# M205 Topology and Groups Notes

## Based on the 2013 autumn lectures by Dr H Wilton

Topology and Groups 1st October 2013

weeks 1–2 : Introduction to topology
weeks 3+ : The fundamental group, covering spaces etc.

Alex Cioba: Problem classes, time has yet to be set.

Definition

A topological space is a pair $(X, \tau)$ where $X$ is some set and $\tau \subseteq P(X)$ ($P(X)$ is the set of all subsets of $X$), where $\tau$ satisfies the following conditions:

- $\{\emptyset, X\} \subseteq \tau$
- $\tau$ is closed under arbitrary unions: $\{U_\alpha : \alpha \in \Lambda\} \subseteq \tau$ $\Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau$.
- $\tau$ is closed under finite intersections: $U, V \in \tau \Rightarrow U \cap V \in \tau$.

$\tau$ is called a topology on $X$, and the $U \in \tau$ are called open sets.

A subset $F \subseteq X$ is called closed if $X \setminus F$ is open.

Example 1

A subset $U \subseteq \mathbb{R}^n$ is called open if $\forall x \in U$, $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.

Exercise: this defines the "usual" or "standard" topology on $\mathbb{R}^n$.

The same definition works in metric spaces.

Example 2

For any set $X$, take $\tau = P(X)$. This is called the discrete topology.

Every point is separated from every other point. – It is the largest possible topology. . . . . . . . . .

Example 3

For any set $X$, we may take $\tau = \{\emptyset, X\}$. This is the smallest possible topology. It is called the indiscrete topology.

→ "the topology doesn't see the difference between any of the points in the space!"

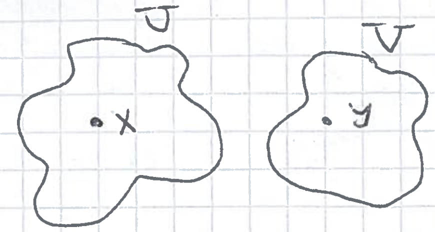
→ A very bad non-metric space topology!

Definition

A topological space $(X, \tau)$ is called Hausdorff if whenever

$x \neq y \in X$, $\exists$ open sets $U \ni x$, $V \ni y$, such that:

$U \cap V = \emptyset$




(This stops "the bad" topologies from happening)

→ Metric space topologies are Hausdorff. (This is left as an exercise.

→ Here, we would say that $U$ is an open neighbourhood, or just neighbourhood, of $x$.

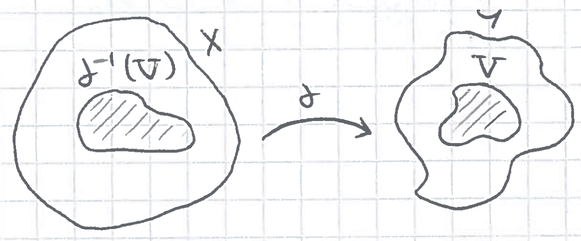
→ $U$ and $V$ are called disjoint ($U \cap V = \emptyset$)

**Definition**

Let $(X, \tau)$ and $(Y, \sigma)$ are topological spaces. A map $f$ $f: X \to Y$ is called continuous if:

$\forall V \in \sigma, \; f^{-1}(V) = \{x \in X : f(x) \in V\} \in \tau$

EXERCISE: $X$ and $Y$ are metric spaces, then this definition coincides with the usual one.

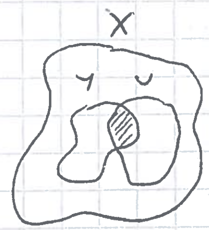



**Definition**

Let $(X, \tau)$ and $(Y, \sigma)$ are topological spaces. Suppose $f: X \longrightarrow Y$ is a bijection, and $f$ and $f^{-1}$ are both continuous. Then $f$ is called a homeomorphism, and $(X, \tau)$ and $(Y, \sigma)$ are homeomorphic.

→ this is the notion of "sameness" for topological spaces.
→ this is a "bendy notion of sameness", there are other, more rigid notions of "sameness" : isometries in metric spaces for example.

**Definition**




Let $(X, \tau)$ be a topological space and $Y \subseteq X$. Then:

$\sigma = \{U \cap Y : U \in \tau\}$

forms a topology on $Y$ (or induces it) called the subspace topology. $Y$ is called a subspace of $X$.

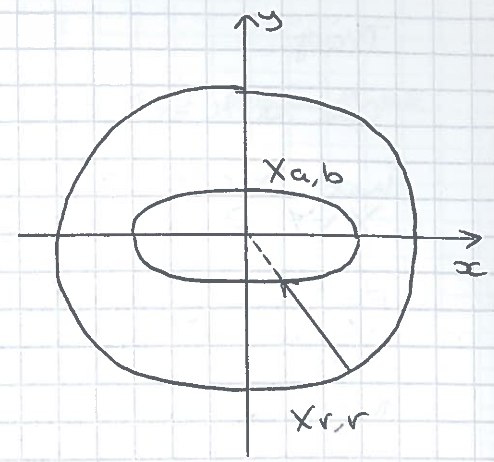
**Example**

Let $a, b > 0$.

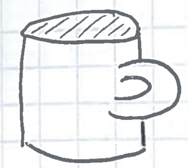
$X_{a,b} = \{(x,y) \in \mathbb{R}^2 : (\frac{x}{a})^2 + (\frac{y}{b})^2 = 1\}$ equipped with the subspace topology from $\mathbb{R}^2$.

Then $\forall a, b$, $X_{a,b}$ is homeomorphic to $X_{1,1}$ : the standard circle. The proof is as follows:

Scale $X_{1,1}$ to $X_{R,R}$ for $R \gg a, b$.

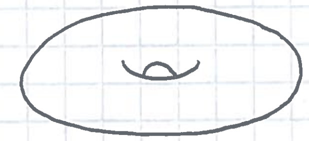



**Classic example**



coffee mug

$\cong$ homeomorphic



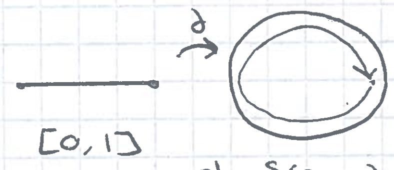
Donut

**Cautionary example**

$X = [0,1)$

Define $f$ where:

$[0,1) \to \; t \longmapsto (\cos 2\pi t, \sin 2\pi t)$




$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

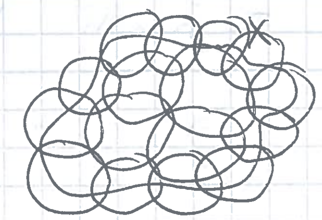
$f$ is a continuous bijection, but not a homeomorphism. The problem is that $f^{-1}$ is not continuous

**Compactness**

**Definition**

Let $(X, \tau)$ be a topological space. An open cover of $X$ is a subset $\mathcal{U} \subseteq \tau$ such that $\forall x \in X$, $\exists U \in \mathcal{U}$ such that $x \in U$

**Definition**




$(X, \tau)$ is compact if every open cover has a finite subcover.

Equivalently:
$\forall$ open covers $\mathcal{U}$ $\exists$ a finite $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{U}'$ is an open cover of $X$.

EXERCISE: For $(X, \tau)$ discrete or indiscrete, when is $(X, \tau)$ compact?

**Example**

Which subsets of $\mathbb{R}^n$ are compact?

1. Suppose $X \subseteq \mathbb{R}^n$ is unbounded. Then let $U_n = X \cap B(0,n)$. Now $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover without a finite subcover
   $\therefore$ compact subsets of $\mathbb{R}^n$ are bounded.

2. **Lemma**

   If $(X, \tau)$ is a Hausdorff topological space and $Y \subseteq X$ with the subspace topology is compact then $Y$ is closed.

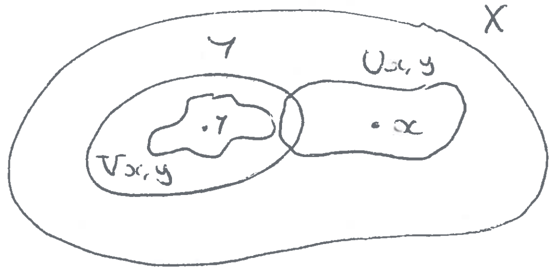
Proof

we need to prove that $X \setminus Y$ is open.

Let $x \in X \setminus Y$. we need to find an open neighbourhood $X \setminus Y \supseteq U_x \ni x$. (i.e. $U_x \cap Y = \emptyset$)

For every $y \in Y$, $\exists$ open neighbourhoods $U_{x,y} \ni x$, $V_{x,y} \ni y$ which are disjoint (because $X$ is Hausdorff)

The set $\{V_{x,y} : y \in Y\}$ is an open cover of $Y$. By compactness, it has a finite subcover:




$$\{V_{x,y_1}, \dots, V_{x,y_n}\}$$

Such that:

$$Y \subseteq \bigcup_{i=1}^{n} V_{x,y_i}$$

Let $U_x = \bigcap_{i=1}^{n} U_{x,y_i}$ Now $U_x \cap Y \subseteq \bigcup_{i=1}^{n} U_x \cap V_{x,y_i}$

$$\subseteq \bigcup_{i=1}^{n} U_{x,y_i} \cap V_{x,y_i} = \emptyset$$

as so, $U_x \cap Y = \emptyset$. $U_x$ is an intersection of finitely many open sets, hence open.

To finish: $X \setminus Y = \bigcup_{x \in X \setminus Y} U_x$

$\Rightarrow$ $X \setminus Y$ is open
$\Rightarrow$ $Y$ is closed $\square$

Answer to question.

compact subsets of $\mathbb{R}^n$ must be bounded and closed.

Theorem (Heine-Borel)

A subspace $X \subseteq \mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof

$\Rightarrow$ done.

$\Leftarrow$ Lemma

$[0,1]^n \subseteq \mathbb{R}^n$ is compact

Proof by induction on $n$.

$n=0$, by convention, for any $X$, $X^0 = \{pt\} = *$

This space is obviously compact.

Inductive step: assume that $[0,1]^n$ is compact and prove that $[0,1]^{n+1}$ is compact.

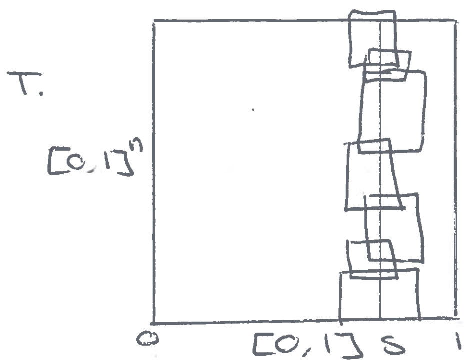
Let $\mathcal{C}$ be an open cover of $[0,1]^{n+1}$

A cube is an open set of the form $(a_1, b_1) \times \dots \times (a_{n+1}, b_{n+1})$

Let's replace every open set in $\mathcal{C}$ with every cube that it contains. No harm done!

Let $T = \{t \in [0,1] : [0,1]^n \times [0,t]$ is contained in a finite subcover of $\mathcal{C}\}$

we'll prove the theorem by showing that $1 \in T$.




(a) $0 \in T$ by induction.
$T \neq \emptyset \Rightarrow s = \sup T$

(b) $s \in T$.
By induction, $\exists$ cubes $C^k = (a_1^k, b_1^k) \times \dots \times (a_{n+1}^k, a_{n+1}^k)$, $k = 1, \dots, m$, such that:

$$[0,1]^n \times \{s\} \subseteq \bigcup_{k=1}^{m} C^k$$

Let $a_{n+1}^k < t < s$ for all $k$. Then $[0,t]$ has a finite cover $\{U_1, \dots, U_p\} \subseteq \mathcal{C}$. Then $\{U_1, \dots, U_p, C_1, \dots, C_m\}$ covers $[0,1]^n \times [0,s]$
$\Rightarrow$ $s \in T$.

(c) $s = 1$
Suppose $s < 1$. choose $s < t < b_{n+1}^k$ $\forall k$. Then we see that $t \in T$, this contradicts $s = \sup T$ ⁂
$\Rightarrow$ $s = 1$.

This proves the lemma $\square$

$\Rightarrow$ $[0,1]^n$ is compact.
$\Rightarrow$ any cube $[-a,a]^n \subseteq \mathbb{R}^n$ is compact.

Lemma

if $(X, \mathcal{T})$ is compact and $Y \subseteq X$ is closed then $Y$ is compact.

Proof

Let $\mathcal{C}$ be an open cover of $Y$. Then $\mathcal{C} \cup \{X \setminus Y\}$ is an open cover of $X$, hence is has a finite subcover $\square$.

If $X \subseteq \mathbb{R}^n$ is any closed bounded subset, then $X \subseteq [-a,a]^n$ for some $a$. $X$ is a closed subset of a compact set.

$\Rightarrow$ $X$ is compact. $\square$

## Lemma

If $f: X \longrightarrow Y$ is a continuous surjection and $X$ is compact, then so is $Y$.

### Proof

Suppose $\mathcal{U}$ is an open cover of $Y$. Then:

$$\{f^{-1}(U): U \in \mathcal{U}\}$$

is an open cover of $X$, so has a finite subcover:

$$\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$$

$\Rightarrow (U_1, \dots, U_n)$ is a finite cover of $Y$. □

## Back to example

$S^1$ is compact (a bounded, closed space)

$[0,1)$ is not compact as it is not closed.

$\Rightarrow$ $S^1 \not\cong [0,1)$ by the above lemma, using $f: X \to Y$ where $t \longmapsto (\cos 2\pi t, \sin 2\pi t)$

Biting

Augmentation ideal (in examples).

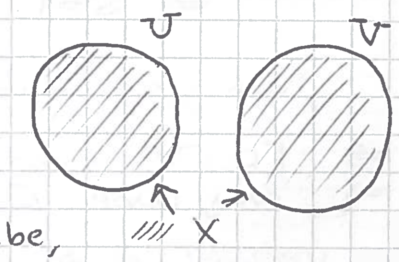
What about $[0,1]$? Is $[0,1] \cong S^1$?
- they are both compact.
- we need another idea:

## Connectedness

### Definition

Let $(X, \tau)$ be a topological space. It is not connected if $\exists$ non-empty, open, disjoint subsets $U, V \in \tau$ such that $U \cup V = X$.

We say that $U, V$ disconnect $X$. Otherwise we say that $X$ is connected.




EXERCISE: when $X$ is discrete, or indiscrete, when is $X$ connected?

### Example

Let $X = [0,1] \cap \mathbb{Q} \subseteq \mathbb{R}$, with the subspace topology.

Then $U = [0, \frac{1}{\pi}) \cap \mathbb{Q}$, $V = (\frac{1}{\pi}, 1] \cap \mathbb{Q}$ disconnect $X$.

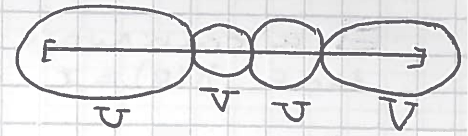
## Lemma

A closed interval $[a,b] \subseteq \mathbb{R}$ is connected. (Put $I = [a,b]$)

### Proof

Suppose $U, V$ disconnect $I$.

Without loss of generality: $a \in U$, $b \in V$




Let $T = \{t \in I : [a,t] \subseteq U\}$.
Clearly $a \in T$, let $s = \sup T$.

We will prove that $s \in U \cap V$, which will provide a contradiction.

For all $t < s$, $[0,t] \subseteq U \Rightarrow t \in U$. But $V$ is open
$\Rightarrow U = I \setminus V$ is closed.
$\Rightarrow s \in U$.

~~Similarly, for all $s < t < s+\varepsilon$, must have $t \in V$.~~

$\forall n \in \mathbb{N}$, $\exists t_n \in [s, s+\frac{1}{n}] \cap V$. Otherwise $[a, s+\frac{1}{n}] \subseteq U$, contradicting that $s = \sup T$.

Now $s = \lim_{n\to\infty} t_n$. But $V = I \setminus U$ is closed $\Rightarrow s \in V$

$\Rightarrow s \in U \cap V$. ※ as $U$ and $V$ are disjoint. □

## Lemma

If $f: X \longrightarrow Y$ is continuous and surjective and $(X, \tau)$ is connected, then $(Y, \sigma)$ is connected.

### Proof

Suppose $U$ and $V$ disconnect $Y$. Then $f^{-1}U$ and $f^{-1}V$ disconnect $X$. □

## Theorem (The Intermediate Value Theorem)

If $f: [a,b] \longrightarrow \mathbb{R}$ is continuous and $f(a) \leq \lambda \leq f(b)$, then $\exists$ $a \leq c \leq b$ such that $f(c) = \lambda$.

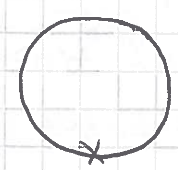
### Proof

$f([a,b])$ is connected. If no $c$ exists, then $f^{-1}([-\infty, \lambda))$ and $f^{-1}((\lambda, \infty])$ disconnect $[a,b]$. □

## Return to question

$[0,1] \cong S^1$?

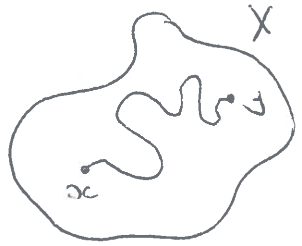
 – not connected ($[0,1] \setminus \{pt\}$)

 – connected ($S^1 \setminus \{pt\}$)

### Definition

A space $(X, \tau)$ is called path connected if $\forall x, y \in X$

$\exists$ a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.




$\gamma$ is called a path.

## Lemma

If $(X, \tau)$ is path connected, then its connected.

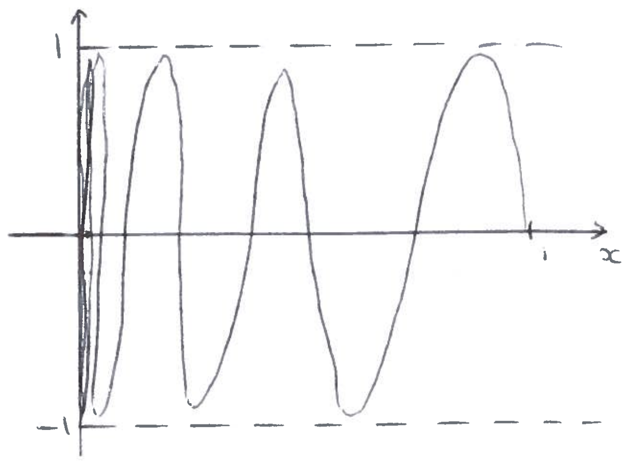
### Proof

Suppose $U$ and $V$ disconnect $(X, \tau)$. Let $x \in U$ and $y \in V$. Let $\gamma$ be a path from $x$ to $y$.

Now $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ will disconnect $[0,1]$.
$\Rightarrow [0,1]$ is not connected $\times$ $\square$

## Example (Topologist's sine curve)

Consider $X = \{(0,0)\} \cup \{(x, \sin\frac{\pi}{x}) : x \in (0,1]\} \subseteq \mathbb{R}^2$ with the subspace topology.




This is connected, but it is NOT path connected.

Therefore, it isn't always the case that connectedness $\Rightarrow$ path connectedness.

## Lemma

$X$ is connected, compact and locally path connected, then $(X, \tau)$ is path connected.

(A space is locally path-connected if every point has an open neighbourhood which is path-connected)

### Proof

Omitted. $\square$

## Sequences and limits

### Definition

Let $(X, \tau)$ be a topological space, $(x_n) \subseteq X$ is a sequence.

A point $x \in X$ is a limit of the sequence $x_n$ if for every open neighbourhood $U \ni x$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n \in U$.

### Example

Suppose $X$ is indiscrete. Then any point is a limit of any sequence, as there is only one open neighbourhood.

– be cautious of limits and sequences!

## Lemma

If $(X, \tau)$ is Hausdorff, then limits are unique, and we may write that:

$$x = \lim_{n \to \infty} x_n.$$

### Proof

Exercise. $\square$

## Problem class

8th October 2013

Alex Ciobo: Problem class (1-2pm Thursday)

### Goal for today

Build lots of new interesting spaces.

## Disjoint Union

### Definition

Suppose $(X_1, \tau_1)$ and $(X_2, \tau_2)$ are topological spaces and $X_1 \cap X_2 = \emptyset$.

The disjoint union, denoted $(X_1 \sqcup X_2, \sigma)$ is the topological space with underlying set $X_1 \cup X_2$ and topology defined by:

$$X_1 \cup X_2 \supseteq V \in \sigma \Leftrightarrow X_1 \cap V \in \tau_1 \text{ and } X_1 \cap V \in \tau_2$$

This is given diagramatically as follows:


//// is $X_1 \sqcup X_2$.

REMARK: we can do this for any indexed set $(x_i : i \in I)$.

This disjoint union is denoted:

$$\coprod_{i \in I} X_i.$$

## Quotient topology

### Definition

Let $(X, \tau)$ be a topological space, and let $\sim$ be some equivalence relation on $X$.

$$X \longrightarrow X/\sim \; = \{\text{equivalence classes}\}$$
$$x \longmapsto [x]$$

This is called the quotient map, call it $q$.

The quotient topology on $X/\sim$ is defined as follows:

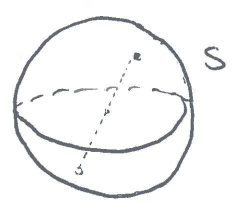
$U \subseteq X/\sim$ is open if and only if $q^{-1}U$ is open in $X$.
(this is the topology on the quotient space with the most open sets such that $q$ is continuous).

EXERCISE: If $f: X \longrightarrow Y$ and $Y$ is discrete, when is $f$ continuous? what about when $Y$ is indiscrete?

Example: the $n$-sphere.

Let us consider $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$, the "$n$-sphere".

For $x, y \in S^n$, then $x \sim y$ if and only if $x = \pm y$.




Define $\mathbb{RP}^n = S^n/\sim$ : "the set of lines through the origin in $\mathbb{R}^{n+1}$!" we can now give this space a quotient topology.


what does $\mathbb{RP}^0$ look like?

$S^0$: $\cdot$ $\cdot$ ($-1$, $1$) $\quad 1 \sim -1 \Rightarrow \mathbb{RP}^0$ is: $\cdot$

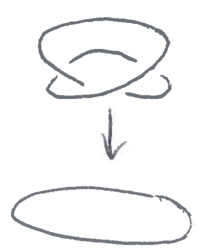
$\cdot \quad \cdot \; S^0$

what about $\mathbb{RP}^1$?

$S^1$:  $\mathbb{RP}^1$: 

$\Rightarrow \mathbb{RP}^1$ is a circle. consider the quotient map:

Let $q: S^1 \longrightarrow \mathbb{RP}^2$
$q$ wraps $S^1$ twice around $\mathbb{RP}^1$ as follows:

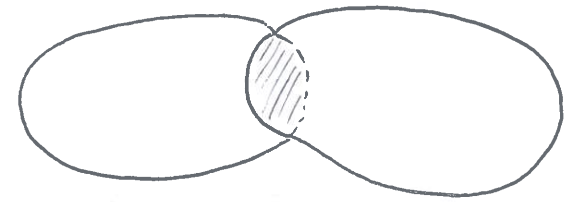


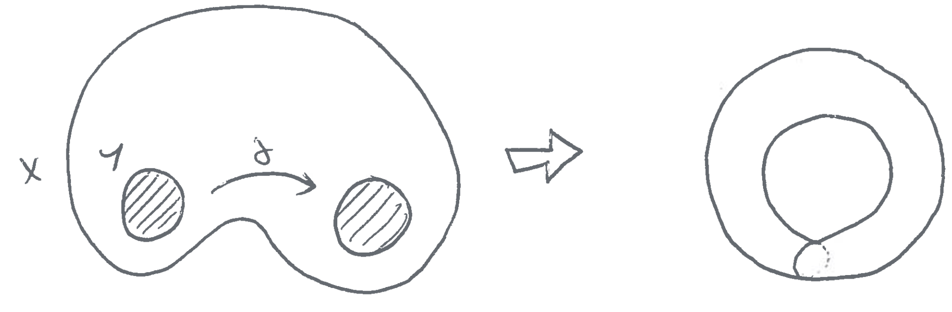
what about $\mathbb{RP}^2$? it cannot be embedded into space in $\mathbb{R}^3$. Northern hemisphere equivalent to southern hemisphere. what about the equator?

+ + + are identified with $\cdot \; \cdot \; \cdot$.

## Glueing

$f: Y \to X$




### Definition

Let $X$ be a topological space and $Y \subseteq X$ be a subspace (a subset with the induced topology). We can glue $X$ to itself along $Y$ as follows:

Define $\sim$ on $X$ to be the smallest equivalence relation such that $y \sim f(y)$ $\forall y \in Y$.

