Task-driven dictionary learning

Presented by Brian McFee for CSE254, W2012

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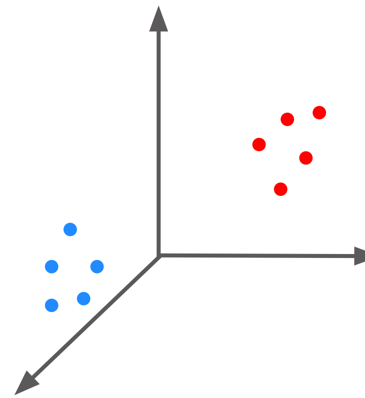
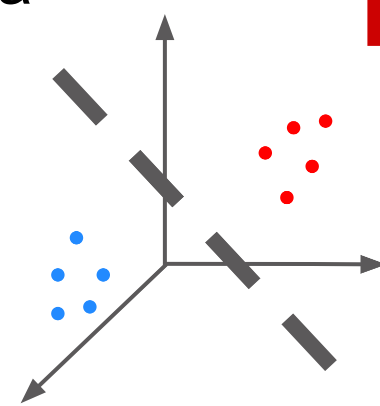
A typical machine learning pipeline

1. Collect data

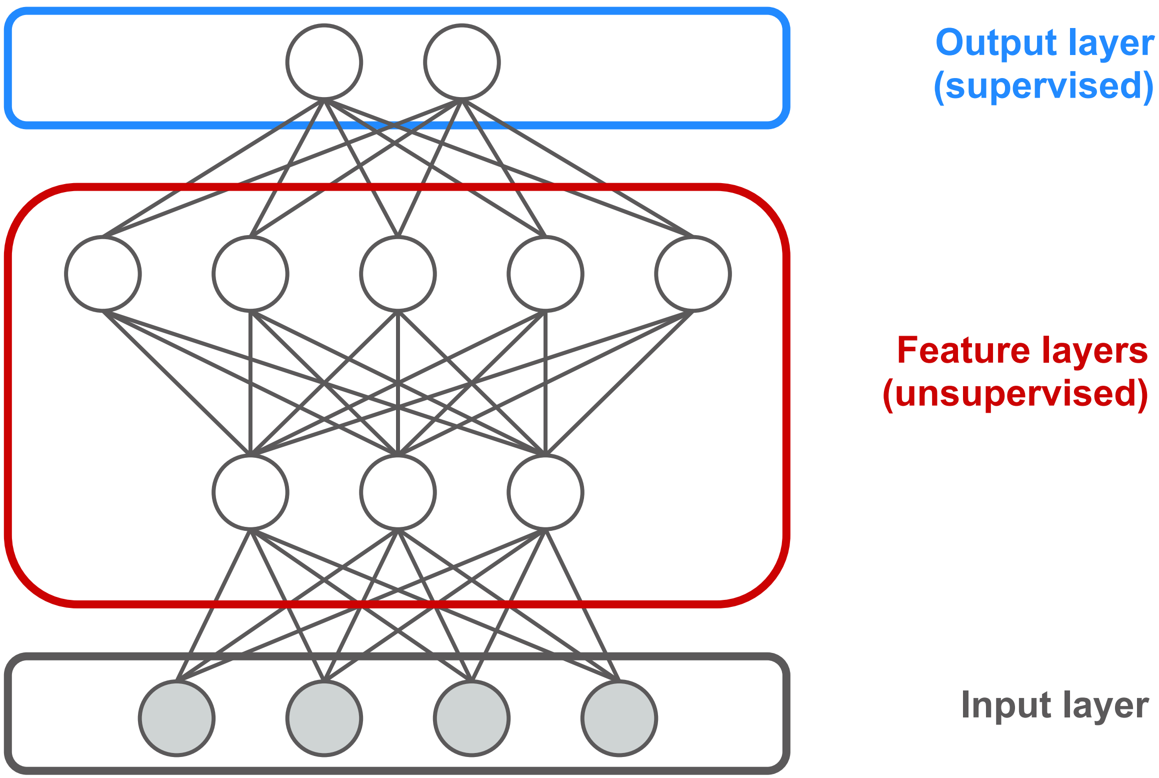


2. Extract features



3. Train predictor

A deep learning pipeline



[Olshausen & Field, 1997]

Background: sparse coding

Given data $X \in \mathbb{R}^{d \times n}$

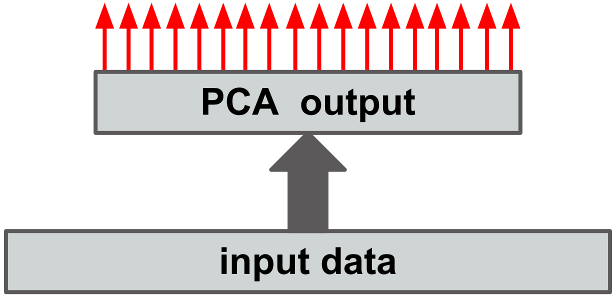
$$X = \begin{bmatrix} | & | & & | \\ x^1 & x^2 & \cdots & x^n \\ | & | & & | \end{bmatrix}$$

