \boxed{\mathrm{median}\lvert x_i - \mathrm{median}(x)\rvert}$

skewness $\text{skew}(x) = \frac{1}{n \cdot s[x]^3} \sum_{i=1}^{n} (x_i - \bar{x})^3$




Density
0
5
10
15
20
25
0
2
4
6
8
10
12
clast size [cm]

Statistics for geoscientists

Probability

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$\frac{\{H\}}{\{H\}\{T\}} = \frac{1}{2} = 0.5$$

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$P(2 \times H \cap 1 \times T) = \frac{\{THH\}\{HTH\}\{HHT\}}{\{HHH\}\{THH\}\{HTH\}\{HHT\}\{TTH\}\{THT\}\{HTT\}\{TTT\}} = \frac{3}{8}$$

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total number of outcomes}}$$

$$P\left(1 \times \text{⚁} \cap 1 \times \text{⚅}\right) = \frac{\{\text{⚁ ⚅}\}\{\text{⚅ ⚁}\}}{\begin{array}{cccccc} \{\text{⚀ ⚀}\} & \{\text{⚁ ⚀}\} & \{\text{⚂ ⚀}\} & \{\text{⚃ ⚀}\} & \{\text{⚄ ⚀}\} & \{\text{⚅ ⚀}\} \\ \{\text{⚀ ⚁}\} & \{\text{⚁ ⚁}\} & \{\text{⚂ ⚁}\} & \{\text{⚃ ⚁}\} & \{\text{⚄ ⚁}\} & \{\text{⚅ ⚁}\} \\ \{\text{⚀ ⚂}\} & \{\text{⚁ ⚂}\} & \{\text{⚂ ⚂}\} & \{\text{⚃ ⚂}\} & \{\text{⚄ ⚂}\} & \{\text{⚅ ⚂}\} \\ \{\text{⚀ ⚃}\} & \{\text{⚁ ⚃}\} & \{\text{⚂ ⚃}\} & \{\text{⚃ ⚃}\} & \{\text{⚄ ⚃}\} & \{\text{⚅ ⚃}\} \\ \{\text{⚀ ⚄}\} & \{\text{⚁ ⚄}\} & \{\text{⚂ ⚄}\} & \{\text{⚃ ⚄}\} & \{\text{⚄ ⚄}\} & \{\text{⚅ ⚄}\} \\ \{\text{⚀ ⚅}\} & \{\text{⚁ ⚅}\} & \{\text{⚂ ⚅}\} & \{\text{⚃ ⚅}\} & \{\text{⚄ ⚅}\} & \{\text{⚅ ⚅}\} \end{array}} = \frac{2}{36} = \frac{1}{18}$$

multiplicative rule

$$P\left(\{2 \times H \cap 1 \times T\} \cap \{1 \times \square \cap 1 \times \square\}\right) = \frac{3}{8}\frac{1}{18} = \frac{3}{144} = 0.021$$

additive rule

$$P\left(\{2 \times H \cap 1 \times T\} \cup \{1 \times \square \cap 1 \times \square\}\right) = \frac{3}{8} + \frac{1}{18} = \frac{31}{72} = 0.43$$

Permutations

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
sampling with replacement							5	2	12	19	10	5	3	19	11	4			

$$\overbrace{n \times n \times \ldots \times n}^{k \text{ times}} = n^k$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

sampling with replacement $\boxed{5}$ 2 12 $\boxed{19}$ 10 $\boxed{5}$ 3 $\boxed{19}$ 11 4

$$\overbrace{n \times n \times \ldots \times n}^{k \text{ times}} = n^k$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

sampling without replacement 15 8 13 5 4 19 7 1 20 10

$$n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$$

what is the probability that two students in a classroom of k

celebrate their birthdays on the same day?

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celebrate their birthdays on the same day?

365^k possible combinations of birthdays (sampling with replacement)

$365!/(365-k)!$ of these combinations do not overlap (sampling without replacement)

what is the probability that two students in a classroom of k

celebrate their birthdays on the same day?

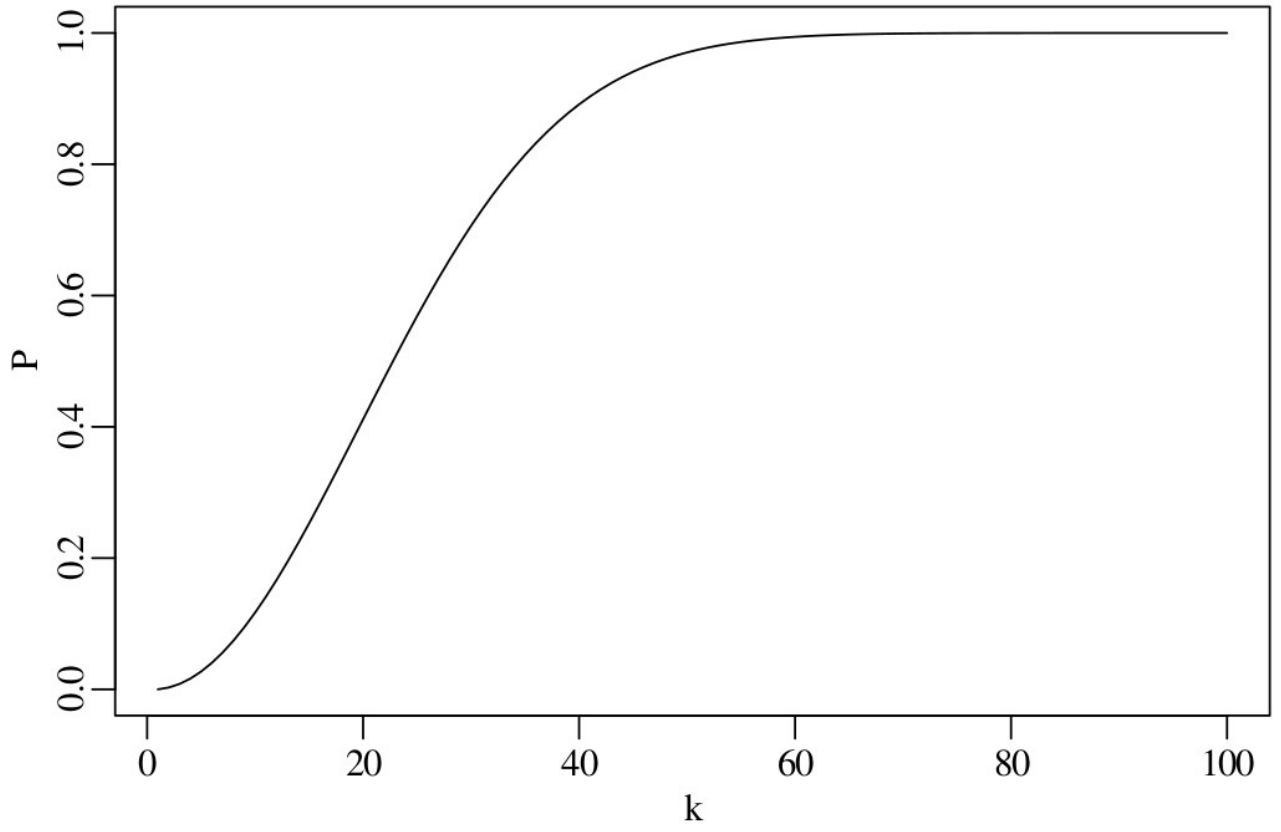
365^k possible combinations of birthdays (sampling with replacement)

$365!/(365-k)!$ of these combinations do not overlap (sampling without replacement)

$$P(\text{no overlapping birthdays}) = \frac{365!}{(365-k)!365^k}$$

$$P(>1 \text{ overlapping birthdays}) = 1 - \frac{365!}{(365-k)!365^k}$$

$$P(> 1 \text{ overlapping birthdays}) = 1 - \frac{365!}{(365-k)!365^k}$$

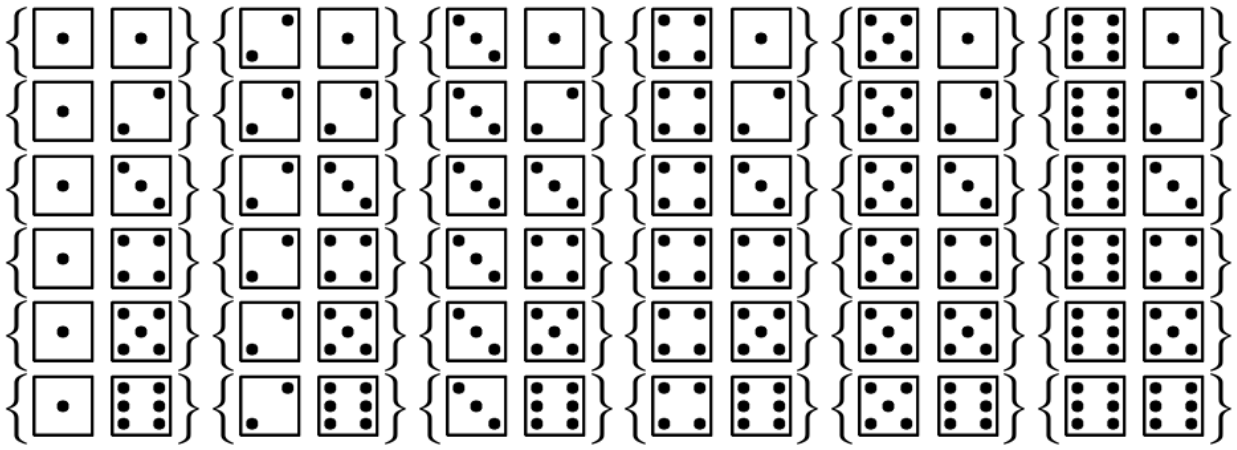


If $k = 23$, then $P(> 1 \text{ overlapping birthdays}) = 0.507$

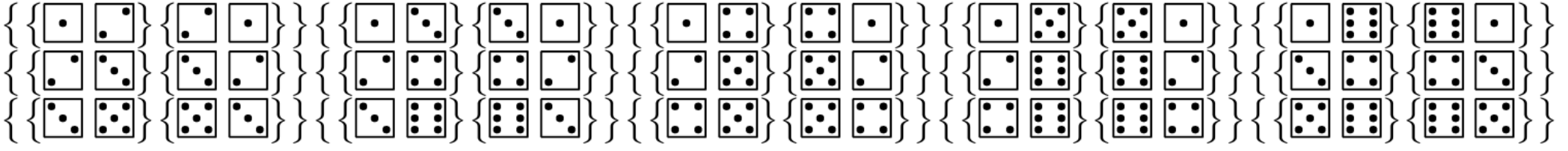
Combinations

$$\{HHH\}\{THH\}\{HTH\}\{HHT\}\{TTH\}\{THT\}\{HTT\}\{TTT\}$$

duplicates: $\{\{HTT\}\{THT\}\{HHT\}\}$ and $\{\{THH\}\{HTH\}\{TTH\}\}$



duplicates:



$$(\#\text{ ordered samples}) = (\#\text{ unordered samples}) \times (\#\text{ ways to order the samples})$$

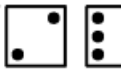
$$(\#\text{ unordered samples}) = \frac{(\#\text{ ordered samples})}{(\#\text{ ways to order the samples})}$$

$$= \frac{n!/(n-k)!\text{ ways to select } k \text{ objects from a collection of } n}{k!\text{ ways to order these } k \text{ objects}} = \frac{n!}{(n-k)!k!}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$\{THH\}\{HTH\}\{HHT\}$

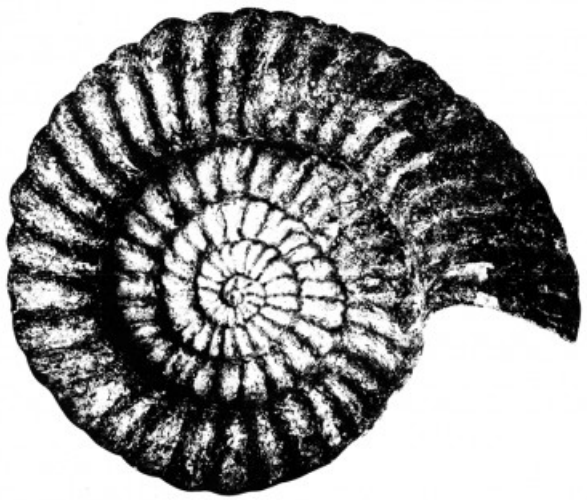
$\{$  $\}\{$  $\}$

$$\binom{3}{2} = \frac{3!}{1!2!} = \frac{6}{2} = 3$$

$$\binom{2}{1} = \frac{2!}{1!1!} = 2$$

Conditional probability

$P(A|B)$ = "The conditional probability of A given B"



JURASSIC			
	LATE	TITHONIAN	144
			151
		KIMMERIDGIAN	154
		OXFORDIAN	159
	MIDDLE	CALLOVIAN	164
		BATHONIAN	169
		BAJOCIAN	176
		AALENIAN	180
	EARLY	TOARCIAN	190
		PLIENSBACHIAN	195
		SINEMURIAN	202
		HETTANGIAN	206

multiplication law

$$P(A|B) = \text{“The conditional probability of } A \text{ given } B\text{”}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$$

$$\left.\begin{aligned} P(B) &= 0.7 \\ P(A|B) &= 0.2 \end{aligned}\right\} \; 0.7 \times 0.2 = 14\%$$

law of total probability

$$P(A|B) = \text{“The conditional probability of } A \text{ given } B\text{”}$$

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

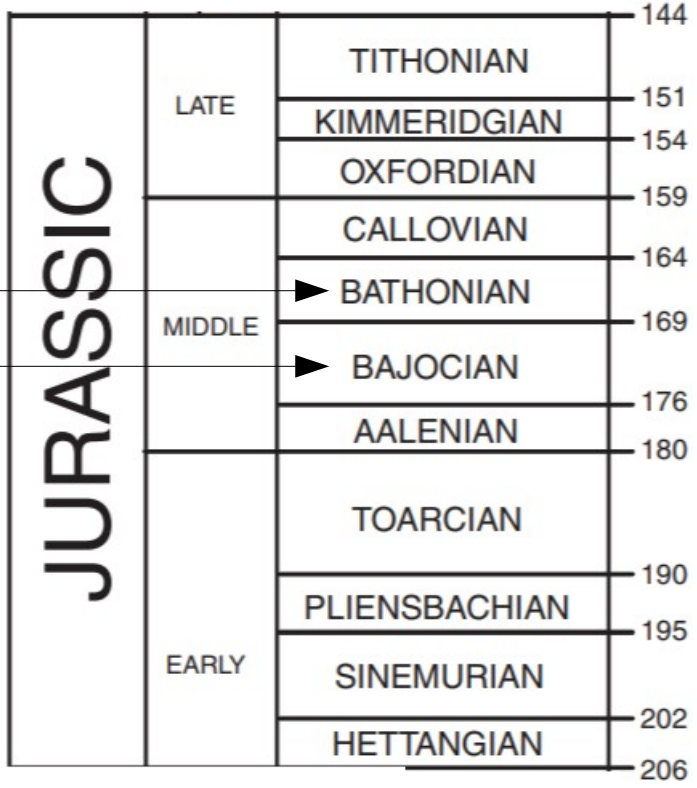
law of total probability

$P(A|B)$ = "The conditional probability of A given B"

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

$$P(B_2) = 0.3$$
$$P(B_1) = 0.7$$

$$P(A|B_2) = 0.5$$
$$P(A|B_1) = 0.2$$



$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = 0.2 \times 0.7 + 0.5 \times 0.3 = 0.29$$

multiplication law $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$

Bayes' Rule $P(B|A) = \dfrac{P(A|B)P(B)}{P(A)}$

multiplication law $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) = P(B \cap A)$

law of total probability $P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Bayes' Rule $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Bayes' Rule

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Suppose that we have found an ammonite fossil in the river bed. What is its likely age?

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.5 \times 0.3} = 0.48$$

The binomial distribution

Bernoulli variable

1. a coin may land on its head (1) or tail (0);
2. a die may land on ⚅ (1) or not (0);
3. a 'wildcat' exploration well may find petroleum (1) or be dry (0).

five gold claims

$$p = 2/3$$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$

five gold claims $\qquad p = 2/3$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$

$$P(1 \times \text{gold}) = P(10000) + P(01000) + P(00100) + P(00010) + P(00001)$$

$$P(10000) = (2/3)(1/3)^4 = 0.0082$$
$$P(01000) = (1/3)(2/3)(1/3)^3 = 0.0082$$
$$P(00100) = (1/3)^2(2/3)(1/3)^2 = 0.0082$$
$$P(00010) = (1/3)^3(2/3)(1/3) = 0.0082$$
$$P(00001) = (1/3)^4(2/3) = 0.0082$$

$$P(1 \times \text{gold}) = \binom{5}{1}(2/3)(1/3)^4 = 5 \times 0.0082 = 0.041$$

five gold claims $\qquad p = 2/3$

$$P(0 \times \text{gold}) = P(00000) = (1/3)^5 = 0.0041$$

$$P(1 \times \text{gold}) = \binom{5}{1}(2/3)(1/3)^4 = 5 \times 0.0082 = 0.041$$

$$P(2 \times \text{gold}) = \binom{5}{2}(2/3)^2(1/3)^3 = 10 \times 0.016 = 0.16$$

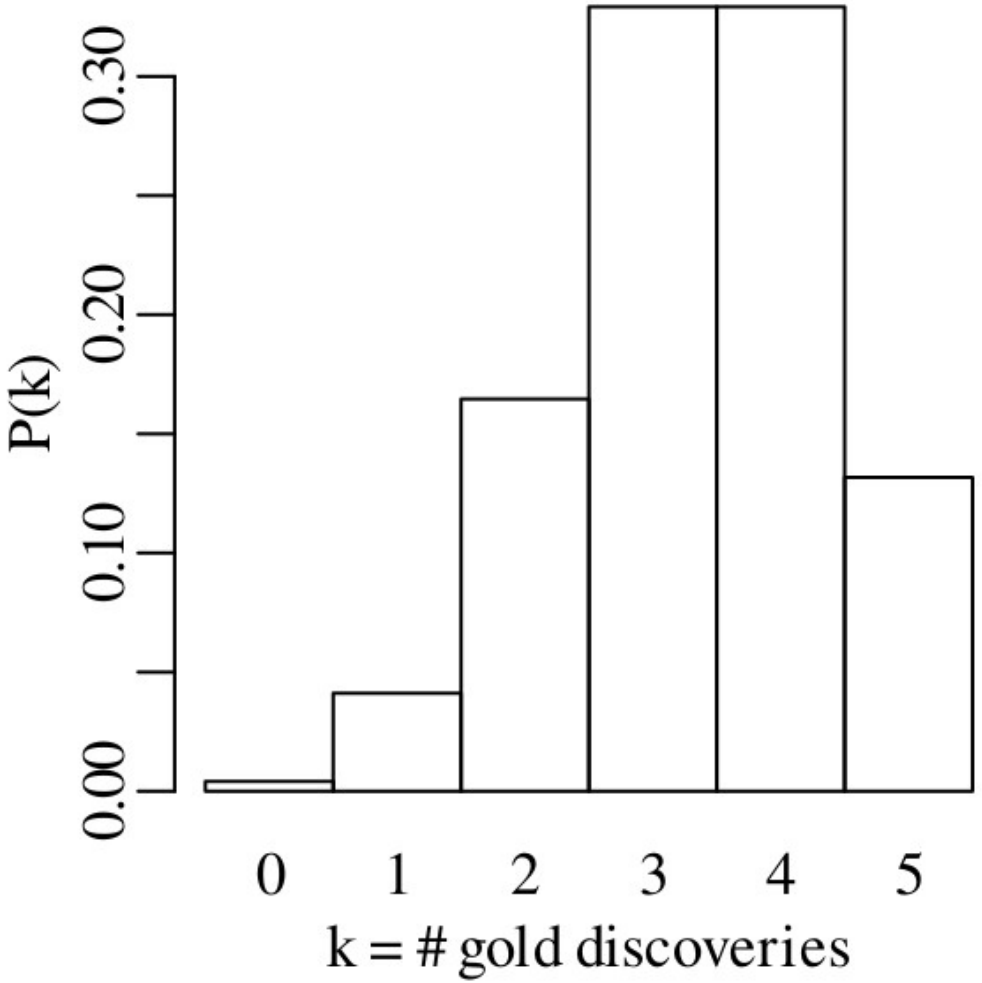
$$P(3 \times \text{gold}) = \binom{5}{3}(2/3)^3(1/3)^2 = 10 \times 0.033 = 0.33$$

$$P(4 \times \text{gold}) = \binom{5}{4}(2/3)^4(1/3) = 5 \times 0.066 = 0.33$$

$$P(5 \times \text{gold}) = (2/3)^5 = 0.13$$

probability mass function (PMF)

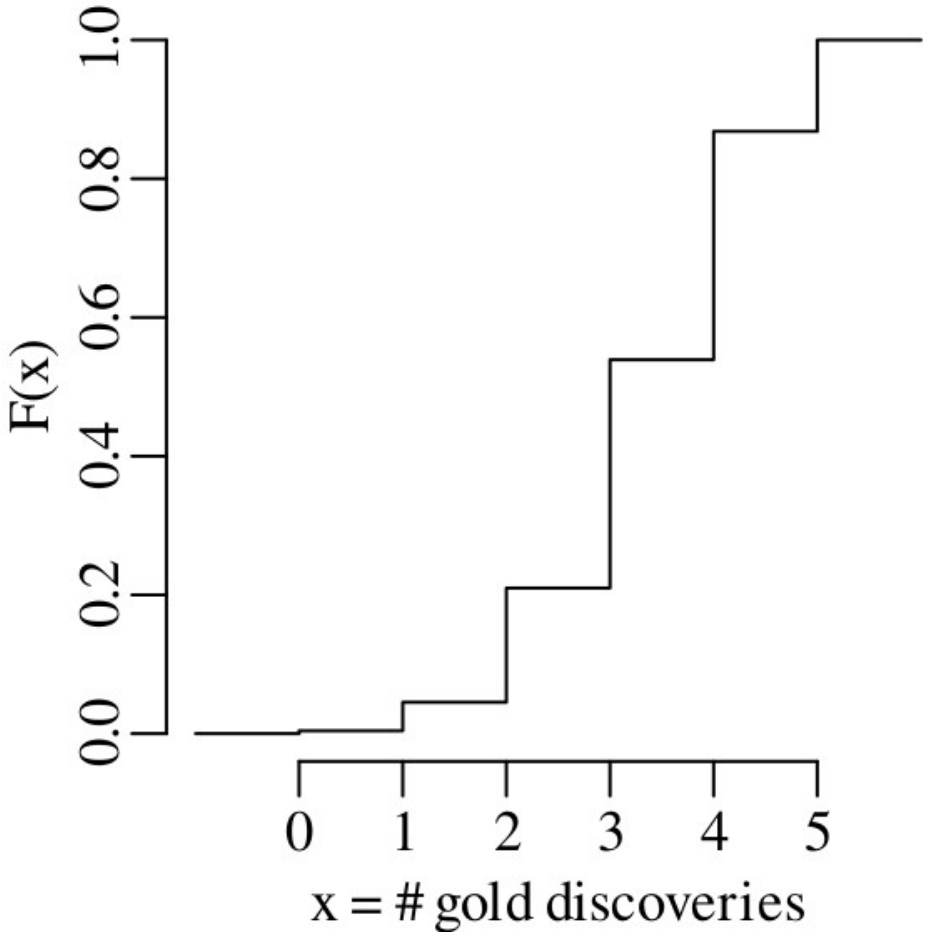
$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$



cumulative distribution function (CDF)

$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$F(x) = P(X \leq x)$$



method of maximum likelihood

$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathcal{L}(p|n,k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\hat{p} = \frac{k}{n} \qquad \text{e.g.} \qquad \hat{p} = \frac{2}{5} = 0.4$$

Hypothesis tests

$$\hat{p} = 2/5 \quad \longleftrightarrow \quad p = 2/3$$

1. Formulate two hypotheses:

   $H_\circ$ (**null hypothesis**) $\qquad p = 2/3$

   H_a (**alternative hypothesis**): $\qquad p < 2/3$

2. Calculate the **test statistic** T (k)

3. Determine the **null distribution** of T under $H_\circ$.

k	0	1	*2*	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	0.0041	0.0453	*0.2099*	0.5391	0.8683	1.0000

p-value

4. Choose a **significance level** ($\alpha = 0.05$)

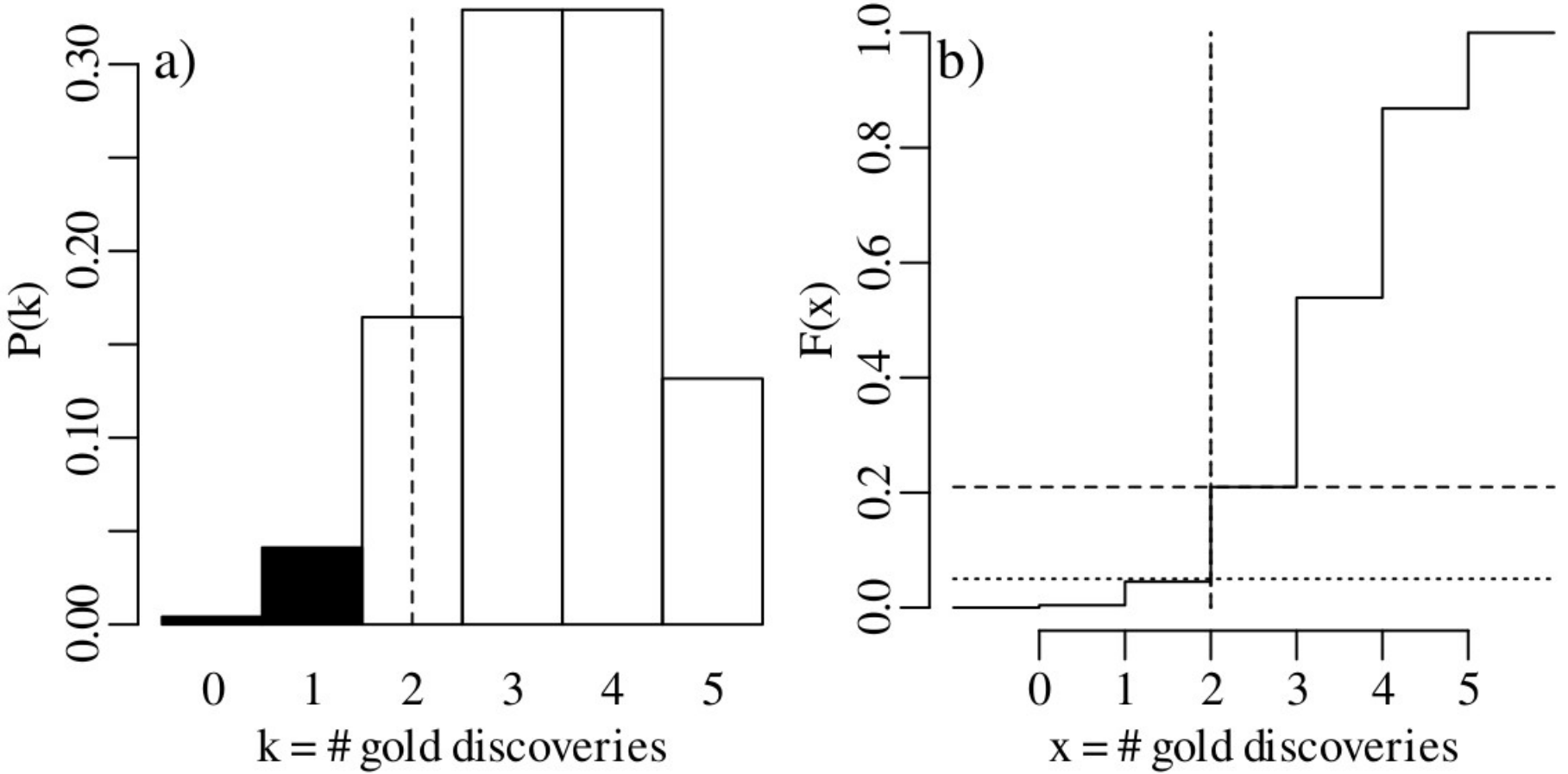
5. Mark the **rejection region**

k	**0**	**1**	*2*	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	**0.0041**	**0.0453**	*0.2099*	0.5391	0.8683	1.0000

rejection region $R = \{0, 1\}$

6. Reach a **decision** $k = 2 \notin R$

7. Alternatively, $0.2099 > \alpha$


a)
P(k)
0.00
0.10
0.20
0.30
0
1
2
3
4
5
k = # gold discoveries
b)
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
x = # gold discoveries

$$\hat{p} = 2/5 \quad \longleftrightarrow \quad p = 2/3$$

one-sided hypothesis test

$H_\circ$ (null hypothesis) $\qquad p = 2/3$

H_a (alternative hypothesis): $\qquad p < 2/3$

two-sided hypothesis test

$H_\circ$ (null hypothesis) $\qquad p = 2/3$

H_a (alternative hypothesis): $\qquad p \neq 2/3$

2. Calculate the **test statistic** $T \quad (k)$

3. Determine the **null distribution** of T under $H_\circ$.

k	0	1	*2*	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	0.0041	0.0453	*0.2099*	0.5391	0.8683	1.0000
$P(T \geq k)$	1.000	0.9959	*0.9547*	0.7901	0.4609	0.1317

4. Choose a **significance level** ($\alpha = 0.05$)

but evaluated twice at $\alpha/2$

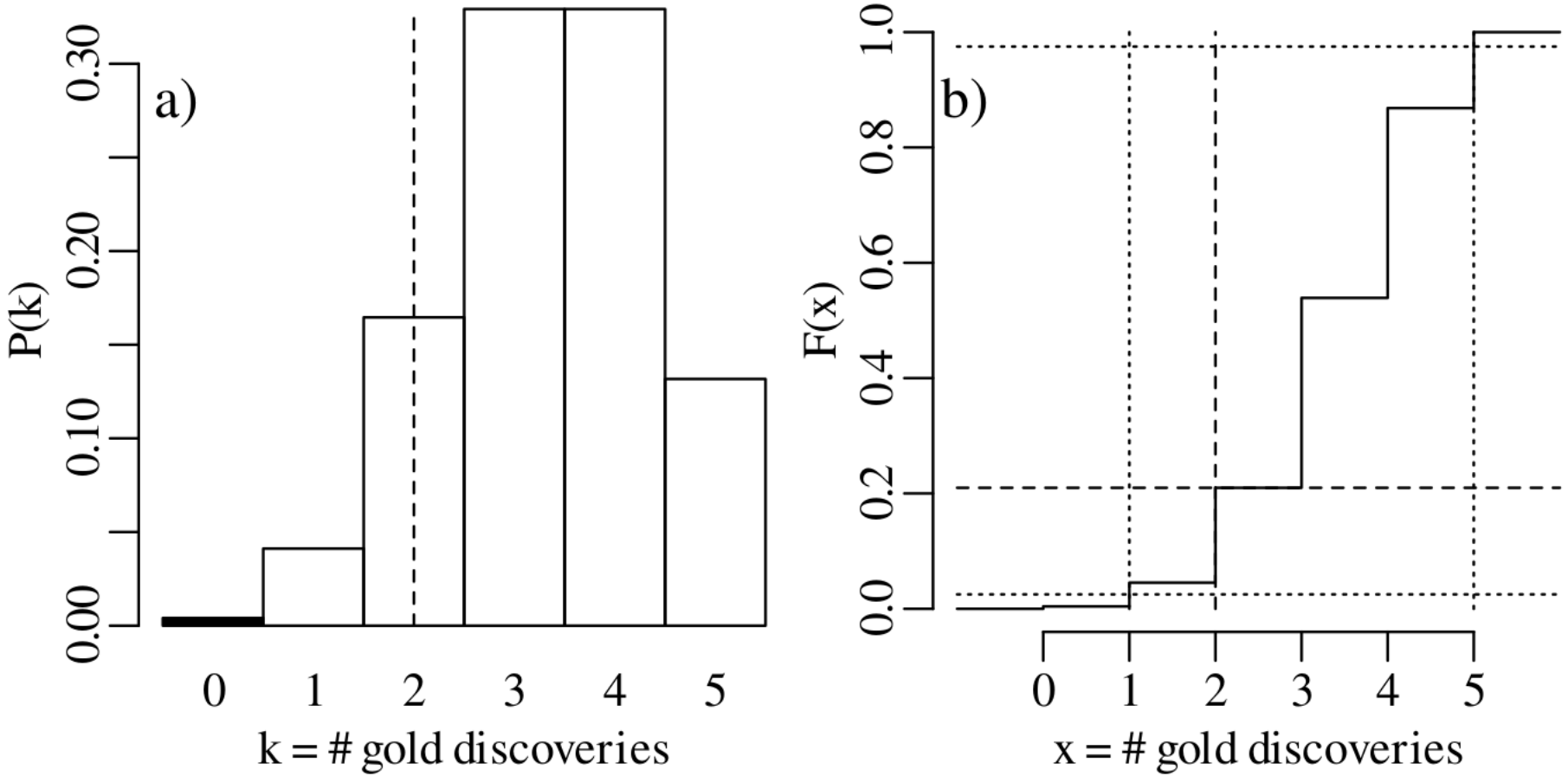
5. Mark the **rejection region**

k	**0**	1	*2*	3	4	5
$P(T = k)$	0.0041	0.0411	0.1646	0.3292	0.3292	0.1317
$P(T \leq k)$	**0.0041**	0.0453	*0.2099*	0.5391	0.8683	1.0000
$P(T \geq k)$	1.000	0.9959	*0.9547*	0.7901	0.4609	0.1317

$$R = \{0\}$$

6. Reach a **decision** $k = 2 \notin R$

7. Alternatively, $2 \times 0.2099 = 0.4198 > 0.05$



Statistical power

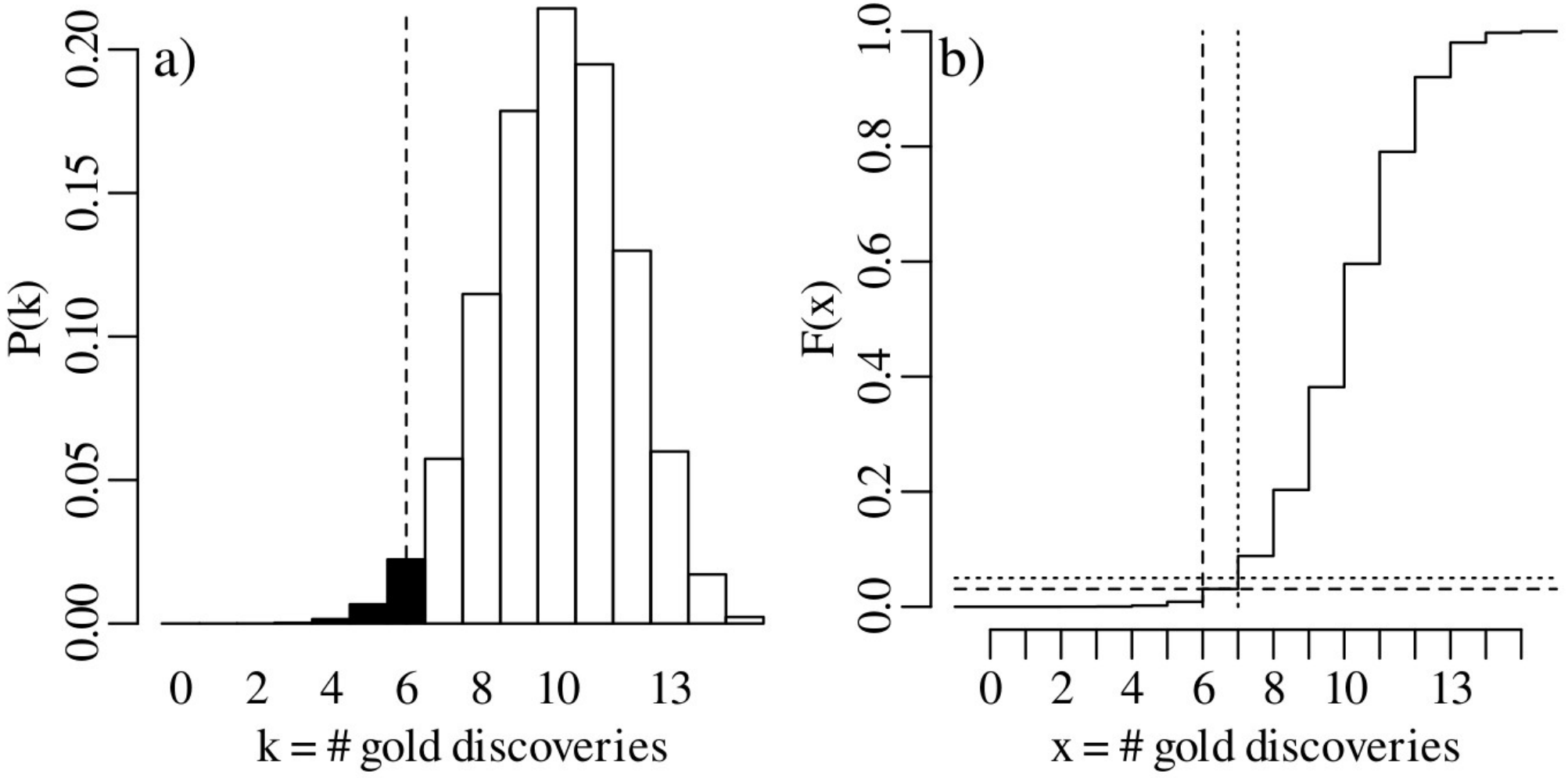
We failed to reject $H_\circ$ for $\hat{p} = \frac{2}{5} = 0.4$. How about $\hat{p} = \frac{6}{15} = 0.4$?

one-sided hypothesis test $\quad (H_\circ : p = 2/3 \text{ vs. } H_a : p < 2/3)$

k	**0**	**1**	**2**	**3**	**4**	**5**	***6***	7
$P(T = k)$	7.0×10^{-8}	2.1×10^{-6}	2.9×10^{-5}	2.5×10^{-4}	0.0015	0.0067	0.0223	0.0574
$P(T \leq k)$	$\mathbf{7.0\times10^{-8}}$	$\mathbf{2.2\times10^{-6}}$	$\mathbf{3.1\times10^{-5}}$	$\mathbf{2.8\times10^{-4}}$	**0.0018**	**0.0085**	***0.0308***	0.0882
k	8	9	10	11	12	13	14	15
$P(T = k)$	0.1148	0.1786	0.2143	0.1948	0.1299	0.0599	0.0171	0.0023
$P(T \leq k)$	0.2030	0.3816	0.5959	0.7908	0.9206	0.9806	0.9977	1.0000

$$R = \{0, 1, 2, 3, 4, 5, 6\}$$

$$k \in R \qquad \text{p-value} = 0.0308 < \alpha$$

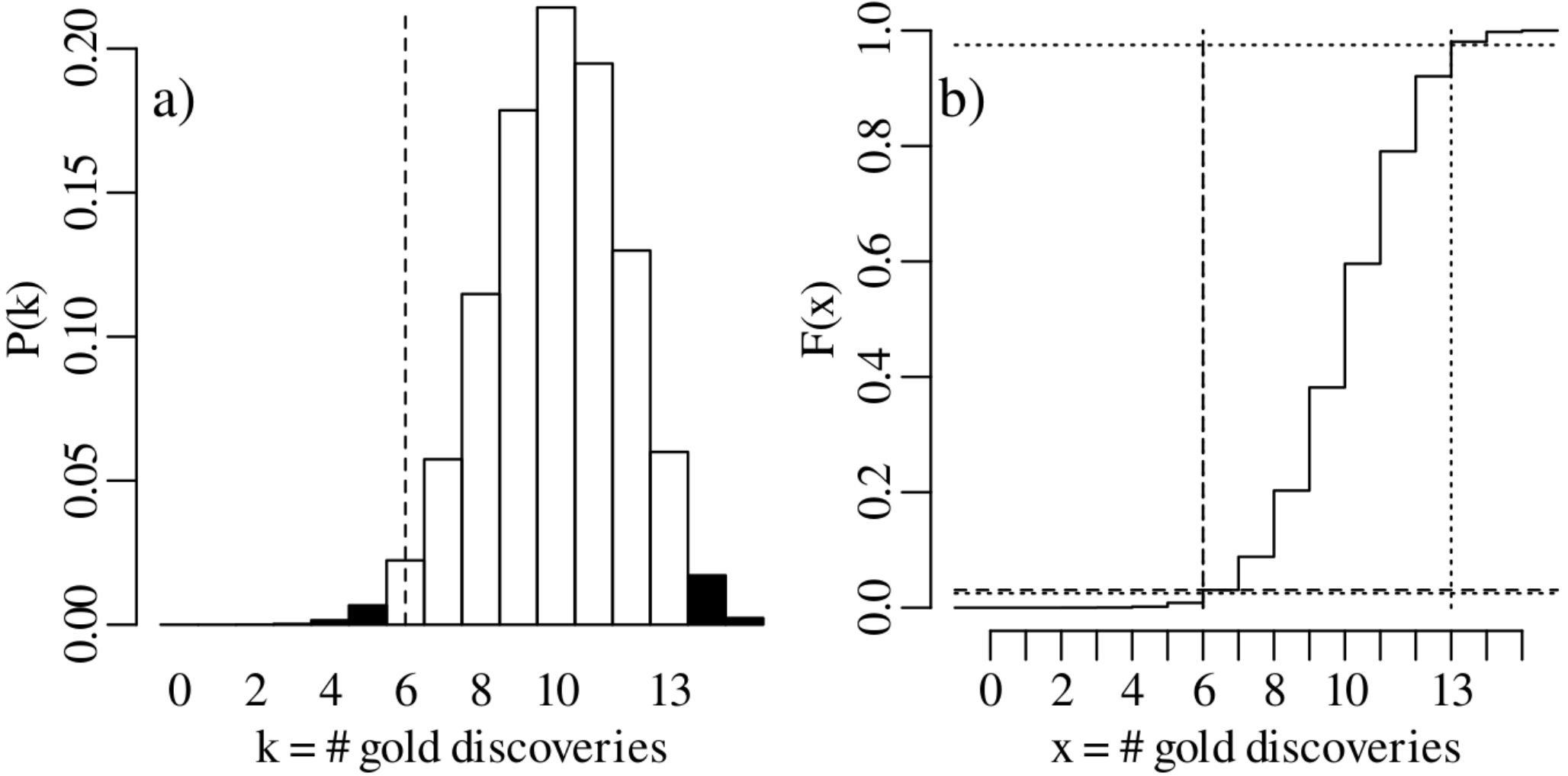

a)
P(k)
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
13
k = # gold discoveries
b)
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
13
x = # gold discoveries

two-sided hypothesis test ($H_\circ : p = 2/3$ vs. $H_a : p \neq 2/3$):

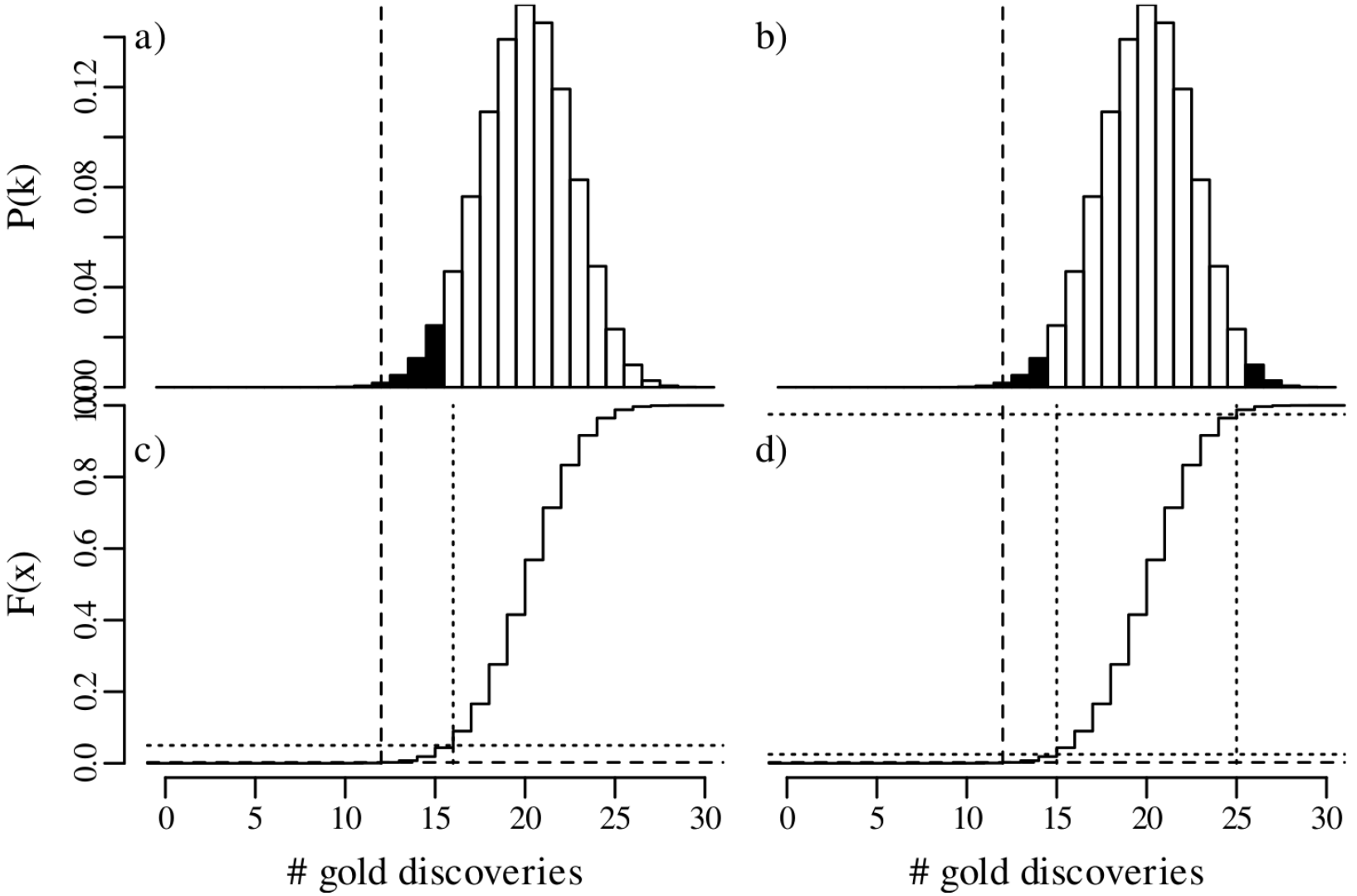
k	**0**	**1**	**2**	**3**	**4**	**5**	*6*	7
$P(T = k)$	7.0×10^{-8}	2.1×10^{-6}	2.9×10^{-5}	2.5×10^{-4}	0.0015	0.0067	0.0223	0.0574
$P(T \leq k)$	$\mathbf{7.0\times10^{-8}}$	$\mathbf{2.2\times10^{-6}}$	$\mathbf{3.1\times10^{-5}}$	$\mathbf{2.8\times10^{-4}}$	**0.0018**	**0.0085**	*0.0308*	0.0882
$P(T \geq k)$	1.0000	1-7.0×10^{-8}	1-2.2×10^{-6}	1-3.1×10^{-5}	1-2.8×10^{-4}	0.9982	0.9915	0.9692
k	8	9	10	11	12	13	**14**	**15**
$P(T = k)$	0.1148	0.1786	0.2143	0.1948	0.1299	0.0599	0.0171	0.0023
$P(T \leq k)$	0.2030	0.3816	0.5959	0.7908	0.9206	0.9806	0.9977	1.0000
$P(T > k)$	0.9118	0.7970	0.6184	0.4041	0.2092	0.0794	**0.0194**	**0.0023**

$$R = \{0, 1, 2, 3, 4, 5, 14, 15\}$$

$k \notin R$ p-value $= (2 \times 0.0308 =) 0.0616 > \alpha$


a)
P(k)
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
13
k = # gold discoveries
b)
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
13
x = # gold discoveries

Next, how about $\hat{p} = \frac{12}{30} = 0.4$?



$H_\circ$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

the accused is ...	guilty	innocent
sentenced	correct decision	Type-I error
acquitted	Type-II error	correct decision

depends on the **confidence level** α

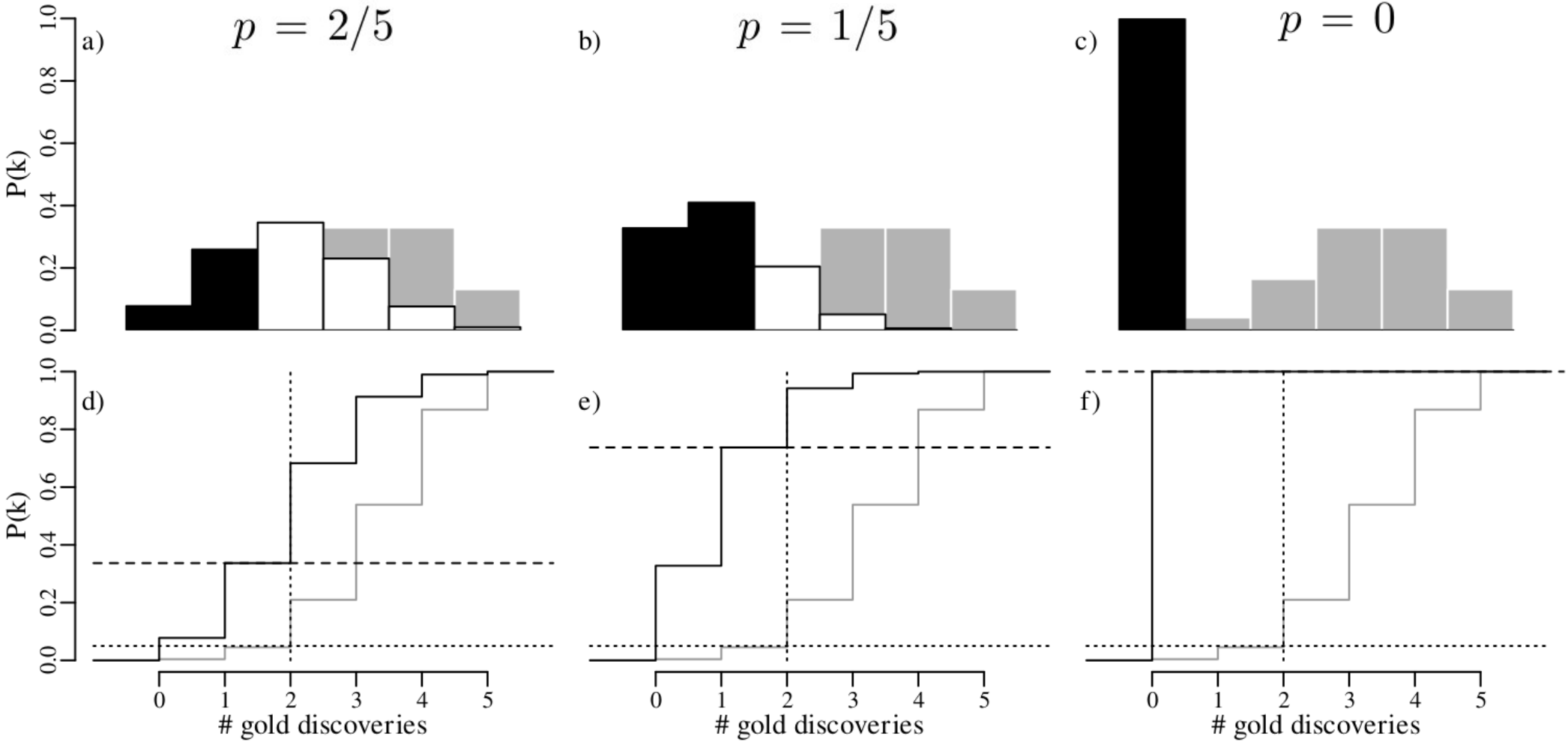
$H_\circ$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$\beta = 1 - \text{power}$

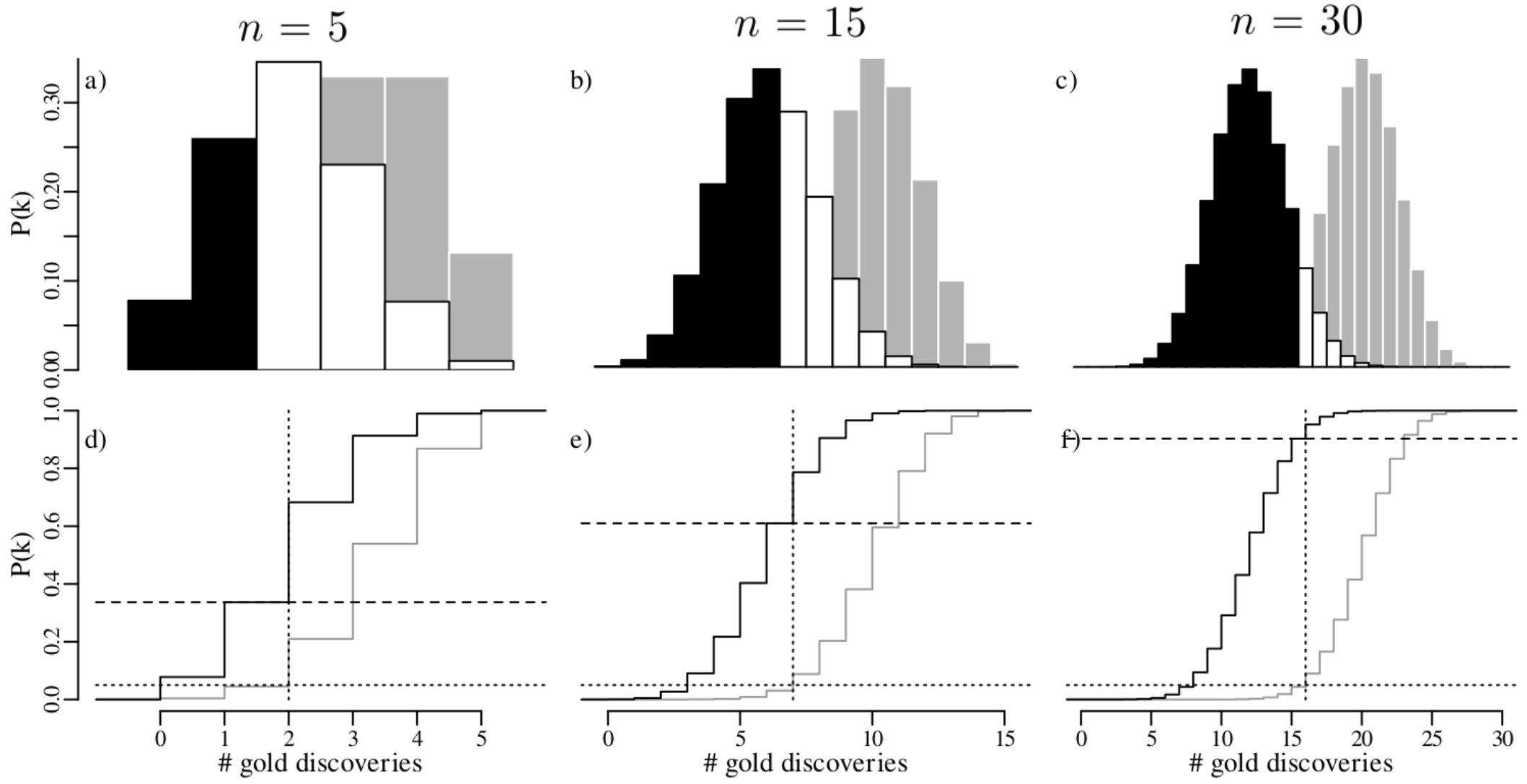
depends on 1. **The degree to which $H_\circ$ is false**

2. **Sample size**

1. The degree to which $H_\circ$ is false



2. Sample size



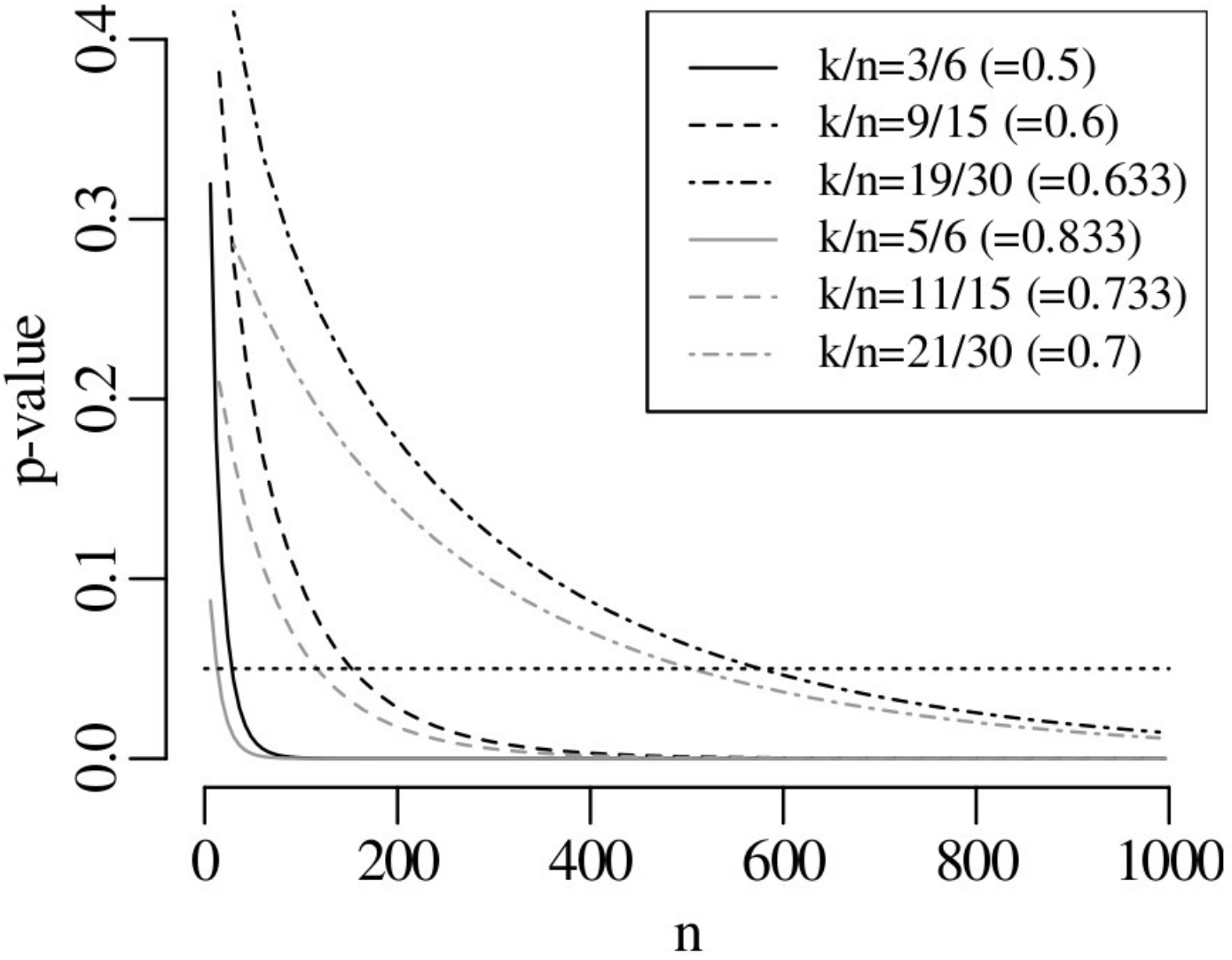
The scientific method

1. Formulate a hypothesis.
2. Design an experiment to test said hypothesis.
3. Carry out the experiment and check to see if it matches the prediction.

1. Hypothesis: Earth's lower mantle is made of olivine.
2. Test: Study the stability of olivine at lower mantle pressures (24-136 GPa).
3. Result: Olivine is not stable at lower mantle pressures.

1. Hypothesis: Earth's lower mantle is made of perovskite.
2. Test: Study the stability of perovskite at lower mantle pressures.
3. Result: Perovskite is stable at lower mantle pressures.

$p = 2/3$?


k/n=3/6 (=0.5)
k/n=9/15 (=0.6)
k/n=19/30 (=0.633)
k/n=5/6 (=0.833)
k/n=11/15 (=0.733)
k/n=21/30 (=0.7)
p-value
0.0
0.1
0.2
0.3
0.4
0
200
400
600
800
1000
n

Pitfalls

All hypotheses are wrong ... in some decimal place – John Tukey (paraphrased)

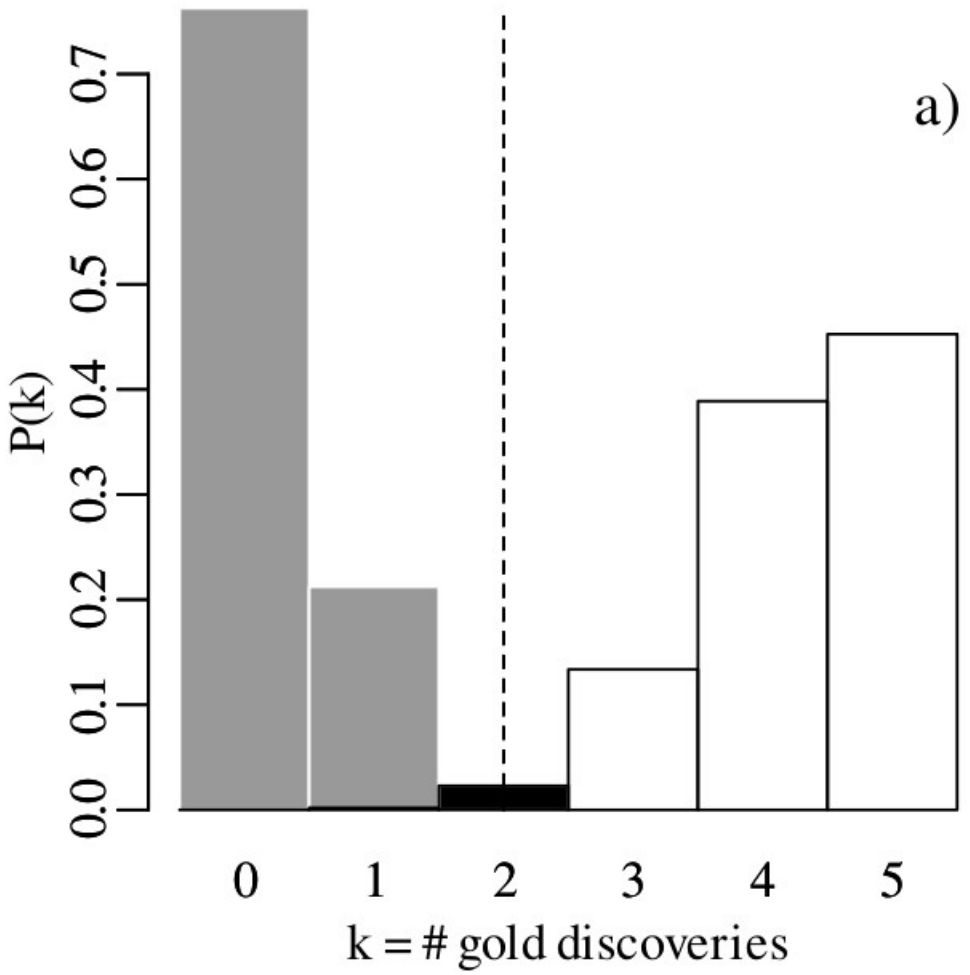
All models are wrong, but some are useful – George Box

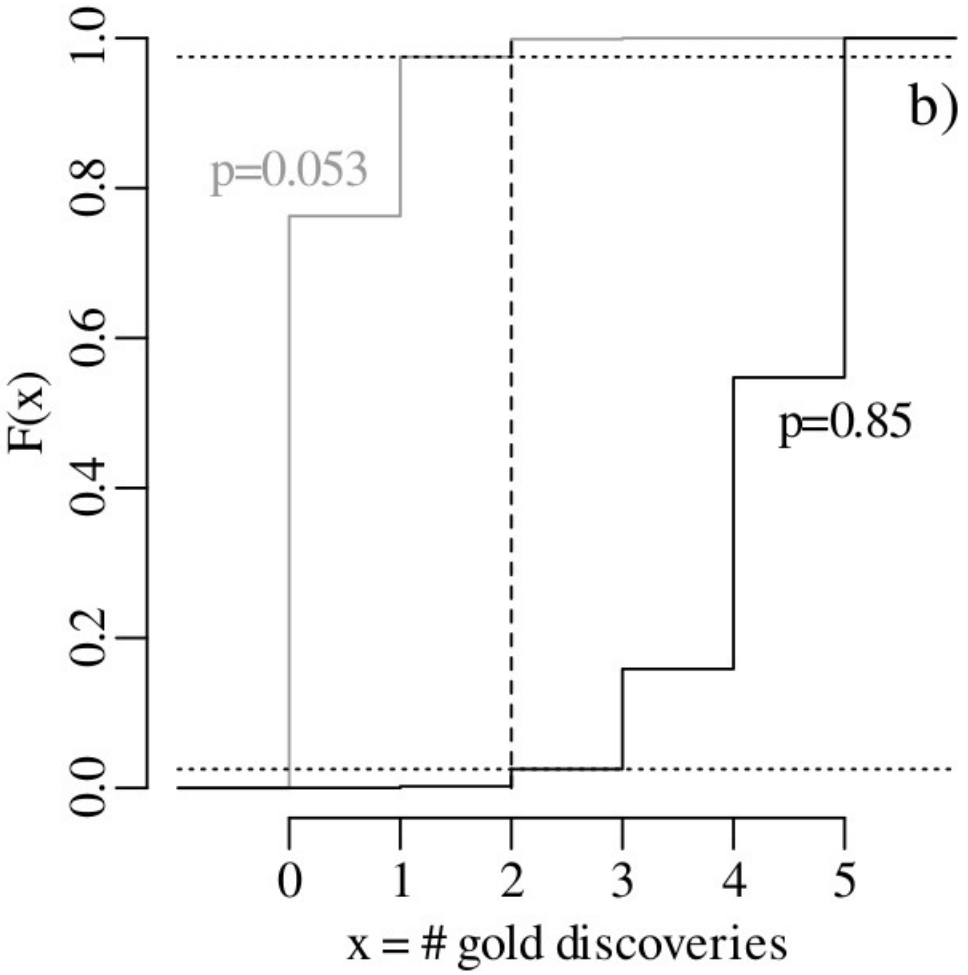
Confidence intervals

Which values of p are compatible with the observation of $k = 2$ successes out of $n = 5$ claims?