Then $X \cup_f = X/\sim$. As long as $Y \cap f(Y) = \emptyset$, this is fairly well behaved: $x \sim x' \Leftrightarrow$ either $x = x'$ or $x \in Y$, $x' = f(x)$, or $x' \in Y$, and $x = f(x')$.

If $X = X_1 \sqcup X_2$, and $Y \subseteq X_1$, $f(Y) \subseteq X_2$, then we write:

$$X \cup_f = X_1 \cup_f X_2.$$

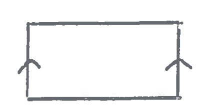
### Example

Let $X =$  $= [-1,1]^2 = \mathbb{R}^2$.

i) 

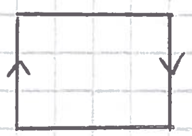

$Y = \{-1\} \times [-1,1]$, $f: (-1, y) \longmapsto (1, y)$

what is $X \cup_f$? A cylinder!


also denoted: 

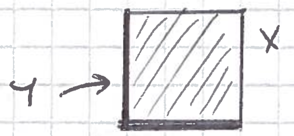
NOTE: the above notation for a cylinder by identifying sides.

2) $X, Y$ is the same: but $f(-1,y) = (1, 1-y)$. This gives:

The mobius strip!

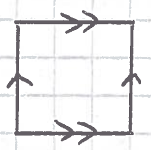
3) $X$ is the same. $Y = \{-1\} \times [-1,1] \cup [-1,1] \times \{-1\}$

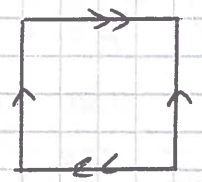



$f(-1,y) = (1,y)$

$f(x,-1) = (x,1)$

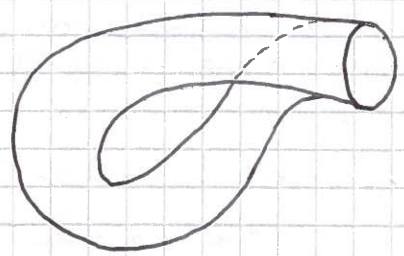
This gives the torus: (or a ring doughnut!)

4) 

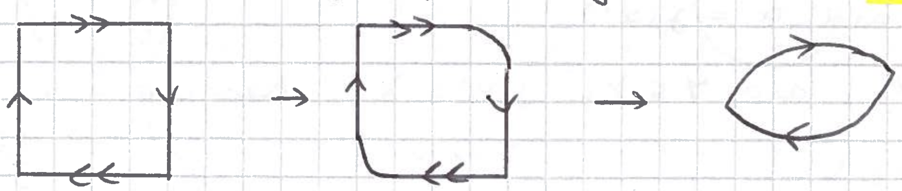
Same gluing as the torus, but sides switched.

This is called the klein bottle:



Choking
Pushout | Pullback

5) one obvious thing left to try:



This is $\mathbb{R}P2$!

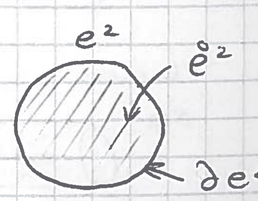
## Cell complexes

### Definition

The n-cell or n-disk is the topological space

$e^n = \{x \in \mathbb{R}^n : \|x\|_2 \le 1\}$

(these are "closed" n-cells).




Then the "open" n-cell is $\mathring{e} = \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$

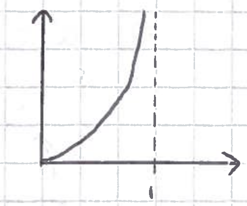
Then $\partial e^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$

NOTE: Topologically, these are fairly boring: $\mathring{e}^n \cong \mathbb{R}^n$.

Sketch proof

$[0,1) \cong [0,\infty)$

$t \longmapsto \tan(\frac{\pi}{2}t)$ provides a homeomorphism.



Now: define $\mathring{e}^n \longrightarrow \mathbb{R}^n$

$$x \longmapsto \begin{cases} \tan(\frac{\pi}{2}\|x\|)\cdot\frac{x}{\|x\|} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Provides homeomorphism. $\triangle$

NOTE: $\partial e^n \cong S^{n-1}$. Also, a 0-cell is a point.

### Definition

A cell complex (or CW-complex) is a topological space $X$ defined inductively as follows:

- $n=0$ $X^{(0)}$ = a disjoint union of 0-cells: $\{e_i^0 : i \in I_0\}$

$\Rightarrow X^{(0)} = \coprod_{i \in I} e_i^0$ usually: $|I_0| < \infty$.

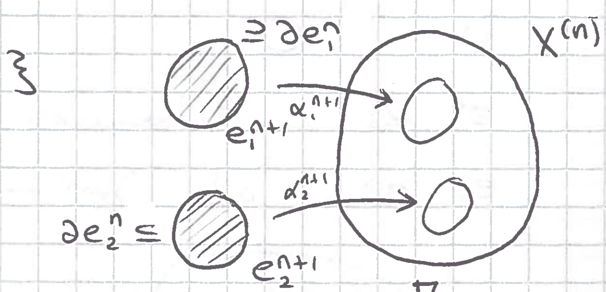
(There are a couple of instances in this case where it will be convenient to think of $e_i$ as (countably) infinite.

$X^{(0)}$ is called the 0-skeleton.

- Suppose we have constructed the n-skeleton $X^{(n)}$. We will define $X^{(n+1)}$ as follows:

Given $(n+1)$-cells: $\{e_i^{n+1} : i \in I_{n+1}\}$ and its map:

$\alpha_i^{n+1} : \partial e_i^n \to X^{(n)}$




(called the "attaching map").

glue them in to get:

Define: $X^{(n+1)} = X^{(n)} \cup_A \coprod_{i \in I} e_i^{n+1}$

where $A : \coprod_{i \in I_{n+1}} \partial e_i^{n+1} \to X^{(n)}$

$\partial e_i^{n+1} \ni x \longmapsto \alpha_i^{n+1}(x)$.

i.e. glues the cells $e_i^{n+1}$ to $X^{(n)}$ using the attaching maps.

- Take $X = \bigcup_n X^{(n)}$.

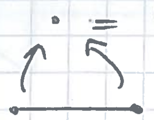
One technical requirement: the image of each $\alpha_i$ should intersect at most finitely many cells in $X^{(n)}$.

Example

$e^0, e^1$, $\alpha^1 : \partial e^1 \longrightarrow X^{(0)} = \{e^0\}$

$\Rightarrow \alpha^1_i : \partial e^1_i \longrightarrow X^{(0)}$ : only one possible map.

$\bullet = X^{(0)}$



$= e^1_i$

$\Downarrow$

$\Rightarrow X \cong S^1$.

There is a homeomorphism that can say this explicitly.

2) $e^0_i$, $e^1_i$, $\alpha^1_i : \partial e^1_i \to X^{(0)}$ (maps everything to a point.

$e^2_i$, $\alpha^2_i : \partial e^2_i \longrightarrow X^{(1)}$ is a homeomorphism (they're both circles).

$X^{(1)} =$  $e^2_i =$ 

we glue $e^2_i$ onto $X^{(1)}$ to get a disk.

$\longrightarrow X^{(2)}$ is a disk.

3) The Spheres $S^0, S^1, S^2, \dots$

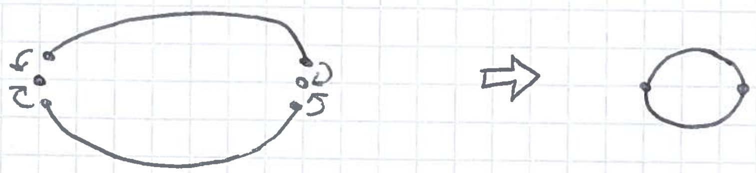
$S^0 = \bullet \quad \bullet$

$\rightarrow$ an easy cell composition of this is $e^0_1, e^0_2$.

$S^1 =$ 

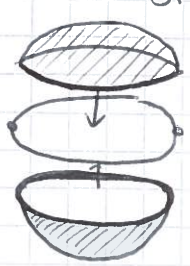
$e^0_1, e^0_2$ : 0-skeleton $e^0_1 \sqcup e^0_2 \cong S^0$

$e^1_1, e^1_2$ : $\alpha^1_i : \partial e^1_i \longrightarrow e^0_1 \sqcup e^0_2$ is a homeomorphism.



This is not the smallest possible one, but it is convenient!

$S^2$ :



$e^0_1, e^0_2 \quad X^{(0)} = e^0_1 \sqcup e^0_2$

$e^1_1, e^1_2$ : $\alpha^1_i : \partial e^1_i \xrightarrow{\cong} X^{(0)}$

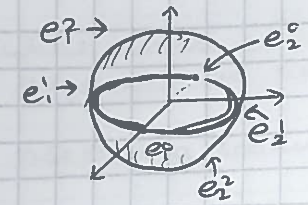
$\Rightarrow X^{(1)} = (X^{(0)} \cup_{\alpha^1_1} e^1_1) \cup_{\alpha^1_2} e^1_2 \cong S^1$

$e^2_1, e^2_2$. $\alpha^2_i : \partial e^2_i \longrightarrow X^{(1)}$, $X^{(2)} = (X^{(1)} \cup_{\alpha^2_1} e^2_1) \cup_{\alpha^2_2} e^2_2$

To prove this rigorously, we need to give these explicitly.

How to prove the cell complex is homeomorphic to the usual $S^2$.

Think of the usual $S^2$




$\subseteq \mathbb{R}^3$.

This exhibits an explicit map from the cell complex to the usual $S^2$. This map is a continuous bijection. To finish the proof you'd need to prove that it's a homeomorphism.

The cell decompositions above respect the antipodal map. It follows that these cell decompositions descend to $\mathbb{R}P^n$.

## $\mathbb{R}P^n$

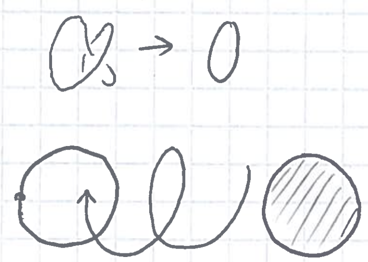
$\mathbb{R}P^0$ : $e^0$ : (one-zero cell)

$\mathbb{R}P^1$ : $e^0_1, e^1_1$ : $\alpha^1_1 : \partial e^1_1 \to e^0_1$

$= \bigcirc$

$\mathbb{R}P^2$ : $\underbrace{\qquad\qquad\qquad}_{\text{all of this}}$

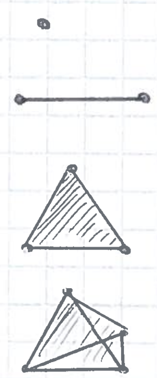
plus $e^2_1$, $\alpha^2_1 : \partial e^2_1 \to \mathbb{R}P^1$. This will be the 2:1 map described before:



## ASIDE - Simplicial complexes

These are defined very similarly. Instead we use n-simplices.

- 0-simplex :
- 1-simplex :
- 2-simplex :
- 3-simplex :



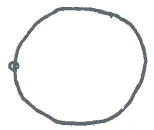
In the definition of simplicial complexes, we proceed as above, with two extra features:

1) attaching are combinatorial isomorphisms.
n-simplices $\mapsto$ n-simplices.

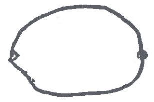
2) Simplices are determined by their vertices. No two simplices can have the same set of vertices.

Example

Decompositions of $S^1$:



: cell decomposition, not a simplicial complex.
– violates (1)



: cell complex, but not simplicial.
– violates (2)

: this is a cell and simplicial complex!

vertex = 0-simplex.
edge = 1-simplex.

Cell complex's are FLEXIBLE
Simplicial complex's are RIGID!

Question 15th October 2013

is $\mathbb{R}^m \not\cong \mathbb{R}^n$? (for $m \neq n$)

Aside – Abstract topological spaces that show up in the real world.

- The space of phylogenetic trees.
- Protein molecules $\hookrightarrow$ a cell.
- Big data sets. (including the internet!)

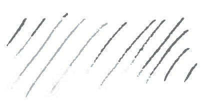
– These are really growth area.

Answer to question

Suppose $n=1$. is $\mathbb{R}^m = \mathbb{R} \not\cong \mathbb{R}^1$?

$\mathbb{R}^1$: ________....

$\mathbb{R}^2$:



If I remove a point from $\mathbb{R}^1$, it will have the same effect as removing a point from $\mathbb{R}^2$.

$\mathbb{R}^1/\{pt\}$ ________ ________ ... – clearly no longer connected.

$\mathbb{R}^2/\{pt\}$:


– $\mathbb{R}^2/\{pt\}$ is pathconnected.
$\Rightarrow$ $\mathbb{R}^2/\{pt\}$ is connected!

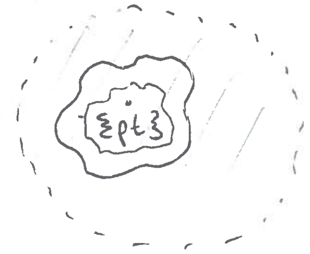
So $\mathbb{R}^2 \not\cong \mathbb{R}$. □

Similarly $\mathbb{R}^m \not\cong \mathbb{R}$ for $m>1$.

What about if $n=2$? clearly we can't look at removing points as before. We need another idea.

Instead of looking at points, we are going to look at loops:

$\mathbb{R}^2$:




$\mathbb{R}^3$:


→ we can shrink a loop down in $\mathbb{R}^3$ down to a point, even if a point is removed: we can slip over the top of the removed point. we cannot do this in $\mathbb{R}^2$.
→ thinking about loops made of elastic!

we can convince ourselves that there are topological

"After removing a point in $\mathbb{R}^2$, some loops are permanently snagged on that point, while others aren't. No loop in $\mathbb{R}^3 \setminus \{pt\}$ are snagged."

Definition

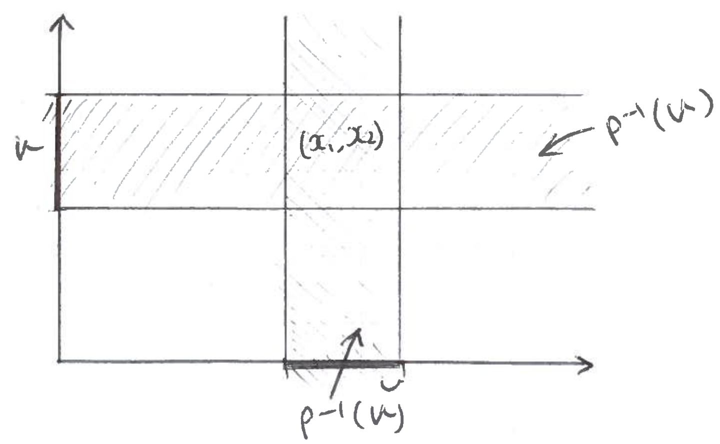
If $(X_1, \tau_1)$ and $(X_2, \tau_2)$ are topological spaces, then the product topology, denoted $\sigma$ on $X_1 \times X_2$, is as follows:

Let $p_i: X_1 \times X_2 \longrightarrow X_i$ $(i=1,2)$ be coordinate projection.

$$p_1(x_1, x_2) = x_1 \quad \text{and} \quad p_2(x_1, x_2) = x_2$$

$\sigma$ is the smallest topology on $X_1 \ X_2$ such that $p_1^{-1}(U)$ is open $\forall U \in \tau_1$ and $p_2^{-1}(V)$ is open $\forall V \in \tau_2$.

Picture




– Every point has open neighbourhoods which look like:

$u \times v$

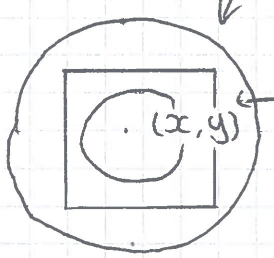
for some: $u \in \tau_1$
$v \in \tau_2$

Exercise

Show that the usual topology and the product topology coincide on $\mathbb{R}^2$.

Proof by picture

circles are the usual topology




squares are the product topology

Show that they fit inside each other.

## Definition

Two continuous maps $f_0: X \longrightarrow Y$ and $f_1: X \longrightarrow Y$ are said to be homotopic if there exists a continuous map, called a homotopy, $F: X \times [0,1] \longrightarrow Y$ such that:

$$F(x,0) = f_0(x)$$
$$F(x,1) = f_1(x)$$

Homotopy means that we can "continuously deform the map $f_0$ to $f_1$"

We will often write $f_t(x) = F(x,t)$.

If we have the above, we write $f_0 \sim f_1$.

NOTE: homotopy is an equivalence relation.

## Examples

1) If $f: X \longrightarrow \mathbb{R}^n$, then $f \sim$ the constant map $0: X \longrightarrow \mathbb{R}^n$, $x \longmapsto 0$.

Indeed let $F(x,t) = t f(x)$.

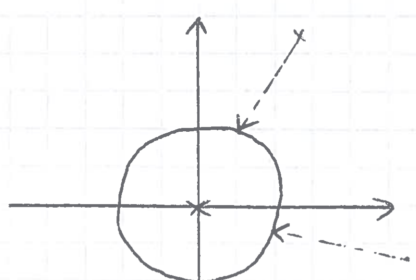
$t=1$ gives $F(x,1) = f(x)$
$t=0$ gives $F(x,0) = 0(x) = 0$

So $f \sim 0$.

2) Consider $f: \mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$
In polar coordinates $(r,\theta) \longmapsto (1,\theta)$

Consider the homotopy:

$$F: (\mathbb{R}^2 \setminus \{0\}) \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$$
$$((r,\theta), t) \longmapsto (r^{1-t}, \theta)$$



when $t=0$, $F((r,\theta),0) = (r^{1-0}, \theta) = \mathrm{id}$
when $t=1$, $F((r,\theta),1) = (1,\theta) = f$

## Definition

If there exist continuous maps $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then we say that $X$ and $Y$ are homotopy equivalent. We write $X \simeq Y$.

By the above examples:

- $\mathbb{R}^n \simeq \{pt\}$
(Take $X = \mathbb{R}^n$, $f = \mathrm{id}_{\mathbb{R}^n}$, then $\mathbb{R}^n$ is contractible)
- $\mathbb{R}^2 \setminus \{0\} \simeq S^1$

## Strategy:

Define (algebraic) invariants of topological spaces, up to homotopy equivalence.

Then if $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\} \Rightarrow \mathbb{R}^m \setminus \{0\} \simeq \mathbb{R}^n \setminus \{0\}$, but our invariants will be different.

We are going to study a space $X$ by studying homotopy classes of loops in $X$. We want to consider based loops.

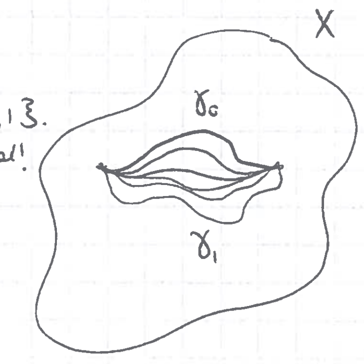
## Definition

Fix a base point $x_0 \in X$. Then a based loop is a continuous map $\gamma: [0,1] \rightarrow X$ such that $\gamma(0) = \gamma(1) = x_0$. Also called a pointed map.

## Definition

If $\gamma_0, \gamma_1: [0,1] \rightarrow X$ are paths with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ then a homotopy between $\gamma_0$ and $\gamma_1$ rel $\{0,1\}$ is a homotopy $F: [0,1] \times [0,1] \rightarrow X$ between $\gamma_0$ and $\gamma_1$ such that:

$$\left.\begin{array}{l} F(0,t) = \gamma_0(0) = \gamma_1(0) \\ F(1,t) = \gamma_0(1) = \gamma_1(1) \quad \forall t. \end{array}\right\} \begin{array}{l} \gamma_1 \sim \gamma_0 \text{ rel } \{0,1\}. \\ \text{- equivalence rel!} \end{array}$$

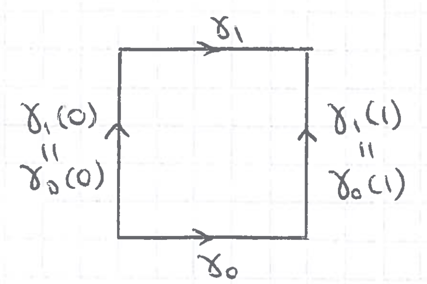



"Slowly deform from one curve to another while keeping the basepoints as they are".

This is called a based homotopy.

## Definition

The fundamental group is defined as follows. Let $(X, x_0)$ be a topological space with a basepoint.



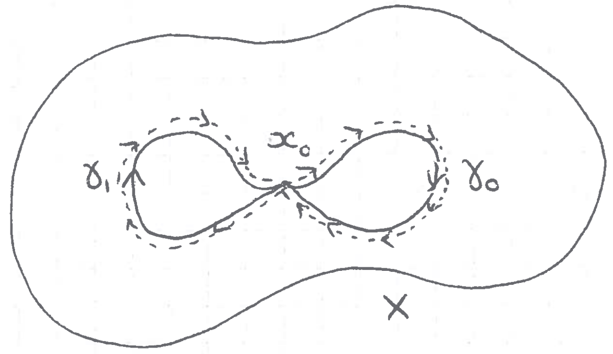

The fundamental group of $X$ at $x_0$ is the set of equivalence classes of based loops in $X$ at $x_0$.

(where the equivalence relation is homotopy rel $\{0,1\}$)

We will write elements as $[\gamma]$.

The fundamental group is denoted by $\pi_1(X, x_0)$.
($\pi_1$ for Poincaré!)

$[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$
How do we compose them?




We will concatenate them.

The concatenation of $\gamma_1$ and $\gamma_2$ is given as follows:

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

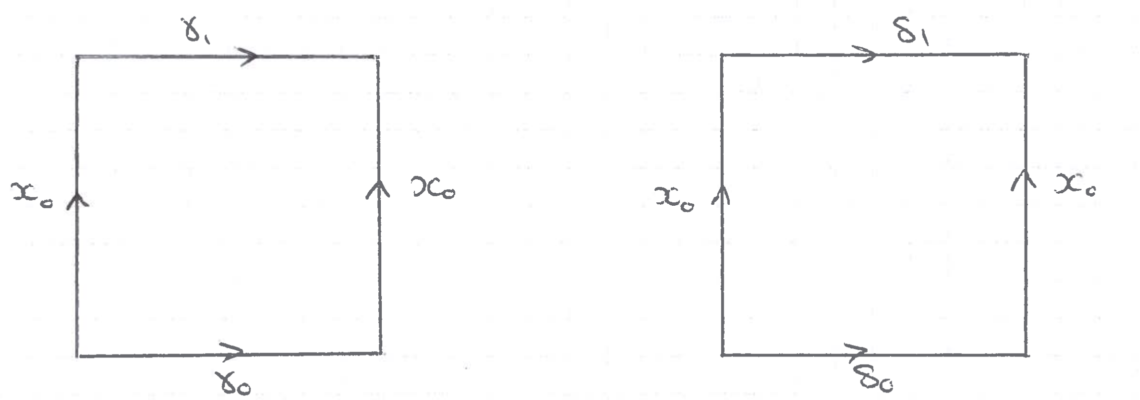
So the group operation is as follows: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$. We need to check that this is a well-defined group operation.

well-defined?