Learn a **dictionary** $D \in \mathbb{R}^{d \times k}$ **encoding** $A \in \mathbb{R}^{k \times n}$ $(k > d)$

$$X \approx DA$$

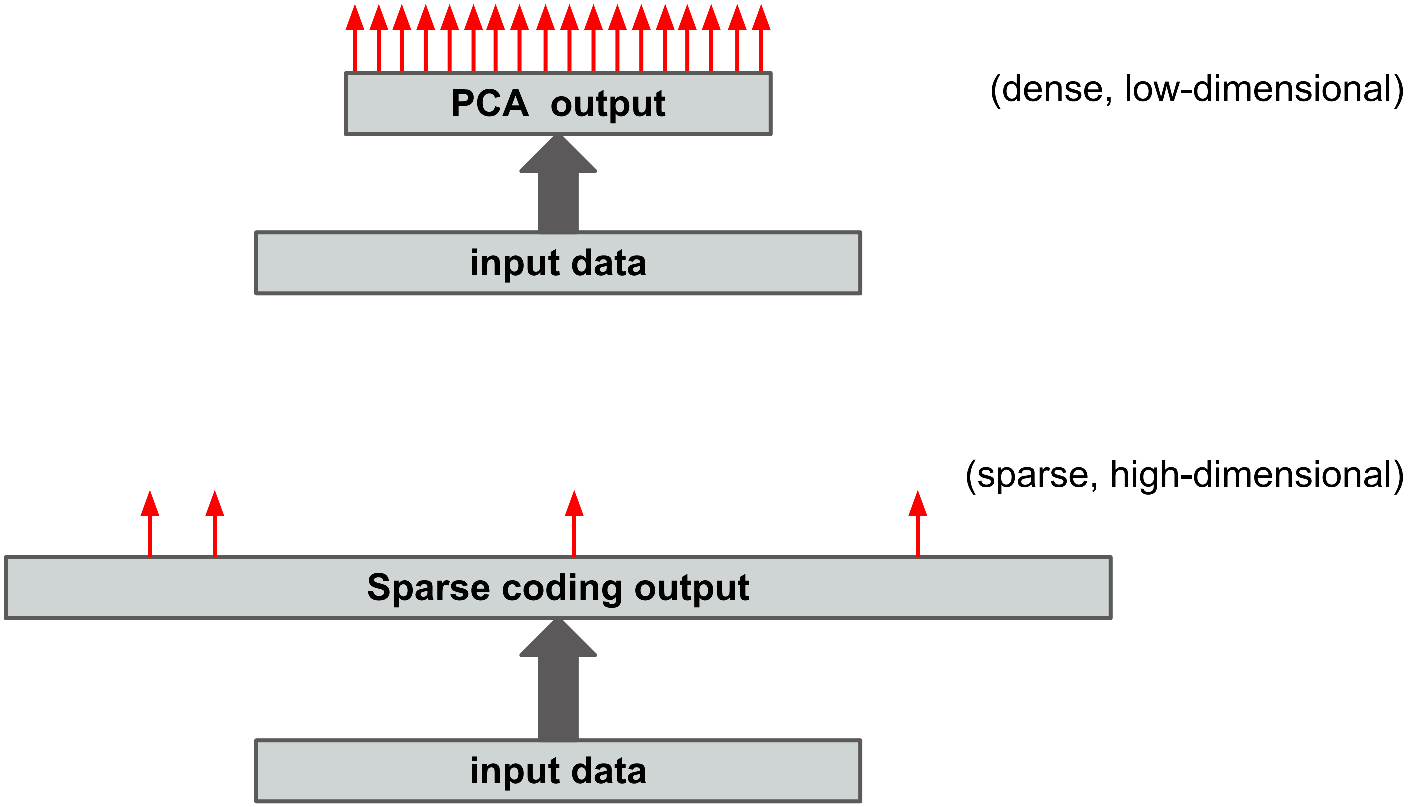
Sparse A = each x^i explained by few codewords

Sparse coding vs PCA



(dense, low-dimensional)

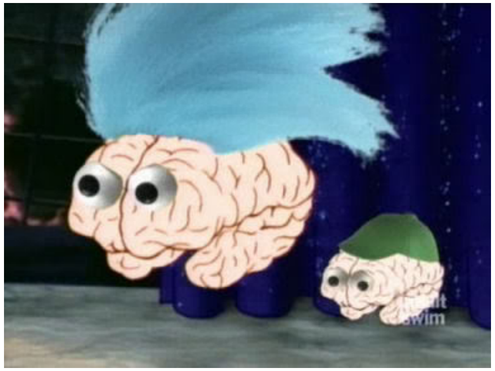
Sparse coding vs PCA



Why code sparsely?

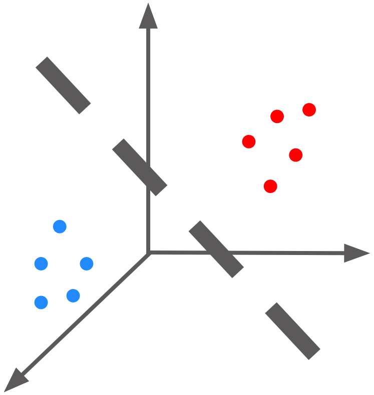
Biologically:

Neurons don't all fire at once (conserve energy)



Practically:

Works well for classification



How to code sparsely

Given x and D, define energy function

$$E(a; x, D) := \overset{\text{Accuracy}}{\frac{1}{2}\|x - Da\|^2} + \overset{\text{Sparsity}}{\lambda\|a\|_1}$$