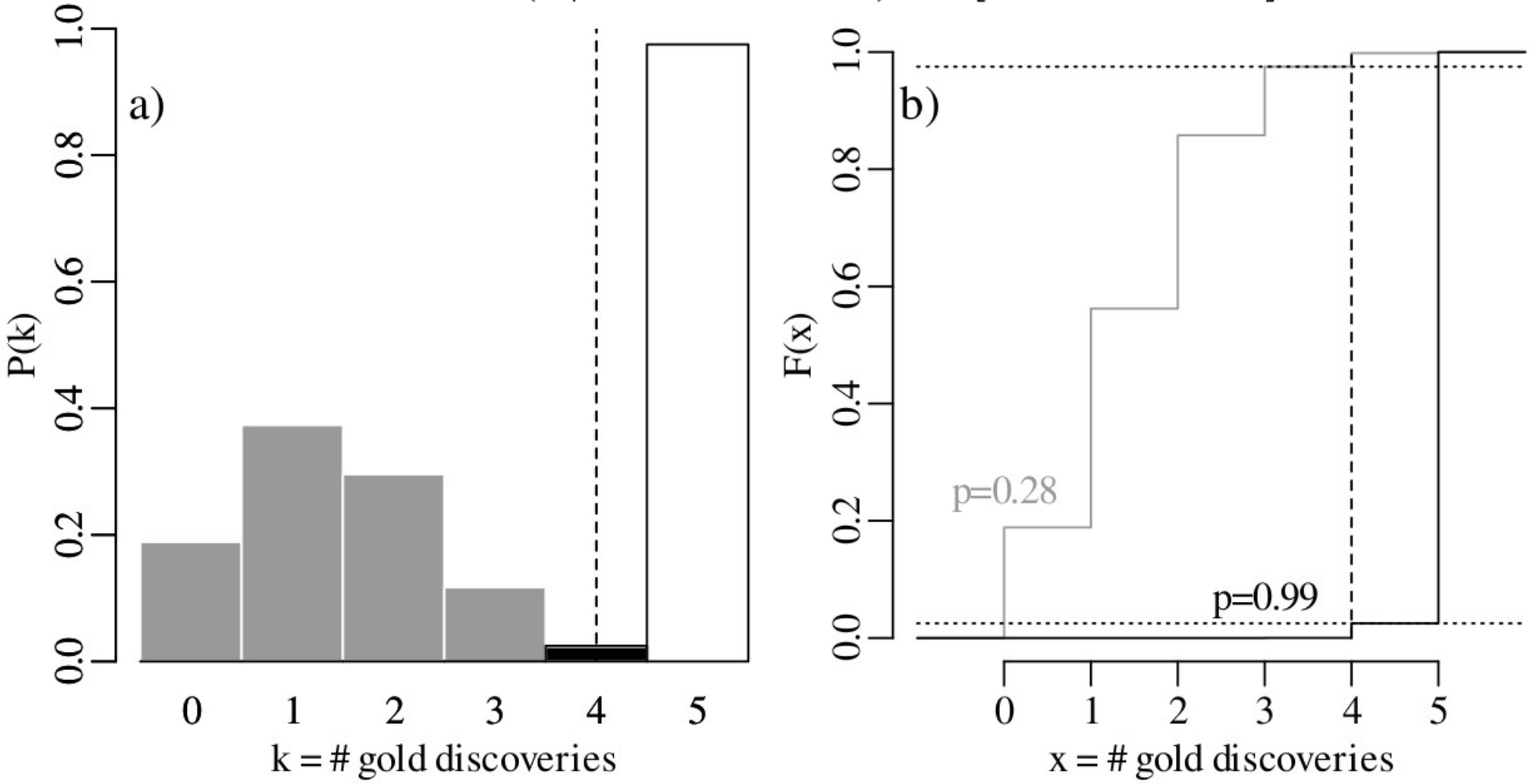
1. ~~**p=0?**~~
2. **p=0.1?** $P(k \geq 2|p = 0.1, n = 5) = \sum_{i=2}^{5} \binom{5}{i} (0.1)^i (0.9)^{n-i} = 0.081 \geq \alpha/2$
3. **p=2/5?** $= \hat{p}$
4. **p=2/3?** $P(k \leq 2|p = 2/3, n = 5) = \sum_{i=0}^{2} \binom{5}{i} (2/3)^i (1/3)^{n-i} = 0.21 \geq \alpha/2$
5. ~~**p=0.9?**~~ $P(k \leq 2|p = 0.9, n = 5) = \sum_{i=0}^{2} \binom{5}{i} (0.9)^i (0.1)^{n-i} = 0.0086 \leq \alpha/2$
6. ~~**p=1?**~~

$$\text{C.I.}(p|k=2,n=5) = [0.053, 0.85]$$

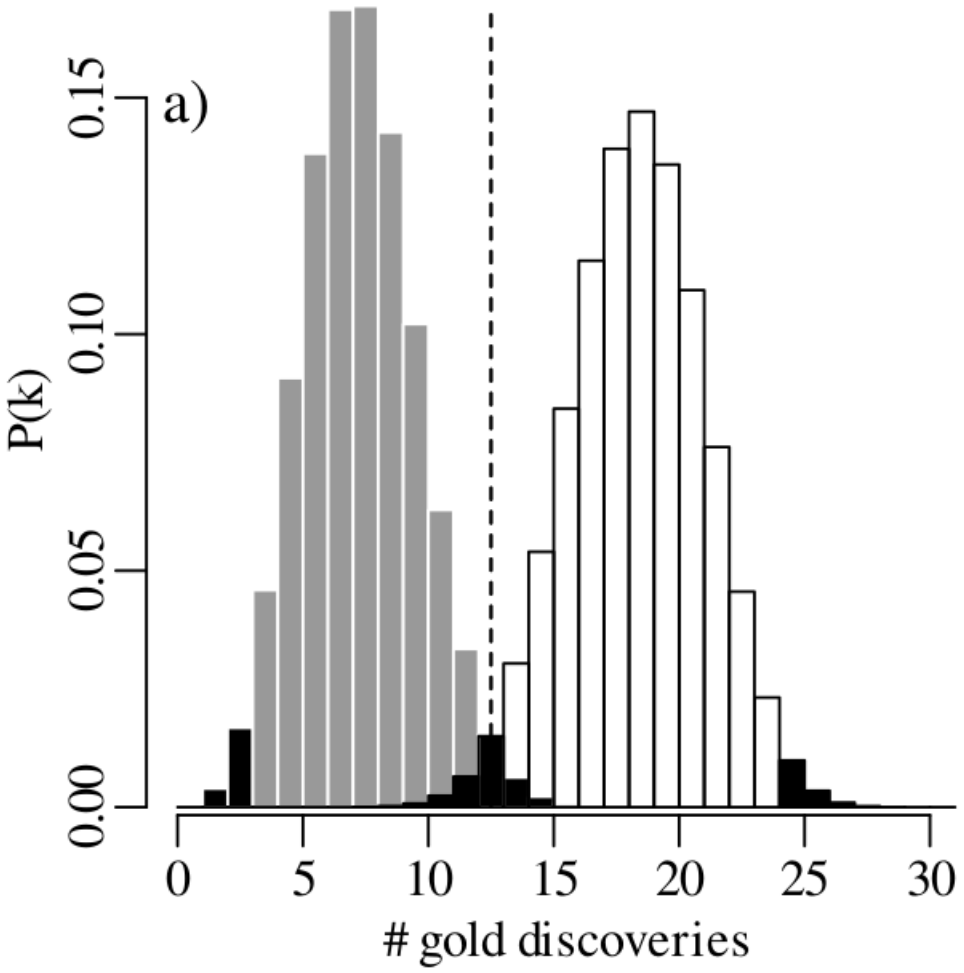


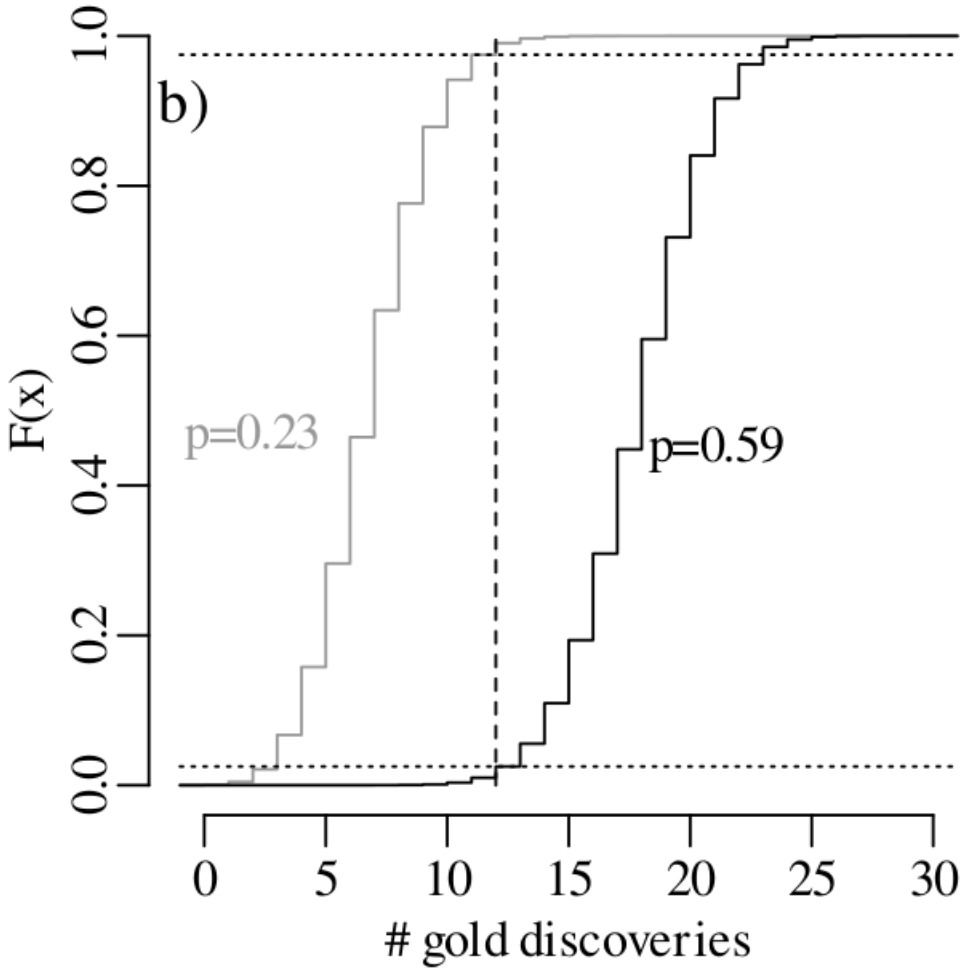


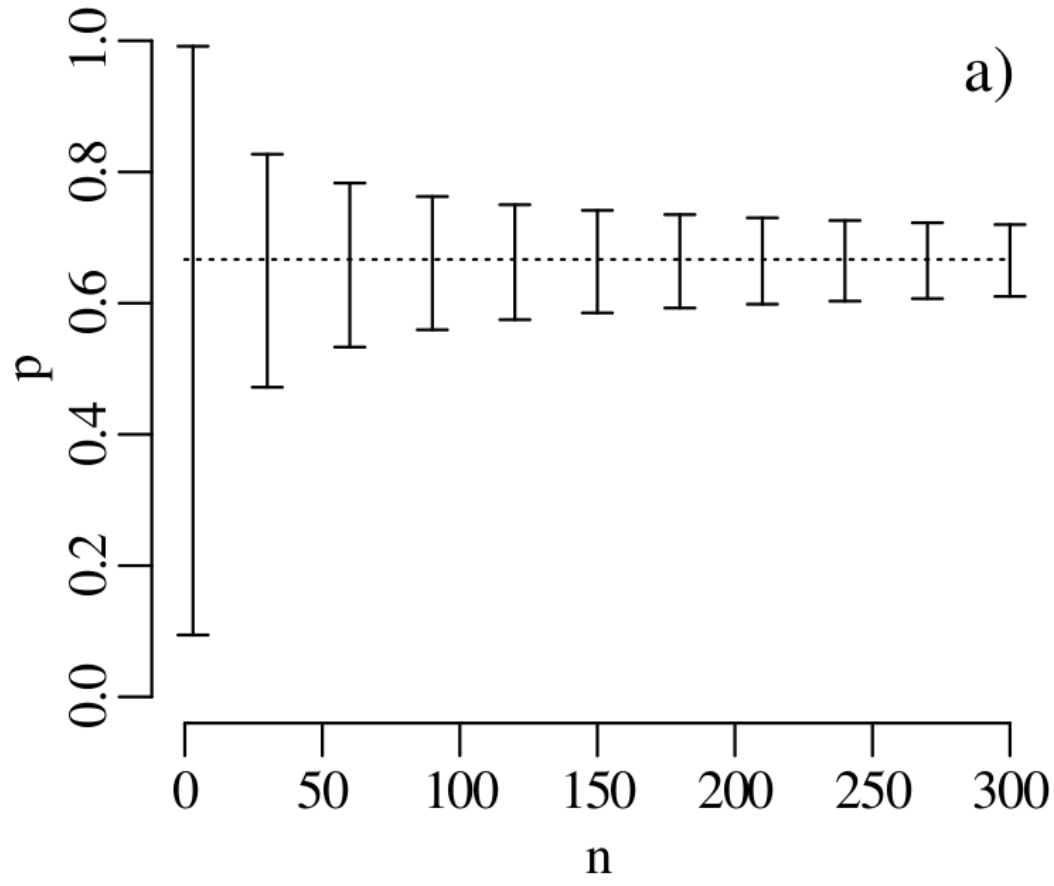
$$\text{C.I.}(p|k=4,n=5) = [0.284, 0.995]$$

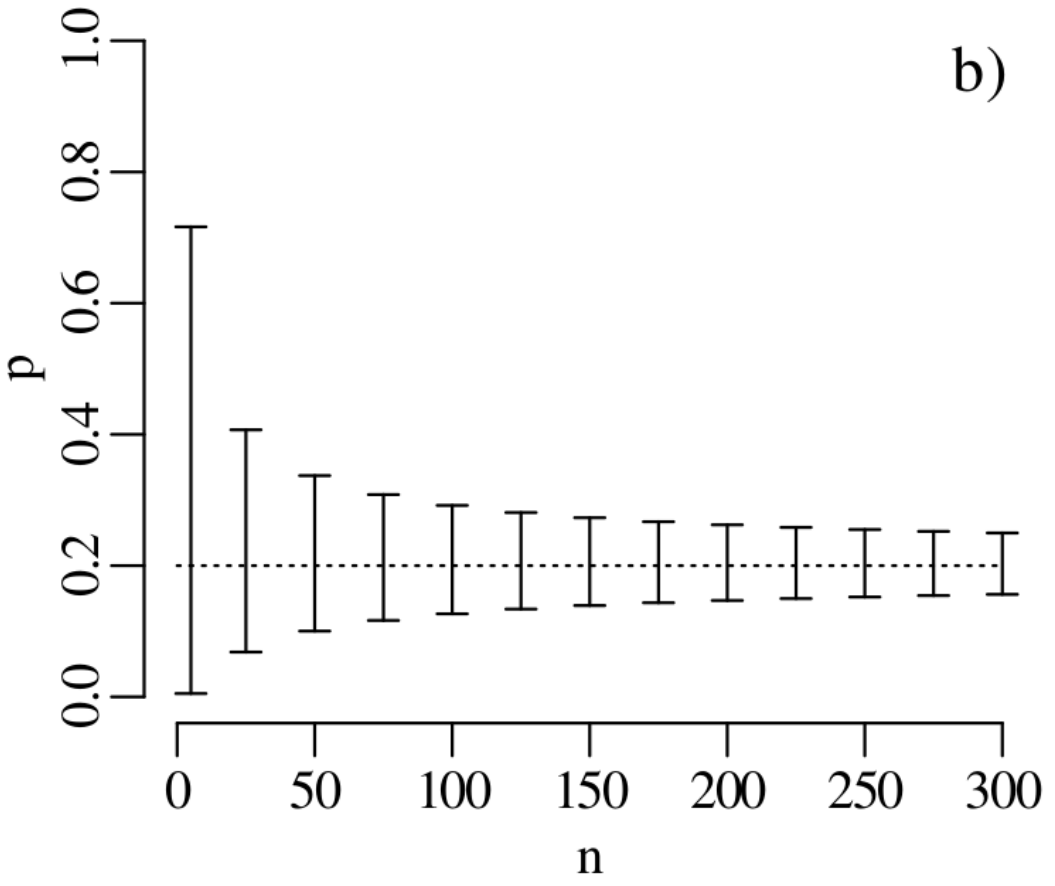


$$\text{C.I.}(p|k=12, n=30) = [0.23, 0.59]$$

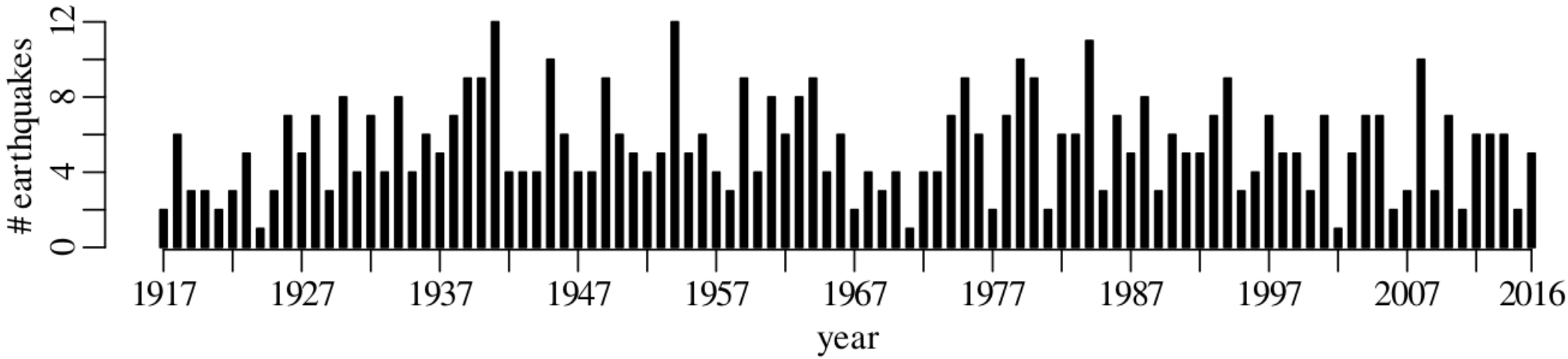


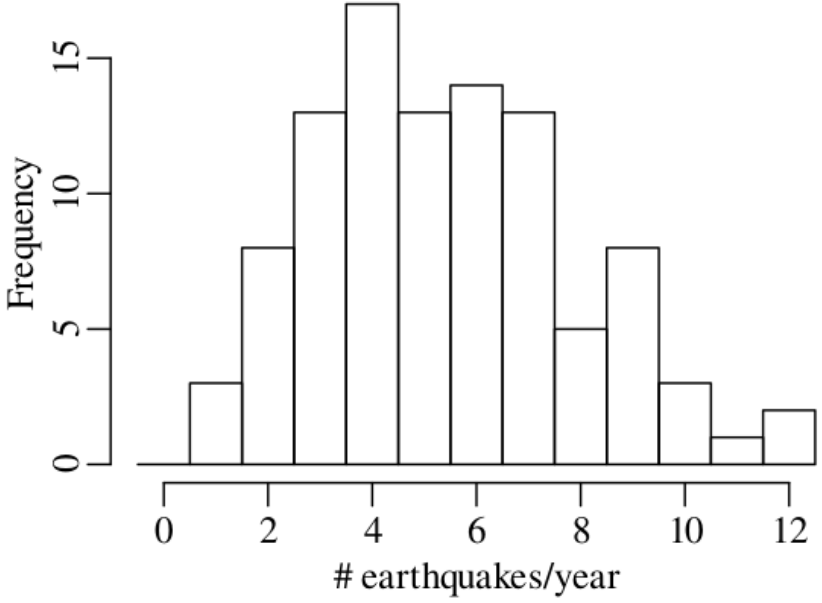



a)
p
0.0
0.2
0.4
0.6
0.8
1.0
0
50
100
150
200
250
300
n


b)
p
0.0
0.2
0.4
0.6
0.8
1.0
0
50
100
150
200
250
300
n

The Poisson distribution

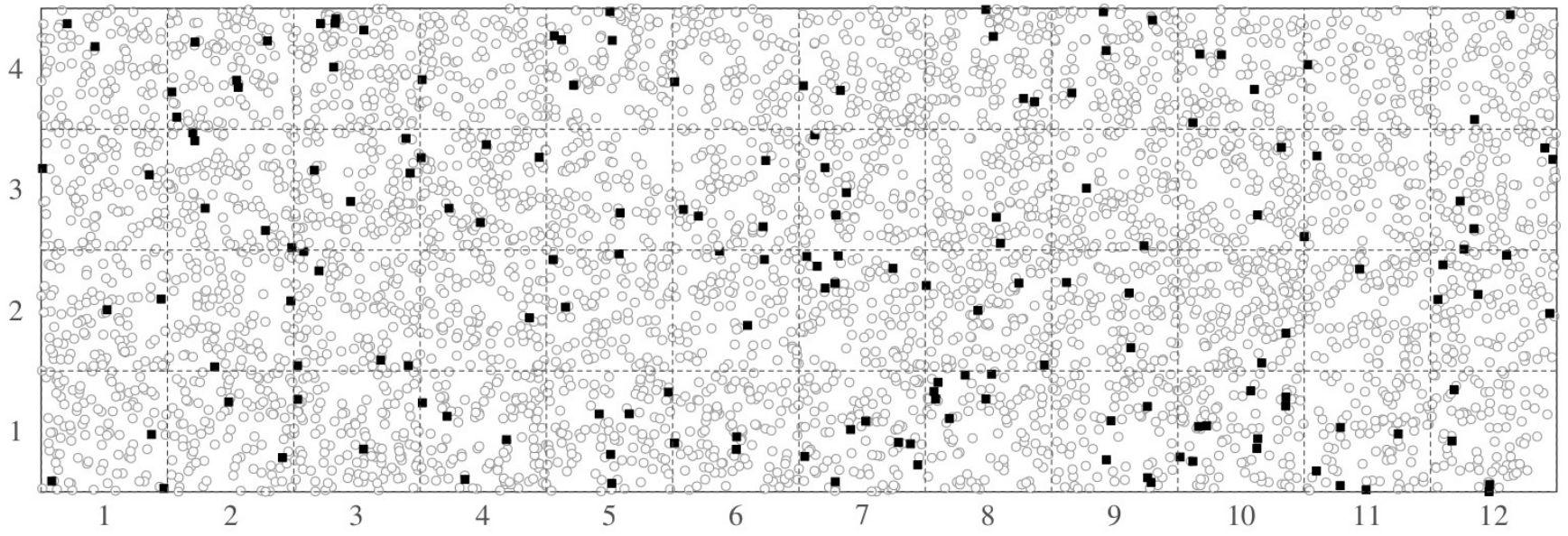




mean: 5.43

standard deviation: 2.50

variance: 6.24



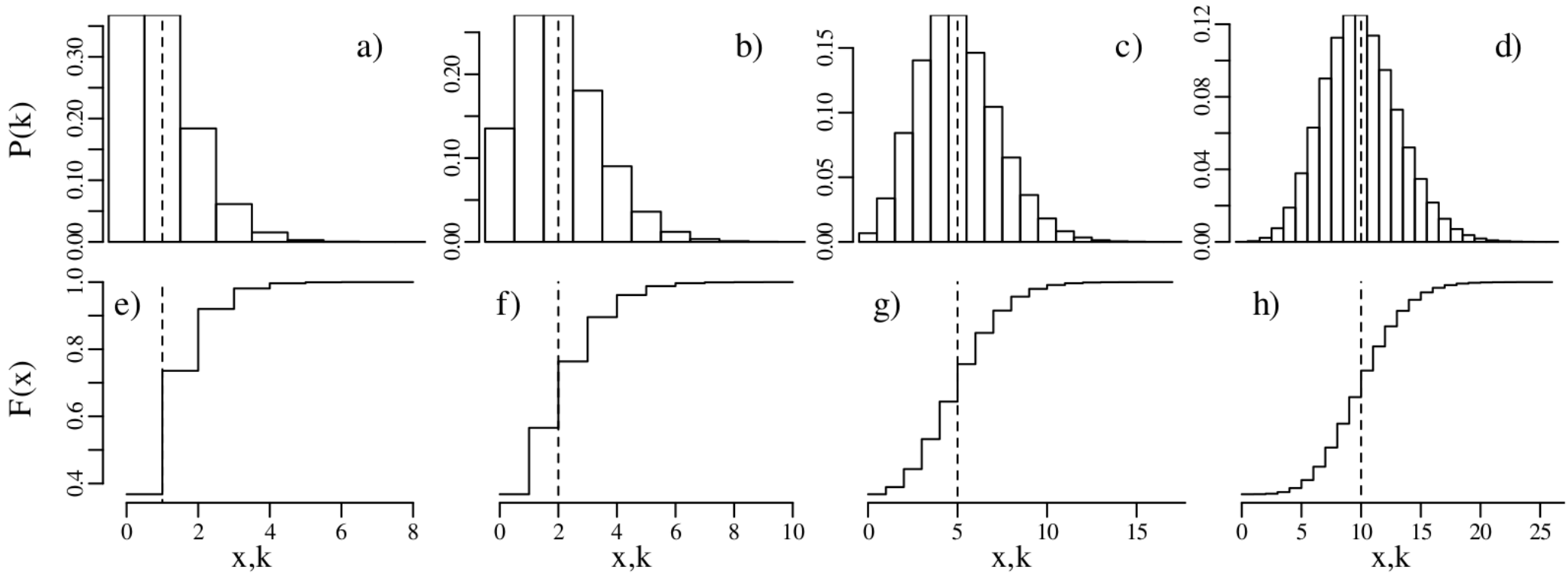
4	2	6	5	1	5	1	2	4	4	4	1	2
3	2	5	4	5	1	4	4	2	2	2	2	5
2	2	2	5	1	3	3	6	4	3	2	1	5
1	3	2	2	4	5	3	7	7	5	9	5	4
	1	2	3	4	5	6	7	8	9	10	11	12

mean: 3.50

standard deviation: 1.85

variance: 3.40

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$



$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k} \quad \xrightarrow[p \to 0]{n \to \infty} \quad P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{where} \quad \lambda = np$$

n	10	20	50	100	200	500	1000	2000	5000	10000
p	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
$P(k \leq 2)$	0.0547	0.0913	0.112	0.118	0.121	0.1234	0.124	0.1243	0.1245	0.1246

Examples of Poisson variables

1. people killed by lightning per year;
2. mutations in DNA per generation;
3. radioactive disintegrations per unit time;
4. mass extinctions per 100 million years.

~~earthquakes including aftershocks~~

~~floods per year~~

Parameter estimation

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathcal{L}(\lambda|k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda = k$$

Hypothesis tests

$k = 9$ zircons given $\lambda = 3.5$

1. $H_\circ$ (**null hypothesis**) $\lambda = 3.5$

   H_a (**alternative hypothesis**): $\lambda > 3.5$

2. test statistic : $k = 9$ successes

3. The null distribution

k	0	1	2	3	4	5	6	7	8	*9*	10
$P(T = k)$	0.030	0.106	0.185	0.216	0.189	0.132	0.077	0.038	0.017	0.007	0.002
$P(T \geq k)$	1.000	0.970	0.864	0.679	0.463	0.275	0.142	0.065	0.027	*0.010*	0.003

4. significance level $\alpha = 0.05$

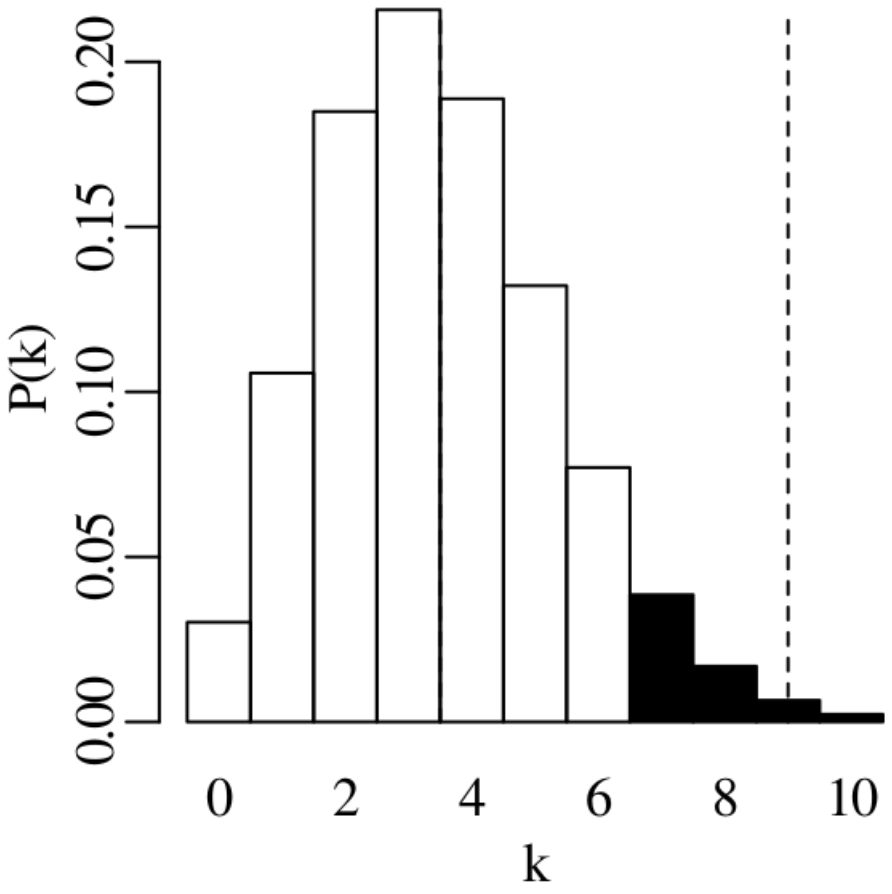
5. rejection region

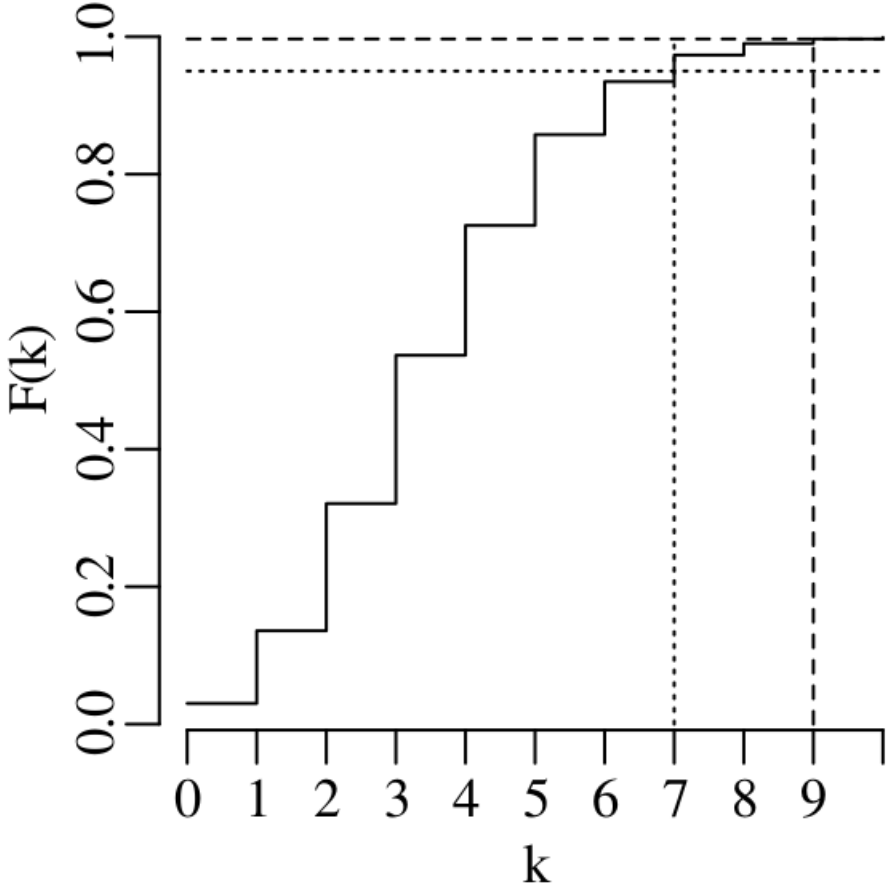
k	0	1	2	3	4	5	6	7	**8**	***9***	**10**
$P(T = k)$	0.030	0.106	0.185	0.216	0.189	0.132	0.077	0.038	0.017	0.007	0.002
$P(T \geq k)$	1.000	0.970	0.864	0.679	0.463	0.275	0.142	0.065	**0.027**	***0.010***	**0.003**

$R = \{8, 9, 10, \ldots, \infty\}$

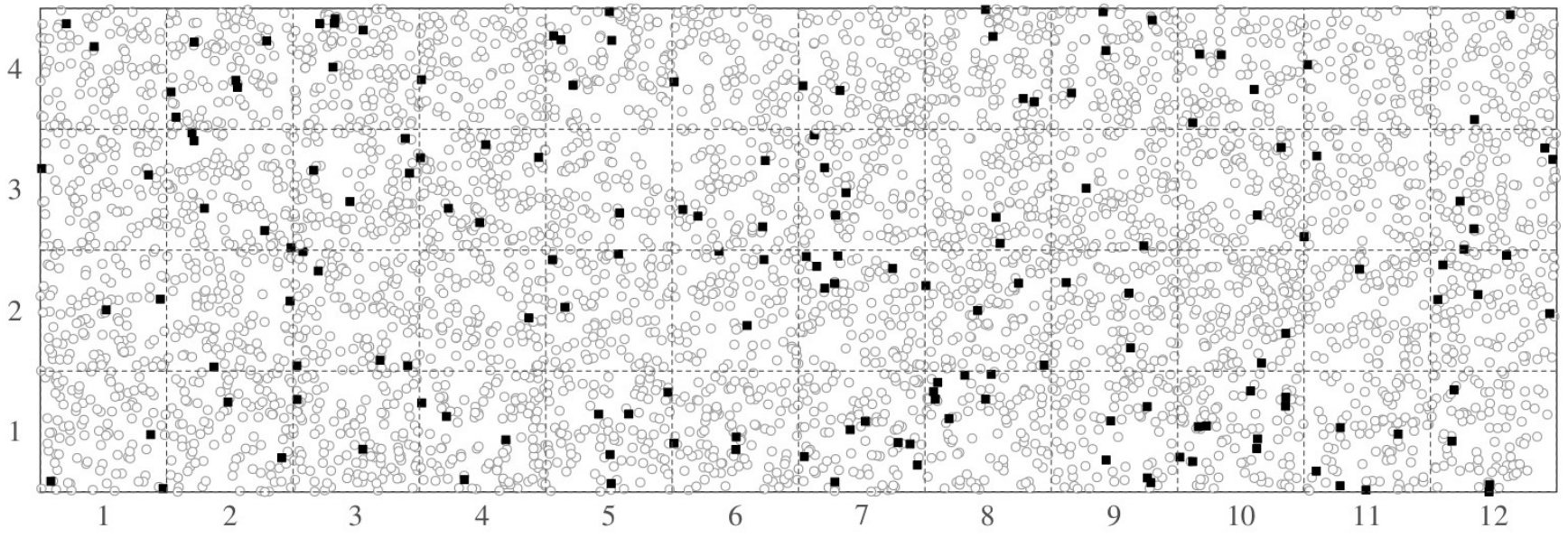
6. $k \in R$

7. p-value $= 0.010 < \alpha$


P(k)
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
k


F(k)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6
7
8
9
k

Multiple testing



4	2	6	5	1	5	1	2	4	4	4	1	2
3	2	5	4	5	1	4	4	2	2	2	2	5
2	2	2	5	1	3	3	6	4	3	2	1	5
1	3	2	2	4	5	3	7	7	5	9	5	4
	1	2	3	4	5	6	7	8	9	10	11	12

Assuming that $H_\circ$ is true, the probability of incurring a type-I error is:

$$1 \text{ experiment: } \alpha = 5\%$$

$$2 \text{ experiments: } 1 - (1-\alpha)^2 = 9.75\%$$

$$3 \text{ experiments: } 1 - (1-\alpha)^3 = 14\%$$

$$48 \text{ experiments: } 1 - (1-\alpha)^{48} = 91.5\%$$

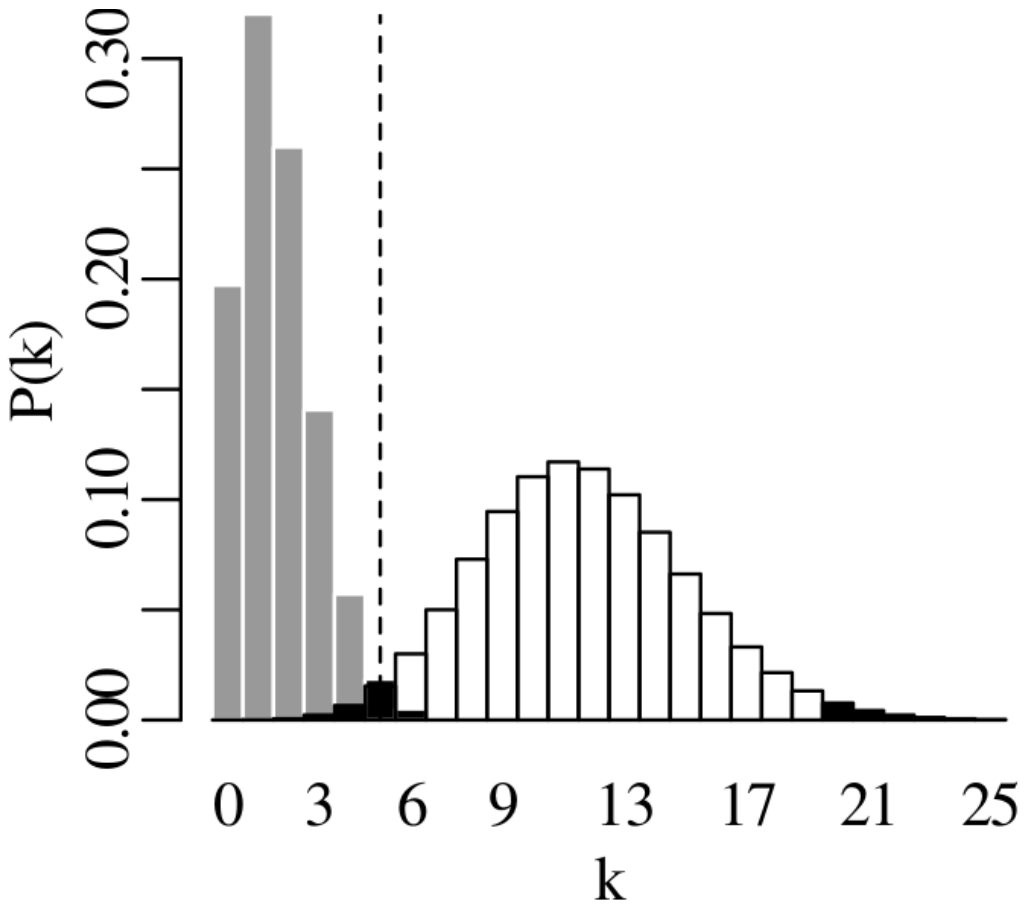
$$N \text{ experiments: } 1 - (1-\alpha)^N\%$$

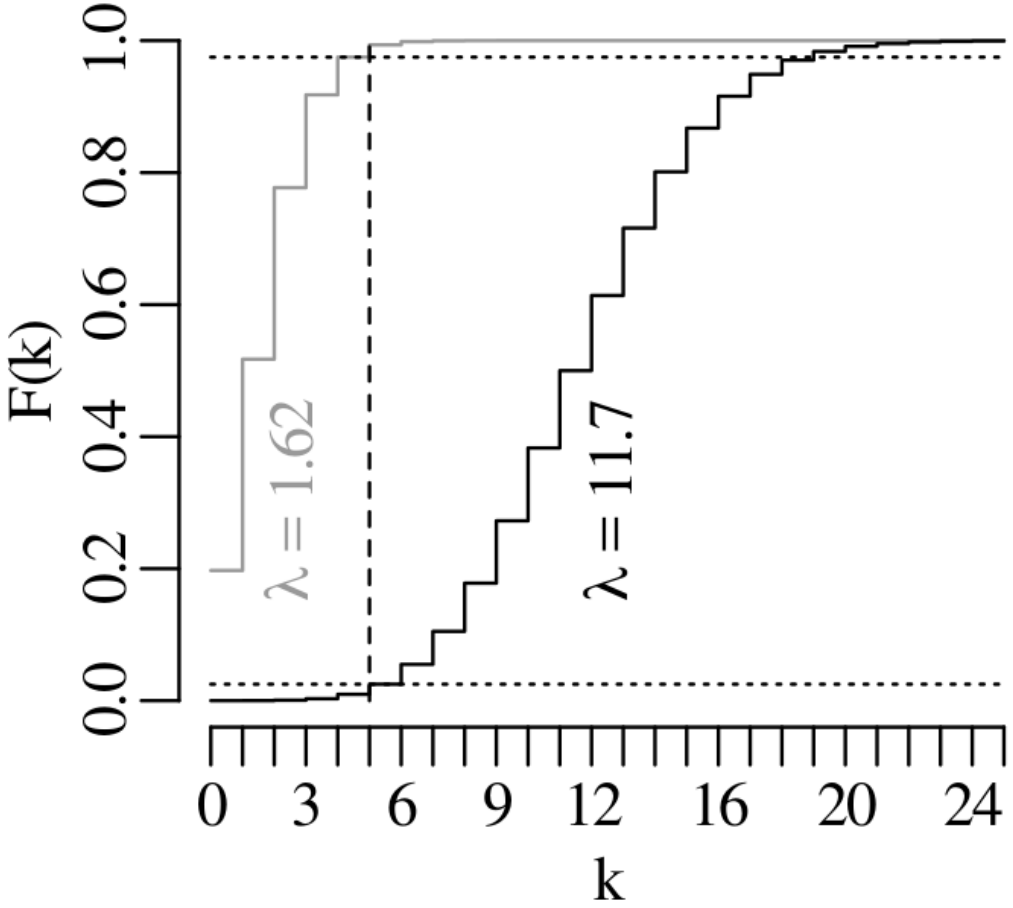
Bonferroni correction: α/N

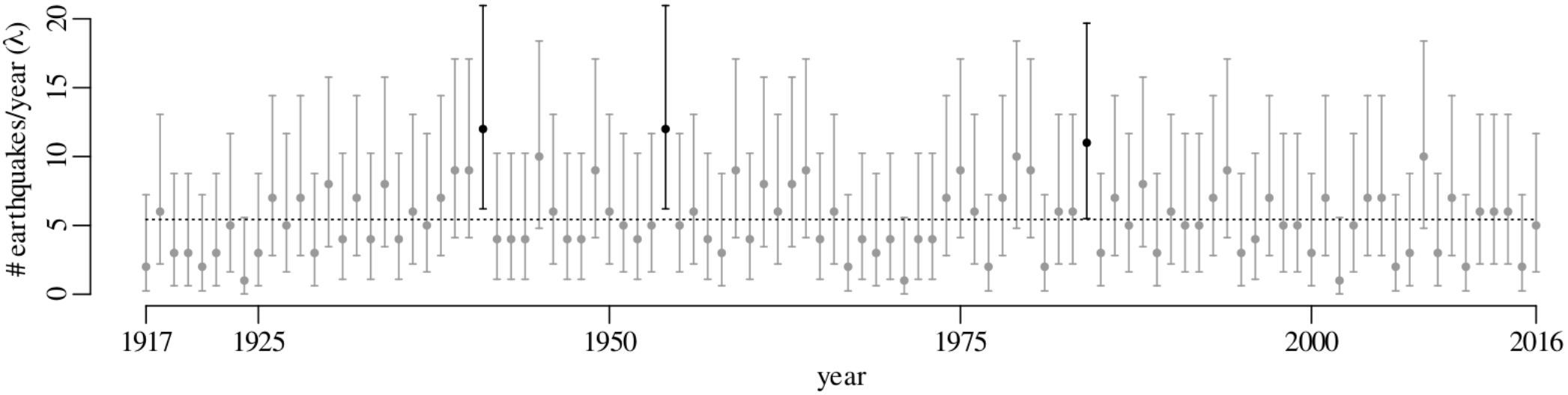
Confidence intervals

5 magnitude 5.0 or greater earthquakes occurred in the US in 2016

What is λ?






earthquakes/year (λ)
0
5
10
15
20
1917
1925
1950
1975
2000
2016
year

The normal distribution

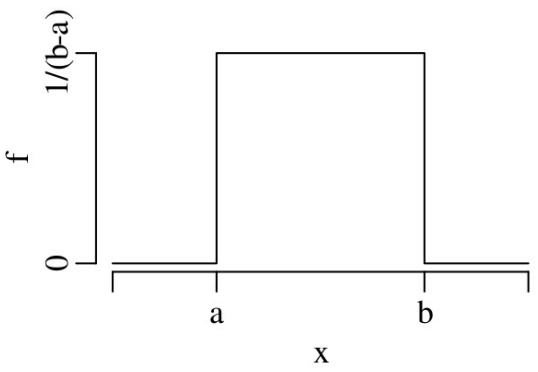
the binomial distribution $$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

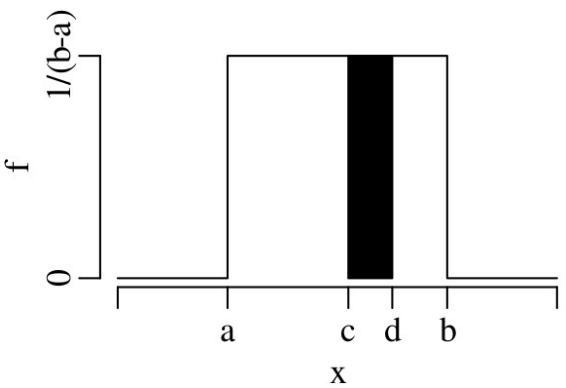
the Poisson distribution $$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

the negative binomial distribution $$P(x|r,p) = \binom{r+x-1}{x} (1-p)^x p^r$$

the multinomial distribution $$P(k_1, k_2, \ldots, k_m | p_1, p_2, \ldots, p_m) = \frac{n!}{\prod_{i=1}^{m} k_i!} \prod_{i=1}^{m} p_i^{k_i}$$

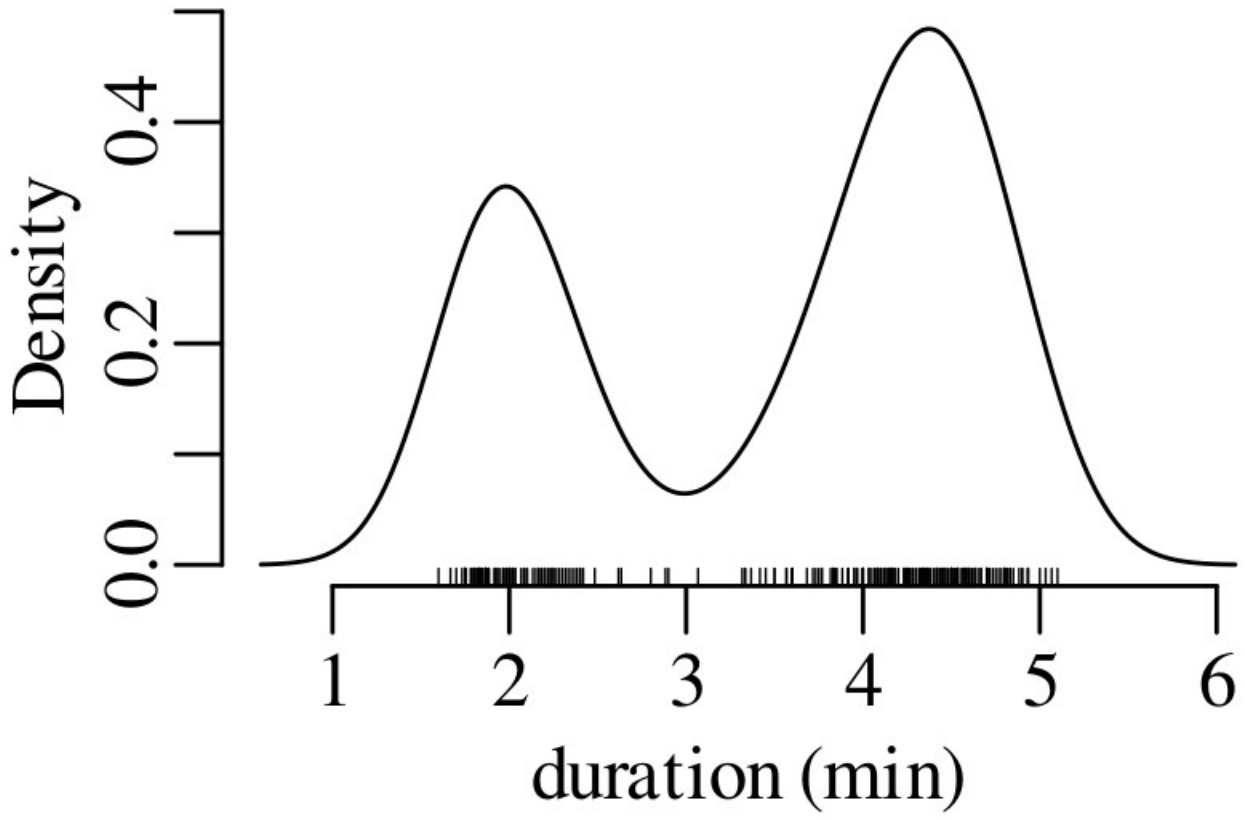
the uniform distribution $$f(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

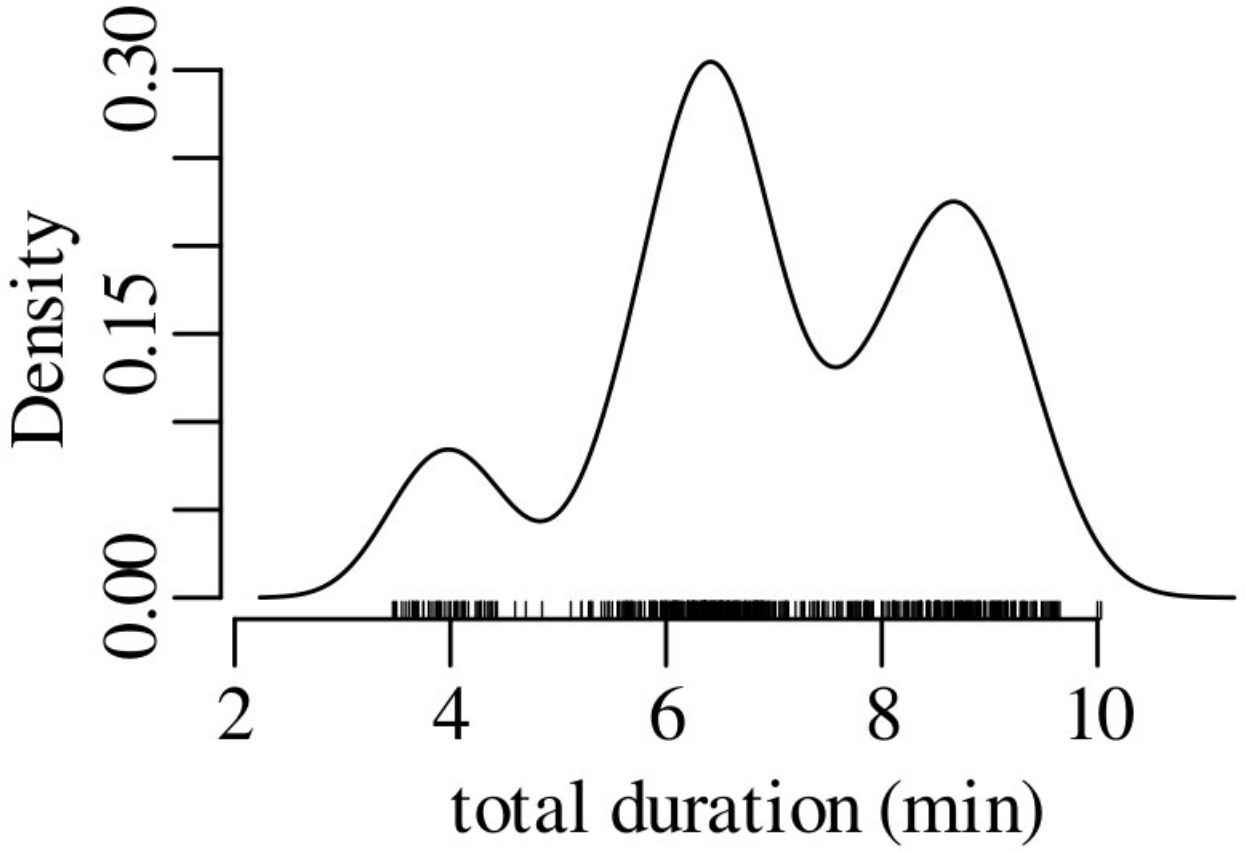


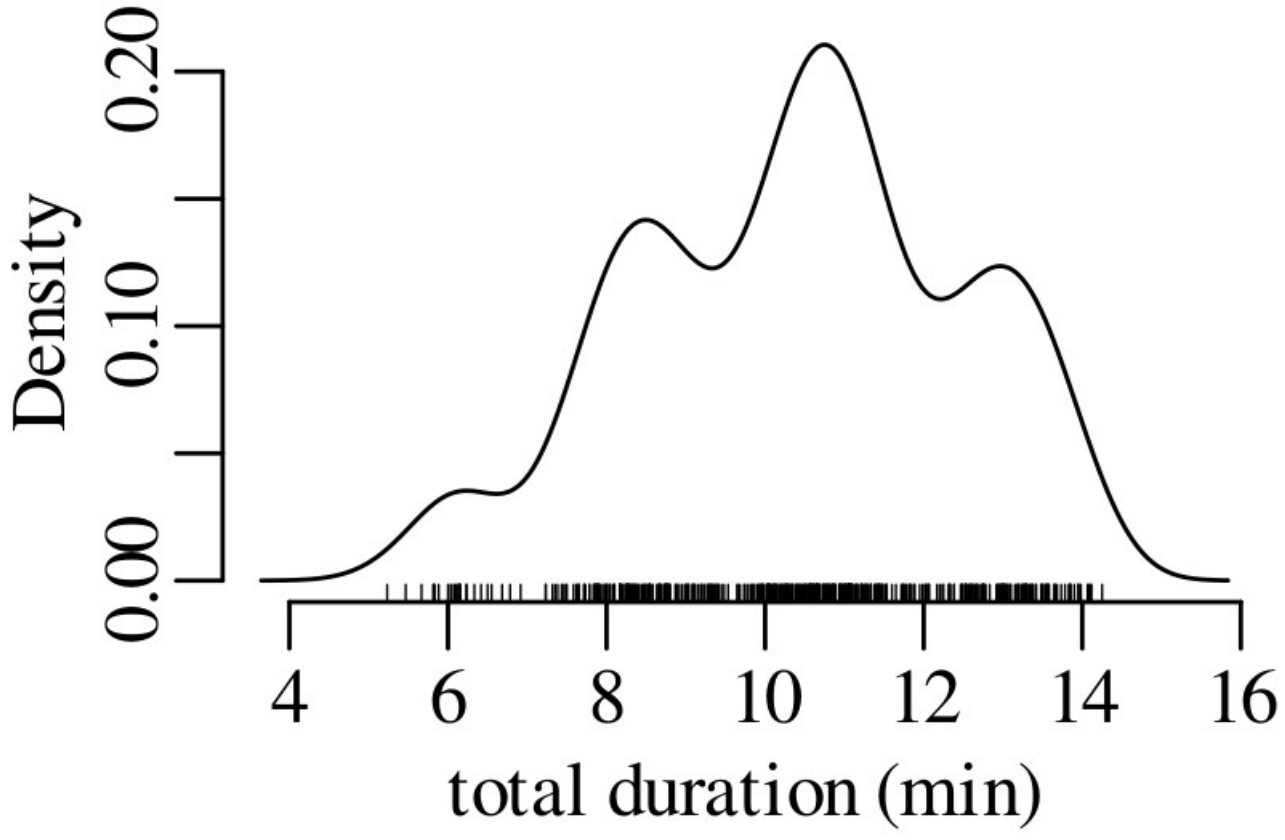


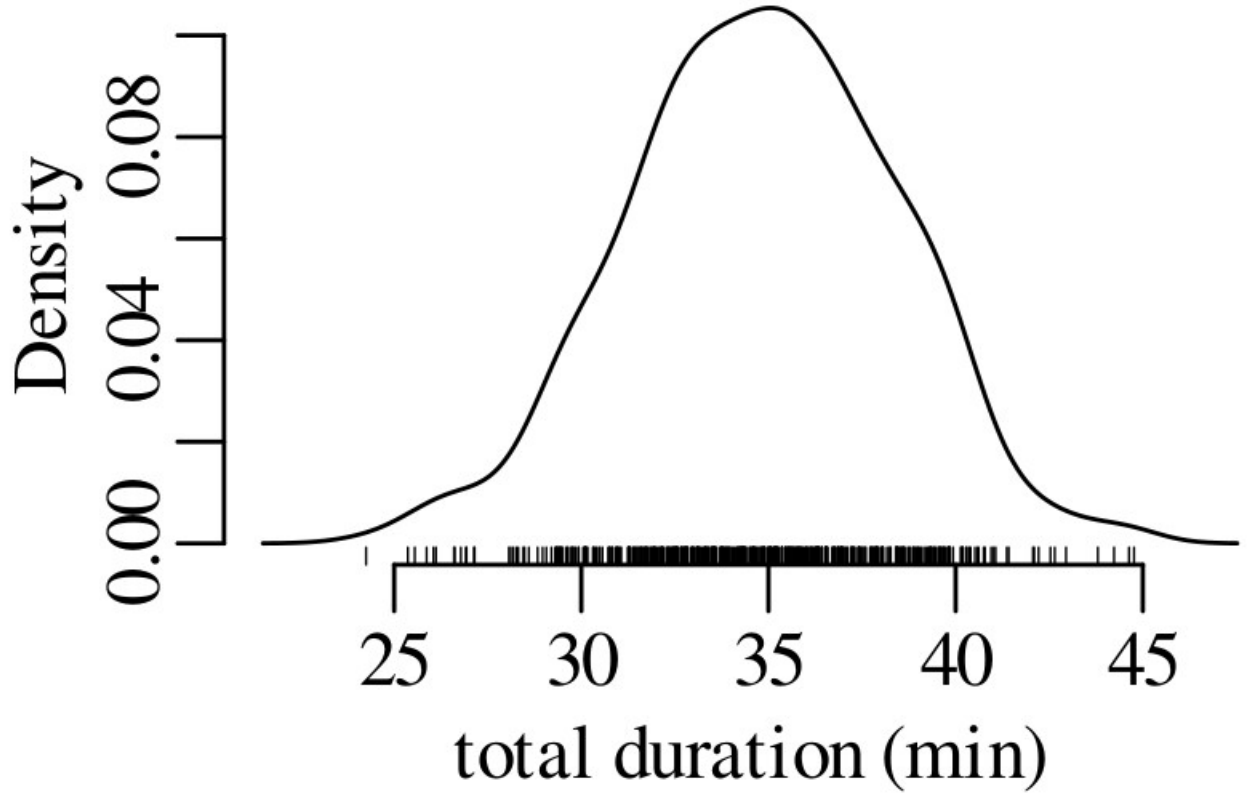
$$P(c \le x \le d) = \int_c^d f(x|a,b)\,dx$$

What is so normal about the Gaussian distribution?

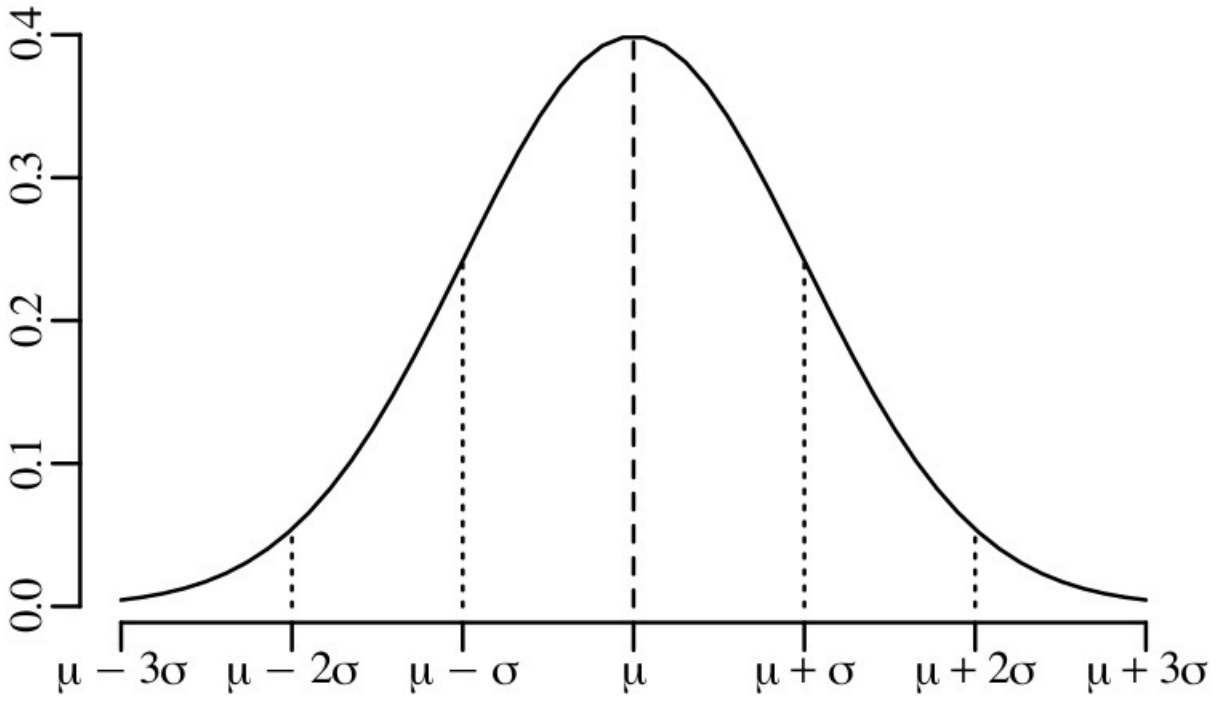


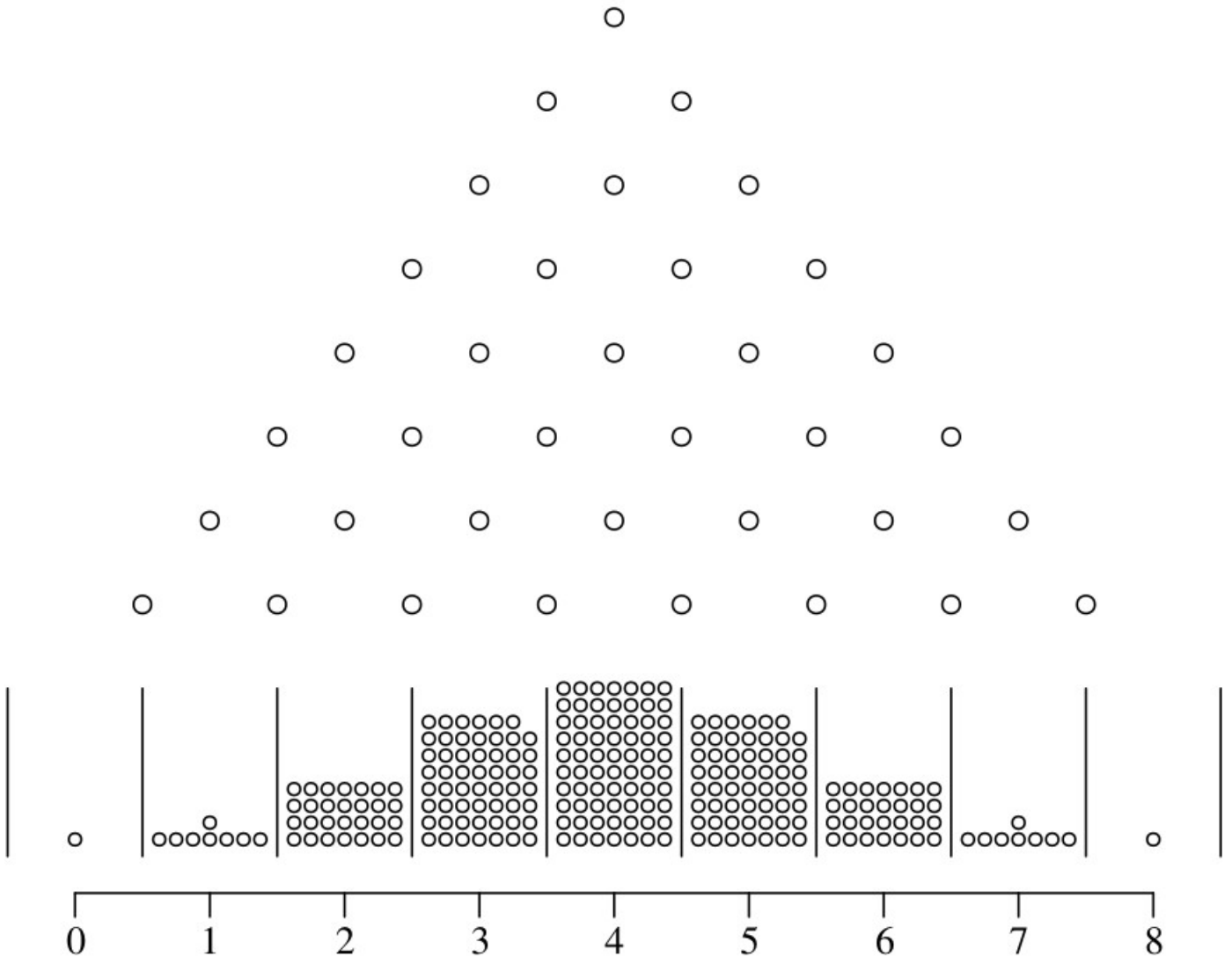

Density
0.00
0.15
0.30
2
4
6
8
10
total duration (min)

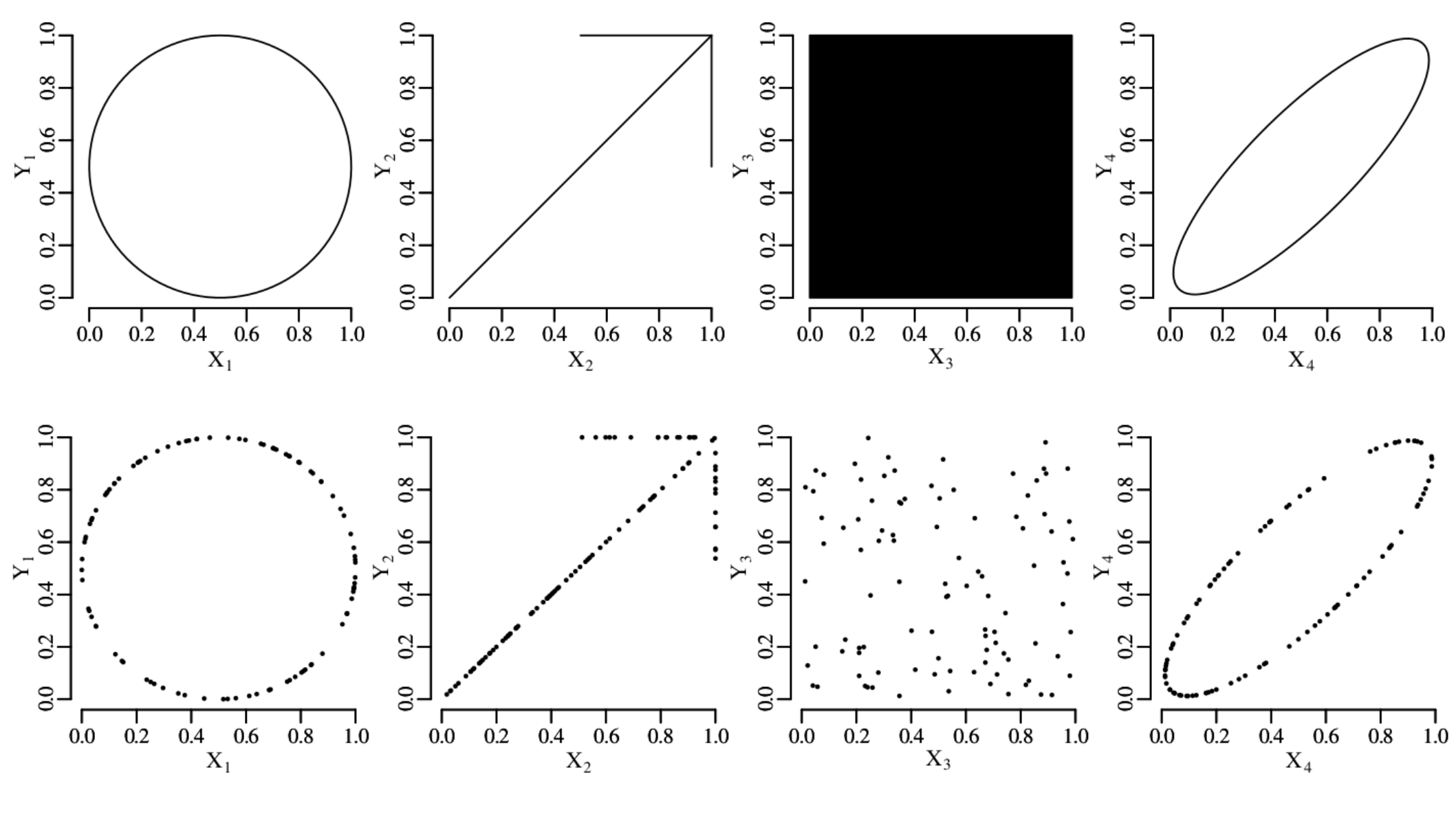

Density
0.00
0.10
0.20
4
6
8
10
12
14
16
total duration (min)

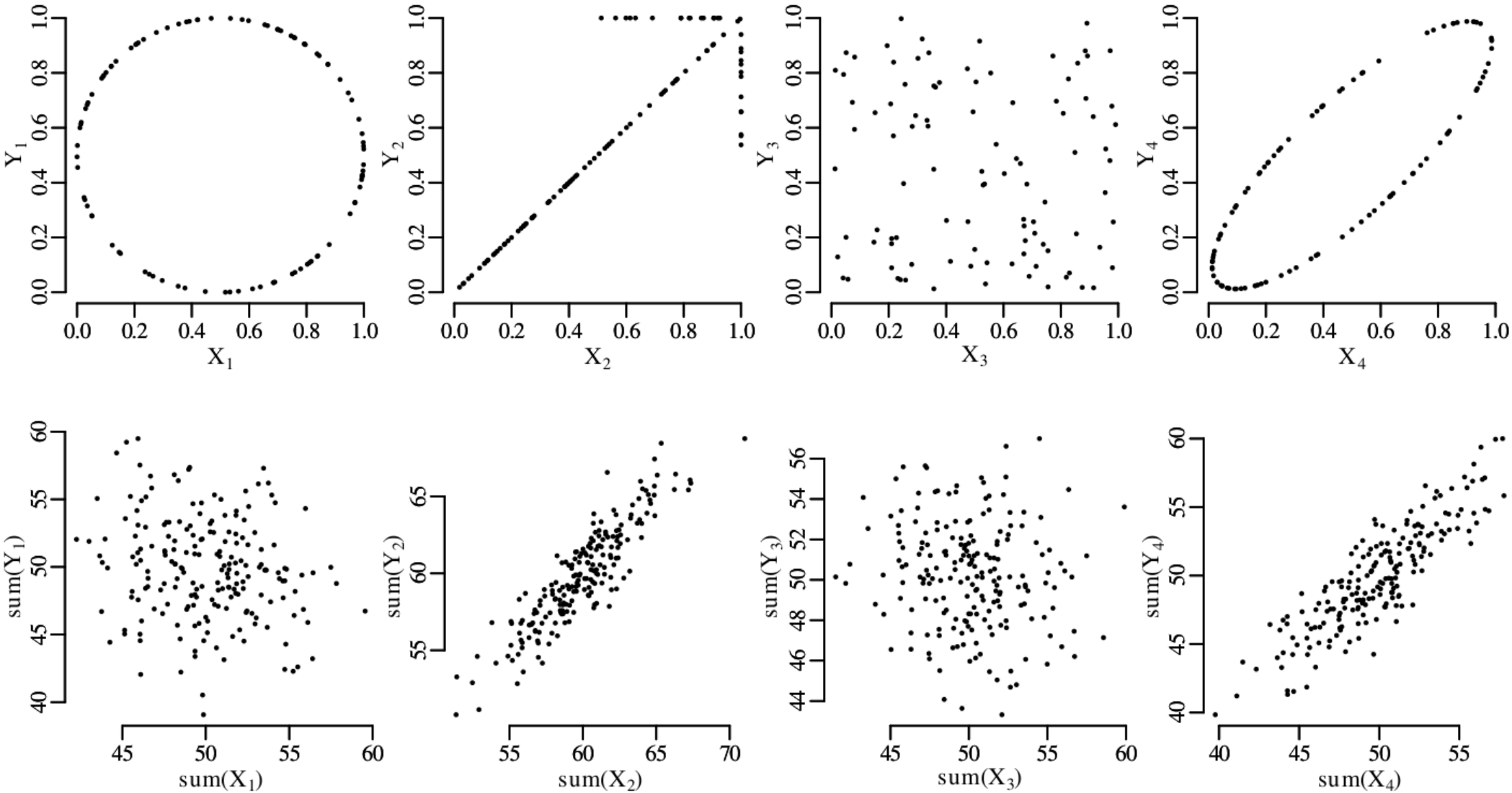

Density
0.00
0.04
0.08
25
30
35
40
45
total duration (min)

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

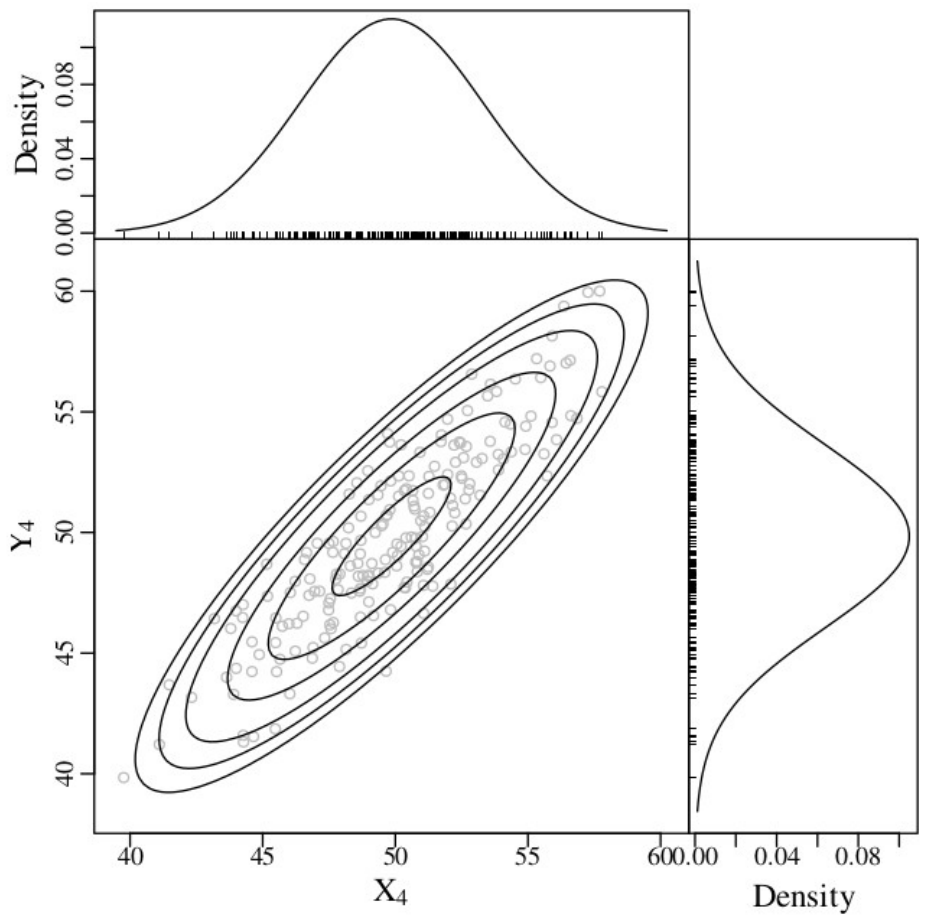



0
1
2
3
4
5
6
7
8

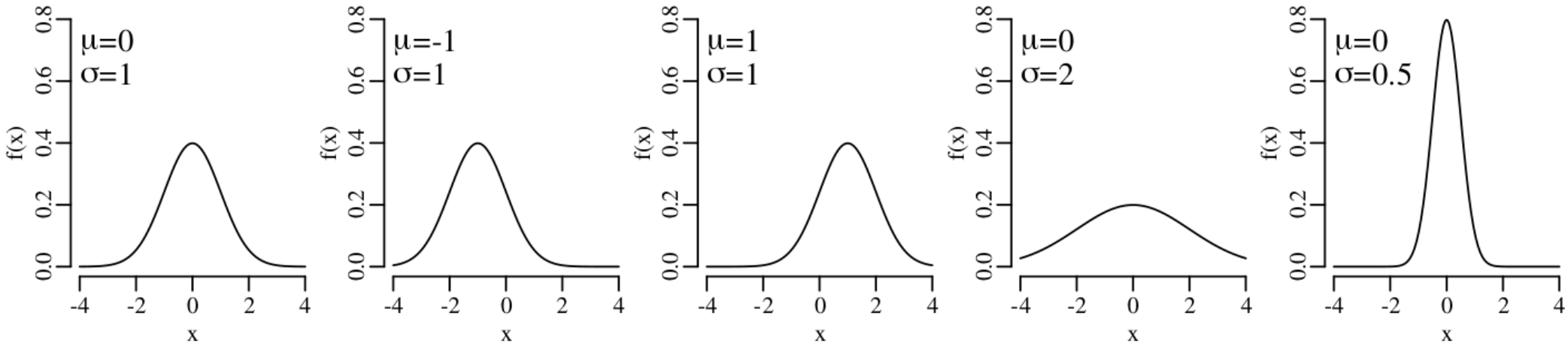

Y1
X1
Y2
X2
Y3
X3
Y4
X4
0.0
0.2
0.4
0.6
0.8
1.0


Y1
X1
Y2
X2
Y3
X3
Y4
X4
sum(Y1)
sum(X1)
sum(Y2)
sum(X2)
sum(Y3)
sum(X3)
sum(Y4)
sum(X4)

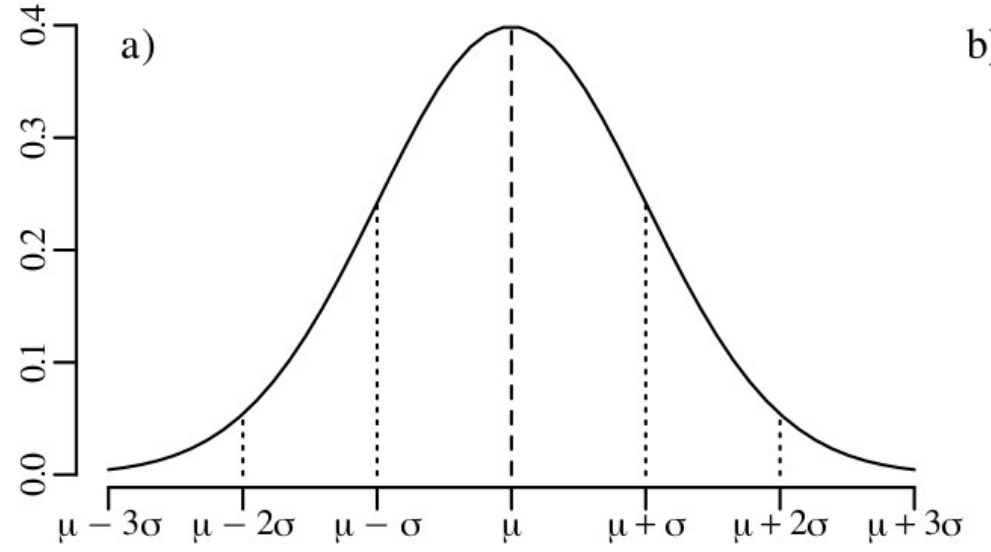
$$f(x, y|\mu_x, \mu_y, \sigma_x, \sigma_y, \sigma_{x,y}) = \frac{\exp\left(-\begin{bmatrix}(x-\mu_x) & (y-\mu_y)\end{bmatrix}\begin{bmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{bmatrix}^{-1}\begin{bmatrix}x-\mu_x\\ y-\mu_y\end{bmatrix}\Big/2\right)}{2\pi\sqrt{\begin{vmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{vmatrix}}}$$

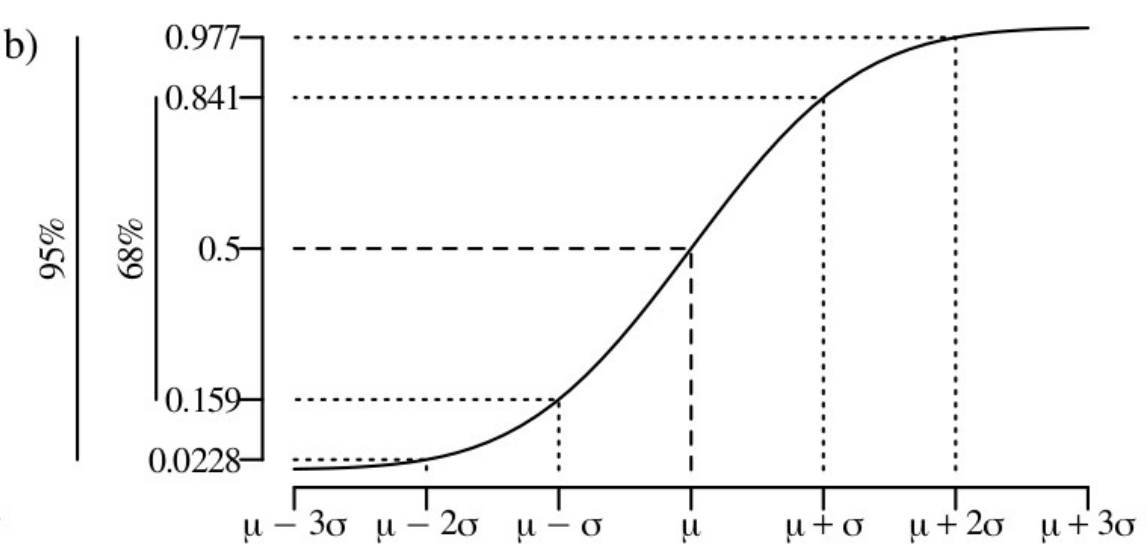


$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

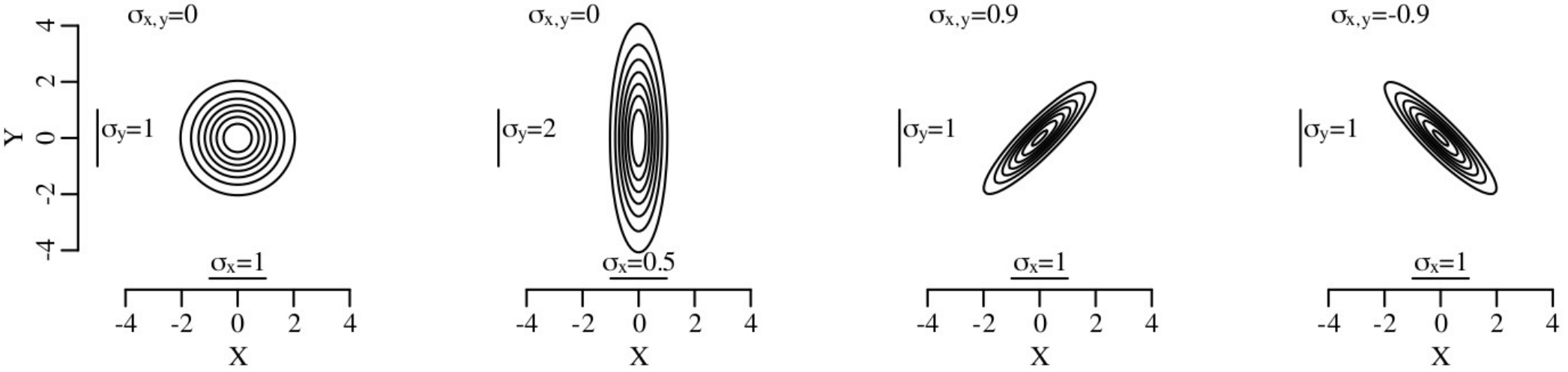


$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$





$$f(x,y|\mu_x,\mu_y,\sigma_x,\sigma_y,\sigma_{x,y}) = \frac{\exp\left(-\begin{bmatrix}(x-\mu_x) & (y-\mu_y)\end{bmatrix}\begin{bmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{bmatrix}^{-1}\begin{bmatrix}x-\mu_x\\ y-\mu_y\end{bmatrix}\Big/2\right)}{2\pi\sqrt{\begin{vmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{vmatrix}}}$$



Parameter estimation

$$\mathcal{L}(\mu, \sigma | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i | \mu, \sigma) \quad \text{with} \quad f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

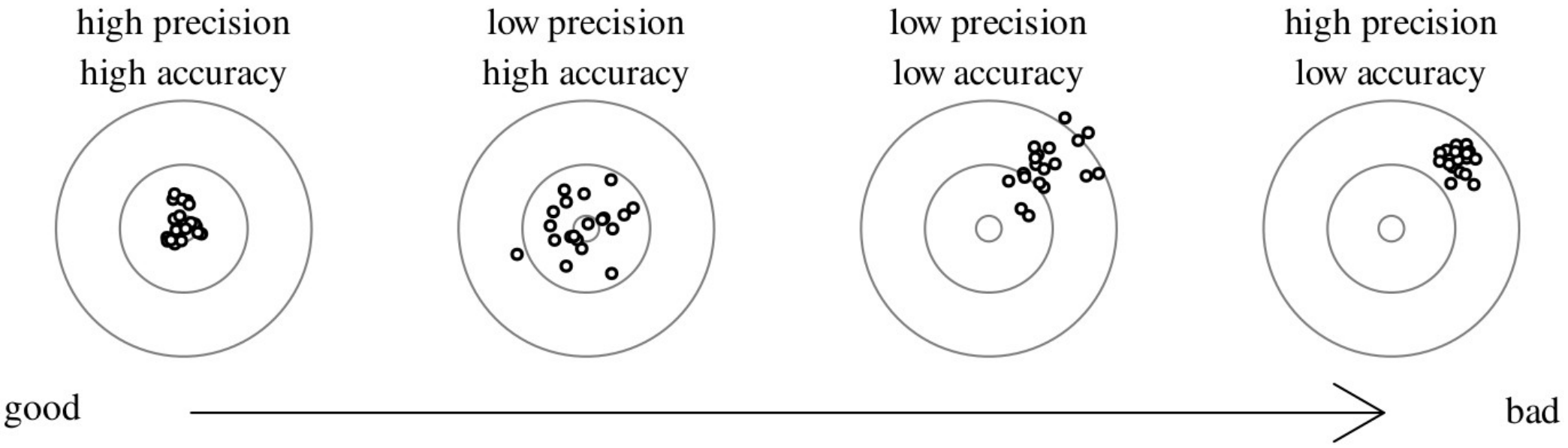
$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \longleftrightarrow \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2} \quad \longleftrightarrow \quad s[x] = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\hat{\sigma}_{x,y} = \sum_{i=1}^{n} \frac{1}{n}(x_i - \mu_x)(y_i - \mu_y) \quad \longleftrightarrow \quad s[x,y] = \sum_{i=1}^{n} \frac{1}{n-1}(x_i - \bar{x})(y_i - \bar{y})$$

Statistics for geoscientists

Error propagation


high precision
high accuracy
low precision
high accuracy
low precision
low accuracy
high precision
low accuracy
good
bad

$z = g(x) \longrightarrow$ given x_i, for $i = 1...n$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s[x] = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$\bar{z} = g(\bar{x}) \longrightarrow$ What is $s[z]$?

