appropriate for large plant uncertainties, whereas nonlinear forms have shown better response when measurement perturbation is high. Finally, initial condition choice should ensure convergent behavior. Computationally, the ANLF burden increases as a function of the number of conditional models.

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FIR Filter Representations of Reduced-Rank Noise Reduction

Per Christian Hansen and Søren Holdt Jensen

Abstract— We show that the reduced-rank output signal computed via truncated (Q)SVD is identical to that from an array of parallelly connected analysis-synthesis finite impulse response (FIR) filter pairs. The filter coefficients are determined by the (Q)SVD, and the filters provide an explicit description of the reduced-rank noise reduction algorithm in the frequency domain.

I. Introduction

The central idea in rank-reducing techniques is to approximate a matrix derived from the data with another matrix of lower rank from which the reconstructed signal is then computed. Reduced-rank techniques are used in many areas such as regularization [4], signal estimation [10], [11], and noise reduction of, e.g., speech signals [2], [6] and nuclear magnetic resonance data [12]. In this correspondence, we focus on rank reduction for signal modeling, or synthesis, of stationary time series and, in particular, as a means for noise reduction. See [5, Sec. IV.3], [7], and [8] for more details about these issues.

The standard rank-reduction algorithm for noise reduction involves the following four steps.

1) Form a Hankel (or Toeplitz) matrix from the input signal.
2) Compute the singular value decomposition (SVD) of this matrix.
3) Discard the small singular values to obtain a matrix with reduced rank.
4) Construct the output signal from this generally unstructured matrix by arithmetic averaging along its antidiagonals (or diagonals).

This algorithm, which we shall refer to as the *truncated SVD* (TSVD) algorithm, has been used successfully in connection with estimation of frequencies of multiple sinusoids [10] and for reduction of white noise in speech signals [2].

Prewhitening of the signal—corresponding to a filtering operation where the signal matrix is multiplied with a filtering matrix—is sometimes used if the noise cannot be considered to be white. The prewhitening operation can be included as an integral part of the algorithm, which then requires the computation of the *quotient SVD* (QSVD) of the signal-prewhitener matrix pair. This is the *truncated*

QSVD (TQSVD) algorithm [6], in which the reduced-rank matrix is obtained by discarding small quotient singular values of the matrix pair. In [6], the TQSVD algorithm is used for reduction of nonwhite broad-band noise in speech signals.

Until now, the spectral properties of rank-reducing techniques have been studied by means of asymptotic results for infinite-length signals and their spectra. A very illuminating discussion of these issues is given in [8]. This analysis shows that the SVD yields an "energy decomposition" of the signal and that the signal components retained in the reconstructed signal—corresponding to the large singular values—are those with high power. However, thus far, no explicit results have been derived for the spectral behavior of rank reduction applied to finite-duration (short-time) signals.

The main contribution of this correspondence is to derive such explicit results for finite-duration signals. In particular, we develop *finite-duration impulse response* (FIR) filter representations of the TSVD and TQSVD algorithms, and we derive closed-form expressions for the FIR filter coefficients. Via these filters, the behavior of the TSVD and TQSVD algorithms can be explicitly studied in the frequency domain.

This correspondence is based on ideas presented in [3], where it is concluded that the TSVD algorithm corresponds to subtracting from the input signal information contained in eigenresiduals corresponding to the smallest singular values. Our analysis, we hope, is more intuitive in that it leads to an array of filters that explicitly describe the TSVD/TQSVD filtering. Moreover, our analysis is more general since we include in a natural way the case of prewhitening (i.e., the TQSVD algorithm).

II. Definitions and Notation

To present our results, it is convenient to introduce a compact notation for working with Hankel and Toeplitz matrices, which we will present in this section. Given a vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_N)^T$ of length N, we define the following two Hankel matrices of dimension $m \times n$ and $(N + n - 1) \times n$, respectively, where $m + n - 1 = N$.

$$\mathcal{H}(\boldsymbol{x}) \equiv \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & & \vdots \\ x_m & x_{n+1} & \cdots & x_N \end{pmatrix} \qquad (1)$$

and

$$\mathcal{H}_p(\boldsymbol{x}) \equiv \begin{pmatrix} 0 & 0 & \cdots & x_1 \\ \vdots & \vdots & & \vdots \\ 0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & & \vdots \\ x_m & x_{m+1} & \cdots & x_N \\ x_{m+1} & x_{m+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_N & 0 & \cdots & 0 \end{pmatrix}. \qquad (2)$$

The subscript p in $\mathcal{H}_p(\boldsymbol{x})$ symbolizes that this matrix is obtained from $\mathcal{H}(\boldsymbol{x})$ by *pre-* and *post-augmentation* with triangular Hankel matrices.