How to code sparsely

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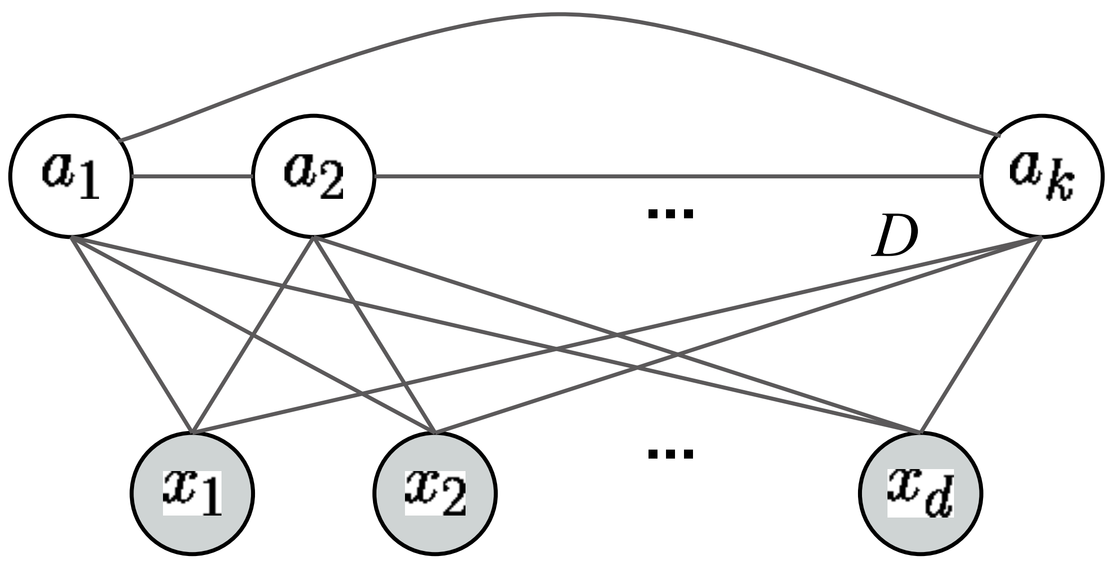
$$E(a; x, D) := \underbrace{\frac{1}{2}\|x - Da\|^2}_{\text{Accuracy}} + \underbrace{\lambda\|a\|_1}_{\text{Sparsity}}$$



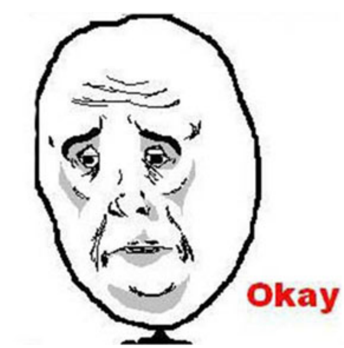
Learning the dictionary

Given n samples $x^i \in \mathbb{R}^d$, minimize total energy:

$$\min_{D,A} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \|x^i - Da^i\|^2 + \lambda \|a^i\|_1$$

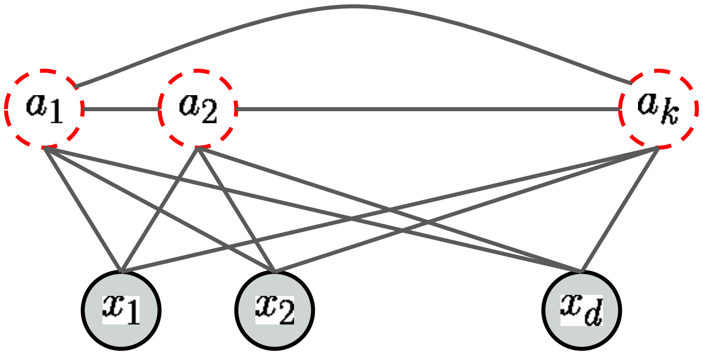
$$\text{s.t. } \forall j: \ \|d^j\|^2 \leq 1$$

Problem: not jointly convex in D,A



Learning by alternating minimization

But it is convex in D and A individually

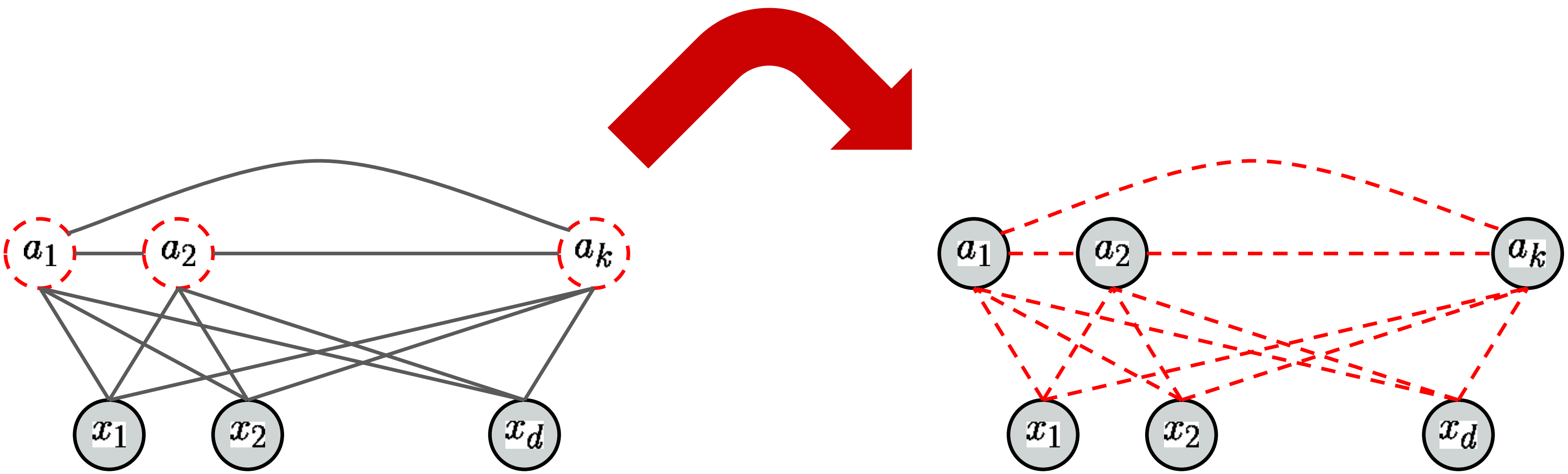


Fix D, optimize A

(Lasso * n)

Learning by alternating minimization

But it is convex in D and A individually



Fix D, optimize A
(Lasso * n)

Fix A, optimize D
(QCQP)