Linear approximation

$$z = g(x)$$

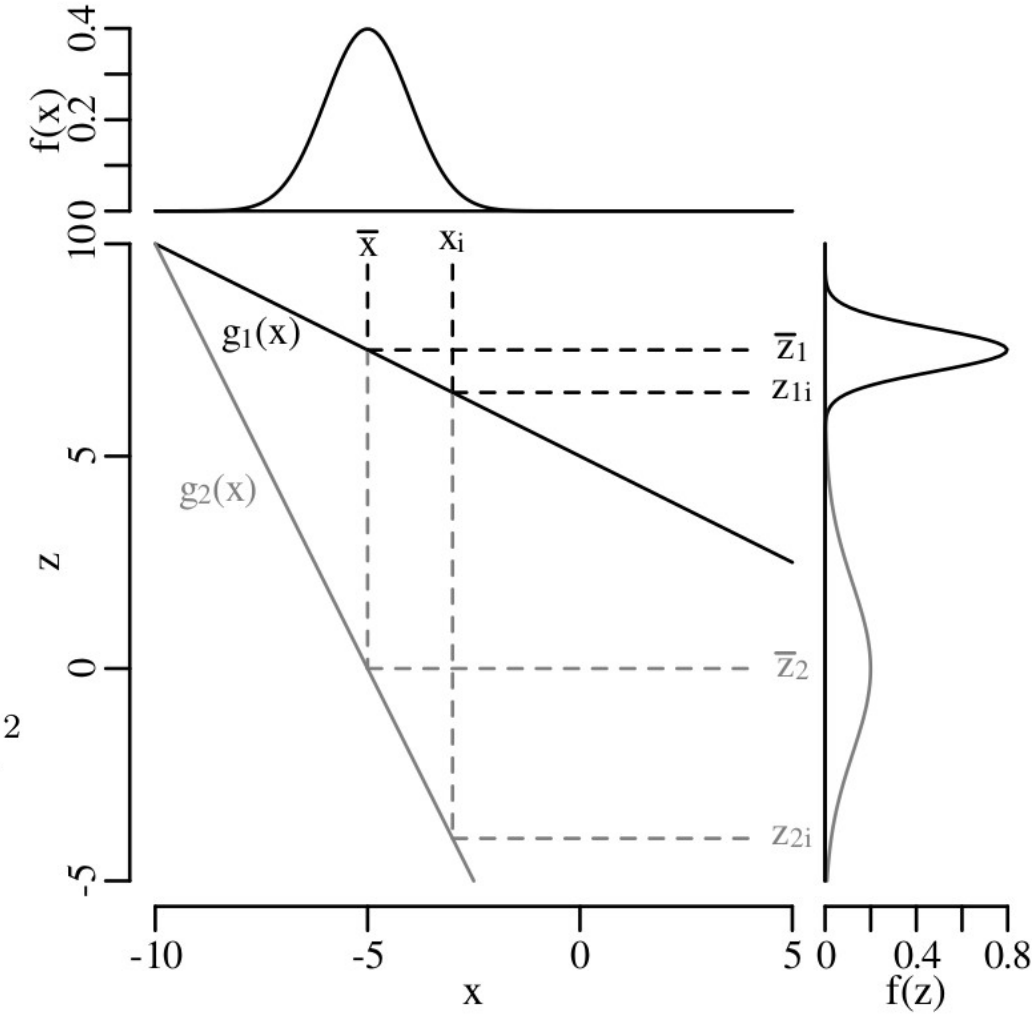
$$(z_i - \bar{z}) \approx \frac{\partial z}{\partial x}(x_i - \bar{x})$$

$$s[z]^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(z_i - \bar{z})^2$$

$$s[z]^2 \approx \frac{1}{n-1}\sum_{i=1}^{n}\left[(x_i - \overline{x})\frac{\partial z}{\partial x}\right]^2$$

$$= \left[\frac{\partial z}{\partial x}\right]^2 \left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2\right] = \left[\frac{\partial z}{\partial x}\right]^2 s[x]^2$$

$$\Rightarrow s[z] = \left|\frac{\partial z}{\partial x}\right| s[x]$$

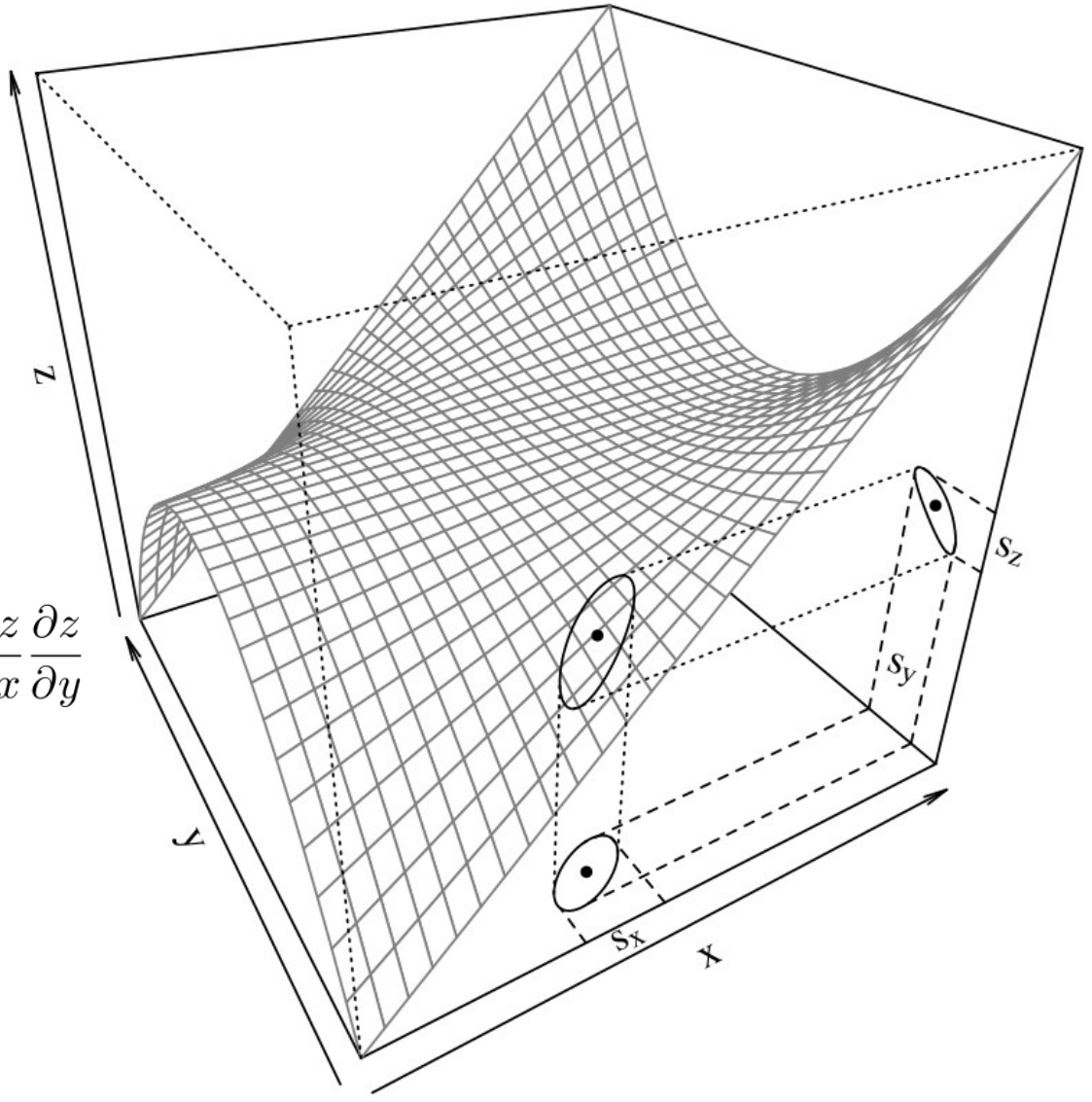


$$z = g(x, y)$$

$$s[z]^2 \approx \frac{1}{n-1} \sum_{i=1}^{n} \left[(x_i - \overline{x}) \frac{\partial z}{\partial x} + (y_i - \overline{y}) \frac{\partial z}{\partial y} \right]^2$$

$$s[z]^2 \approx s[x]^2 \left(\frac{\partial z}{\partial x} \right)^2 + s[y]^2 \left(\frac{\partial z}{\partial y} \right)^2 + 2\, s[x, y] \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

$$s[z]^2 \approx \begin{bmatrix} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} s[x]^2 & s[x, y] \\ s[x, y] & s[y]^2 \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix}$$



$$s[z]^2 \approx s[x]^2 \left(\frac{\partial z}{\partial x}\right)^2 + s[y]^2 \left(\frac{\partial z}{\partial y}\right)^2 + 2\ s[x,y]\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

addition

$$z = a + bx + cy$$

$$\frac{\partial z}{\partial x} = b \text{ and } \frac{\partial z}{\partial y} = c$$

$$s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2 + 2bc\ s[x,y]$$

$$z = x + y \longrightarrow s[z]^2 = s[x]^2 + s[y]^2$$

$$s[z]^2 \approx s[x]^2 \left(\frac{\partial z}{\partial x}\right)^2 + s[y]^2 \left(\frac{\partial z}{\partial y}\right)^2 + 2\ s[x,y]\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

subtraction

$$z = ax - by$$

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = -b$$

$$s[z]^2 = a^2 s[x]^2 + b^2 s[y]^2 - 2ab\ s[x,y]$$

$$z = x - y \longrightarrow s[z]^2 = s[x]^2 + s[y]^2$$

$$s[z]^2 \approx s[x]^2 \left(\frac{\partial z}{\partial x}\right)^2 + s[y]^2 \left(\frac{\partial z}{\partial y}\right)^2 + 2\ s[x,y]\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

multiplication

$$z = axy$$

$$\frac{\partial z}{\partial x} = ay \text{ and } \frac{\partial z}{\partial y} = ax$$

$$s[z]^2 = (ay)^2 s[x]^2 + (ax)^2 s[y]^2 + 2(ay)(ax)\ s[x,y]$$

$$\left(\frac{s[z]}{z}\right)^2 = \left(\frac{ay}{axy}s[x]\right)^2 + \left(\frac{ax}{axy}s[y]\right)^2 + 2\left(\frac{ay}{axy}\right)\left(\frac{ax}{axy}\right)s[x,y]$$

$$\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 + 2\frac{s[x,y]}{xy}$$

addition	$z = a + bx + cy$	$s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2 + 2bc\ s[x,y]$
subtraction	$z = ax - by$	$s[z]^2 = a^2 s[x]^2 + b^2 s[y]^2 - 2ab\ s[x,y]$
multiplication	$z = axy$	$\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 + 2\frac{s[x,y]}{xy}$
division	$z = a\frac{x}{y}$	$\left(\frac{s[z]}{z}\right)^2 = \left(\frac{s[x]}{x}\right)^2 + \left(\frac{s[y]}{y}\right)^2 - 2\frac{s[x,y]}{xy}$
exponentiation	$z = ae^{bx}$	$\left(\frac{s[z]}{z}\right)^2 = b^2 s[x]^2$
logarithms	$z = a\ln[bx]$	$s[z]^2 = a^2 \left(\frac{s[x]}{x}\right)^2$
power	$z = ax^b$	$\left(\frac{s[z]}{z}\right)^2 = b^2 \left(\frac{s[x]}{x}\right)^2$

the **chain rule**

$$d = d_\circ + v_\circ t + gt^2$$

can be written as

$$d = x + y \qquad \text{where} \quad x \equiv d_\circ + v_\circ t \qquad \text{and} \qquad y \equiv gt^2$$

$$s[x]^2 = (v_\circ s[t])^2 \qquad s[y]^2 = g^2 \left(\frac{s[t]}{t}\right)^2$$

$$s[d] = \sqrt{s[x]^2 + s[y]^2}$$

$$s[d] = \sqrt{(v_\circ s[t])^2 + g^2 \left(\frac{s[t]}{t}\right)^2}$$

$$z = g(x)$$

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

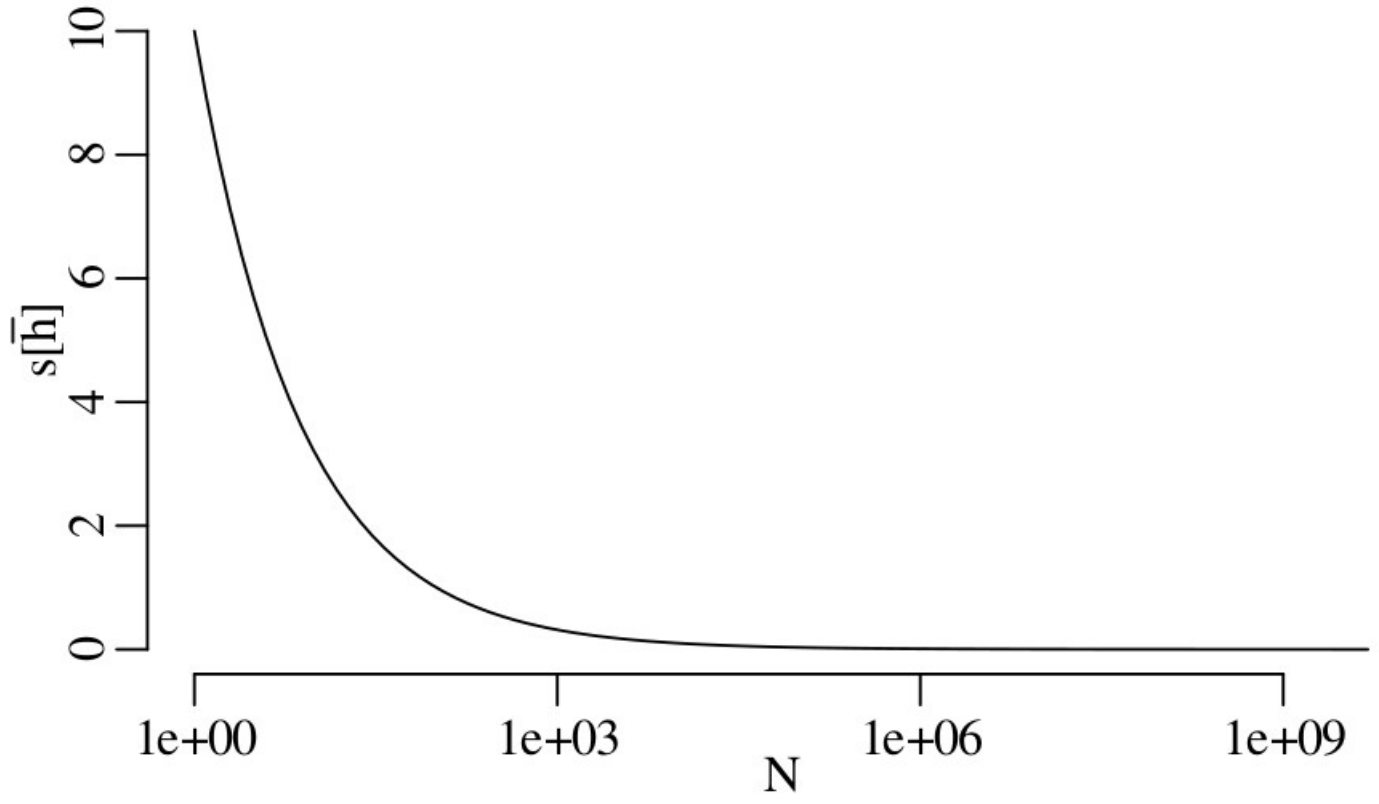
---

addition $z = a + bx + cy$ $s[z]^2 = b^2 s[x]^2 + c^2 s[y]^2$

$$s[\bar{x}]^2 = \sum_{i=1}^{n} \left(\frac{s[x_i]}{n} \right)^2 = \left(\frac{1}{n} \right)^2 \sum_{i=1}^{n} s[x_i]^2$$

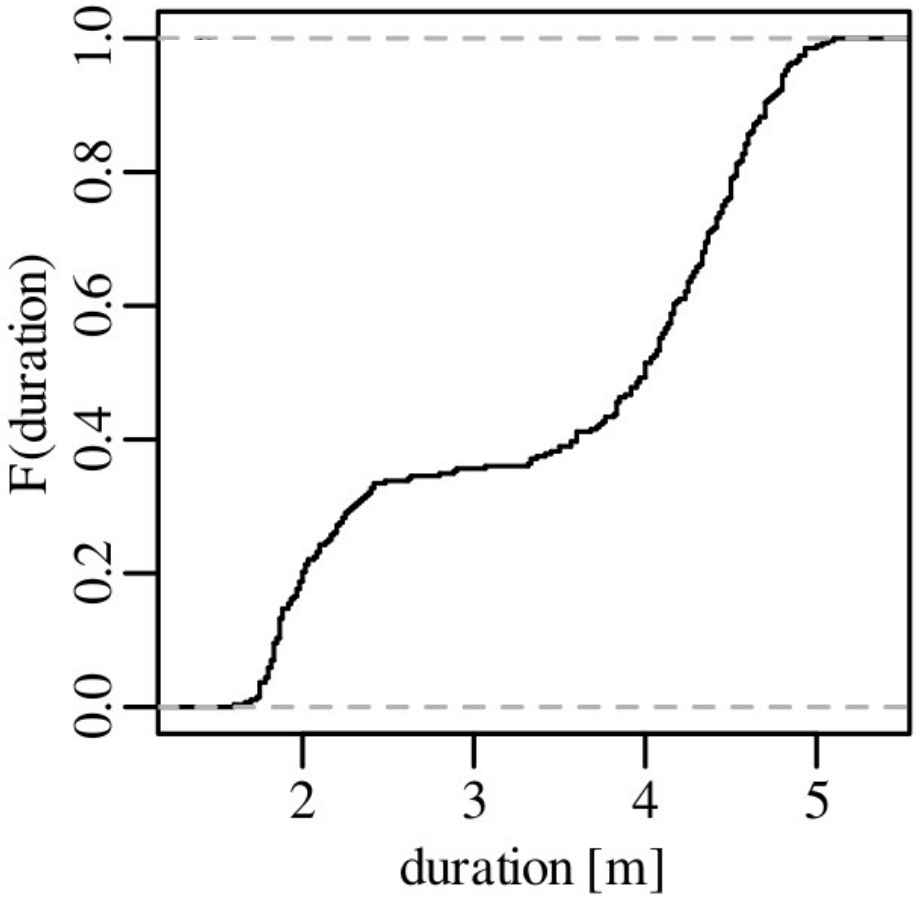
$$s[\bar{x}]^2 = \left(\frac{1}{n} \right)^2 \sum_{i=1}^{n} s[x]^2 = n \left(\frac{1}{n} \right)^2 s[x]^2$$

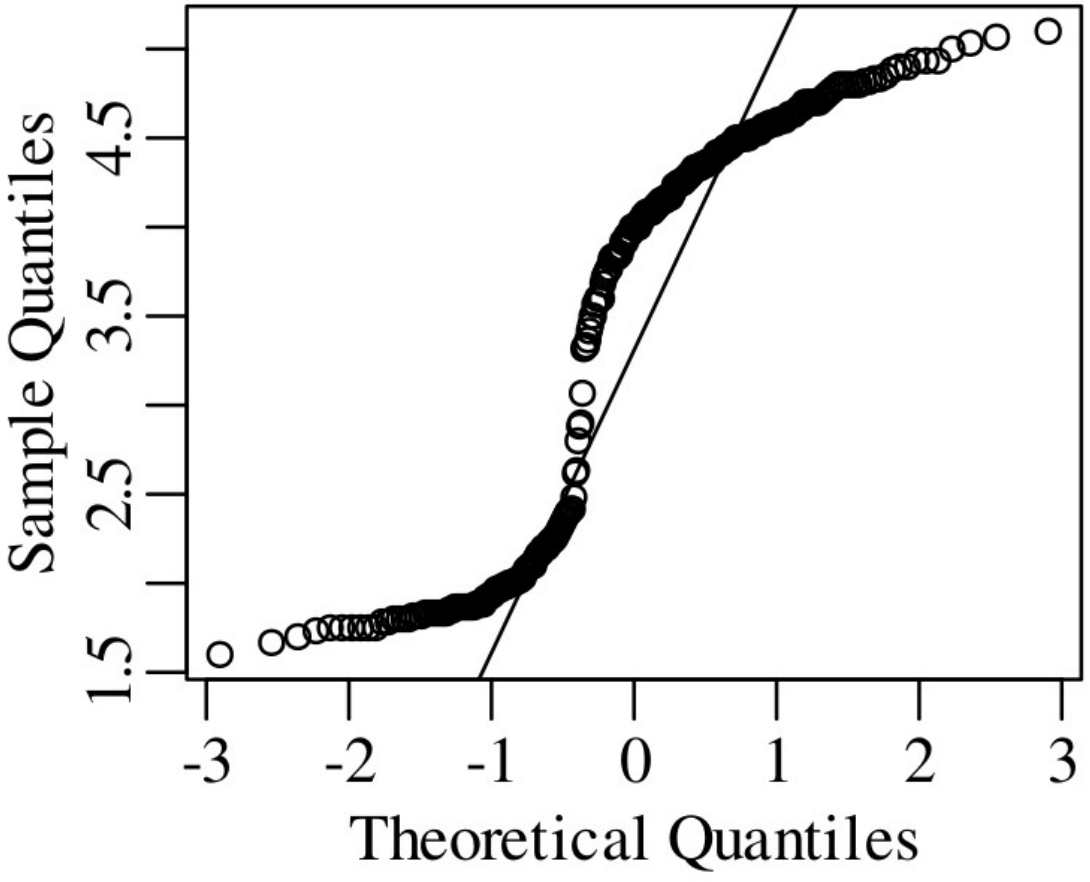
$$s[\bar{x}] = \frac{s[x]}{\sqrt{n}}$$

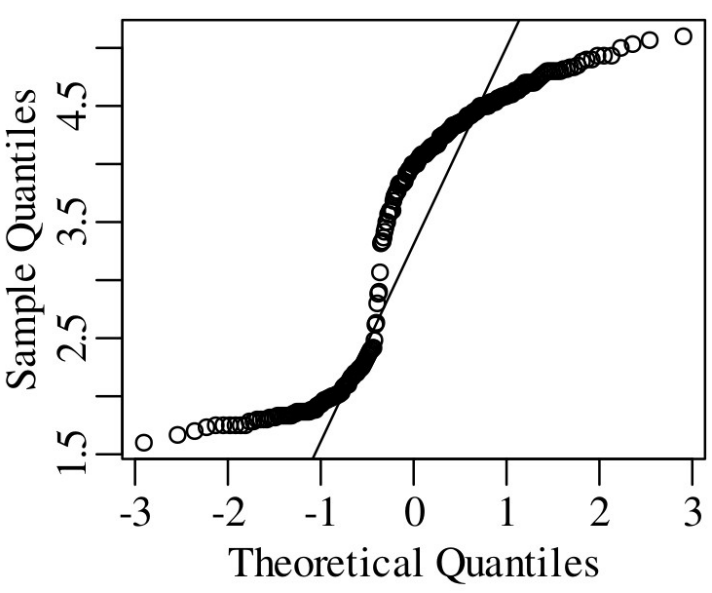

0
2
4
6
8
10
s[h̄]
1e+00
1e+03
1e+06
1e+09
N

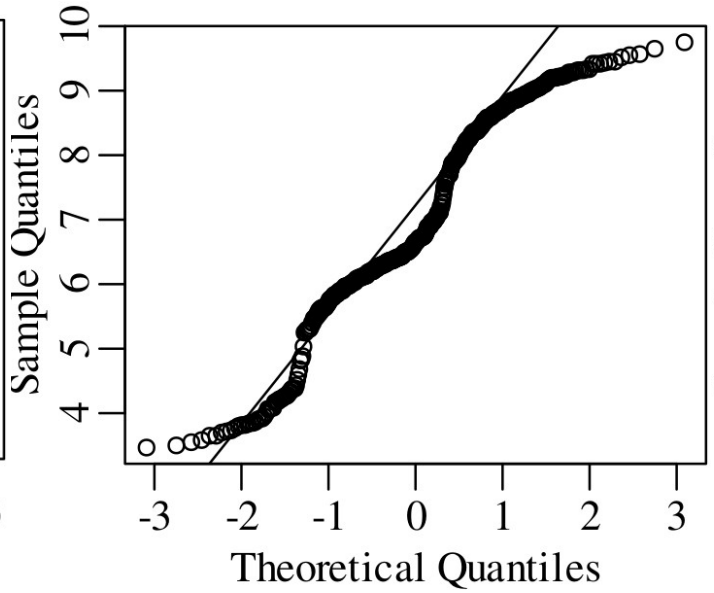
Comparing distributions

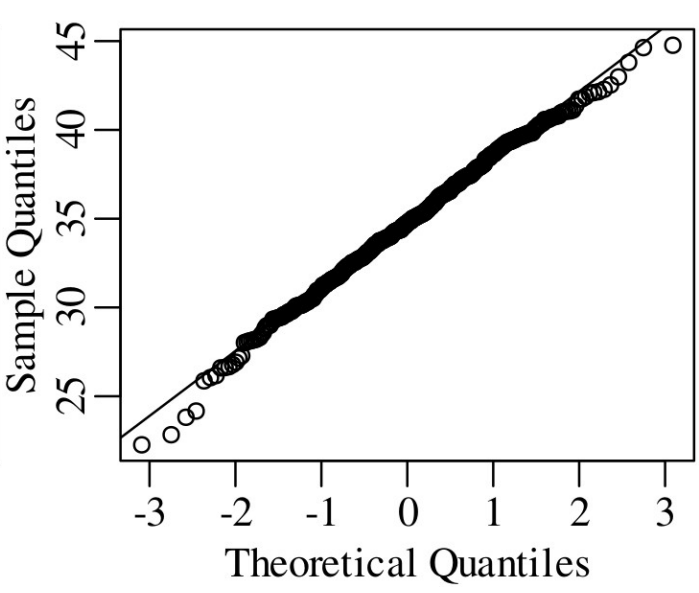
Q-Q plots

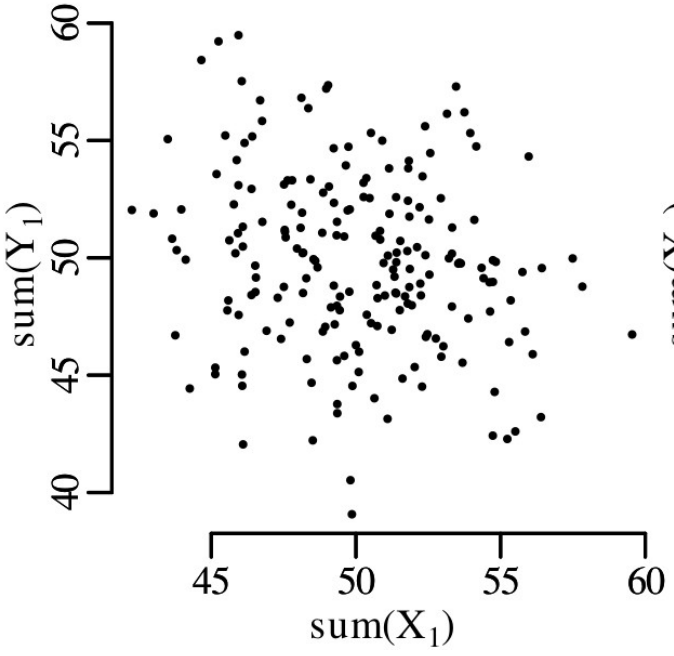

F(duration)
0.0
0.2
0.4
0.6
0.8
1.0
2
3
4
5
duration [m]

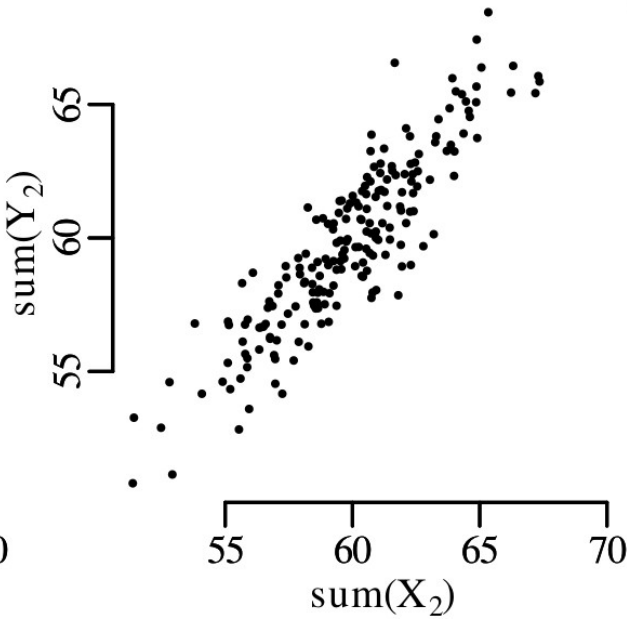

Sample Quantiles
1.5
2.5
3.5
4.5
-3
-2
-1
0
1
2
3
Theoretical Quantiles

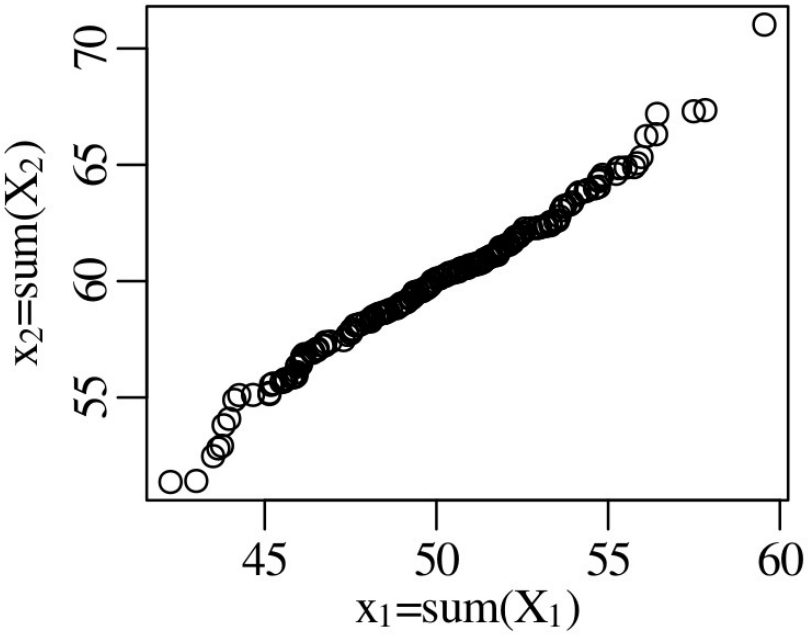

Sample Quantiles
1.5
2.5
3.5
4.5
-3
-2
-1
0
1
2
3
Theoretical Quantiles


Sample Quantiles
4
5
6
7
8
9
10
-3
-2
-1
0
1
2
3
Theoretical Quantiles


Sample Quantiles
25
30
35
40
45
-3
-2
-1
0
1
2
3
Theoretical Quantiles


sum(Y1)
40
45
50
55
60
45
50
55
60
sum(X1)


sum(Y2)
55
60
65
55
60
65
70
sum(X2)


x2=sum(X2)
55
60
65
70
45
50
55
60
x1=sum(X1)

The t-test

coin #	1	2	3	4	5
density ($\mathrm{g/cm^3}$)	19.07	19.09	19.17	19.18	19.31

$$\bar{x} = 19.164$$

$$\text{density of pure gold} = 19.30 \ \mathrm{g/cm^3}$$

coin #	1	2	3	4	5
density ($\mathrm{g/cm^3}$)	19.07	19.09	19.17	19.18	19.31

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$ follows a **standard normal distribution**

$$t = \frac{\bar{x} - \mu}{s[x]/\sqrt{n}}$$ follows a **Student t-distribution** with $(n-1)$ **degrees of freedom**

1. $H_\circ$ (**null hypothesis**) $\mu = 19.30$

   H_a (**alternative hypothesis**): $\mu < 19.30$

2. $t = \dfrac{\bar{x} - \mu_\circ}{s[x]/\sqrt{n}}$ where $\bar{x} = 19.164$, $\mu_\circ = 19.30$, $s[x] = 0.0948$, $n = 5$ so that $t = -3.2091$

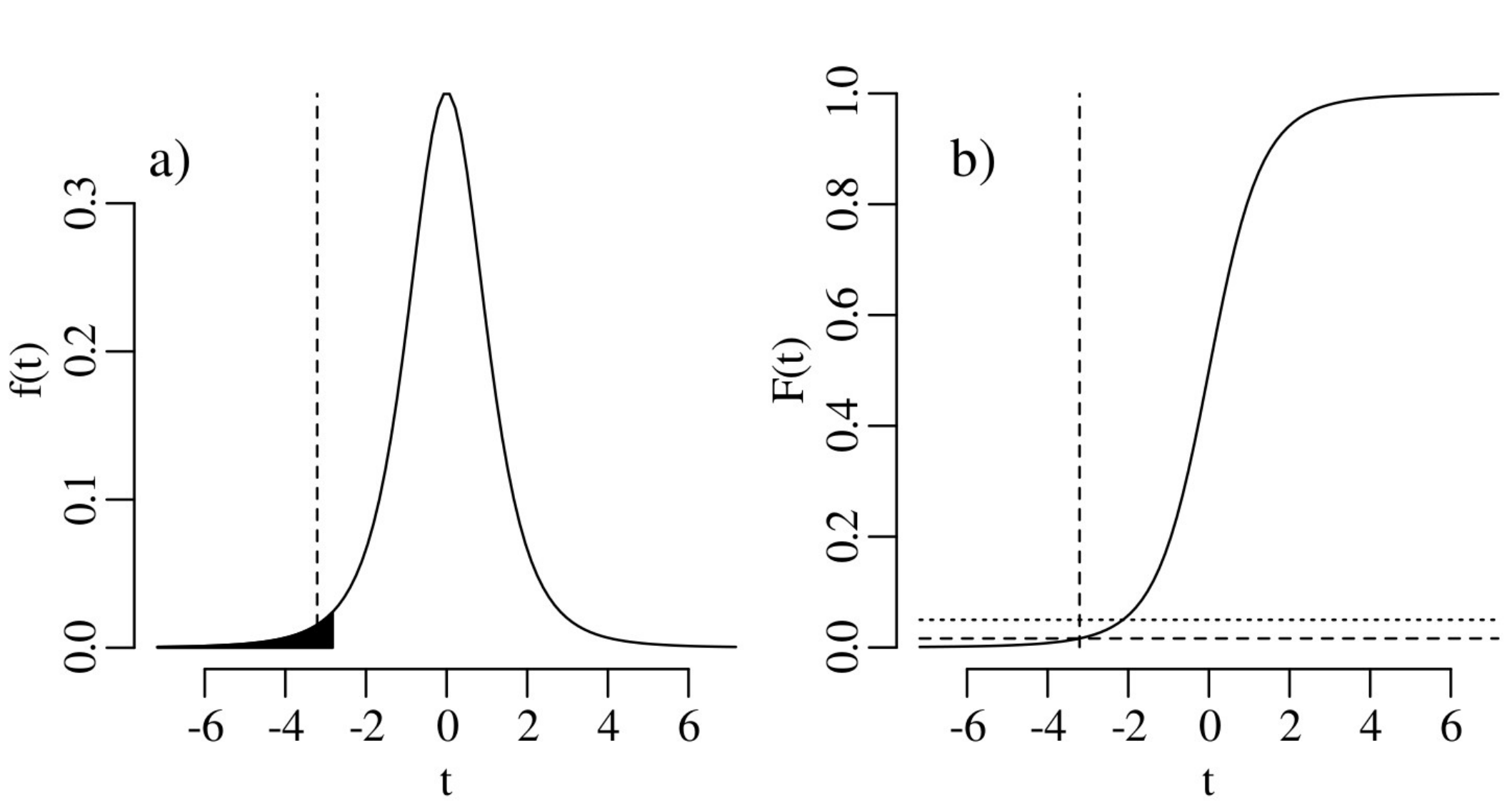
3.

t	-3.70	*-3.2091*	-2.80	-2.10	-1.50	-0.74	0.00	0.74	1.50	2.10	2.80	3.70
$P(T \le t)$	0.01	*0.0163*	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99

4. $\alpha = 0.05$

5.

t	**-3.70**	***-3.2091***	**-2.80**	**-2.10**	-1.50	-0.74	0.00	0.74	1.50	2.10	2.80	3.70
$P(T \le t)$	**0.01**	***0.0163***	**0.025**	**0.05**	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99


a)
f(t)
0.0
0.1
0.2
0.3
-6
-4
-2
0
2
4
6
t
b)
F(t)
0.0
0.2
0.4
0.6
0.8
1.0
-6
-4
-2
0
2
4
6
t

two sample t-test

coin #	1	2	3	4	5
density (1^{st} collection)	19.07	19.09	19.17	19.18	19.31
density (2^{nd} collection)	19.17	19.30	19.31	19.32	

$$\bar{x}_1 = 19.164 \longleftrightarrow \bar{x}_2 = 19.275$$

1. $H_\circ$ (**null hypothesis**) $\mu_1 = \mu_2$

   H_a (**alternative hypothesis**): $\mu_1 \neq \mu_2$

2. $$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{where} \quad s_p = \sqrt{\frac{(n_1-1)s[x_1]^2 + (n_2-1)s[x_2]^2}{n_1+n_2-2}}$$
   $$= -2.014$$

3. Student t-distribution with (n_1+n_2-2) degrees of freedom

t	-3.00	-2.40	*-2.014*	-1.90	-1.40	-0.71	0.00	0.71	1.40	1.90	2.40	3.00
$P(T \leq t)$	0.01	0.025	*0.042*	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(T \geq t)$	0.99	0.975	*0.958*	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.01

4. $\alpha = 0.05$

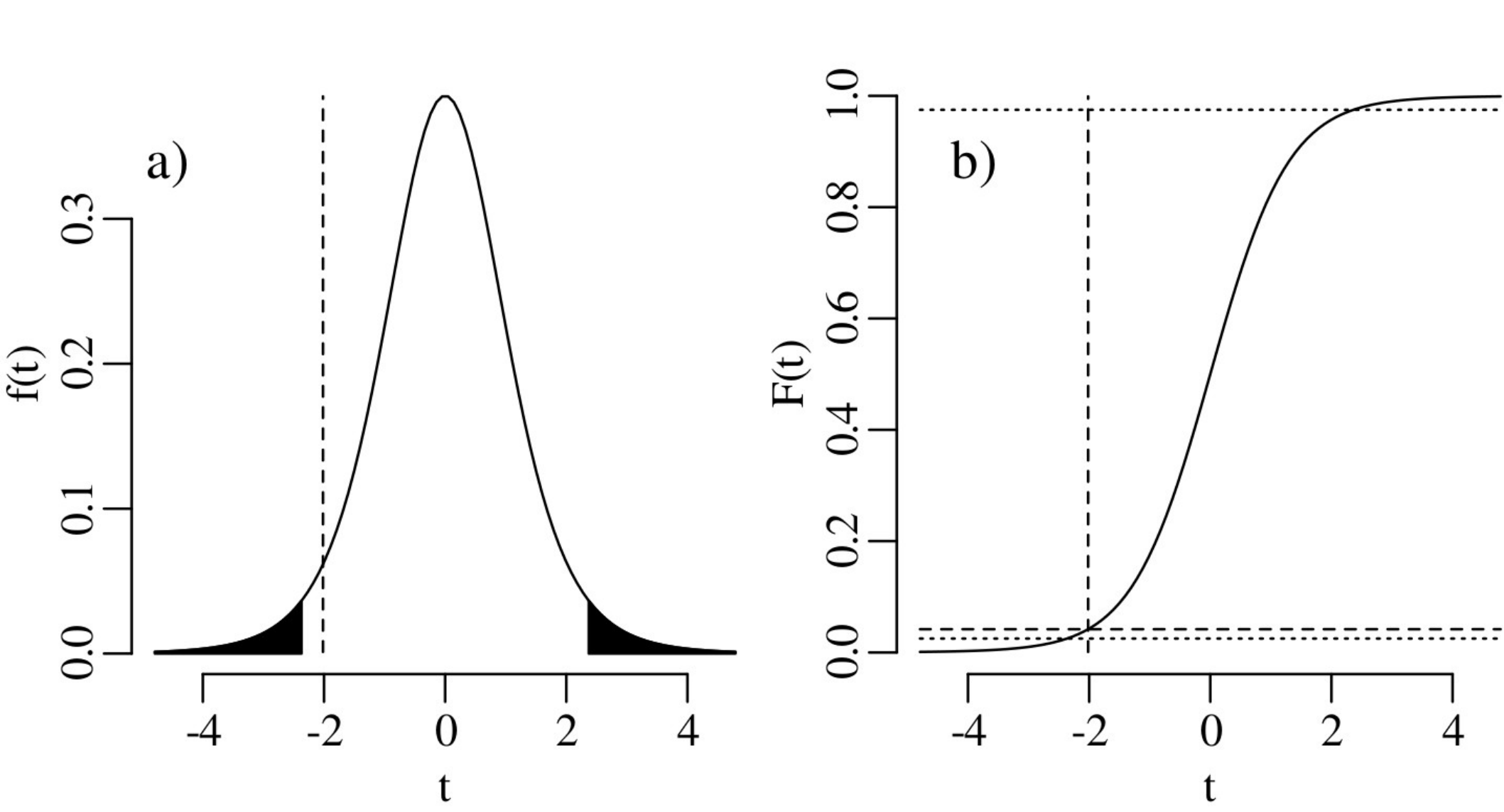
5.

t	**-3.00**	**-2.40**	*-2.014*	-1.90	-1.40	-0.71	0.00	0.71	1.40	1.90	**2.40**	**3.00**
$P(T \leq t)$	**0.01**	**0.025**	*0.042*	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(T \geq t)$	0.99	0.975	*0.958*	0.95	0.9	0.75	0.5	0.25	0.1	0.05	**0.025**	**0.01**

$$R = \{t < -2.40 \quad \text{and} \quad t > 2.40\}$$

6. $t = -2.014 \quad \notin R$

7. p-value $= 0.084 \ (= 2 \times 0.042) > 0.05$


a)
f(t)
0.0
0.1
0.2
0.3
-4
-2
0
2
4
t
b)
F(t)
0.0
0.2
0.4
0.6
0.8
1.0
-4
-2
0
2
4
t

Confidence intervals

$$t = \frac{\bar{x} - \mu}{s[x]/\sqrt{n}}$$

$$t_{df,\alpha/2} \leq t \leq t_{df,1-\alpha/2}$$

$$t_{df,\alpha/2} \leq \frac{\bar{x} - \mu}{s[x]/\sqrt{n}} \leq t_{df,1-\alpha/2}$$

$$\bar{x} - t_{df,\alpha/2}\frac{s[x]}{\sqrt{n}} \geq \mu \geq \bar{x} - t_{df,1-\alpha/2}\frac{s[x]}{\sqrt{n}}$$

with $t_{df,1-\alpha/2} = -t_{df,\alpha/2}$

$$\bar{x} + t_{df,\alpha/2}\frac{s[x]}{\sqrt{n}} \leq \mu \leq \bar{x} - t_{df,\alpha/2}\frac{s[x]}{\sqrt{n}}$$

$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2}\frac{s[x]}{\sqrt{n}} \right\}$$

$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\}$$

coin #	1	2	3	4	5
density ($\mathrm{g/cm^3}$)	19.07	19.09	19.17	19.18	19.31

$$\bar{x} = 19.164$$

$$s[x] = 0.0948$$

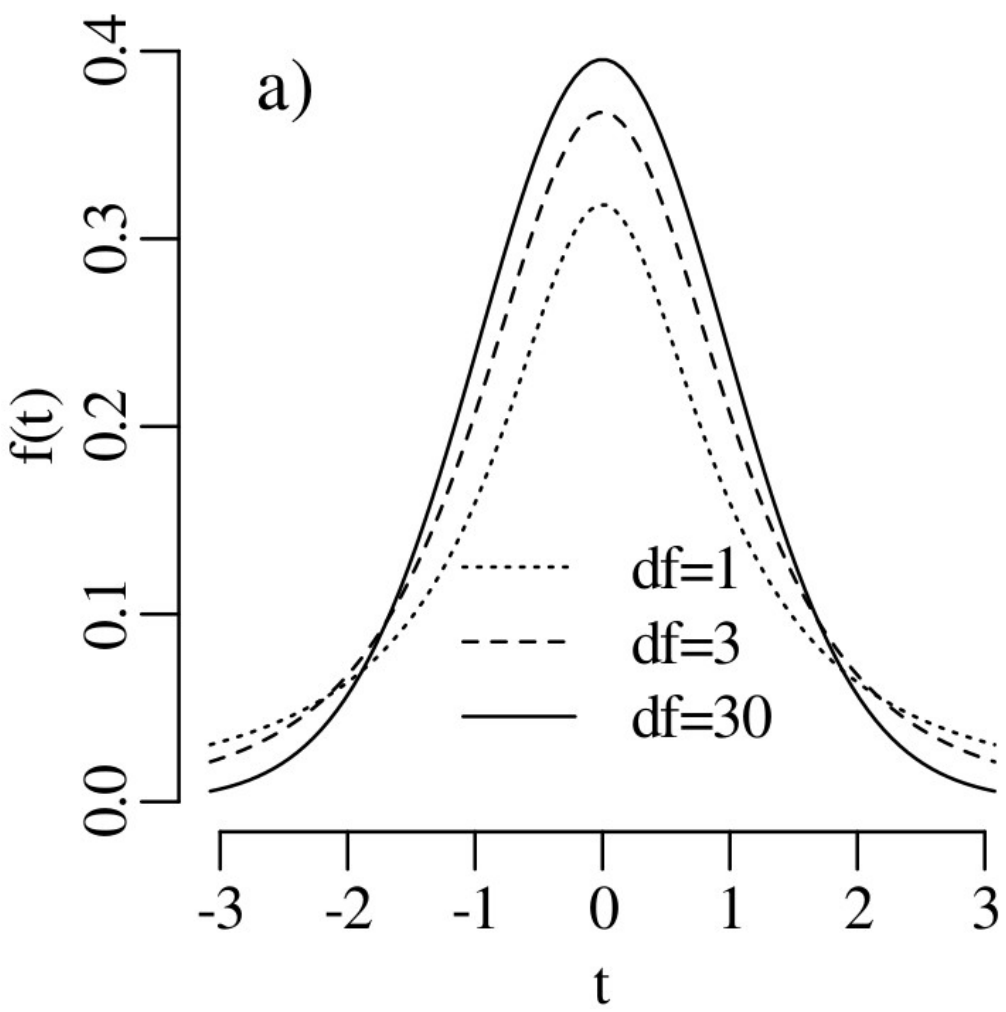
$$df = 4$$

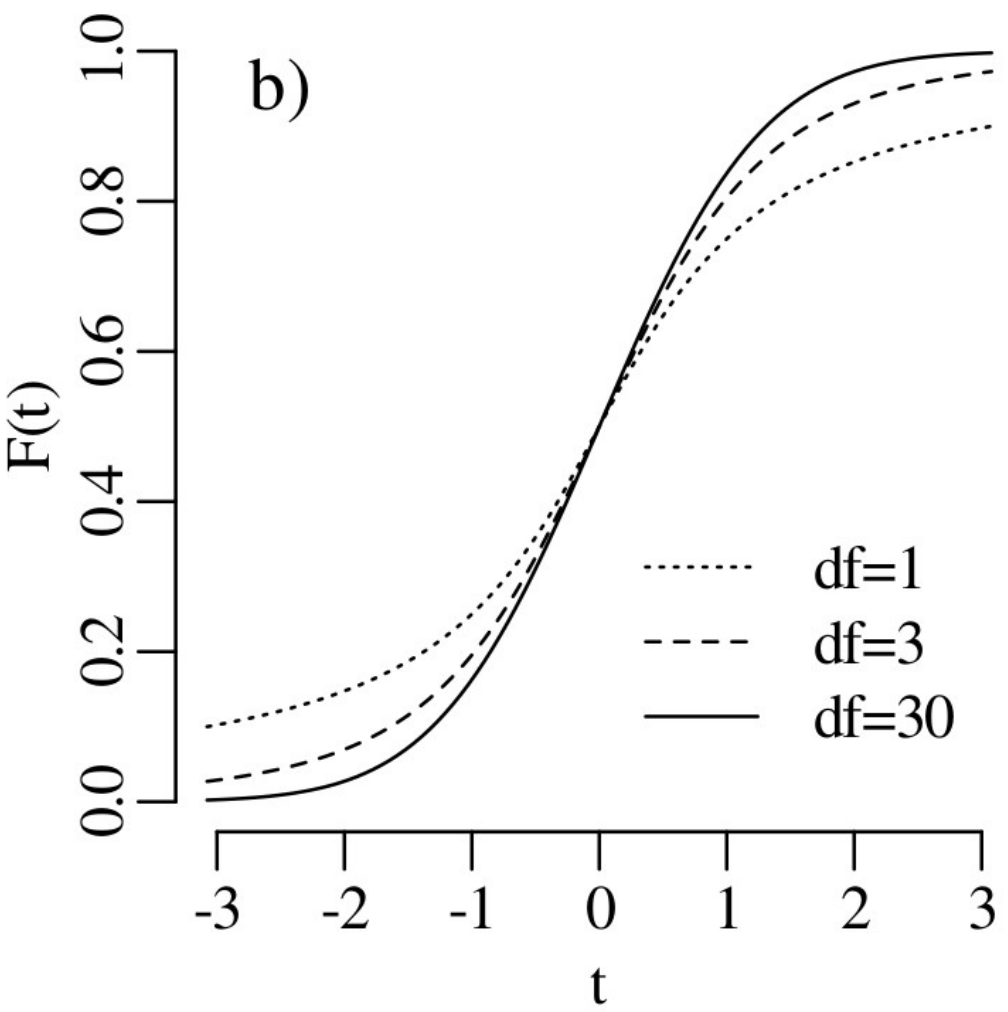
$$t_{4,0.025} = -2.776$$

$$\mu = 19.16 \pm 0.12 \ \mathrm{g/cm^3}$$

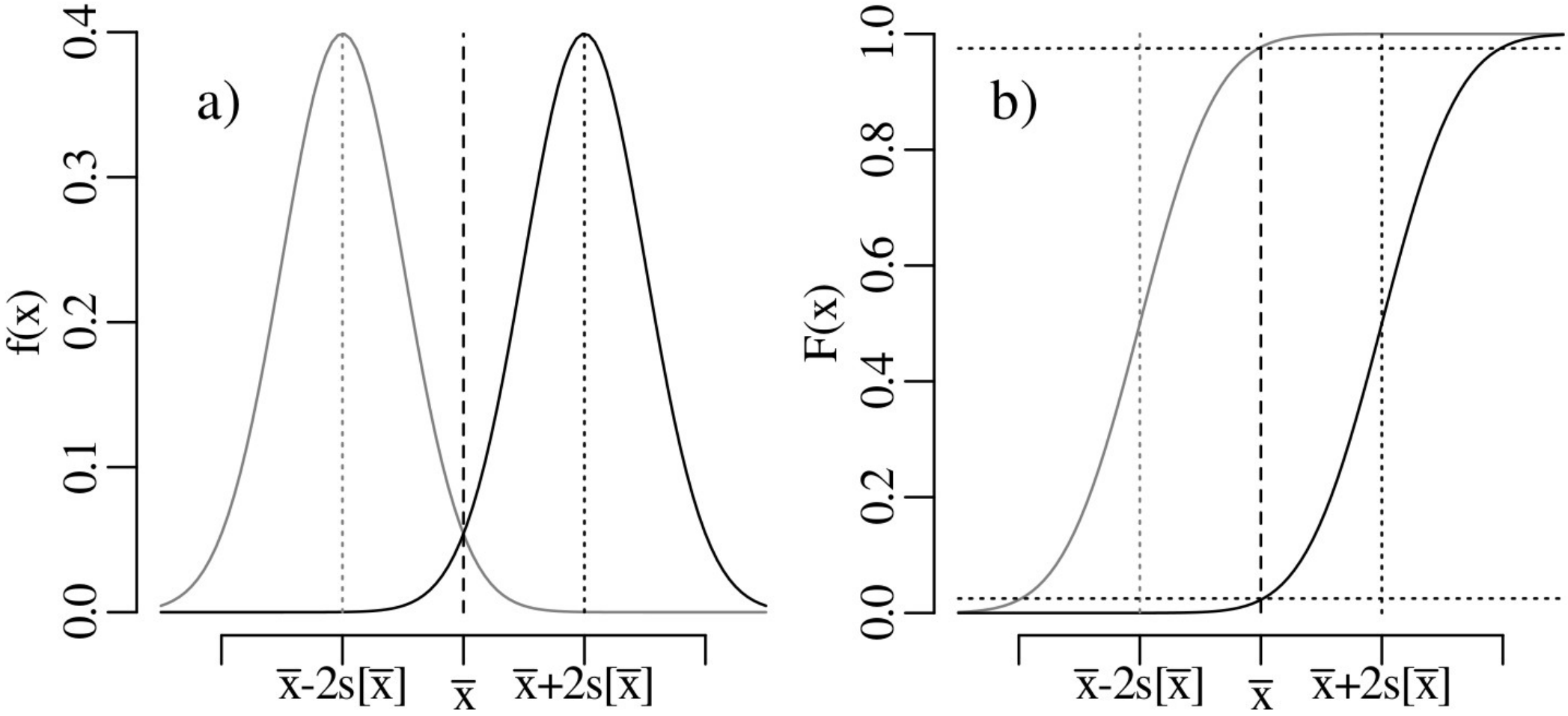
df	1	2	3	4	5	6	7	8	9	10	30	100	1000
$t_{df,0.975}$	12.710	4.303	3.182	2.776	2.571	2.447	2.365	2.306	2.262	2.228	2.042	1.984	**1.962**

$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\} \longrightarrow \mu \in \{\bar{x} \pm 2s[\bar{x}]\}$$

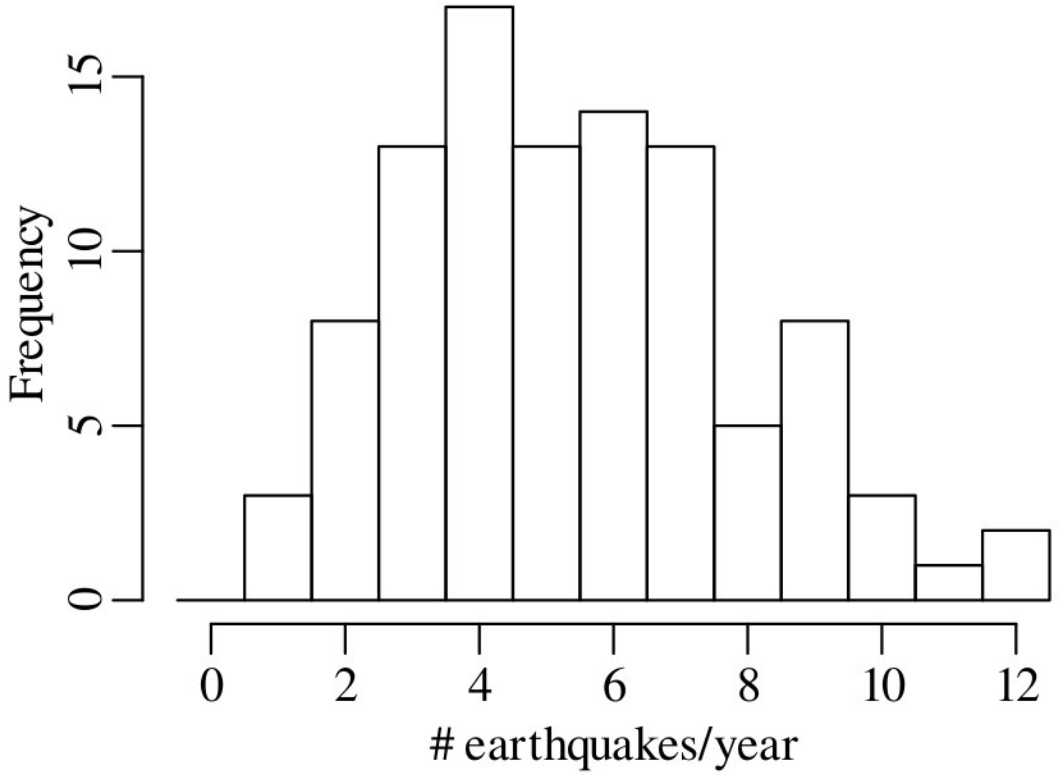




$$\mu \in \left\{ \bar{x} \pm t_{df,\alpha/2} \frac{s[x]}{\sqrt{n}} \right\} \longrightarrow \mu \in \{\bar{x} \pm 2s[\bar{x}]\}$$


a)
f(x)
0.0
0.1
0.2
0.3
0.4
x̄-2s[x̄]
x̄
x̄+2s[x̄]
b)
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
x̄-2s[x̄]
x̄
x̄+2s[x̄]

The χ^2-test



mean: 5.43
standard deviation: 2.50
variance: 6.24

number of earthquakes per year	0	1	2	3	4	5	6	7	8	9	10	11	12
number of years	0	3	8	13	17	13	14	13	5	8	3	1	2

observed

number of earthquakes per year	0	1	2	3	4	5	6	7	8	9	10	11	12
number of years	0	3	8	13	17	13	14	13	5	8	3	1	2

number of earthquakes per year	≤ 2	3	4	5	6	7	8	9	≥ 10
number of years	11	13	17	13	14	13	5	8	6

expected $$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

number of earthquakes per year (k)	≤ 2	3	4	5	6	7	8	9	≥ 10
$N \times P(k\|\lambda = 5.43)$	9.28	11.7	15.9	17.2	15.6	12.1	8.22	4.96	5.02

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

1. $H_\circ$ (**null hypothesis**): the earthquake data follow a Poisson distribution

   H_a (**alternative hypothesis**): the earthquake data do not follow a Poisson distribution

2. $$\chi^2 = \frac{(11-9.28)^2}{9.28} + \frac{(13-11.7)^2}{11.7} + \frac{(17-15.9)^2}{15.9} + \frac{(13-17.2)^2}{17.2} + \frac{(14-15.6)^2}{15.6} + \frac{(13-12.1)^2}{12.1} + \frac{(5-8.22)^2}{8.22} + \frac{(8-4.96)^2}{4.96} + \frac{(6-5.02)^2}{5.02} = 5.14$$

3. χ^2 distribution with $n-2$ degrees of freedom

χ^2	1.24	1.69	2.17	2.83	4.25	*5.14*	6.35	9.04	12.0	14.1	16.0	18.5
$P(X \leq \chi^2)$	0.01	0.025	0.05	0.1	0.25	*0.36*	0.5	0.75	0.9	0.95	0.975	0.99
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	*0.743*	0.5	0.25	0.1	0.05	0.025	0.01

4. $\alpha = 0.05$

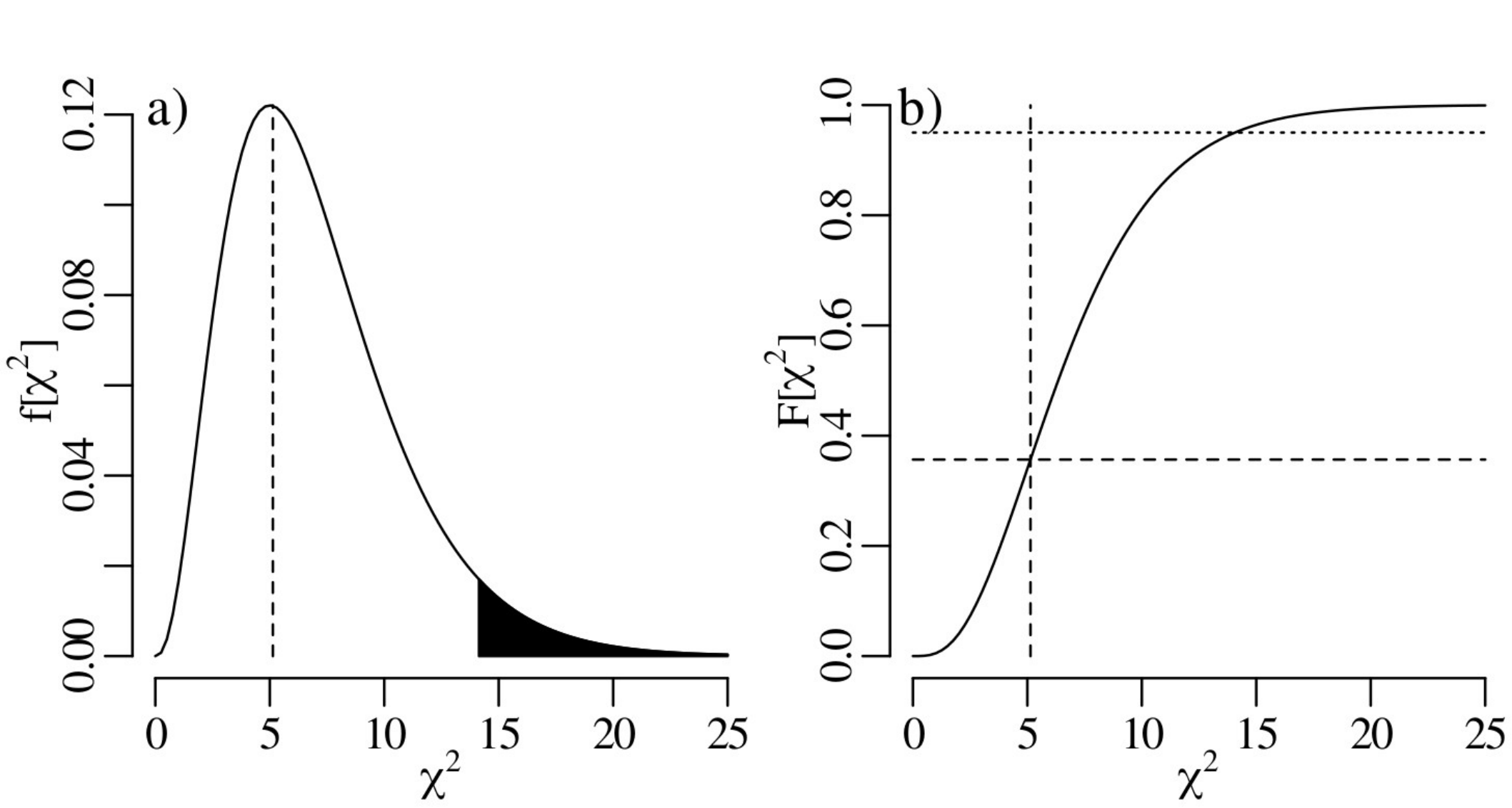
5.

χ^2	1.24	1.69	2.17	2.83	4.25	*5.14*	6.35	9.04	12.0	**14.1**	**16.0**	**18.5**
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	*0.743*	0.5	0.25	0.1	**0.05**	**0.025**	**0.01**

$$R = \{\chi^2 \geq 14.1\}$$

6. $\chi^2 = 5.14 \notin R$

7. p-value $= 0.743 > 0.05$


a)
b)
f[χ²]
F[χ²]
0.00
0.04
0.08
0.12
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
χ²

Comparing two or more samples

lithology	granite	basalt	gneiss	quartzite
sample A	10	5	6	20
sample B	25	12	10	35

lithology	granite	basalt	gneiss	quartzite	row sum
sample A	10	5	6	20	41
sample B	25	12	10	35	82
column sum	35	17	16	55	123

$$\text{expected counts of bin } (i,j) = \frac{(\text{sum of row } i) \times (\text{sum of column } j)}{(\text{sum of all the cells})}$$

lithology	granite	basalt	gneiss	quartzite
sample A	11.7	5.67	5.33	18.3
sample B	23.3	11.30	10.70	36.7

1. $H_\circ$ (**null hypothesis**): samples A and B have the same composition

   H_a (**alternative hypothesis**): samples A and B do not have the same composition

2. $$\chi^2 = \frac{(10-11.7)^2}{11.7} + \frac{(5-5.67)^2}{5.67} + \frac{(6-5.33)^2}{5.33} + \frac{(20-18.3)^2}{18.3} + \frac{(25-23.3)^2}{23.3} + \frac{(12-11.30)^2}{11.30} + \frac{(10-10.70)^2}{10.70} + \frac{(35-36.7)^2}{36.7} = 0.86$$

3. χ^2-distribution with $(2-1)\times(4-1) = 3$ degrees of freedom

χ^2	0.115	0.216	0.352	0.584	*0.86*	1.21	2.37	4.11	6.25	7.81	9.35	11.3
$P(X \le \chi^2)$	0.01	0.025	0.05	0.1	*0.157*	0.25	0.5	0.75	0.9	0.95	0.975	0.99
$P(X \ge \chi^2)$	0.99	0.975	0.95	0.9	*0.843*	0.75	0.5	0.25	0.1	0.05	0.025	0.01

4. $\alpha = 0.05$

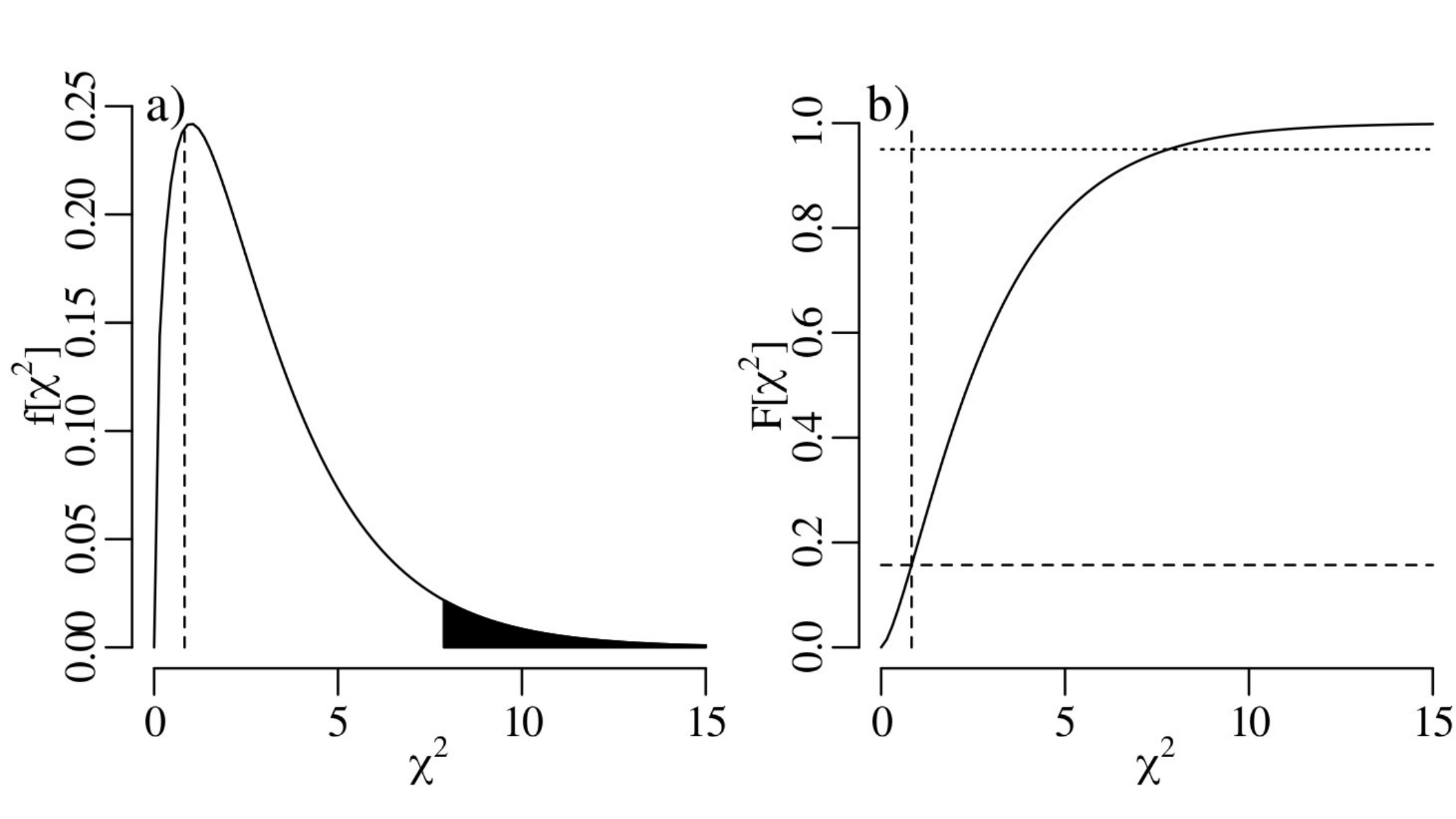
5.

