DIP Homework 3

3.28

(a)

- By the convolution theorem, Fourier transform the convolution of 2 functions in the space domain equals to the product in the frequency domain of the Fourier transforms of the two functions.
- The Fourier transform of a Gaussian

$$g\left(x\right) = N\left(x; \mu, \sigma^2\right)$$
$$F\left\{g\left(x\right)\right\} = F_g\left(\omega\right) = exp\left[-j\omega\mu\right] exp\left[-\frac{\sigma^2\omega^2}{2}\right]$$

- Therefore, given 2 Gaussian function g_1, g_2

$$\begin{aligned} g_{12}\left(x\right) &= g_1 \star g_2 \\ &= F^{-1}\left\{F\left\{g_1\right\} \cdot F\left\{g_2\right\}\right\} \\ &= F^{-1}\left\{exp\left[-j\omega\mu_1\right] exp\left[-\frac{\sigma_1^2\omega^2}{2}\right] exp\left[-j\omega\mu_2\right] exp\left[-\frac{\sigma_2^2\omega^2}{2}\right]\right\} \\ &= F^{-1}\left\{exp\left[-j\omega\left(\mu_1 + \mu_2\right)\right] exp\left[-\frac{\left(\sigma_1^2 + \sigma_2^2\right)\omega^2}{2}\right]\right\} \\ &= N\left(x; \mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right) \end{aligned}$$

- By the equations above, the convolution of 2 Gaussian functions is still a Gaussian function

(b)

- We denote the composite filter's standard deviation as σ'

$$\begin{aligned} \sigma' &= \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \\ &= \sqrt{1.5^2 + 2^2 + 4^2} = \frac{\sqrt{89}}{2} \end{aligned}$$

(c)

- Kernel size changes
  - 3x3 and 5x5 convolution results into 7x7 kernel

- 7x7 and 7x7 convolution results into 13x13 kernel
- The size of the final filter is 13x13

3.38

- Convolution has associativity property, the following is the proof

$$(f \star g)(t) = \int_0^t f(s)g(t-s)ds$$

$$\begin{aligned}
((f \star g) \star h)(t) &= \int_0^t (f \star g)(s)h(t-s)\,ds \\
&= \int_{s=0}^t \left(\int_{u=0}^s f(u)g(s-u)\,du \right) h(t-s)\,ds \\
&= \int_{s=0}^t \int_{u=0}^s f(u)g(s-u)h(t-s)\,du\,ds \\
&= \int_{u=0}^t \int_{s=u}^t f(u)g(s-u)h(t-s)\,ds\,du \\
&= \int_{u=0}^t \int_{s=0}^{t-u} f(u)g(s)h(t-s-u)\,ds\,du \\
&= \int_{u=0}^t f(u) \left(\int_{s=0}^{t-u} g(s)h(t-u-s)\,ds \right) du \\
&= \int_{u=0}^t f(u)(g \star h)(t-u)\,du \\
&= (f \star (g \star h))(t)
\end{aligned}$$

- Therefore, the result should be the same no matter the order of smoothing and Laplacian operations is.

4.2

- Apply Fourier transform to $f(t)$

$$\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\,dt \\
&= \int_0^T A e^{-j2\pi\mu t}\,dt \\
&= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_0^T \\
&= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu T} - 1\right] = \frac{A}{j2\pi\mu}\left[1 - e^{-j2\pi\mu T}\right] \\
&= \frac{A}{j2\pi\mu}\left[1 - \cos(2\pi\mu T) + j\sin(2\pi\mu T)\right]
\end{aligned}$$