Learning by alternating minimization

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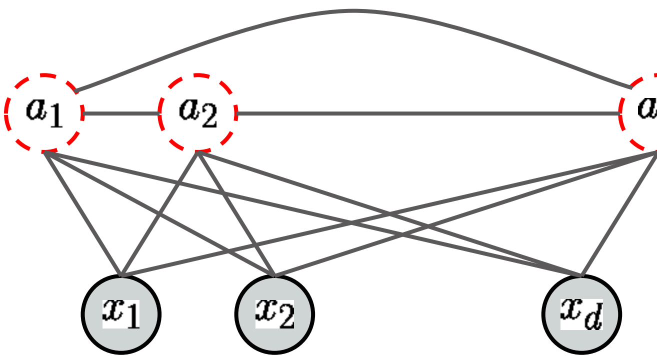
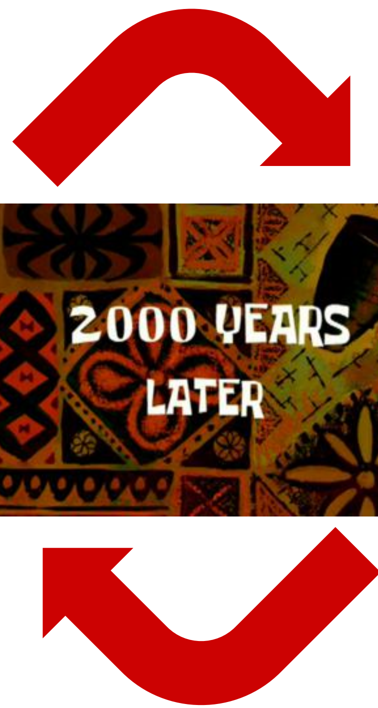
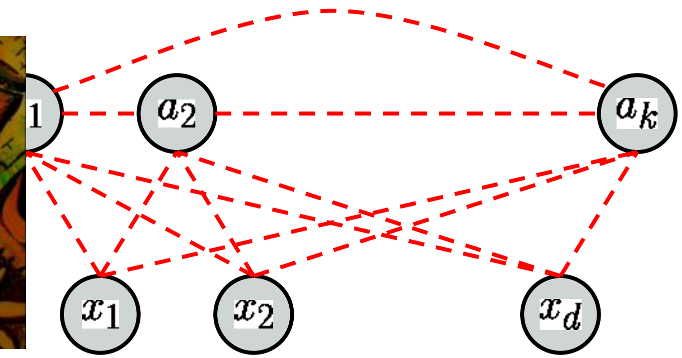




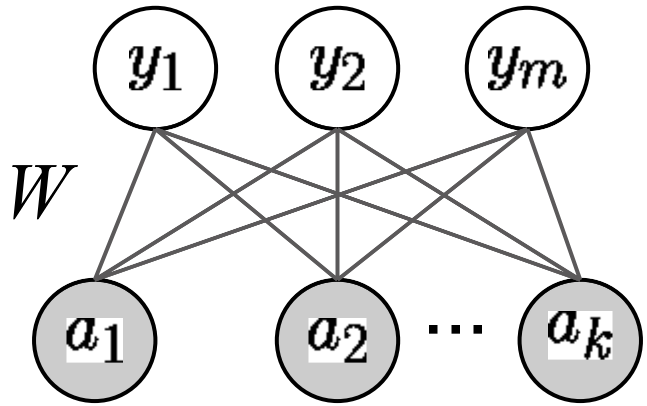


Fix D, optimize A
(Lasso * n)

Fix A, optimize D
(QCQP)

Supervised learning

Idea: use encodings a as input to a predictor



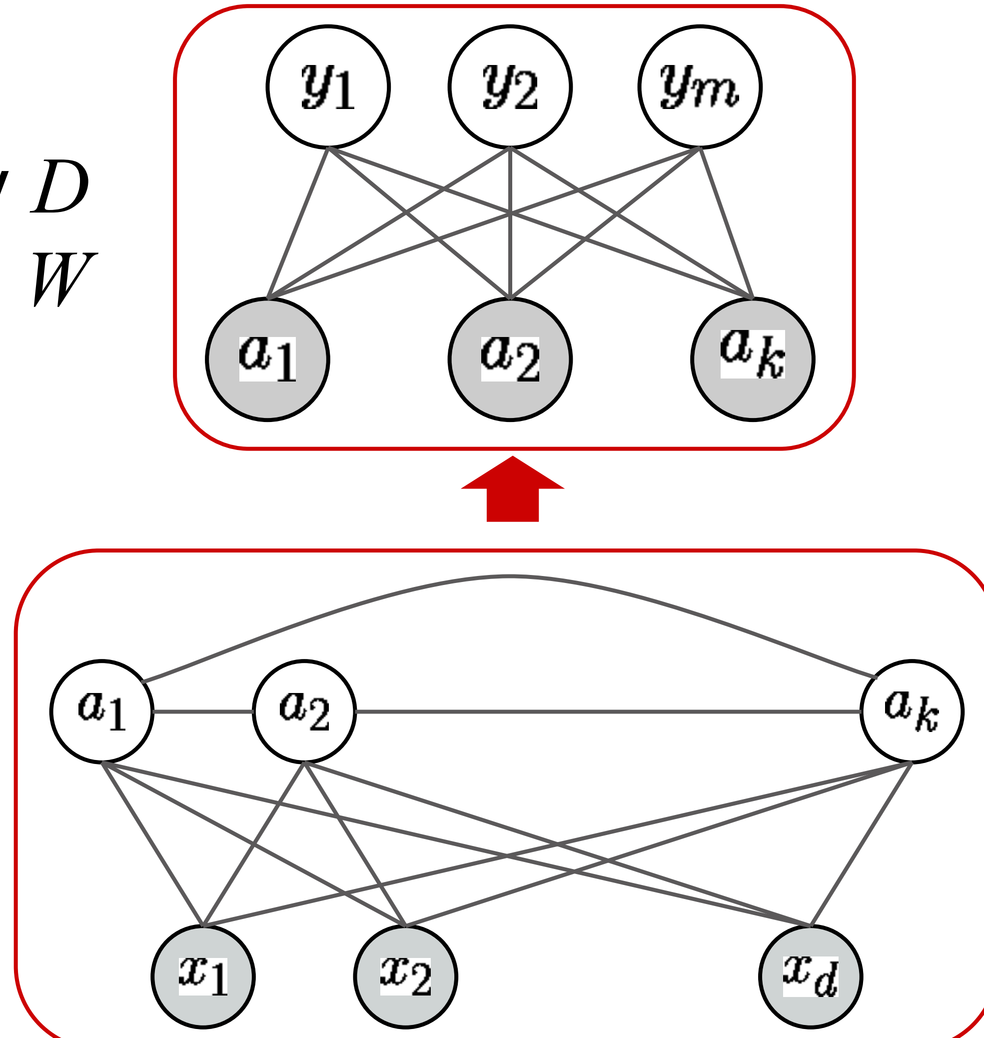
Ex: linear regression, $y \in \mathbb{R}^m$ $W \in \mathbb{R}^{m \times k}$