Next, let $\boldsymbol{J}$ denote the "exchange matrix," such that multiplication of a matrix from the right by $\boldsymbol{J}$ reverses the order of its columns. We define two Toeplitz matrices derived from $\mathcal{H}(\boldsymbol{x})$ and $\mathcal{H}_p(\boldsymbol{x})$ by reversing their columns, i.e., $\mathcal{T}(\boldsymbol{x}) \equiv \mathcal{H}(\boldsymbol{x})\boldsymbol{J}$ and $\mathcal{T}_p(\boldsymbol{x}) \equiv \mathcal{H}_p(\boldsymbol{x})\boldsymbol{J}$. In these relations, $\boldsymbol{J}$ is $n \times n$.

Given a signal vector $\boldsymbol{x}$ of length N, we now consider the matrix-vector products $\mathcal{H}(\boldsymbol{x})\boldsymbol{y}$ and $\mathcal{H}_p(\boldsymbol{x})\boldsymbol{y}$, where the Hankel matrices have

n columns, and the vector $\boldsymbol{y}$ has length n. Clearly, premultiplication of $\boldsymbol{y}$ with either $\mathcal{H}(\boldsymbol{x})$ or $\mathcal{H}_p(\boldsymbol{x})$ corresponds to filtering the signal $\boldsymbol{x}$ with an FIR filter whose coefficients are the elements of the vector $\boldsymbol{y}$. The vector $\mathcal{H}(\boldsymbol{x})\boldsymbol{y}$ has length $m < N$ and contains no "end effects," whereas the vector $\mathcal{H}_p(\boldsymbol{x})\boldsymbol{y}$ has length $N + n - 1$ and contains "end effects."

Similarly, premultiplication of $\boldsymbol{y}$ with either $\mathcal{T}(\boldsymbol{x})$ or $\mathcal{T}_p(\boldsymbol{x})$ corresponds to filtering $\boldsymbol{x}$ with an FIR filter whose coefficients are the elements of $\boldsymbol{y}$ in reverse order.[1] To see this, note that $\boldsymbol{J}\boldsymbol{J}$ is the identity, and therefore

$$\mathcal{T}_p(\boldsymbol{x})\boldsymbol{y} = (\mathcal{T}_p(\boldsymbol{x})\boldsymbol{J})(\boldsymbol{J}\boldsymbol{y}) = \mathcal{H}_p(\boldsymbol{x})(\boldsymbol{J}\boldsymbol{y}) \qquad (3)$$

and similarly $\mathcal{T}(\boldsymbol{x})\boldsymbol{y} = \mathcal{H}(\boldsymbol{x})(\boldsymbol{J}\boldsymbol{y})$.

Finally, we define the averaging operator $\mathcal{A}$ that transforms a given $m \times n$ matrix $\boldsymbol{M}$ into a signal vector $\boldsymbol{s} \equiv \mathcal{A}(\boldsymbol{M})$ of length N by arithmetic averaging along the $N = m + n - 1$ antidiagonals of $\boldsymbol{M}$, i.e., the ith component of $\boldsymbol{s}$ is given by

$$s_i = \frac{1}{\beta - \alpha + 1} \sum_{k=\alpha}^{\beta} \boldsymbol{M}_{i-k+1, k}$$

where $\alpha = \max(1, i - m + 1)$ and $\beta = \min(n, i)$. Averaging along the (anti)diagonals is commonly used in signal processing [11]. For the special case where $\boldsymbol{M}$ is a rank-one matrix, i.e., where we can write $\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v}^T$, the averaging operation can be expressed in the simple form