χ^2	0.115	0.216	0.352	0.584	*0.86*	1.21	2.37	4.11	6.25	**7.81**	**9.35**	**11.3**
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	*0.843*	0.75	0.5	0.25	0.1	**0.05**	**0.025**	**0.01**

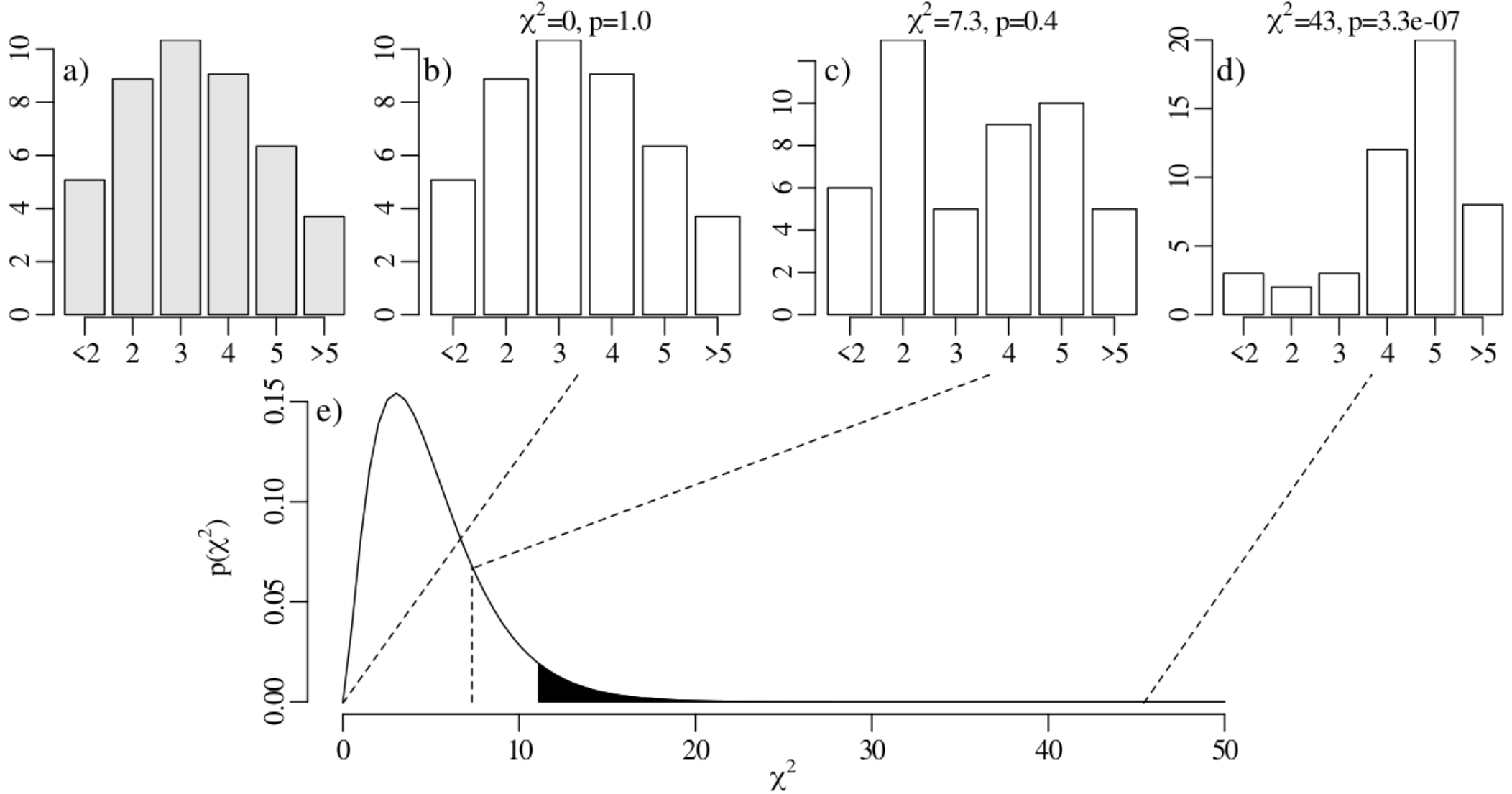
$$R = \{\chi^2 > 7.81\}$$

6. $\chi^2 = 0.86 \notin R$

7. p-value $= 0.843 > 0.05$


a)
b)
f[χ²]
F[χ²]
0.00
0.05
0.10
0.15
0.20
0.25
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
χ²

Cherry picking (Type-I errors revisited)



Effect size (Type-II errors revisited)

lithology	quartz	plagioclase	alkali feldspar	lithics
sample A	29544	14424	13706	47864
sample B	29454	14788	13948	47311

lithology	quartz	plagioclase	alkali feldspar	lithics	row sum
sample A	29544	14424	13706	47864	105538
sample B	29454	14788	13948	47311	105501
column sum	58998	29212	27654	95175	211039

1. $H_\circ$ (**null hypothesis**): samples A and B have identical compositions

   H_a (**alternative hypothesis**): samples A and B have different compositions

2.

lithology	quartz	plagioclase	alkali feldspar	lithics
sample A	29504	14609	13829	47596
sample B	29494	14603	13825	47579

$\longrightarrow \chi^2 = 10.0$

3. χ^2-distribution 3 degrees of freedom

χ^2	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.81	9.35	*10.0*	11.3
$P(X \leq \chi^2)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	*0.9814*	0.99
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	*0.0186*	0.010

4. $\alpha = 0.05$

5.

χ^2	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	**7.81**	**9.35**	***10.0***	**11.3**
$P(X \geq \chi^2)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	**0.05**	**0.025**	***0.0186***	**0.010**

$$R = \{\chi^2 > 7.81\}$$

6. $\chi^2 = 10.0 \in R$

7. p-value $= 0.0186 < \alpha$

depends on the **confidence level** α

$H_\circ$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$\beta = 1 - \text{power}$

depends on 1. **The degree to which $H_\circ$ is false**

2. **Sample size**

depends on the **confidence level** α

$H_\circ$ is ...	false	true
rejected	correct decision	Type-I error
not rejected	Type-II error	correct decision

$\beta = 1 - \text{power}$

depends on

1. **The degree to which $H_\circ$ is false**
2. **Sample size**

effect size

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$w = \sqrt{\sum_{i=1}^{n} \frac{(o_i - e_i)^2}{e_i}}$$

where $o_i = O_i/N$ and $e_i = E_i/N$

with $N = \sum_i^n O_i = \sum_i^n E_i$

effect size	small	medium	large
w	0.1	0.3	0.5

effect size	small	medium	large
w	0.1	0.3	0.5

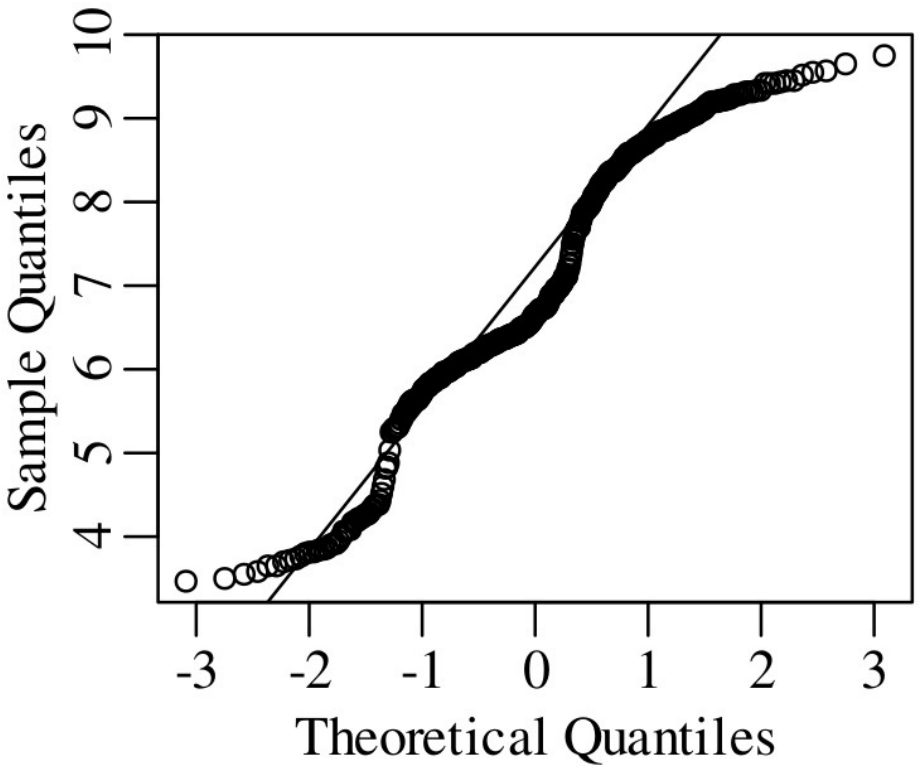
$o =$

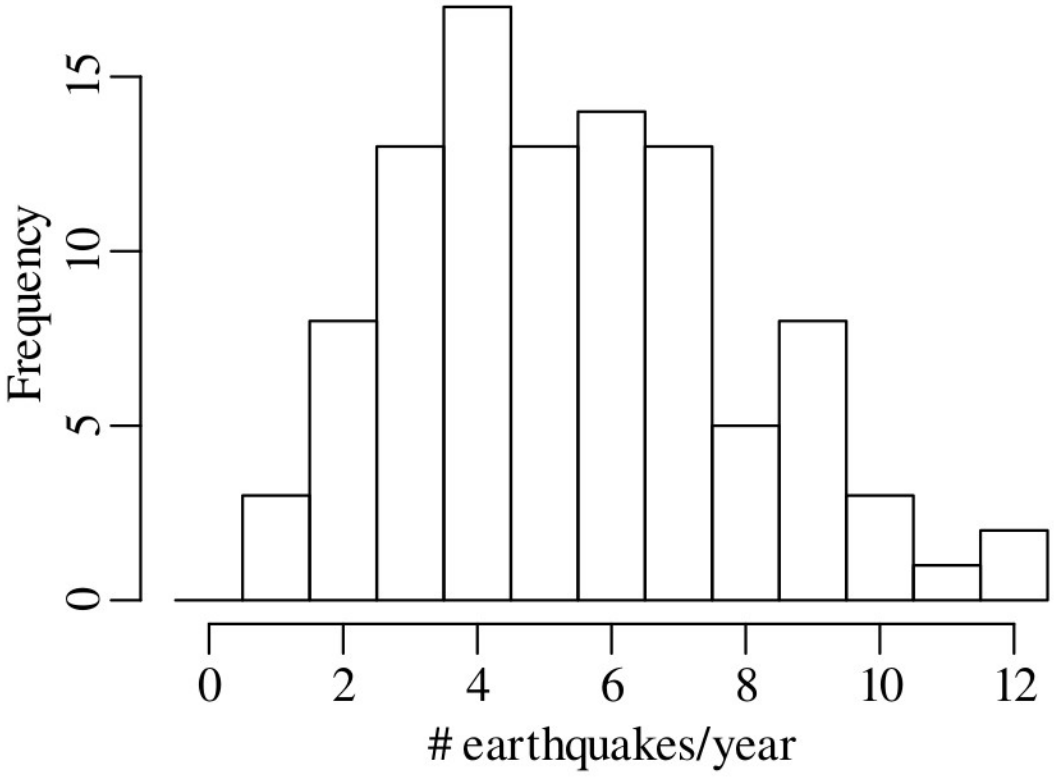
lithology	quartz	plagioclase	alkali feldspar	lithics
sample A	0.1400	0.06835	0.06495	0.2268
sample B	0.1396	0.07007	0.06609	0.2242

$e =$

lithology	quartz	plagioclase	alkali feldspar	lithics
sample A	0.1398	0.06922	0.06553	0.2255
sample B	0.1398	0.06920	0.06551	0.2255

$$w = 0.00688$$


Sample Quantiles
4
5
6
7
8
9
10
-3
-2
-1
0
1
2
3
Theoretical Quantiles


Frequency
0
5
10
15
0
2
4
6
8
10
12
earthquakes/year

Non-parametric tests

1. Wilcoxon test

coin #	1	2	3	4	5
density (1^{st} collection)	19.07	19.09	19.17	19.18	19.31
density (2^{nd} collection)	19.17	19.30	19.31	19.32	

sample	1	2	3	4	5
A	A_1	A_2	A_3	A_4	A_5
B	B_1	B_2	B_3	B_4	

sample	1	2	3	4	5
A	A_1	A_2	A_3	A_4	A_5
B	B_1	B_2	B_3	B_4	

suppose that

$$A_4 < A_1 < A_2 < B_3 < A_5 < B_1 < B_4 < A_3 < B_2$$

then

rank	1	2	3	4	5	6	7	8	9
value	A_4	A_1	A_2	B_3	A_5	B_1	B_4	A_3	B_2

$$W = 4 + 6 + 7 + 9 = 26$$

e.g. $W = 10$

arrangement 1:	B_1	B_2	B_3	B_4	A_1	A_2	A_3	A_4	A_5
arrangement 2:	B_2	B_1	B_3	B_4	A_1	A_2	A_3	A_4	A_5
arrangement 2:	B_1	B_2	B_3	B_4	A_2	A_1	A_3	A_4	A_5
etc.									

$$P(W = 10|n_A = 5, n_B = 4) = \frac{4!5!}{9!} = 0.00794$$

coin #	1	2	3	4	5
density (1st collection)	19.07	19.09	19.17	19.18	19.31
density (2nd collection)	**19.17**	**19.30**	**19.31**	**19.32**	

1. $H_\circ$ (**null hypothesis**) median(sample 1) = median(sample 2)

   H_a (**alternative hypothesis**): median(sample 1) $\neq$ median(sample 2)

2.

rank	1	2	3.5	**3.5**	5	**6**	7.5	**7.5**	**9**
density	19.07	19.09	19.17	**19.17**	19.18	**19.30**	**19.31**	19.31	**19.32**

$$W = 3.5 + 6 + 7.5 + 9 = 26$$

3.

W	10	11	12	14	16	19	22	24	*26*	27	28
$P(w \leq W)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	*0.95*	0.975	0.99
$P(w \geq W)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	*0.05*	0.025	0.010

4. $\alpha = 0.05$

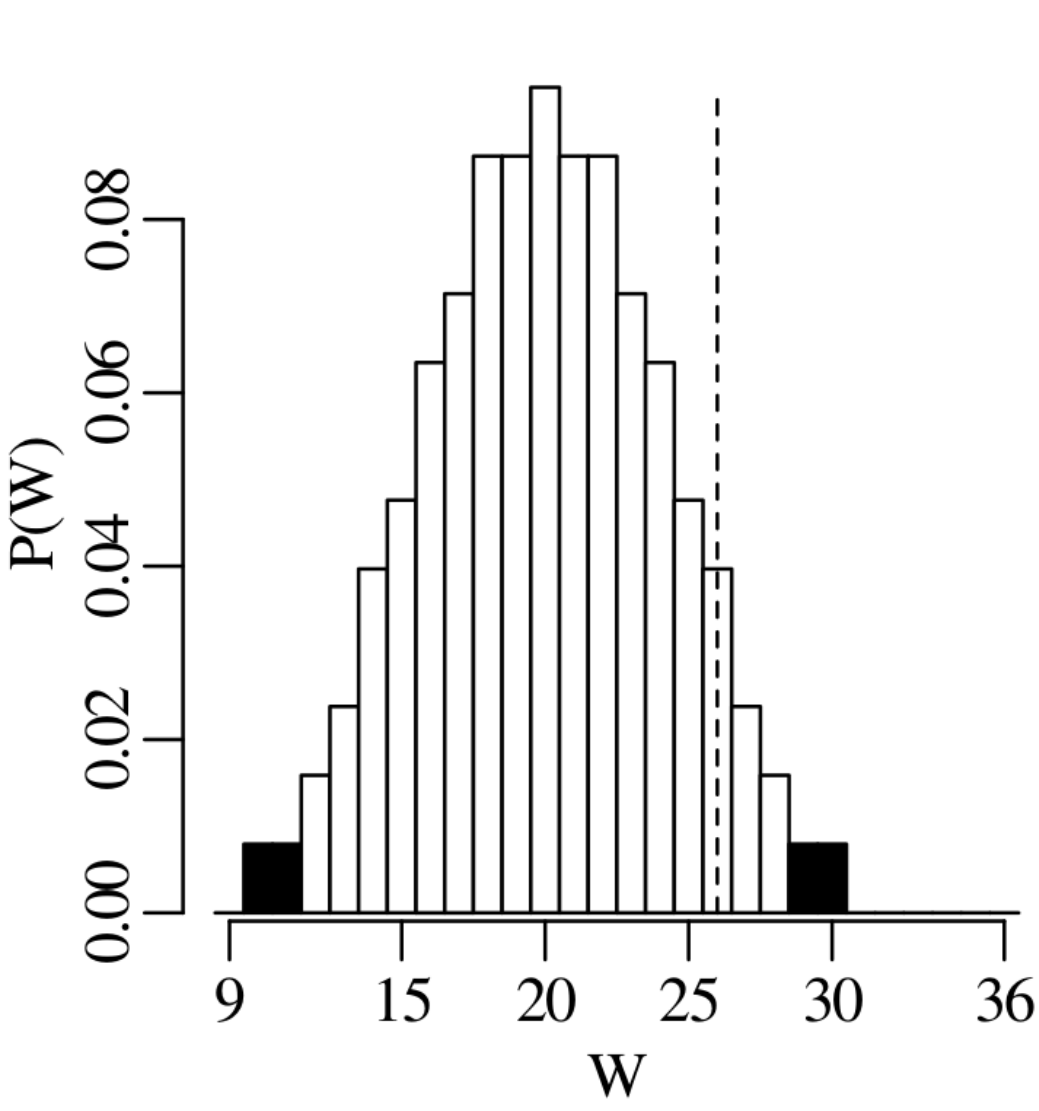
5.

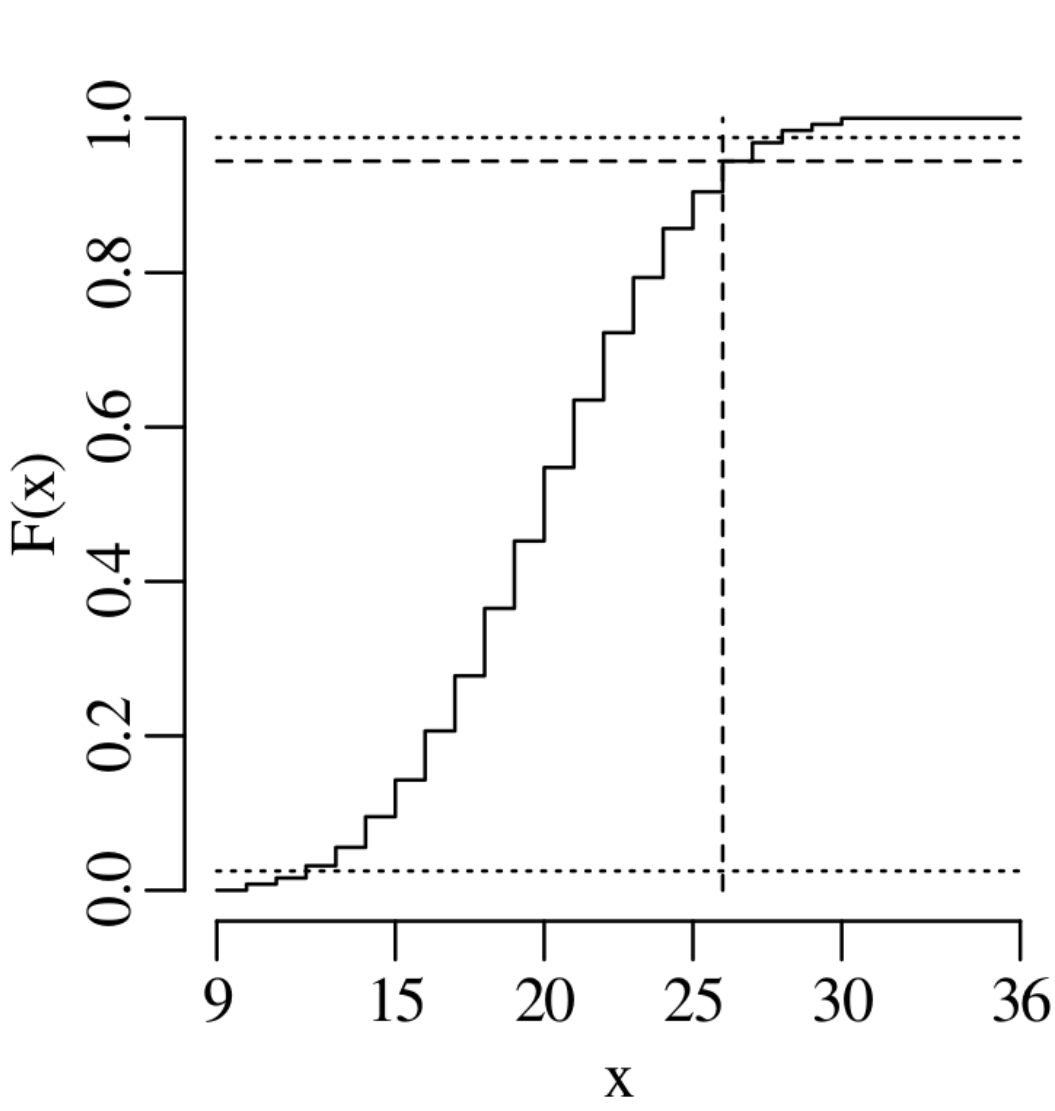
W	**10**	11	12	14	16	19	22	24	26	27	**28**
$P(w \leq W)$	**0.01**	0.025	0.05	0.1	0.25	0.5	0.75	0.9	*0.95*	0.975	0.99
$P(w \geq W)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	*0.05*	0.025	**0.01**

$$R = \{ W = 10 \text{ and } W > 27 \}$$

6. $W = 26 \quad \notin R$

7. p-value $= \ 0.10 > \alpha$


P(W)
0.00
0.02
0.04
0.06
0.08
9
15
20
25
30
36
W


F(x)
0.0
0.2
0.4
0.6
0.8
1.0
9
15
20
25
30
36
x

2. **Kolmogorov-Smirnov test**

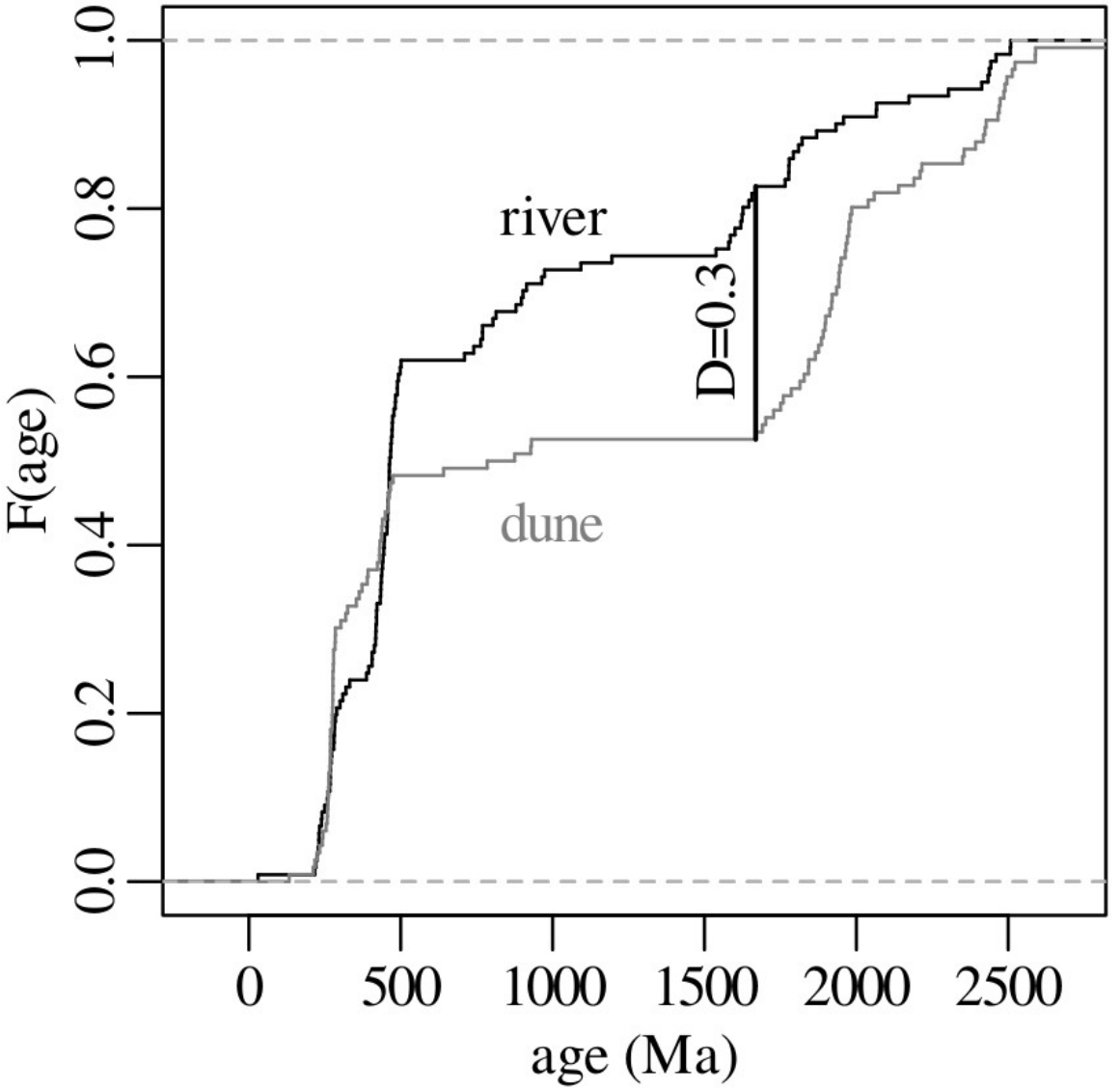
Given

$x = \{x_1, x_2, \ldots, x_n\}$

and

$y = \{y_1, y_2, \ldots, y_m\}$

$D = \max_z |F_x(z) - F_y(z)|$



1. $H_\circ$ (**null hypothesis**): samples 1 and 2 were drawn from the same distribution

   H_a (**alternative hypothesis**): samples 1 and 2 were drawn from different distributions

2. $D = 0.30$

3.

D	0.061	0.061	0.070	0.078	0.087	0.104	0.130	0.157	0.174	0.191	0.209	*0.301*
$P(d \leq D)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99	*0.99996*
$P(d \geq D)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.010	4.5×10^{-5}

4. $\alpha = 0.05$

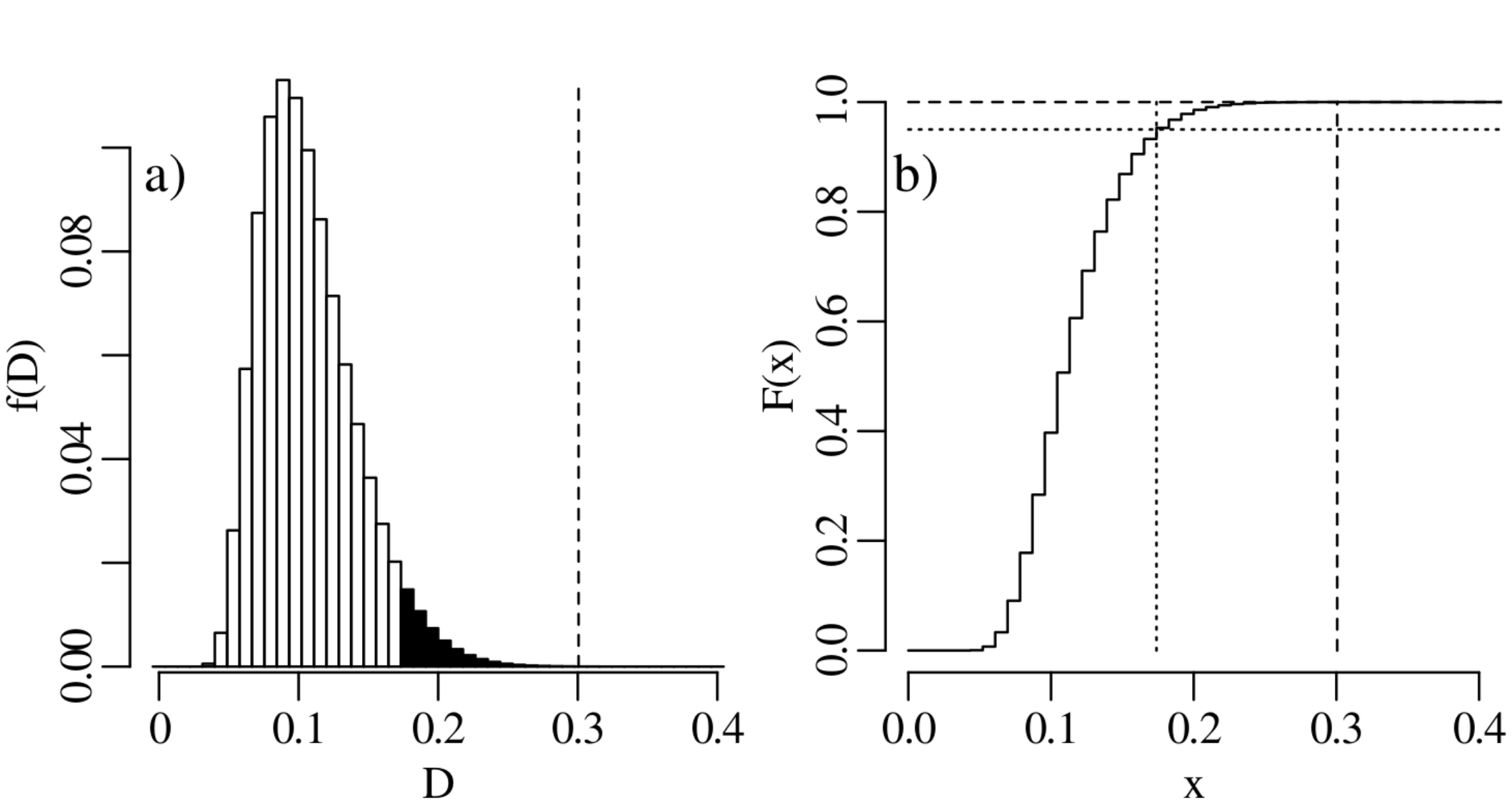
5.

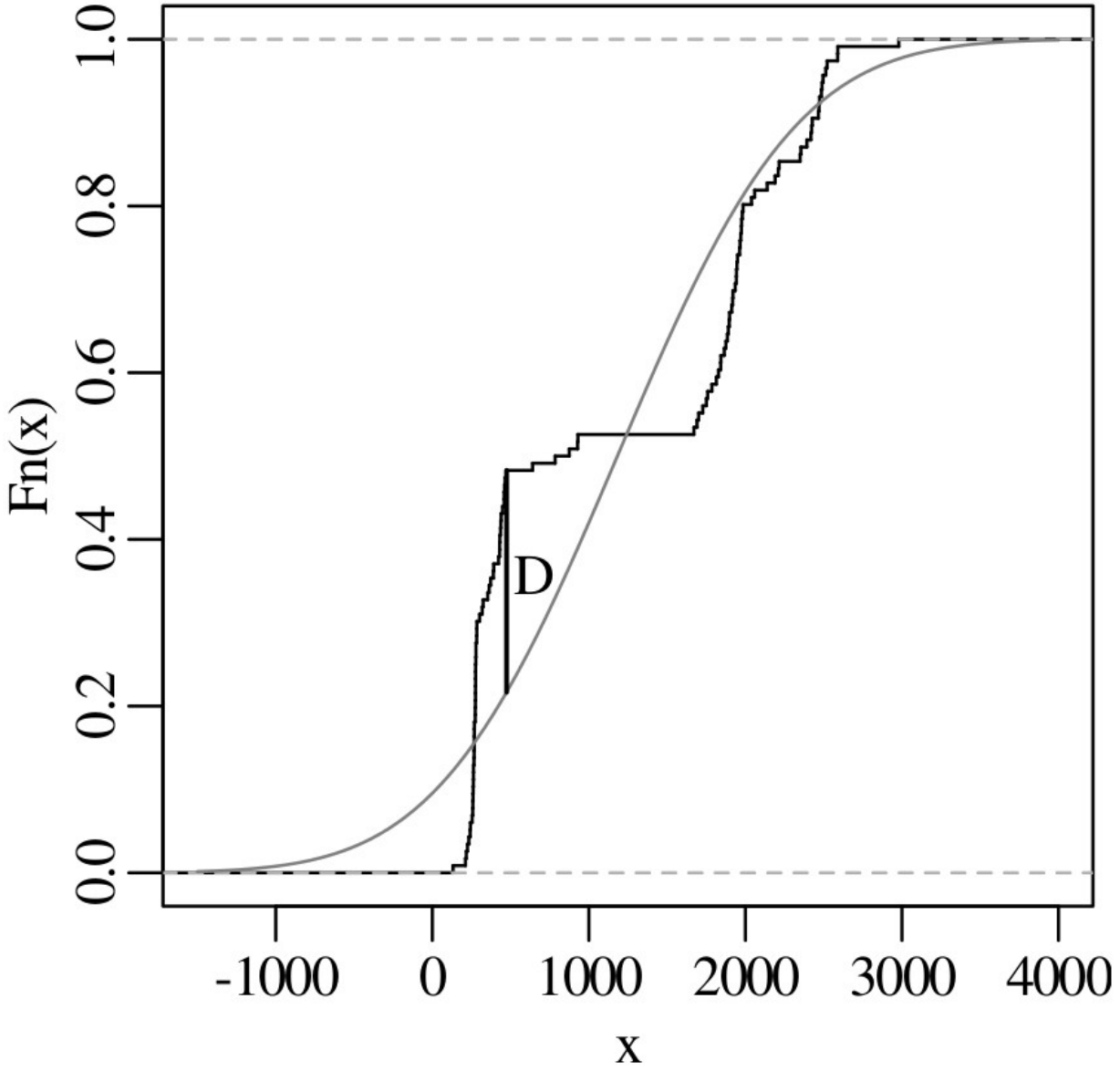
D	0.061	0.061	0.070	0.078	0.087	0.104	0.130	0.157	**0.174**	**0.191**	**0.209**	***0.301***
$P(d \geq D)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	**0.05**	**0.025**	**0.010**	$\boldsymbol{4.5 \times 10^{-5}}$

$R = \{D > 0.174\}$

6. $D = 0.301 \in R$

7. p-value $= 4.5 \times 10^{-5} < \alpha$


a)
f(D)
0.00
0.04
0.08
0
0.1
0.2
0.3
0.4
D
b)
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
x


Fn(x)
0.0
0.2
0.4
0.6
0.8
1.0
D
-1000
0
1000
2000
3000
4000
x

Statistics for geoscientists

Regression

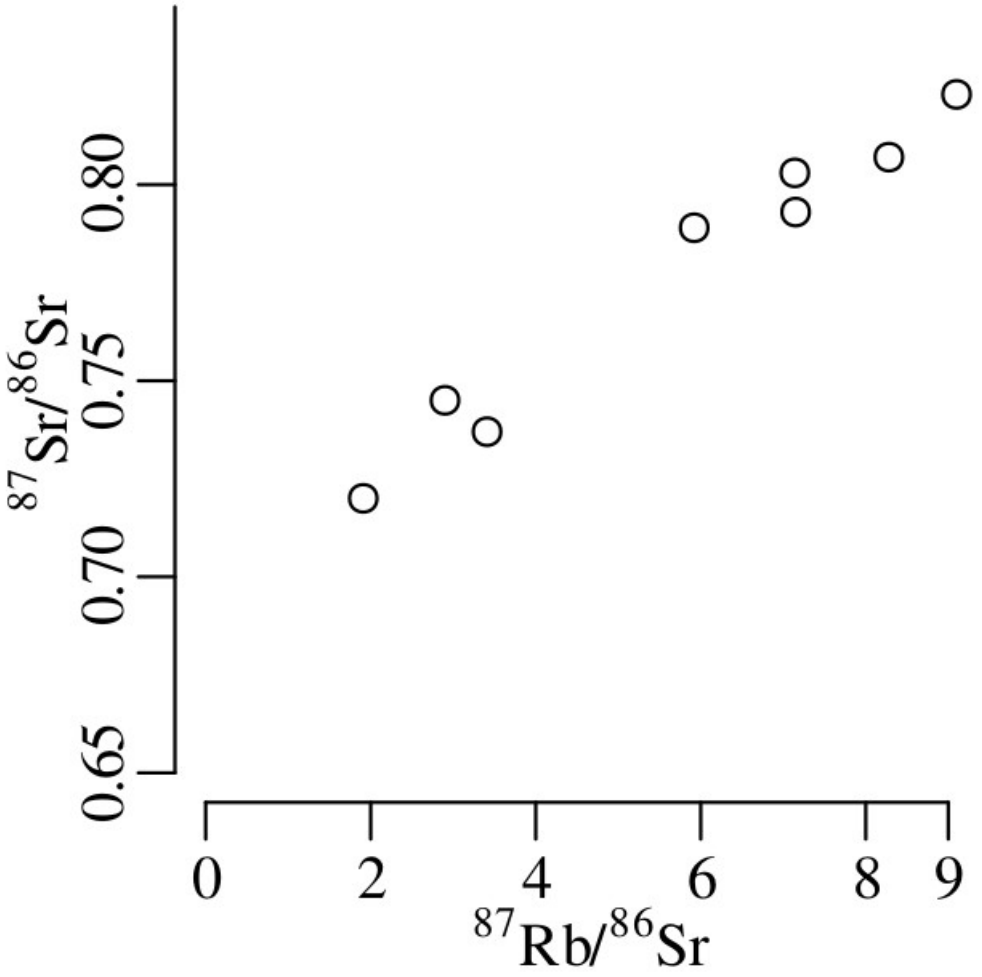
$$\left[\frac{^{87}Sr}{^{86}Sr}\right] = \left[\frac{^{87}Sr}{^{86}Sr}\right]_{\circ} + \left[\frac{^{87}Rb}{^{86}Sr}\right]\left(e^{\lambda t} - 1\right)$$

$$y_i = \beta_0 + \beta_1\, x_i$$

i	1	2	3	4	5	6	7	8
$[^{87}Rb/^{86}Sr] = x_i$	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28
$[^{87}Sr/^{86}Sr] = y_i$	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807

$$\left[\frac{^{87}Sr}{^{86}Sr}\right] = \left[\frac{^{87}Sr}{^{86}Sr}\right]_{\circ} + \left[\frac{^{87}Rb}{^{86}Sr}\right]\left(e^{\lambda t} - 1\right)$$

$$y_i = \beta_0 + \beta_1\, x_i + \epsilon_i$$



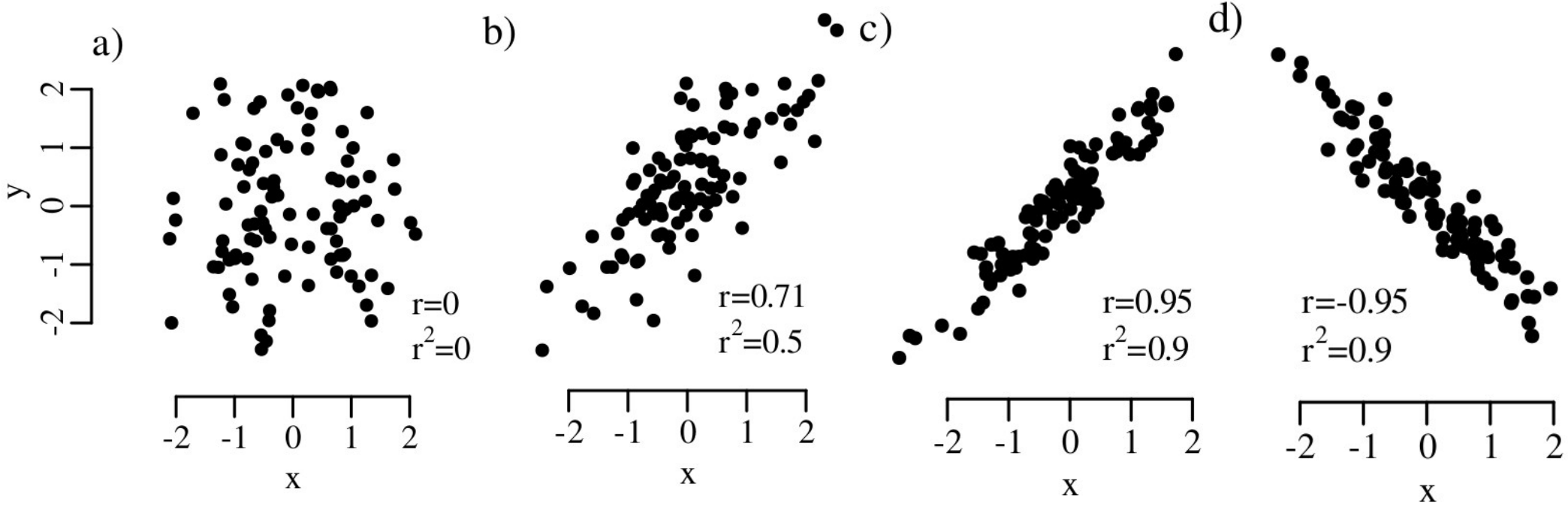
$$\rho = \frac{\sigma_{x,y}}{\sigma_x \sigma_y}$$

$$s[x] = \sqrt{\sum_{i=1}^{n} \frac{1}{n-1}(x_i - \bar{x})^2}$$

$$s[x,y] = \sum_{i=1}^{n} \frac{1}{n-1}(x_i - \bar{x})(y_i - \bar{y})$$

$$r = \frac{s[x,y]}{s[x]s[y]}$$

$$r = \frac{s[x,y]}{s[x]s[y]}$$



$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

1. $H_\circ$ (**null hypothesis**) $\rho = 0$

   H_a (**alternative hypothesis**): $\rho \neq 0$

2. $$t = \frac{0.985\sqrt{8-2}}{\sqrt{1-0.985^2}} = 13.98$$

3.

t	-3.10	-2.40	-1.90	-1.40	-0.72	0	0.72	1.40	1.90	2.40	3.10	*13.98*
$P(t \leq T)$	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99	*0.9999958*
$P(t \geq T)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.010	*0.0000042*

4. $\alpha = 0.05$

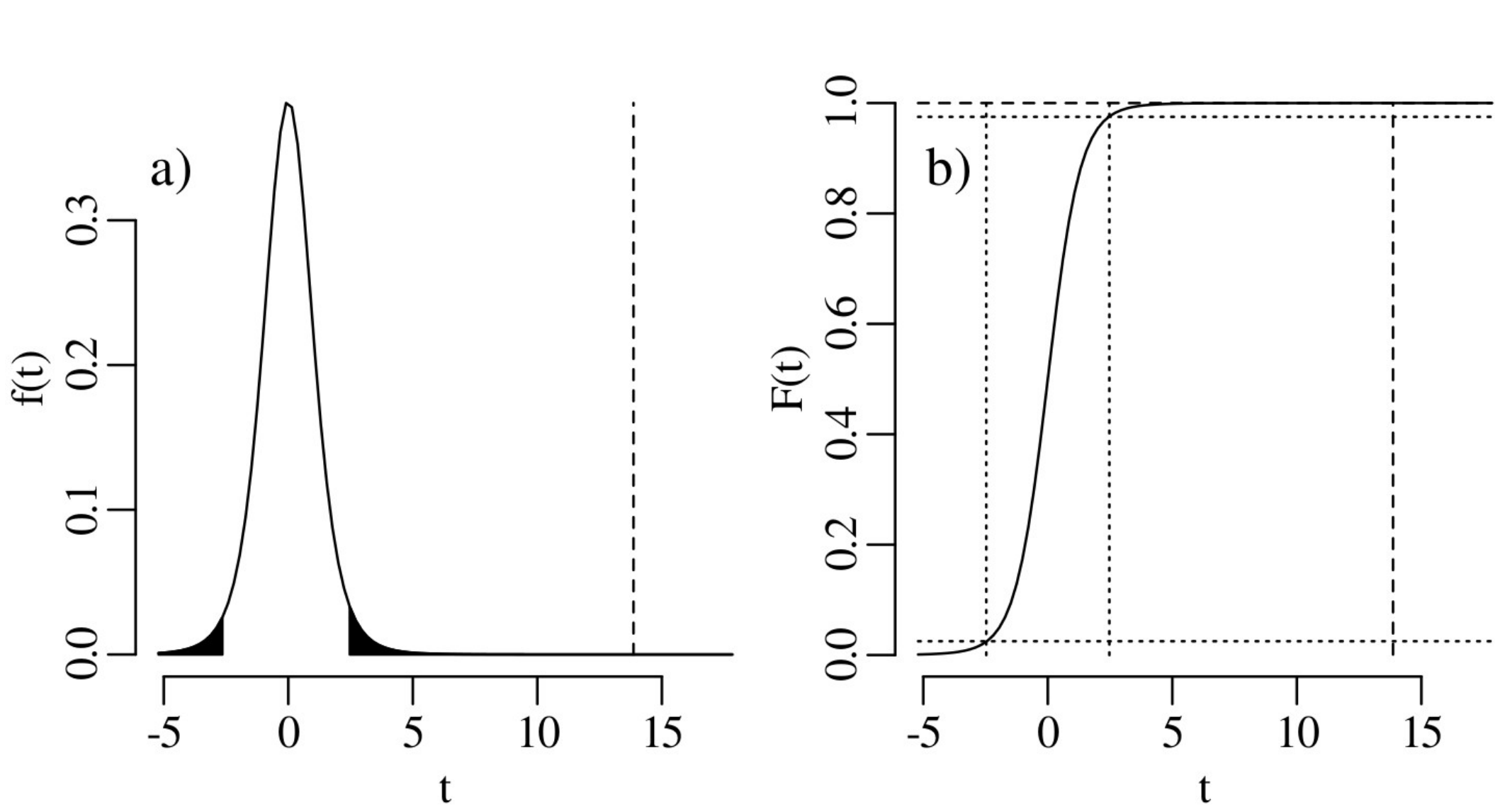
5.

t	**-3.10**	**-2.40**	-1.90	-1.40	-0.72	0	0.72	1.40	1.90	**2.40**	**3.10**	***13.98***
$P(t \leq T)$	**0.01**	**0.025**	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99	*0.9999958*
$P(t \geq T)$	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	**0.025**	**0.010**	***0.0000042***

$$R = \{|t| > 2.40\}$$

6. $t = 13.98 \quad \in R$

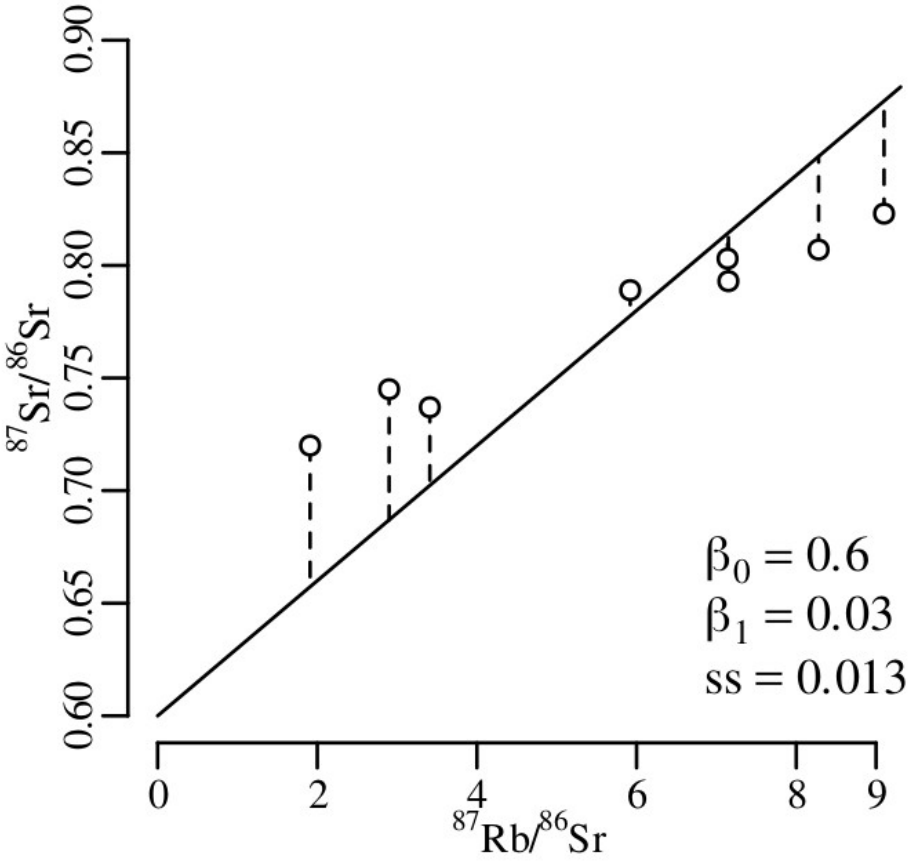
7. p-value $= 2 \times 0.0000042 = 0.0000084 < \alpha$


a)
f(t)
0.0
0.1
0.2
0.3
-5
0
5
10
15
t
b)
F(t)
0.0
0.2
0.4
0.6
0.8
1.0
-5
0
5
10
15
t

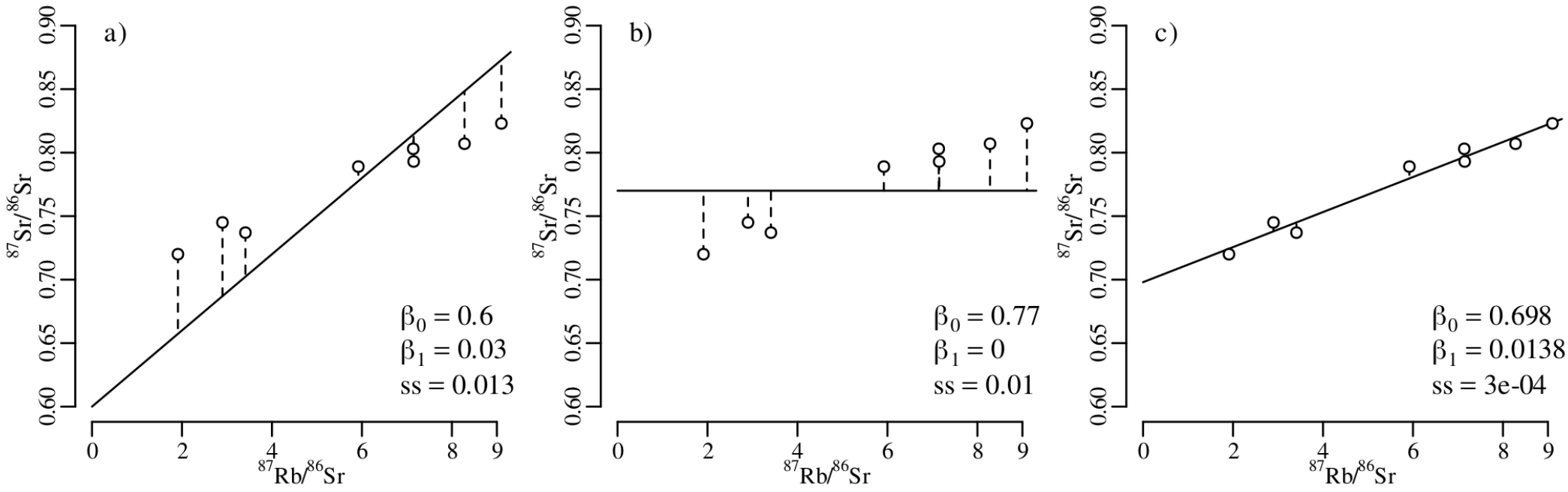
Least Squares

$$y_i = \beta_0 + \beta_1 \, x_i + \epsilon_i$$

$$ss \equiv \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)^2$$



i	1	2	3	4	5	6	7	8
x_i	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28
y_i	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807
$\beta_0 + \beta_1 x_i$	0.687	0.8142	0.873	0.7023	0.6573	0.8145	0.7776	0.8484
ϵ_i	-0.058	0.0112	0.05	-0.0347	-0.0627	0.0215	-0.0114	0.0414

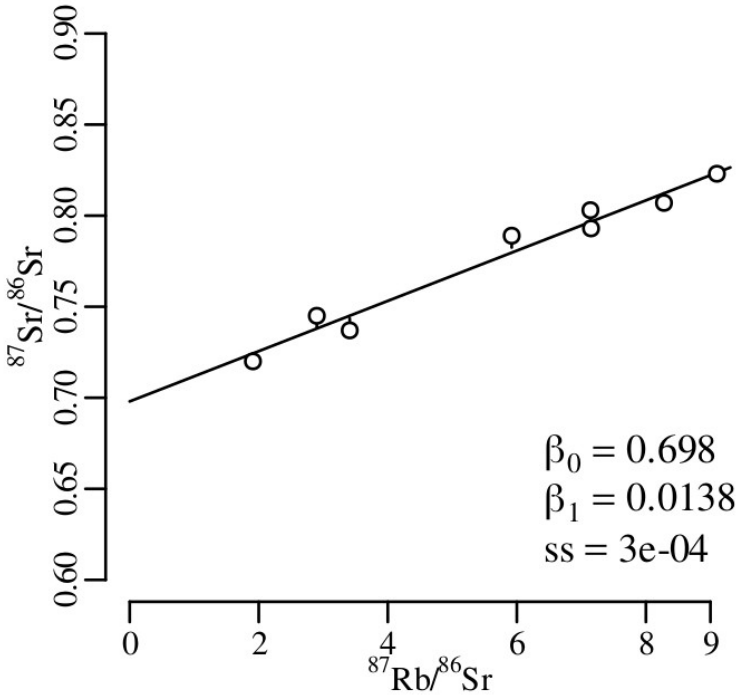

a)
b)
c)
$^{87}Sr/^{86}Sr$
$^{87}Rb/^{86}Sr$
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0
2
4
6
8
9
$\beta_0 = 0.6$
$\beta_1 = 0.03$
ss = 0.013
$\beta_0 = 0.77$
$\beta_1 = 0$
ss = 0.01
$\beta_0 = 0.698$
$\beta_1 = 0.0138$
ss = 3e-04

$$\begin{cases} \frac{\partial ss}{\partial \beta_0} = 2\sum(\beta_0 + \beta_1 x_i - y_i) = 0 \\ \frac{\partial ss}{\partial \beta_1} = 2\sum(\beta_0 + \beta_1 x_i - y_i)x_i = 0 \end{cases}$$

$$\begin{cases} \hat{\beta}_0 = \left(\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i\right) / \left(n\sum x_i^2 - \left(\sum x_i\right)^2\right) \\ \hat{\beta}_1 = \left(n\sum x_i y_i - \sum x_i \sum y_i\right) / \left(n\sum x_i^2 - \left(\sum x_i\right)^2\right) \end{cases}$$

i	1	2	3	4	5	6	7	8	$\sum_{i-1}^{8}$
x_i	2.90	7.14	9.10	3.41	1.91	7.15	5.92	8.28	45.81
y_i	0.745	0.803	0.823	0.737	0.720	0.793	0.789	0.807	6.217
x_i^2	8.41	50.98	82.81	11.63	3.648	51.12	35.05	68.56	312.2
$x_i y_i$	2.160	5.733	7.489	2.513	1.375	5.670	4.671	6.682	36.29

$$\begin{cases} \hat{\beta}_0 = (312.2 \times 6.217 - 45.81 \times 36.29) \,/\, \left(8 \times 312.2 - 45.81^2\right) = 0.698 \\ \hat{\beta}_1 = (8 \times 36.29 - 45.81 \times 6.217) \,/\, \left(8 \times 312.2 - 45.81^2\right) = 0.0138 \end{cases}$$



Maximum Likelihood

$$y_i = \beta_0 + \beta_1 \, x_i + \epsilon_i$$

$$f(\epsilon_i|\beta_0, \beta_1, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\epsilon_i^2}{2\sigma^2}\right], \text{ where } \epsilon_i = \beta_0 + \beta_1 x_i - y_i$$

$$\mathcal{L}(\beta_0, \beta_1, \sigma|\{x_1, y_1\}, \ldots, \{x_n, y_n\}) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\epsilon_i^2}{2\sigma^2}\right]$$

$$\max_{\beta_0,\beta_1}\left[\prod_{i=1}^{n}\mathcal{L}\right] = \max_{\beta_0,\beta_1}\left[\sum_{i=1}^{n}\ln\mathcal{L}\right] = \max_{\beta_0,\beta_1}\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \sum_{i=1}^{n}\left(\frac{\epsilon_i^2}{2\sigma^2}\right)\right]$$

$$= \max_{\beta_0,\beta_1}\left[-\sum_{i=1}^{n}\epsilon_i^2\right] = \min_{\beta_0,\beta_1}\left[\sum_{i=1}^{n}\epsilon_i^2\right] = \min_{\beta_0,\beta_1}(ss)$$

$$\Sigma_{\hat{\beta}} = \begin{bmatrix} s[\hat{\beta}_0]^2 & s[\hat{\beta}_0, \hat{\beta}_1] \\ s[\hat{\beta}_0, \hat{\beta}_1] & s[\hat{\beta}_1]^2 \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2 \sum_{i=1}^n x_i^2}{n \sum_{i=1}^n (x_i^2 - \bar{x}^2)} & -\frac{\sigma^2 \bar{x}}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} \\ -\frac{\sigma^2 \bar{x}}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} & \frac{\sigma^2}{\sum_{i=1}^n (x_i^2 - \bar{x}^2)} \end{bmatrix}$$

$$\beta_0 \in \left\{ \hat{\beta}_0 \pm t_{df,\alpha/2} \; s[\hat{\beta}_0] \right\}$$

$$\beta_1 \in \left\{ \hat{\beta}_1 \pm t_{df,\alpha/2} \; s[\hat{\beta}_1] \right\}$$

where $df = n - 2$

$$y = \beta_0 + \beta_1 x$$

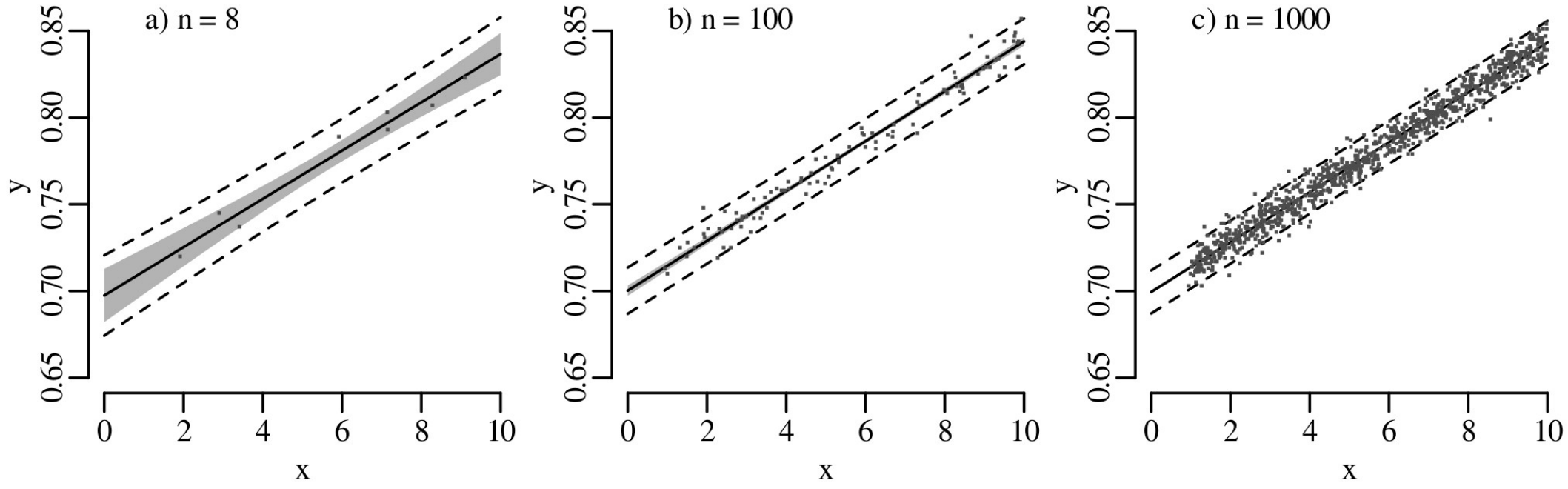
$$\Sigma_{\hat{\beta}} = \begin{bmatrix} s[\hat{\beta}_0]^2 & s[\hat{\beta}_0, \hat{\beta}_1] \\ s[\hat{\beta}_0, \hat{\beta}_1] & s[\hat{\beta}_1]^2 \end{bmatrix}$$

$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df,\alpha/2} \sqrt{s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1] x} \right\}$$

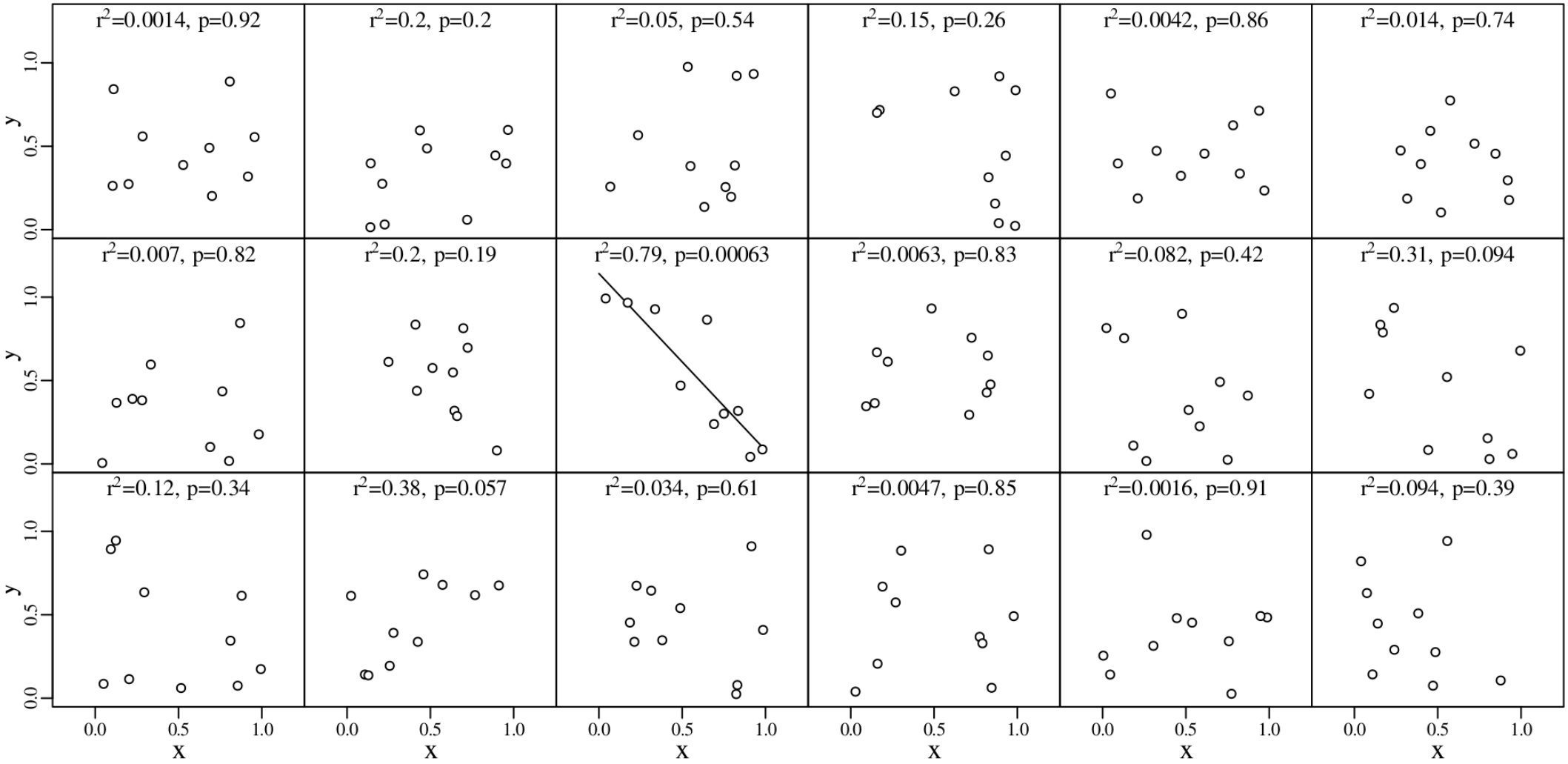
$y = {}^{87}Sr/{}^{86}Sr$
0.65
0.70
0.75
0.80
0.85
0
2
4
6
8
10
$x = {}^{87}Rb/{}^{86}Sr$

$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df,\alpha/2} \sqrt{s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1] x} \right\}$$

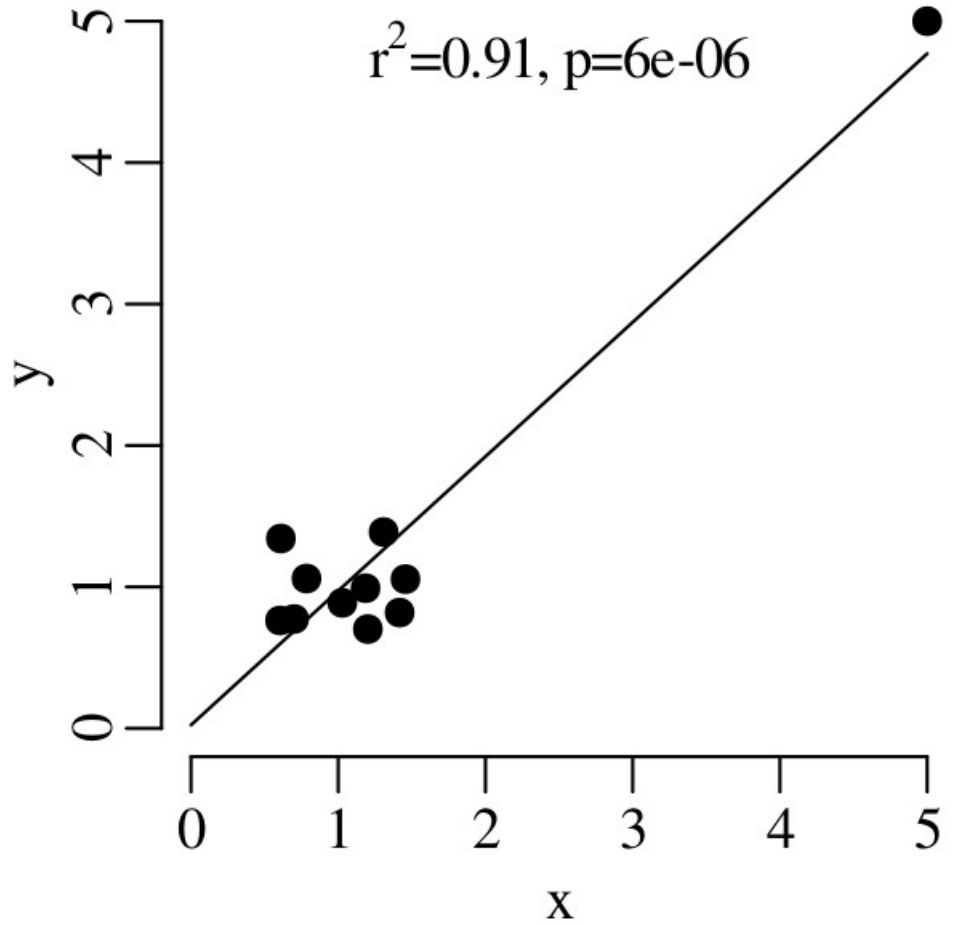
$$y \in \left\{ \hat{\beta}_0 + \hat{\beta}_1 x \pm t_{df,\alpha/2} \sqrt{\hat{\sigma}^2 + s[\hat{\beta}_0]^2 + s[\hat{\beta}_1]^2 x^2 + 2s[\hat{\beta}_0, \hat{\beta}_1] x} \right\}$$



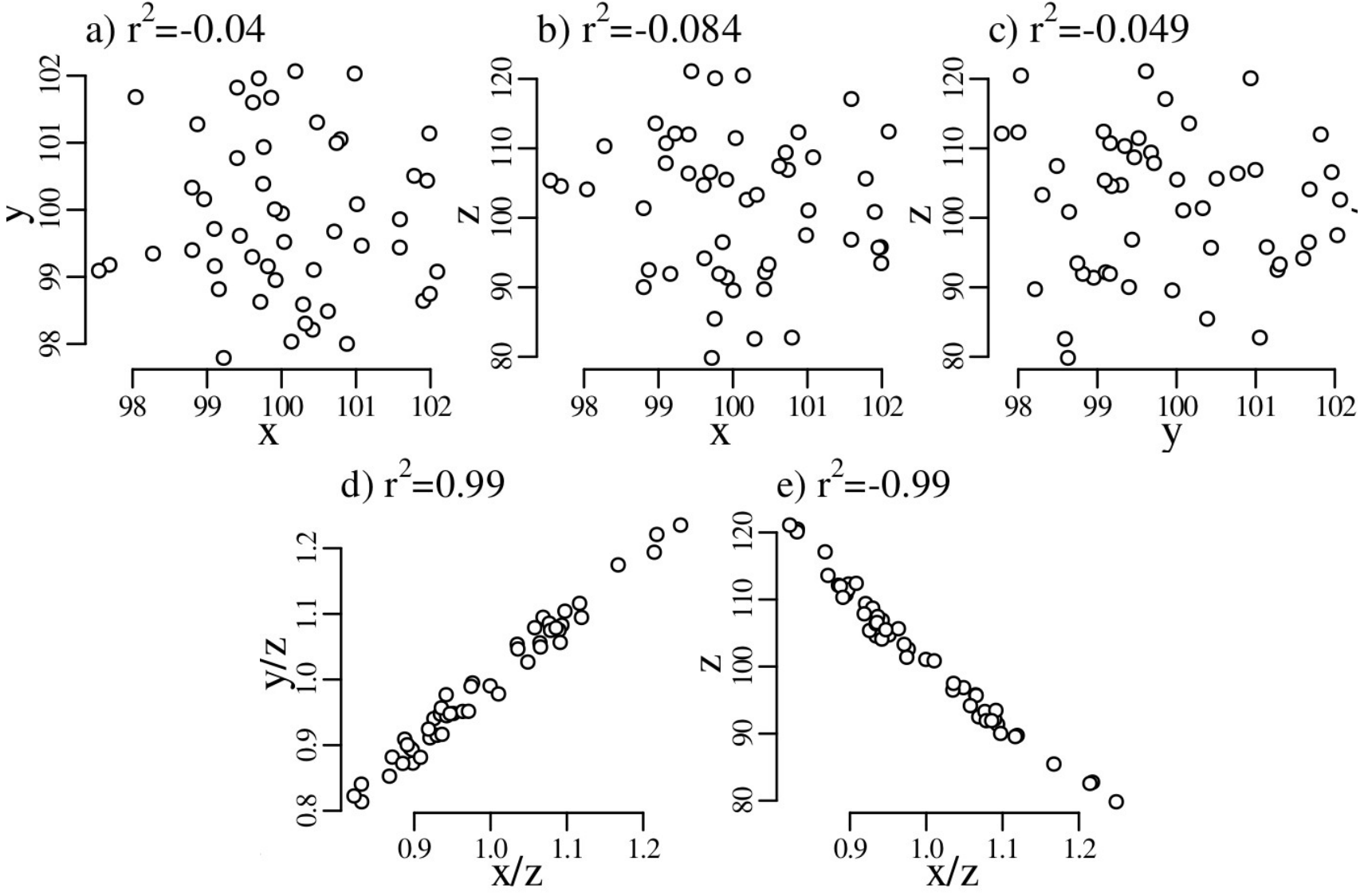
p-hacking

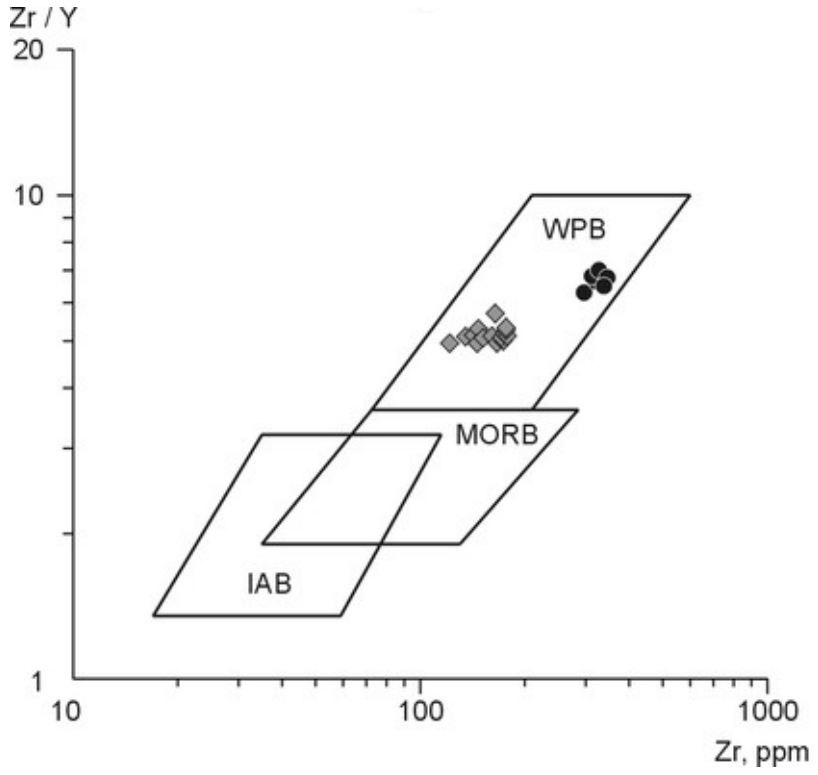

r²=0.0014, p=0.92
r²=0.2, p=0.2
r²=0.05, p=0.54
r²=0.15, p=0.26
r²=0.0042, p=0.86
r²=0.014, p=0.74
r²=0.007, p=0.82
r²=0.2, p=0.19
r²=0.79, p=0.00063
r²=0.0063, p=0.83
r²=0.082, p=0.42
r²=0.31, p=0.094
r²=0.12, p=0.34
r²=0.38, p=0.057
r²=0.034, p=0.61
r²=0.0047, p=0.85
r²=0.0016, p=0.91
r²=0.094, p=0.39
y
x
0.0
0.5
1.0

