Data-Driven Methods: Theory and Practice

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State data

- Dynamic Mode Decomposition `pymor.algorithms.dmd.dmd (phdmd)`
- Operator Inference

System-invariant data

- AAA Algorithm `pymor.reductors.aaa.PAAAReductor`
- Loewner Framework `pymor.reductors.loewner.LoewnerReductor`
- Eigensystem Realization Algorithm `pymor.reductors.era.ERAReductor`
- QuadBT
- TF-IRKA `pymor.reductors.interpolation.TFBHIReductor`

Input-output data

- MOESP
- N4SID

State Equation — **continuous-time**

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

State Equation — **discrete-time**

$$
\begin{aligned}
Ex_{t+1} &= Ax_t + Bu_t \\
y_t &= Cx_t + Du_t
\end{aligned}
$$

State Equation (for dummies) **continuous-time**

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

State Equation (for dummies) **discrete-time**

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

State Equation (for dummies) — **continuous-time**

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

State Equation (for dummies) — **discrete-time**

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

Solution — **continuous-time**

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(t)d\tau$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(t)d\tau$$

Solution — **discrete-time**

$$x_t = A^t x_0 + \sum_{k=0}^{\infty} A^{t-k}Bu_k$$

$$y_t = CA^t x_0 + \sum_{k=0}^{\infty} CA^{t-k}Bu_k$$

State Equation (for dummies) continuous-time

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

State Equation (for dummies) discrete-time

$$\begin{aligned}x_{t+1} &= Ax_t + Bu_t\\ y_t &= Cx_t\end{aligned}$$

Solution continuous-time

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System Invariants

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Impulse response **continuous-time**

$$h(t) = Ce^{At}B$$

Impulse response **discrete-time**

$$h_t = CA^{t-1}B$$

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Impulse response — **discrete-time**

$$h_t = CA^{t-1}B$$

Transfer function — **continuous-time**

$$H(s) = C(sI - A)^{-1}B$$

Transfer function — **discrete-time**

$$H(z) = C(zI - A)^{-1}B$$

System Invariants

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Markov parameters — continuous-time

$$h_i = CA^{i-1}B = \frac{d^i}{dt^i}h(t)\bigg|_{t=0} = \frac{d^i}{ds^i}H(s)\bigg|_{s=\infty}$$

Markov parameters — discrete-time

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Markov parameters — **discrete-time**

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Hankel Operator **continuous-time**

$$\mathcal{H} : L_2^m(\mathbb{R}_-) \to L_2^p(\mathbb{R}_+),$$

where

$$y^+(t) = \int_{-\infty}^{0} h(t-\tau)u^-(\tau)d\tau$$

Hankel Operator **discrete-time**

$$\mathcal{H} : \ell_2^m(\mathbb{Z}_-) \to \ell_2^p(\mathbb{Z}_+),$$

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Hankel singular values

$$\sigma_k(\mathcal{H}) = \sqrt{\lambda_k(PQ)}$$

Hankel Matrix

$$\mathcal{H} = \begin{bmatrix} h_1 & h_2 & \cdots & \\ h_2 & h_3 & \cdots & \\ \vdots & \vdots & & \end{bmatrix} = \begin{bmatrix} CB & CAB & \cdots \\ CAB & CA^2B & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$= \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & \cdots \end{bmatrix} = \mathcal{O}\mathcal{C}$$

Hankel with care! — **continuous-time**

Hankel operator $\neq$ Hankel matrix

Hankel matrix — **discrete-time**

$$y^+ = \mathcal{H}u^- = \begin{bmatrix} h_1 & h_2 & \dots \\ h_2 & h_3 & \\ \vdots & & \ddots \end{bmatrix} u^-$$

Gramians — **continuous-time**

$$P = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$$

$$Q = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau$$

Gramians — **discrete-time**

$$P = \sum_{k=0}^\infty A^k B B^T {A^T}^k = \mathcal{C}\mathcal{C}^T$$

$$Q = \sum_{k=0}^\infty {A^T}^k C^T C A^k = \mathcal{O}^T \mathcal{O}$$

Recap Balanced Truncation (square-root method) — pyMOR: BTReductor

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,

$$P = SS^T, \quad Q = RR^T.$$

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2. Compute singular value decomposition

$$R^T S = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix}.$$

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3. Define

$$W := R Z_1 \Sigma_1^{-1/2}, \qquad V := S Y_1 \Sigma_1^{-1/2}.$$

4. Then the reduced order model is $(W^T A V, W^T B, C V)$.

$$
\begin{aligned}
\mathcal{H} &=
\begin{bmatrix}
h_1 & h_2 & \cdots & h_s \\
h_2 & h_3 & \cdots & h_{s+1} \\
\vdots & \vdots & \iddots & \vdots \\
h_s & h_{s+1} & \cdots & h_{2s-1}
\end{bmatrix} \\
&=
\begin{bmatrix}
CB & CAB & \cdots \\
CAB & CA^2B & \cdots \\
\vdots & \vdots &
\end{bmatrix}
=
\underbrace{\begin{bmatrix}
C \\
\vdots \\
CA^{s-1}
\end{bmatrix}}_{\mathcal{O}=U\Sigma^{1/2}}
\underbrace{\begin{bmatrix}
B & \cdots & A^{s-1}B
\end{bmatrix}}_{\mathcal{C}=\Sigma^{1/2}V^H}
= U\Sigma V^H \in \mathbb{R}^{ps \times ms}
\end{aligned}
$$

$$\mathcal{O}^f A = \begin{bmatrix} C \\ \vdots \\ CA^{s-2} \end{bmatrix} A = \begin{bmatrix} CA \\ \vdots \\ CA^{s-1} \end{bmatrix} = \mathcal{O}^l$$

$$A = \left(\mathcal{O}^f\right)^{\dagger} \mathcal{O}^l$$

$$B = \mathcal{C} \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} I_p & 0 \end{bmatrix} \mathcal{O}$$

Gramians **continuous-time**

$$P = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\imath\omega I - A)^{-1} B B^T (-\imath\omega I - A^T)^{-1} d\omega$$

$$Q = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\imath\omega I - A^T)^{-1} C^T C (\imath\omega I - A)^{-1} d\omega$$

Gramians **discrete-time**

$$P = \sum_{k=0}^{\infty} A^k B B^T A^{T^k} = \frac{1}{2\pi} \int_0^{2\pi} (e^{\imath\omega} I - A)^{-1} B B^T (e^{-\imath\omega} I - A^T)^{-1} d\omega$$

$$Q = \sum_{k=0}^{\infty} A^{T^k} C^T C A^k = \frac{1}{2\pi} \int_0^{2\pi} (e^{-\imath\omega} I - A^T)^{-1} C^T C (e^{\imath\omega} I - A)^{-1} d\omega$$

Questions?