$$\mathcal{A}(\boldsymbol{M}) = \mathcal{A}(\boldsymbol{u}\boldsymbol{v}^T) = \boldsymbol{D}\mathcal{T}_p(\boldsymbol{u})\boldsymbol{v} \qquad (4)$$

where $\boldsymbol{D}$ is an $N \times N$ diagonal matrix given by

$$\boldsymbol{D} = \mathrm{diag}(1, 2^{-1}, 3^{-1}, \cdots, \mu^{-1}, \mu^{-1}, \cdots, \mu^{-1}$$
$$\cdots, 3^{-1}, 2^{-1}, 1) \qquad (5)$$

and where $\mu = \min(m, n)$. The first $\mu - 1$ and last $\mu - 1$ diagonal elements account for the fact that the corresponding antidiagonals of $\boldsymbol{M}$ do not have full length μ. Equation (4) can easily be verified by inserting the elements of $\boldsymbol{D}, \boldsymbol{u}$, and $\boldsymbol{v}$.

III. The Filter Representation of the TSVD Algorithm

Given an input signal vector $\boldsymbol{s}_{\mathrm{in}}$ of length N, consisting of a pure signal plus additive *white* noise, we first form the associated $m \times n$ Hankel matrix

$$\boldsymbol{H} = \mathcal{H}(\boldsymbol{s}_{\mathrm{in}}). \qquad (6)$$

To simplify our exposition, we assume that $m \geq n$, which is usually the case in signal processing applications. After computing the SVD $\boldsymbol{H} = \Sigma_{i=1}^n \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$, the next step is to approximate $\boldsymbol{H}$ by a rank-k matrix $\boldsymbol{H}_k$ with $k \leq n$. There are several possibilities here, and in a unified notation, we can write $\boldsymbol{H}_k$ as

$$\boldsymbol{H}_k = \sum_{i=1}^{k} w_i \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T, \qquad k \leq n. \qquad (7)$$

The *least squares* approximation is obtained with $w_i = 1, i = 1, \cdots, k$. The *minimum variance* approximation [1] is obtained with $w_i = 1 - \sigma_{\mathrm{noise}}^2 / \sigma_i^2, i = 1, \cdots, k$, where $\sigma_{\mathrm{noise}}^2$ is the variance of the white noise (σ_{noise} can, e.g., be estimated from the smallest singular values of $\boldsymbol{H}$). In addiiton, see [9] for more about these weights.

The final step is to compute the output signal vector $\boldsymbol{s}_{\mathrm{out}}$ of length N from $\boldsymbol{H}_k$. This is done by arithmetic averaging along the antidiagonals of $\boldsymbol{H}_k$, i.e., $\boldsymbol{s}_{\mathrm{out}} = \mathcal{A}(\boldsymbol{H}_k)$.

[1]This operation is identical to a convolution of the signals represented by $\boldsymbol{x}$ and $\boldsymbol{y}$.
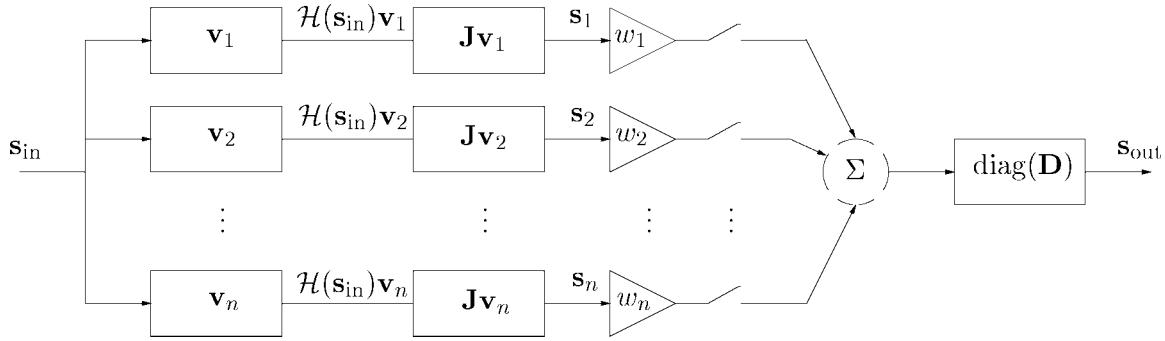
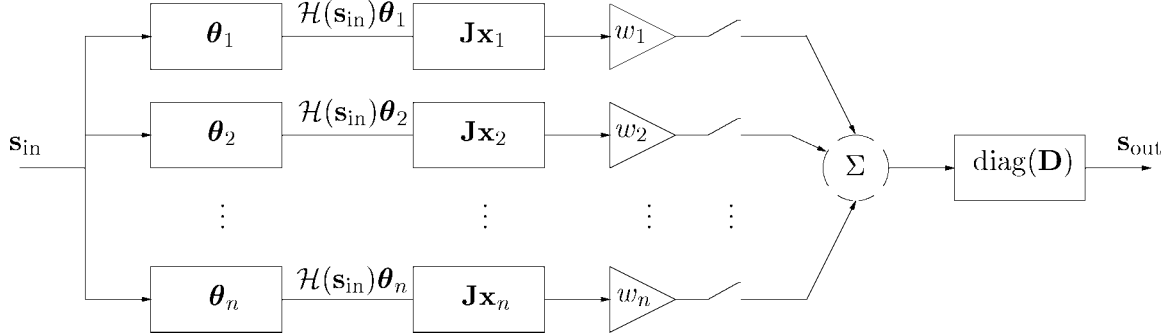Fig. 1. FIR filter representation of the TSVD algorithm.