$$E(y; W, a) := \frac{1}{2}\|y - Wa\|^2$$

A disconnected architecture

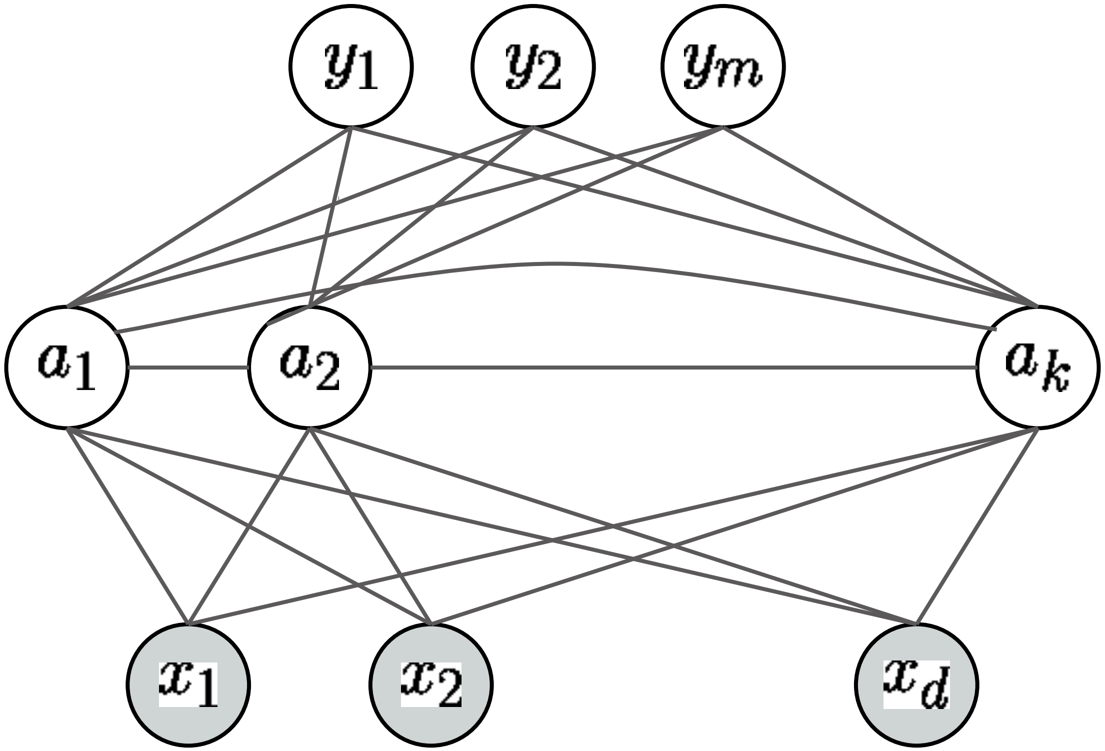
The usual plan:

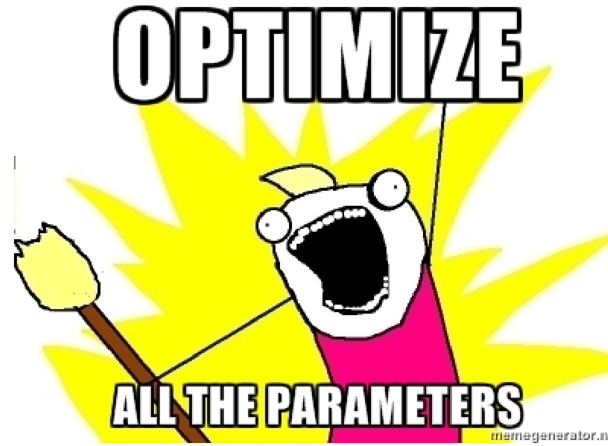
1. Learn dictionary D
2. Learn predictor W
3. Fame and glory



But these are tasks not independent!