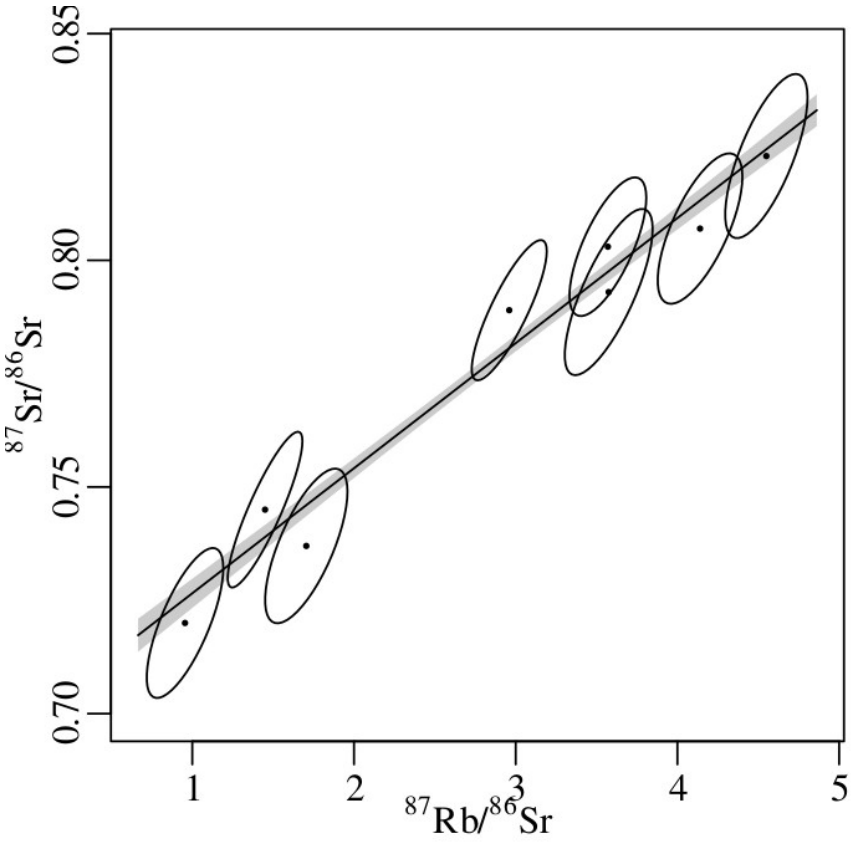
outliers



Spurious correlation



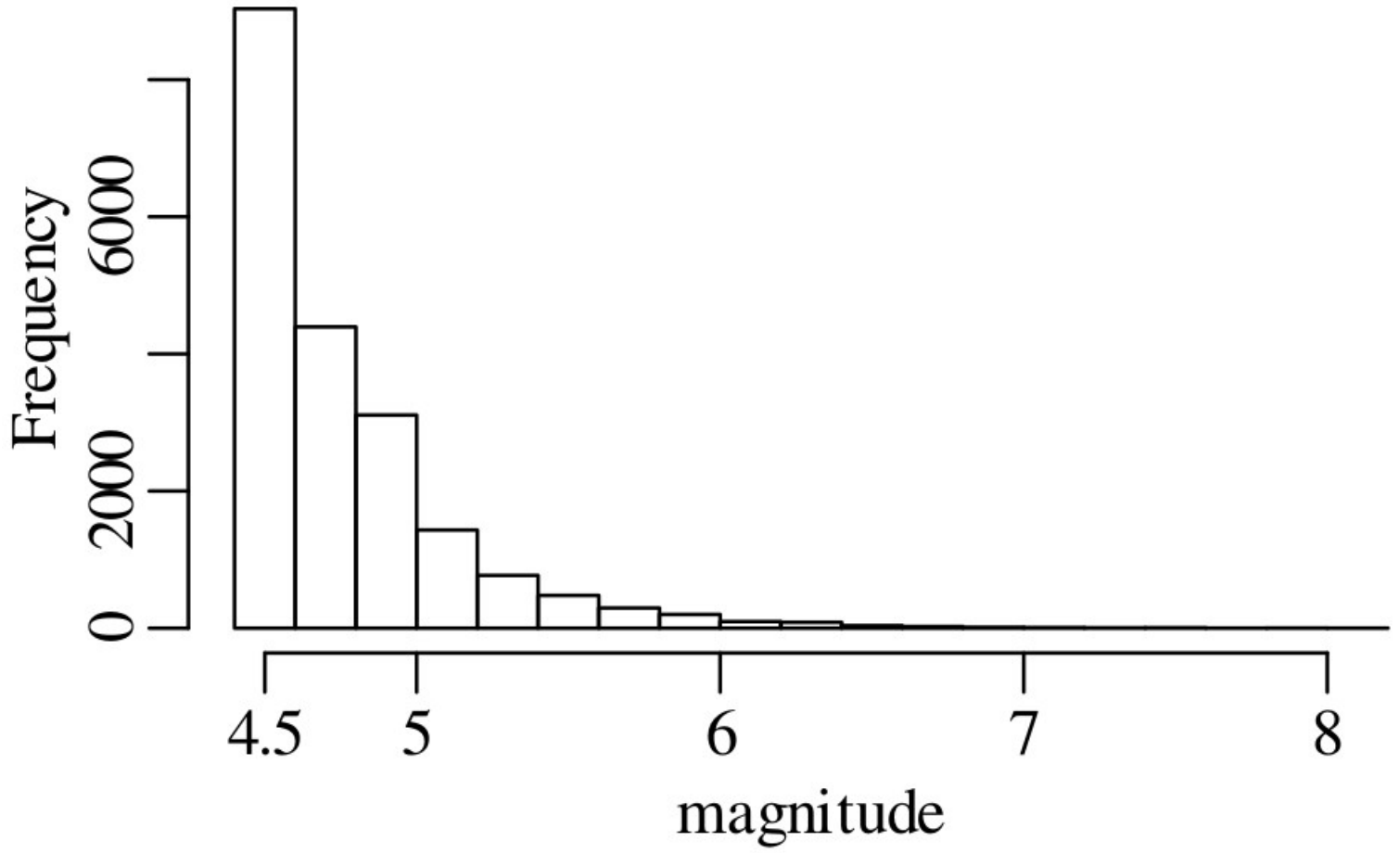

Zr / Y
20
10
1
10
100
1000
Zr, ppm
WPB
MORB
IAB

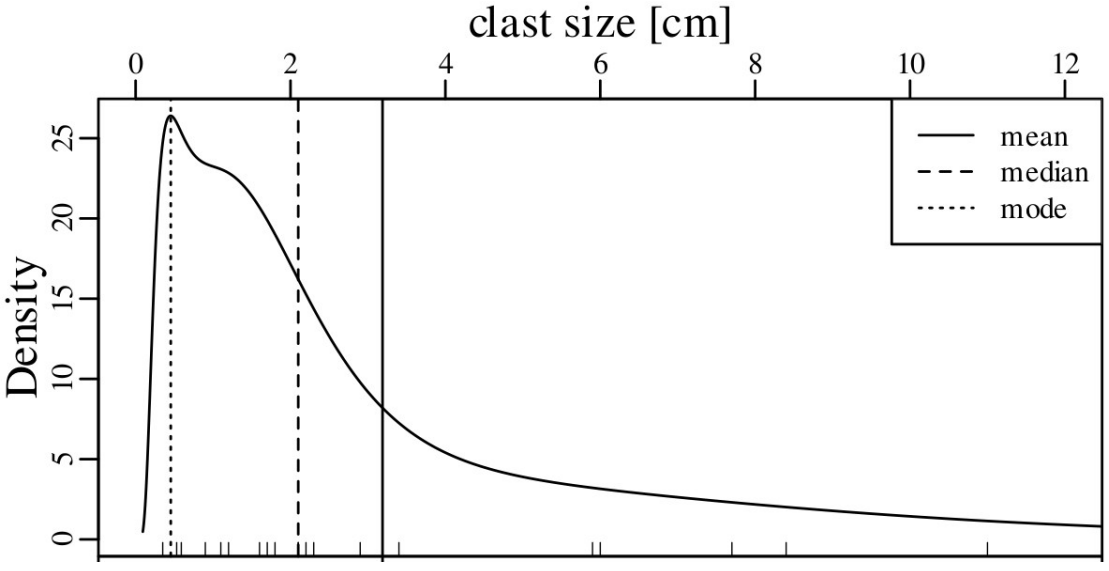

0.85
0.80
0.75
0.70
$^{87}Sr/^{86}Sr$
1
2
3
4
5
$^{87}Rb/^{86}Sr$

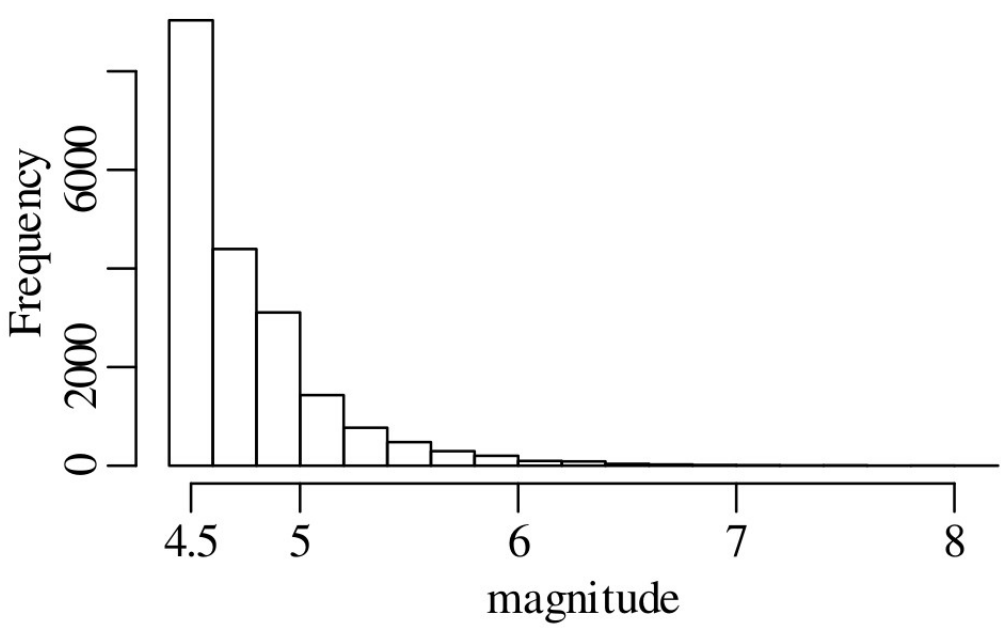
$$\rho_{\frac{w}{y},\frac{x}{z}} \approx \frac{\rho_{w,x}\left[\frac{\sigma_w}{\mu_w}\right]\left[\frac{\sigma_x}{\mu_x}\right] - \rho_{w,z}\left[\frac{\sigma_w}{\mu_w}\right]\left[\frac{\sigma_z}{\mu_z}\right] - \rho_{x,y}\left[\frac{\sigma_x}{\mu_x}\right]\left[\frac{\sigma_y}{\mu_y}\right] + \rho_{y,z}\left[\frac{\sigma_y}{\mu_y}\right]\left[\frac{\sigma_z}{\mu_z}\right]}{\sqrt{\left[\frac{\sigma_w}{\mu_w}\right]^2 + \left[\frac{\sigma_y}{\mu_y}\right]^2 - 2\rho_{w,y}\left[\frac{\sigma_w}{\mu_w}\right]\left[\frac{\sigma_y}{\mu_y}\right]}\sqrt{\left[\frac{\sigma_x}{\mu_x}\right]^2 + \left[\frac{\sigma_z}{\mu_z}\right]^2 - 2\rho_{x,z}\left[\frac{\sigma_x}{\mu_x}\right]\left[\frac{\sigma_z}{\mu_z}\right]}}$$

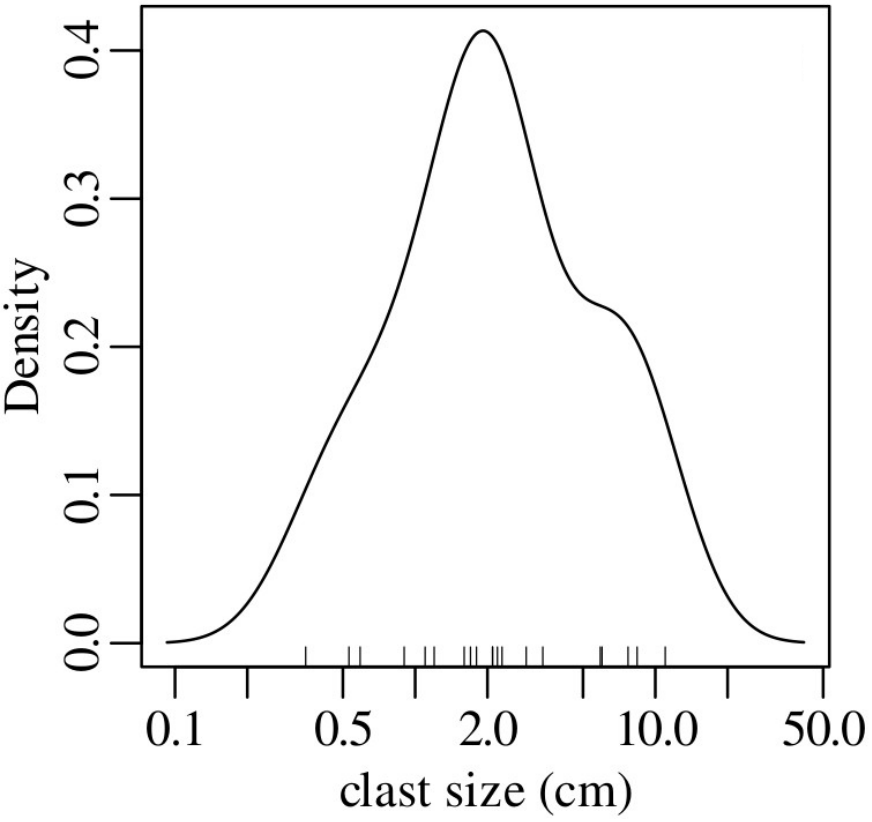
$$\rho_{\frac{y}{z},\frac{x}{z}} \approx \frac{\left[\frac{\sigma_z}{\mu_z}\right]^2}{\sqrt{\left[\frac{\sigma_y}{\mu_y}\right]^2 + \left[\frac{\sigma_z}{\mu_z}\right]^2}\sqrt{\left[\frac{\sigma_x}{\mu_x}\right]^2 + \left[\frac{\sigma_z}{\mu_z}\right]^2}}$$

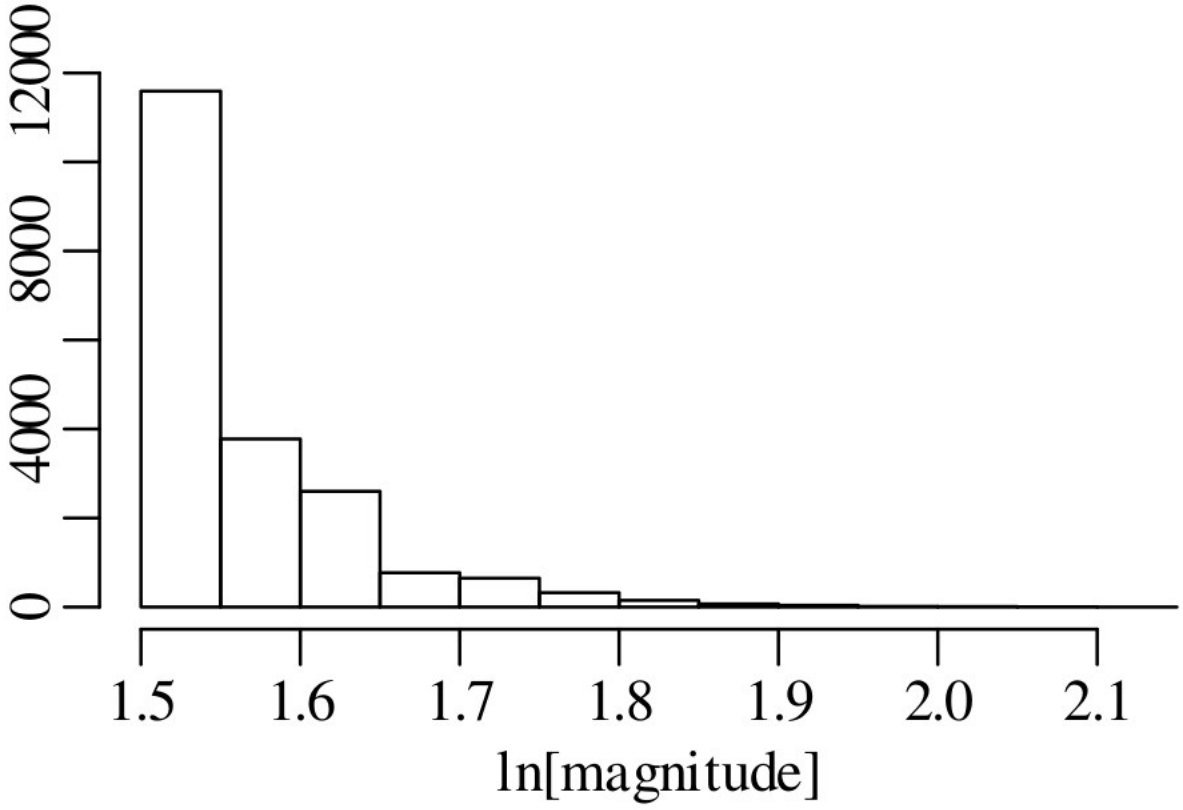
Fractals and chaos

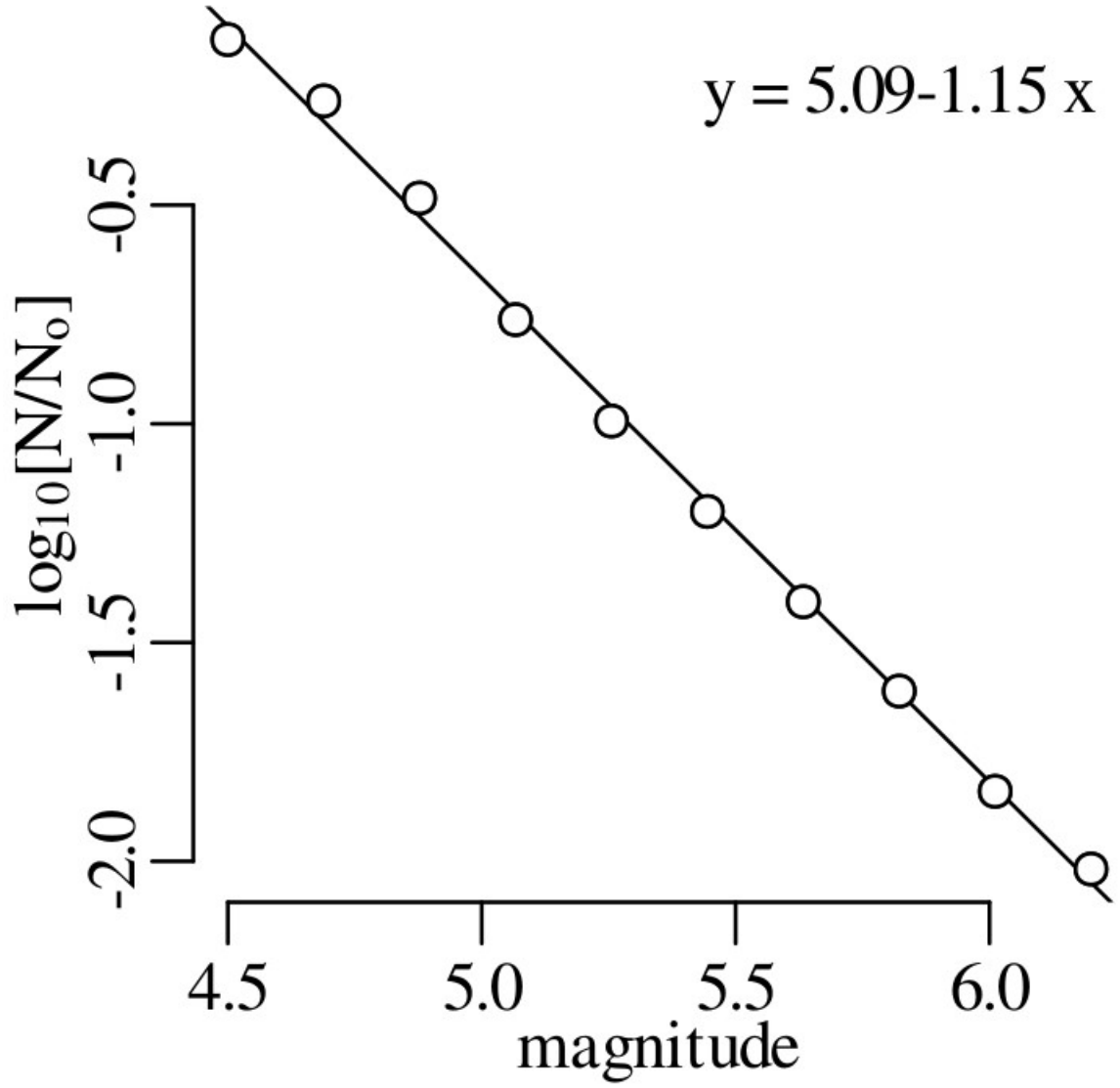

Frequency
0
2000
6000
4.5
5
6
7
8
magnitude


clast size [cm]
0
2
4
6
8
10
12
Density
0
5
10
15
20
25
mean
median
mode

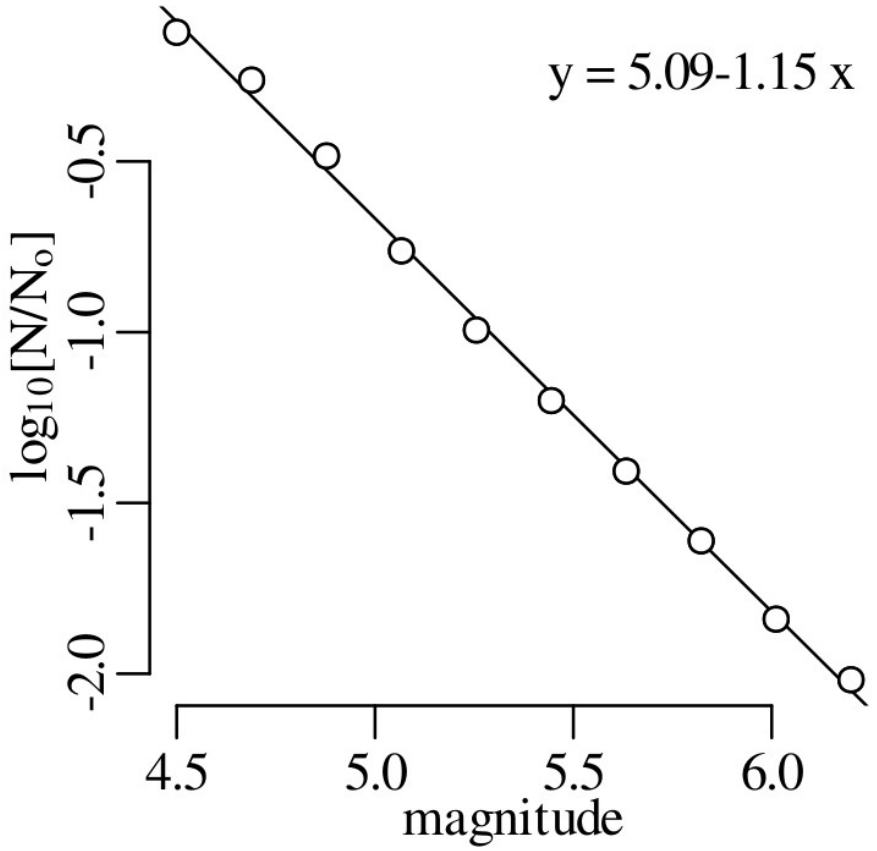

Frequency
0
2000
6000
4.5
5
6
7
8
magnitude


Density
0.0
0.1
0.2
0.3
0.4
0.1
0.5
2.0
10.0
50.0
clast size (cm)


0
4000
8000
12000
1.5
1.6
1.7
1.8
1.9
2.0
2.1
ln[magnitude]


y = 5.09-1.15 x
log10[N/No]
-0.5
-1.0
-1.5
-2.0
4.5
5.0
5.5
6.0
magnitude

P($\geq$ 9.0 earthquake)?



$$\log_{10}[N/N_\circ] = a + b \text{ magnitude}$$

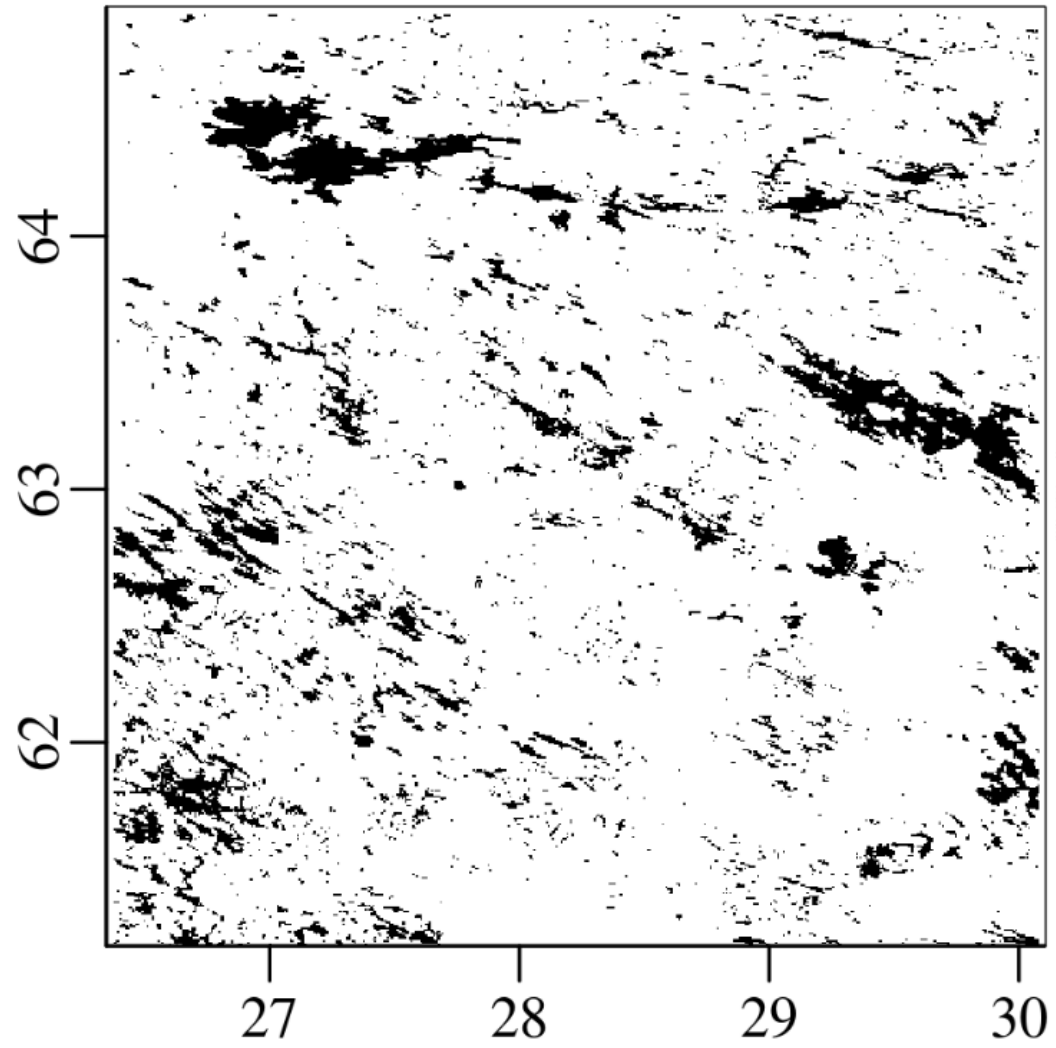
$$N_\circ = \frac{20000 \text{ earthquakes}}{1031 \text{ days}} \times 365 \text{ days}$$

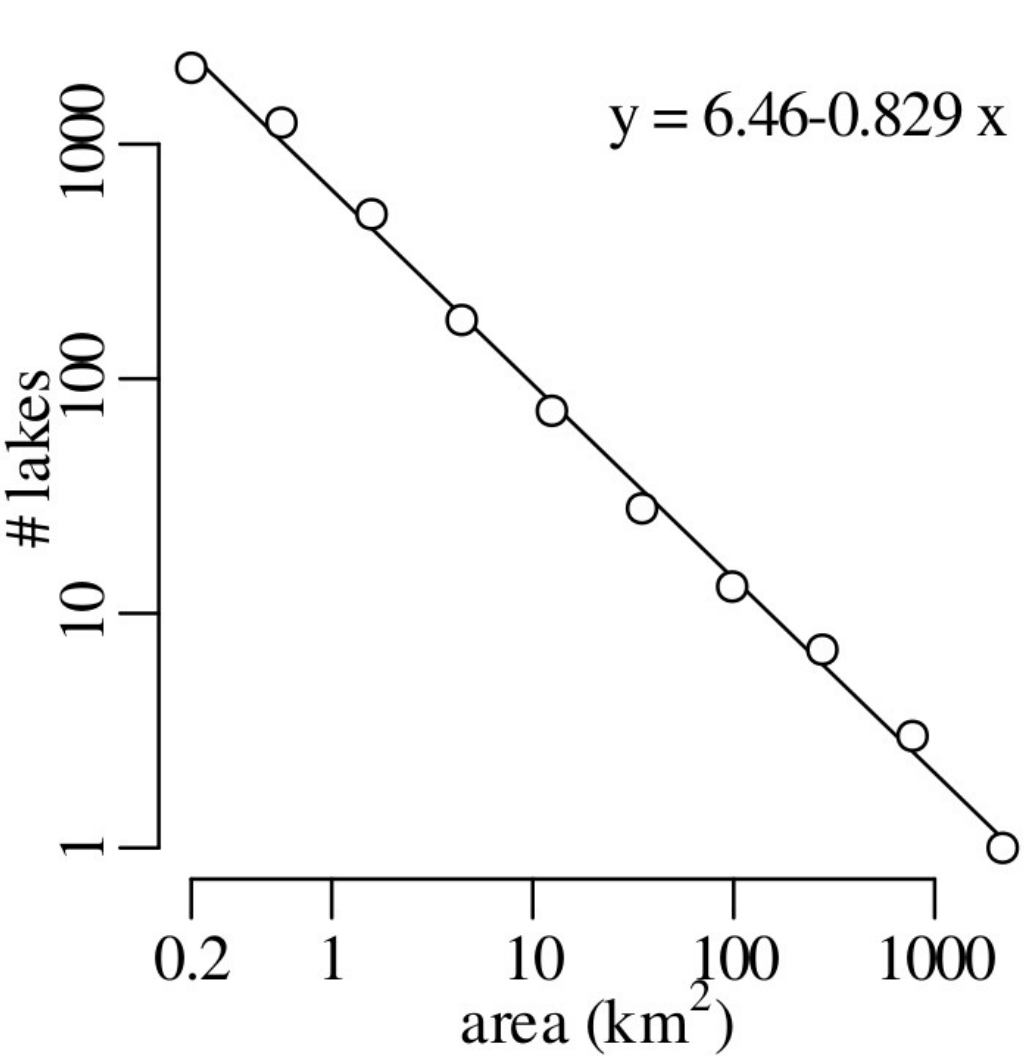
$$= 7080 \text{ earthquakes}$$

$$\log_{10}[N/N_\circ] = 5.09 - 1.15 \times 9.0 = -5.26$$

$$N = 7080 \times 10^{-5.26} = 0.039$$

$$= 3.9\%$$


64
63
62
27
28
29
30

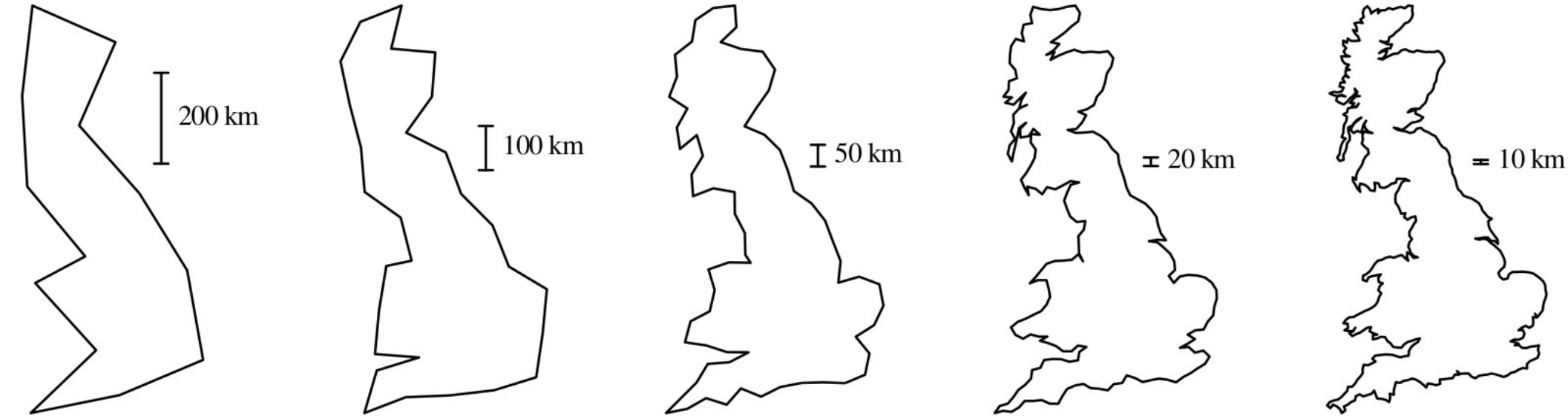

y = 6.46-0.829 x
# lakes
1
10
100
1000
0.2
1
10
100
1000
area (km2)

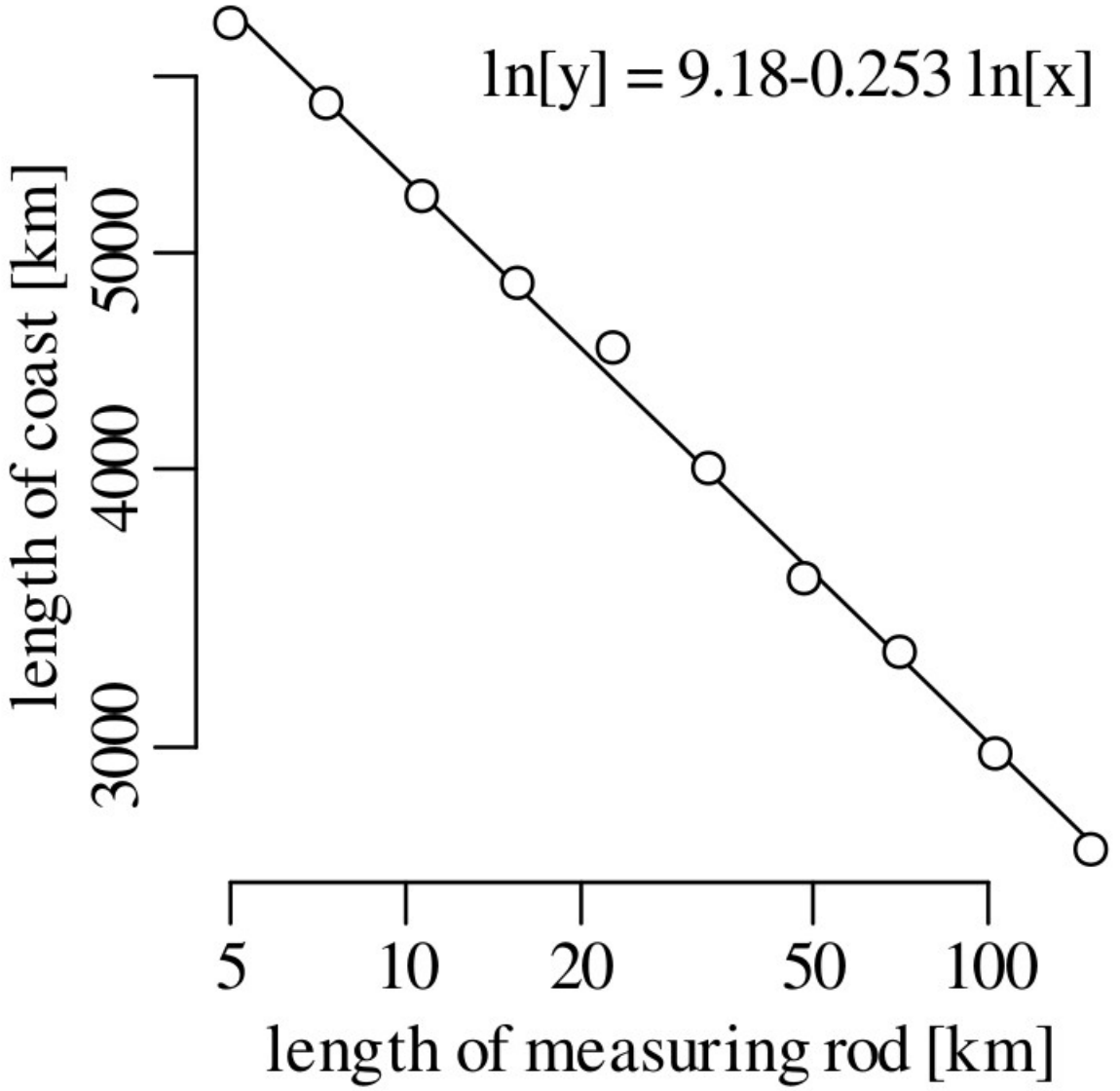
SCIENCE, VOL. 156

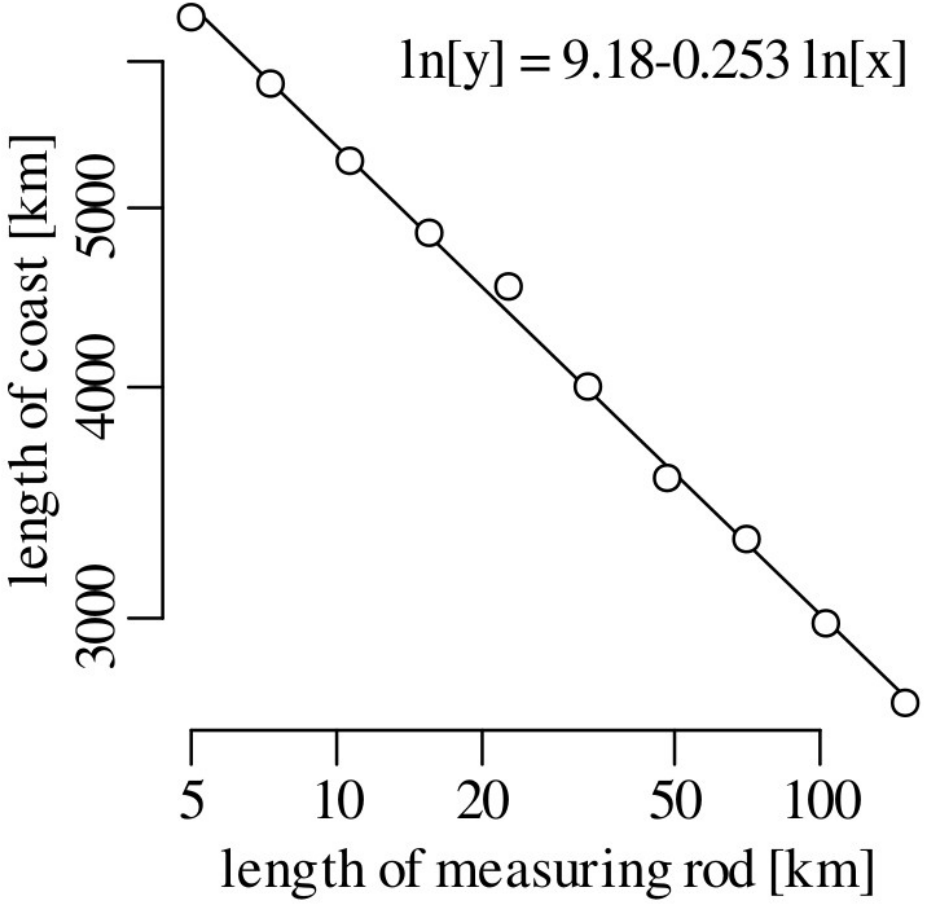
Reports

How Long Is the Coast of Britain?
Statistical Self-Similarity and Fractional Dimension

BENOIT MANDELBROT
International Business Machines,
Thomas J. Watson Research Center,
Yorktown Heights, New York 10598



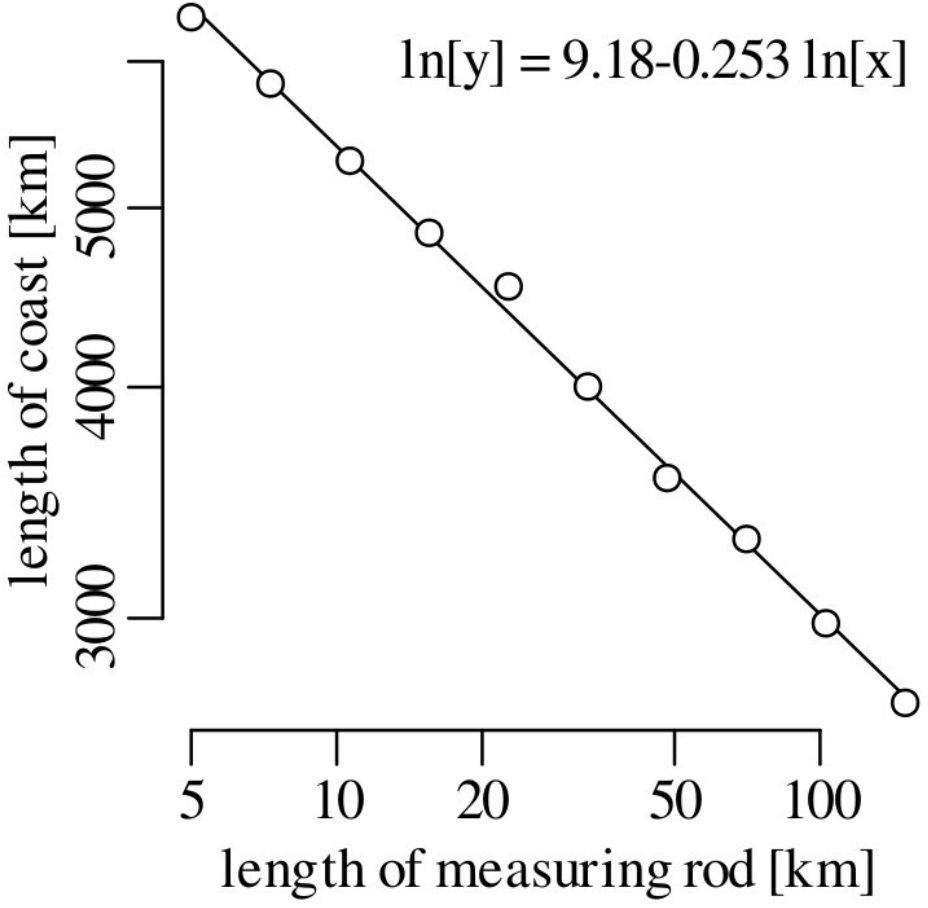

ln[y] = 9.18-0.253 ln[x]
length of coast [km]
3000
4000
5000
5
10
20
50
100
length of measuring rod [km]

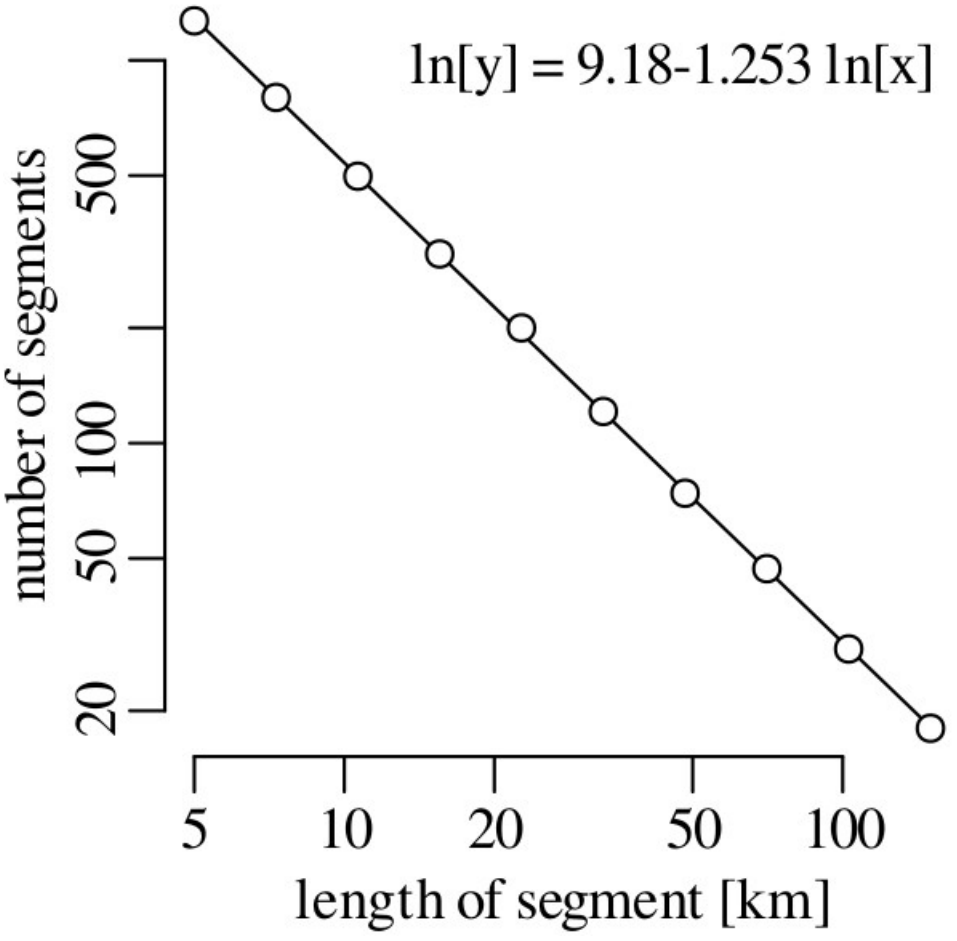


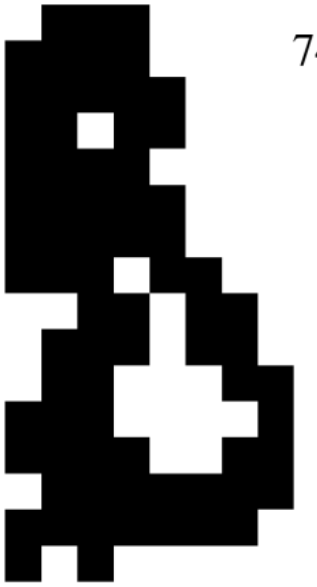
30 cm long ruler

$$\exp(9.18 - 0.253 \ln[3 \times 10^{-4}])$$

$$= 42,060 \text{ km!}$$


ln[y] = 9.18-0.253 ln[x]
length of coast [km]
3000
4000
5000
5
10
20
50
100
length of measuring rod [km]


ln[y] = 9.18-1.253 ln[x]
number of segments
20
50
100
500
5
10
20
50
100
length of segment [km]

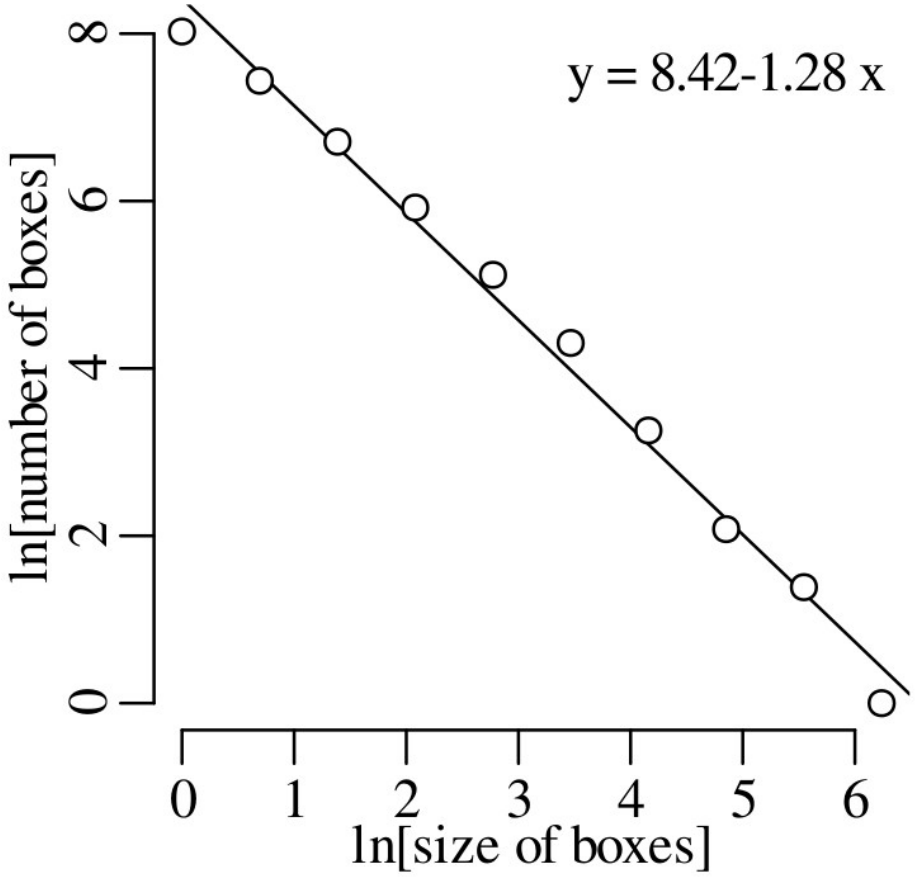


74/256

167/1024

819/16384

3058/262144

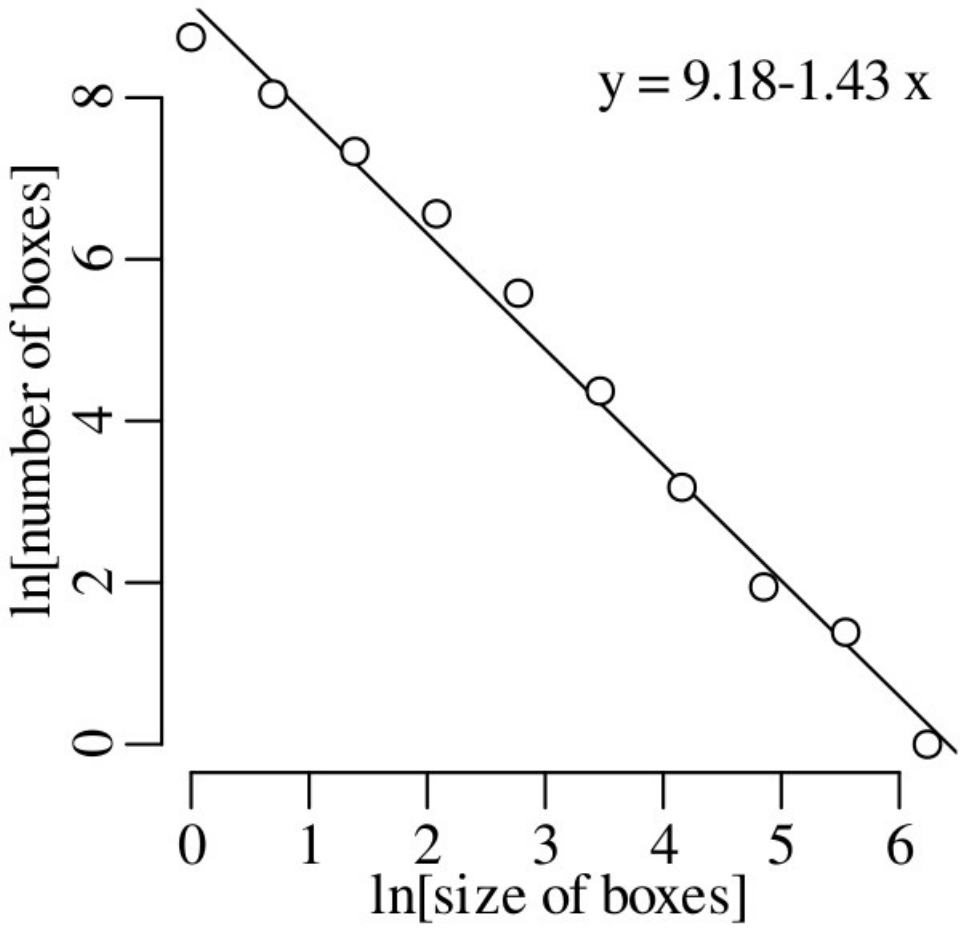



708/4096

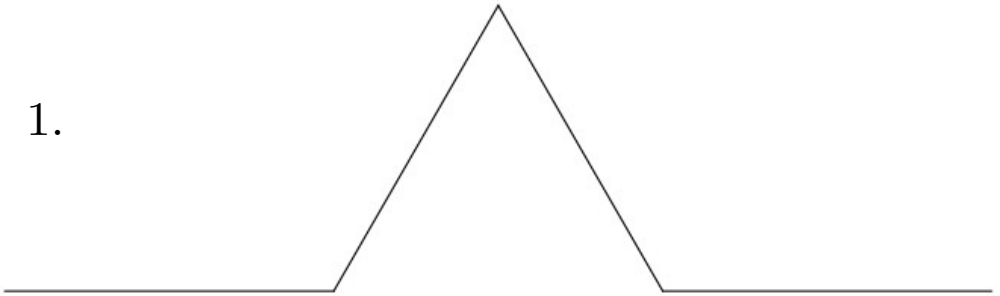

1532/16384


3115/65536

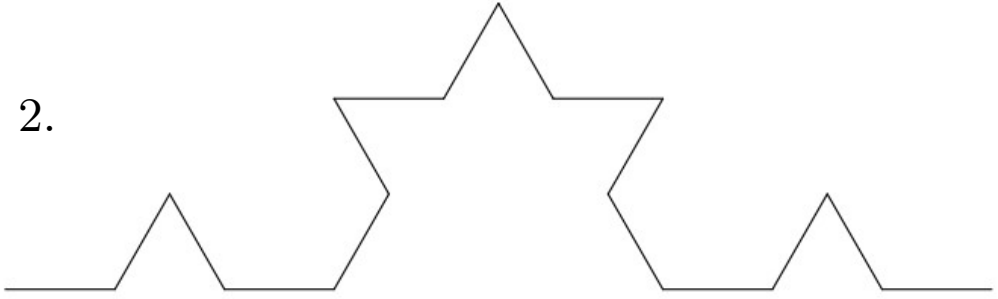

6299/262144


y = 9.18-1.43 x
ln[number of boxes]
0
2
4
6
8
0
1
2
3
4
5
6
ln[size of boxes]

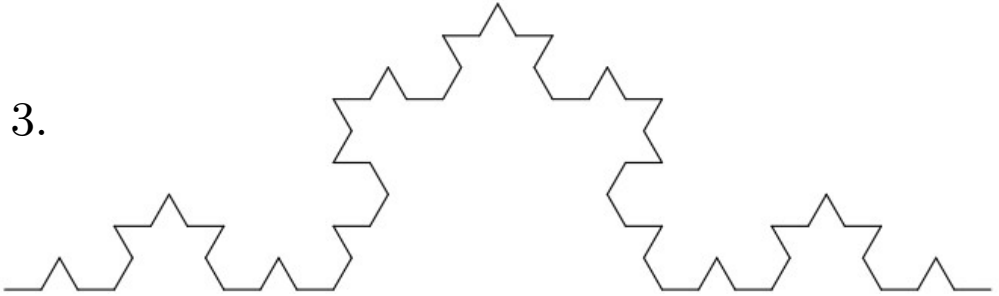
1.



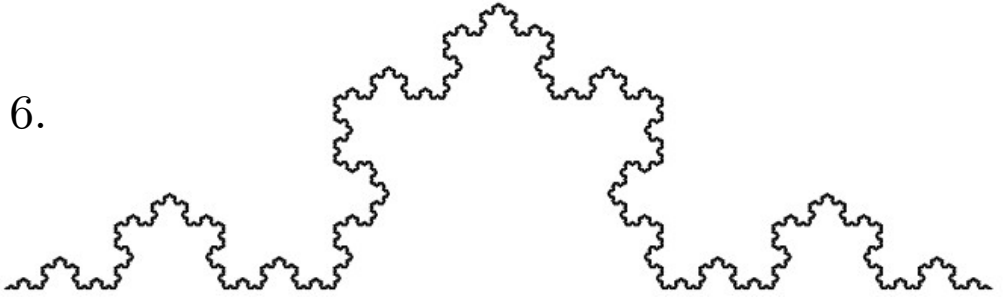
2.

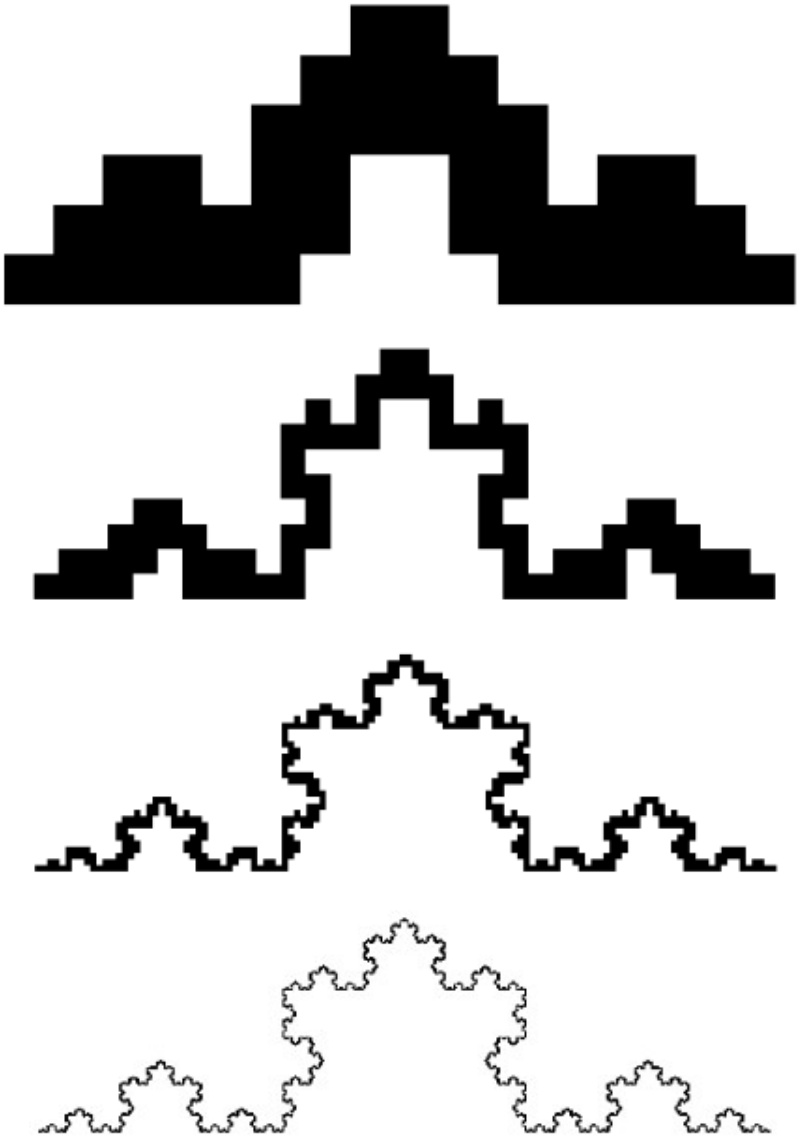


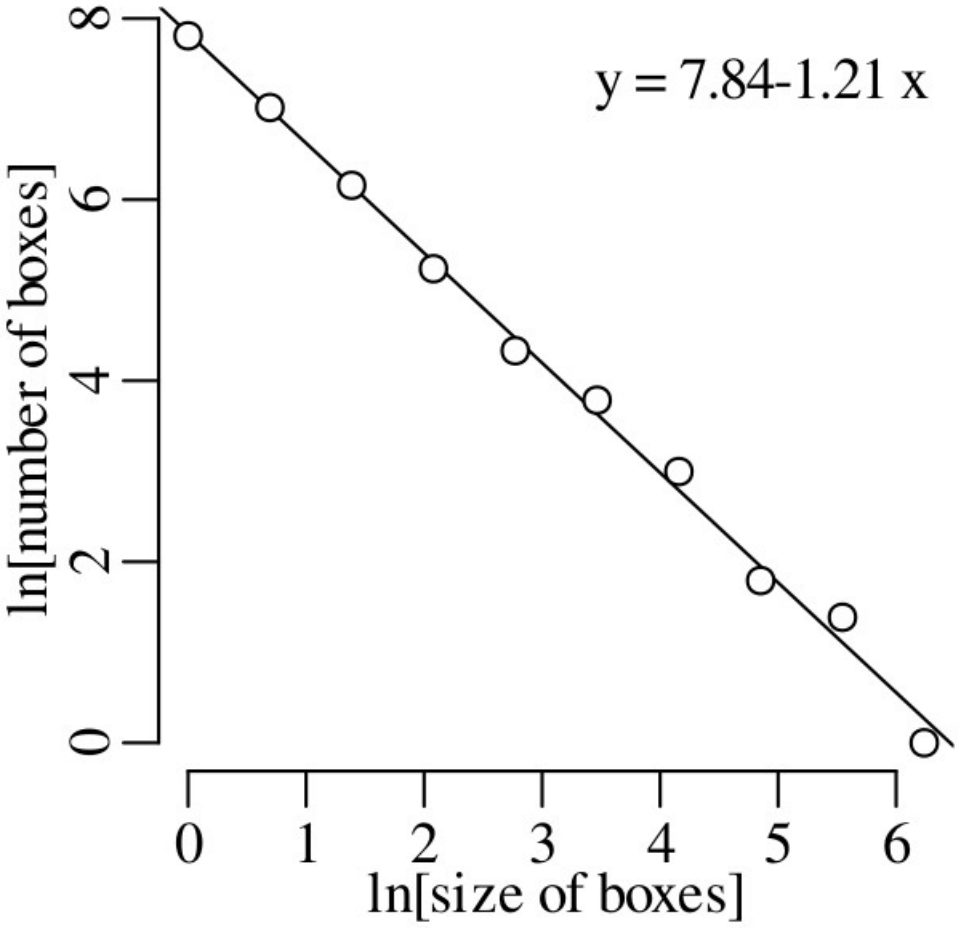
3.

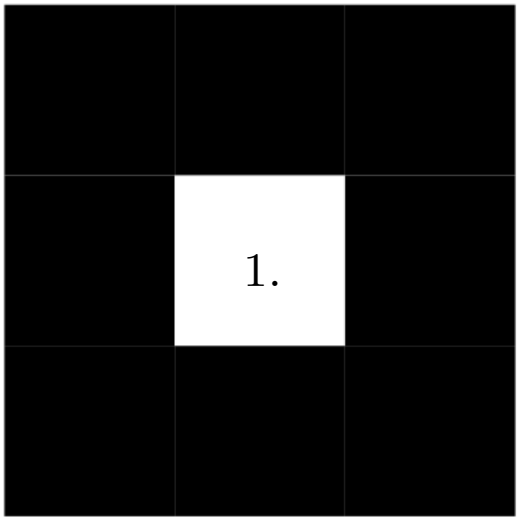


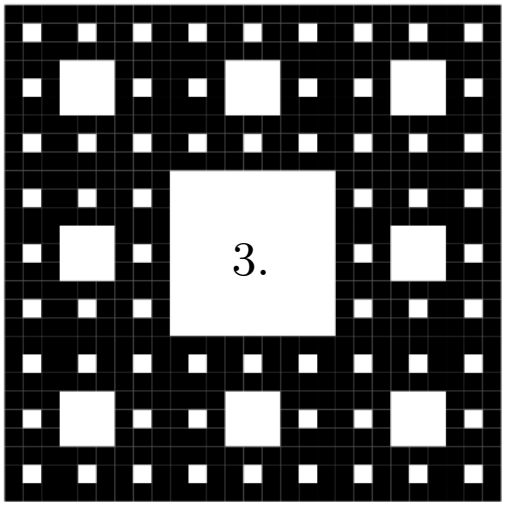
6.

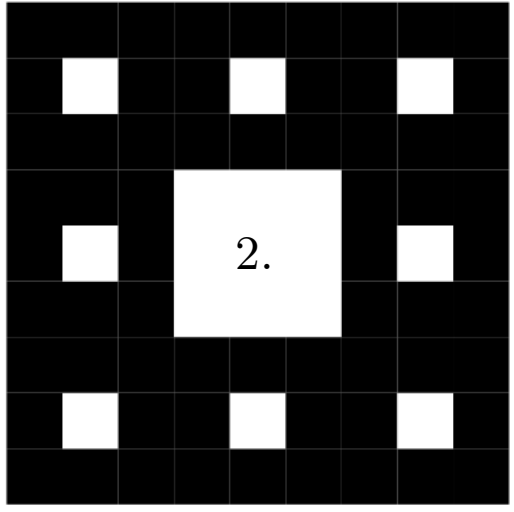


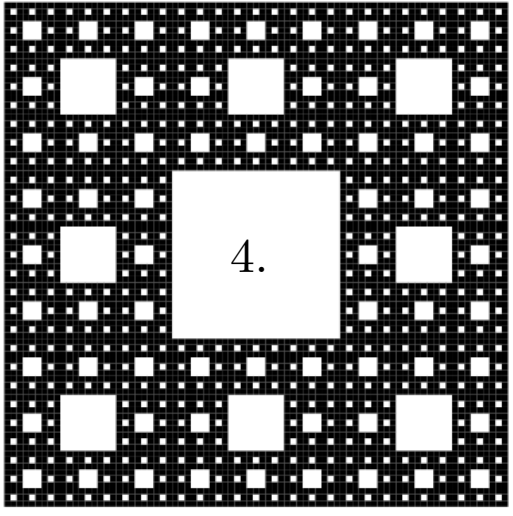


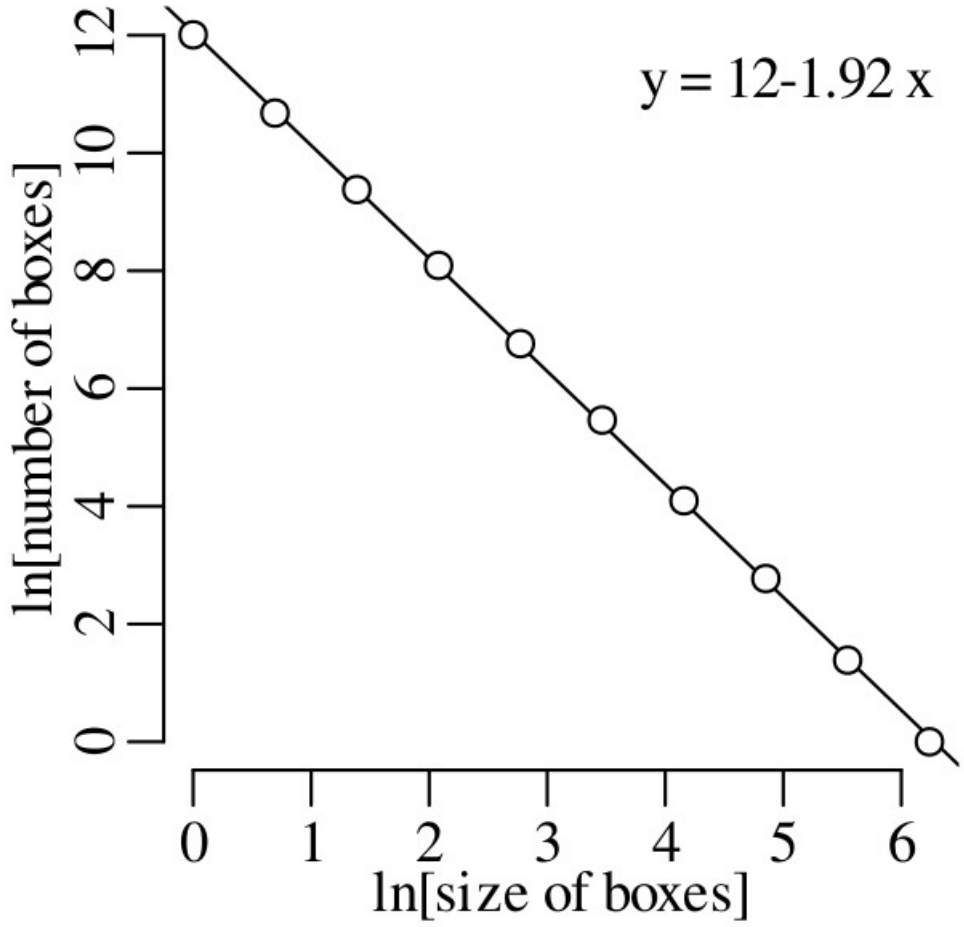

y = 7.84-1.21 x
ln[number of boxes]
0
2
4
6
8
0
1
2
3
4
5
6
ln[size of boxes]


1.


3.

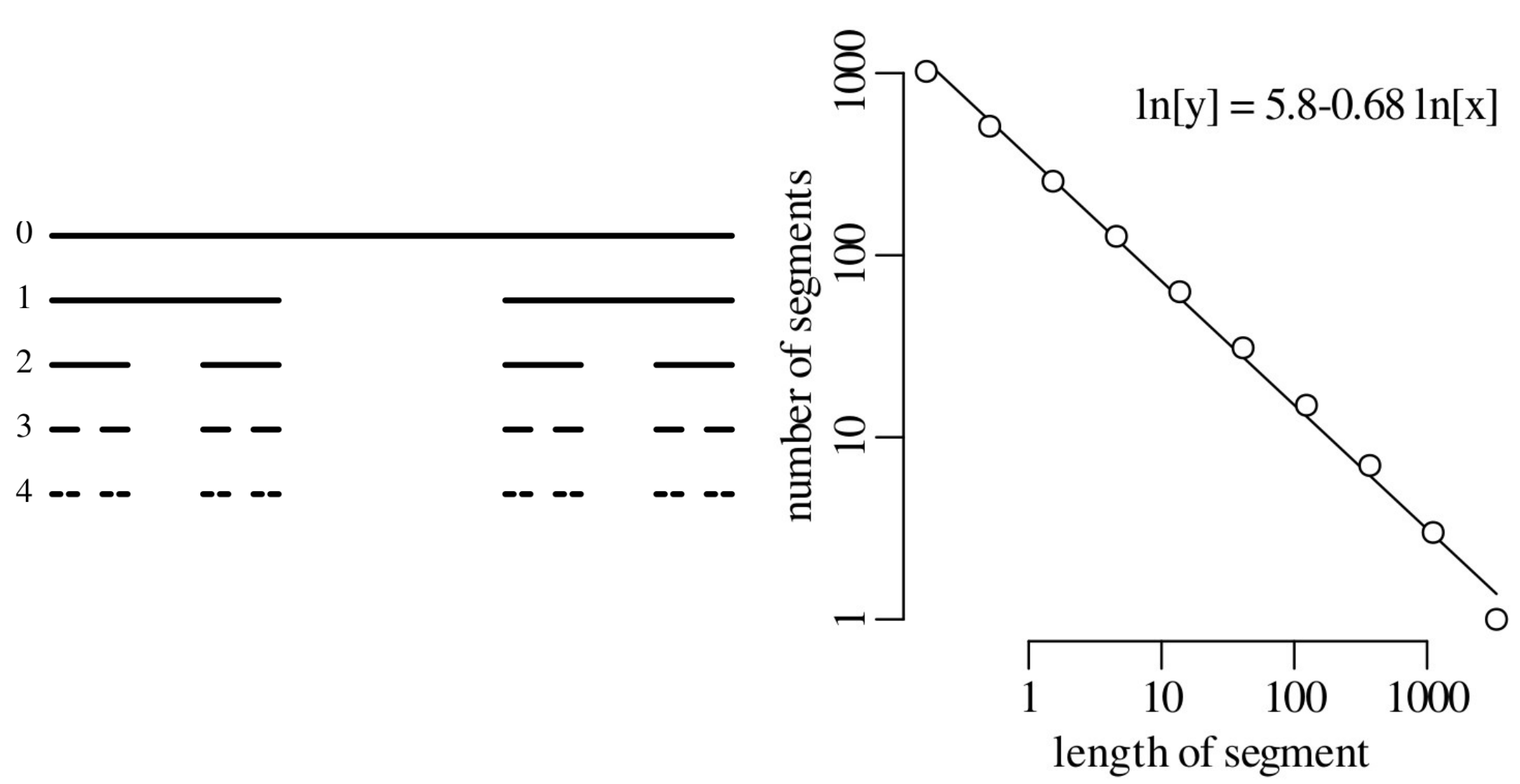

2.