Fig. 2. FIR filter representation of the TQSVD algorithm.

In order to derive our filter representation of the TSVD algorithm, we use (7) and the fact that averaging and summation are interchangeable

$$s_{\text{out}} = \mathcal{A}\left(\sum_{i=1}^{k} w_i \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T\right) = \sum_{i=1}^{k} w_i \mathcal{A}(\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T).$$

Since $\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$ is a rank-one matrix, we can now use (4) to obtain

$$s_{\text{out}} = \sum_{i=1}^{k} w_i \boldsymbol{D} \mathcal{T}_p(\sigma_i \boldsymbol{u}_i) \boldsymbol{v}_i.$$

Here, $\boldsymbol{D}$ is the constant matrix (5), and we use the identity $\boldsymbol{H}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ to obtain

$$s_{\text{out}} = \boldsymbol{D} \sum_{i=1}^{k} w_i \mathcal{T}_p(\boldsymbol{H}\boldsymbol{v}_i)\boldsymbol{v}_i = \boldsymbol{D} \sum_{i=1}^{k} w_i \mathcal{H}_p(\boldsymbol{H}\boldsymbol{v}_i)(\boldsymbol{J}\boldsymbol{v}_i)$$

where we used (3) in the last equation. By means of (6), we thus arrive at

$$s_{\text{out}} = \boldsymbol{D} \sum_{i=1}^{k} w_i \mathcal{H}_p(\mathcal{H}(\boldsymbol{s}_{\text{in}})\boldsymbol{v}_i)(\boldsymbol{J}\boldsymbol{v}_i). \qquad (8)$$

This equation defines the precise relation between the input vector s_{in} and the output vector s_{out}.

If we define the n intermediate signals s_i by

$$s_i = \mathcal{H}_p(\mathcal{H}(\boldsymbol{s}_{\text{in}})\boldsymbol{v}_i)(\boldsymbol{J}\boldsymbol{v}_i), \qquad i = 1, \cdots, n$$

then the output signal essentially consists of a weighted sum of the first k intermediate signals $s_1, \cdots, s_k$. For each s_i, we see that $\mathcal{H}(\boldsymbol{s}_{\text{in}})\boldsymbol{v}_i$ is a signal obtained by passing s_{in} through an FIR filter with filter coefficients $\boldsymbol{v}_i$, and s_i is a signal obtained by passing $\mathcal{H}(\boldsymbol{s}_{\text{in}})\boldsymbol{v}_i$ through an FIR filter with filter coefficients $\boldsymbol{J}\boldsymbol{v}_i$, i.e., the coefficients of the first filter in reverse order. It is well known that this results in a zero-phase filtered version of s_{in}.