Unified architecture





Jointly learn codebook D and predictor W

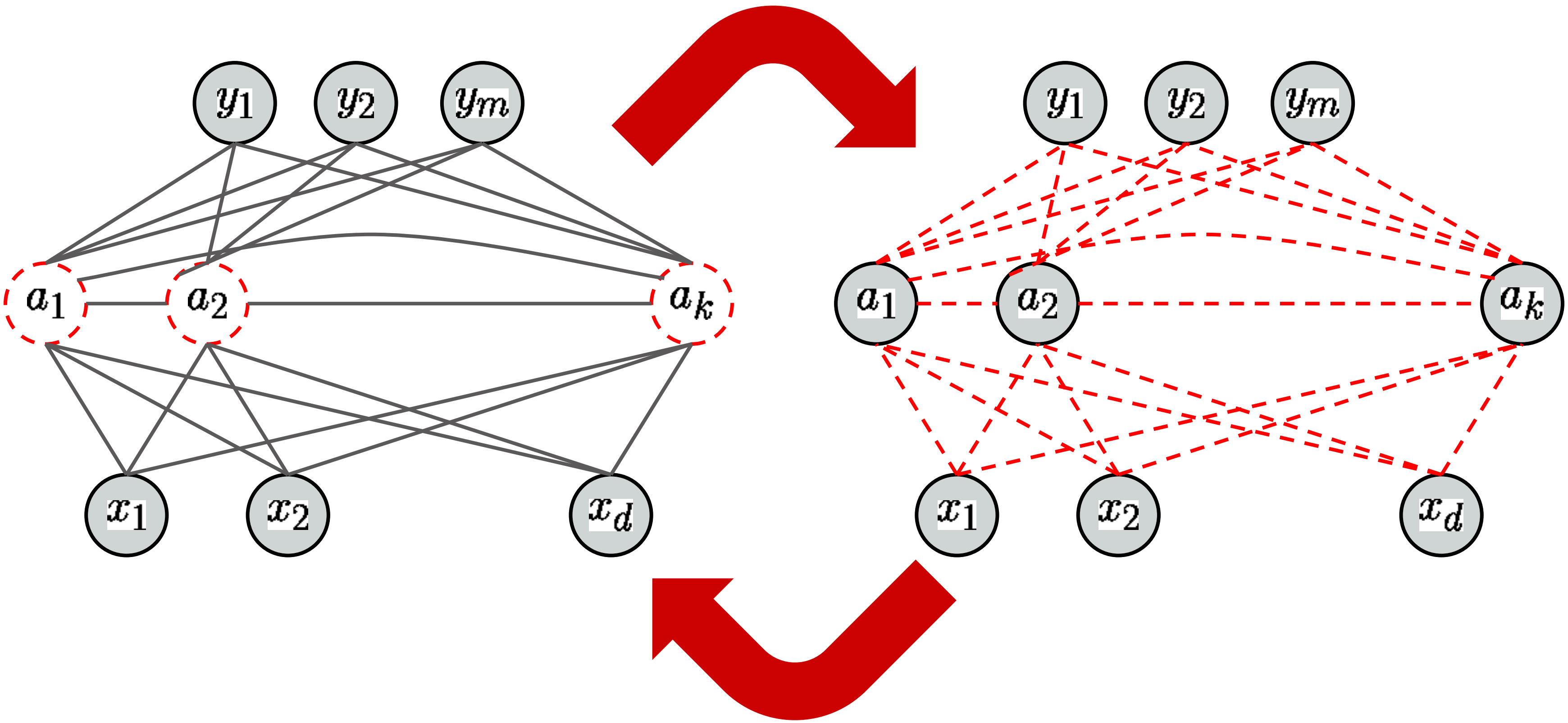
[Mairal, Bach, Ponce, Sapiro, and Zisserman, 2009]

Joint optimization: round 1

Given n samples (x^i, y^i), minimize energy:

$$\min_{D,A,W} \left(\sum_{i=1}^{n} \overbrace{E(y^i; W, a^i)}^{\text{Prediction cost}} + \lambda_0 \overbrace{E(a^i; x^i, D)}^{\text{Coding cost}} \right) + \overbrace{\frac{\nu}{2} \|W\|_{\mathrm{F}}^2}^{\text{Regularizer}}$$

Alternating minimization



Fix (W, D), update A

Fix A, update (W,D)

At prediction time...

For fixed W, D and test input x, compute

$$\arg\min_{y,a} E(y;W,a) + \lambda_0 E(a;x,D)$$

Easy for linear regression:

$$E(y;W,a) := \frac{1}{2}\|y - Wa\|^2$$

$$y = Wa \;\Rightarrow\; E(y;W,a) = 0$$

At prediction time...

What about categorical output? (e.g. $y \in \{0, 1\}$)

Encoding a depends on **input AND output**

Seems unnatural, complicates prediction

[Mairal, Bach and Ponce, 2011]

Code first, predict later

Old learning problem (sum of energies):

$$\min_{D,A,W} \left(\sum_{i=1}^{n} E(y^i; W, a^i) + \lambda_0 E(a^i; x^i, D) \right) + \frac{\nu}{2} \|W\|_{\mathrm{F}}^2$$

[Mairal, Bach and Ponce, 2011]

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New formulation (optimal encoding):

$$\min_{D,W} \sum_{i=1}^{n} E\left(y^i; W, a^*(x^i, D)\right) + \frac{\nu}{2} \|W\|_{\mathrm{F}}^2$$

$$a^*(x, D) = \arg\min_{a} E(a; x, D)$$

Optimal encoding

Output is now a function of optimal encoding

$$\min_{D,W} \sum_{i=1}^{n} E\left(y^i; W, a^*(x^i, D)\right) + \frac{\nu}{2}\|W\|_F^2$$

Prediction is feed-forward:

a. Encode x as $a*$
b. Predict y
c. Done.

But how to learn parameters W, D?

Stochastic gradient descent

Minimize objective in expectation:

$$\min_{D,W} \; f(D,W)$$

$$f(D,W) := \; \mathbb{E}_{(x,y)}\left[E(y;W,a^*(x,D))\right] + \frac{\nu}{2}\|W\|_{\mathrm{F}}^2$$