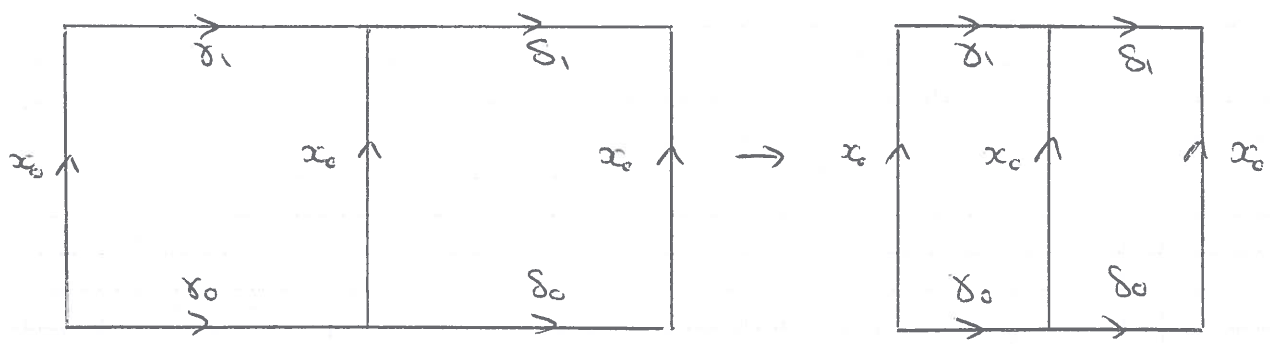
$\gamma_0 \sim \gamma_1$ rel $\{0,1\}$ and $\delta_0 \sim \delta_1$ rel $\{0,1\}$

$\Rightarrow \gamma_0 \cdot \delta_0 \sim \gamma_1 \cdot \delta_0$ rel $\{0,1\}$

Proof by picture




These follow from the hypothesis. We can glue these together to get a new square:




□

Group axioms

operation $\cdot$ (multiplication) satisfies

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Identity: $\exists 1$ such that $a \cdot 1 = 1 \cdot a = a$
- Inverses: $\forall a, \exists a^{-1}$ such that $a^{-1} \cdot a = 1 = a \cdot a^{-1}$

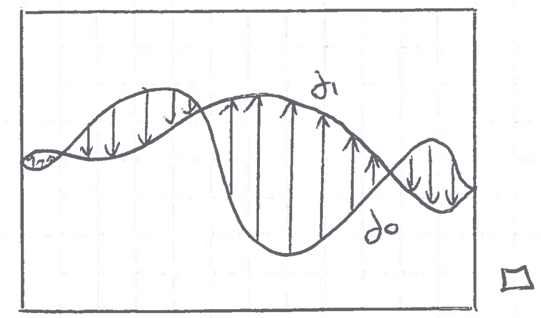
NOTE: in general we don't have commutivity

we need to check these things for $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$

Lemma

If $f_0, f_1 : [0,1] \to [0,1]$ with $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$ then $f_0 \sim f_1$ rel $\{0,1\}$

Proof by picture

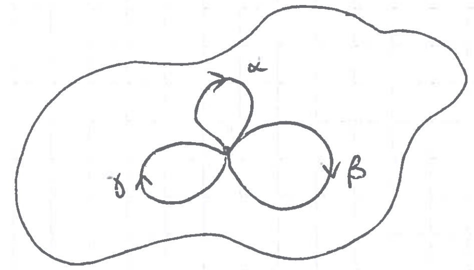



□

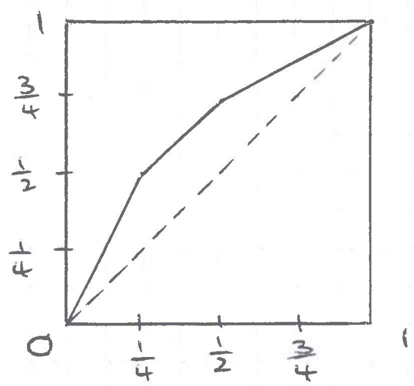
Proof of group axioms

Let $\alpha, \beta, \gamma$ be based loops in $(X, x_0)$. Associativity:

1) $\alpha \cdot (\beta \cdot \gamma) \sim (\alpha \cdot \beta) \cdot \gamma$ rel $\{0,1\}$.




Let $f : [0,1] \to [0,1]$ be the following function with graph:




Then $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot (\beta \cdot \gamma)) \circ f$.

By the lemma above: $f \sim id_{[0,1]}$ rel $\{0,1\}$

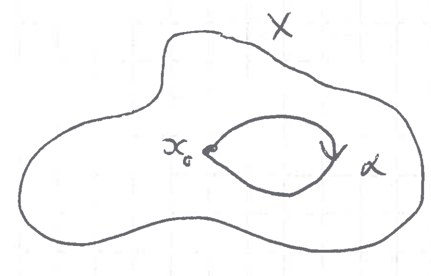
— $f$    - - - $id$

Composition of this homotopy with $(\alpha \cdot (\beta \cdot \gamma))$ we see that:

$$\begin{aligned} (\alpha \cdot \beta) \cdot \gamma &= (\alpha \cdot (\beta \cdot \gamma)) \circ f \\ &\sim (\alpha \cdot (\beta \cdot \gamma)) \circ id_{[0,1]} \text{ rel } \{0,1\} \\ &= (\alpha \cdot (\beta \cdot \gamma)) \end{aligned}$$

$\Rightarrow (\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$ rel $\{0,1\}$.

2) Identity

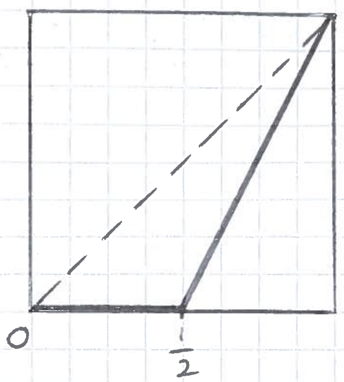



The ident. element $1 = [x_0]$

$x_0 \cdot \alpha \sim \alpha \sim \alpha \cdot x_0$

Intuitively this is true.

Let $f$ be:




Then $x_0 \cdot \alpha = \alpha \circ f$

$f \sim id_{[0,1]}$ rel $\{0,1\}$

$x_0 \cdot \alpha \sim \alpha \circ id_{[0,1]}$ rel $\{0,1\}$ by the Lemma.

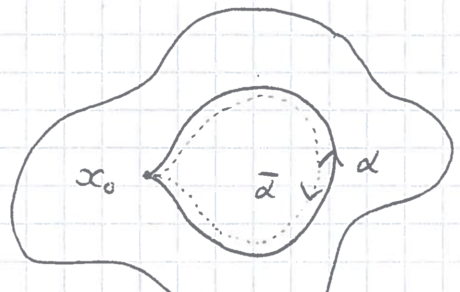
$= \alpha$. ($\alpha \cdot \alpha_0 = \alpha$ similarly)

/ $f$    ⁄ id.

3) Inverses

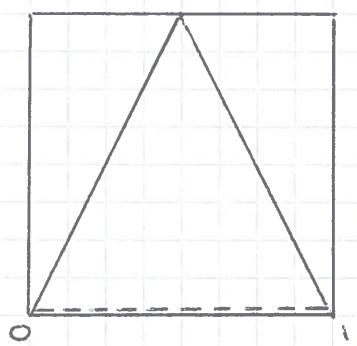
$[\alpha]^{-1} = [\bar{\alpha}]$ where:

$\bar{\alpha}(t) = \alpha(1-t)$




Need to check that $\alpha \cdot \bar{\alpha} \sim x_0 \sim \bar{\alpha} \cdot \alpha$ rel $\{0,1\}$.

Let $f$ be




– $f$ will not be homotopic to the identity, but it is homotopic to zero!

$\alpha \cdot \bar{\alpha} = \alpha \circ f \sim \alpha \circ 0$ rel $\{0,1\}$
$= \alpha(0) = x_0$

The same way for $\bar{\alpha} \cdot \alpha = x_0$ concludes this proof. □

$\Rightarrow \pi_1(X, x_0)$ really is a group! □

Example – $\pi_1(\mathbb{R}^n, 0)$

Let $\gamma: [0,1] \longrightarrow \mathbb{R}^n$ be a based loop.

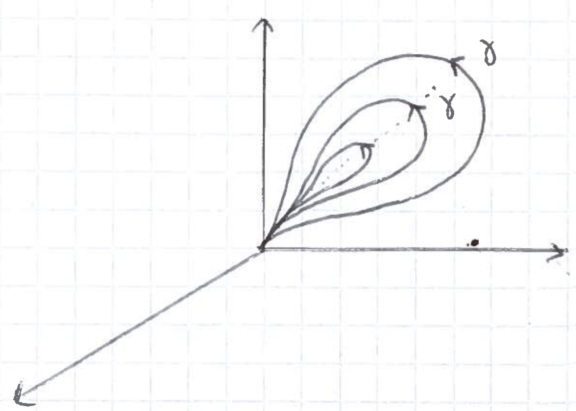
Consider $F: [0,1] \times [0,1] \longrightarrow \mathbb{R}^n$
$(s,t) \longmapsto t\gamma(s)$

Now $F(s,1) = \gamma(s)$
$F(s,0) = 0(s) = 0$

So we have a homotopy between $\gamma$ and $0$, and:

$F(0,t) = F(1,t) = 0$ for all $t$.

$\therefore \pi_1(\mathbb{R}^n, 0) \cong \{id\}$
– $\pi_1(\mathbb{R}^n, 0)$ is trivial.




Note

Note that the choice of basepoint is not too important.

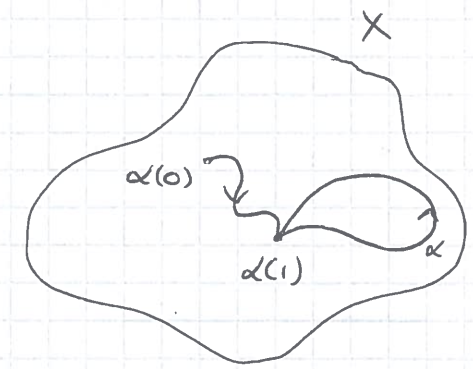
Lemma

If $\alpha: [0,1] \to X$ is any path (not necessarily a based point), then there is an isomorphism:

$\alpha_\#: \pi_1(X, \alpha(\ )) \xrightarrow{\cong} \pi_1(X, \alpha(\ ))$

As long as our spaces are path connected, the isomorphism class of $\pi_1$ is NOT dependant on the base point.

Proof




If $\gamma$ is a loop based at $\alpha(1)$, define:

$\alpha_\#[\gamma] = [\alpha \cdot \gamma \cdot \bar{\alpha}]$

Need to check that $\alpha_\#$ is a well-defined group homomorphism, and that $\bar{\alpha}_\#$ is its inverse.

This is left up to us to finish the proof! □

Summary

Path-connected topological spaces → Groups $\pi_1(X, x_0)$

Continuous maps $f: X \to Y$ → $f_\#: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$

we need to define →

Definition

If $x_0 \in X$ and $f: X \longrightarrow Y$ is a continuous map. The induced map

$f_\#: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ (✲)

is defined by: $[\gamma] \longmapsto [f \circ \gamma]$

CHECK that $f_\#$ is well-defined and is a group homomorphism (If $\gamma \sim \delta$ rel $\{0,1\}$ then $f \circ \gamma \sim f \circ \delta$ rel $\{0,1\}$, and that it is a group homomorphism as usual.)

Theorem

Suppose $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ is a homotopy equivalence. For any $x_0 \in X$,

$f_\#: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$

is an isomorphism.

Key Property      22nd October

If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_\# = g_\# \circ f_\#$. This is called functoriality.

Proof.

## Lemma

Suppose $f_0, f_1 : X \longrightarrow Y$ are homotopic maps, via the homotopy $F(x,t)$.

Let $x_0$ be a base point for $X$, and let $\alpha(t) = F(x_0, t)$. Then:

$$f_{0*} : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f_0(x_0))$$

decomposes as follows:

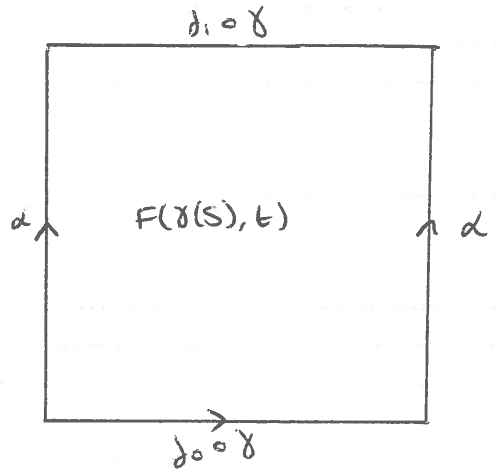
$$\pi_1(X, x_0) \xrightarrow{f_{1*}} \pi_1(Y, f_1(x_0)) \xrightarrow{\alpha_*} \pi_1(Y, f_0(x_0))$$

i.e. $f_{0*} = \alpha_* \circ f_{1*}$.

## Proof of lemma

Let $[\gamma] \in \pi_1(X, x_0)$. We need to prove that:

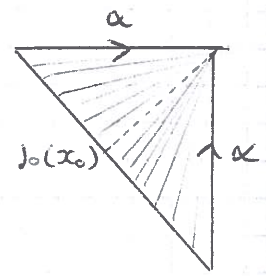
$$f_0 \circ \gamma \sim \alpha \cdot (f_1 \circ \gamma) \cdot \bar{\alpha} \quad \text{rel } 0,1.$$



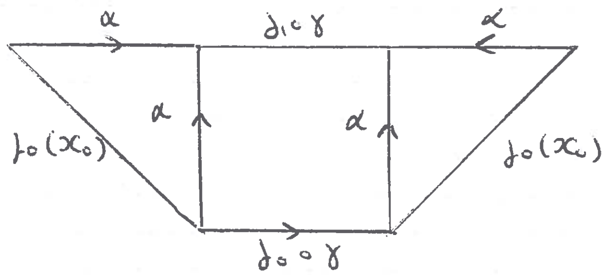

consider the following map:

crush the triangle to the line:

$\equiv [0,1] \xrightarrow{\alpha} Y.$

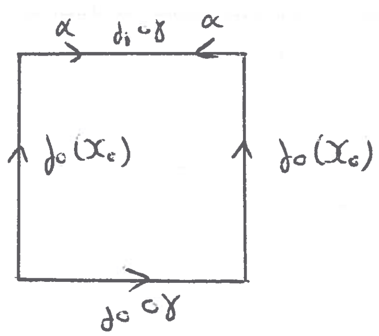



We will then attach these triangles onto a square:




We can build a continuous map  $\longrightarrow Y$.

But  is homeomorphic to a square.




This provides a homotopy! □

## Proof of the Theorem

Let $F$ be a homotopy between $g \circ f$ and $id_X$. Let $\gamma(t) = F(x_0, t)$

By the lemma $(g \circ f)_* = \alpha_* \circ (id_X)_* = \alpha_*$

By functoriality $g_* \circ f_* = (g \circ f)_* = \alpha_*$
$\Rightarrow f_*$ is injective.

Let $G$ be a homotopy between $f \circ g$ and $id_Y$, and let $\beta(t) = G(f(x_0), t)$.

Then, as before:

$f_* \circ g_* = \beta_*$ is an isomorphism.
$\Rightarrow f_*$ is surjective.
$\Rightarrow f_*$ is an isomorphism. □

## Non-trivial example

$$X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}, \quad x_0 = 1.$$

$[0,1] \longrightarrow$  1

We have an intuitive idea of the winding number $w(\gamma)$ = the number of times you go round $S^1$ clockwise − the number of times you go round anti-clockwise.

FACT: $\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$
$[\gamma] \longmapsto w(\gamma)$

This map is an isomorphism of groups, we will prove this later today.

## Applications of the fundamental group

- $\mathbb{R}^2 \not\cong \mathbb{R}^m$ for $m > 2$
- Brouwer's Fixed Point Theorem.
- The fundamental theorem of algebra.

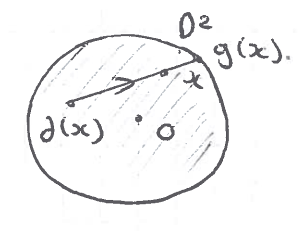
## Theorem (Brouwer's Fixed Point Theorem)

Let $D^2 = \{ x \in \mathbb{R}^2 : \|x\|_2 \leq 1 \}$ is a continuous map. Any continuous map

$$f : D^2 \to D^2$$

has a fixed point (i.e. a point such that $f(x) = x$).

## Proof




Suppose $f(x) \neq x \ \forall x \in D^2$. Define $g : D^2 \to \partial D^2$ as in the picture. This is a continuous map.

So we have:

$$g_* : \pi_1(D^2, 1) \longrightarrow \pi_1(S^1, 1)$$

$\pi_1(S^1, 1) \cong \mathbb{Z}$. $D^2$ is clearly contractible to a point
$\Rightarrow \pi_1(D^2, 1) \cong 1$.

Let $i : S_1 \longrightarrow D^2$ be inclusion.

Because $g \circ i = id_{S^1}$, we have that:

$$\underset{\cong\,\mathbb{Z}}{\pi_1(S^1,1)} \xrightarrow{i_*} \underset{\cong\,1}{\pi_1(D^2,1)} \xrightarrow{g_*} \underset{\cong\,\mathbb{Z}}{\pi_1(S^1,1)}$$

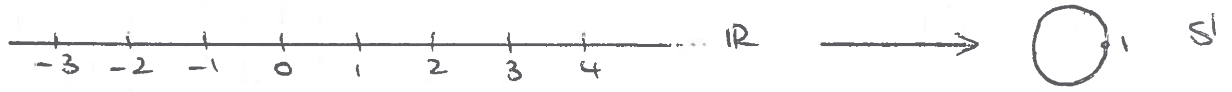
$(id_S)_* = id_{\pi_1(S^1,1)}$
$\Rightarrow id_{\mathbb{Z}} = 0$ ※ □

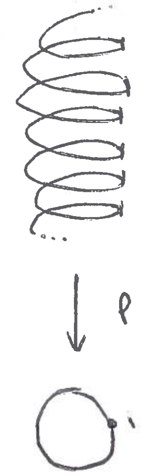
Goal

Define winding number rigorously.

Our main tool is the following map:

$$p : \mathbb{R} \longrightarrow S^1 \subseteq \mathbb{C}$$
$$t \longmapsto e^{2\pi i t}$$



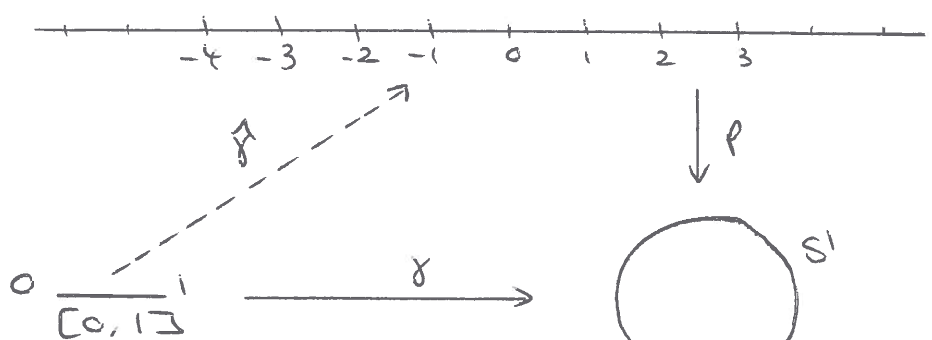




Another way to think about $g$: $\mathbb{Z} \subseteq \mathbb{R}$, as a subgroup under addition. $\mathbb{Z}$ is a normal subgroup; we have a quotient map:

$$\mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z}$$
$$S^1 \cong \mathbb{R}/\mathbb{Z}$$

Idea

The idea for defining the winding number:




Define $\hat{\gamma} : [0,1] \longrightarrow \mathbb{R}$ such that:

i) $\hat{\gamma}(0) = 0$
ii) $p \circ \hat{\gamma} = \gamma$

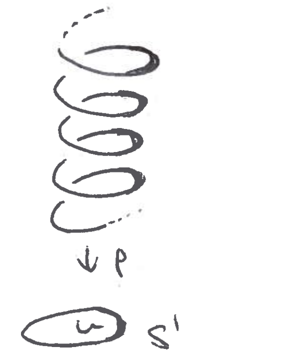
Set $\omega(\gamma) = \hat{\gamma}(1)$.

Key property of $p$ is called the covering map property. Every point in the circle $x \in S^1$ has a neighbourhood $U$ such that

$$p^{-1}U \cong U \times \Delta$$

where $\Delta$ is a discrete topological space, and the following diagram commutes:

$$\begin{tikzcd} p^{-1}U \arrow[r, phantom, "\cong"] \arrow[dr, "p"'] & U \times \Delta \arrow[d, "\text{projection onto the first factor.}"] \\ & U \end{tikzcd}$$




$U = \{ \,|\; |\; |\; |\; |\; |\; |\, \}$

Path-lifting lemma

If $\gamma : [0,1] \longrightarrow S^1$ is a path with $\gamma(0) = 1$, then there is a unique path $\hat{\gamma} : [0,1] \to \mathbb{R}$ such that:

1) $\hat{\gamma}(0) = 0$
2) $p \circ \hat{\gamma} = \gamma$

Proof

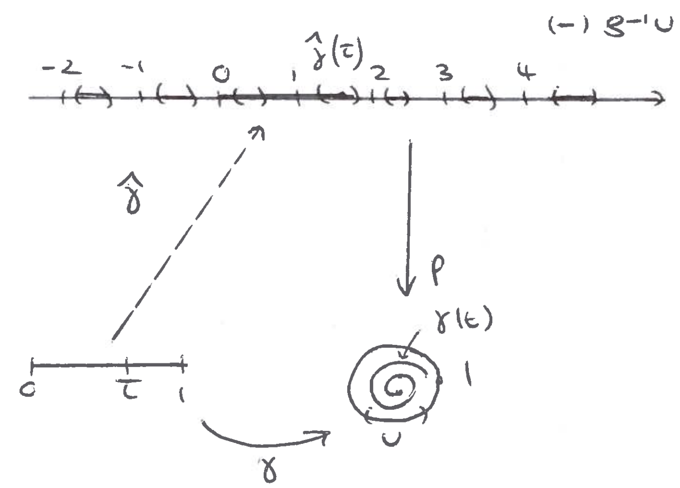
Part 1) Existence

Let $T = \{ \tau \in [0,1] : \gamma|_{[0,\tau]}$ has a lift, $\hat{\gamma}$ as in statement$\}$. Clearly $0 \in T$. So $T \neq \emptyset$.

GOALS: Prove that $T$ is open and $[0,1] \setminus T$ is open. $\Rightarrow T = [0,1]$ $\Rightarrow 1 \in T$ as required.

Claim that $T$ is open. Suppose $\tau \in T$ and let $x = \gamma(\tau)$.

Let $U$ be the neighbourhood of $x$ provided by the covering map property. Then $\exists\, \delta \in \Delta$ unique such that $\hat{\gamma}(\tau) \in U \times \{\delta\}$. $U \times \{\delta\} \subseteq \mathbb{R}$.




Let $W$ be a connected neighbourhood of $\tau$ such that $\gamma(W) \subseteq U$. So

Now $(p|_{\hat{U}})^{-1} \circ \gamma|_W$ extends $\hat{\gamma}$ across $W$. That is:

$$t \longmapsto \begin{cases} (p|_{\hat{U}})^{-1} \circ \gamma|_W & (t \in W) \\ \hat{\gamma}(t) & t \leq \tau \end{cases}$$

is a lift of $\gamma$ over $[0,\tau] \cup W$. $\therefore W \subseteq T$

This proves that $T$ is open.

Claim that $[0,1]\setminus T$ is open. Suppose $\tau \notin T$. Let $U$ be a "covering map property" neighbourhood of $\gamma(\tau)$, and let $\gamma(W) \subseteq U$.

Then if $\exists \in T \cap W$, then just as above:

$$t \longmapsto \begin{cases} ((p|_{\hat{U}})^{-1} \circ \gamma|_W)(t) & t \in W \\ \hat{\gamma}(t) & t \leq S \end{cases}$$

extends $\hat{\gamma}$ across $W$. ($\hat{U}$ is the component of $p^{-1}U$ that contains $\hat{\gamma}(S)$)

$\Rightarrow \tau \in T$ $\#$

$\therefore W \subseteq [0,1]\setminus T. \Rightarrow [0,1]\setminus T$ is open

$\therefore$ $\hat{\gamma}$ extends, as required. This proves existance.

Part 2) Uniqueness

Suppose $\hat{\gamma}_1, \hat{\gamma}_2$ are both lifts of $\gamma$ with $\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = 0$. i.e. $p \circ \hat{\gamma}_i = \gamma$

Let $T' = \{\tau \in [0,1] : \hat{\gamma}_1(\tau) = \hat{\gamma}_2(\tau)\}$

Again, we'll be done if we can prove that non-empty, open, and closed.

$0 \in T' \Rightarrow T'$ is non-empty. Claim that $T'$ is closed.

If $t_n \longrightarrow \tau$ as $n \to \infty$ and $t_n \in T'$, then:

$$\hat{\gamma}_1(t) = \lim_{n\to\infty} \hat{\gamma}_1(t_n) = \lim_{n\to\infty} \hat{\gamma}_2(t_n) = \hat{\gamma}_2(t)$$

So $T'$ is closed. Claim that $T'$ is open. As in the existence proof, if $\tau \in T'$, so $\hat{\gamma}_1(t) = \hat{\gamma}_2(t)$. Then for same $\hat{U} \subseteq p^{-1}U$ and same $W$ a neighbourhood of $\tau$:

$$\hat{\gamma}_i(t) = (p|_{\hat{U}})^{-1} \circ \gamma(t) \quad \forall t \in W, \; i = 1,2.$$

$\therefore \hat{\gamma}_1(t) = \hat{\gamma}_2(t) \Rightarrow t \in T! \Rightarrow T'$ is open.