From the above discussion, it is evident that the FIR filters $\boldsymbol{v}_i$ and $\boldsymbol{J}\boldsymbol{v}_i$ are *analysis* and *synthesis* filter pairs connected in parallel. Fig. 1 summarizes this interpretation of (8). The matrix $\boldsymbol{D}$ represents an N-point window whose elements are the diagonal elements of $\boldsymbol{D}$. For completeness, we have included all n filter branches corresponding to the n SVD components of $\boldsymbol{H}$ plus a switch in each branch. The TSVD output signal s_{out} is then obtained by closing the first k switches corresponding to the largest k singular values used in (7). Exact reconstruction of s_{in} is obtained with $k = n$ and all $w_i = 1$.

The FIR filters $\boldsymbol{v}_i$ are related to the *eigenfilters* whose coefficients are the elements of the eigenvectors of the $n \times n$ correlation matrix for s_{in}; see [5, Sec. IV.4]. As the signal length $N \to \infty$ and the number of columns n is fixed, and provided that the signal s_{in} is wide-sense stationary, the singular vectors $\boldsymbol{v}_i$ converge to the eigenvectors of the true $n \times n$ correlation matrix of the partial signals $[\boldsymbol{s}_{\text{in}}(\ell+1), \cdots, \boldsymbol{s}_{\text{in}}(\ell+n)]$ for any ℓ. Consequently, the FIR filters characterized by $\boldsymbol{v}_i$ converge to the eigenfilters associated with these length-n signals.

IV. The Filter Representation of the TQSVD Algorithm

If the noise in the input signal s_{in} is *colored*, then it is common to apply prewhitening and dewhitening in the TSVD algorithm. If e denotes the pure-noise component of s_{in} and if we compute the QR-factorization of the Hankel matrix associated with $e, \mathcal{H}(e) = \boldsymbol{Q}\boldsymbol{R}$, then $\mathcal{H}(e)^T\mathcal{H}(e) = \boldsymbol{R}^T\boldsymbol{R}$ is an estimate of the noise correlation matrix, and the prewhitened matrix that we work with is $\tilde{\boldsymbol{H}} = \boldsymbol{H}\boldsymbol{R}^{-1}$. We omit the case where $\boldsymbol{R}$ is singular. After truncation has been applied to $\tilde{\boldsymbol{H}}$, yielding the rank-k matrix $\tilde{\boldsymbol{H}}_k$, the dewhitened output signal is computed as $s_{\text{out}} = \mathcal{A}(\tilde{\boldsymbol{H}}_k \boldsymbol{R})$.