4.


y = 12-1.92 x
ln[number of boxes]
0
2
4
6
8
10
12
0
1
2
3
4
5
6
ln[size of boxes]

$$\ln[\text{number of boxes}] = C + D \times \ln[\text{size of boxes}]$$

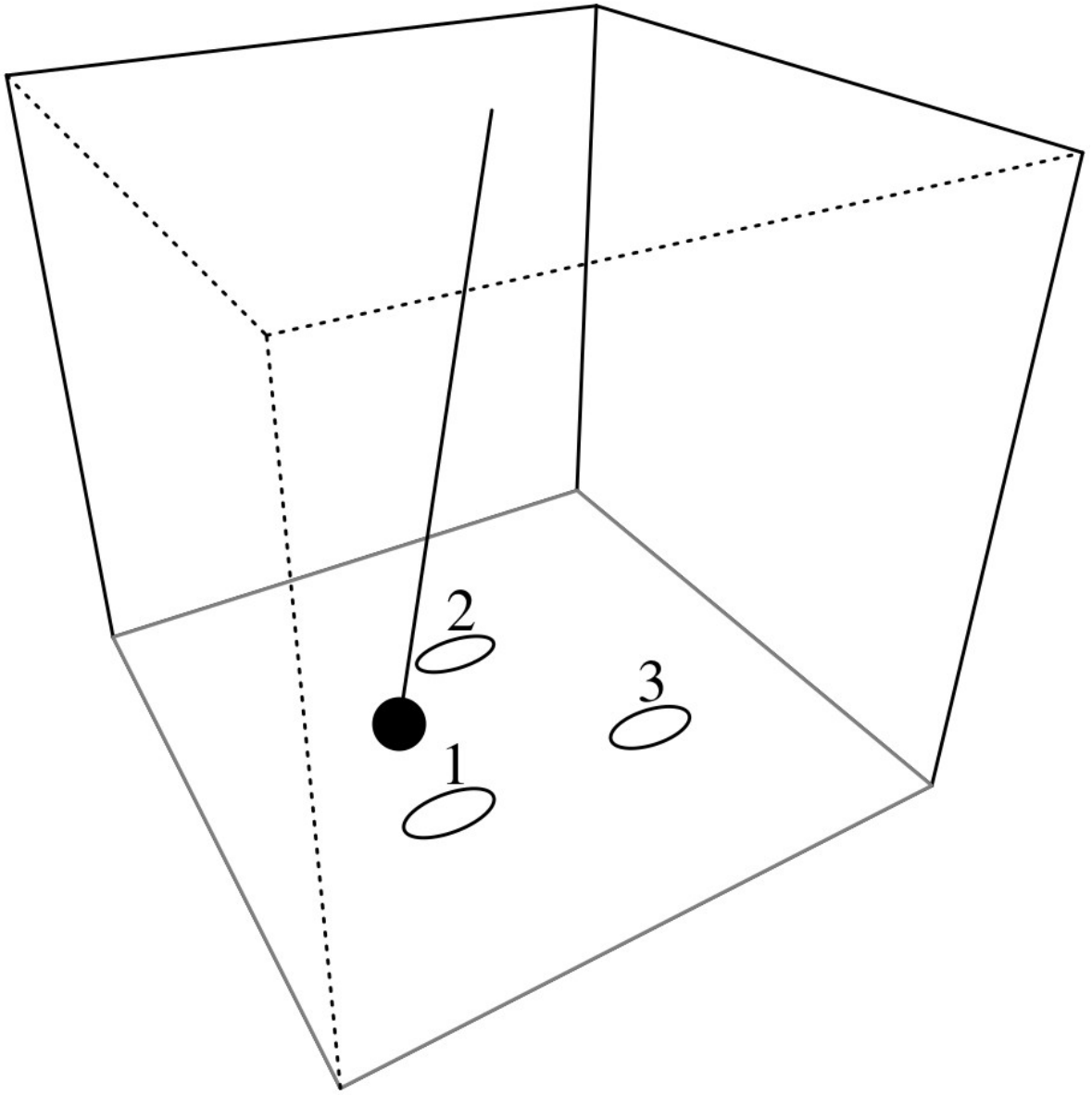
fractal dimension

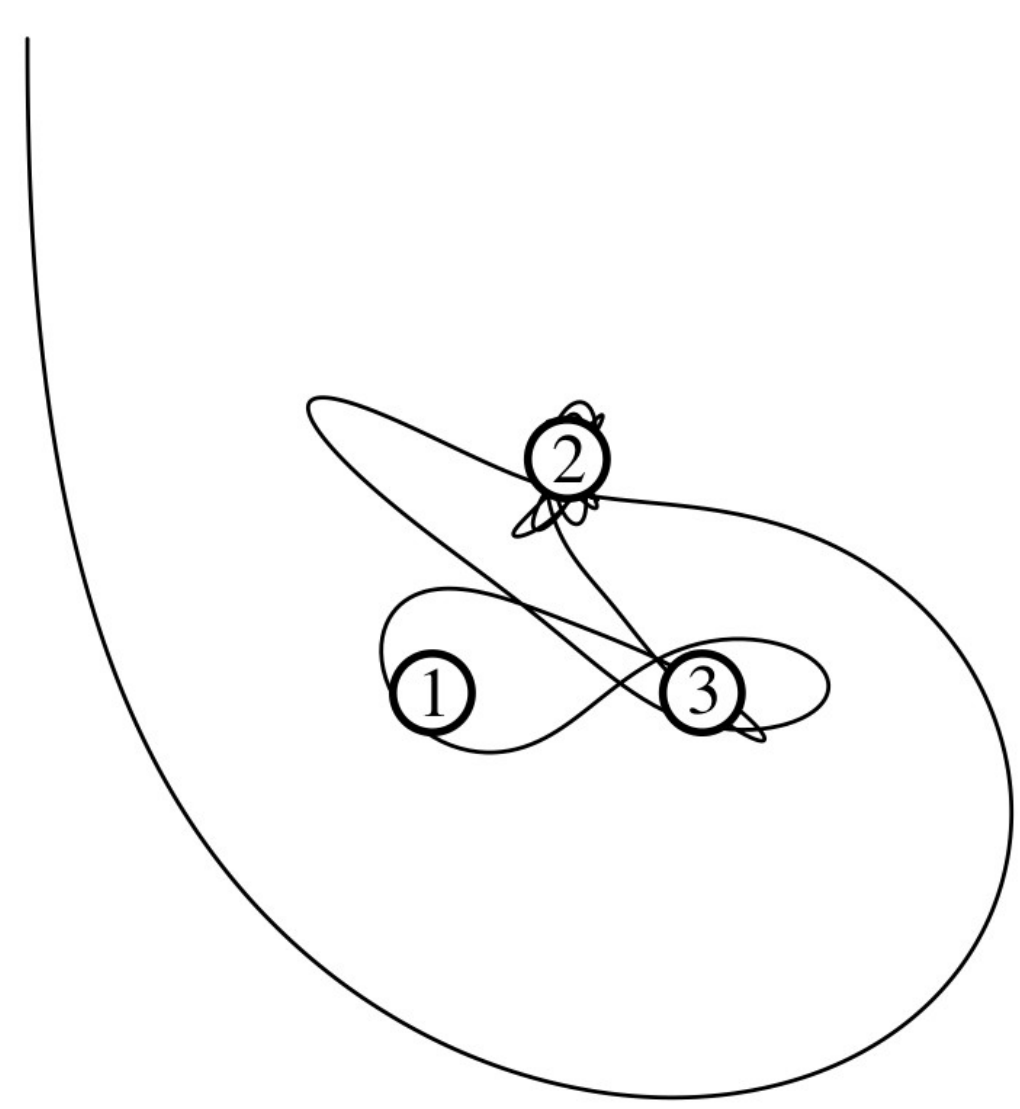

0
1
2
3
4
ln[y] = 5.8-0.68 ln[x]
number of segments
1000
100
10
1
1
10
100
1000
length of segment

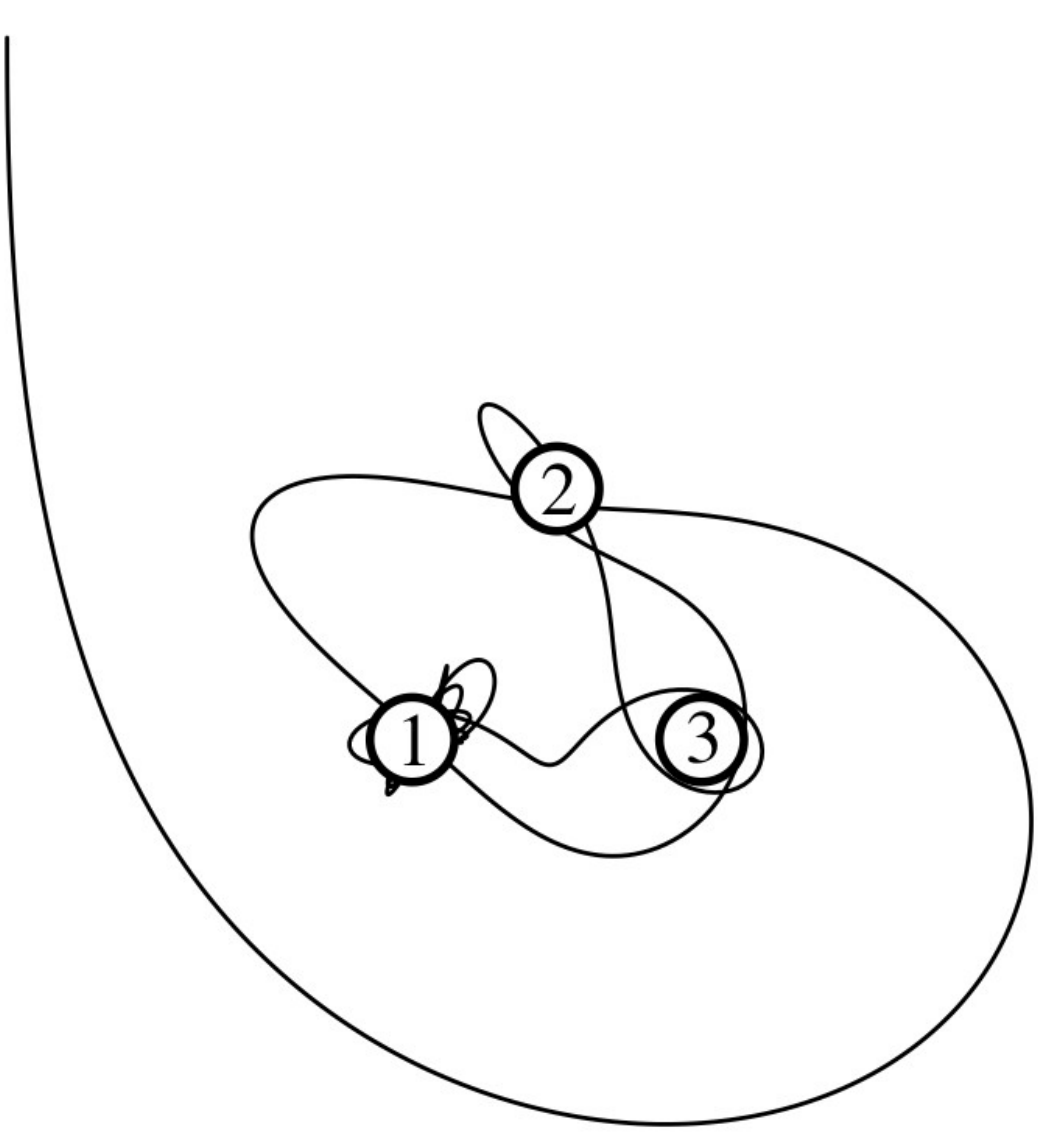
Chaos

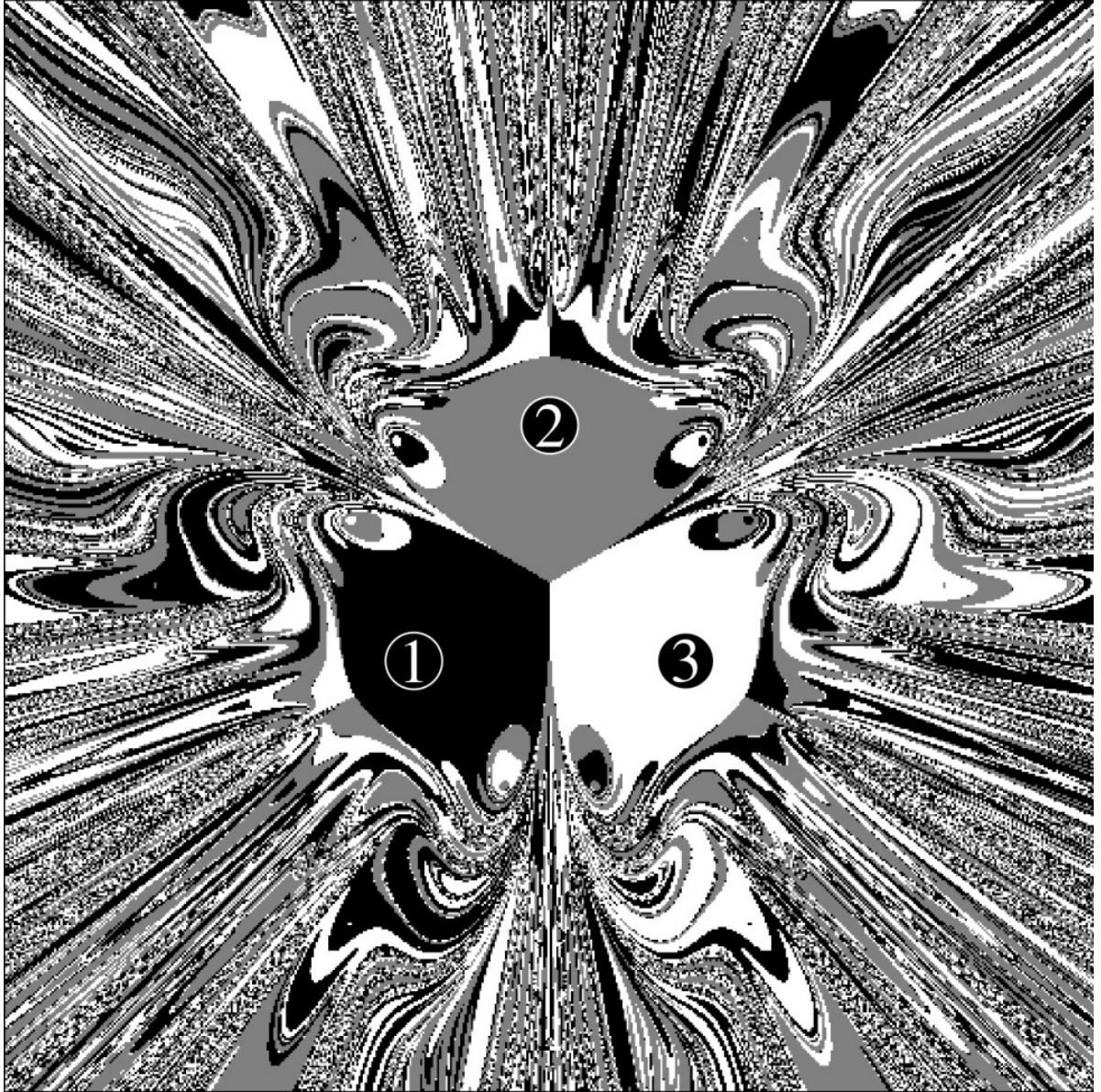
$$F_m(i) \propto 1/|d(i)|^2$$

$$F_f(i) \propto v$$




2
1
3


2
1
3

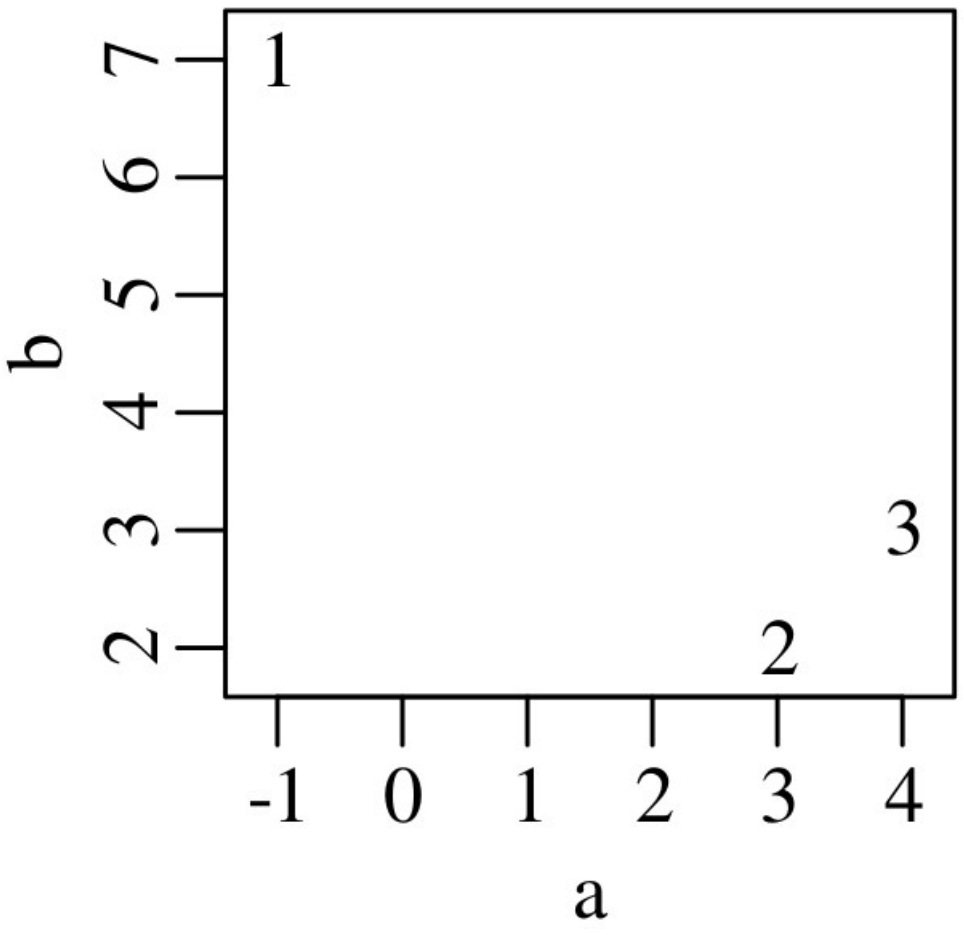

②
①
③

Statistics for geoscientists

Unsupervised learning

Principal Component Analysis

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \end{array}$$



$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \end{array}$$

$$X = 1_{3,1}\ C + S\ V\ D$$

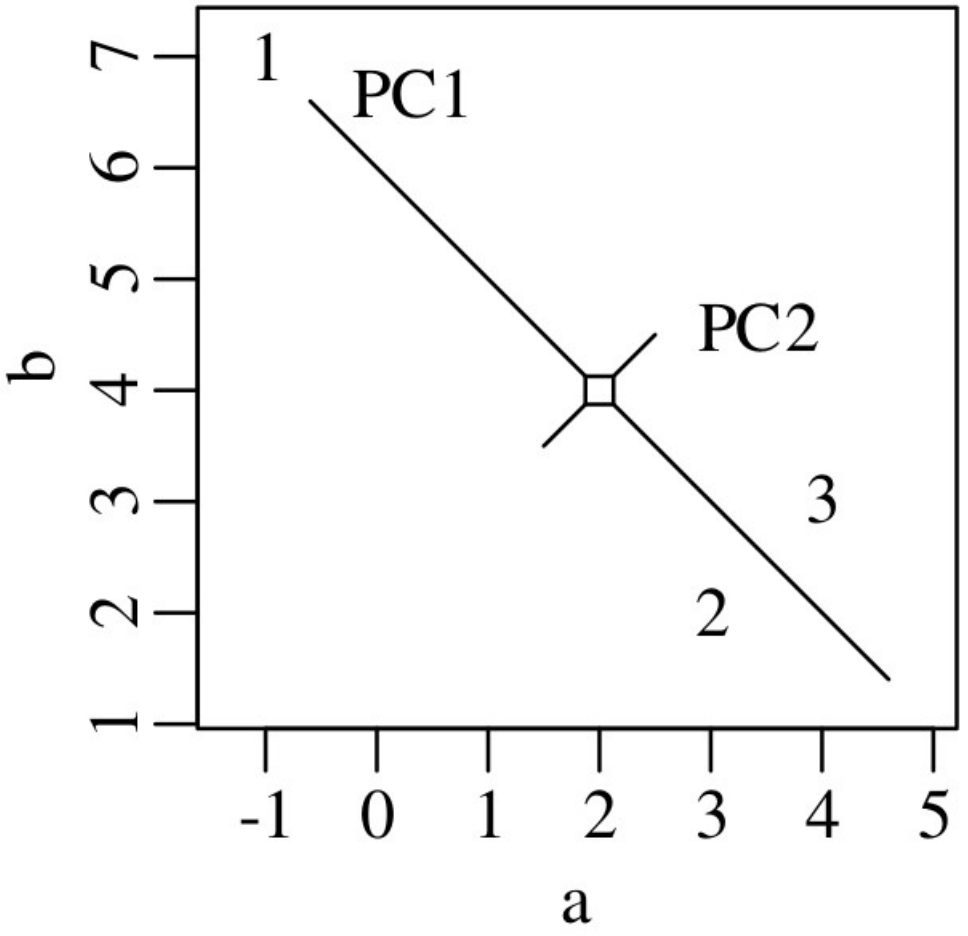
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} -1.15 & 0 \\ 0.58 & -1 \\ 0.58 & 1 \end{bmatrix} \begin{bmatrix} 3.67 & 0 \\ 0 & 0.71 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix}$$

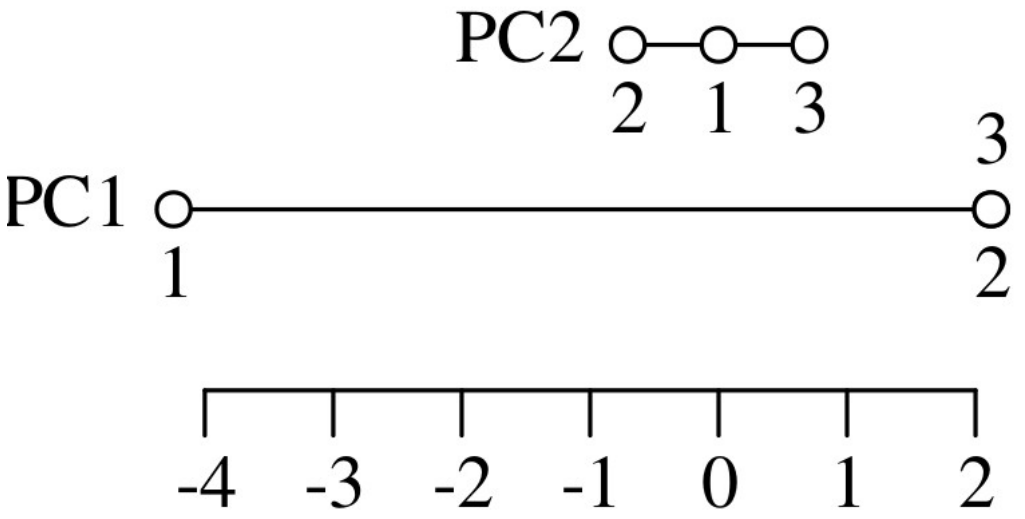
$$X = 1_{3,1}\ C + S\ V\ D$$

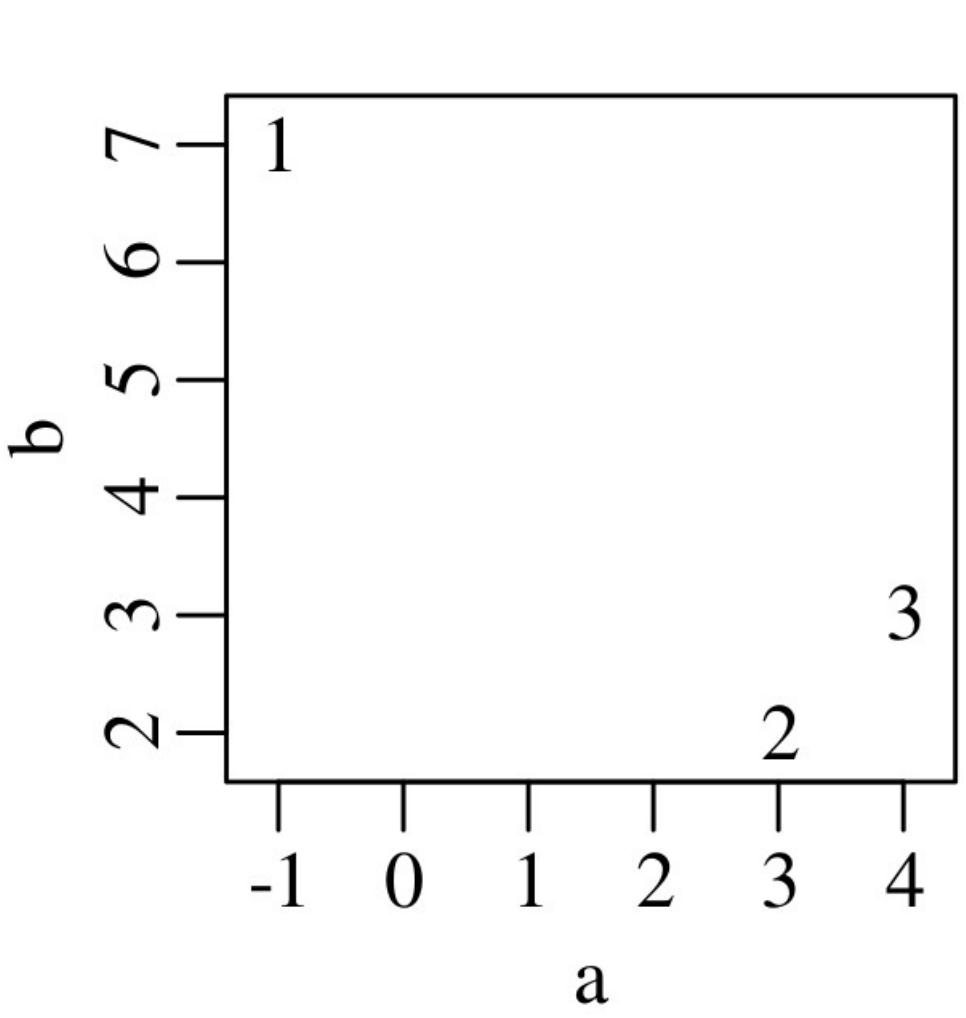
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} -1.15 & 0 \\ 0.58 & -1 \\ 0.58 & 1 \end{bmatrix} \begin{bmatrix} 3.67 & 0 \\ 0 & 0.71 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix}$$

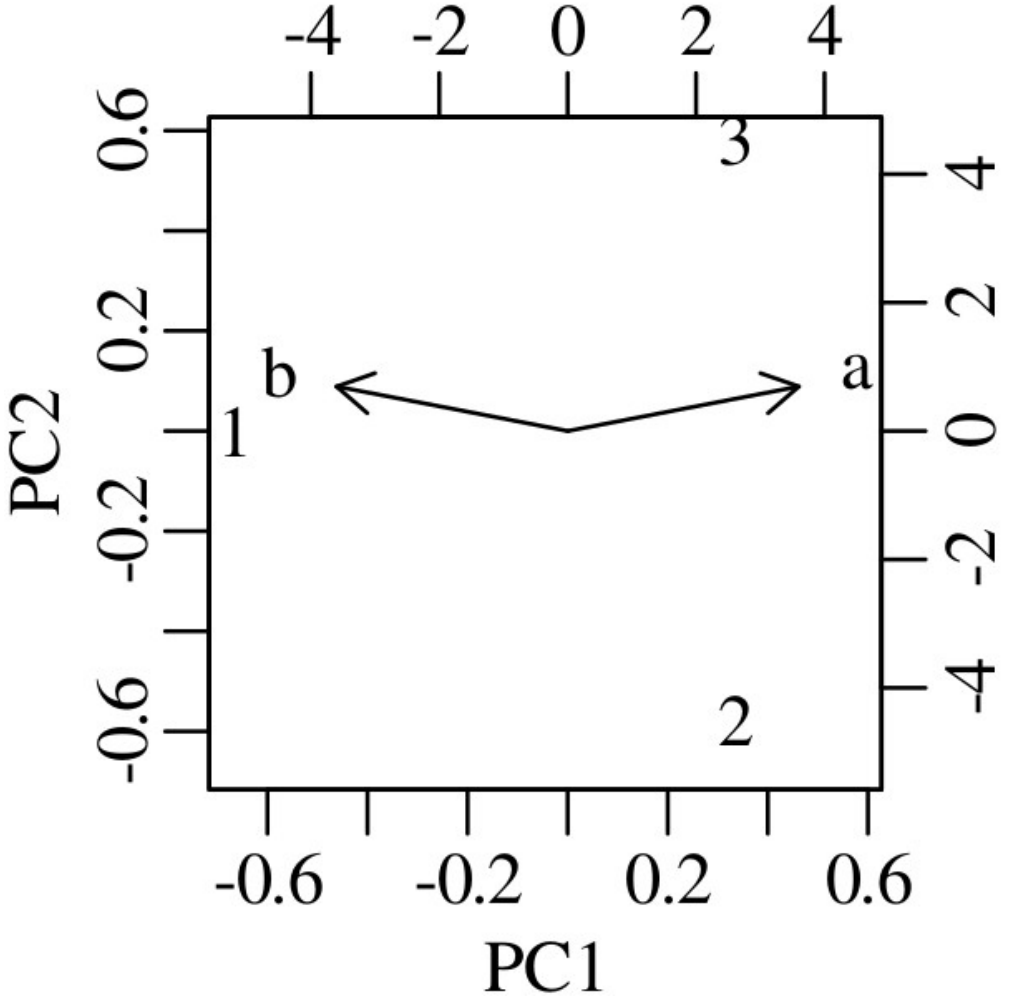
$$P = S\ V = \begin{bmatrix} -4.24 & 0 \\ 2.12 & -0.71 \\ 2.12 & 0.71 \end{bmatrix},$$

and $L = V\ D = \begin{bmatrix} 2.6 & -2.6 \\ 0.5 & 0.5 \end{bmatrix}$

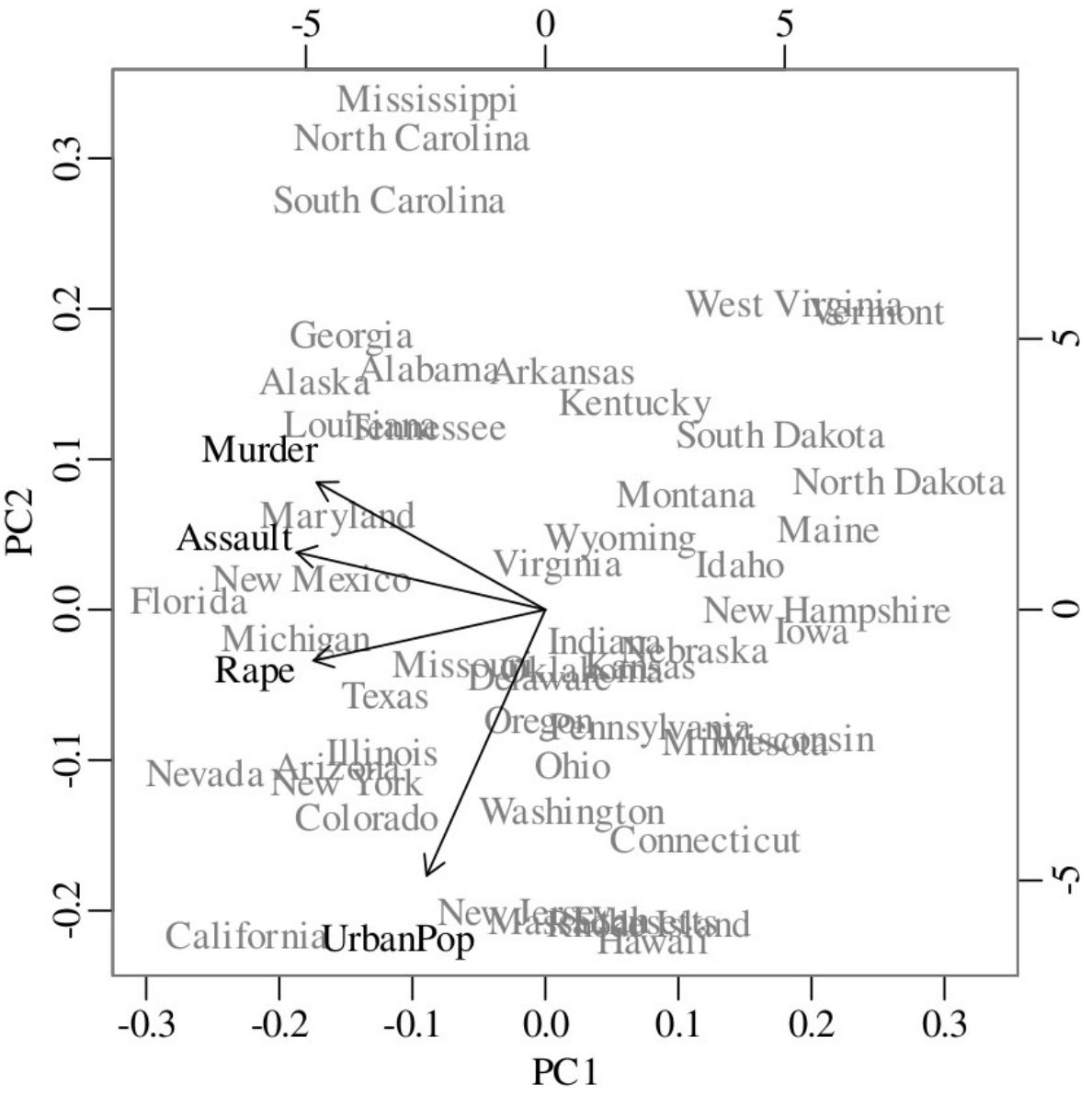

1
PC1
PC2
3
2
b
a
1 2 3 4 5 6 7
-1 0 1 2 3 4 5


PC2
2 1 3
PC1
1
3
2
-4 -3 -2 -1 0 1 2


b
a
1
2
3
-1
0
1
2
3
4
2
3
4
5
6
7

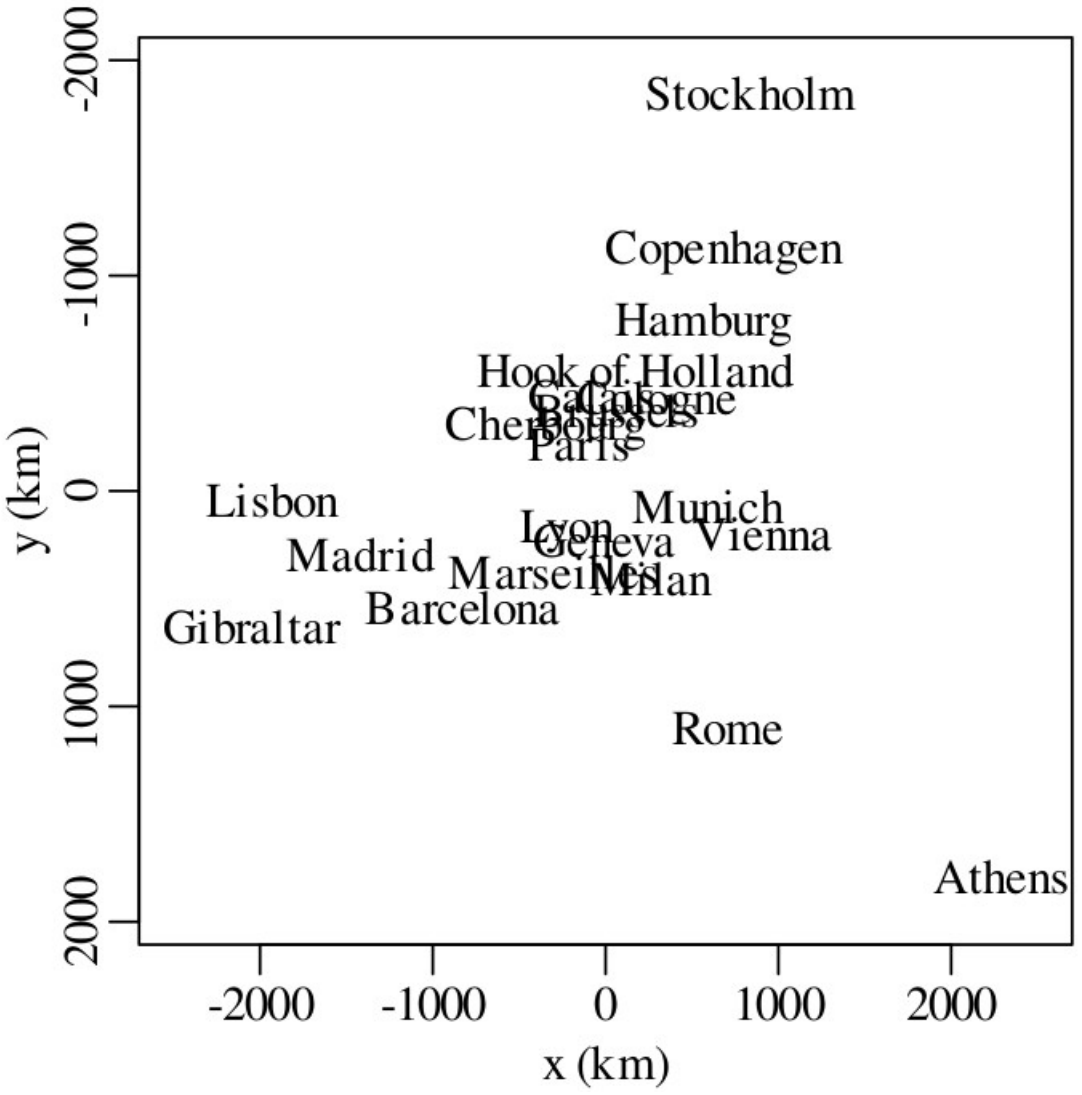

PC2
PC1
a
b
1
2
3
-0.6
-0.2
0.2
0.6
-4
-2
0
2
4

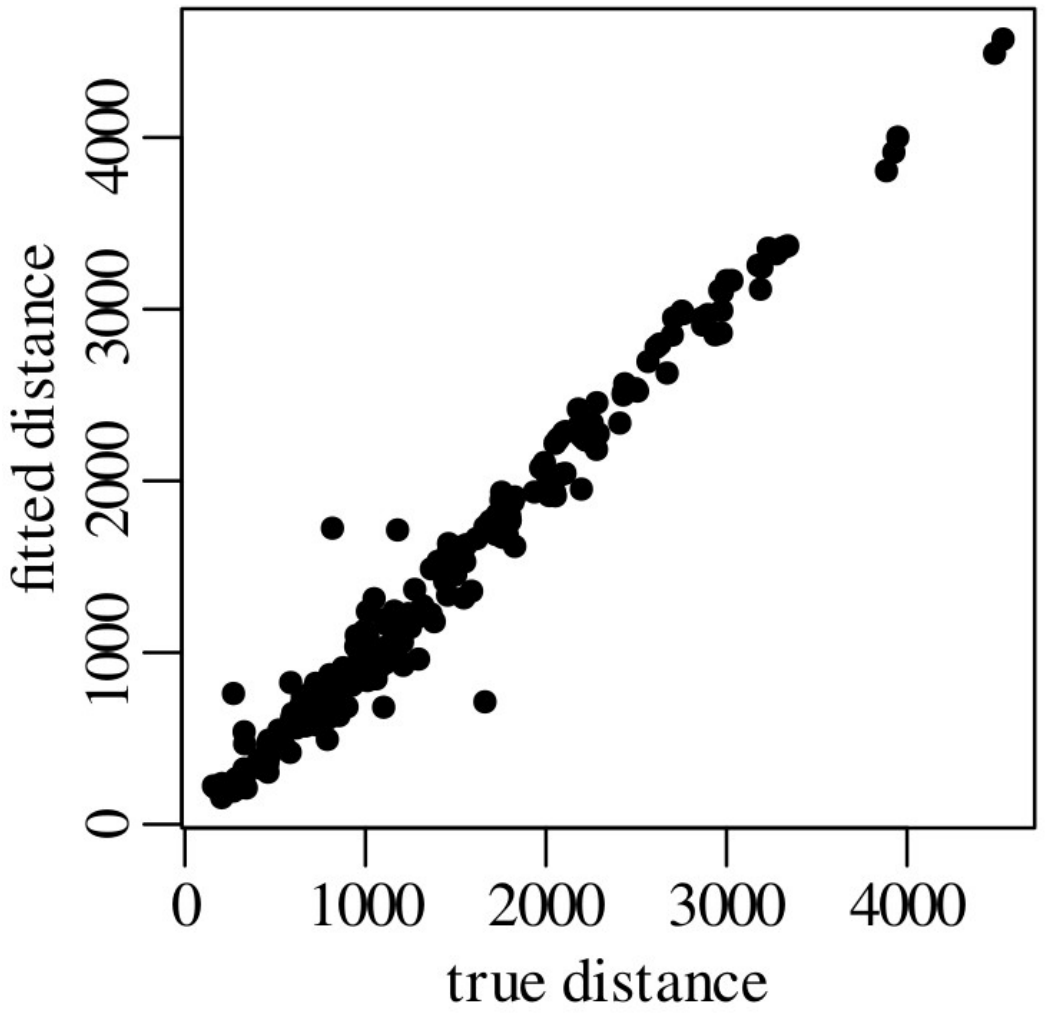
	Murder	Assault	Rape	UrbanPop
Alabama	13.2	236	21.2	58
Alaska	10.0	263	44.5	48
Arizona	8.1	294	31.0	80
Arkansas	8.8	190	19.5	50
California	9.0	276	40.6	91
Colorado	7.9	204	38.7	78
⋮	⋮	⋮	⋮	⋮
Wisconsin	2.6	53	10.8	66
Wyoming	6.8	161	15.6	60


-5
0
5
Mississippi
North Carolina
South Carolina
West Virginia
Vermont
Georgia
Alabama
Arkansas
Alaska
Kentucky
Louisiana
Tennessee
South Dakota
Murder
North Dakota
Montana
Maryland
Assault
Wyoming
Maine
Virginia
Idaho
New Mexico
Florida
New Hampshire
Iowa
Michigan
Indiana
Nebraska
Rape
Missouri
Oklahoma
Kansas
Delaware
Texas
Oregon
Pennsylvania
Wisconsin
Minnesota
Illinois
Nevada
Arizona
Ohio
New York
Colorado
Washington
Connecticut
New Jersey
Rhode Island
California
UrbanPop
Hawaii
0.3
0.2
0.1
0.0
-0.1
-0.2
PC2
5
0
-5
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
PC1

Multidimensional Scaling

	Athens	Barcelona	Brussels	...	Rome	Stockholm	Vienna
Athens	0	3313	2963	...	817	3927	1991
Barcelona	3313	0	1326	...	1460	2868	1802
Brussels	2963	1318	0	...	1511	1616	1175
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
Rome	817	1460	1511	...	0	2707	1209
Stockholm	3927	2868	1616	...	2707	0	2105
Vienna	1991	1802	1175	...	1209	2105	0


Stockholm
Copenhagen
Hamburg
Hook of Holland
Paris
Lisbon
Munich
Vienna
Madrid
Milan
Barcelona
Gibraltar
Rome
Athens
-2000
-1000
0
1000
2000
y (km)
x (km)



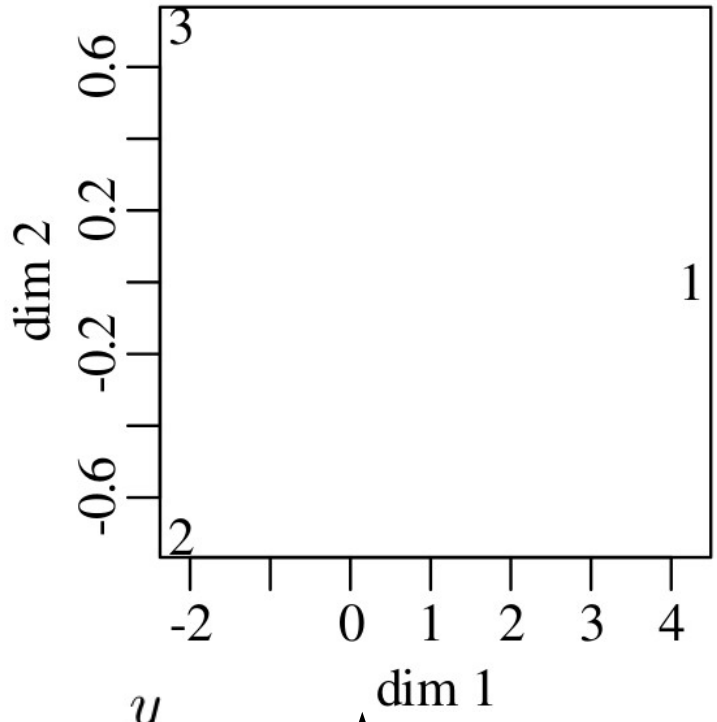
$$S = \sqrt{\frac{\sum_{i=1}^{n} \sum_{j=i+1}^{n} (f(d[i,j]) - \delta[i,j])^2}{\sum_{i=1}^{n} \sum_{j=i+1}^{n} \delta[i,j]^2}}$$

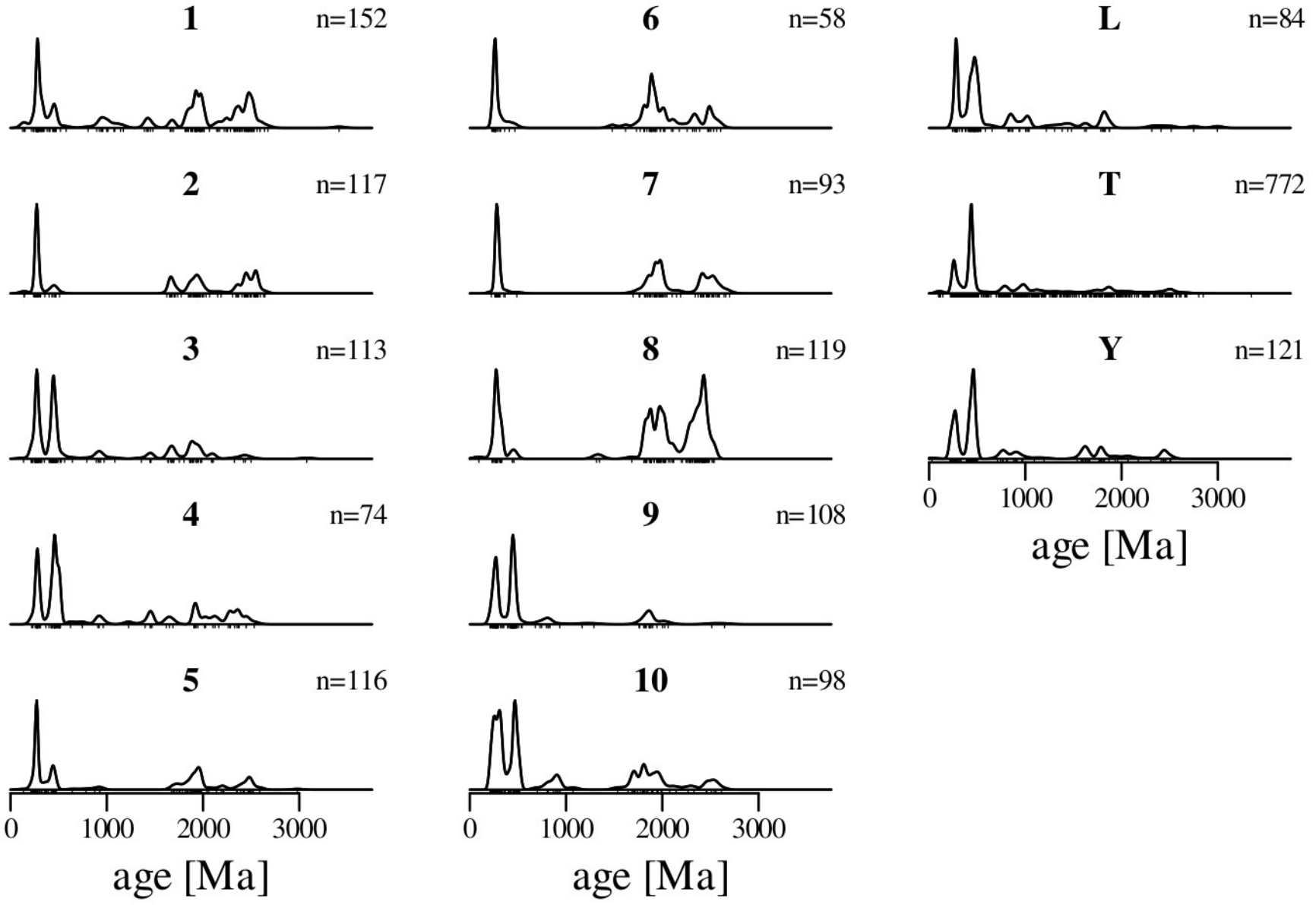
fit	poor	fair	good	excellent	perfect
S	0.2	0.1	0.05	0.025	0

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \end{array}$$

$$d[i,j] = \sqrt{(a[i]-a[j])^2 + (b[i]-b[j])^2}$$

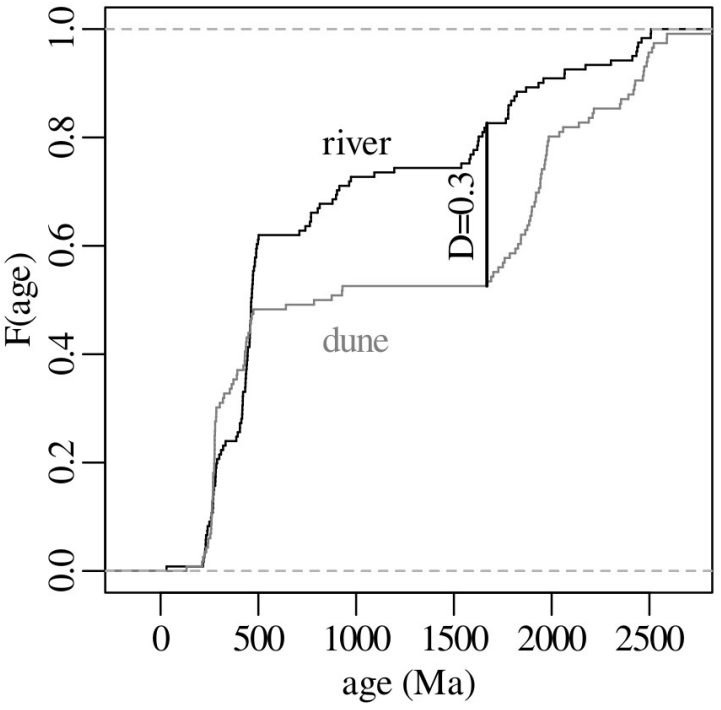
$$d = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 6.4 & 6.4 \\ 6.4 & 0 & 1.4 \\ 6.4 & 1.4 & 0 \end{bmatrix} \end{array} \longrightarrow m = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 4.24 & 0 \\ -2.12 & -0.71 \\ -2.12 & 0.71 \end{bmatrix} \end{array}$$

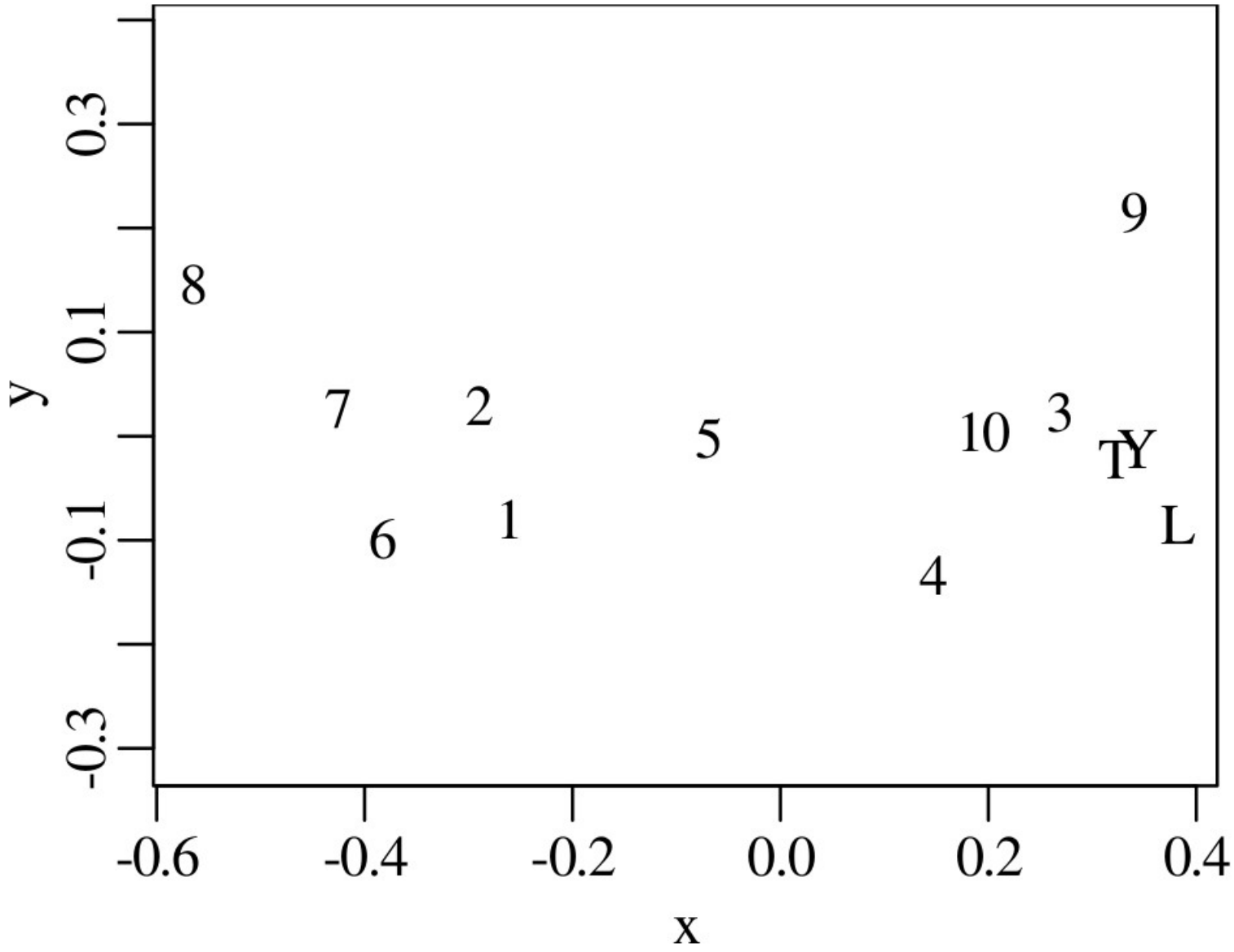



1
n=152
2
n=117
3
n=113
4
n=74
5
n=116
6
n=58
7
n=93
8
n=119
9
n=108
10
n=98
L
n=84
T
n=772
Y
n=121
0
1000
2000
3000
age [Ma]
0
1000
2000
3000
age [Ma]

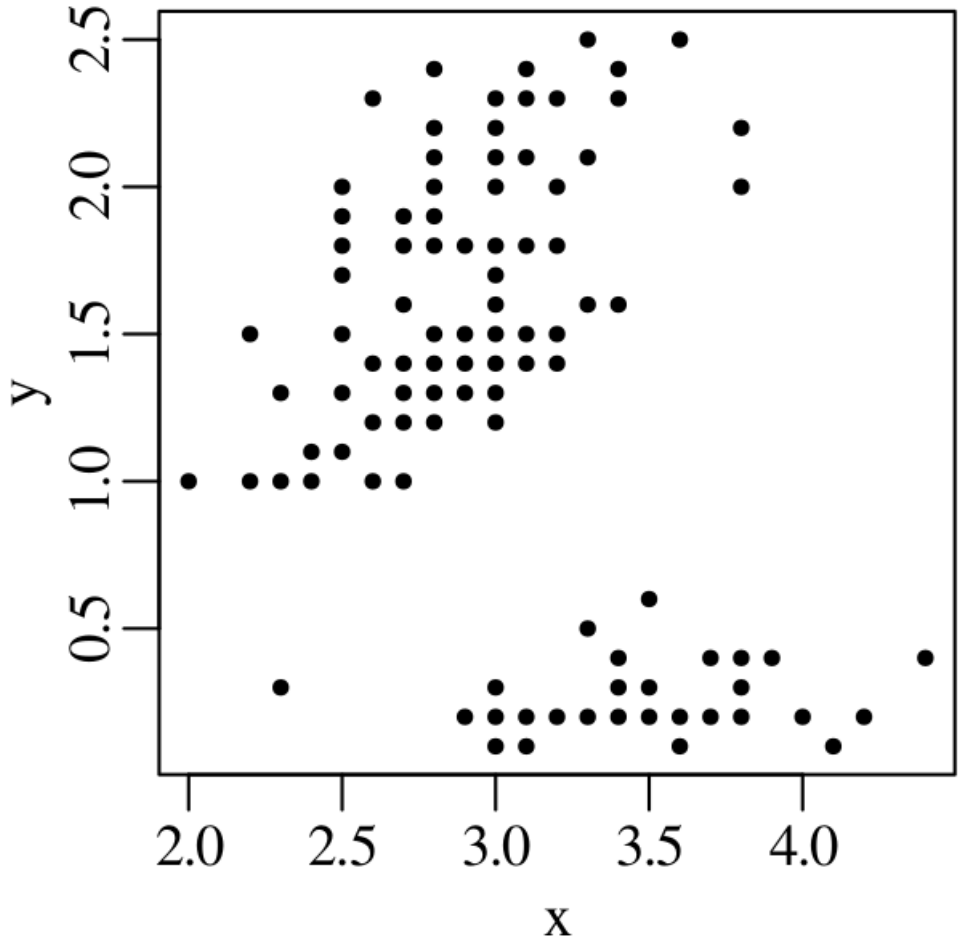
$d =$

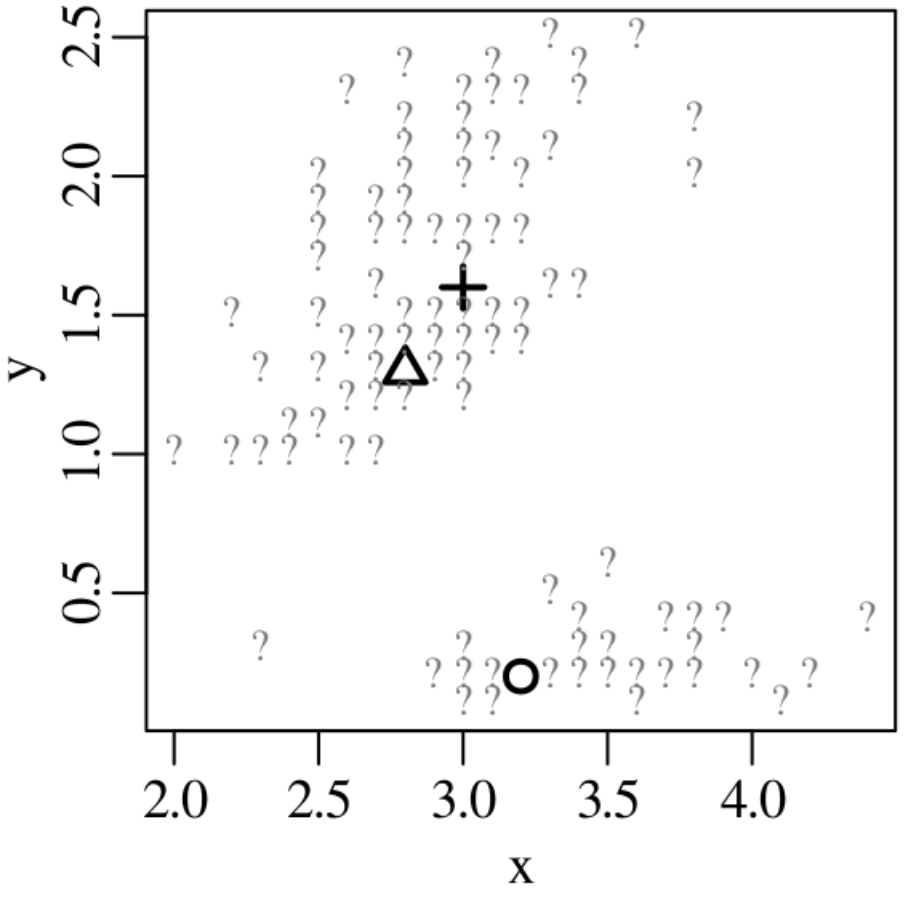
	1	2	3	4	5	6	7	8	9	10	L	T	Y
1	0	14	33	27	18	14	15	22	48	32	42	37	40
2	14	0	36	33	16	14	15	24	46	32	47	42	43
3	33	36	0	19	24	44	47	55	17	10	13	12	8
4	27	33	19	0	20	38	41	48	28	14	21	17	16
5	18	16	24	20	0	22	24	33	31	20	33	28	30
6	14	14	44	38	22	0	14	24	52	41	52	48	49
7	15	15	47	41	24	14	0	16	51	43	54	49	52
8	22	24	55	48	33	24	16	0	61	53	63	59	62
9	48	46	17	28	31	52	51	61	0	20	22	18	16
10	32	32	10	14	20	41	43	53	20	0	17	15	13
L	42	47	13	21	33	52	54	63	22	17	0	10	11
T	37	42	12	17	28	48	49	59	18	15	10	0	7
Y	40	43	8	16	30	49	52	62	16	13	11	7	0

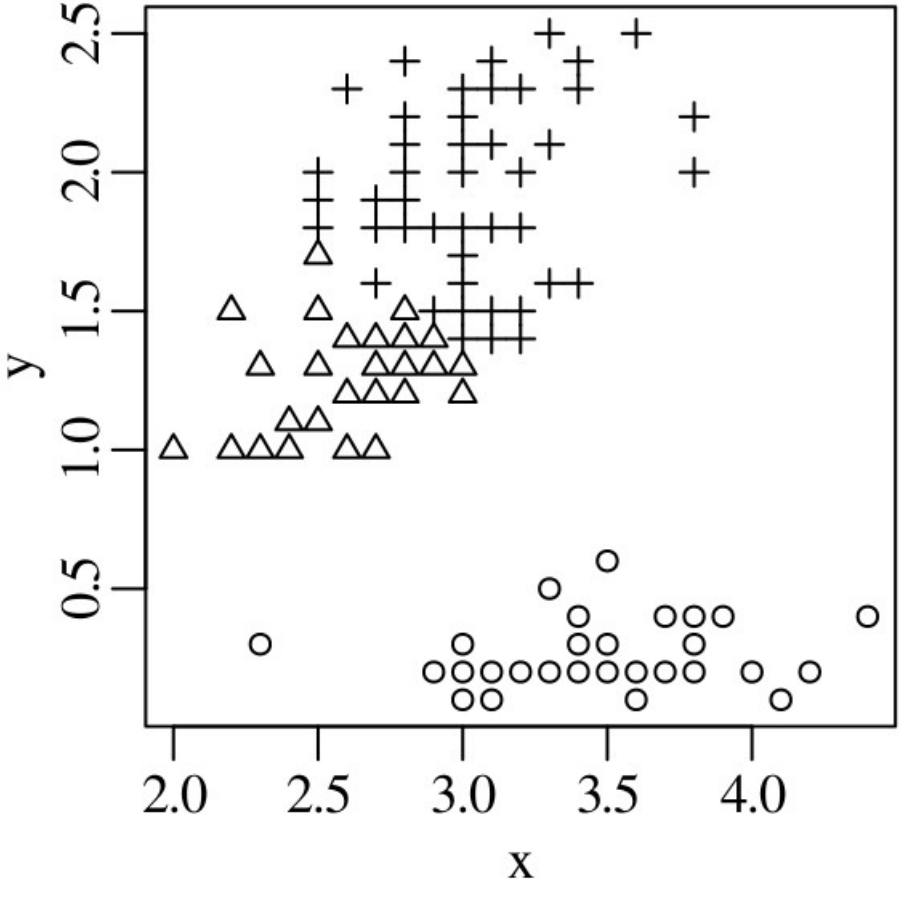


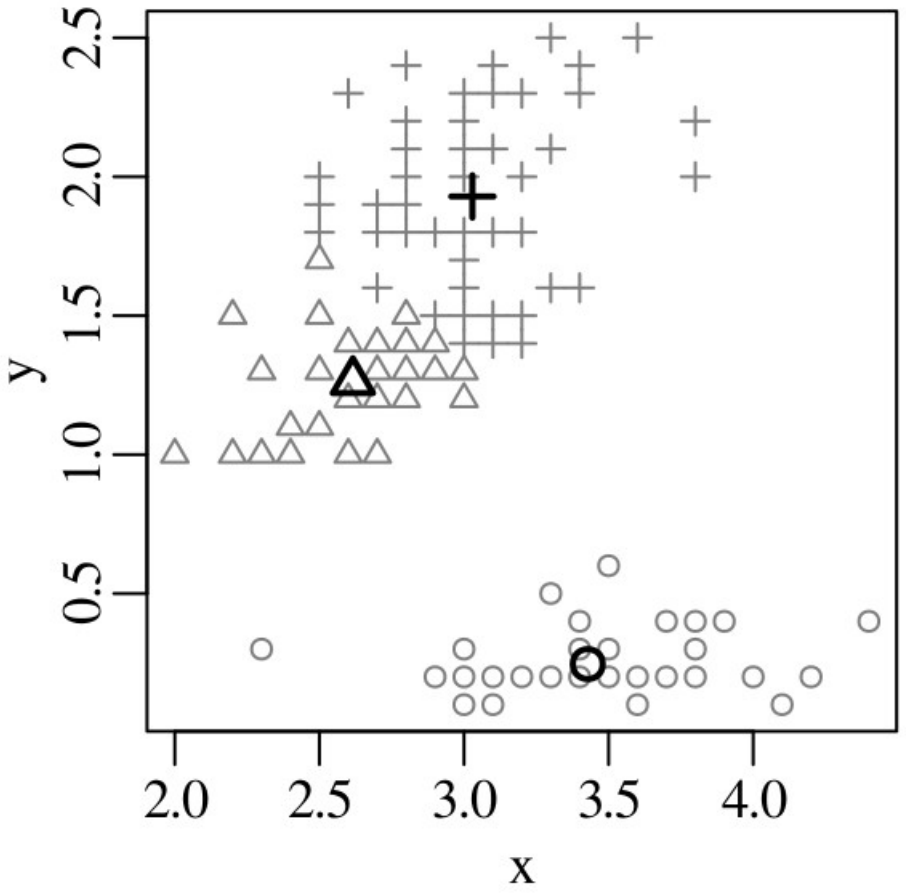

y
0.3
0.1
-0.1
-0.3
9
8
7
2
5
10
3
T
Y
1
6
L
4
-0.6
-0.4
-0.2
0.0
0.2
0.4
x

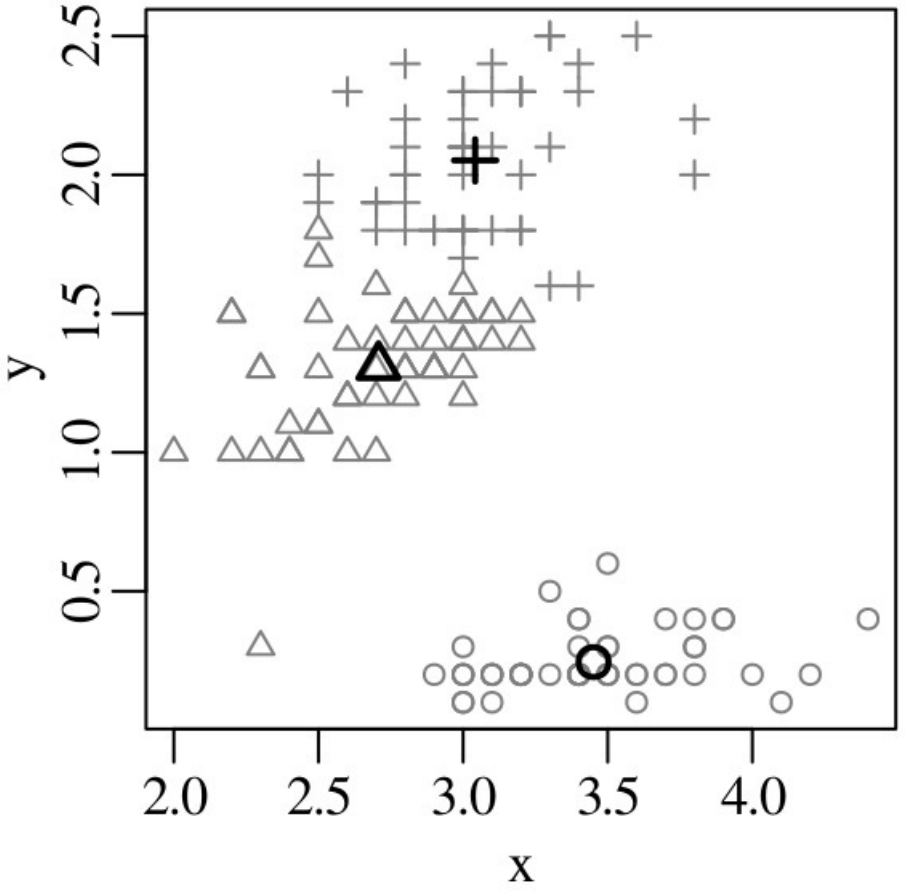
K-means clustering

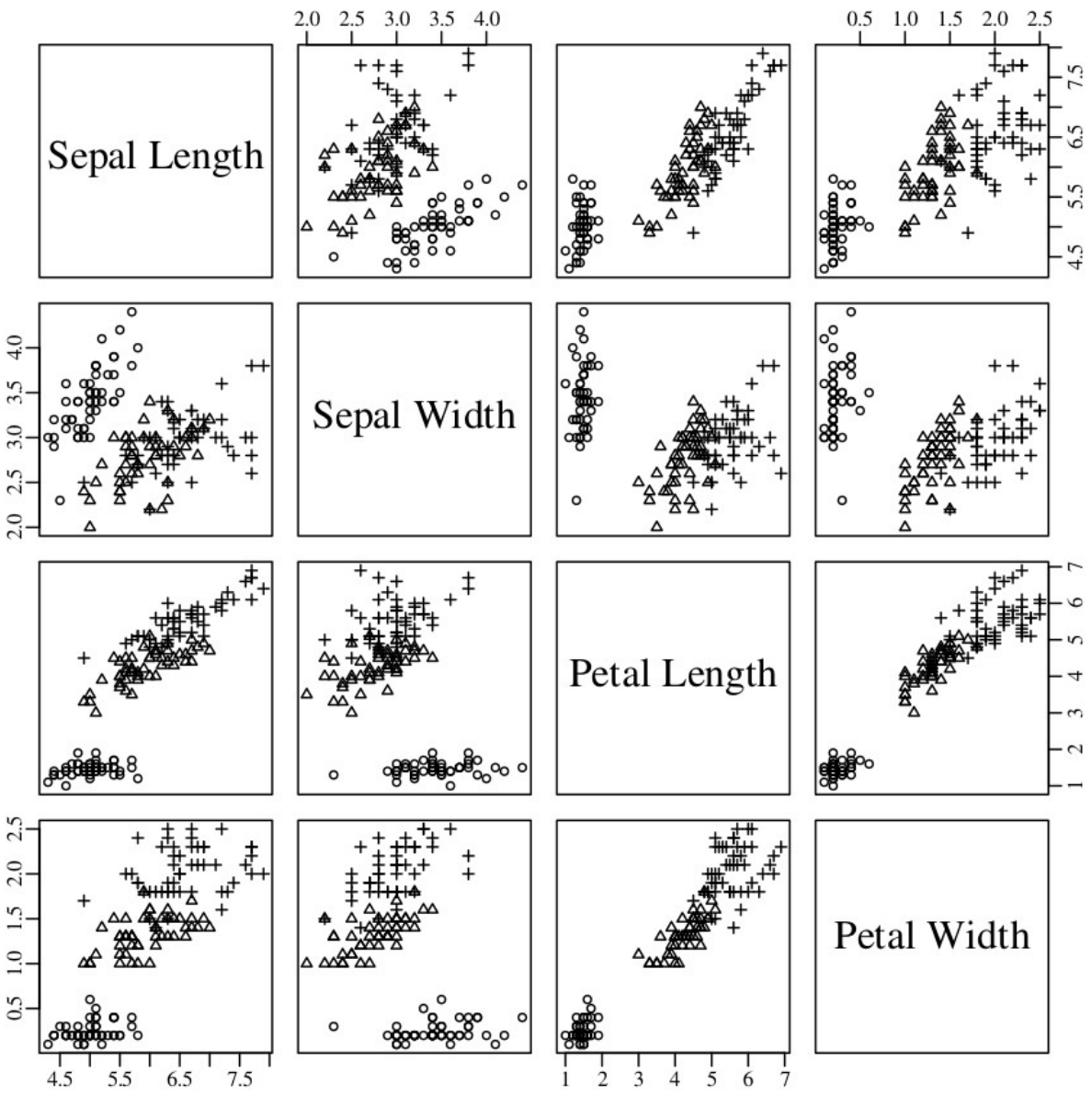



y
2.5
2.0
1.5
1.0
0.5
2.0
2.5
3.0
3.5
4.0
x


y
2.5
2.0
1.5
1.0
0.5
2.0
2.5
3.0
3.5
4.0
x


y
2.5
2.0
1.5
1.0
0.5
2.0
2.5
3.0
3.5
4.0
x

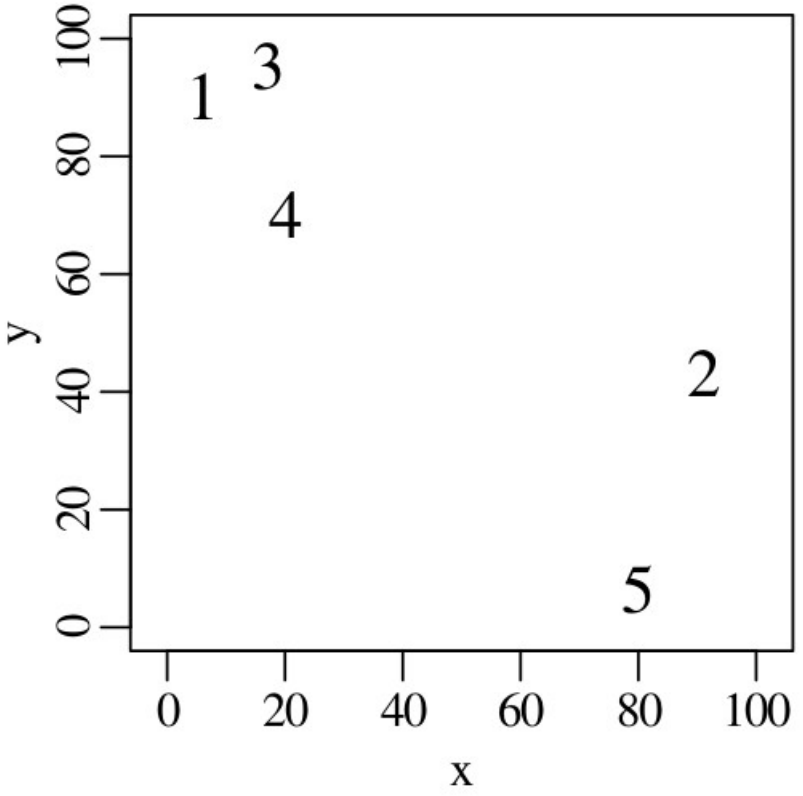

y
2.5
2.0
1.5
1.0
0.5
2.0
2.5
3.0
3.5
4.0
x



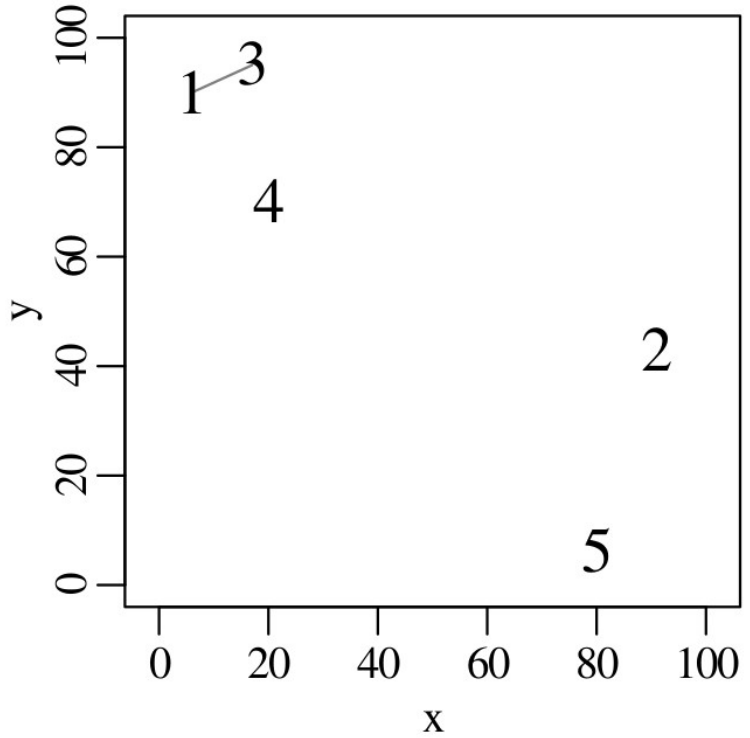
cluster	setosa	versicolor	virginica
1	50	0	0
2	0	48	14
3	0	2	36