This proves uniqueness and hence concludes the proof of the path-lifting lemma. □

Definition

If $\gamma : [0,1] \longrightarrow S^1$ is a based loop ($\gamma(0) = \gamma(1) = 1$) then we define the winding number of $\gamma$, $w(\gamma)$, as:

$$w(\gamma) = \hat{\gamma}(1)$$

where $\hat{\gamma}$ is the unique lift of $\gamma$ to $\mathbb{R}$ with $\hat{\gamma}(0) = 0$.

Remark

$p \circ \hat{\gamma}(1) = \gamma(1) = 1$

$\Rightarrow w(\gamma) = \hat{\gamma}(1) \in p^{-1}(1) = \mathbb{Z}$

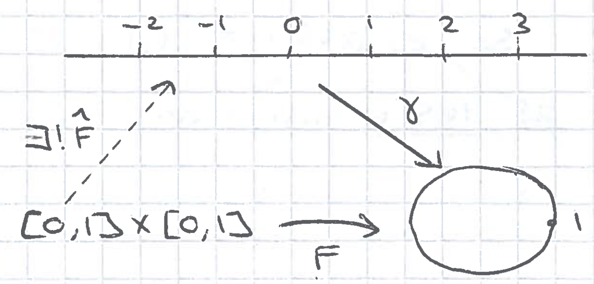
$\Rightarrow$ the winding number $w(\gamma)$ is an integer as expected.

We'd like to define a map $\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ where $[\gamma] \longmapsto w(\gamma)$. To show that this is well defined, we need an extension of the path lifting lemma.

The Homotopy Lifting Lemma

If $F : [0,1] \times [0,1] \longrightarrow S^1$ be a continuous map with $F(0,t) = 1$ $\forall t$, then $\exists!$ continuous map $\hat{F} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ such that:

1) $\hat{F}(0,t) = 0$ $\forall t$
2) $p \circ \hat{F} = F$




Proof

Omitted. For the PLL we used that $[0,1]$ is connected, ($[0,1] \times [0,1]$ is connected also) and $p$ still has this covering map property. So the proof of the HLL is very similar to that of the PLL, hence is omitted. □

Theorem

The winding number defines an isomorphism of groups:

$$\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$$
$$[\gamma] \longmapsto w(\gamma)$$

Proof

The map is well defined by the Homotopy-Lifting Lemma.

Suppose $\gamma_1, \gamma_2$ are based loops in $S^1$ and $\gamma_1 \simeq \gamma_2$ rel $\{0,1\}$. We need to prove that $w(\gamma_1) = w(\gamma_2)$.

Let $F$ be a based homotopy from $\gamma_1$ to $\gamma_2$ rel $\{0,1\}$, and let $\hat{F}$ be its unique lift with $\hat{F}(0,t) = 0$, guaranteed by the Homotopy-Lifting Lemma.

Then $\hat{F}(1, \cdot) : [0,1] \longrightarrow \mathbb{R}$ is a continuous map. For any $t$,

$p(\hat{F}(1,t)) = F(1,t) = 1$ as $F$ is a based homotopy.

$\Rightarrow \hat{F}(1,t) \in p^{-1}1 = \mathbb{Z} \subseteq \mathbb{R}$.

$\therefore \hat{F}(1, \cdot)$ is a continuous map $[0,1] \to \mathbb{Z}$

$\Rightarrow \hat{F}(1, \cdot)$ is constant.

In particular, $\hat{\gamma}_1(1) = \hat{F}(1,0) = \hat{F}(1,1) = \hat{\gamma}_2(1)$

$\Rightarrow w(\gamma_1) = w(\gamma_2)$, as required.

It remains to prove that $[\gamma] \longmapsto w(\gamma)$ is an isomorphism.

1) It's a homomorphism.

Let $[\alpha], [\beta] \in \pi_1(S^1, 1)$ with $w(\alpha) = m$, $w(\beta) = n$.

Need to prove that $w(\alpha \cdot \beta) = m + n$.

Let $\hat{\gamma} = \hat{\alpha} \cdot T_m(\hat{\beta})$ where $T_m(\hat{\beta}) = \hat{\beta} + m$. By uniqueness of lifts $\hat{\gamma} = \widehat{\alpha \cdot \beta}$.

So $w(\alpha \cdot \beta) = \hat{\gamma}(1) = T_m(\hat{\beta})(1) = m + \hat{\beta}(1) = m + n$.

2) It's a bijection.

SURJECTIVITY: Let $\gamma_n(t) = p \circ \hat{\gamma}_n(t)$ where $\hat{\gamma}_n(t) = nt$. Then the winding number $w(\gamma_n) = n$ (easy check!)

INJECTIVITY: Suppose $w(\gamma)$. Then $\hat{\gamma}(1) = 0 \Rightarrow \hat{\gamma}$ is a loop in $\mathbb{R}$ based at $0$. Now

$\hat{F}(s,t) = t\hat{\gamma}(s)$ is a homotopy in $\mathbb{R}$ that shrinks $\hat{\gamma}$ to $0$ rel $\{0,1\}$.

Now $p \circ \hat{F} = F$ is a homotopy in $S^1$ that shrinks $\gamma$ to $1$ rel $\{0,1\}$ which proves injectivity.

This proves the theorem. $\square$

29th October

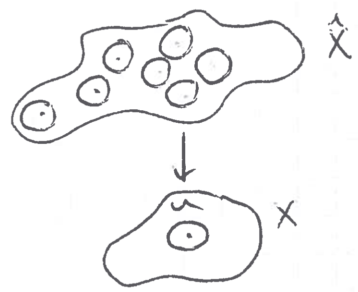
Last time, we only used the "covering map property" $p$.

**Definition**

A continuous map* $p: \hat{X} \longrightarrow X$ is called a covering map if: $\forall x \in X$, $x$ has a neighbourhood $U$ such that $p^{-1}U \cong U \times \Delta$, where $\Delta$ is a discrete space.

(i.e. $p^{-1}$ is a disjoint union of copies of $U$). and we have:

$$\begin{tikzcd} p^{-1}U \arrow[r, phantom, "\cong"] \arrow[dr, "p"'] & U \times \Delta \arrow[d, "\text{projection}"] \\ & U \end{tikzcd}$$




$\hat{X}$ is called a covering space.

* NOTE: $p$ must be surjective.

We will almost always assume that $X$ and $\hat{X}$ are path-connected.

**Path-lifting lemma**

Let $p$ be a covering map: $(\hat{X}, \hat{x}_0) \xrightarrow{p} (X, x_0)$. Let $\alpha: ([0,1], 0) \longrightarrow (X, x_0)$ be a path.

Then there exists a unique lift $\hat{\alpha}$ such that:

1) $p \circ \hat{\alpha} = \alpha$
2) $\hat{\alpha}(0) = \hat{x}_0$

$$\begin{tikzcd} & \hat{X} \arrow[dr, "p"] & \\ {[0,1]} \arrow[ur, "\hat{\alpha}"] \arrow[rr, "\alpha"'] & & X \end{tikzcd}$$

**Homotopy-lifting lemma**

Let $(\hat{X}, \hat{x}_0) \xrightarrow{p} (X, x_0)$ be a covering map. Let $F: [0,1] \times [0,1] \longrightarrow X$ be a homotopy with $F(0,t) = x_0 \ \forall t$.

Then there is a unique lift $\hat{F}$ such that:

1) $p \circ \hat{F} = F$
2) $\hat{F}(0,t) = \hat{x}_0 \ \forall t$

$$\begin{tikzcd} & \hat{X} \arrow[dr, "p"] & \\ {[0,1] \times [0,1]} \arrow[ur, "\hat{F}"] \arrow[rr, "F"'] & & X \end{tikzcd}$$

**Definition**

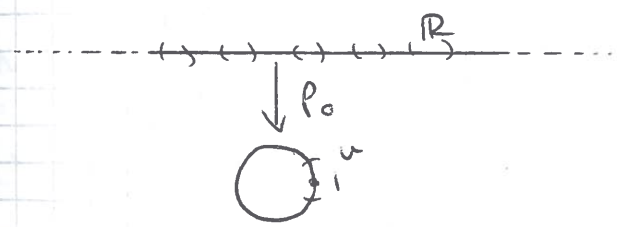
Covering spaces $\hat{X}_1, \hat{X}_2 \xrightarrow{p_1, p_2} X$ are isomorphic if there exists a covering transformation $\varphi: \hat{X}_1 \longrightarrow \hat{X}_2$ which is a homeomorphism such that:

$p_1 = p_2 \circ \varphi \quad (\Leftrightarrow p_2 = p_1 \circ \varphi^{-1})$

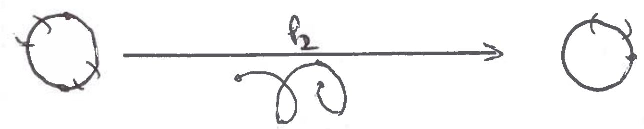
$$\begin{tikzcd} \hat{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \hat{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

**Examples**

1) Let $X = S^1$. We already have one example of a covering space: $\mathbb{R}$ with $p_0$ as follows:

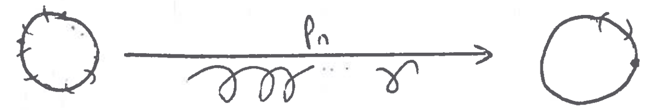



2) $X \cong S^1$. Let $\hat{X} \cong S^1$.

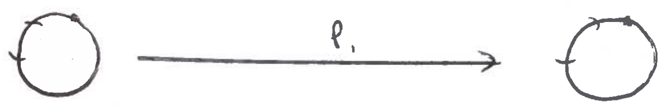



$p_1$ wraps the circle twice.

3) $X \cong S^1$, let $\hat{X} \cong S^1$
Let $p_n$ wrap around $n$ times. Then:




4) $X \cong S^1$, $\hat{X} \cong S^1$.
$p_1$ is the trivial covering map:




For each $n \in \mathbb{N}$ we have a covering map $p_n: \hat{X}_n \longrightarrow X$ where $X \cong \hat{X}_n \cong S^1$ which can be thought of as the restriction of the map $z \longmapsto z^n$ in $\mathbb{C}$.

Each point has $n$ pre-images under $p_n^{-1}$ for $n\in\mathbb{N}$, and $|p_0^{-1}(p(t))| = \infty$.

What kind of homomorphisms do covering maps induce?

**Lemma**

Let $p: (\hat{X}, \hat{x}_0) \longrightarrow (X, x_0)$ is a covering map. Then:

$$p_* : \pi_1(\hat{X}, \hat{x}_0) \longrightarrow \pi_1(X, x_0)$$

is injective.

**Proof**

If $\hat{\gamma}$, a based loop in $\hat{X}$ has $\gamma = p\circ\hat{\gamma}$ homotopic to $x_0$, then by the Homotopy lifting lemma, the homotopy lifts to a homotopy from $\hat{\gamma}$ to $\hat{x}_0$. □

e.g. in the example above: $p_{n*} : \pi_1(\hat{X}_n, \hat{x}_0) = \langle n \rangle \leq \mathbb{Z}$.

**Theorem** Classification of covering spaces.

Let $X$ be a path-connected cell complex with $x_0 \in X$ a basepoint.

$$\left\{\begin{array}{l}\text{Path-connected covering} \\ \text{spaces with base-point} \\ (\hat{X}, \hat{x}_0) \longrightarrow (X, x_0) \text{ up} \\ \text{to isomorphism.}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Subgroups of} \\ \pi_1(X, x_0)\end{array}\right\}$$

$$(\hat{X}, x_0) \longmapsto p_* \pi_1(\hat{X}, \hat{x}_0)$$

NOTE: Addendum to the definition of covering transformation: if $(\hat{X}_1, \hat{x}_1)$ and $(\hat{X}_2, \hat{x}_2)$ are based covering spaces, we insist that a covering transformation $\varphi$ has $\varphi(x_1) = x_2$.

Then we have that:

1. The map $(\hat{X}, \hat{x}_0) \longmapsto p_*\pi_1(\hat{X}, \hat{x}_0)$ is bijective. If $H \leq \pi_1(X, x_0)$, then we will usually denote the corresponding covering space by $(\hat{X}_H, \hat{x}_H) \xrightarrow{p_H} (X, x_0)$

2. If $H, K \leq \pi_1(X, x_0)$, we have that:

$$H \leq K \iff \exists q : \hat{X}_H \longrightarrow \hat{X}_K \text{ such that } p_H = p_K \circ q.$$

3. A subgroup $H \trianglelefteq \pi_1(X, x_0)$ if and only if $\hat{X}_H \longrightarrow X$ is a "normal" covering and the quotient:

$$\pi_1(X, x_0)/H = Q$$

acts on $\hat{X}_H$ by covering transformations, so

$$\hat{X}_H / Q \cong X.$$

**Definition**

Covariant & Contravariant functor.
Gagging

A path-connected covering space $\tilde{X} \xrightarrow{p} X$ with $\pi_1(\tilde{X}, \tilde{x}_0) \cong 1$ is called the universal cover.

NOTE: If $\pi_1(\tilde{X}, \tilde{x}_0) = 1$, it is called simply connected.

Because $1 \trianglelefteq \pi_1(X, X_0)$, by part 3 of the theorem there is an action of $\pi_1(X, x_0)$ on $\tilde{X}$ by covering transformations, and:

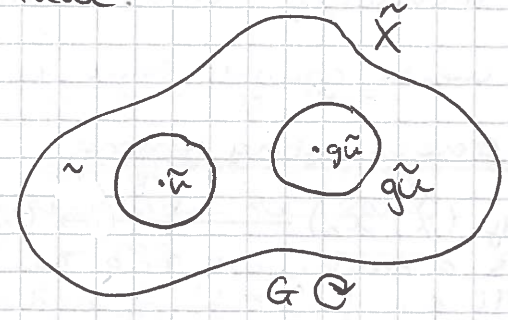
$$\tilde{X}/\pi_1(X, x_0) \cong X.$$

**Definition**

Suppose a group $G$ acts on a topological space $\tilde{X}$ by homeomorphisms. The action is called freely discontinuous if $\forall \tilde{x} \in \tilde{X}$ has a neighbourhood $\tilde{U}$ such that:

$$g\cdot\tilde{U} \cap \tilde{U} = \emptyset$$

whenever $g \in G \setminus 1$.



**Lemma**

If $G \curvearrowright \tilde{X}$ freely discontinuously then the quotient map

$$\tilde{X} \longrightarrow \tilde{X}/G$$

is a covering map, and $G$ acts by covering transformations.

This enables us to compute many more examples of the fundamental group.

**Proof**

Exercise ☺ □

**Proposition**

If $\tilde{X}$ is simply connected and $G \curvearrowright X$ freely discontinuously, then:

$$\tilde{X} \longrightarrow \tilde{X}/G \equiv X$$

is the universal cover and $G \cong \pi_1(X, x_0)$ for any choice of basepoint $x_0$.

**Proof**

Immediate from lemma above □

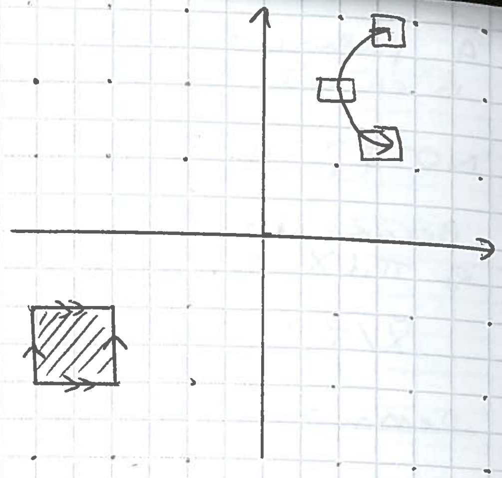
This enables us to write down lots more examples of fundamental groups.

**Example**

$\tilde{X} \cong \mathbb{R}^n$. Let $\mathbb{Z}^n$ act on $\mathbb{R}^n$ via $g\cdot x = g + x$. ($g \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$)

It's easy to see that this action is freely discontinuous, and $\mathbb{R}^n$ is simply connected.

So $\pi_1(\mathbb{R}^n/\mathbb{Z}^n) \cong \mathbb{Z}^n$



In fact, the quotient:

$$\mathbb{R}^n/\mathbb{Z}^n \cong \text{n-dimensional torus } \underbrace{S^1 \times \dots \times S^1}_{\text{n-copies.}}$$

NOTE: Friday afternoon: Alex Cioba, special session on homology. (11 or 12 ish?)

We're going to start to prove the classification theorem.

General lifting lemma

If $(\hat{X}, \hat{x}_0) \xrightarrow{p} (X, x_0)$ is a covering map and $f:(Y,y_0)\to(X,x_0)$ is a continuous map then there exists a lift $\hat{f}:(Y,y_0)\to(\hat{X},\hat{x}_0)$ (i.e. $p\circ\hat{f} = f$) if and only if:

$$f_*(\pi_1(Y,y_0)) \le p_*(\pi_1(\hat{X}, x_0))$$

Proof

$\Rightarrow$ If $\hat{f}$ exists then $f_*\pi_1(Y,y_0) = (p_* \circ \hat{f}_*)(\pi_1(Y,y_0))$

$= p_*(\hat{f}_*\pi_1(Y,y_0)) \le p_*\pi_1(\hat{X},\hat{x}_0)$.

$\Leftarrow$ Need to define $\hat{f}(y)$ for $y \in Y$. Choose $\alpha$ a path in $Y$ from $y_0$ to $y$. Let $\hat{\alpha}$ be the unique lift of $f\circ\alpha$ to $\hat{X}$ with $\hat{\alpha}(0) = \hat{x}_0$.

Define $\hat{f}(y) = \hat{\alpha}(1)$. Check that $\hat{f}$ is well-defined. Suppose $\alpha_1, \alpha_2$ are both paths in $Y$ from $y_0$ to $y$. Need to show that $\hat{\alpha}_1(1) = \hat{\alpha}_2(1)$

Then $\alpha_1 \cdot \bar{\alpha}_2$ is a based loop in $(Y, y_0)$

$\Rightarrow [\alpha_1 \cdot \bar{\alpha}_2] \in \pi_1(Y, y_0)$

$\Rightarrow [f\circ(\alpha_1\cdot\bar{\alpha}_2)] \in f_*\pi_1(Y,y_0) \le p_*\pi_1(\hat{X}, x_0)$

$\Rightarrow \exists$ a based loop $\hat{\delta}$ in $(\hat{X}, x_0)$ such that $f\circ(\alpha_1\cdot\bar{\alpha}_2) \simeq p\circ\hat{\delta}$ rel $\{0,1\}$.

By the Homotopy lifting lemma, $\hat{\delta} \simeq \hat{\alpha}_1 \cdot \bar{\hat{\alpha}}_2$ rel $\{0,1\}$

So $\hat{\alpha}_1 \cdot \bar{\hat{\alpha}}_2$ is a loop.

$\Rightarrow \hat{\alpha}_1(1) = \hat{\alpha}_2(1)$

Continuity: Locally, $\hat{f} = p' \circ f$ for $p'$ a local inverse of $p$ (i.e. on an open neighbourhood of any point $y \in Y$)

So $\hat{f}$ is a composition of continuous maps, hence continuous. □

When lifts exist, they are unique:

Lemma

For $(\hat{X}, \hat{x}_0) \xrightarrow{p} (X, x_0)$ as above and continuous map $f:(Y,y_0)\to(X,x_0)$ as above. If:

$$\hat{f}_1, \hat{f}_2 : Y \to \hat{X}$$

are both lifts of $f$ (i.e. $p\circ\hat{f}_i = f$) and $\hat{f}_1(y) = \hat{f}_2(y)$ for some $y \in Y$, then $\hat{f}_1 = \hat{f}_2$ everywhere.

Proof

Let $\eta \in Y$, and let $\alpha$ be a path in $Y$ from $y$ to $\eta$. Let:

Note that $p\circ\hat{f}_i\circ\alpha = f\circ\alpha$ for $i=1,2$. So $\hat{f}_i\circ\alpha$ is a lift of $f\circ\alpha$ with $\hat{f}_1 = \hat{f}_2$ because the two lifts have the same start, they are equal by the path lifting lemma.

Therefore: $\hat{f}_1(\eta) = \hat{f}_1\circ\alpha(1) = \hat{f}_2\circ\alpha(1) = \hat{f}_2(\eta)$ □

Proof of the injectivity part 1 of the classification theorem

Suppose $(\hat{X}_1, \hat{x}_1) \xrightarrow{p_1} (X, x_0)$ and $(\hat{X}_2, \hat{x}_2) \xrightarrow{p_2} (X, x_0)$ are covering maps with

$$p_{1*}\pi_1(\hat{X}_1, \hat{x}_1) = p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)$$

$\exists \hat{p}_1$ by the General lifting lemma: $\hat{p}_1 : (\hat{X}_1, \hat{x}_1) \to (\hat{X}_2, \hat{x}_2)$ and similarly, $\exists \hat{p}_2 : (\hat{X}_2, \hat{x}_2) \to (\hat{X}_1, \hat{x}_1)$.

Need to show that $\hat{p}_2 \circ \hat{p}_1 = \mathrm{id}_{\hat{X}_1}$

We have:

$$p_1 \circ \hat{p}_2 \circ \hat{p}_1 = p_2 \circ \hat{p}_1 = p_1$$

and $\hat{p}_2 \circ \hat{p}_1(\hat{x}_1) = \hat{x}_1$

$\therefore$ $\hat{p}_2 \circ \hat{p}_1$ is a lift of $p_1$ (diagram)

$$\begin{tikzcd} \hat{X}_1 \arrow[rr, "\mathrm{id}X_1"] \arrow[dr] & & \hat{X}_1 \arrow[dl] \\ & X & \end{tikzcd}$$

But $\mathrm{id}\hat{X}_1$ is also a lift, and they agree on $\hat{x}_1$, so by uniqueness of lifts:

$$\hat{p}_2 \circ \hat{p}_1 = \mathrm{Id}X_1. \quad □$$

Likewise, $\hat{p}_1 \circ \hat{p}_2 = \mathrm{id}X_2$, so $\hat{p}_1$ and $\hat{p}_2$ are covering transformations

$\Rightarrow (\hat{X}_1, \hat{x}_1)$ and $(\hat{X}_2, \hat{x}_2)$ are the same.

Recall 13th November 13

$H \trianglelefteq G \Rightarrow gHg^{-1} = H \quad \forall g \in G$

Proof of part 2 of the classification theorem

This follows from the General lifting lemma as follows:

$\Rightarrow \quad H = p_{H*}\pi_1(\hat{X}_H, \hat{x}_H) = (p_K \circ q)_* \pi_1(\hat{X}_H, \hat{x}_H)$

$= (p_{K*} \circ q_*)\pi_1(\hat{X}_H, \hat{x}_H)$ by functoriality

$\subseteq p_{K*}\pi_1(\hat{X}_K, \hat{x}_K) = K.$

$\Leftarrow$ Suppose $H \subseteq K$.

By the general lifting lemma $\exists \hat{p}_K$

$$\begin{array}{ccc} & & (\hat{X}_K, \hat{x}_K) \\ & \nearrow^{\hat{p}_K} & \downarrow p_K \\ (\hat{X}_H, \hat{x}_H) & \xrightarrow{p_H} & (X, x_0) \end{array}$$

Put $q = \hat{p}_K$.

So we have proved 2.

Degrees

Definition

Let $X$ be path connected and let $p: \hat{X} \longrightarrow X$ be a covering map.

Then $\deg(p) = \# p^{-1}(x) \quad \forall x \in X$ (number of pre-images of every point in $x$).

Why does this not depend on $x$? Consider the map $X \to \mathbb{N}$ (with the discrete topology), $x \mapsto \# p^{-1}(x)$. This map is continuous, because $p$ is a covering map. But $X$ is connected. The only way this is possible is if the map is constant.

$\Rightarrow \deg(p)$ does not depend on $x$.

Proposition

If $p_H: (\hat{X}_H, \hat{x}_H) \longrightarrow (X, x_0)$ is a covering map (with $H = p_{H*}\pi_1(\hat{X}_H, \hat{x}_H)$) then:

$$\deg p_H = [\pi_1(X, x_0) : H]$$

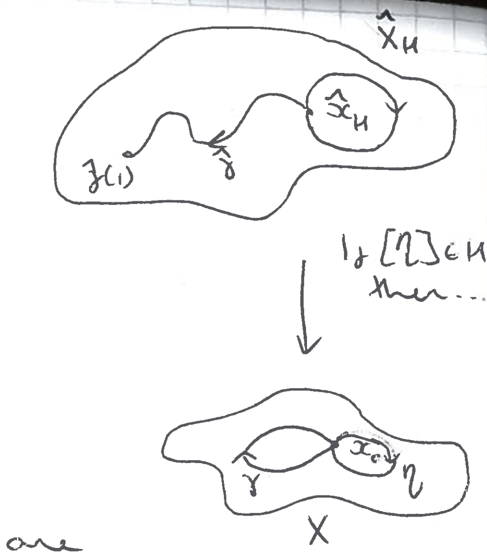
Lemma

With the same notation as above, the map:

$$\begin{array}{ccc} H \backslash \pi_1(X, x_0) & \longrightarrow & p_H^{-1}(x_0) \\ H[\gamma] & \longmapsto & \hat{\gamma}(1) \end{array}$$

is a bijection.