In [6], it was shown how the complete process can be formulated in a compact notation as well as in a simple algorithm based on the QSVD of matrix pairs. In the same paper, it was also shown that this algorithm can use the matrix pair $(\mathcal{H}(\boldsymbol{s}_{\text{in}}), \mathcal{H}(e))$ as input, thus avoiding the computation of $\boldsymbol{R}$, which is convenient when e is
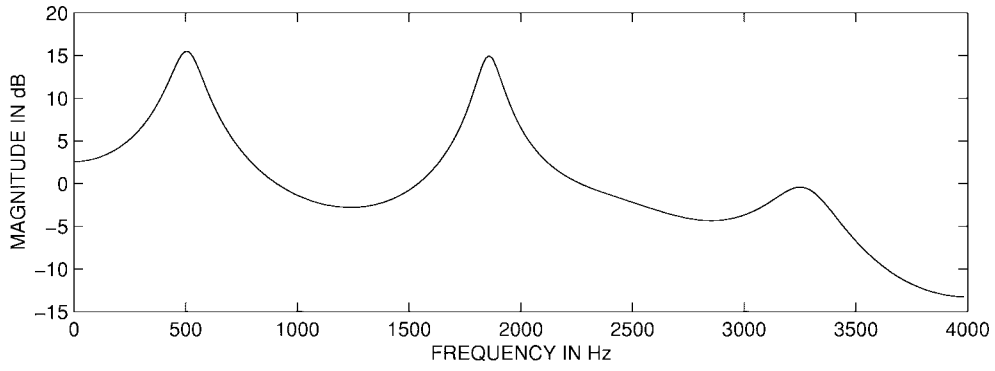
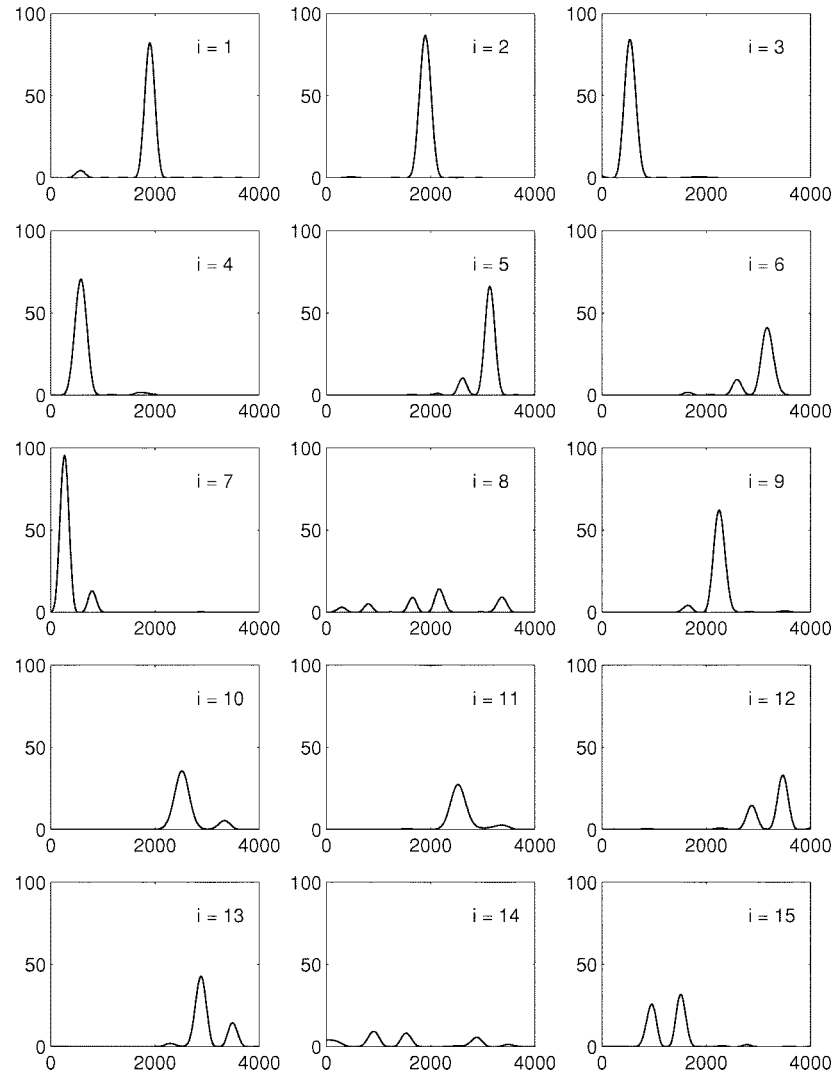Fig. 3.   LPC model spectrum of the pure pre-emphasized speech segment.

Fig. 4.   Frequency responses of the first 15 QSVD analysis/synthesis filters.

available. The resulting TQSVD algorithm [6] thus incorporates the prewhitening and dewhitening as an integral part of the algorithm.

The QSVD[2] of the matrix pair $(\mathcal{H}(s_{\text{in}}), \mathcal{H}(e))$ is given by

$$\mathcal{H}(s_{\text{in}}) = \sum_{i=1}^{n} \delta_i \hat{u}_i x_i^T \qquad (9)$$

[2] The QSVD is available in MATLAB Version 5.2 as the function gsvd.

$$\mathcal{H}(e) = \sum_{i=1}^{n} \mu_i \hat{v}_i x_i^T \qquad (10)$$

where $\hat{u}_i$ and $\hat{v}_i$ are orthonormal vectors, the vectors x_i are linearly independent, and $\delta_i^2 + \mu_i^2 = 1$ for $i = 1, \cdots, n$. The quotient singular values are δ_i/μ_i, and they satisfy $\delta_1/\mu_1 \geq \cdots \geq \delta_n/\mu_n \geq 0$. In the white-noise case, the QSVD yields the ordinary SVD of $H = \mathcal{H}(s_{\text{in}})$.

We shall not derive the TQSVD algorithm here but instead refer to [6]. Assuming that the noise is wide-sense stationary and that a

pure noise signal e is available, the algorithm can be summarized as follows.