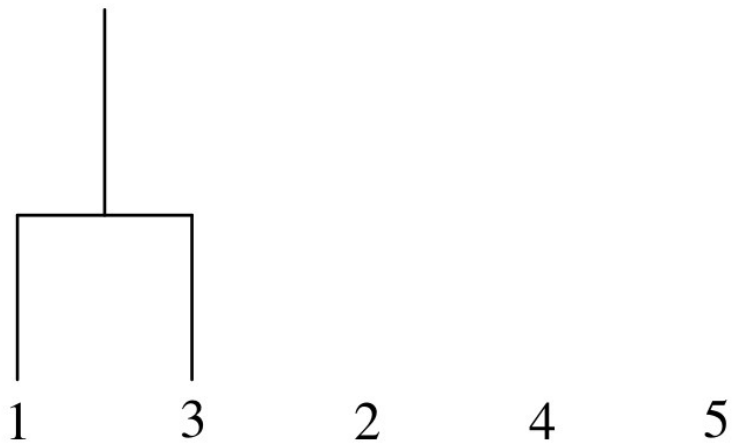
Hierarchical clustering

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 6 & 90 \\ 91 & 43 \\ 17 & 95 \\ 20 & 70 \\ 80 & 6 \end{bmatrix} \end{array}$$



$$
d = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0 & 97.1 & \boxed{12.1} & 24.4 & 112 \\
97.1 & 0 & 90.4 & 76.0 & 38.6 \\
\boxed{12.1} & 90.4 & 0 & 25.2 & 109 \\
24.4 & 76.0 & 25.2 & 0 & 87.7 \\
112 & 38.6 & 109 & 87.7 & 0
\end{bmatrix}
\end{array}
$$

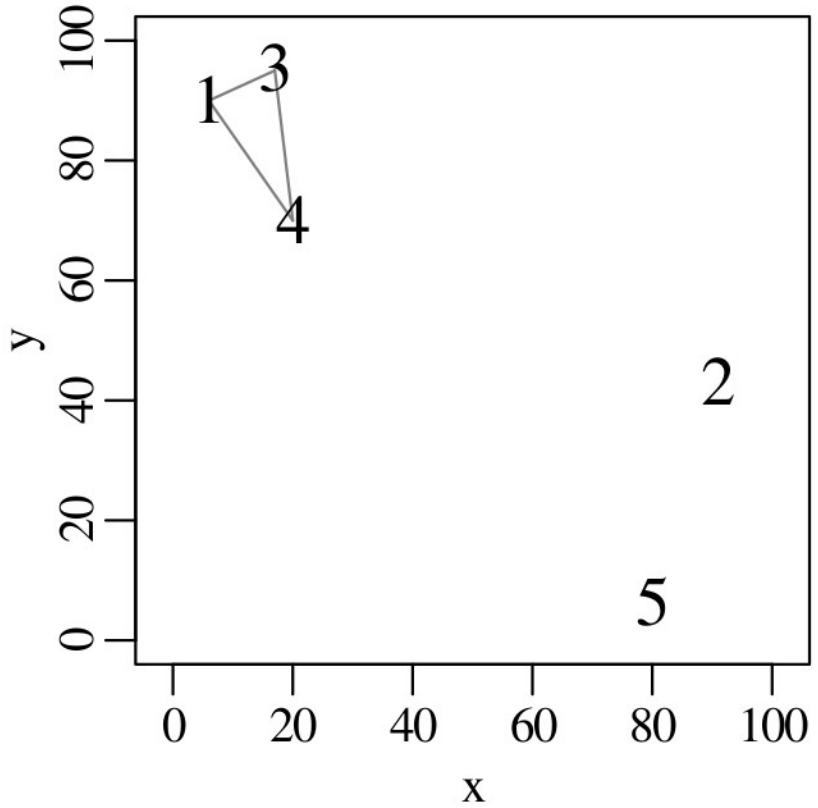


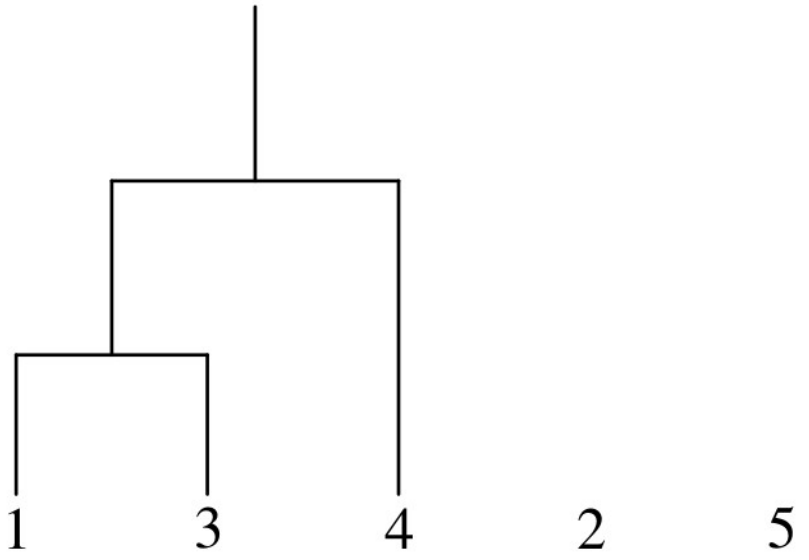


$$d = \begin{array}{c} \\ 13 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 13 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 97.1 & \boxed{25.2} & 112 \\ 97.1 & 0 & 76.0 & 38.6 \\ \boxed{25.2} & 76.0 & 0 & 87.7 \\ 112 & 38.6 & 87.7 & 0 \end{bmatrix} \end{array}$$

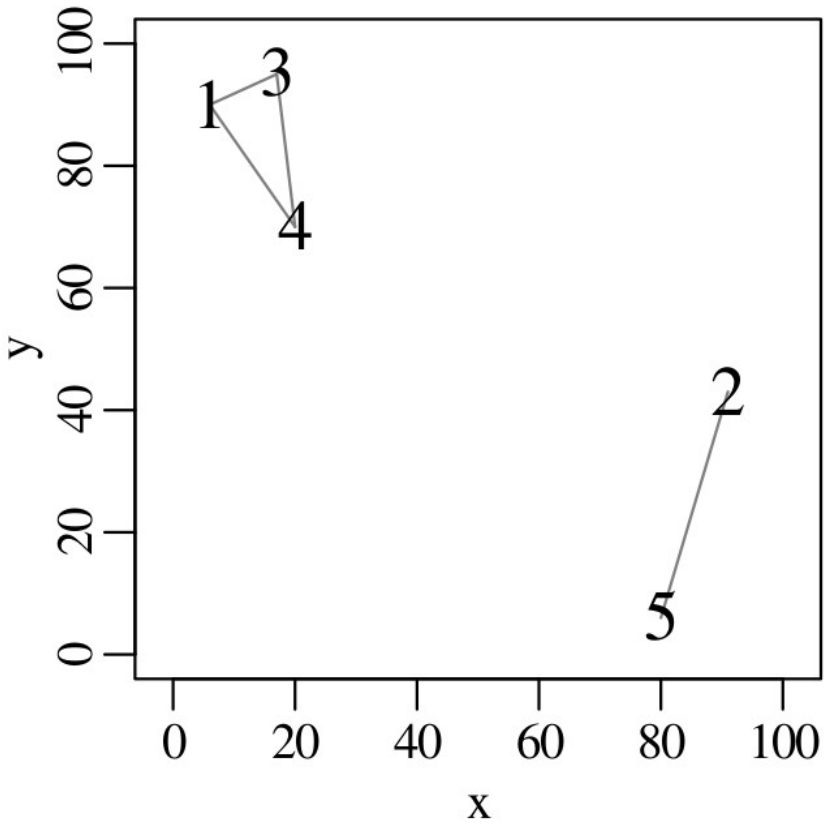
where

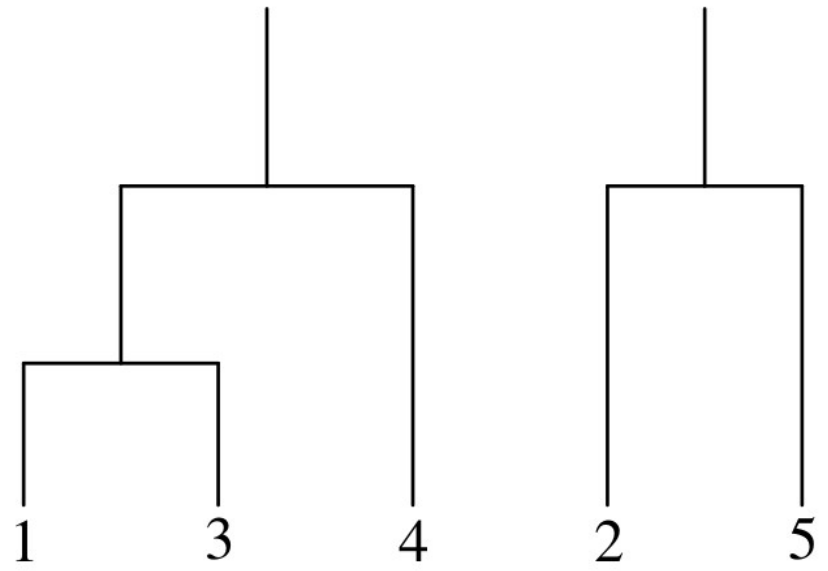
$$d[13, i] = \max(d[1, i], d[3, i]) \text{ for } i \in \{2, 4, 5\}$$

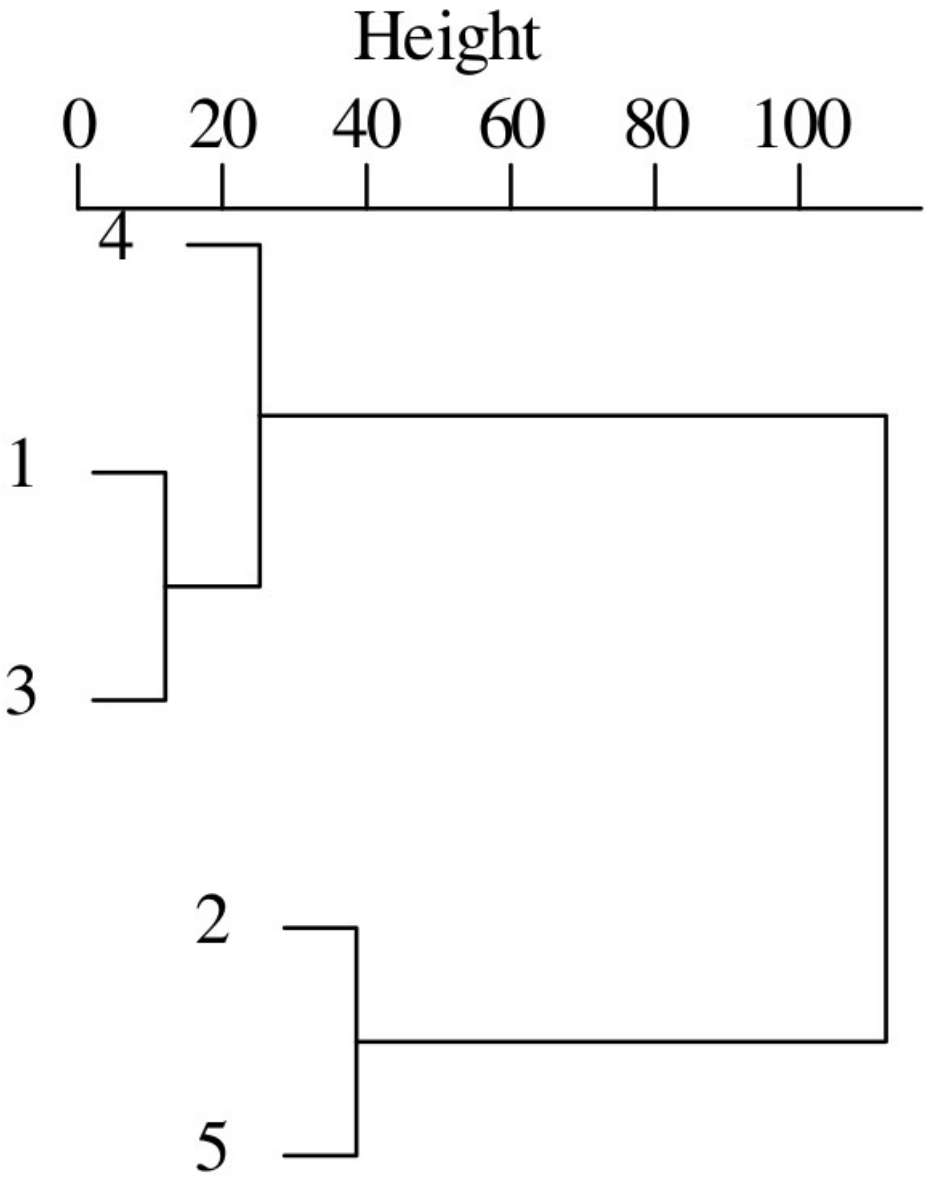


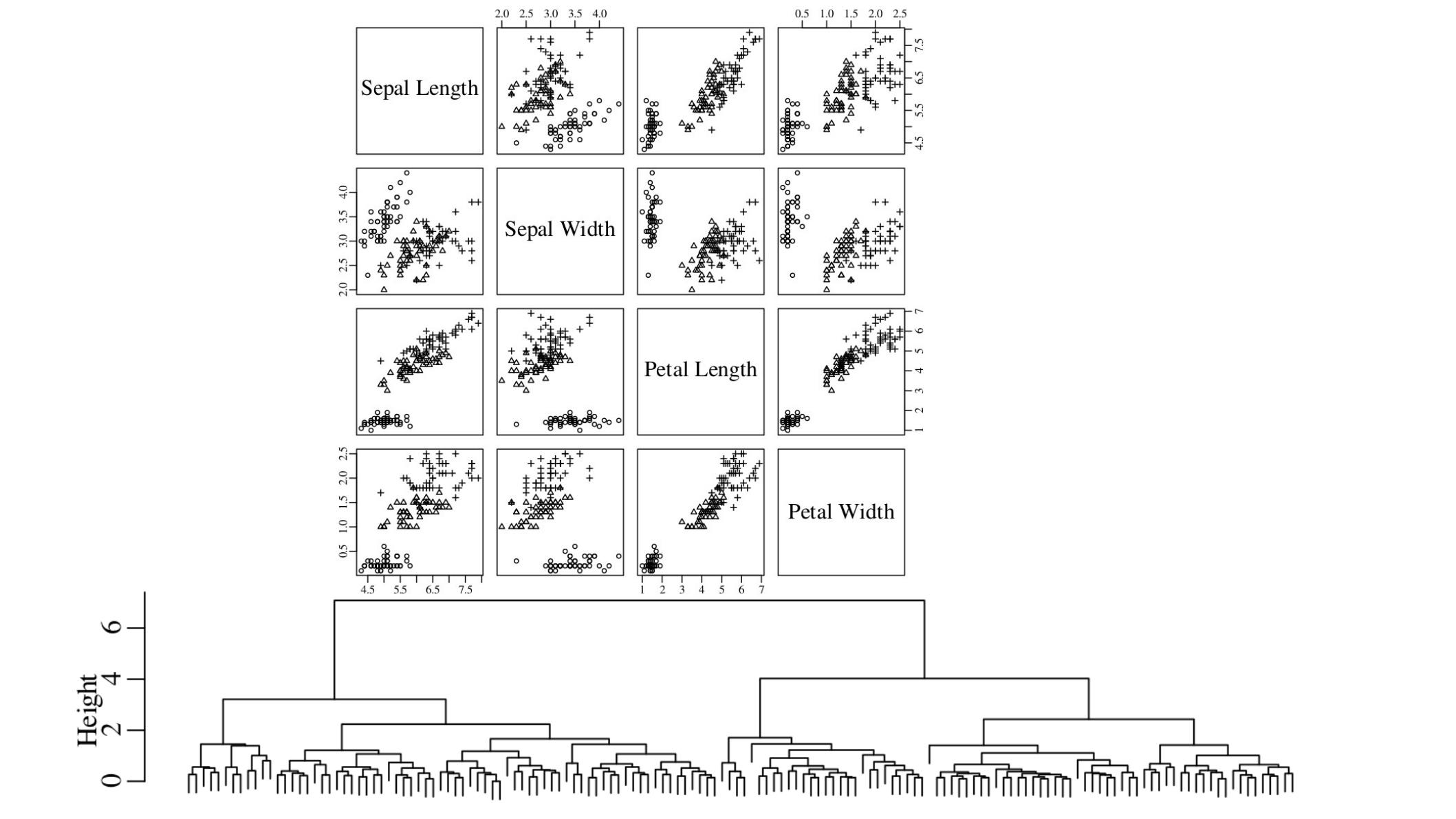


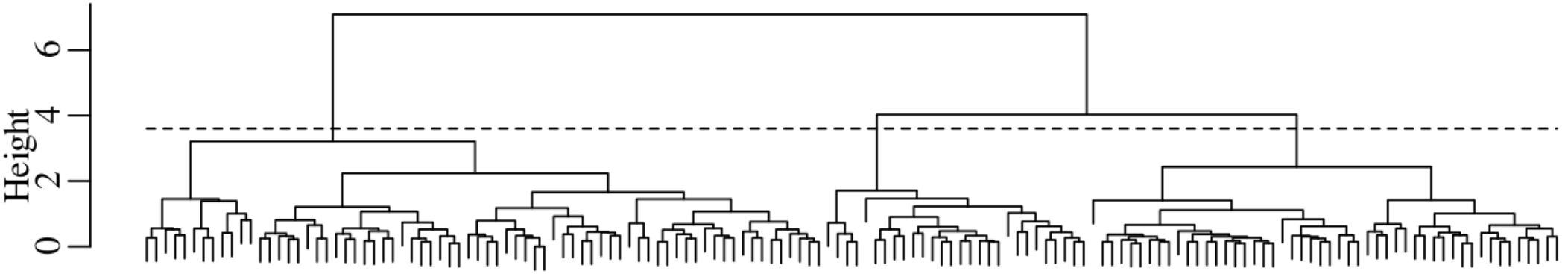
$$
d = \begin{array}{c} \\ 134 \\ 2 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccc} 134 & 2 & 5 \end{array} \\
\begin{bmatrix} 0 & 97.1 & 112 \\ 97.1 & 0 & \boxed{76.0} \\ 112 & \boxed{76.0} & 0 \end{bmatrix}
\end{array}
$$






Height
0
20
40
60
80
100
4
1
3
2
5


Sepal Length
Sepal Width
Petal Length
Petal Width
Height
0
2
4
6

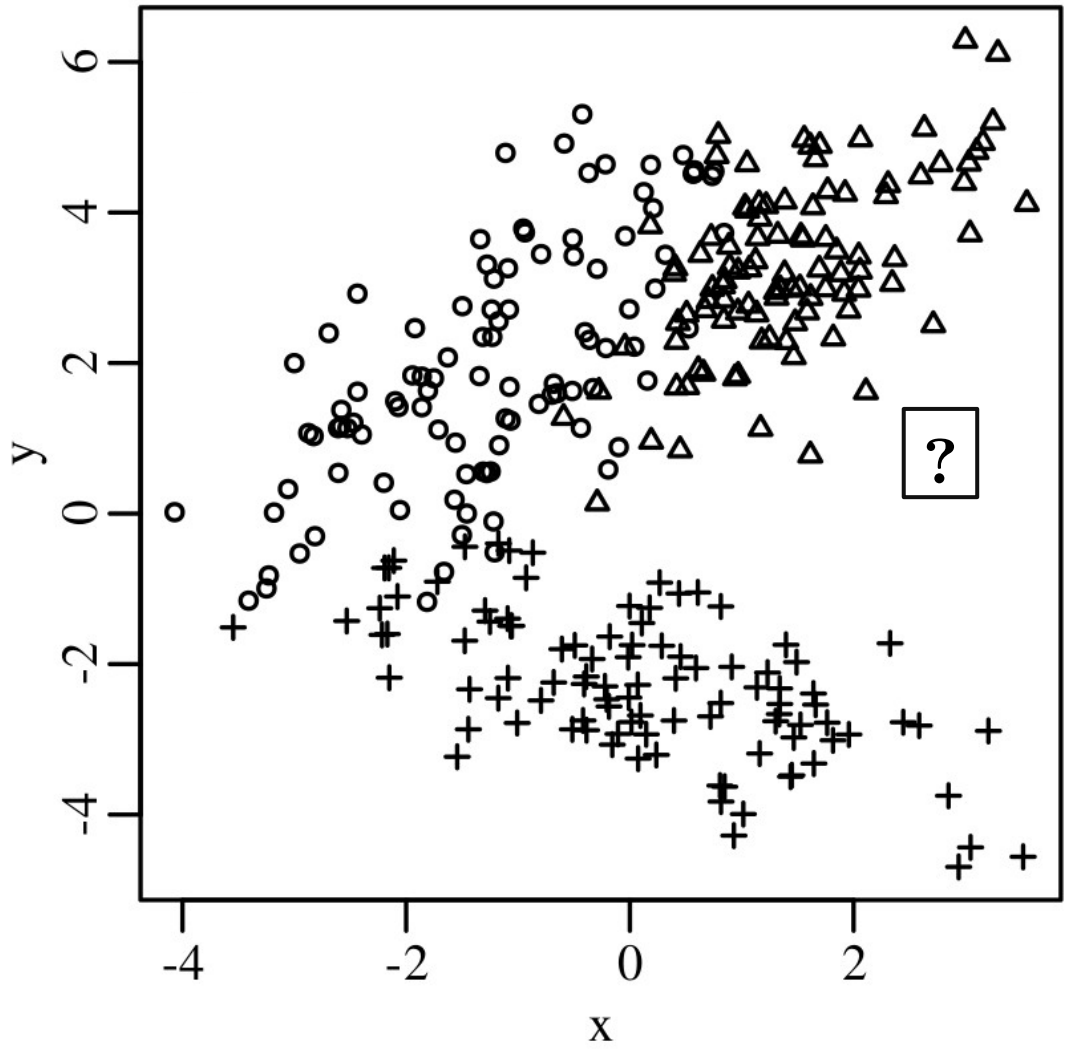


cluster	setosa	versicolor	virginica
1	50	0	0
2	0	23	49
3	0	27	1

Statistics for geoscientists

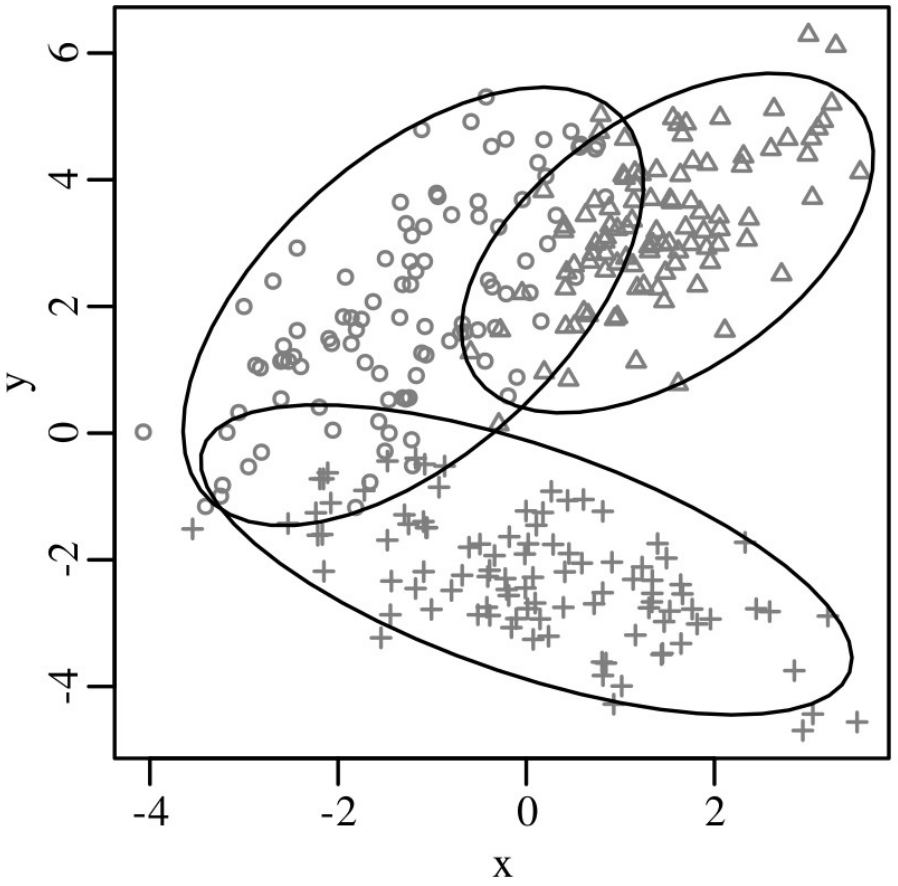
Supervised learning

Discriminant Analysis



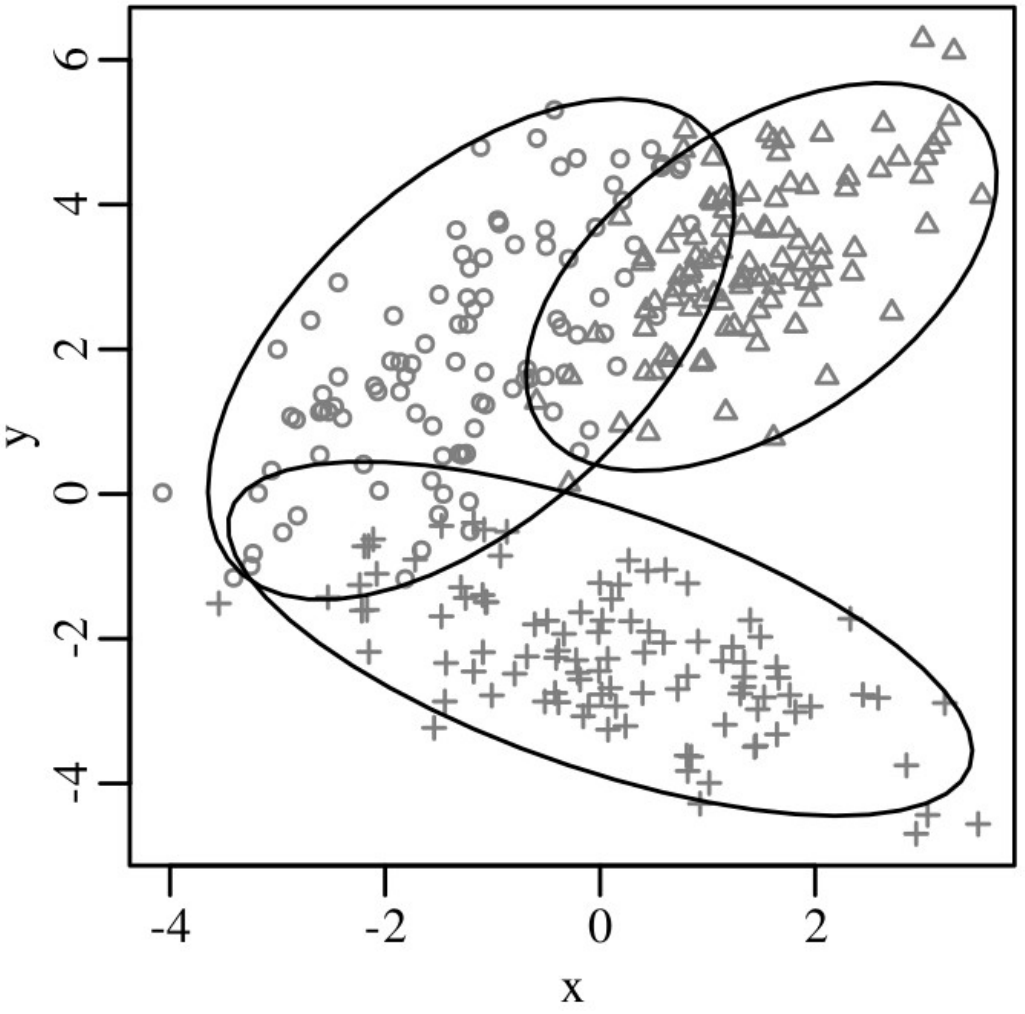
$$d = \max_{k=1,\ldots,K} P(G = k|X = x)$$

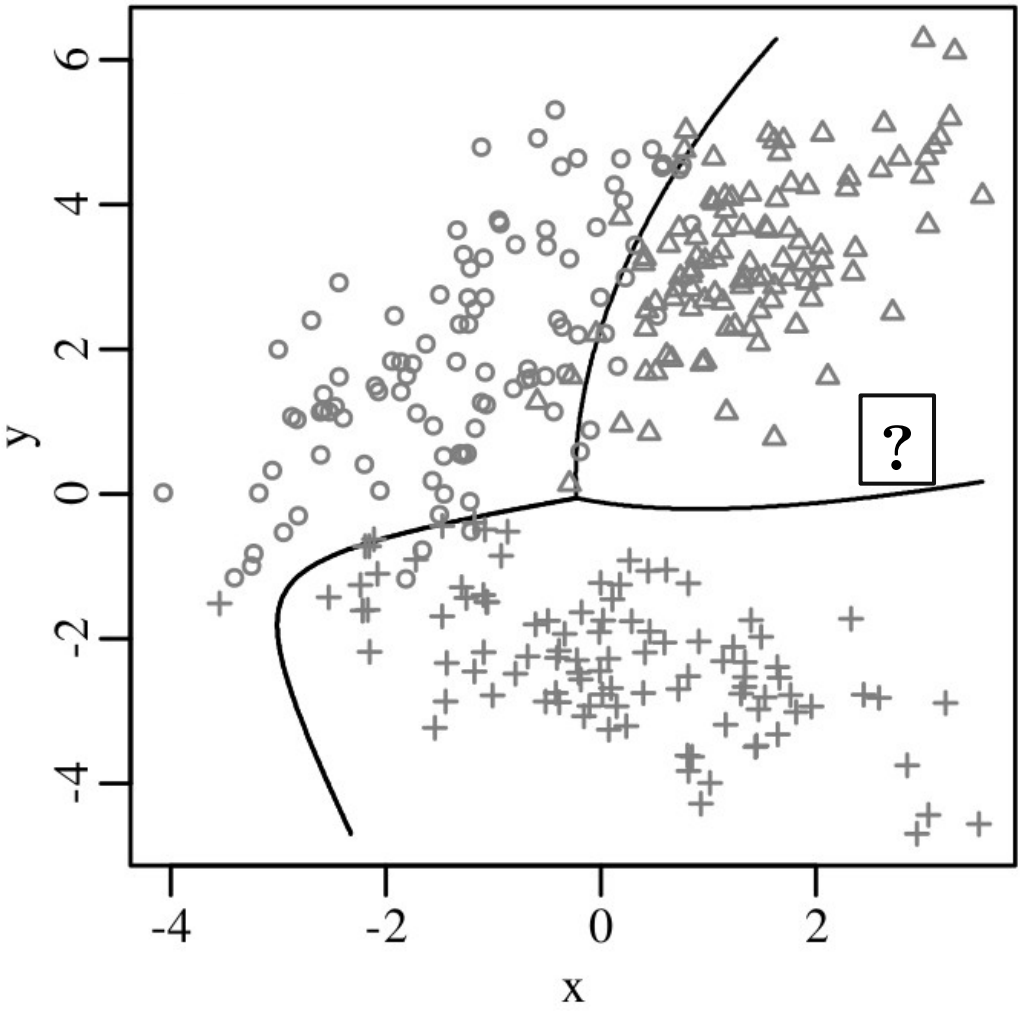
$$P(G|X) \propto P(X|G)P(G)$$



$$P(X = x|G = k) = \frac{\exp\left(-\frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k)\right)}{\sqrt{(2\pi)^N|\Sigma_k|}}$$

$$d = \max_{k=1,\ldots,K}\left[-\frac{1}{2}\ln|\Sigma_k| - \frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k)\right]$$


y
6
4
2
0
-2
-4
-4
-2
0
2
x

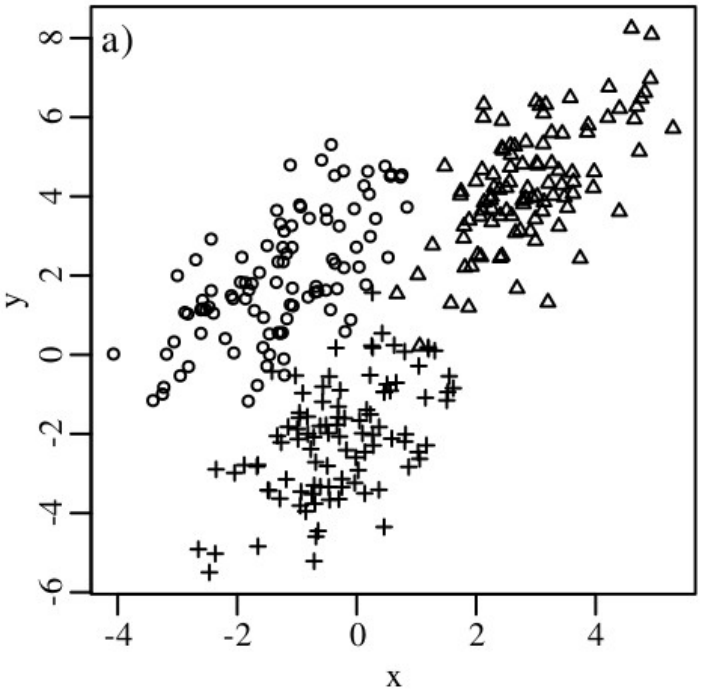

y
6
4
2
0
-2
-4
-4
-2
0
2
x
?

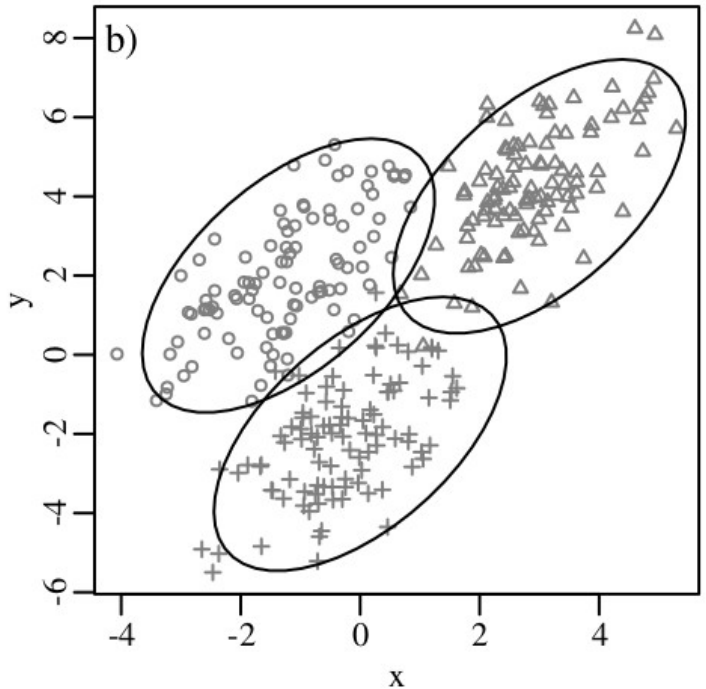
QDA

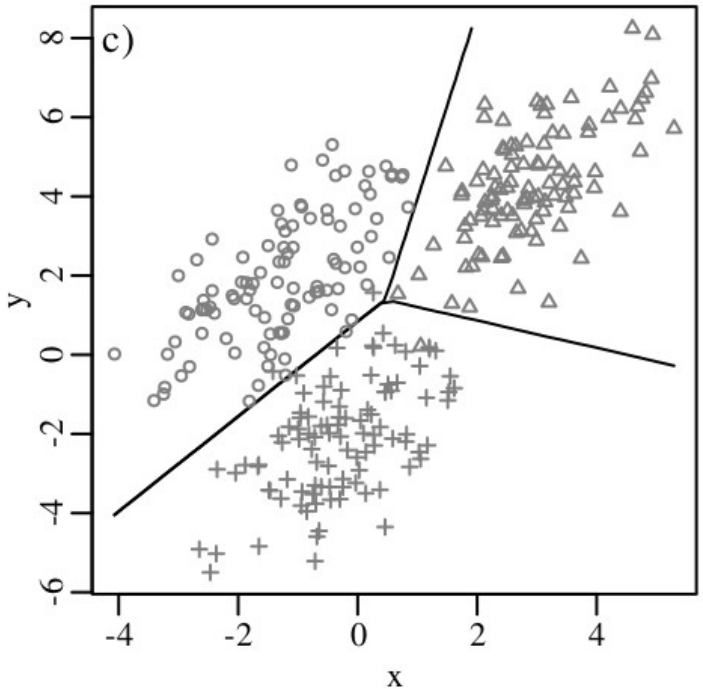
$$d = \max_{k=1,\ldots,K} \left[-\frac{1}{2} \ln |\Sigma_k| - \frac{1}{2}(x-\mu_k)^T \Sigma_k^{-1} (x-\mu_k) \right]$$

LDA

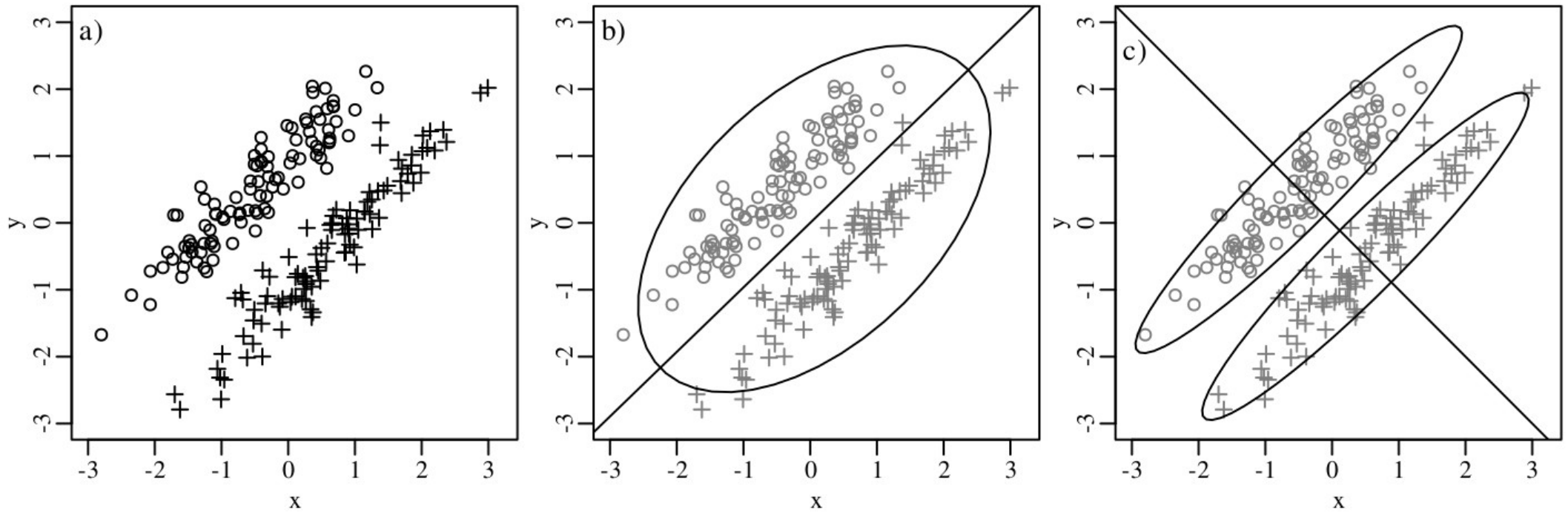
$$d = \max_{k=1,\ldots,K} \left[x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k \right]$$

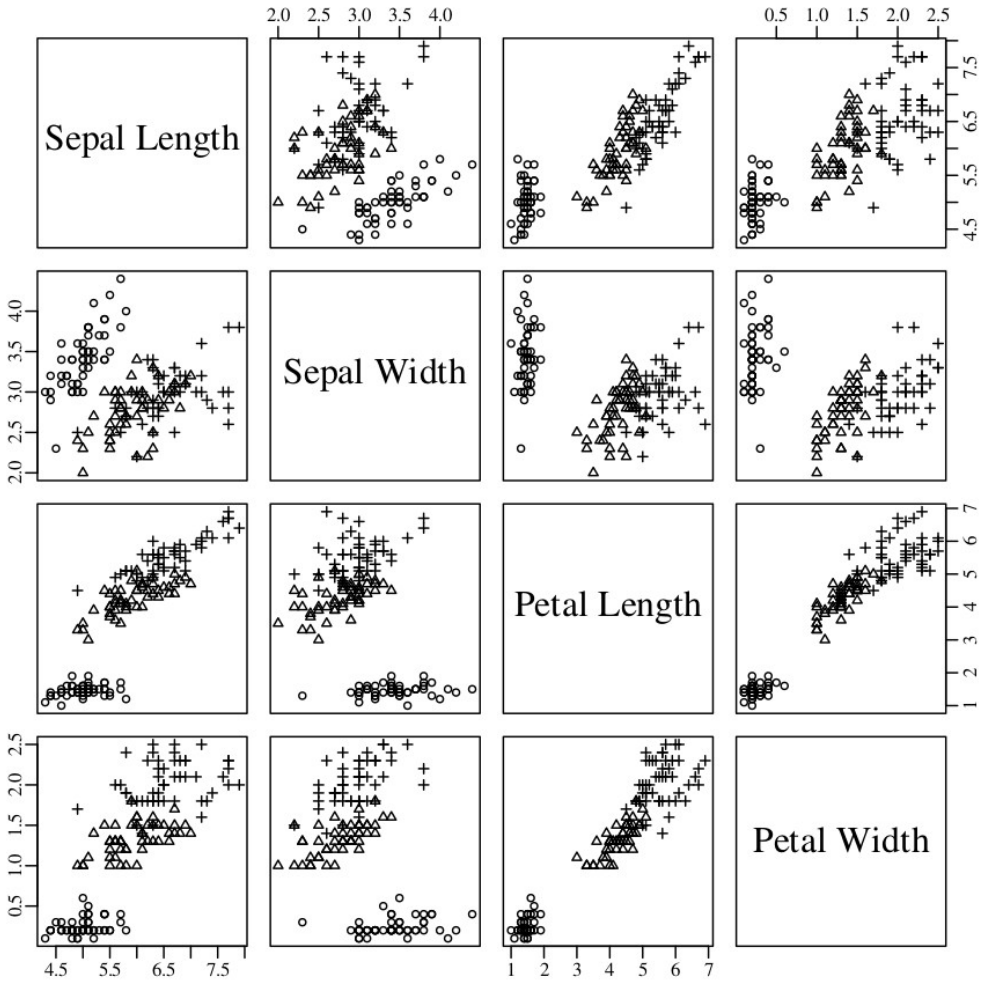


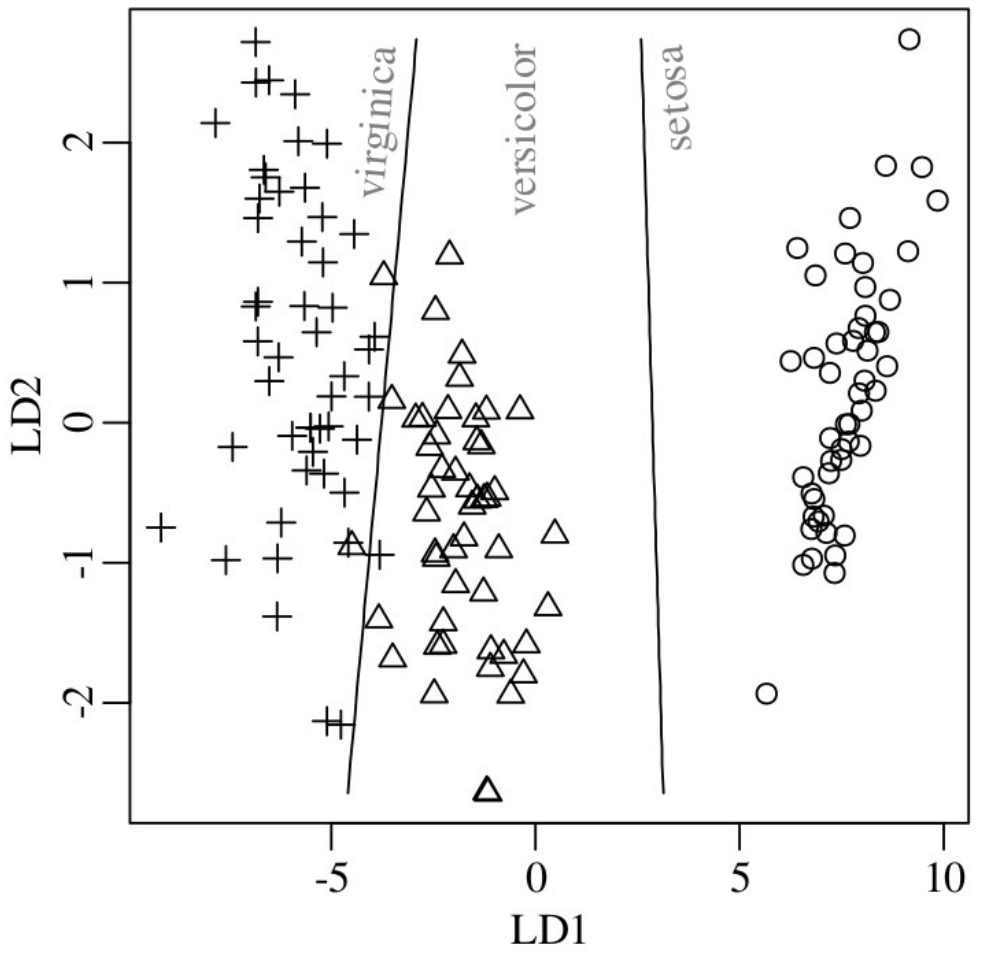




PCA vs. LDA



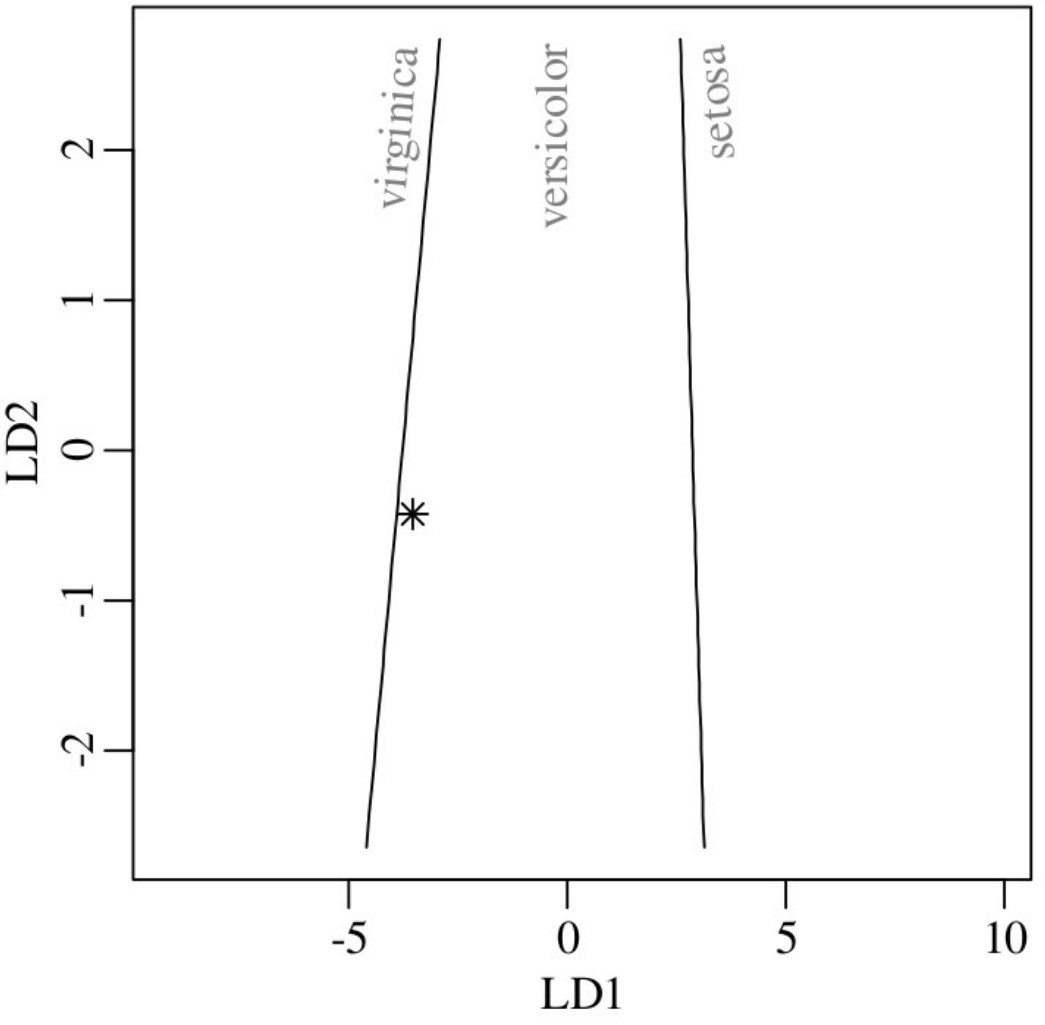

Sepal Length
Sepal Width
Petal Length
Petal Width


virginica
versicolor
setosa
LD1
LD2

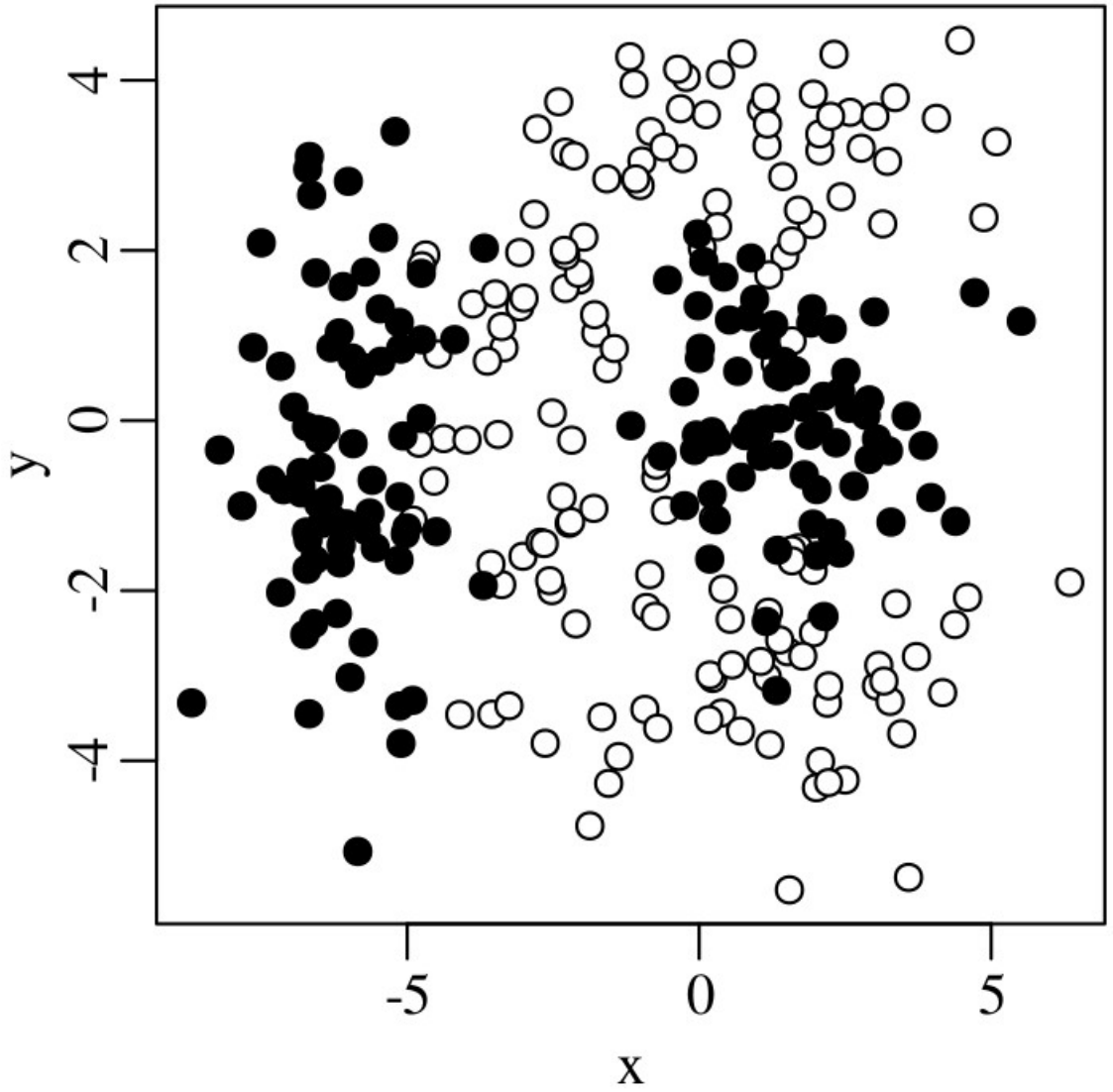
$$\begin{aligned}\text{LD1} = 0.83 \times &\text{ (sepal length - 5.84)}\\ + 1.53 \times &\text{ (sepal width - 3.06)}\\ - 2.20 \times &\text{ (petal length - 3.76)}\\ - 2.81 \times &\text{ (petal width - 1.20)}\end{aligned}$$

$$\begin{aligned}\text{LD2} = 0.024 \times &\text{ (sepal length - 5.85)}\\ + 2.16 \times &\text{ (sepal width - 3.06)}\\ - 0.93 \times &\text{ (petal length - 3.76)}\\ + 2.84 \times &\text{ (petal width - 1.20)}\end{aligned}$$

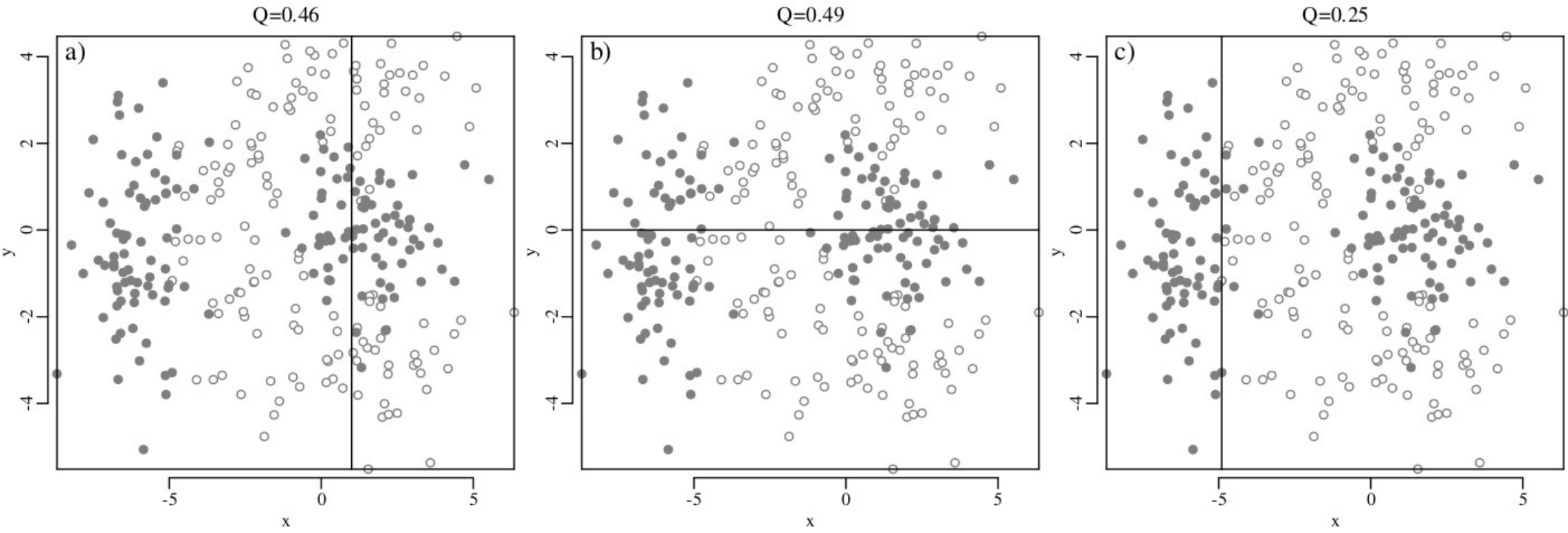
	virginica	*versicolor*	*setosa*
$P(G\|X)$	0.19	0.81	3.2×10^{-27}

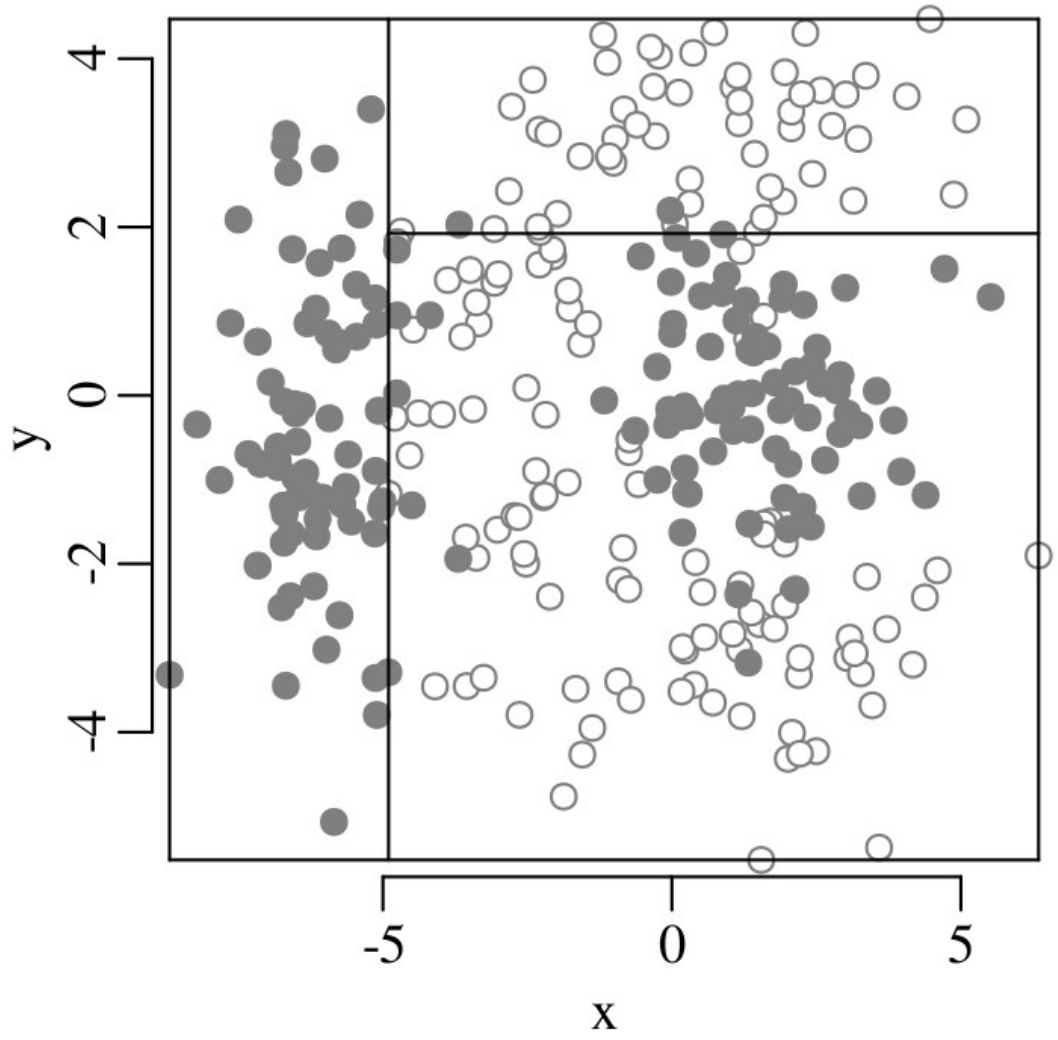


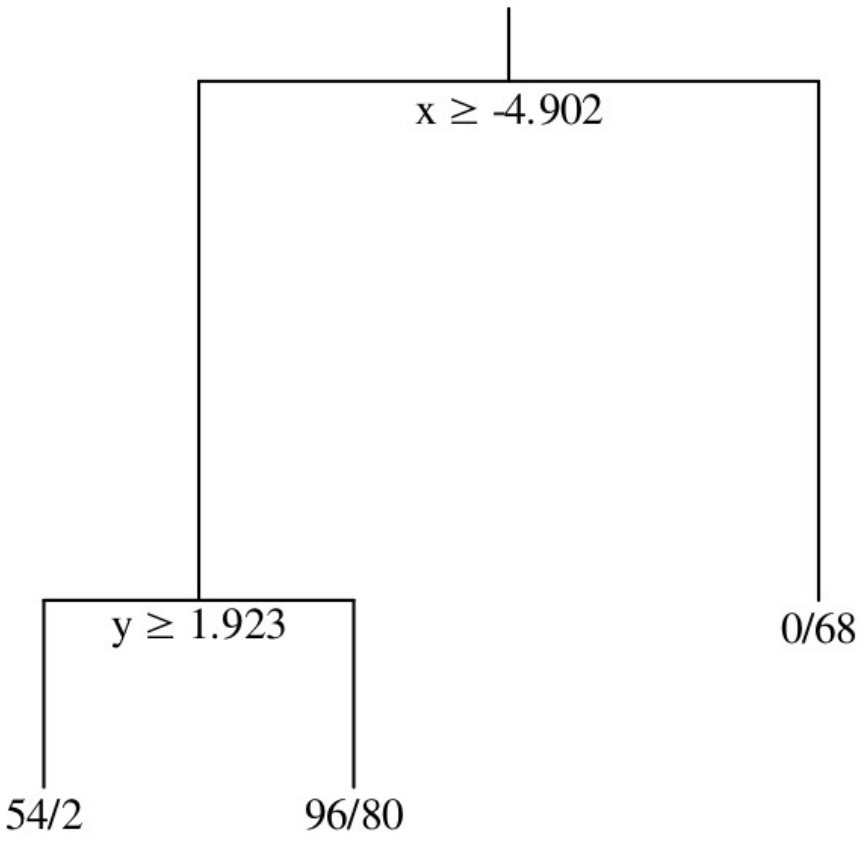
Decision trees

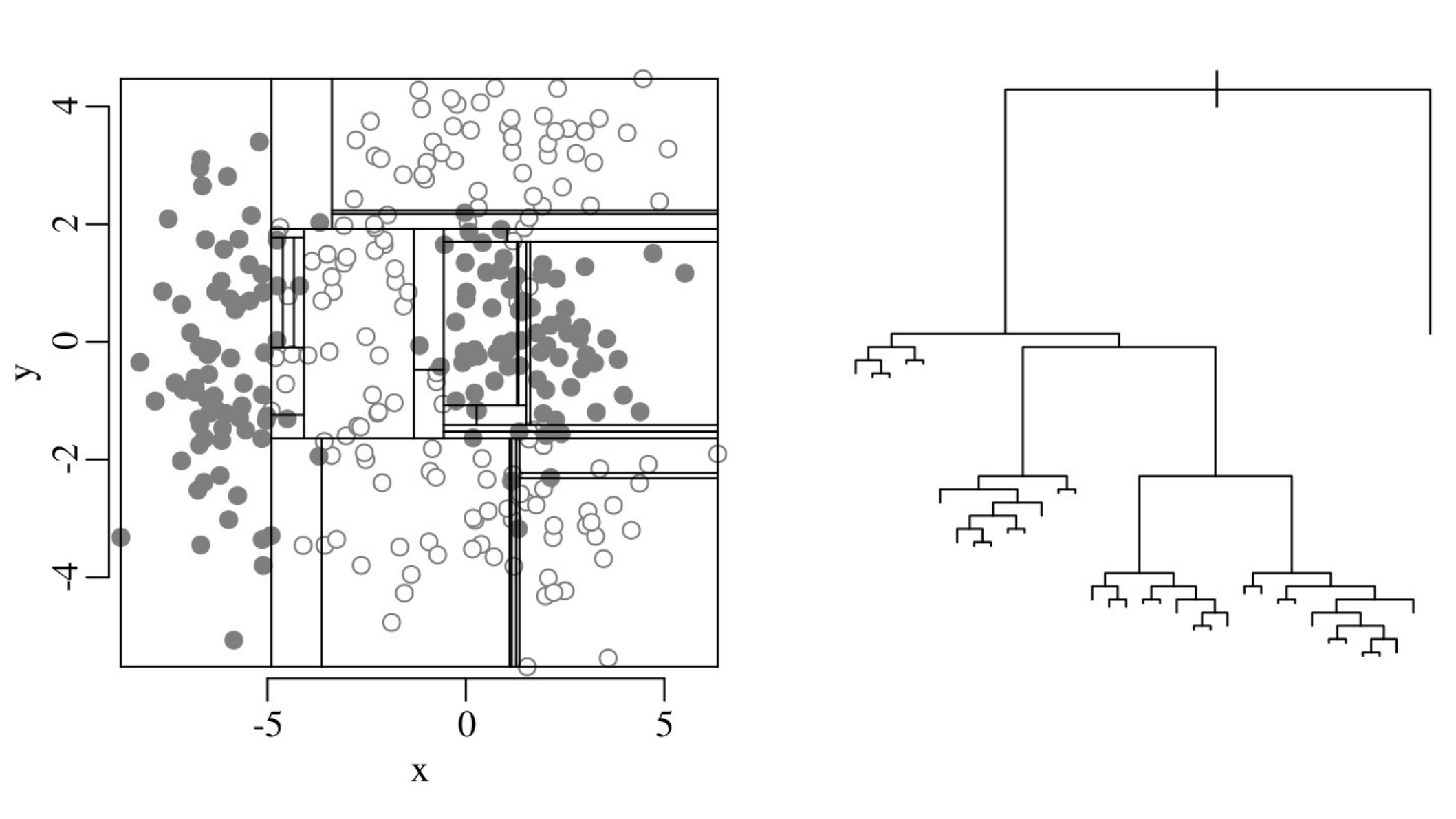


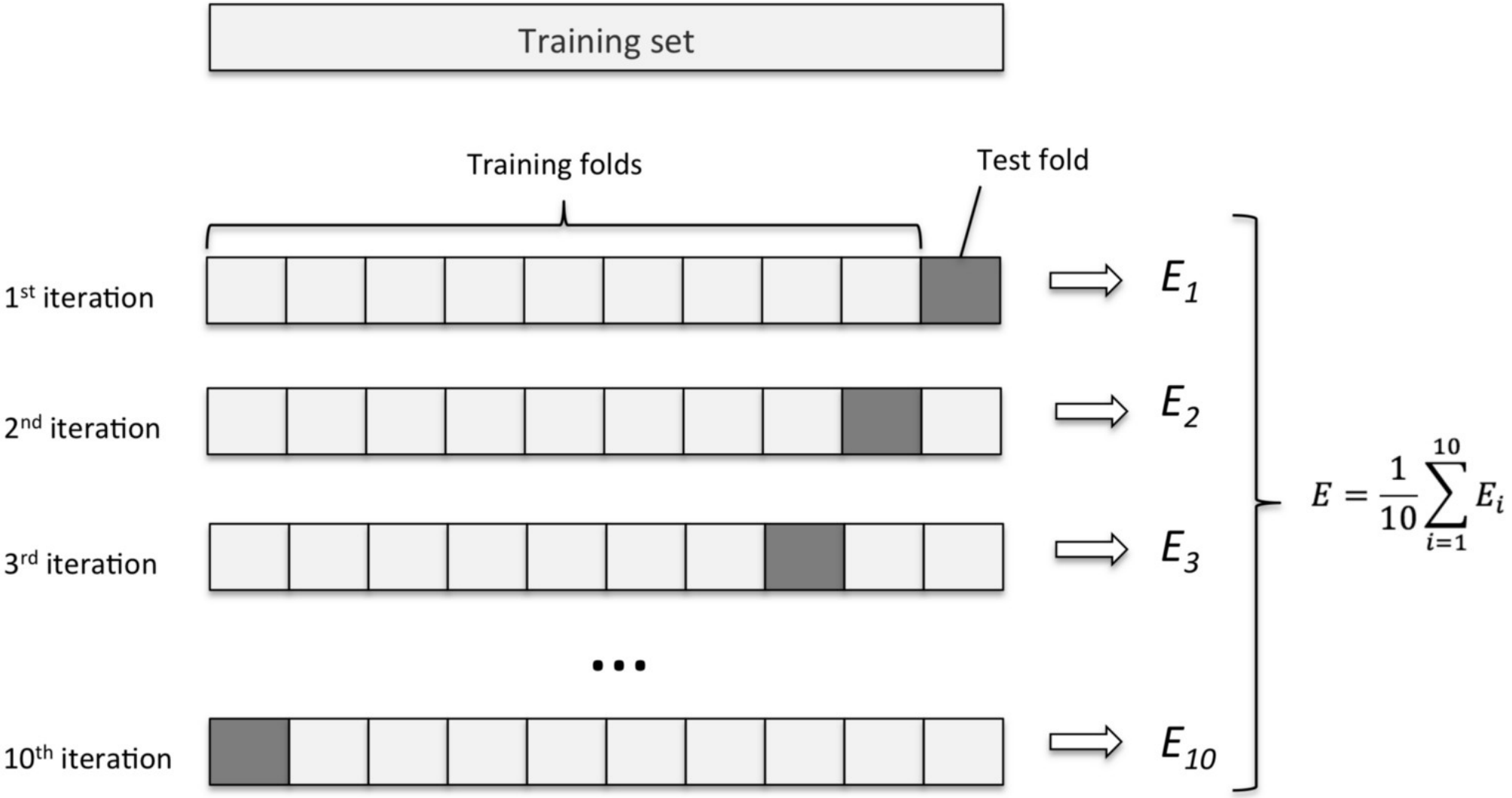
$$Q = p_{\circ}(1 - p_{\circ}) + p_{\bullet}(1 - p_{\bullet})$$