Proof




First we need to prove that is well-defined.
Suppose $\exists \gamma'$ such that $H[\gamma'] = H[\gamma]$
(i.e. $[\gamma'][\gamma]^{-1} \in H$)

Need to prove that $\hat{\gamma}'(1) = \hat{\gamma}(1)$.

Well, $[\gamma'][\gamma]^{-1} = [\gamma' \cdot \bar{\gamma}] \in p_H \pi_1(\hat{X}_H, \hat{x}_H)$

$\Leftrightarrow \hat{\gamma}' \cdot \hat{\bar{\gamma}}$ is a loop in $\hat{X}_H$

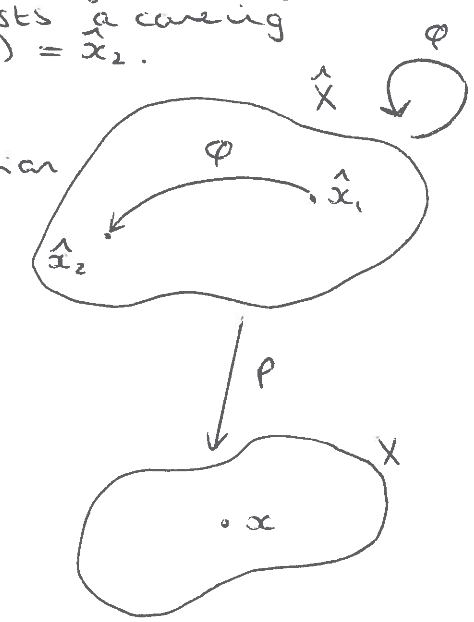
$\Leftrightarrow \hat{\gamma}'(1) = \hat{\gamma}(1)$ as required.

Injective because all the above implications are reversible.

Surjective: Let $\hat{y} \in p_H^{-1}(x_0)$. If $\hat{\gamma}$ is any path from $\hat{x}_H$ to $\hat{y}$, then setting $\gamma = p \circ \hat{\gamma}$, $H[\gamma] = \hat{\gamma}(1)$ by definition. $\square$

Next we will prove part 3 of the classification theorem.

Definition




A covering space $p: \hat{X} \longrightarrow X$ is called normal or regular if for every $x \in X$, and $\forall \hat{x}_1, \hat{x}_2 \in p^{-1}(x)$, there exists a covering transformation $\varphi: \hat{X} \longrightarrow \hat{X}$ such that $\varphi(\hat{x}_1) = \hat{x}_2$.

Recall that a covering transformation means that $p \circ \varphi = p$. This covering transformation is necessarily unique.

Lemma

We can now start to prove part 3: with the usual notation:

$$H \trianglelefteq \pi_1(X, x_0) \Leftrightarrow \hat{X}_H \xrightarrow{p_H} X$$

is a normal covering.

Proof

$\Leftarrow$ Let $\hat{X}_H \xrightarrow{p_H} X$ be normal, and let $g = [\gamma] \in \pi_1(X, x_0)$. We need to prove that $gHg^{-1} = H$.

Let $h = [\eta] \in H$. We need to prove that:

$$ghg^{-1} = [\gamma \cdot \eta \cdot \bar{\gamma}] \in H \qquad \text{(i.e. that } gHg^{-1} \subseteq H\text{)}$$

Let $\varphi$ be the unique covering transformation such that:

$$\varphi(\hat{x}_H) = \hat{\gamma}(1)$$

Now $\hat{\gamma} \cdot (\varphi \circ \hat{\eta}) \cdot \hat{\bar{\gamma}}$ is a loop in $\hat{X}_H$, based at $\hat{x}_H$, and:

$$(p_H) \circ (\hat{\gamma} \cdot (\varphi \circ \hat{\eta}) \cdot \hat{\bar{\gamma}}) = \gamma \cdot \eta \cdot \bar{\gamma}$$

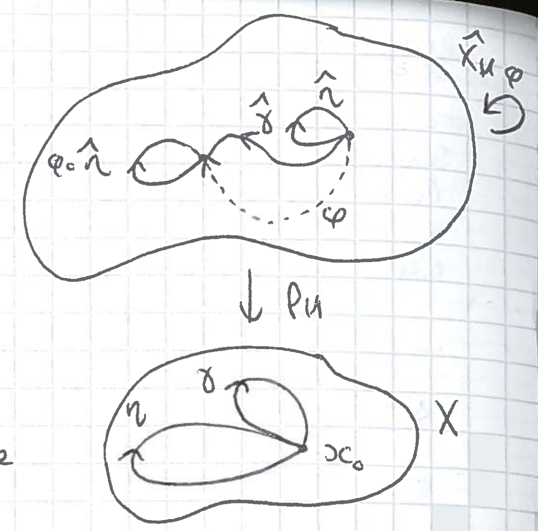
It was the lift of $\gamma\cdot\eta\cdot\bar\gamma$

$\therefore [\gamma\cdot\eta\cdot\bar\gamma]\in H$, so $ghg^{-1}\subseteq H$.

But $g$ was arbitrary, so we also have:

$$g^{-1}Hg\subseteq H$$
$$\iff H\subseteq gHg^{-1}$$

So they're equal as required.




$\Rightarrow$ Let $x\in X$ and $\hat x,\hat y\in p_H^{-1}(x)$. We need to construct a covering transformation $\varphi$ such that $\varphi(\hat x_1)=\hat x_2$.

First, given $g=[\gamma]$, we will construct a corresponding covering transformation $\varphi_g\colon \hat X_H\longrightarrow X_H$.

We will do this by defining $\varphi_g(\hat x)$.

Choose a path $\hat\alpha$ from $\hat x_H$ to $\hat x$. Define $\varphi_g(\hat x)$ to be:

$$\varphi_g(\hat x)=\widehat{\gamma\cdot\alpha}(1)\quad\text{where } \alpha=p\circ\hat\alpha$$

Claim that
$$\begin{aligned}\pi_1(X,x_0)&\longrightarrow\{\text{covering transformations}\}\\ [\gamma]&\longmapsto\varphi_g\end{aligned}$$

Suppose $\beta$ is another path from $\hat x_H$ to $\hat x_g$. We need to prove that

$$\widehat{\gamma\cdot\alpha}(1)=\widehat{\delta\cdot\beta}(1)$$

where $\beta=p_H\circ\hat\beta$

$\iff \hat\gamma\cdot(\hat\alpha\cdot\hat\beta)\cdot\bar{\hat\delta}$ is a loop

If we set $\eta=\alpha\cdot\bar\beta$, then we are asking if $g[\eta]g^{-1}\in H$.

Yes! because $H$ is normal! This proves the claim.
(Except, need to check that it really is a covering transformation.)

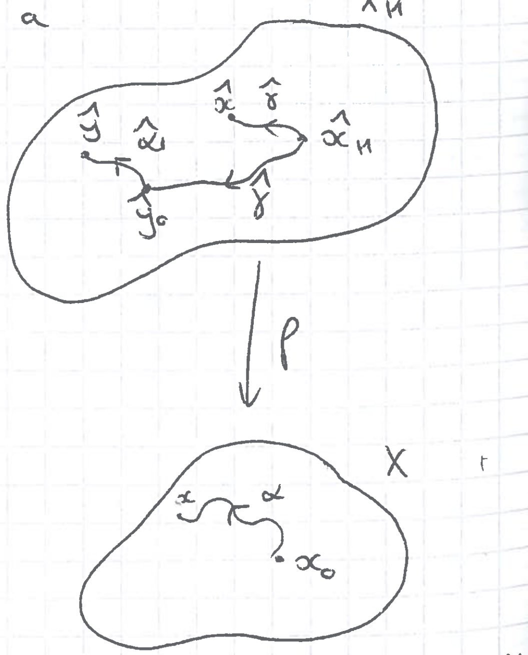
Choose a path $\hat\alpha$ from $\hat x_y$ to $\hat x$. Let $\alpha=p_H\circ\hat\alpha$ such and let $\alpha'$ be the (backwards) lift of $\alpha$ at $\hat y$.

Let $\hat y_0=\hat\alpha'(0)\in p_H^{-1}(x_0)$.

Let $\hat\gamma$ be a path from $\hat x_H$ to $\hat y_0$ and let $\gamma=p_H\circ\hat\gamma$.

Then by construction:

$$\varphi_{[\gamma]}(\hat x)=\hat y.$$




$\alpha=p_H\circ\hat\alpha,\ x=p_H(\hat x)=p_H(\hat y)$

This proves the lemma □

To finish off, we need to prove that

the map $\pi_1(X,x_0)\longrightarrow\{\text{covering transformations}\}$ induces an isomorphism:

$$H\backslash\pi_1(X,x_0)\longrightarrow\{\text{covering transformations}\}$$

<u>Step 1</u>

Check its an isomorphism, left as an exercise.

<u>Step 2</u>

Check that the kernel is $H$:

<u>Step 3</u>

Check that it's surjective.

<u>Proof of Step 2</u>

Suppose $\varphi_{[\gamma]}=\mathrm{id}$. Then $\varphi_{[\gamma]}(\hat x_H)=\hat x_H$.

$\iff \hat\gamma(1)=\hat x_H$

$\iff \hat\gamma$ is a based loop in $\hat X_H$.

$\iff [\gamma]\in H$.

<u>Proof of Step 3</u>

Let $\varphi$ be a covering transformation, and let $\hat y_0=\varphi(\hat x_H)$. Let $\hat\gamma$ be a path from $\hat x_H$ to $\hat y_0$, and $\gamma=p_H\circ\hat\gamma$. By construction,

$$\varphi_{[\gamma]}(\hat x_H)=\hat y_0=\varphi(\hat x_H)=\varphi_{[\gamma]}=\varphi.\ \square$$

So $H/\pi_1(X,x_0)\cong\{\text{covering transformations}\}$

Finally, we want to prove that

$$\hat X_H/(\pi_1(X,x_0)/H)\cong X.$$

Put $Q=\pi_1(X,x_0)/H$.

$$\begin{tikzcd} Q\circlearrowright & \hat X_H \arrow[dl] \arrow[d] \\ \hat X_H/Q \arrow[r] & X \end{tikzcd}$$

Note: the action of $Q$ is freely discontinuous

But each point of $\hat X_H/Q$ is in bijection with a point of $X$.

So this map exists as a bijection. It's easy to check that its continuous with continuous inverse.

This proves part 3 of the classification theorem □.

<u>Proof of part 1, surjectivity of the classification theorem</u>

$$(\hat X_H,\hat x_H)\longmapsto H=p_{H*}\pi_1(\hat X_H,\hat x_H)$$

$$\{\text{covering spaces}\}\longrightarrow\{\text{subgroups}\}$$

we will use the following Theorem which will be proven later.

**Theorem**

If $X$ is a cell-complex (connected), then $X$ has a universal cover: $(\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$

**Proof of part 1**

By part 3, $\pi_1(X, x_0)$ acts by covering transformations on $\tilde{X}$.

Define $\hat{X}_H = \tilde{X}/H$

Because $H$ acts freely discontinuously, we can let $q: \tilde{X} \to \hat{X}_H$ be the quotient map which is a covering map.

It's easy to see that the universal covering map $\tilde{p}: \tilde{X} \to X$ descends to a covering map $\hat{p}_H: \hat{X}_H \to X$.

It remains to check that:

$$H = p_{H\#}\pi_1(\hat{X}_H, \hat{x}_H)$$

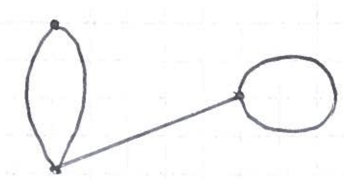
This is easy with the tools from proof of part 3. $\square$

**Graphs and free groups** — 19th November 13

**Definition**

A graph is a 1-dimensional cell complex. i.e. a cell complex $X$ such that $X = X^{(1)}$.

0-cells: vertices
1-cells: edges



– a finite graph.

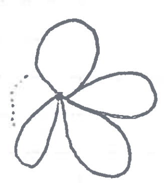
**Example**

The rose with $r$ petals is with a graph with 1 vertex and $r$ edges:

$X$ = 

$X_2$ = 

⋮

$X_r$ = 

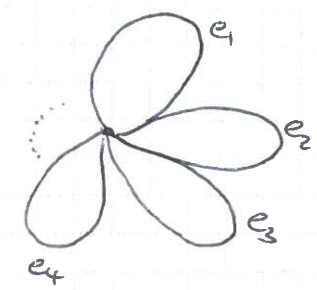
Alternatively, if $(X, x_0)$, $(Y, y_0)$ the wedge product is defined as:

$$X \vee Y = X \sqcup Y / x_0 \sim y_0$$


so $X_r = S^1 \vee S^1 \vee \dots \vee S^1$ ← $r$ copies of the circle glued together at the basepoint $x_0$.

NOTE: $r$ does not necessarily have to be finite! $r$ can be infinite, but we will usually think of it as finite.

**Definition**

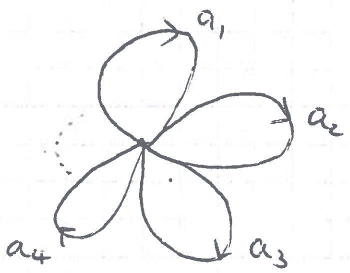



The free group of rank $r$ is defined to be:

$$F_r = \pi_1(X_r, x_0)$$

Let $\alpha_i: [0,1] \longrightarrow X_r$ be a path such that $\alpha_i$ identifies $(0,1)$ homeomorphically with $e_i$.

Let $a_i = [\alpha_i] \in F_r$.




(often where $r$ is small, $a = a_1$, $b = a_2$, $c = a_3$ and so on...)

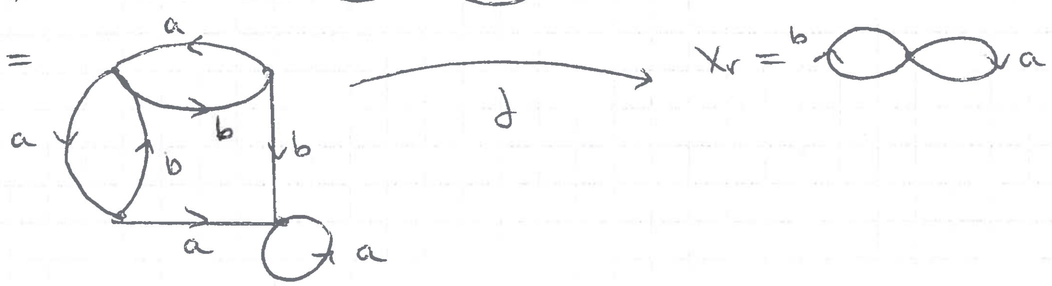
construct the universal cover $\tilde{X}_r$: this is our goal!

Given a graph $Y$, we can describe a map $Y \to X_r$ by labelling each edge of $Y$ with an arrow and one of the $a_i$.

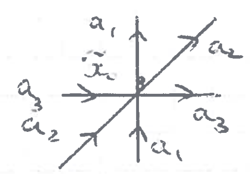
e.g.

if $r = 2$, then $X_r$ = 

Let $Y$ =



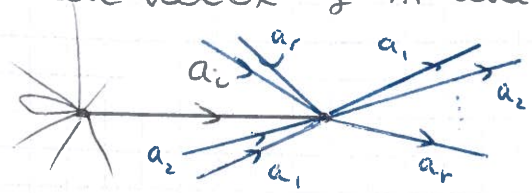

Now let's construct $\tilde{X}_r$ inductively:

we start with $Y_1$ =  ($r = 3$ here)


$Y_1$ has $2r$ edges: $r$ point outwards away from $\tilde{x}_0$, while $r$ point inwards towards $\tilde{x}_0$.

Construct $Y_{n+1}$ from $Y_n$ as follows:

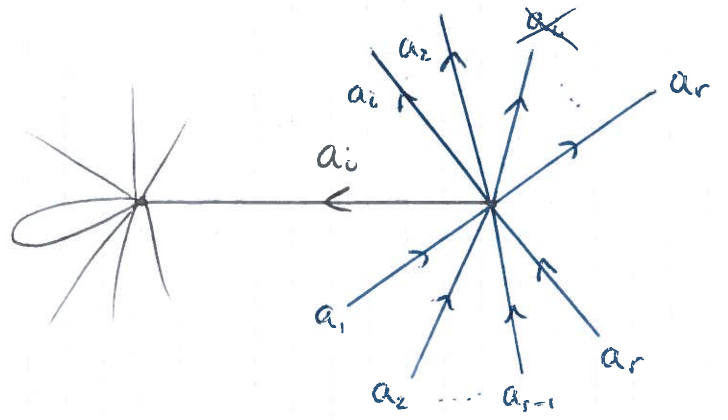
For each vertex of $Y_n$ like this:




but not $a_i$ going in!

(vertices with only $a_i$ coming in). we add the vertices in blue.

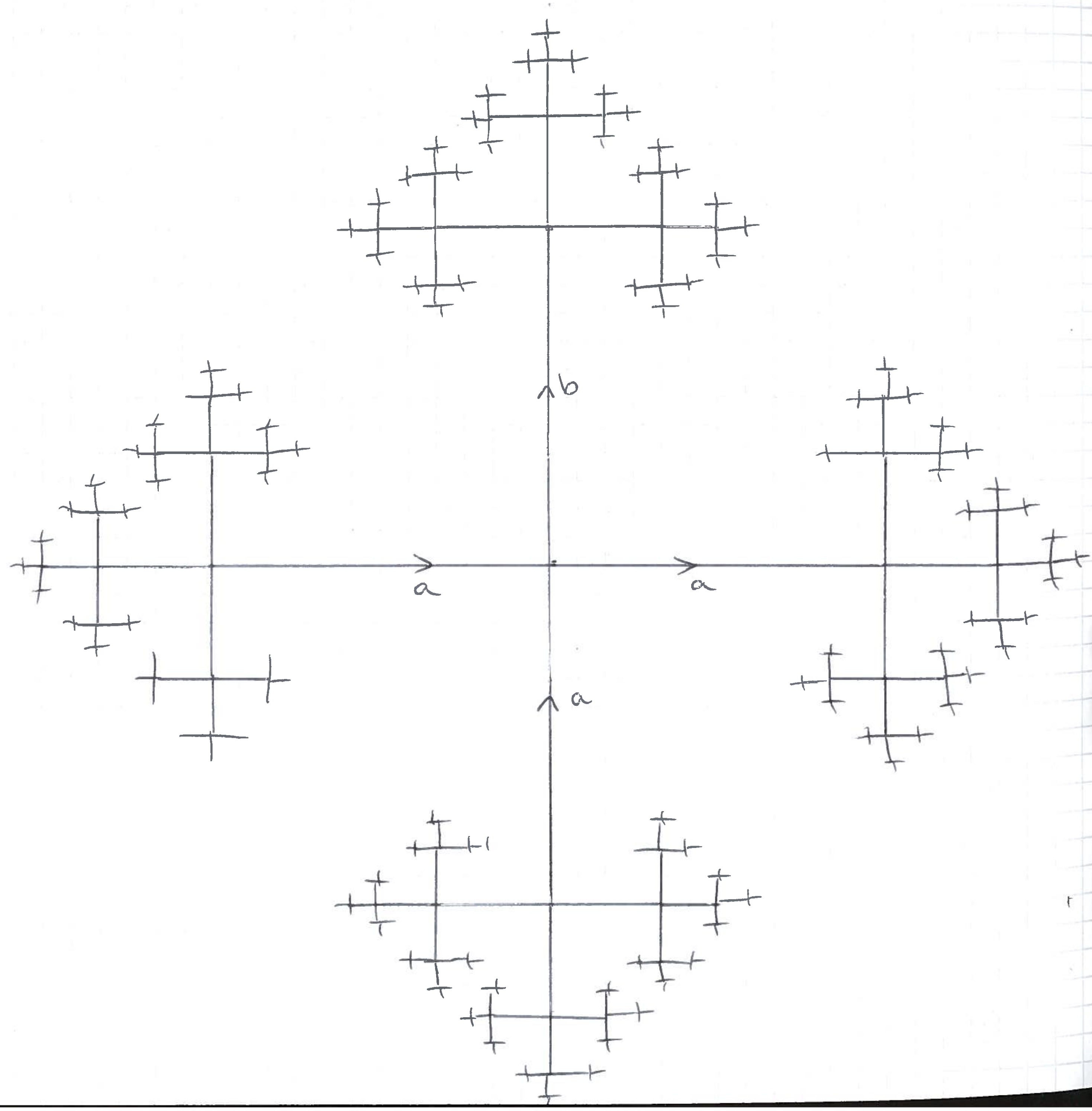
For each vertex as follows:




At each stage, $Y_n$ is a tree (a graph with no loops, i.e. connected and simply connected) and $Y_n \hookrightarrow Y_{n+1}$.

For $X_2 = \infty$

we get something like:




Let $\tilde{X}_r = \bigcup_{i=1}^{\infty} Y_i$

The resulting graph has the following properties:

1) Path connected

2) Simply connected

Let $\gamma$ be a loop in $\tilde{X}_r$. Then $\operatorname{im}\gamma \subseteq Y_n$ for some $n < \infty$, so homotopic to a point.

3) Regular 2-valent graph.

4) The map $\tilde{X}_r \longrightarrow X_r$ is a covering map.

$\Rightarrow$ $\tilde{X}_r$ is the universal cover of $X_r$, and $F_r$ acts on $\tilde{X}_r$ by covering transformations.

Basic geometry of trees

- every pair of points is joined by a "unique" path (a unique sensible path!).

Lemma

Every pair of distinct points in $\tilde{X}_r$ is joined by a unique injective path. (up to reparametrization)

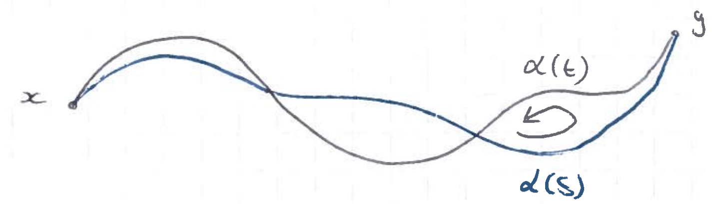
Proof

Take $n$ big enough that $x, y \in Y_n$. Existence is clearly true by construction.

For uniqueness, suppose $\alpha, \beta$ are distinct injective paths.

Take $s < t$ such that:

$\alpha(s) \in \operatorname{im}\beta$
$\alpha(t) \in \operatorname{im}\beta$




but $\forall\, s < u < t$, $\alpha(u) \notin \operatorname{im}(\beta)$.

Now we have an embedded $S^1 \hookrightarrow Y_n$ which contradicts the fact that $Y_n$ is a tree.

Lemma

1) Every path from $x$ to $y$ is homotopic rel $\{0,1\}$ to the embedded path.

2) If $\alpha$ is locally injective, then $\alpha$ is injective. i.e. $\forall t \in [0,1]$ $\exists \varepsilon > 0$ such that $\alpha|_{(t-\varepsilon, t+\varepsilon)}$ is injective.

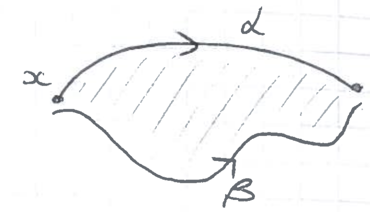
Proof

1) Follows because $\tilde{X}_\sigma$ is simply connected.

Let $\alpha$ be an injective path from $x$ to $y$, and let $\beta$ be any path from $x$ to $y$.

Then $\alpha \cdot \bar{\beta}$ is a loop in $\pi_1(\tilde{X}_r, x)$, hence homotopic to a point.

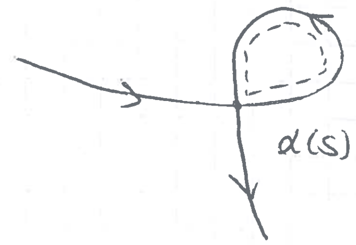
$\therefore$ the path $\alpha, \beta$ together bound the image of a disk and we can push $\beta$ over the disk to $\alpha$.