1) Form the two Hankel matrices $\mathcal{H}(s_{\text{in}})$ and $\mathcal{H}(e)$.
2) Compute the QSVD of the pair $(\mathcal{H}(s_{\text{in}}), \mathcal{H}(e))$, cf. (9) and (10).
3) Form the rank-k matrix

$$Z_k = \sum_{i=1}^{k} w_i \delta_i \hat{u}_i x_i^T \qquad (11)$$

where $w_1, \cdots, w_n$ correspond to the TSVD weights in (7).
4) Compute the TQSVD output signal as $s_{\text{out}} = \mathcal{A}(Z_k)$.

The rank-k matrix approximation that corresponds to H_k in (7) is now given by (11). The weights w_i are computed by the same formulas as in the TSVD algorithm but with the singular values σ_i replaced by the quotient singular values δ_i/μ_i, and σ_{noise}^2 being the noise variance of the prewhitened signal.

Following exactly the same procedure as in Section III, we obtain an expression for $s_{\text{out}} = \mathcal{A}(Z_k)$ that leads to a filter representation for the TQSVD algorithm. The main difference is that we need an additional set of vectors $\theta_1, \cdots, \theta_n$ defined such that $(\theta_1, \cdots, \theta_n)^T = (x_1, \cdots, x_n)^{-1}$. The two sets of vectors are biorthonormal, i.e., $\theta_i^T x_j = 1$ for $i = j$ and zero otherwise. To derive our filter representation, we use the identity $H\theta_i = \delta_i \hat{u}_i$ and obtain

$$s_{\text{out}} = D \sum_{i=1}^{k} w_i \mathcal{H}_p(\mathcal{H}(s_{\text{in}})\theta_i)(Jx_i).$$

We have skipped the obvious details of the derivation. Fig. 2 summarizes our filter representation of the TQSVD algorithm.

The filters that arise in the TQSVD filter representation have filter coefficients θ_i and Jx_i, and they are not zero-phase filters as in the white-noise case. Perfect reconstruction is still obtained when $k = n$ and all $w_i = 1$.

V. AN EXAMPLE

We consider a segment s_{in} of voiced speech in additive colored noise. The speech signal was sampled at 8 kHz and filtered by a first-order FIR pre-emphasis filter $P(z) = 1 - 0.95z^{-1}$. Fig. 3 shows the *linear predictive coding* (LPC) model spectrum of the pure speech segment. The noise was generated such that its *power spectral density* (PSD) was

$$S(\omega) = \frac{1}{(1 + a^2) - 2a\cos(\omega T_s)}, \quad a = 0.5, \quad T_s = 125 \ \mu\text{s}$$

and scaled such that the signal-to-noise ratio (SNR) of the signal s_{in} was 15 dB. The segment length N was set to 160 samples corresponding to 20 ms, which is the typical block length in speech processing because this length is short enough for the segment to be nearly stationary. The size of both $\mathcal{H}(s_{\text{in}})$ and $\mathcal{H}(e)$ was set to 141×20.

Fig. 4 shows the frequency response of the first 15 combined analysis/synthesis filters (cf., Fig. 2) associated with the QSVD of $(\mathcal{H}(s_{\text{in}}), \mathcal{H}(e))$. It is clearly seen that the first six analysis/synthesis filters have bandpass characteristics and capture the three formants (i.e., the maxima in the LPC model spectrum) of the speech segment.

The remaining 14 filters (of which the first nine are shown in Fig. 4) capture those spectral components of the signal that lie in between the formants (including the noise), and these components are much less important when reconstructing the signal. In this way,

we achieve a good noise reduction of voiced speech sounds in the TQSVD method.

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The MDL Criterion for Rank Determination Via Effective Singular Values

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Abstract—Konstantinides and Yao have considered the problem of rank determination by use of effective singular values. In this correspondence, we show how to use the minimum description length criterion of Rissanen to provide an alternative means of estimating the index of the smallest nonzero singular value of a matrix when given estimates of the singular values.

I. INTRODUCTION

The subject of matrix rank determination by consideration of effective singular values has been considered by Konstantinides and Yao [1] and many others in various contexts. However, the approaches considered in [1] do not work well in such problems as

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