Stochastic gradient descent

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Algorithm: repeat

a. Pick a random (x,y) from training data
b. Update D, W

Stochastic gradient descent

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Algorithm: repeat

a. Pick a random (x,y) from training data
b. Update D, W

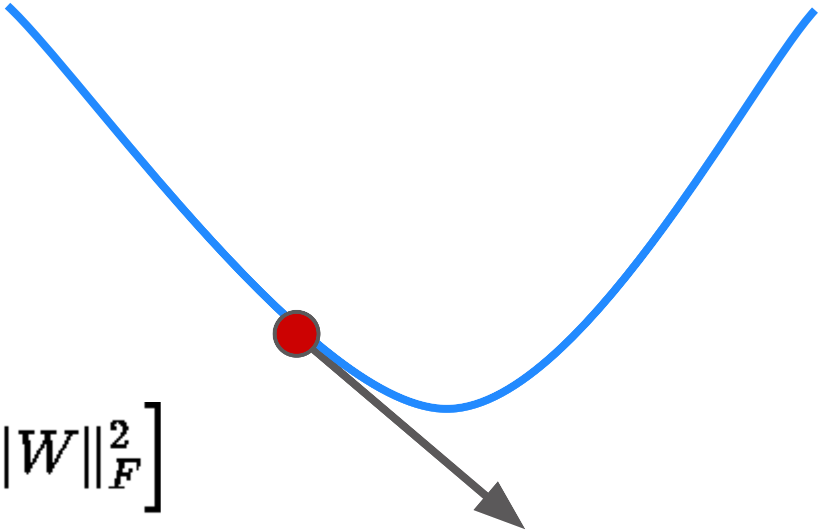
Need two quantities:

$$\nabla_W f(D,W)$$

$$\nabla_D f(D,W)$$

W update

Easy for differentiable E:



$$W \leftarrow W - \eta \nabla_W \left[E(y; W, a^*) + \frac{\nu}{2} \|W\|_F^2 \right]$$

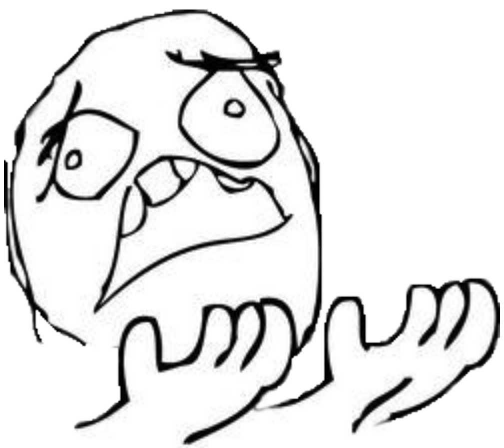
for learning rate η

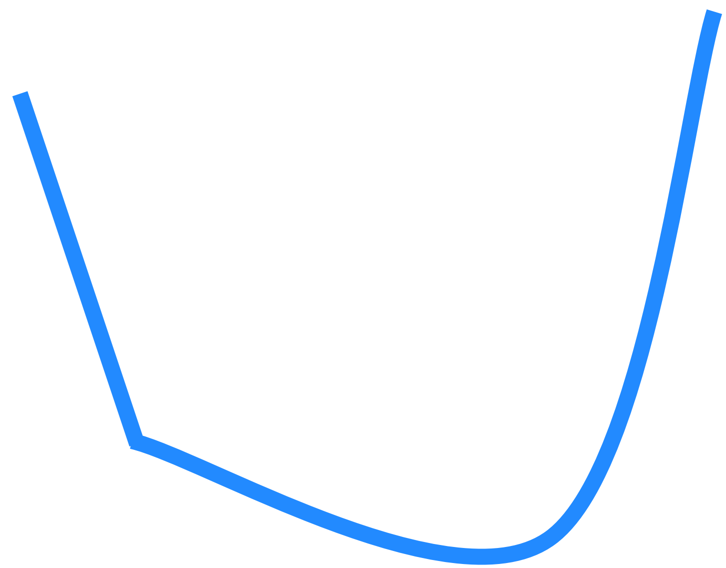
D update?