2) Suppose $\alpha$ is locally injective and let $s, t$ be such that $\alpha(s) = \alpha(t)$.

But $\alpha|_{(s,t)}$ is injective.

Now $\alpha([s,t])$ is an embedded circle, which contradicts the fact that $T_n$ was a tree. □




## Free Groups

Let $\mathfrak{X} = \{x_1, \dots, x_r\}$
$\mathfrak{X}^{\pm} = \{x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}\}$

Then $\mathfrak{X}^* = \{$all strings on the letters $\mathfrak{X}^{\pm}\}$
e.g. $x_{27} x_2^{-1} x_6^{-1} x_5 x_{52}$, $x_3 x_3 x_3 \cdots x_3 = x_3^n$, $\emptyset$

Elements of $\mathfrak{X}^*$ are called words: $w(\underline{x})$

A pinch is a subword of the form $x_i^{\pm 1} x_i^{\mp 1}$.

An elementary reduction removing a pinch from a word is $u x_i^{\pm 1} x_i^{\mp 1} v = uv$.

A word is called reduced if it contains no pinches.

e.g.

$x_1^{-1} x_2 x_2^{-1} x_1 x_1^{-1}$

two pinches, first cancel the $x_2$'s:

$\rightsquigarrow x_1^{-1} x_1 x_1^{-1}$

two pinches, first cancel the $x_1$'s on the left:

$\rightsquigarrow x_1^{-1}$

How do we know we always get the same answer?

It's clear that a finite number of elementary reductions can always be applied to reach a reduced word. It's not obvious that the result is independent of choices made.

There's a map:

$$\Phi : \mathfrak{X}^* \longrightarrow F_r$$
$$w(\underline{x}) \longmapsto w(\underline{a})$$

e.g. $x_1 x_2 x_1^{-1} x_2^{-1} \longmapsto a_1 a_2 a_1^{-1} a_2^{-1}$

NOTE: if $w_1(\underline{x}) \rightsquigarrow w_2(\underline{x})$ is an elementary reduction, then:

$$\Phi(w_1(\underline{x})) = \Phi(w_2(\underline{x}))$$

**Theorem**

$\forall g \in F_r$, $\Phi^{-1}(g)$ contains exactly one reduced word.

**Proof**

For $w(\underline{x}) = x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \cdots x_{i_n}^{\pm 1} \in \mathfrak{X}^*$, we have an obvious representative for $\Phi(w(\underline{x}))$

$$\Phi(w(\underline{x})) = [\alpha_{i_1}^{\pm 1} \cdot \alpha_{i_2}^{\pm 1} \cdots \alpha_{i_n}^{\pm 1}]$$

For $g \in F_r$ represent $g$ by some loop $g = [\gamma]$. Now lift $\gamma$ to a path $\hat{\gamma}$ in $\tilde{X}_r$.

Let $\hat{\beta}$ be the unique embedded path from $\hat{x}_0$ to $\hat{\gamma}(1)$. Up to reparametrization;

$$\hat{\beta} = \widehat{\alpha_{i_1}^{\pm 1} \alpha_{i_2}^{\pm 1} \cdots \alpha_{i_n}^{\pm 1}}$$

for some indices.

Therefore, if $w(\underline{x}) = x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1}$ because they're both represented by paths whose lifts then have the same end point in the universal cover.

Suppose $w(\underline{x})$ contains a pinch $x_i \cdot x_i^{-1}$. Then $\hat{\beta}$ contains a subpath which is a lift of $\alpha_i \cdot \bar{\alpha}_i$.

This contradicts that $\hat{\beta}$ is injective.
$\therefore$ $w(\underline{x})$ is reduced.

Similarly, if $w_1(\underline{x})$ and $w_2(\underline{x})$ are two reduced words with $\Phi(w_1(\underline{x})) = \Phi(w_2(\underline{x})) = g$, then, because $\tilde{x}_0$ and $\hat{\gamma}(1) = g x_0$ are joined by a unique injective path, $w_1(\underline{x}) = w_2(\underline{x})$ because reduced words give locally injective (hence injective) paths. □

**Corollary** (Universal property)

Let $H$ be a group and $h_1, \dots, h_r$ be elements. Then there exists a unique homomorphism:

$$F_r \qquad H$$

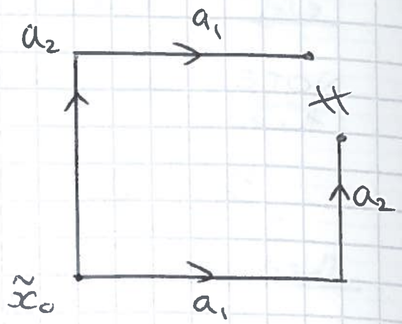
such that $a_i \longmapsto h_i \ \forall i$.

**Corollary**

If $r \geq 2$, $F_r$ is non-abelian.

## Prog

$a_1 a_2 a_1^{-1} a_2^{-1}$ has no pinches
$\therefore$ $a_1 a_2 a_1^{-1} a_2^{-1} \neq 1$. □




## Fact

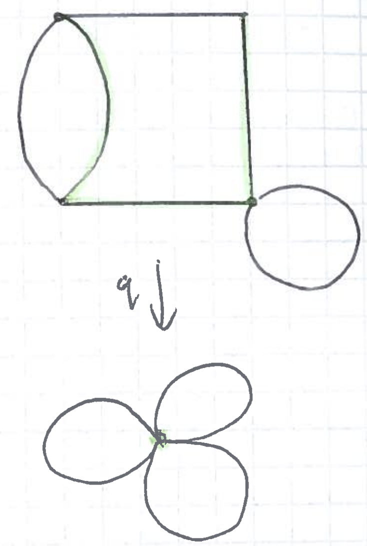
If $r \neq s$, $F_r \not\cong F_s$

## Other graphs

## Recall

Every graph $X$ contains a maximal subtree $T$, which contains every vertex.

Let $X/T = X/\sim$ where $t_1 \sim t_2$ whenever $t_1, t_2 \in T$.




## Proposition

The quotient map $q: X \longrightarrow X/T$ is a homotopy equivalence.

## Prog

See Hatcher's book: "homotopy extension property"! □

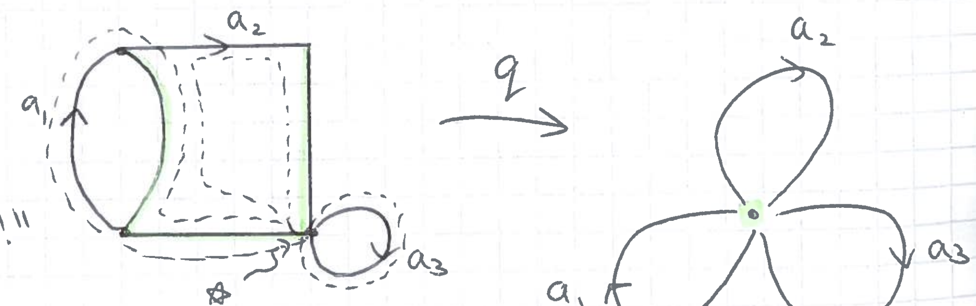
Using this, we can write down:

## Corollary

For any connected graph $X$, $\pi_1 X$ is free.

What's more, we can write down a generating set of loops.

" Go along the maximal tree to $a_i$ and back down the maximal tree!"



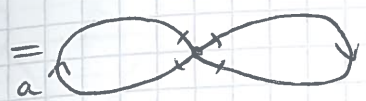

## EXAM Q'S IN THIS!

For a choice of basepoint ✲ in $X$, we can write down loops $\beta_1, \dots, \beta_r$ in $X$ such that:

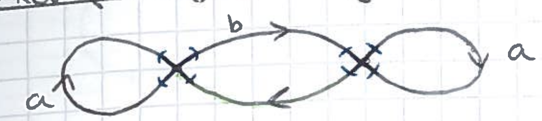
$$q_* [\beta_i] = a_i \quad \text{for each } i.$$

## Subgroups of $F_r / F_2$

We can now write down lots of examples of covering spaces of $X_r$ $\longleftrightarrow$ subgroups of $F_r$.

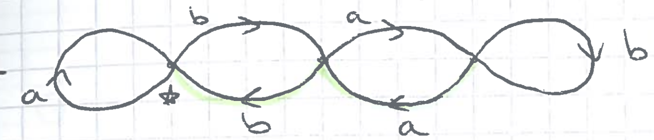
$X_2 =$ 


## Examples of covering spaces

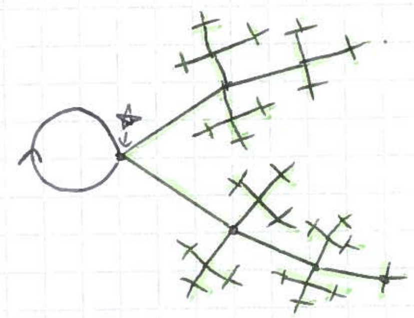
1) 


This is a covering space because each vertex looks the same (with labels and arrows) as the vertex of $X_2$

- it has an $a$ going in and going out
- it has a $b$ going in and going out.

2) 


This is a NON-NORMAL covering space! We do not have the required symmetry.

3) 

This is an infinite-degree covering space.

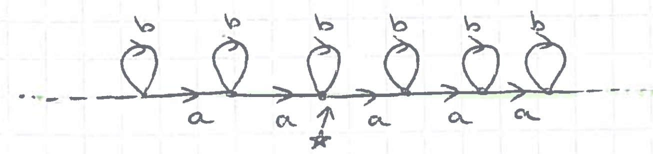
NB: $X_r$, $F_r$, then $r$ is called the rank.

Performing the algorithm on the covers to get a generating set for the subgroup:

1) $\langle a, b^2, b^{-1}ab \rangle$ ↑ (— is maximal tree)

2) $\langle a, b^2, b^{-1}a^2, b^{-1}a^{-1}bab \rangle$

3) $\langle a \rangle$

4) 


This is a covering space (4 edges going in and out each). The maximal tree is —

This is an infinite rank subgroup!

The $\langle a^n b a^{-n} : n \in \mathbb{Z} \rangle$

NOTE: The covering space of a graph is a graph!

__Theorem__

(Nielson and Schrier)

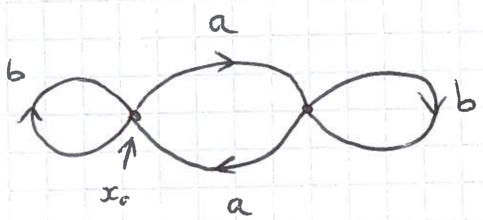
Every subgroup of a free group is free.

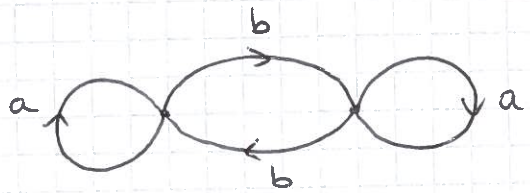
__Proof__

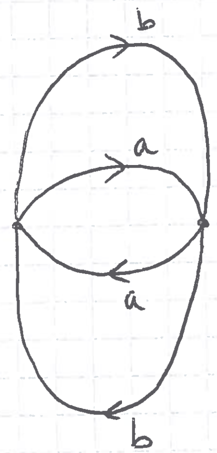
Every covering space of a graph is a graph! □

__Examples of covering spaces__ 　　　　__26th November__

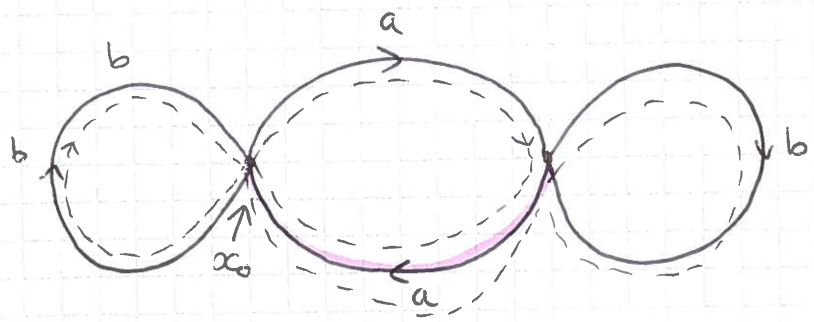
__Deg = 2__ (connected)




- Pick a maximal tree
- generators are given by: start at base point, round the edge and back to the base point, as below:



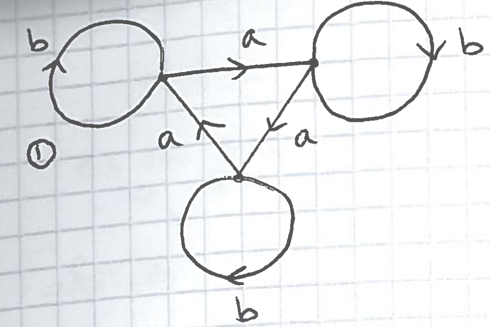

— minimal spanning tree.

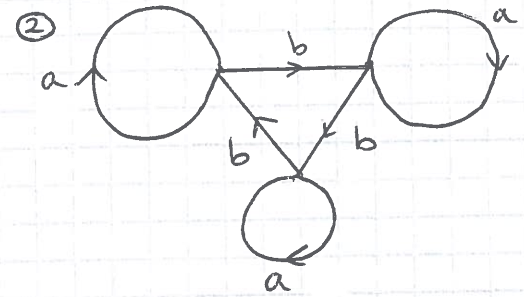
gives $\langle b, a^2, a^{-1}ba \rangle$

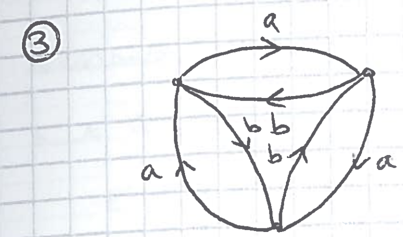
__Recall__

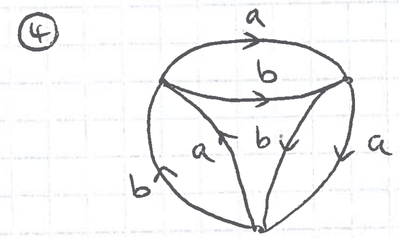
Every covering space of degree two is normal, because every subgroup of index two is normal.

__Deg = 3__

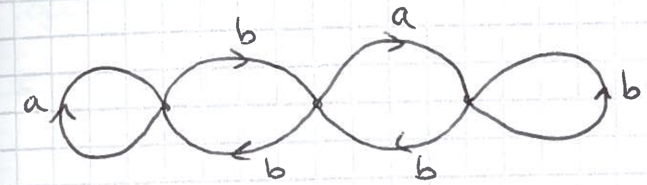


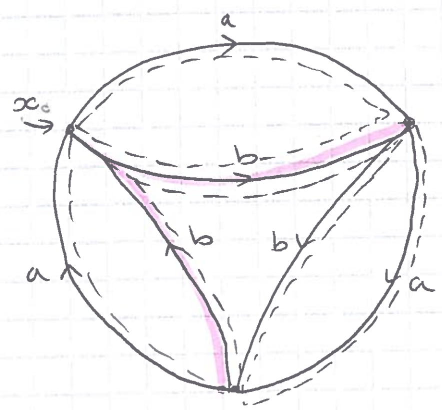




(1) – (4) are all normal.

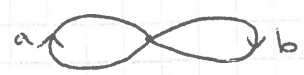



NOT normal! back to #4:




— maximal tree.

gives $\langle ab^{-1}, bab, b^3, b^{-1}a \rangle$

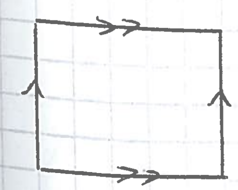
$F_2$: 

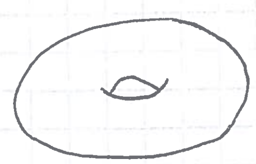
$\exists f : F_2 \longrightarrow \mathbb{Z}^2 = \langle x \rangle \otimes \langle y \rangle$

$a \longmapsto x$

$b \longmapsto y$

__The Torus__

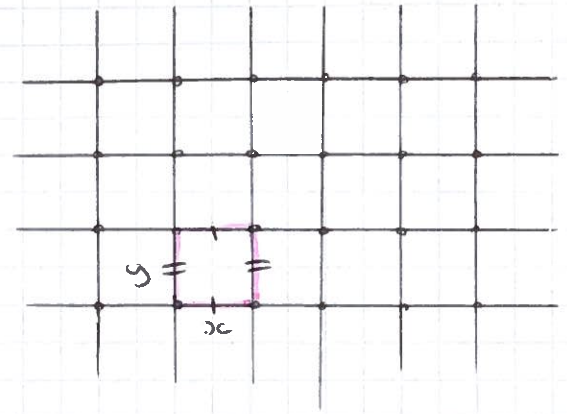




$S^1 \times S^1$.

what is the universal cover of the Torus?

$\mathbb{R}^2$



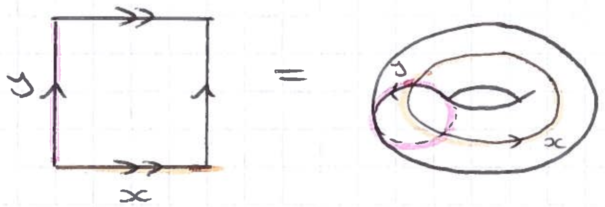

$\mathbb{Z}^2$ acts on ↗ by translation. Let $g \in \mathbb{Z}^2$.

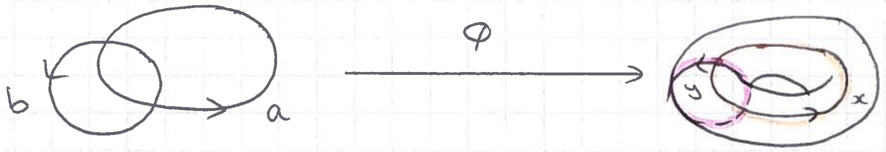
$g \# x \longmapsto x + g$

This is a freely discontinuous.

| | edges get identified to each~~other~~ by the translation
= = identified as above.

So get

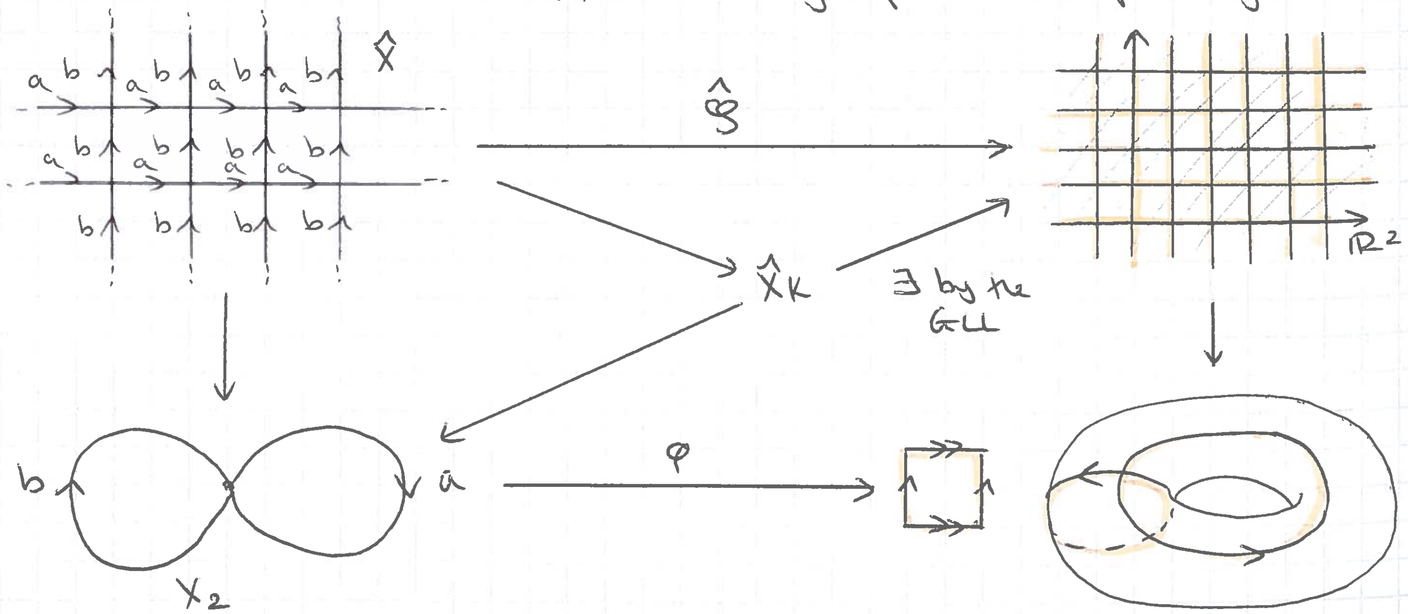



Note that $\varphi_\# = f$

Let $K = \ker(f)$.

QUESTION: what is $\hat{X}_K$, the covering space corresponding to K?




$\hat{\varphi}$ is a lift of $\varphi \circ p$ to $\mathbb{R}^2$. By the General Lifting Lemma:

$$\pi_1 \hat{X} = \varphi_\#^{-1} \pi_1(\mathbb{R}^2) = f^{-1}(1) = \ker(f) = K$$

By the GLL, $\hat{X}_K$ is also identified with the orange lattice on $\mathbb{R}^2$, so $\hat{X} = \hat{X}_K$.

## Rank

Suppose $s < r$, we would like to prove that $F_s \not\cong F_r$.

### Definition

Let G be a group and S a subset. Then:

$$\langle S \rangle = \bigcap \left\{ \begin{matrix} \text{all subgroups} \\ \text{containing } S \end{matrix} \right\} \text{ – "subgroup generated by S"}$$

If $G = \langle S \rangle$ for $|S| < \infty$, we say that G is finitely generated.

### Example

$$(\mathbb{Z}/2)^r = \underbrace{\mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2}_{r \text{ copies}}$$

This can be generated by r elements but NOT by s elements, where $s < r$.

(Proof: if $|S| = s$ and $S \subseteq (\mathbb{Z}/2)^r$, then $\langle S \rangle$ is a vector subspace of dimension $\leq s < r \Rightarrow \langle S \rangle \neq (\mathbb{Z}/2)^r$) □

### Lemma

If $s < r$, then $F_s \not\cong F_r$.

### Proof

Suppose $\varphi : F_s \longrightarrow F_r$ is an isomorphism.

$$\langle a_1, \dots, a_s \rangle \quad \langle b_1, \dots, b_r \rangle$$

Then let $q : F_r \longrightarrow (\mathbb{Z}/2)^r$ be the obvious map that sends each $b_i$ to a different factor.

Then $q \circ \varphi$ is also a surjection.

$\Rightarrow \{ q \circ \varphi(a_i) \mid i = 1, \dots, s$ generate $(\mathbb{Z}/2)^r \}$

This provides a contradiction. □

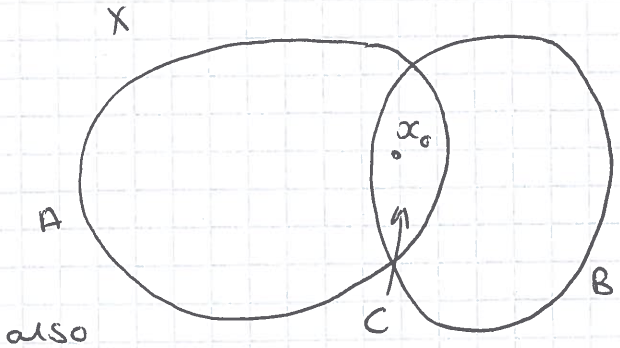
To compute more fundamental groups, we need to know what happens when we glue spaces together.

The Seifert-Van Kampen Theorem addresses this! As follows:

## The Seifert–Van Kampen Theorem

Let $X$ be a path-connected topological space. Suppose that:

$$X = A \cup B$$




where $A$, $B$ are both open and path-connected and $C = A \cap B$ is also path-connected. Let $x_0 \in C$. Then the diagram ☆ is a "push-out"!

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} \pi_1(C, x_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "☆"] & \pi_1(A, x_0) \arrow[d] \\ \pi_1(B, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

### What is a push-out?

We have a commutative diagram, as follows, is a push-out if:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left, "j_A"] & \\ B \arrow[r] \arrow[drr, bend right, "j_B"'] & X & \\ & & \Gamma \end{tikzcd}$$

wherever $j_A : A \longrightarrow \Gamma$, $j_B : B \longrightarrow \Gamma$ are homomorphisms that still make the diagram commute:

$$\exists!\ \varphi : \ \longrightarrow \Gamma$$

making the diagram commute.

### Lemma

If

$$\begin{tikzcd} C \arrow[r, "g_A"] \arrow[d, "g_B"'] & A \arrow[d] \\ B \arrow[r] & G \end{tikzcd}$$

are homomorphisms, then the push-out $G$ is unique up to isomorphism.