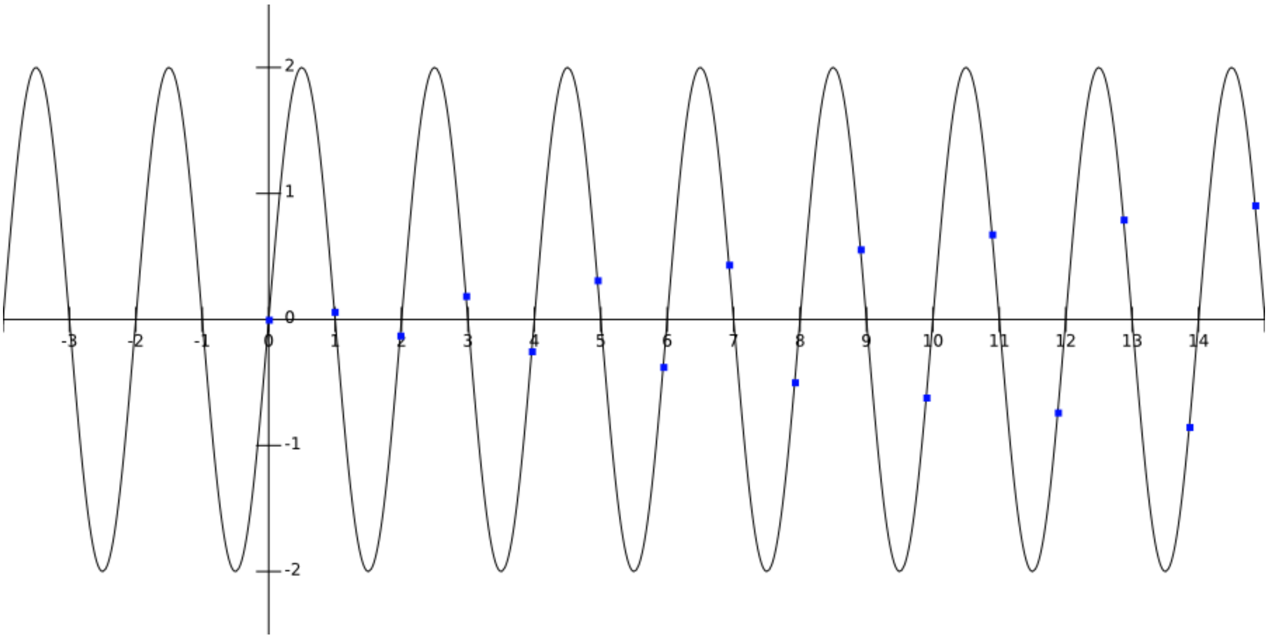
- The result has an additional term

$$\frac{A}{j2\pi\mu}[1-\cos(2\pi\mu T)]$$

4.6

(a)

- Sampling rate exceeds the Nyquist rate slightly.
  - function period = 2, function frequency = 0.5 Hz
  - sampling period = 0.99, sampling frequency ≈ 1.01 Hz (slightly > 1 Hz)



(b)

- The sampling rate in figure 4.11 is slightly less than 0.5 Hz

(c)

- I would choose the sampling rate 1 Hz

4.17

(a)

- Define $F(u)$ is the M point DFT of $f(x)$

$$F(u)=DFT\{f(x)\}=\sum_{x=0}^{M-1}f(x)\,e^{-j2\pi\mu x/M}dx$$

- We can obtain

$$
\begin{aligned}
DFT\left\{f(x)\,e^{j2\pi\mu_0 x/M}\right\} &= \sum_{x=0}^{M-1}\left[f(x)\,e^{j2\pi\mu_0 x/M}\right]e^{-j2\pi\mu x/M}\,dx \\
&= \sum_{x=0}^{M-1} f(x)\,e^{j2\pi(\mu_0-\mu)x/M}\,dx \\
&= F(\mu_0-\mu)
\end{aligned}
$$

(b)

- Define $f(x)$ is the M point IDFT of $F(u)$

$$
f(x) = IDFT\{F(u)\} = \frac{1}{M}\sum_{u=0}^{M-1} F(u)\,e^{j2\pi\mu x/M}\,du
$$

- We can obtain

$$
\begin{aligned}
f(x-x_0) &= \frac{1}{M}\sum_{u=0}^{M-1} F(u)\,e^{j2\pi\mu(x-x_0)/M}\,du \\
&= \frac{1}{M}\sum_{u=0}^{M-1} F(u)\,e^{-j2\pi\mu x_0/M}\,e^{j2\pi\mu x/M}\,du \\
&= \frac{1}{M}\sum_{u=0}^{M-1}\left[F(u)\,e^{-j2\pi\mu x_0/M}\right]e^{j2\pi\mu x/M}\,du \\
&= IDFT\left\{F(u)\,e^{-j2\pi\mu x_0/M}\right\}
\end{aligned}
$$

- Apply DFT on both sides

$$
\begin{aligned}
DFT\{f(x-x_0)\} &= DFT\left\{IDFT\left\{F(u)\,e^{-j2\pi\mu x_0/M}\right\}\right\} \\
&= F(u)\,e^{-j2\pi\mu x_0/M}
\end{aligned}
$$