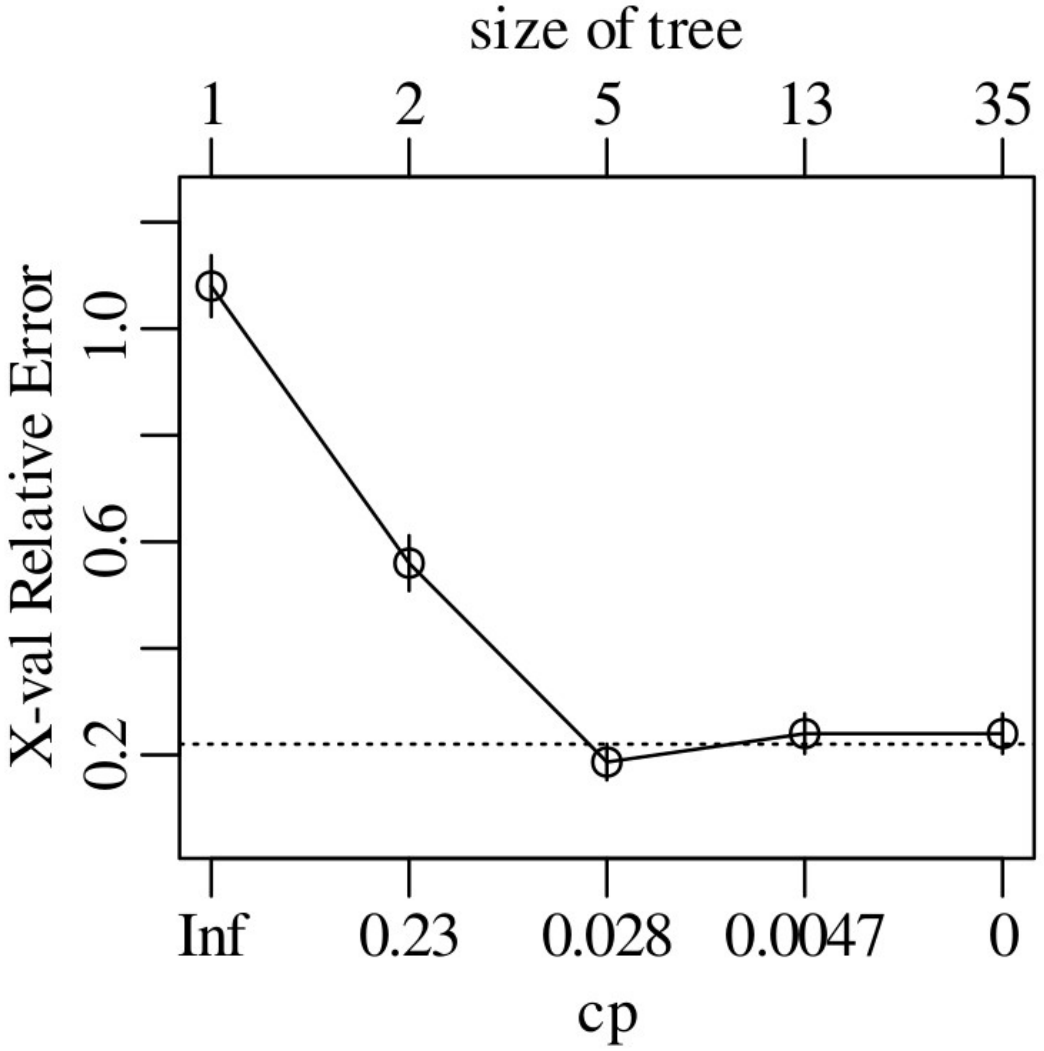



y
4
2
0
-2
-4
-5
0
5
x


x ≥ -4.902
y ≥ 1.923
0/68
54/2
96/80

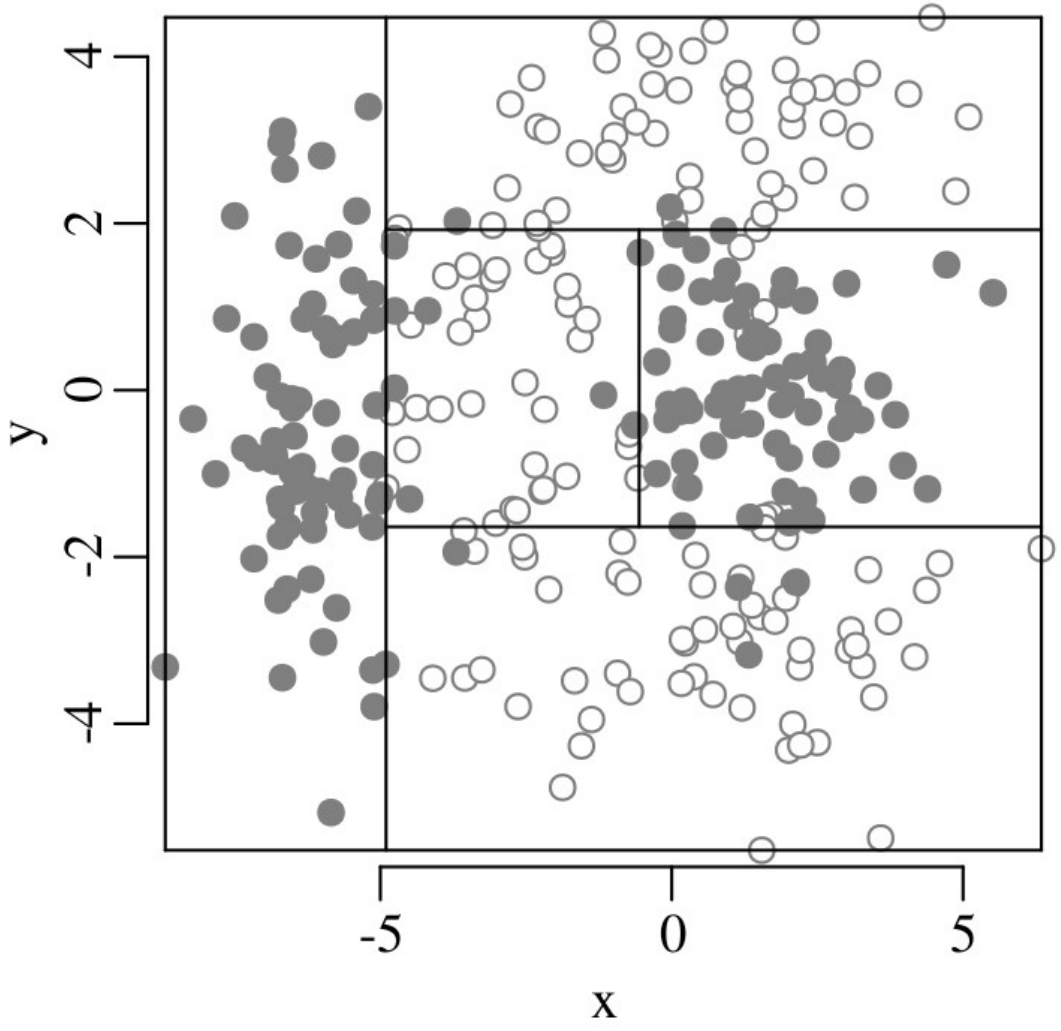

4
2
0
-2
-4
y
-5
0
5
x

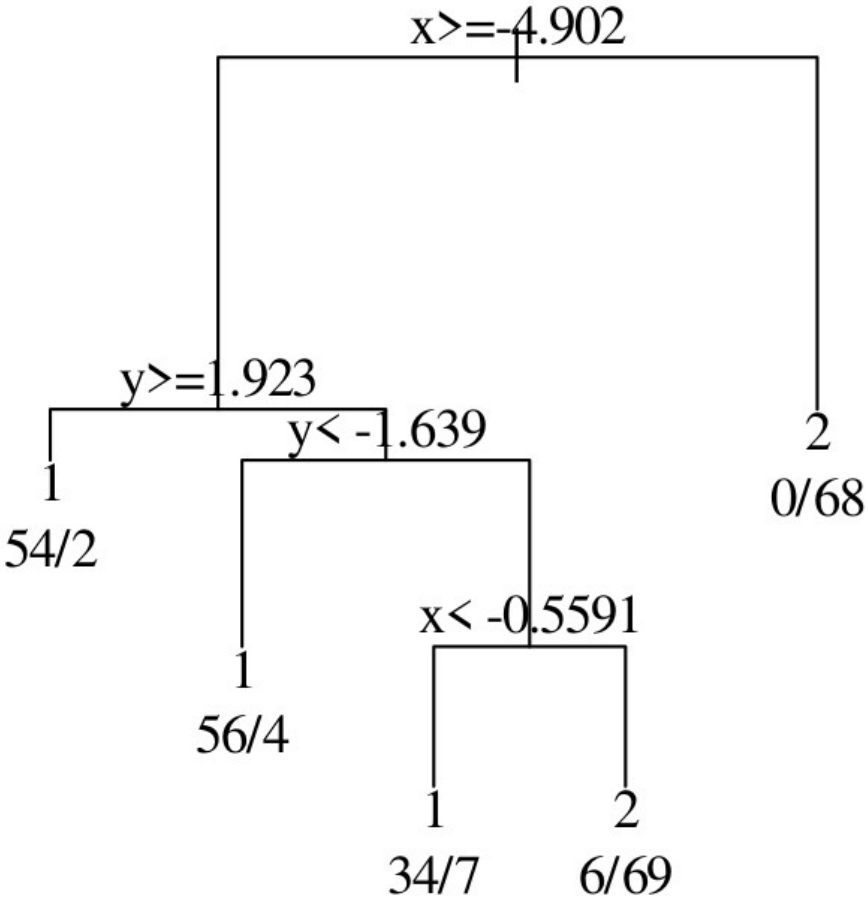

Training set
Training folds
Test fold
1st iteration
2nd iteration
3rd iteration
10th iteration
E1
E2
E3
E10
$E = \frac{1}{10}\sum_{i=1}^{10} E_i$

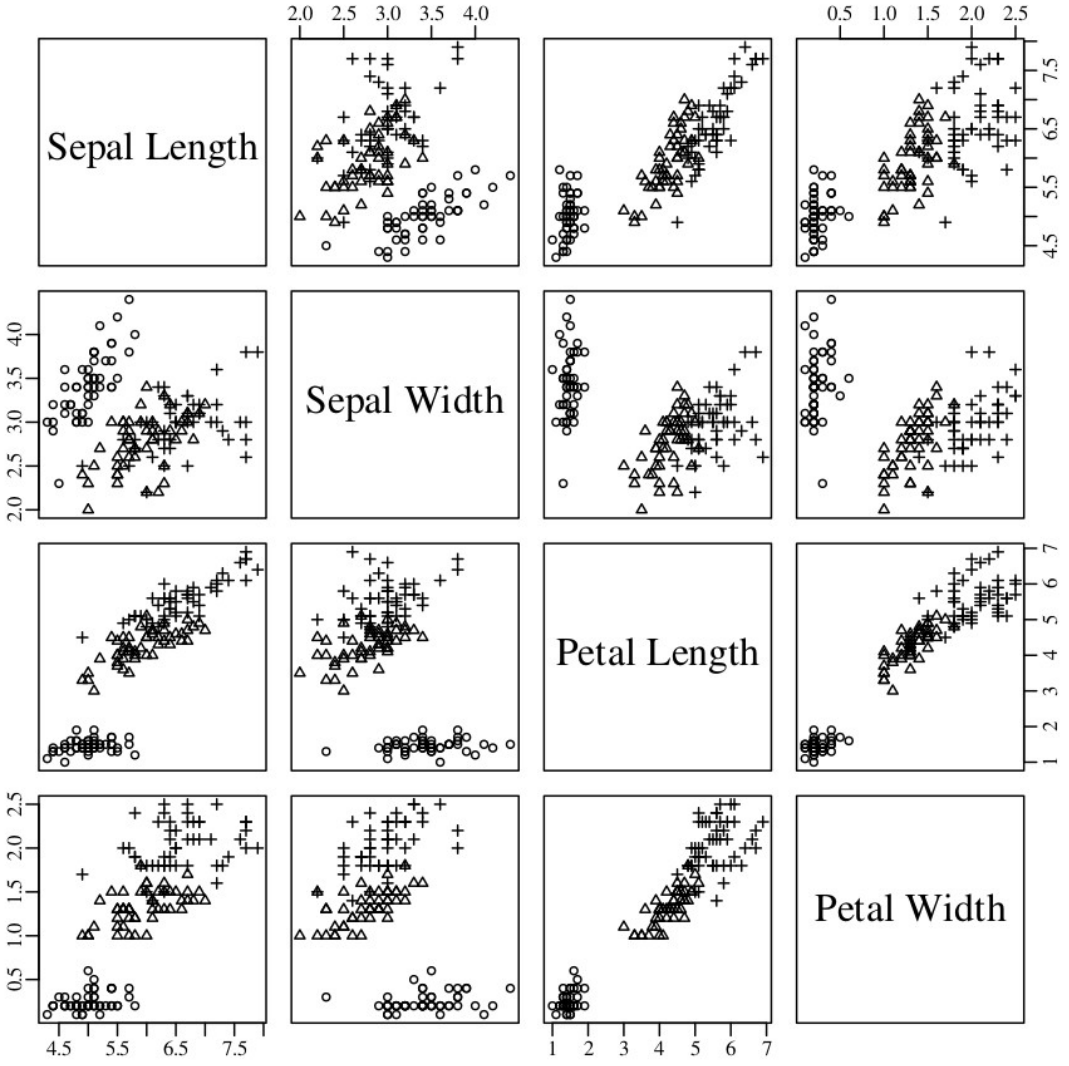


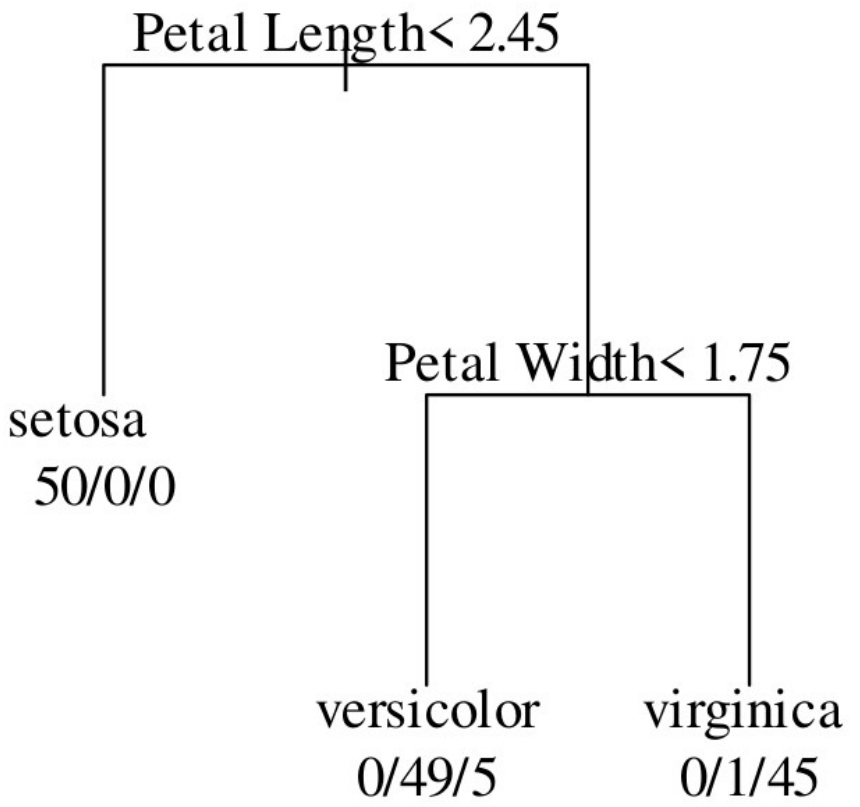
$$cp_\alpha(T) = \sum_{m=1}^{|T|} N_m Q_m + \alpha|T|$$

$$T_\alpha = \min_{T \subset T_\circ} cp_\alpha(T)$$


y
4
2
0
-2
-4
-5
0
5
x


x>=-4.902
y>=1.923
y< -1.639
2
0/68
1
54/2
1
56/4
x< -0.5591
1
34/7
2
6/69


Sepal Length
Sepal Width
Petal Length
Petal Width


Petal Length< 2.45
Petal Width< 1.75
setosa
50/0/0
versicolor
0/49/5
virginica
0/1/45

Statistics for geoscientists

Compositional data

Ratio data

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
A/B	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079
B/A	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10
A/B	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079
B/A	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0
$1/(A/B)$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

	1	2	3	4	5	6	7	8	9	10	mean
A/B	1.30	2.10	0.83	2.40	0.26	1.80	1.30	0.67	1.70	0.079	1.20
B/A	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0	2.30
$1/(A/B)$	0.78	0.47	1.20	0.42	3.80	0.55	0.76	1.50	0.60	13.0	

$$\frac{1}{\overline{A/B}} = \frac{1}{1.20} = 0.81 \neq 2.30 = \overline{B/A}$$

$$\text{and } \frac{1}{\overline{B/A}} = \frac{1}{2.30} = 0.44 \neq 1.20 = \overline{A/B}$$

	1	2	3	4	5	6	7	8	9	10
A	0.27	0.37	0.57	0.91	0.20	0.90	0.94	0.66	0.63	0.062
B	0.21	0.18	0.69	0.38	0.77	0.50	0.72	0.99	0.38	0.780

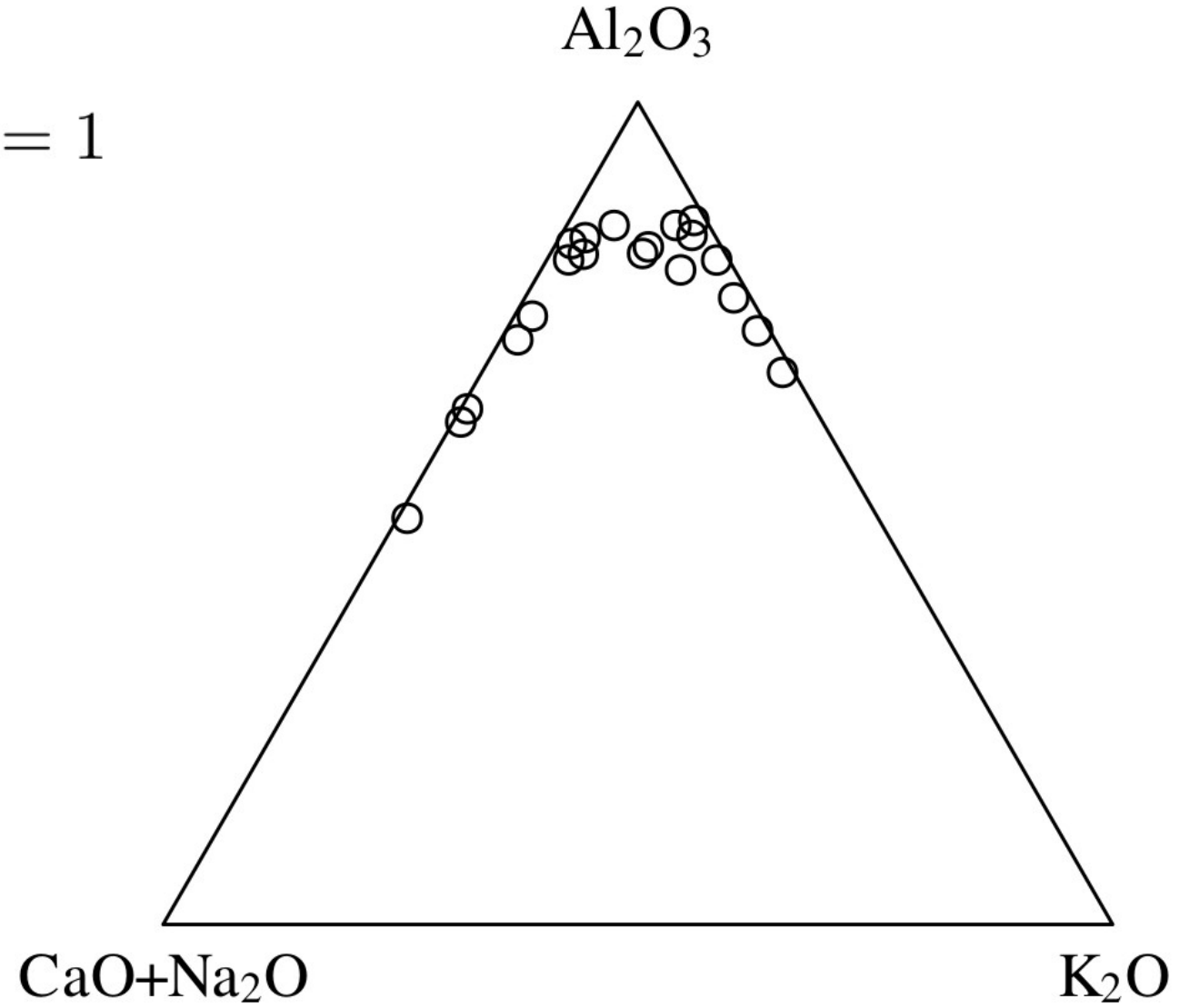
	1	2	3	4	5	6	7	8	9	10
$\ln[A/B]$	0.25	0.75	-0.18	0.86	-1.30	0.59	0.27	-0.41	0.50	-2.50
$\ln[B/A]$	-0.25	-0.75	0.18	-0.86	1.30	-0.59	-0.27	0.41	-0.50	2.50

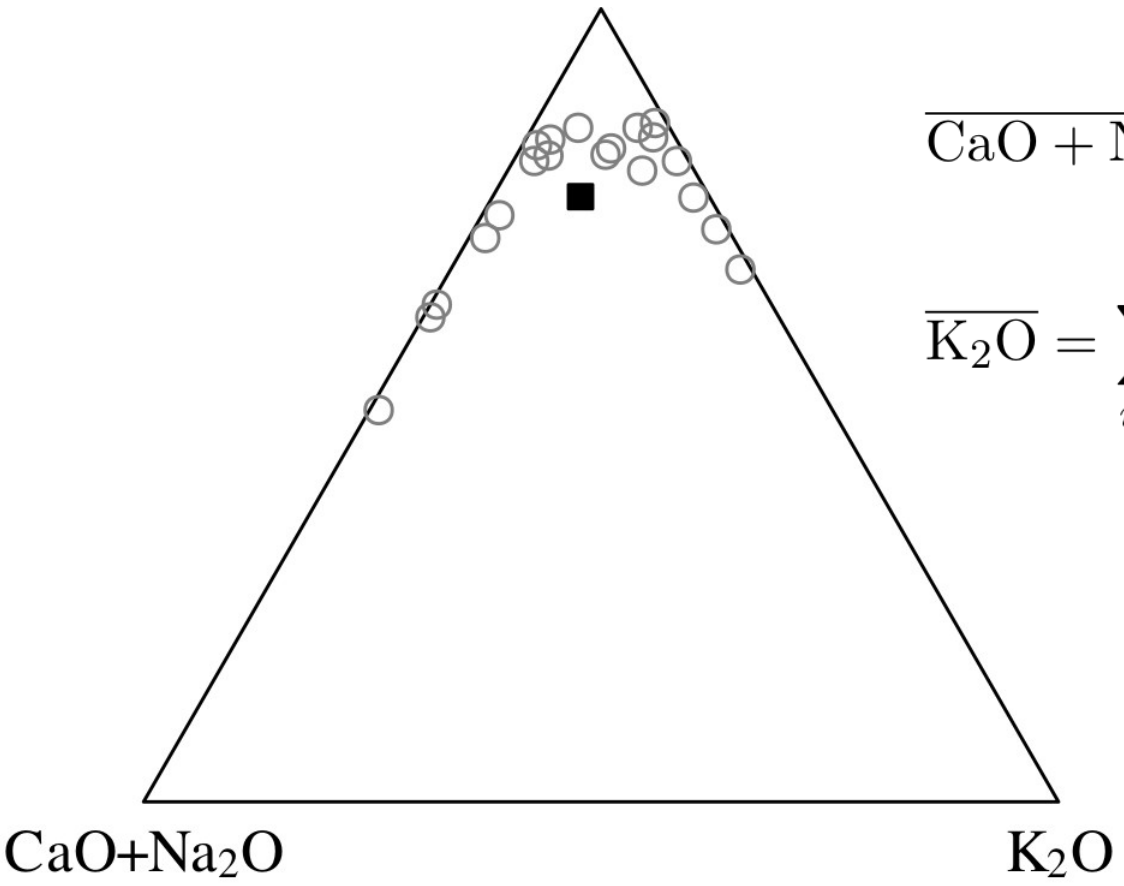
	mean	exp[mean]
$\ln[A/B]$	-0.12	0.88
$\ln[B/A]$	0.12	1.13

$$\frac{1}{g(A/B)} = \frac{1}{0.88} = 1.13 = g(B/A)$$

$$\text{and } \frac{1}{g(B/A)} = \frac{1}{1.13} = 0.88 = g(A/B)$$

$$\sum_{i=1}^{n} C_i = 1$$

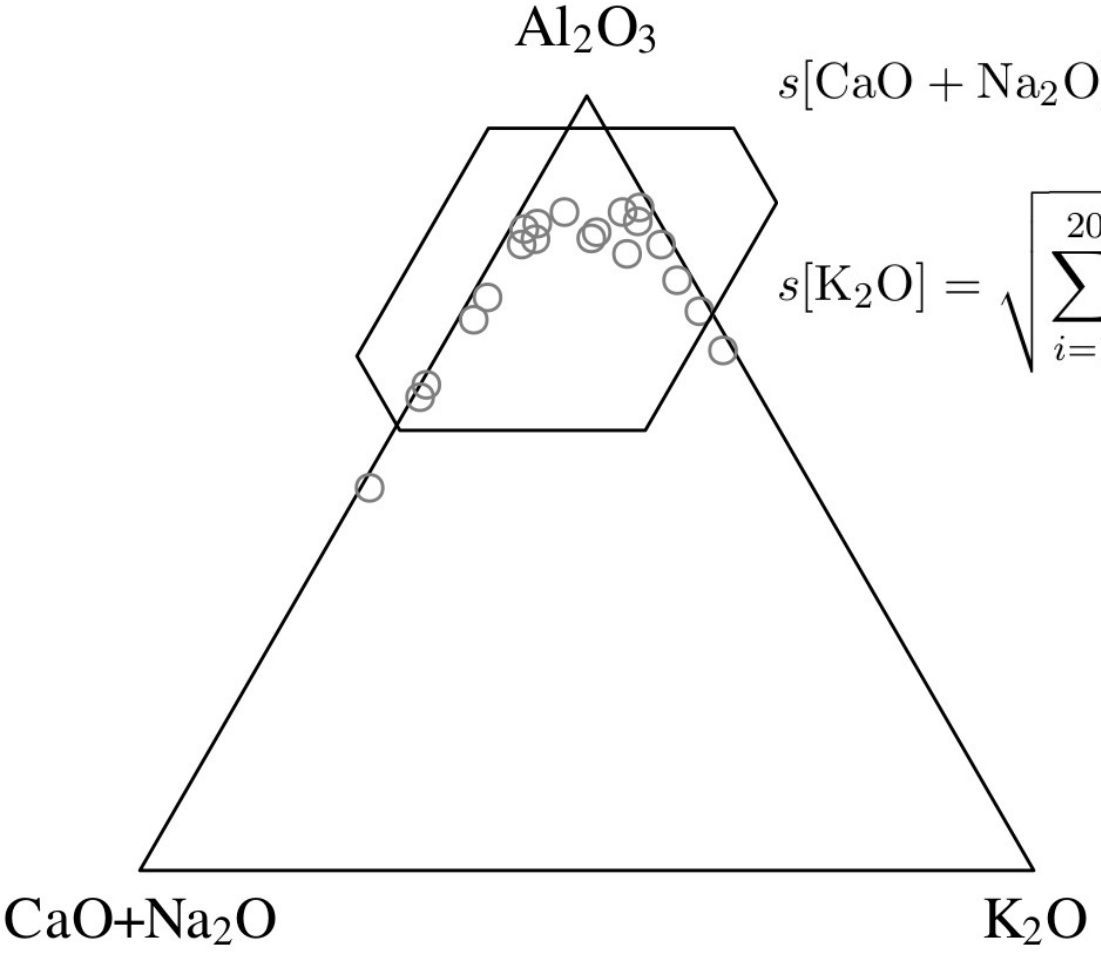




$$\overline{Al_2O_3} = \sum_{i=1}^{20} (Al_2O_3)_i/20 = 0.763$$

$$\overline{CaO + Na_2O} = \sum_{i=1}^{20} (CaO_2 + Na_2O)_i/20 = 0.141$$

$$\overline{K_2O} = \sum_{i=1}^{20} (K_2O)_i/20 = 0.096$$



$$s[\mathrm{Al_2O_3}] = \sqrt{\sum_{i=1}^{20} \frac{((\mathrm{Al_2O_3})_i - 0.763)^2}{19}} = 0.0975$$

$$s[\mathrm{CaO} + \mathrm{Na_2O}] = \sqrt{\sum_{i=1}^{20} \frac{((\mathrm{CaO_2} + \mathrm{Na_2O})_i - 0.141)^2}{19}} = 0.142$$

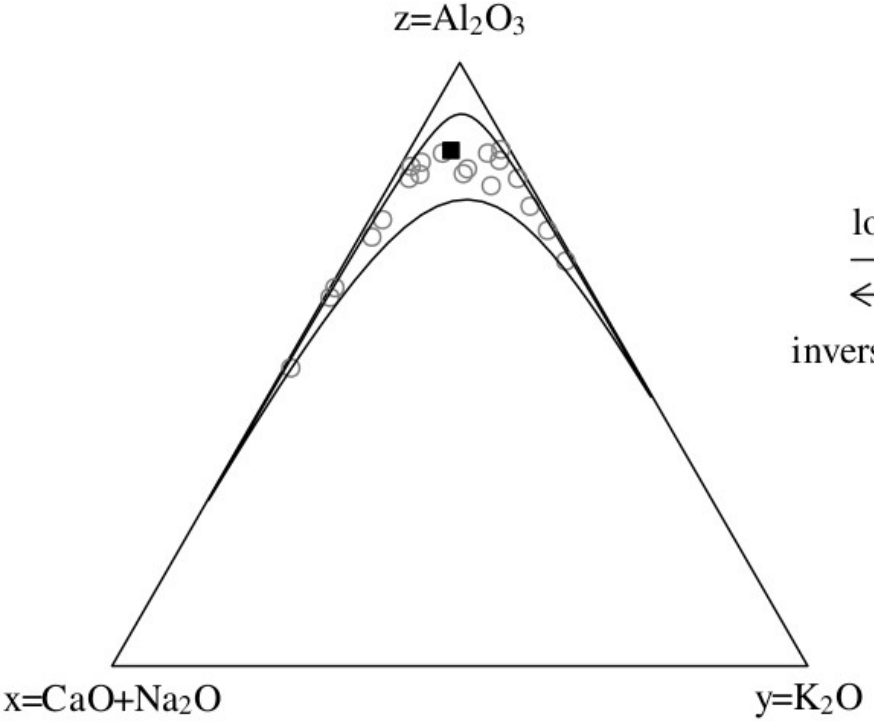
$$s[\mathrm{K_2O}] = \sqrt{\sum_{i=1}^{20} \frac{((\mathrm{K_2O})_i - 0.096)^2}{19}} = 0.0926$$

$$\mathrm{Al_2O_3} : 0.763 \pm 0.195$$

$$\mathrm{CaO} + \mathrm{Na_2O} : 0.141 \pm 0.284$$

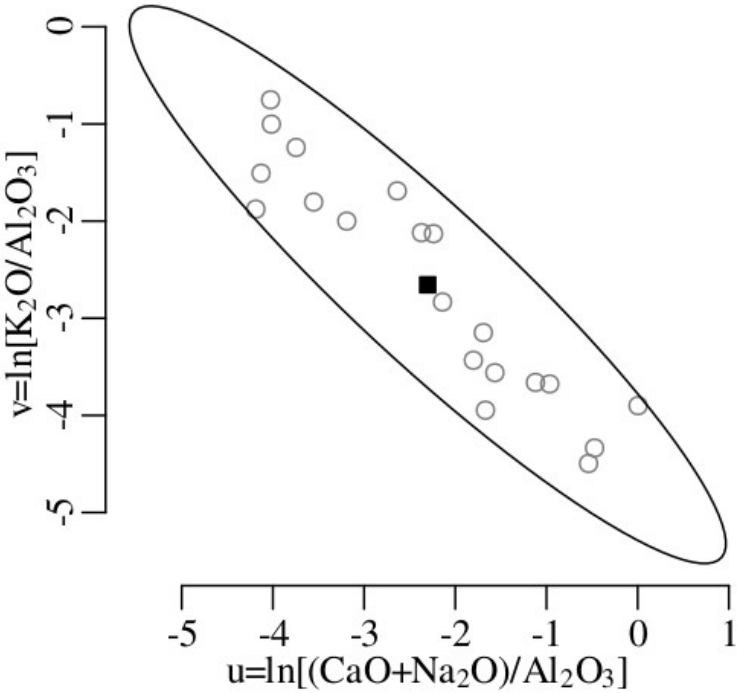
$$\mathrm{K_2O} : 0.096 \pm 0.185$$

$$v = \ln\left[\frac{x}{z}\right],\ w = \ln\left[\frac{y}{z}\right]$$



logratio transformation

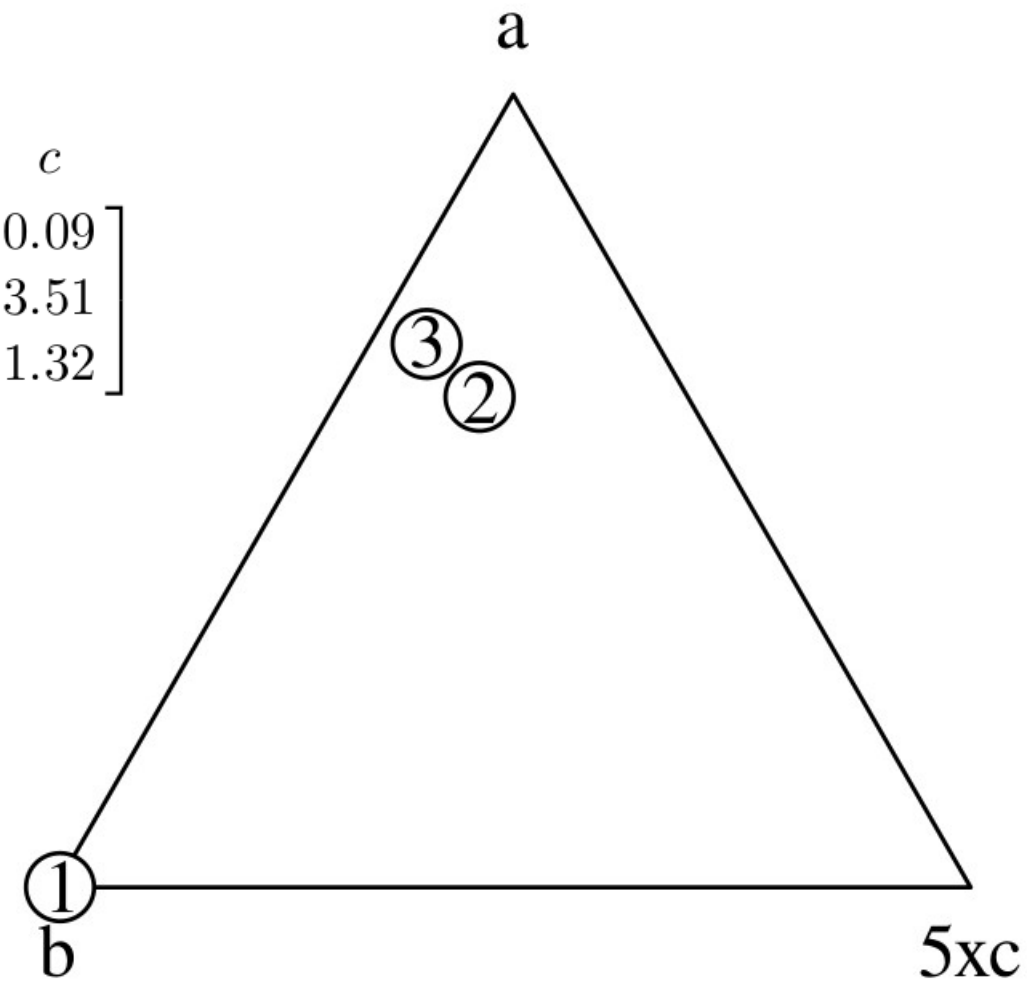
inverse logratio transformation



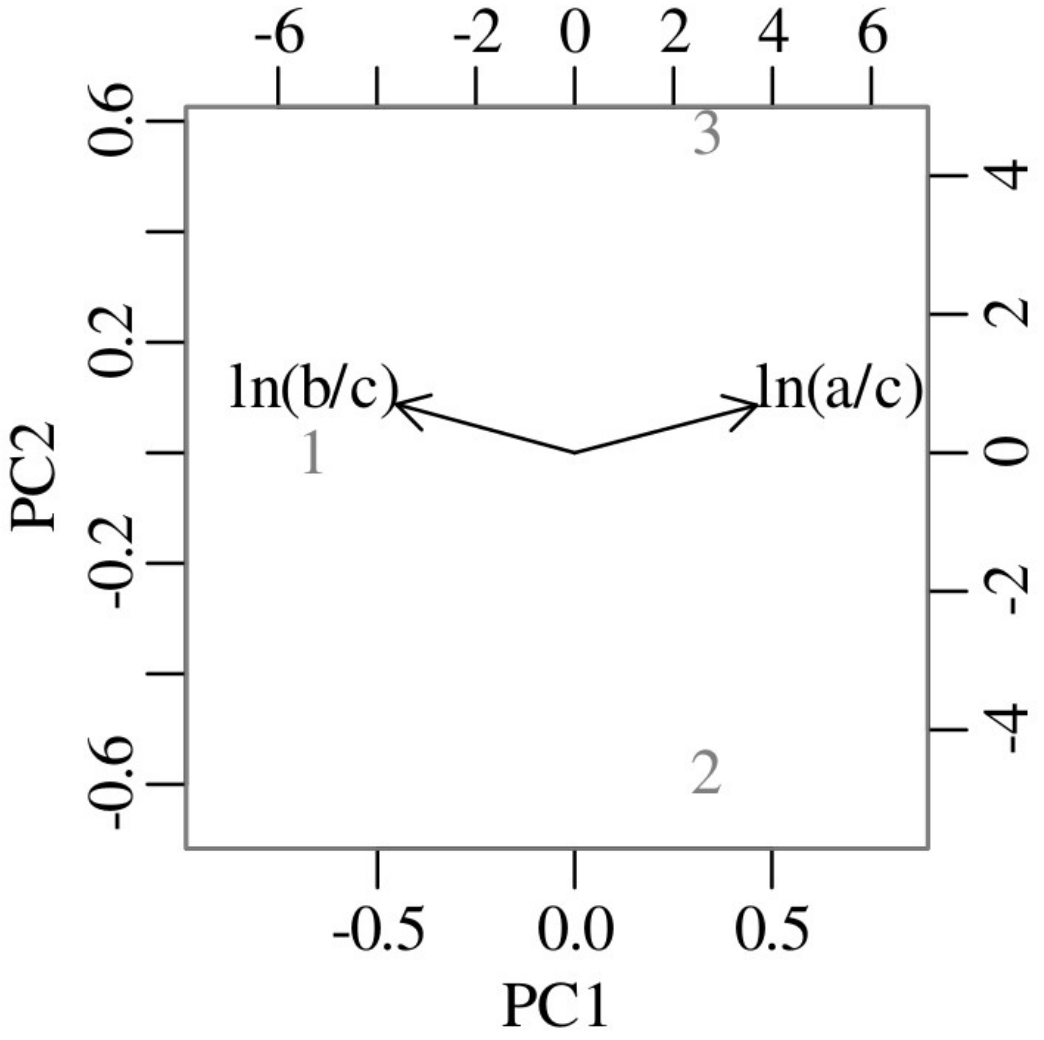
$$x = \frac{\exp[v]}{\exp[v] + \exp[w] + 1},\ y = \frac{\exp[w]}{\exp[v] + \exp[w] + 1},\ z = \frac{1}{\exp[v] + \exp[w] + 1}$$

PCA of compositional data

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0.03 & 99.88 & 0.09 \\ 70.54 & 25.95 & 3.51 \\ 72.14 & 26.54 & 1.32 \end{bmatrix} \end{array}$$



$$X_{\mathrm{a}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} \ln(a/c) & \ln(b/c) \end{array} \\ \begin{bmatrix} -1 & 7 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \end{array}$$



centred logratio transformation

$$u_i = \ln\left[\frac{x_i}{g_i}\right], \; v_i = \ln\left[\frac{y_i}{g_i}\right], \text{ and } w_i = \ln\left[\frac{z_i}{g_i}\right]$$

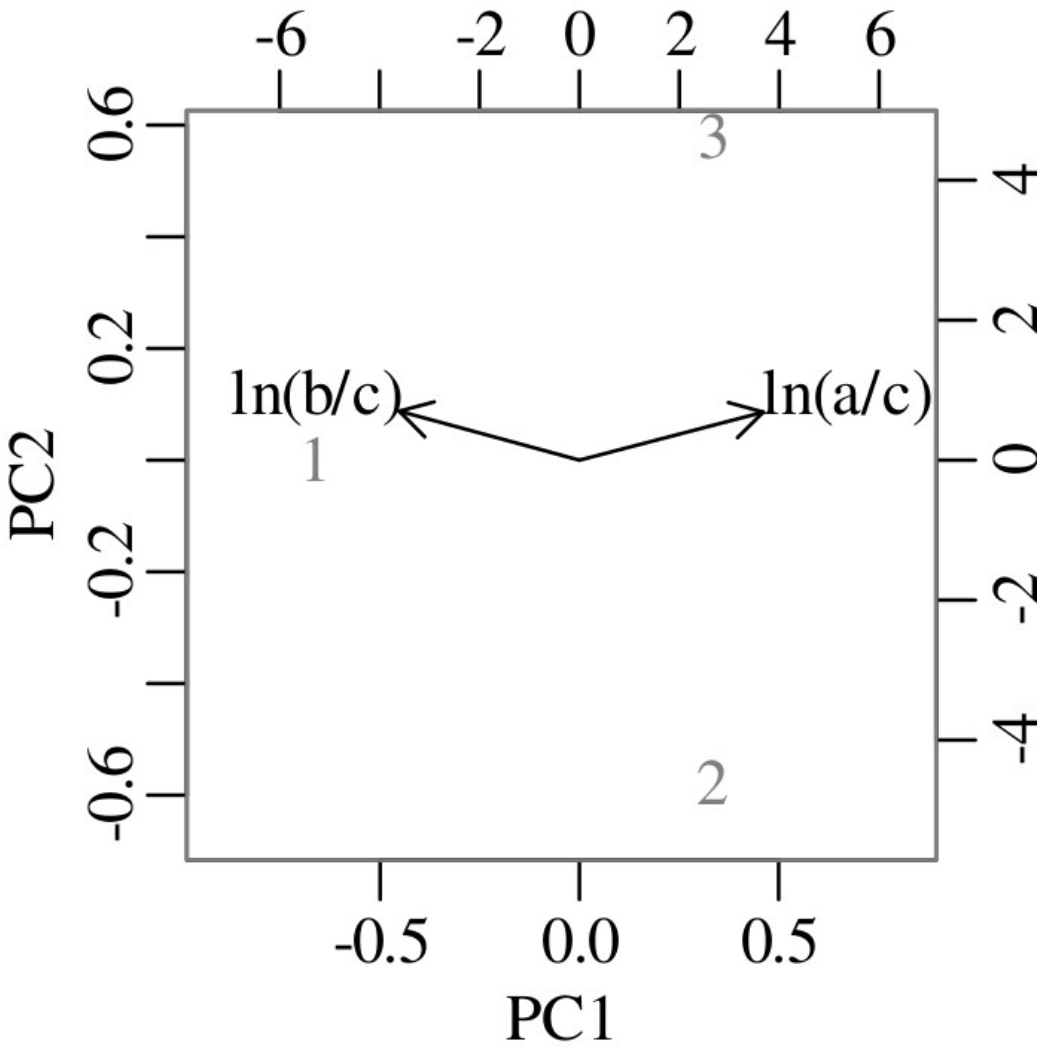
$$\text{where} \quad g_i = \exp\left[\frac{\ln[x_i] + \ln[y_i] + \ln[z_i]}{3}\right]$$

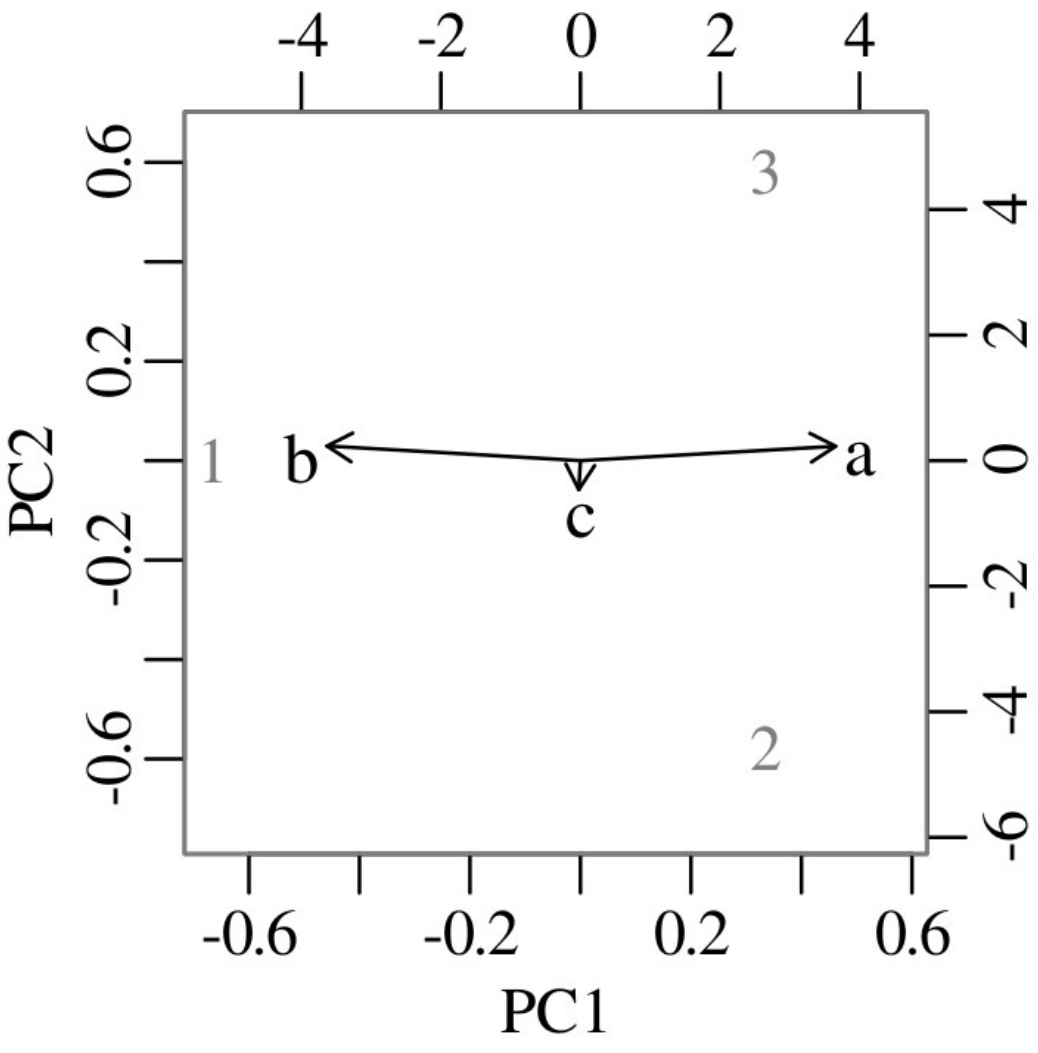
$$X_{\mathrm{c}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \ln(a/g) & \ln(b/g) & \ln(c/g) \end{array} \\ \begin{bmatrix} -3 & 5 & -2 \\ 1.33 & 0.33 & -1.67 \\ 1.67 & 0.67 & -2.33 \end{bmatrix} \end{array}$$

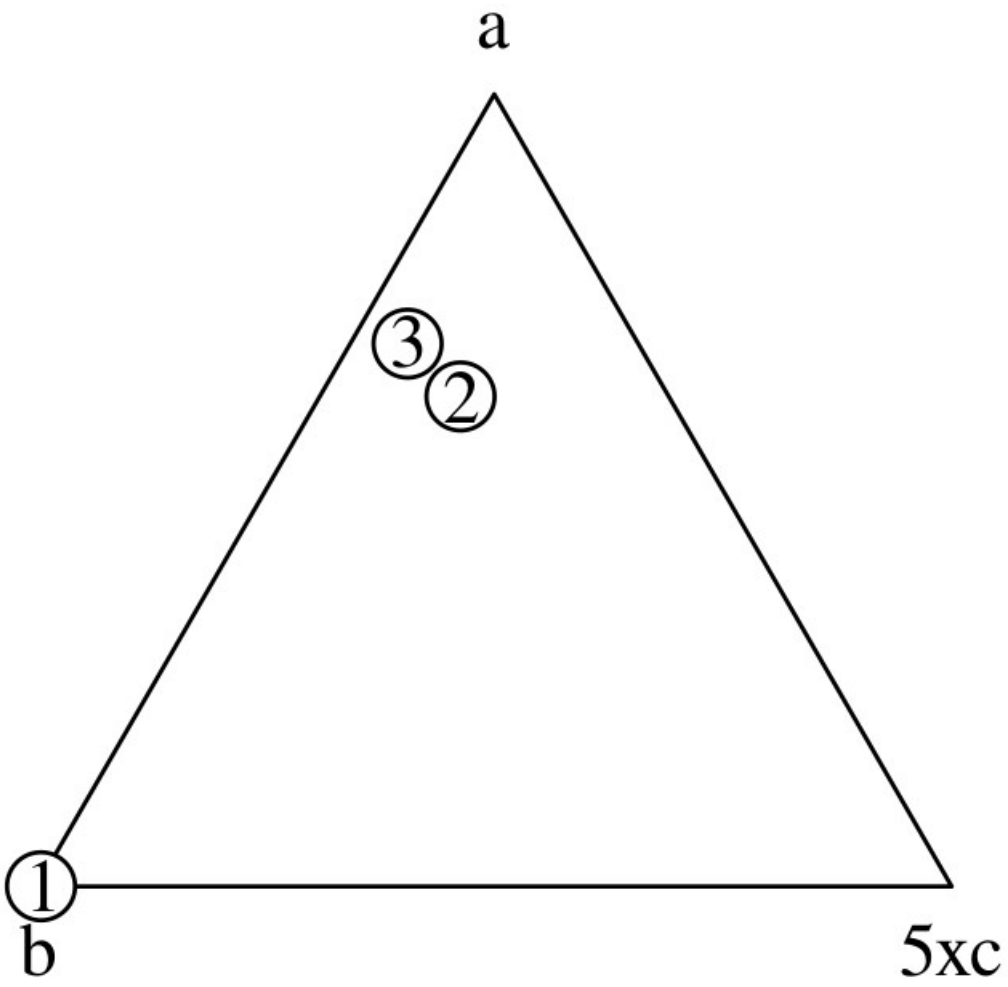
$$
X_{\mathrm{c}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{ccc} \ln(a/g) & \ln(b/g) & \ln(c/g) \end{array} \\
\begin{bmatrix} -3 & 5 & -2 \\ 1.33 & 0.33 & -1.67 \\ 1.67 & 0.67 & -2.33 \end{bmatrix}
\end{array}
$$

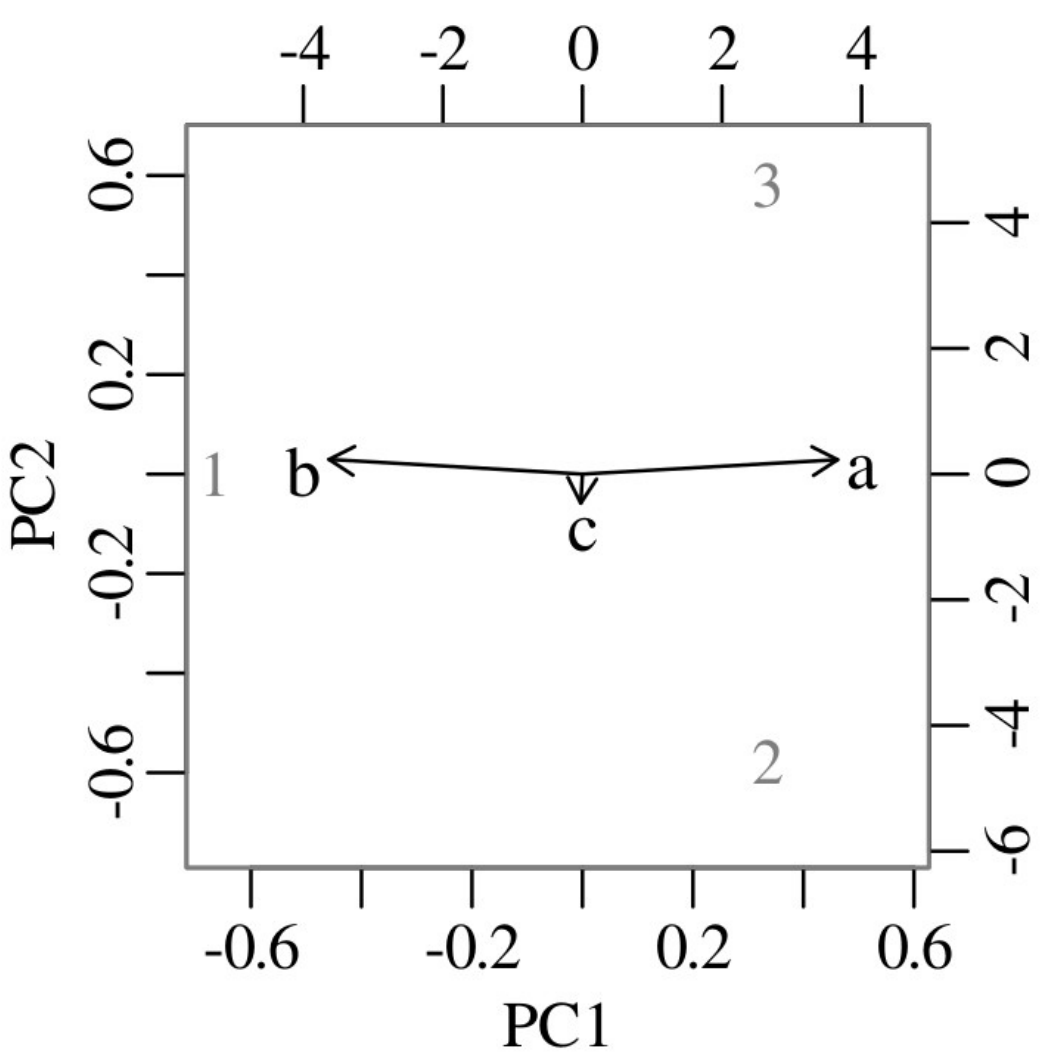
$$
X_c = 1_{3,1}\ C + S\ V\ D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -2 \end{bmatrix} +
$$

$$
\begin{bmatrix} -1.15 & 0 & 0.82 \\ 0.58 & -1 & 0.82 \\ 0.58 & 1 & 0.82 \end{bmatrix}
\begin{bmatrix} 3.67 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 0.71 & -0.71 & 0 \\ 0.41 & 0.41 & -0.82 \\ 0.58 & 0.58 & 0.58 \end{bmatrix}
$$

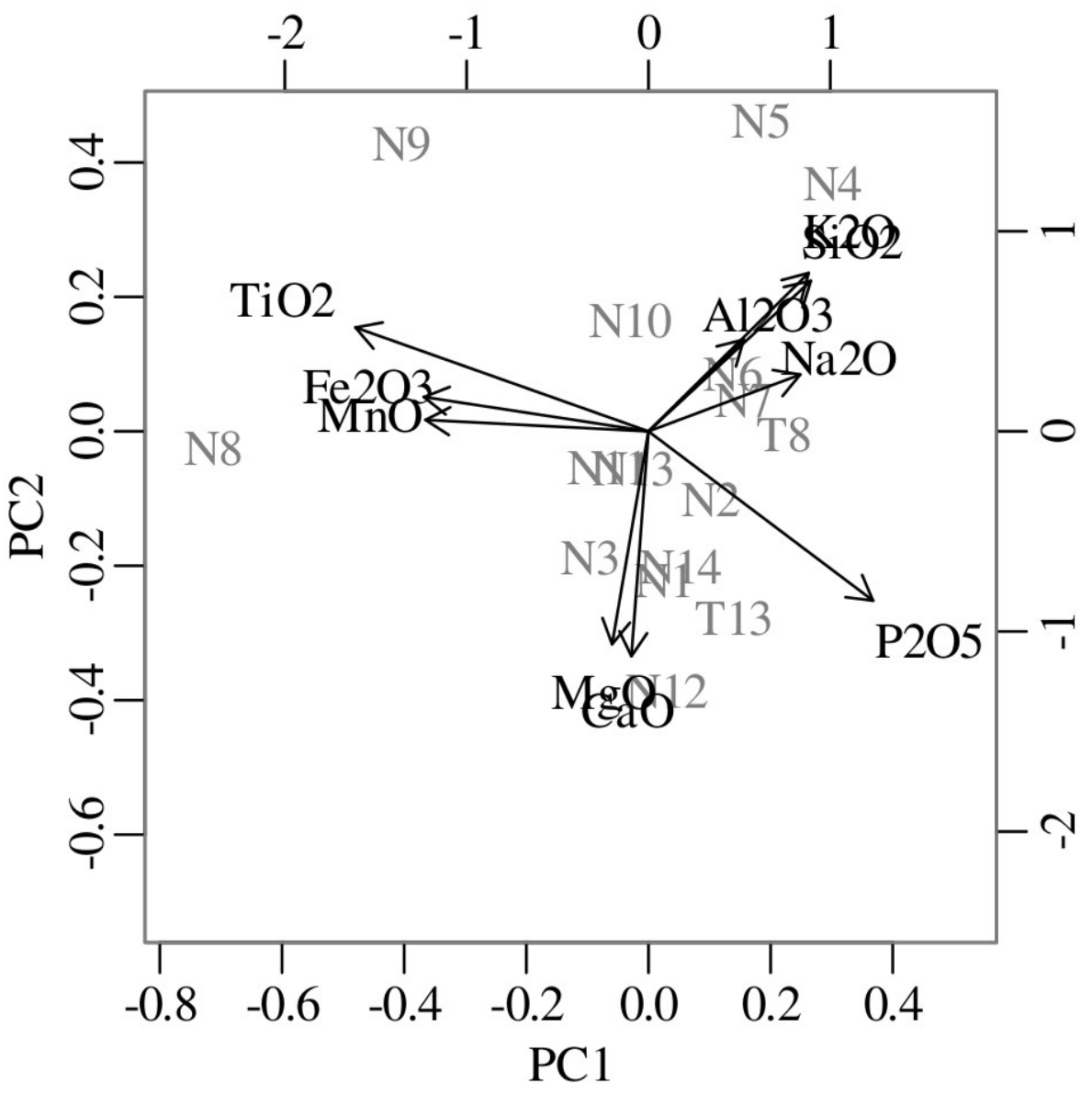

-6
-2
0
2
4
6
0.6
0.2
-0.2
-0.6
PC2
3
ln(b/c)
ln(a/c)
1
2
4
2
0
-2
-4
-0.5
0.0
0.5
PC1


-4
-2
0
2
4
0.6
0.2
-0.2
-0.6
PC2
3
1
b
a
c
2
4
2
0
-2
-4
-6
-0.6
-0.2
0.2
0.6
PC1

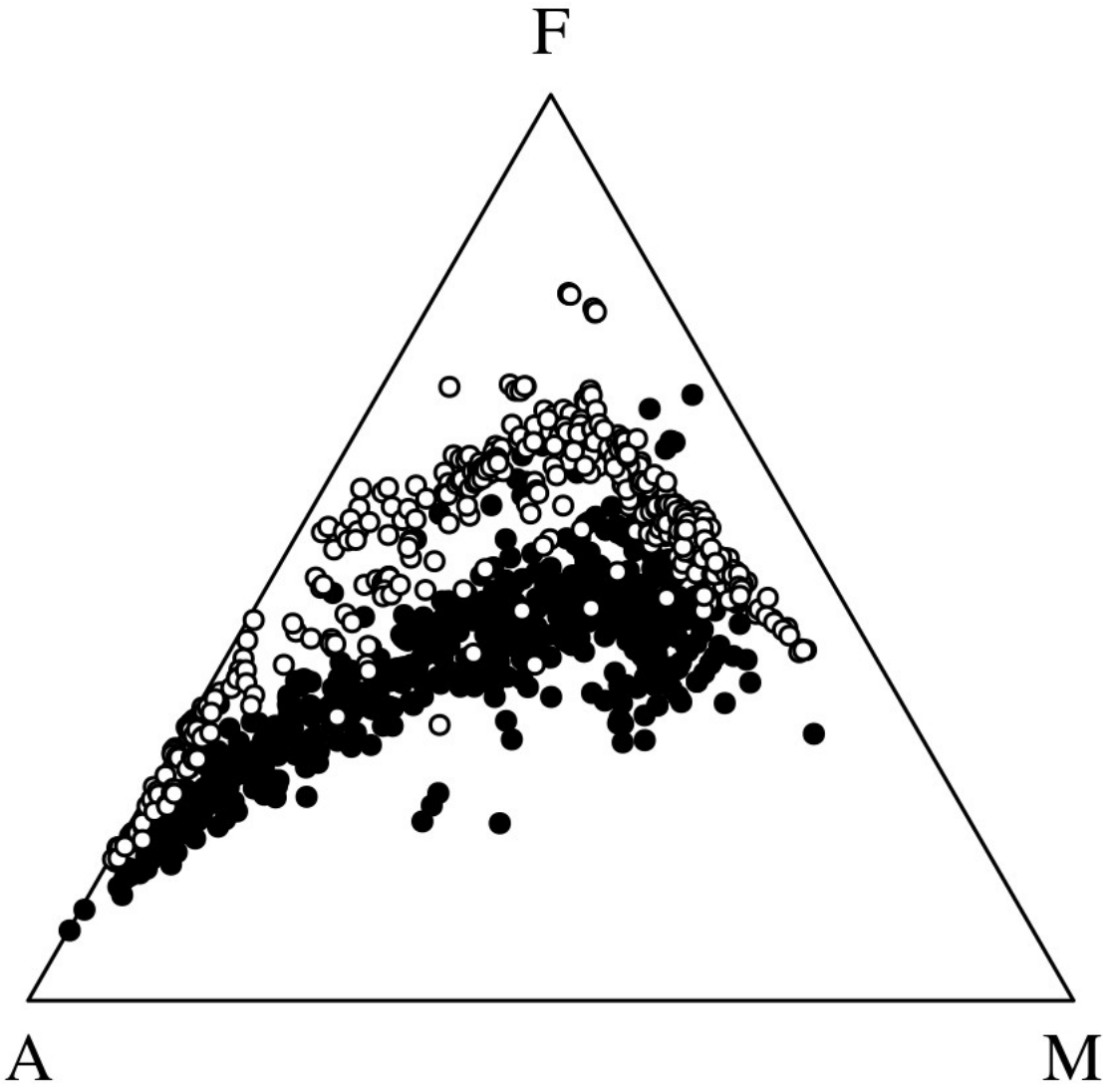

a
3
2
1
b
5xc

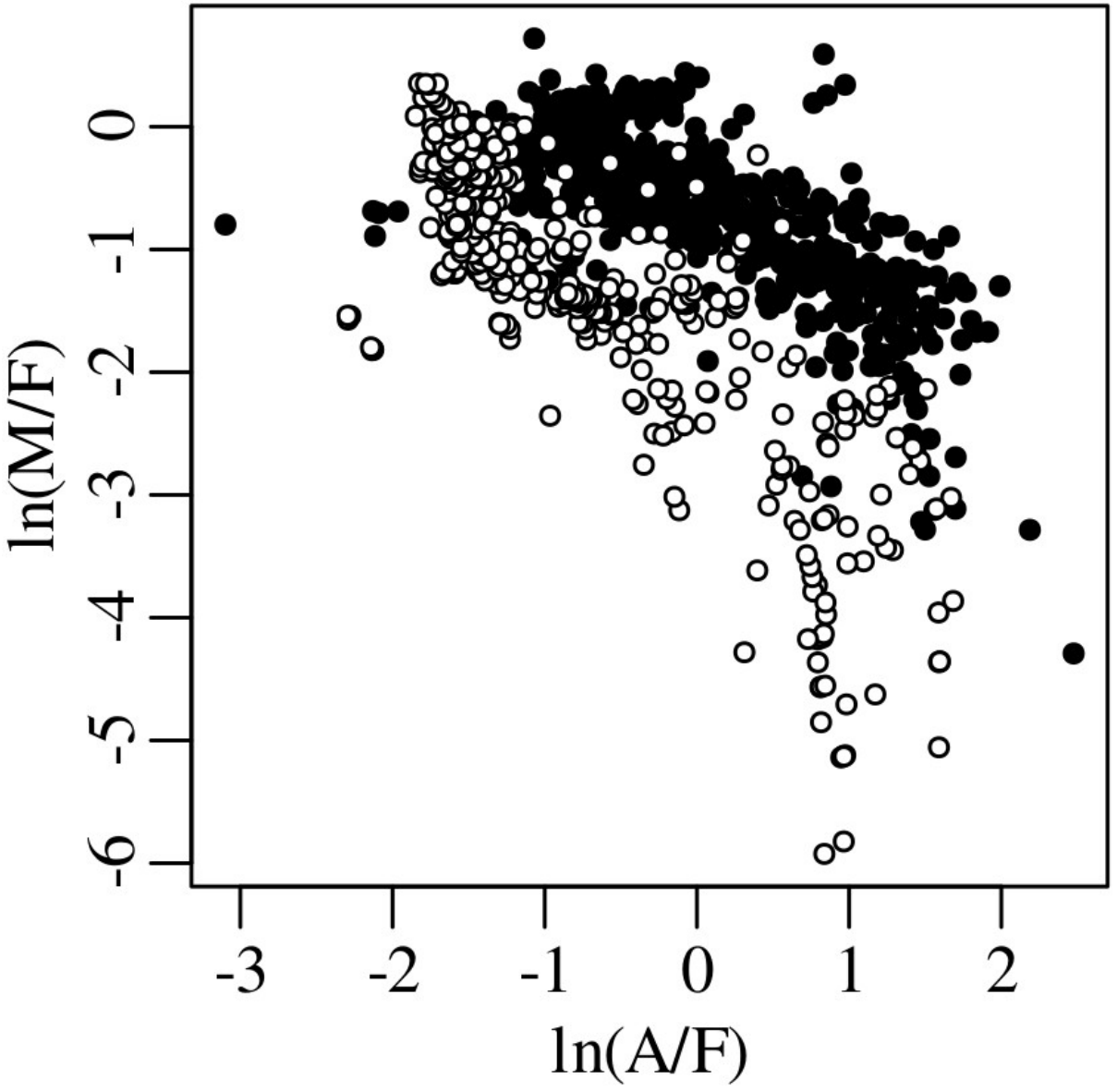

-4
-2
0
2
4
0.6
0.2
-0.2
-0.6
PC2
3
1
b
a
c
2
4
2
0
-2
-4
-6
-0.6
-0.2
0.2
0.6
PC1

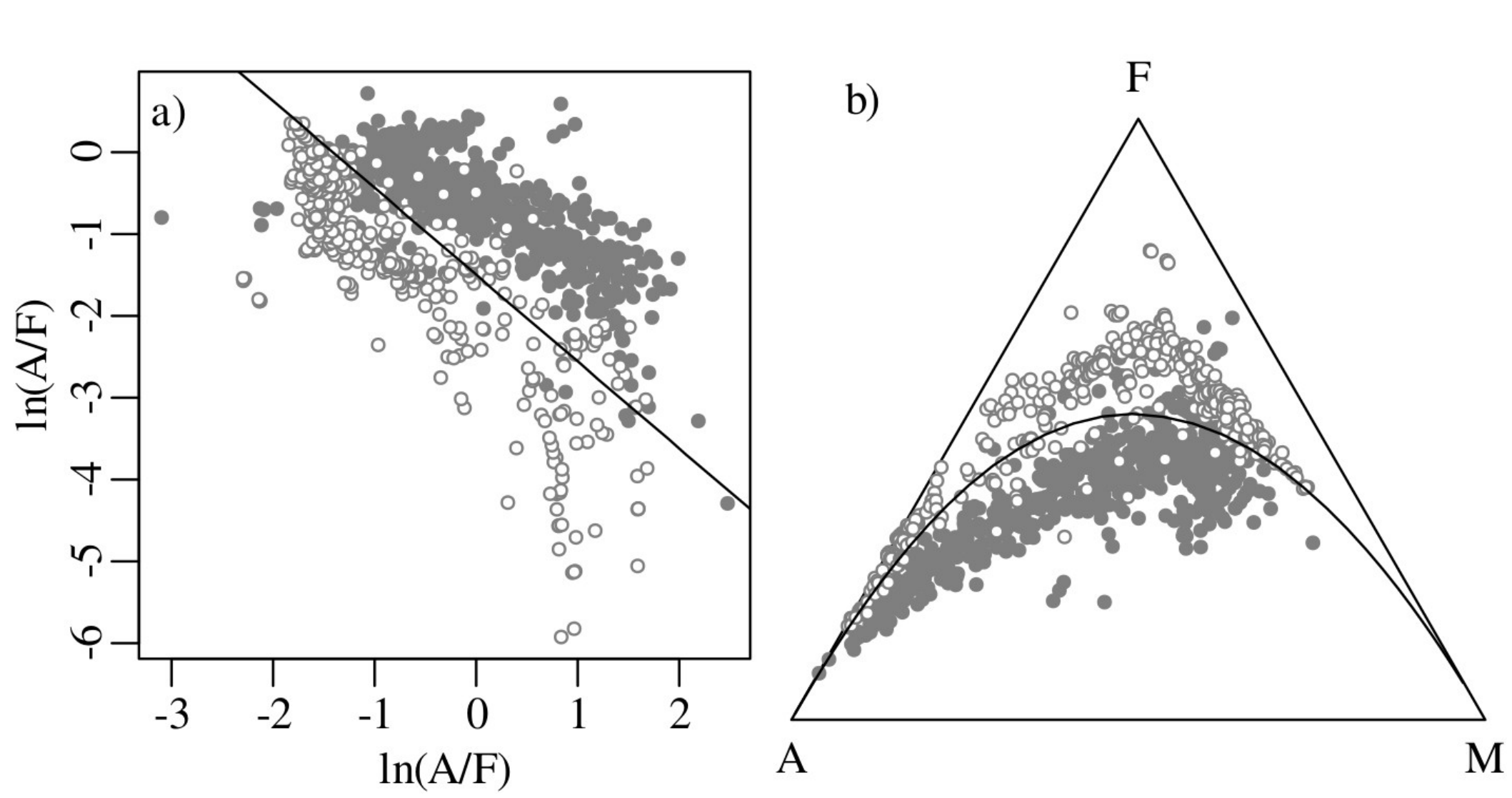
	SiO_2	Al_2O_3	Fe_2O_3	MgO	CaO	Na_2O	K_2O	TiO_2	P_2O_5	MnO
N1	82.54	6.14	3.18	1.65	2.24	1.16	1.35	0.44	0.11	0.06
N2	83.60	6.42	2.55	1.25	1.83	1.21	1.48	0.36	0.08	0.04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
N13	79.96	6.41	3.19	1.31	2.10	1.09	1.62	0.51	0.06	0.05
N14	73.62	9.96	4.07	2.01	3.45	1.86	2.29	0.44	0.10	0.07
T8	85.70	5.89	1.82	0.81	1.44	1.12	1.67	0.23	0.07	0.03
T13	82.54	6.02	2.30	1.42	3.07	1.19	1.46	0.34	0.11	0.04


-2
-1
0
1
N9
N5
N4
K2O
SiO2
TiO2
N10
Al2O3
Na2O
N6
Fe2O3
MnO
N7
T8
N8
N2
N3
N14
T13
P2O5
MgO
N12
CaO
0.4
0.2
0.0
-0.2
-0.4
-0.6
PC2
1
0
-1
-2
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
PC1

LDA of compositional data




ln(M/F)
0
-1
-2
-3
-4
-5
-6
-3
-2
-1
0
1
2
ln(A/F)


a)
ln(A/F)
0
-1
-2
-3
-4
-5
-6
-3
-2
-1
0
1
2
ln(A/F)
b)
F
A
M

Statistics for geoscientists

An introduction to R