### Proof

Suppose $G_1$ and $G_2$ are both pushouts:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl] \\ B \arrow[r] \arrow[ur, phantom, ""] & G_1 \end{tikzcd}$$

$G_2$, $\eta$, $\xi$

— exists because $G_2$ is a push-out.

— exists because $G_1$ is a push-out.

Then $\xi \circ \eta : G_2 \longrightarrow G_2$ is the unique such map that makes:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \arrow[dr, bend left] & \\ B \arrow[r] \arrow[dr, bend right] & G_2 \arrow[r] & G_1 \\ & G_1 & \end{tikzcd}$$

But so does $id_{G_2}$.

By uniqueness, $\xi \circ \eta = id_{G_2}$. Similarly $\eta \circ \xi = id_{G_1}$.

$\Rightarrow \eta = \xi^{-1}$, and we have a bijection.

$\Rightarrow G_1 \cong G_2$.

□

### Example

$$\begin{tikzcd} 1 \arrow[r, "1 \mapsto g"] \arrow[d, "1 \mapsto h"'] & \mathbb{Z} = \langle g \rangle \arrow[d, "g \mapsto a"] \arrow[ddr, bend left, "j_A"] & \\ \mathbb{Z} = \langle h \rangle \arrow[r, "h \mapsto b"'] \arrow[drr, bend right, "j_B"'] & F_2 = \langle a, b \rangle & \\ & & \Gamma \end{tikzcd}$$

By the universal property of $F_2$, $\exists!$ map $F_2 \longrightarrow \Gamma$ where $a \longmapsto j_A(g)$, $b \longmapsto j_B(h)$.

i.e. this diagram is a push-out.

When we start with maps:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & A \\ B & \end{tikzcd}$$

we use the following notation for the push-out:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A ☆ B \end{tikzcd}$$

where $A ☆ B$ is called the free product of $A$ and $B$.

As we saw above, $F_2 \cong \mathbb{Z} ☆ \mathbb{Z}$, and more generally, $F_r \cong \underbrace{\mathbb{Z} ☆ \ldots ☆ \mathbb{Z}}_{r \text{ copies}}$.

Free products enjoy a similar description to the reduced

"reduced word" description we gave for free groups.

**Theorem**

An element of $G * H$ can be expressed uniquely as:

$$g_1 h_2 g_3 h_4, \dots, g_{n-1} h_n$$

where $g_i \in G$, $h_i \in H$ for $i \in \{1, \dots, n\}$

**Example**

The diagram is a push-out:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & 1 \arrow[d] \arrow[dd, bend left] \\
1 \arrow[r] \arrow[dr, bend right] & 1 \arrow[d] \\
 & G
\end{tikzcd}$$

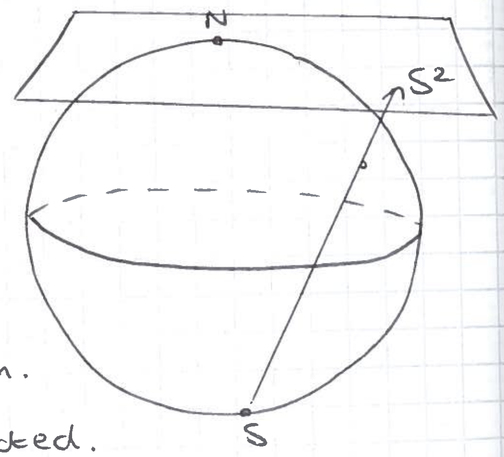
**Proposition**

$\pi_1(S^2)$ is trivial

**Proof**

Think of $S^2 \hookrightarrow \mathbb{R}^3$ in the usual way. Let:

$$A = S_2 \setminus \{N\}$$
$$B = S_2 \setminus \{S\}$$




A and B are certainly open, and their compliments $\{N\}$, $\{S\}$ respectively are closed.

CLAIM: $B \cong \mathbb{R}^2$, hence simply connected. By stereographic projection.

Similarly $A \cong \mathbb{R}^2$, hence simply connected.

Either stereographic projection gives that:

$$C = A \cap B = S^2 \setminus \{N, S\} \cong \mathbb{R}^2 \setminus \{0\} \cong S^1$$

$\Rightarrow \pi_1 C \cong \mathbb{Z}$

Therefore, by the Seifert-Van Kampen Theorem, the diagram

$$\begin{tikzcd}
\pi_1(C, x_0) \cong \mathbb{Z} \arrow[r] \arrow[d] & \pi_1(A, x_0) \cong 1 \arrow[d] \\
\pi_1(B, x_0) \cong 1 \arrow[r] & \pi_1(S^2, x_0)
\end{tikzcd}$$

is a push-out.

So $\pi_1(S^2, x_0) \cong 1$ □

**Corollary**

$\mathbb{R}^3 \not\cong \mathbb{R}^2$

**Proof**

If $\mathbb{R}^3 \cong \mathbb{R}^2$, then $\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\} \simeq S^1$. But $\mathbb{R}^3 \setminus \{0\} \simeq S^2$.

$\Rightarrow \pi_1(S^2, x_0) \cong \pi_1(S^1, x_0)$

$\Rightarrow 1 \cong \mathbb{Z}$

But $1 \not\cong \mathbb{Z}$, contradiction. □

**The Seifert-Van Kampen Theorem**

If A, B and C, $A \cap B = C$ and path connected, then the diagram:

$$\begin{tikzcd}
\pi_1(C, x_0) \arrow[r] \arrow[d] & \pi_1(A, x_0) \arrow[d] \\
\pi_1(B, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

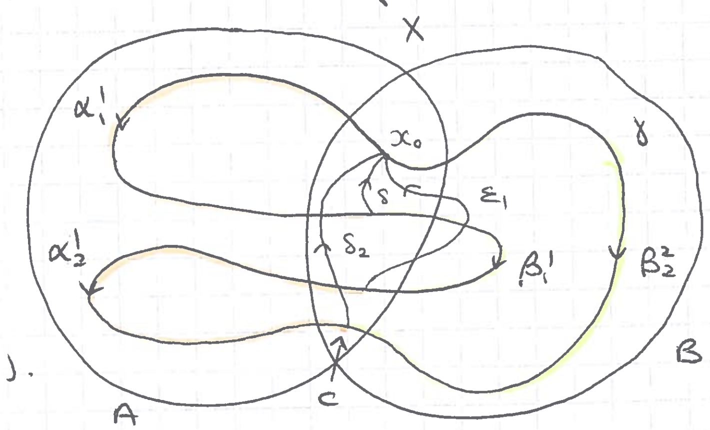
is a push-out

**Lemma 1**

If $\gamma$ is a based loop in $X$ then $\gamma \sim \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdot \beta_2 \cdot \ \cdots \ \cdot \alpha_n \cdot \beta_n$ (rel $\{0, 1\}$) where each $\alpha_i$ is a based loop in $B$.

**Proof**

We have $\gamma = \alpha'_1 \cdot \beta'_1 \cdot \ \cdots \ \cdot \alpha'_n \cdot \beta'_n$ where $\alpha'_i$ is a path in $A$ and $\beta'_i$ is a path in $B$.




(NOTE: that because A, B are open every $t \in [0,1]$ has an open neighbourhood $U = (t-\varepsilon, t+\varepsilon)$ such that $\gamma(U) \subseteq A$ or $B$. By compactness of $[0,1]$, there are finitely many such neighbourhoods, which cover $[0,1]$. This is why $n$ can be taken as finite).

If $\gamma(t_i)$ is the cross-over point from $\alpha'_i$ to $\beta'_i$, let $\delta_i$ be a path in $C$ from $\gamma(t'_i)$ to $x_0$. If $\gamma(s_i)$ is the cross-over point from $\beta'_i$ to $\alpha'_i$, let $\varepsilon_i$ be a path in $C$ from $\gamma(s'_i)$ to $x_0$.

Let $\ _0 = x_0$ and $\varepsilon_n = x_0$.

Set $\alpha_i = \bar{\varepsilon}_{i-1} \cdot \alpha'_i \cdot \delta_i$
$\beta_i = \bar{\delta}_i \cdot \beta'_i \cdot \varepsilon_i$

Each of these is a based loop. Furthermore:

$$\alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdot \beta_2 \cdot \ \cdots \ \cdot \alpha_n \cdot \beta_n$$
$$= \bar{\varepsilon}_0 \cdot \alpha'_1 \cdot \delta_1 \cdot \bar{\delta}_1 \cdot \beta'_1 \cdot \varepsilon_1 \cdot \bar{\varepsilon}_1 \cdot \alpha'_2 \cdot \ \cdots \ \cdot \bar{\delta}_n \cdot \beta'_n \cdot \varepsilon_n$$

But $S_i \cdot \bar{S}_i$ and $\varepsilon_j \cdot \bar{\varepsilon}_j \sim \{pt\}$, so:

$\alpha_1 \cdot \beta_1 \cdot \dots \cdot \alpha_n \cdot \beta_n \sim \alpha_1' \cdot \beta_1' \cdot \dots \cdot \beta_n'$ as $\bar{\varepsilon}_0 = x_0$ and $\varepsilon_n = x_0$.

$= \gamma$ rel $\{0,1\}$ □

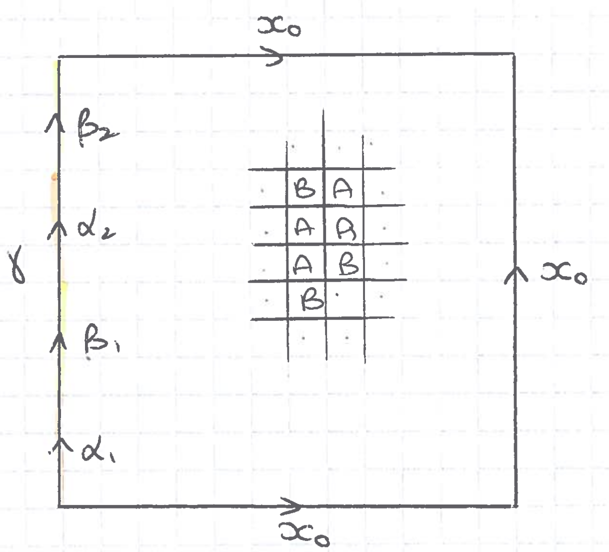
## Lemma 2

If $\gamma = \alpha_1 \cdot \beta_1 \cdot \dots \cdot \alpha_n \cdot \beta_n \sim x_0$ rel $\{0,1\}$. Then, for some $\alpha_i$ or $\beta_i$ either:

- $\alpha_i$ is homotopic to $C$ rel endpoints or
- $\beta_i$ is homotopic to $C$ rel endpoints.

## Sketch proof

Consider the homotopy $F: [0,1] \times [0,1] \longrightarrow X$ between $\gamma$ and $x_0$. ($F$ is based).




By compactness, $\exists \varepsilon > 0$ and finitely many $x_i$ such that:

$$B_\varepsilon(x_i) \subseteq A \text{ or } B$$

and $\bigcup B_\varepsilon(x_i) = [0,1] \times [0,1]$.

$\therefore$ we can subdivide into small squares $R_i$ such that:

$$F(R_i) \subseteq A \text{ or } B \ \forall i.$$

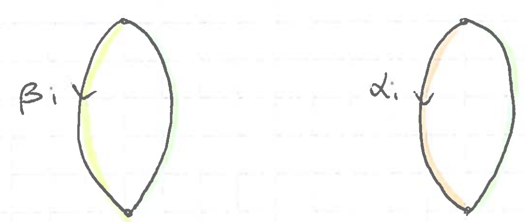
Label $R_i$ by 'A' if $F(R_i) \subseteq A$, and label $R_j$ by 'B' if $F(R_j) \subseteq B$.

Whenever $R_i, R_j$ share a side and have opposite labels, $R_i \cap R_j$ is a path in $C$.

| | |
|---|---|
| A | B |

These paths don't end.

Following these paths, we eventually find a region that looks something like this:




□

## Proof of the Seifert – Van Kampen Theorem

Suppose we have:

$$\begin{tikzcd}
\pi_1 C \arrow[r, "i_{A*}"] \arrow[d, "i_{B*}"'] & \pi_1 A \arrow[d, dash, "j_{A*}"] \arrow[ddr, bend left, "f_A"] & \\
\pi_1 B \arrow[r, dash, "j_{B*}"] \arrow[drr, bend right, "f_B"'] & \pi_1 C \arrow[dr, dashed, "?f"] & \\
& & \Gamma
\end{tikzcd}$$

Let $g \in \pi_1 X$. By Lemma 1, I can write $g = j_{A*}(a_1) j_{B*}(b_1) \cdots (j_{A*}(a_n) \cdot j_{B*}(b_n))$.

Choose such a factorisation with $n$ smallest. We must define:

$$f(g) = f_{A*}(a_1) f_{B*}(b_1) f_{A*}(a_2) \cdots f_{B*}(b_n)$$

We need to prove that this is well-defined, but then it is necessarily unique.

Suppose $g = j_{A*}(a_1) j_{B*}(b_1) \cdots j_{B*}(b_n)$

$= j_{A*}(a_1') \cdots j_{B*}(b_n')$

$\Rightarrow j_{A*}(a_1') \cdot \ j_{B*}(b_n') j_{B*}(b_n^{-1}) \cdots j_{A*}(a_1^{-1}) = 1$

In terms of paths: $\alpha_1' \cdot \dots \cdot \beta_n' \cdot \bar{\beta}_n \cdot \dots \cdot \bar{\alpha}_1 \sim x_0$

By Lemma 2, some subpath contained in $A$ or $B$ is actually homotopic into $C$.

Because $n$ was minimal, it follows that $\beta_n' \cdot \bar{\beta}_n$ is homotopic into $C$. rel $\{0,1\}$ Therefore by induction on $n$, $f$ is well defined. □

## Construction of universal covers

3rd December 2013

### Definition

$G$ acting on $X$ ($G \curvearrowright X$), then define $x \sim y$ if and only if $\exists g \in G$ such that $y = gx$

$$X \backslash G = X/\sim$$

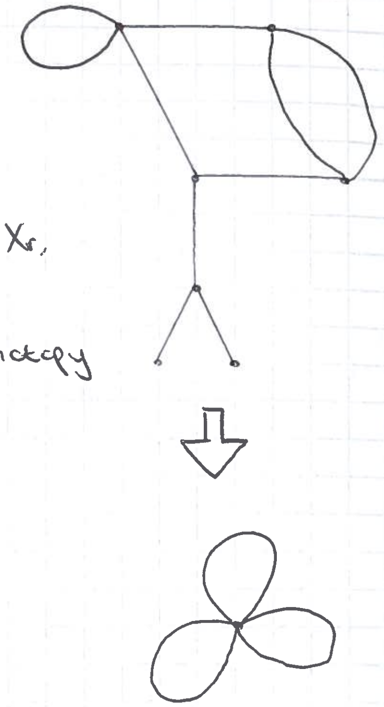
We have used this several times: for example $\pi_1 X \curvearrowright \tilde{X}$: $X = \tilde{X}/\pi_1 X$.

### Theorem

If $X$ is a (finite) cell complex, then $X$ has a universal cover: $\tilde{X} \xrightarrow{p} X$

## Step 1

If $X$ is a graph, i.e. $X = X^{(1)}$.



### Recall

1) we already constructed $\tilde{X}$ by hand when $X = X_r$, the rose with $r$ petals.

2) constructing a maximal tree gives a homotopy equivalance with a rose for any $X$.

### Definition

For maps $f_1 : Y_1 \longrightarrow X$, $f_2 : Y_2 \longrightarrow X$, then the fibre product is:

$$P = \{(y_1, y_2) \in Y_1 \times Y_2 : f_1(y_1) = f_2(y_2)\}$$

we have maps $g_i : P \longrightarrow Y_i$ obtained by forgetting the other coordinate (i.e. by projection).

### Lemma

Suppose $f : Y \longrightarrow X$ is continuous and $p : \hat{X} \longrightarrow X$ is a covering map. Let $\hat{Y} = \{(y, \hat{x}) : f(y) = p(\hat{x})\}$ be the fibre product.

Then $q$ is a covering map.

$$\begin{tikzcd} \hat{Y} \arrow[r, "\hat{f}"] \arrow[d, "q"'] & \hat{X} \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

### Proof

Let $y \in Y$ and let $x = f(y)$

Let $U$ be a neighbourhood of $x$ as guaranteed by the covering space condition, so $p^{-1}U = U \times \Delta$. Let $V = f^{-1}U$.

We need to prove that:

$$q^{-1}V \cong V \times \Delta$$

and that $q$ is projection onto $V$. Now:

$$q^{-1}V = \{(y, \hat{x}) : f(y) \in U \text{ and } \hat{x} \in p^{-1}U \text{ where } p^{-1}U \cong U \times \Delta \text{ with } f(y) = p(\hat{x})\}$$

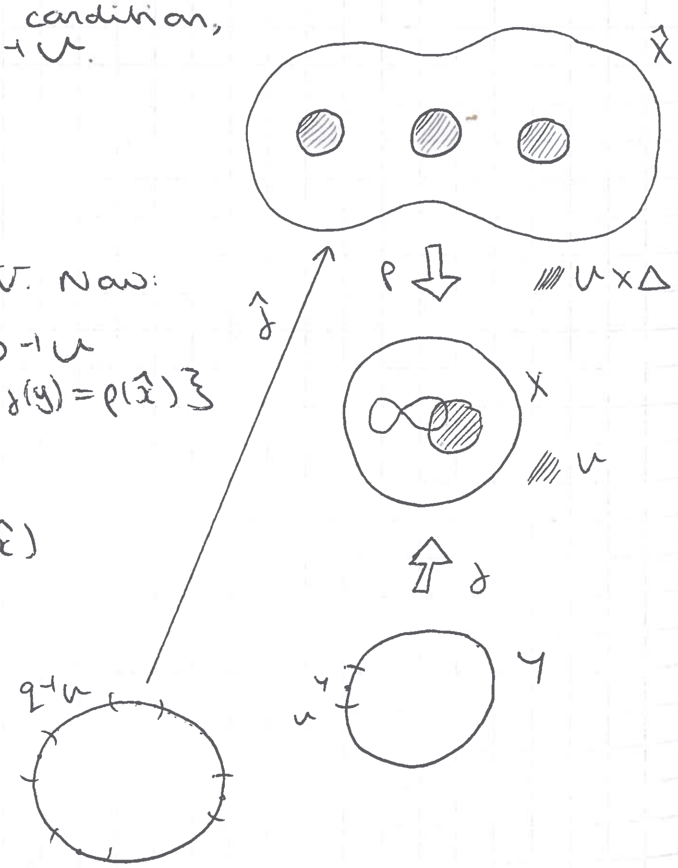
$$= \{(y, x, \delta) : f(y) = x \in U,\ \delta \in \Delta\}$$

writing $\hat{x} = (x, \delta)$ where $x = p(\hat{x})$ and $\delta \in \Delta$.

$$\cong \{(y, \delta) : f(y) \in U,\ \delta \in \Delta\}$$

$$= V \times \Delta$$

and $q$ is projection onto the first factor. $\square$

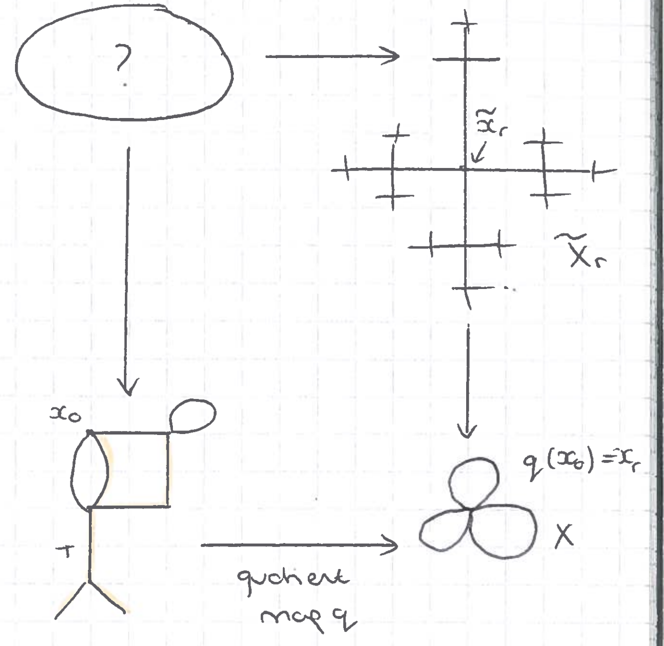



### Proposition

If $X$ is a finite graph, then $X$ has a universal cover.

### Proof

Let $T \subseteq X$ be a maximal tree.




Collapsing $T$ gives a rose $X_r$, and the quotient map $q : X \longrightarrow X_r$ is a homotopy equivalance.

Let $x_0$ be a choice of base vertex in $X$. Let $x_r = q(x_0)$, and let $\tilde{x}_r$ be a lift of $x_r$ to $\tilde{X}_r$

Let $P$ be the fibre product. Let $\tilde{x}_0 = (x_0, \tilde{x}_r) \in P$, and let $\tilde{X} \subseteq P$ be the path component containing $\tilde{x}_0$. (i.e. it is all the points you can join to $\tilde{x}_0$ by a path)

Now $\tilde{X} \to X$ is a path connected covering space. It remains only to prove that $\tilde{X}$ is simply connected.

The map $q \circ p : \tilde{X} \longrightarrow X_r$ has a lift, $\tilde{q} : \tilde{X} \longrightarrow \tilde{X}_r$, such that $\tilde{q}(\tilde{x}_0) = \tilde{x}_r$.

$$\begin{tikzcd} P \subseteq X \times \tilde{X}_r \arrow[r, "\tilde{q}"] \arrow[d, "p"'] & \tilde{X}_r \arrow[d, "p_r"] \\ X \arrow[r, "q"'] & X_r \end{tikzcd}$$

Therefore by the General lifting lemma,

$$(q \circ p)_* \pi_1(\tilde{X}, \tilde{x}_0) \leq p_{r*} \pi_1(\tilde{X}_r, \tilde{x}_r) = 1$$

$$q_* \circ p_* \pi_1(\tilde{X}, \tilde{x}_0) \Rightarrow \pi_1(\tilde{X}, \tilde{x}_0) = 1,$$ because $q_*$ is an isomorphism and $p_*$ is injective. $\square$

As in the proof of the classification theorem, for covers we therefore that:

### Corollary

For any subgroup $H \leq \pi_1(X, x_0)$, there is a path connected covering space

$$(\hat{X}_H, \hat{x}_H) \xrightarrow{p_H} (X, x_0)$$

Such that $H = p_{H*} \pi_1(\hat{X}_H, \hat{x}_H)$

### Proof

Take $\hat{X}_H = \tilde{X}/H$ $\square$

## Step 2

Take arbitrary (finite) cell complexes $X$.

### Idea

1-skeleton + higher dimensions.

$X$ is a complex. $X^{(1)}$

Choose $\hat{X}^{(1)}$ to be a certain covering space of $X^{(1)}$ and glue on lifts of cells to $\hat{X}^{(1)}$.

Consider the inclusion map

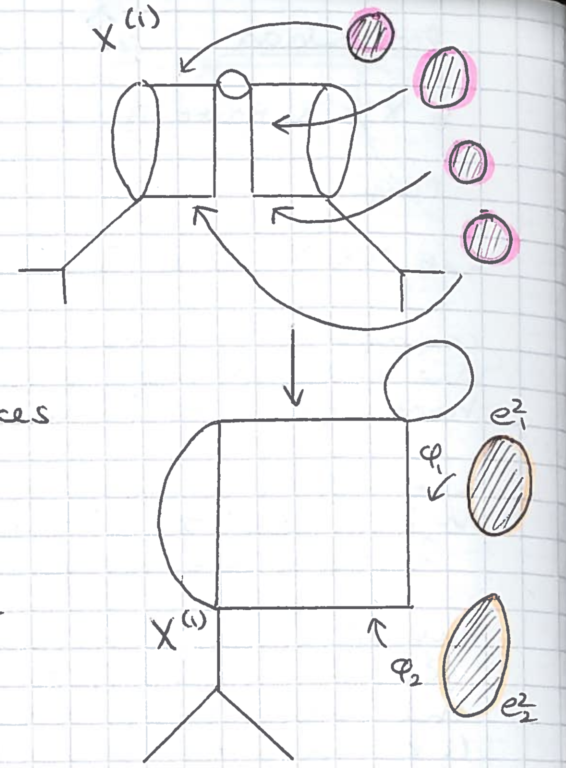
$$i : X^{(1)} \longrightarrow X$$

Pick a base point $x_0 \in X^{(0)}$. Then $i$ induces a map:

$$\pi_1(X^{(1)}, x_0) \xrightarrow{i_*} \pi_1(X, x_0)$$

Let $K = \ker i_*$. Let $\hat{X}^{(1)} = \hat{X}^{(1)}_K$ be the covering space corresponding to $K$.