a(x,D)* is **not differentiable**

$$a^*(x, D) = \arg\min_a \frac{1}{2}\|x - Da\|^2 + \lambda\|a\|_1$$

$$\nabla_D\, a^*(x, D) = ???$$



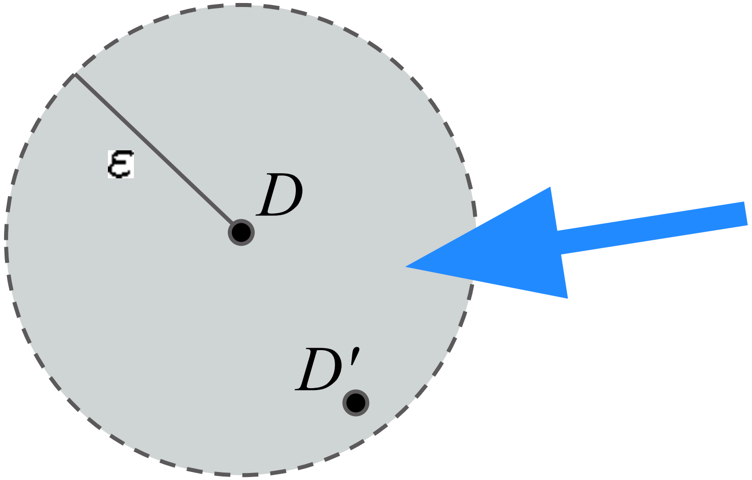


Key observation

Small change in *D*

⇓

Small change in $a^*(x,D)$

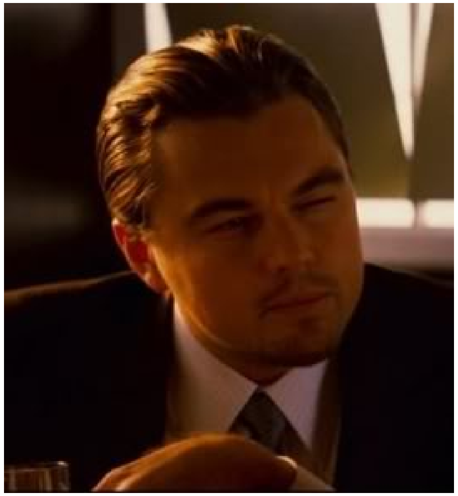


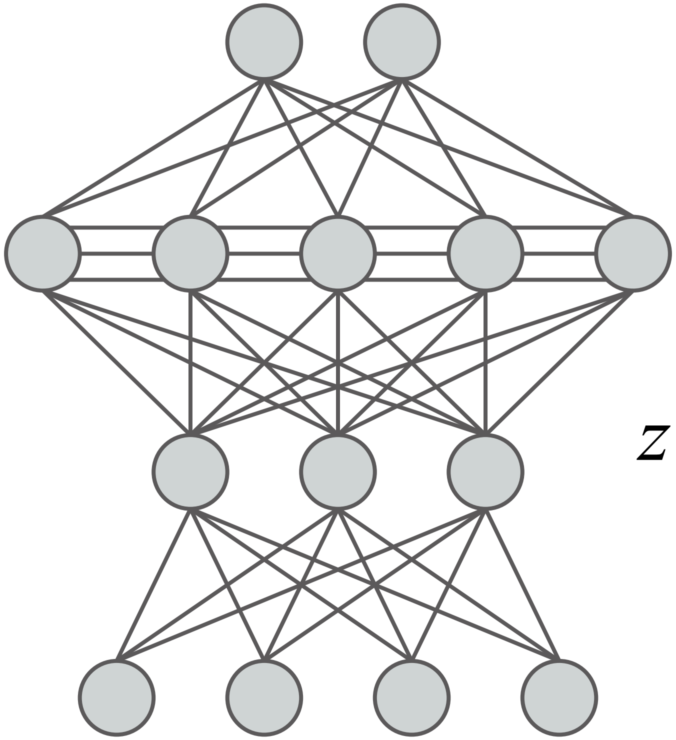
Differentiable region

$$a^*(x, D') \approx a^*(x, D)$$

Deeper?

Add a compression layer: $a^*(Zx, D)$





y: output

a: coding

z: compression

x: input

Experiments

Exp. 1: Hand-written digit recognition

1-vs-all logistic regression (10 classes)

MNIST: 60K train, 10K test, 28x28

USPS: 7.3K train, 2K test, 16x16

D	Unsupervised				Supervised			
k^*	50	100	200	300	50	100	200	300
MNIST	5.27	3.92	2.95	2.36	0.96	0.73	0.57	**0.54**
USPS	8.02	6.03	5.13	4.58	3.64	3.09	2.88	**2.84**

% Error

Exp. 2: Inverse half-tone

Original (8bpp) | Observed (1bpp) | Reconstructed (8bpp)

Predict grayscale image from black&white input

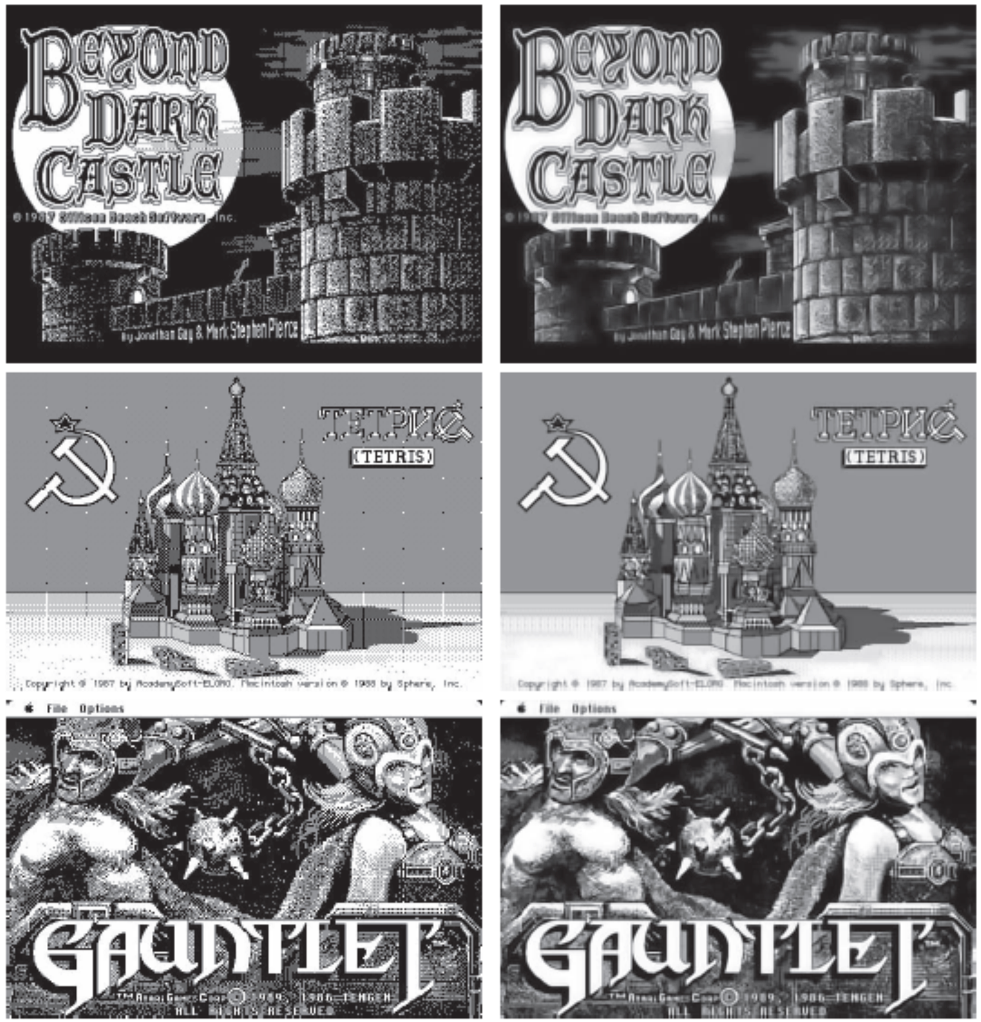
Linear regression

10x10 image patches

k=500 codewords

36 images, 9M patches

Exp. 2: Inverse half-tone results



Conclusion

Supervision can help learn good features