Now we will build-up $\tilde{X}$ inductively from $\hat{X}^{(1)}$ by adding cells.




– 2-cells

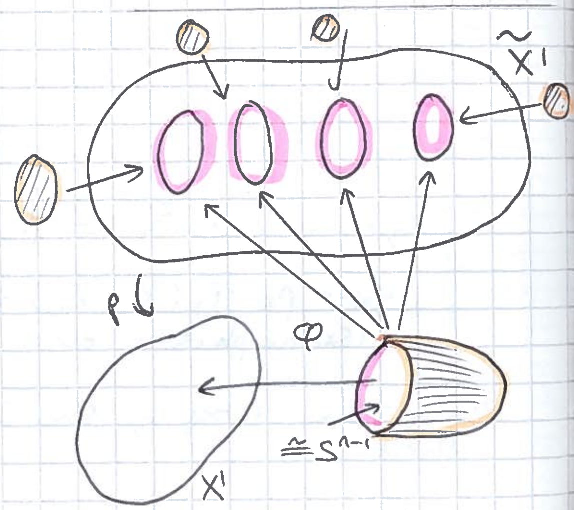
– lifts of 2-cells

By induction, we can assume that $X = X' \cup e^n$ along some attaching map $\varphi$, where we have built a cover which is suitable, $\tilde{X}'$ of $X'$.

Let $\{\tilde{\varphi}_i : i \in I\}$ be ALL the lifts of the map $\varphi : \partial e^n \longrightarrow X'$ to $\tilde{X}'$.

Let $\{e_i^n : i \in I\}$ be $n$-cells. Construct $\tilde{X}$ by attaching $e_i^n$ to $\tilde{X}_1$ along $\tilde{\varphi}_i$

NOTE that the map $\tilde{X}' \to X'$ extends to a map $\tilde{X} \longrightarrow X$.



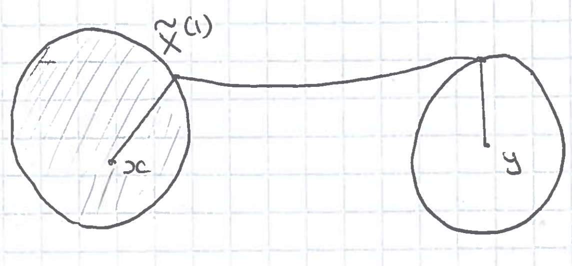

__Theorem__

$\tilde{X} \xrightarrow{p} X$ is the universal cover of $X$.

__Proof__

We need to prove three things:

(1) it's path-connected.
(2) it's a covering space
(3) it's simply connected.




Any $\tilde{x} \in \tilde{X}$ can be joined by a path to $\tilde{X}^{(1)} = \hat{X}^{(1)}_K$, which is path connected.

__(1)__ is easy: as above ↑

__(2)__ why is it a covering space? We'll prove this by induction.

We may assume that $p : \tilde{X}' \longrightarrow X'$ is a covering map.

Let $x \in X$. We need to prove that $x$ has a neighbourhood $U$ such that $p^{-1}(U) \cong U \times \Delta$.

This is clear if $x \in e^{\circ n}$ or if $x \in X' \setminus \varphi(\partial e^n)$.

∴ assume that $x = \varphi(t)$ for $t \in \partial e^n$. Then a neighbourhood of $x$ looks like $U \cup V$ where $U \subseteq X'$ and $V \subseteq e^n$.

The only possible problem is that if $U$ has a lift $\tilde{U}$ not in the image of a lift of $\varphi$, so there's no possible $\tilde{V}$ which corresponds to attach.

So part (3) follows from:

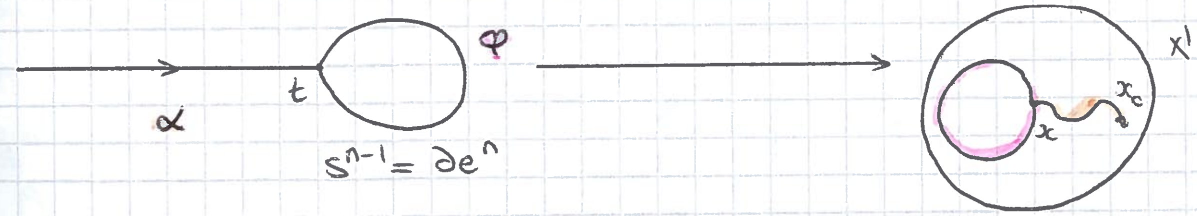
__Claim__

If $\tilde{x} \in p^{-1}(x)$ and $x = \varphi(t)$ then $\exists$ a lift $\tilde{\varphi} : \partial e^n \longrightarrow \tilde{X}'$ such that $\tilde{\varphi}(t) = \tilde{x}$.

__Proof__

Fix basepoints $x_0 \in X^{(0)}$, $\tilde{x}_0 \in (\tilde{X}')^{(0)}$. Let $\tilde{\alpha}$ be a path in $\tilde{X}'$ from $\tilde{x}_0$ to $\tilde{x}$.

Let $\alpha = p \circ \tilde{\alpha}$. Then we can build a lollipop out of $\alpha$ and $\partial e_n$.

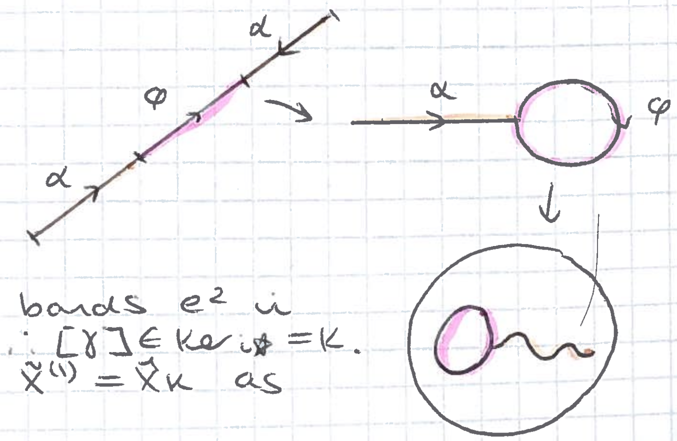



I need to show that the map of this lollipop lifts to $\tilde{X}'$, sending $*$ to $\tilde{x}_0$. There are 2 cases:

__Case 1__: $n \geq 3$

The lollipop is simply connected, so the lift exists by the general lifting lemma. //

__Case 2__: $n = 2$




The lollipop defines a based loop $\gamma$.

Now $[\gamma] \in \pi_1(X^{(1)}, x_0)$.

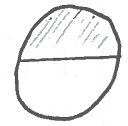
Note that $i_*[\gamma] = 1$ because $\varphi$ bounds $e^2$ in $X$, so $\gamma$ is homotopic to a point. ∴ $[\gamma] \in \ker i_* = K$. So by definition of $\hat{X}_K$, $\gamma$ lifts to $\hat{X}^{(1)} = \hat{X}_K$ as required. //

∴ $\tilde{X} \longrightarrow X$ is a covering space.

__(3)__ Finally, we want to prove that $\tilde{X}$ is simply connected. Let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at $\tilde{x}_0$.

Note that we can homotope $\tilde{\gamma}$ so that:

$$\operatorname{im}\tilde{\gamma} \subseteq \tilde{X}^{(1)}.$$

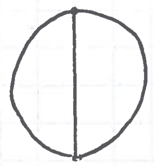
For each $n$-cell with $n \geq 2$ →  by using a homotopy in a ball. Let $\gamma = p \circ \tilde{\gamma}$. Since $\gamma$ lifts to $\tilde{\gamma}$, a based loop, $[\gamma] \in K = \pi_1(X^{(1)}, x_0)$

$\Rightarrow$ $i_*[\gamma] = 1 \in \pi_1(X, x_0)$

$\Rightarrow$ $\gamma$ is homotopic to $x_0$ in $X$.

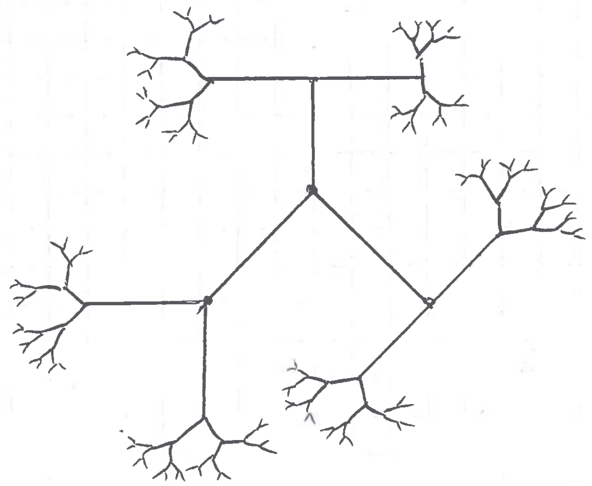
$\therefore$ by the homotopy lifting lemma, $\tilde{\gamma}$ is homotopic to $\tilde{x}_0$ in $\tilde{X}$. □

## Example 1

$X =$ 

The universal cover of $X$, $\tilde{X}$ must be:

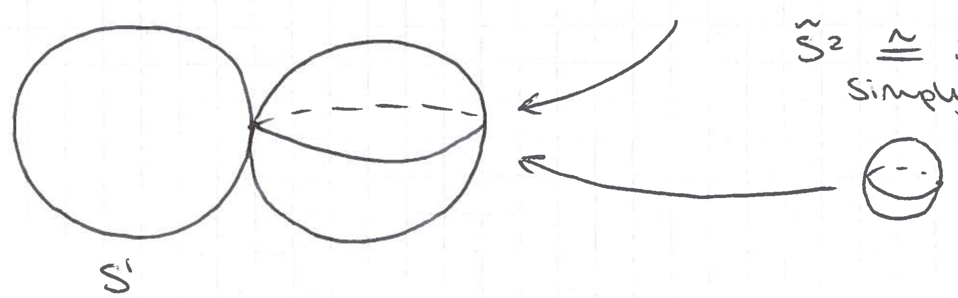
- a graph
- simply connected, i.e. a tree
- it's trivalent (i.e. every vertex has 3 incident edges)



## Example 2

$X = S^1 \vee S^2$ $\quad \tilde{S}^1 \cong \mathbb{R}$

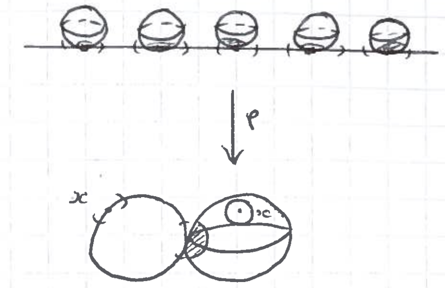
$\tilde{S}^2 \cong S^2$ as $S^2$ is simply connected.




Then the covering space (universal) is:

To prove that this is $\tilde{X}$, we need to prove that:

(1) it's connected.
(2) $p$ is a covering map
(3) $\tilde{X}$ is simply connected.




(1) and (2) is obvious by the picture. ((2) will need a few cases to prove rigorously).

## Question

Which groups arise as fundamental groups of finite cell-complexes?

## Lemma

If $X$ is a finite path-connected cell complex and $x_0 \in X^{(0)}$, then the inclusion map:

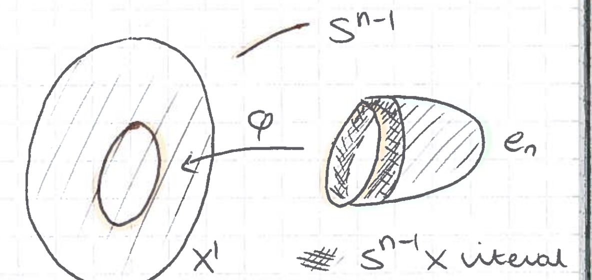
$$i : X^{(2)} \hookrightarrow X$$

induces an isomorphism:

$$i_* : \pi_1(X^{(2)}, x_0) \cong \pi_1(X, x_0)$$

## Proof

By induction, we can take $X = X' \cup_\varphi e^n$, $n \geq 3$, and the lemma holds for $X'$.




$\bar{B}^n \subseteq \mathring{e}^n$ be a large closed ball and let $A^n = e^n \setminus B^n \cong S^{n-1} \times$ interval. Let $X^+ = X' \cup_\varphi A^n$. Note that $X^+ \simeq X'$, and $X^+ \subseteq X$ is open, and $X^+ \cap \mathring{e}^n$ is $S^{n-1} \times$ open interval. Put $C_n = X^+ \cap \mathring{e}^n$.

By the Seifert-van Kampen Theorem, the diagram: Choose $c \in C_n$.

$$\begin{tikzcd}
1 \cong \pi_1(C_n, c) \arrow[r] \arrow[d] & \pi_1(\mathring{e}^n, c) \cong 1 \arrow[d] \\
\pi_1(X^+, c) \arrow[r, "i_*"] & \pi_1(X, c)
\end{tikzcd}$$

is a pushout. But note that the diagram:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 1 \arrow[d] \arrow[ddr, bend left, "f"] & \\
\pi_1(X^+, c) \arrow[r, "id"] \arrow[drr, bend right, "f"'] & \pi_1(X, c) \arrow[dr, "f"] & \\
& & G
\end{tikzcd}$$

is a push-out. $\therefore$ by the uniqueness of push-outs, $i_*$ is an isomorphism, as claimed. □

## Presentations

Let $G$ be a group with a finite generating set, $\{S_1, \dots, S_n\}$.
i.e. no proper subgroup of $G$ contains $S$).

Such a $G$ is called finitely generated.

There's a surjection: (homomorphism)

$$\begin{aligned} F_n &\xrightarrow{\eta} G \\ a_i &\longmapsto S_i \end{aligned}$$

Let $K = \ker\eta \trianglelefteq F_n$.

By the first isomorphism theorem, $G \cong F_n/K$. If we can specify $K$, then we can specify $G$ (up to isomorphism).

Definition

Let $\Gamma$ be a group and $R \subseteq \Gamma$ be a subset. Then the subgroup normally generated by $R$ is:

$$\bigcap_{\substack{N \trianglelefteq \Gamma \\ N \supseteq R}} N = \langle\langle R \rangle\rangle$$

This is denoted $\langle\langle R \rangle\rangle$. This is the smallest normal subgroup containing $R$.

Definition

A presentation (finite) is a set $a_1, \dots, a_n$ of generators for $F_n$ and a (finite) subset $\{r_1, \dots, r_m\} \subseteq F_n$.

This is usually written:

$$\langle \underset{\text{generators}}{a_1, \dots, a_n} \mid \underset{\text{relations}}{r_1, \dots, r_m} \rangle$$

It presents the group $F_n / \langle\langle r_1, \dots, r_m \rangle\rangle$. This is denoted:

$$|\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle| = F_n / \langle\langle r_1, \dots, r_m \rangle\rangle$$

Examples

1) $\mathbb{Z} \cong |\langle a \mid \; \rangle|$ $\qquad \pi_1(\bigcirc) = \mathbb{Z}$

2) $|\langle a \mid a^n \rangle| \cong \mathbb{Z}/n\mathbb{Z}$

3) $|\langle a, b \mid aba^{-1}b^{-1} \rangle|$

$$\Downarrow$$

$aba^{-1}b^{-1} = 1$, so $ab = ba$

so $|\langle a, b \mid aba^{-1}b^{-1} \rangle| \cong \mathbb{Z}^2$

4) $|\langle a, b \mid aba^{-1}b^{-1}, a^2 \rangle| \cong \mathbb{Z}/2 \oplus \mathbb{Z}$

If there are finitely many relations, so if $G$ has a finite presentation, then $G$ is called finitely presentable.

10th December 13

Recall

$F_m = \langle a_1, \dots, a_m \rangle$. If $\{r_1, \dots, r_n\} \subseteq F_m$, then $\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$ is a group presentation, and it presents

$$G = F_m / \langle\langle r_1, \dots, r_n \rangle\rangle \cong |\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle|$$

In this case, a $G$ with a finite presentation, then $G$ is called finitely presentable.

Examples

1) $\mathbb{Z} \cong |\langle a \mid \; \rangle|$

2) $\mathbb{Z}/n \cong |\langle a \mid a^n \rangle|$

3) $\mathbb{Z}^2 \cong |\langle a, b \mid aba^{-1}b^{-1} \rangle|$
i.e. $aba^{-1}b^{-1} = 1 \Leftrightarrow ab = ba$

4) $\mathbb{Z} \times \mathbb{Z}/n \cong |\langle a, b \mid aba^{-1}b^{-1}, b^n \rangle|$

We call the term $aba^{-1}b^{-1}$ the commutator, and denote it $[a, b]$.
These are all abelian groups, what about non-abelian ones?

If we want to present infinite non-abelian groups, the easiest way is with topology.

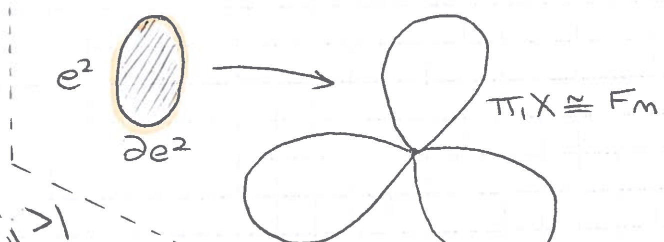
Question (from last time)

Which groups can arise as fundamental groups of finite cell complexes?

Lemma

Let $F_m = |\langle a_1, \dots, a_m \rangle|$ and let $G = |\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$.
Let $S \in F_m$. Then:

$$\begin{tikzcd}
 & F_m \arrow[dr] & \\
\mathbb{Z} \arrow[ur, "1 \mapsto S"] \arrow[rr] \arrow[d] & & G \cong |\langle a_i \mid r_j \rangle| \cong F_m/\langle\langle r_j \rangle\rangle \arrow[d, "\text{quotient map}"] \\
1 \arrow[rr] & & |\langle a_i \mid r_j, S \rangle| \cong F_m/\langle\langle r_j, S \rangle\rangle
\end{tikzcd}$$




$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & F_m \arrow[d] \\
1 \arrow[r] & ?
\end{tikzcd}$$

is a pushout.

## Proof

Consider the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left] & \\
1 \arrow[r] \arrow[drr, bend right] & |\langle a_i, r_j, S \rangle| \arrow[dr, "f'"] & \\
& & \Gamma
\end{tikzcd}$$

We're going to abuse notation and think of $S$ as an element an element of $G$.

By the commutativity of the diagram:

$$f(S) = 1 \iff S \in \operatorname{Ker} f$$

$\therefore$ $f$ factors through the quotient map.

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & G/\langle\langle S \rangle\rangle \arrow[d, "f'"] \\
& \Gamma
\end{tikzcd}$$

$$G/\langle\langle S\rangle\rangle = (F_m/\langle\langle r_1, \dots, r_n\rangle\rangle)/\langle\langle S\rangle\rangle \cong F_m/\langle\langle r_1, \dots, r_n, S\rangle\rangle$$

Why is this?

$$\langle\langle S \rangle\rangle = \left\{ \prod_{i=1}^{n} g_i S g_i^{-1} : g_i \in G \right\}$$

$$f(S) = 1 \Rightarrow f(h) = 1 \ \forall h \in \langle\langle S \rangle\rangle. \ \square$$

This tells us that attaching a 2-cell is the same thing as adding a relation.

### Example

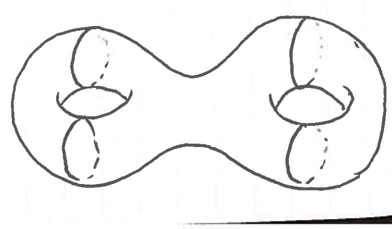
The surface of genus two.

### Definition

A two-dimensional manifold, (A manifold is a space which is locally Euclidean) is a surface.

We will be interested in compact 2-dimensional manifolds.

**Example** (The surface of genus two)

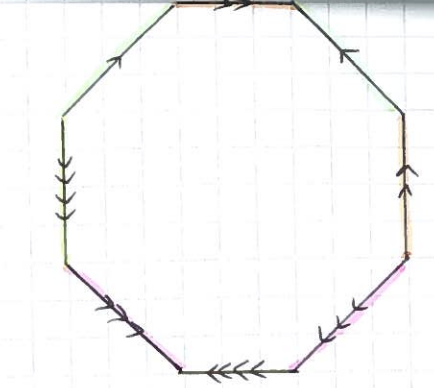


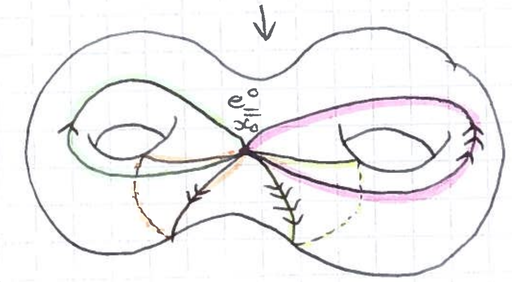
Like the torus, this can be constructed by taking a polygon and identifying

(☆ EXAM ☆)

sides.

So $\Sigma_2 = e^0 \cup e^1_1 \cup e^1_2 \cup e^1_3 \cup e^1_4 \cup e^2$



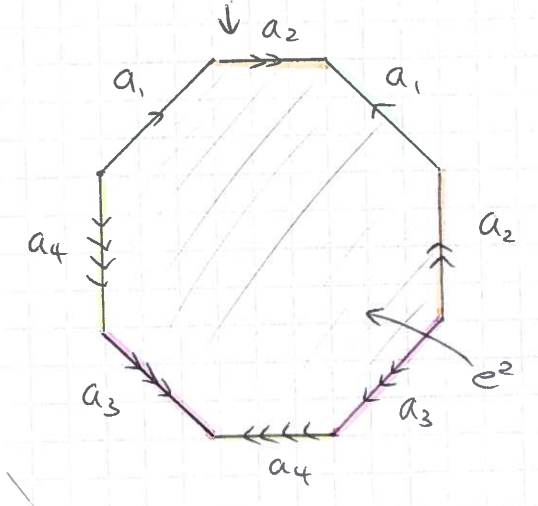



By the Seifert-van-Kampen Theorem and the lemma above:

$$\pi_1(\Sigma_2, x_0) = |\langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_4 a_3^{-1} a_4^{-1} \rangle|$$

(by reading round the boundary of the 2-cells).

Here's the answer to a previous question.



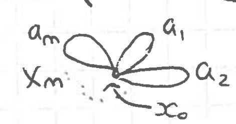

### Theorem

A group is finitely presentable if and only if it is the fundamental group of a finite cell complex.

### Proof

$\Rightarrow$ Let $G \cong |\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle|$ (denote $\mathcal{G} = \langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$)

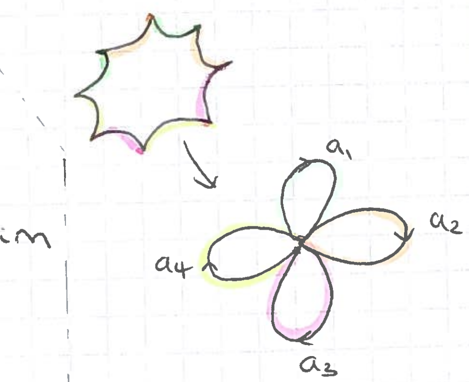
We will build a finite cell complex $X_{\mathcal{G}}$ such that $G \cong \pi_1(X_{\mathcal{G}})$




1) Take $X^{(1)}_{\mathcal{G}} = X_m$

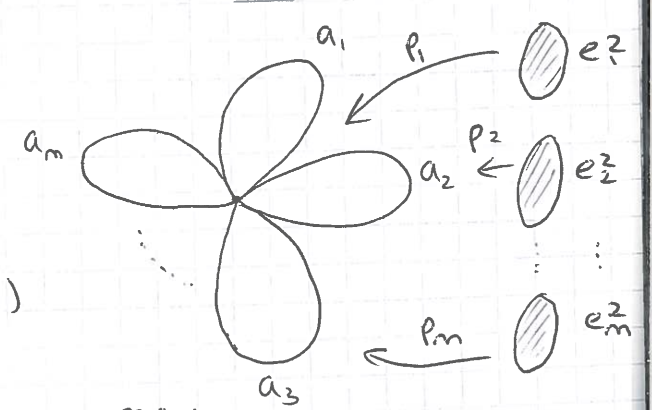
2) For each $r_j \in F_m = \pi_1(X_m, x_0)$, let $\rho_j$ be a based loop in $X_m$ such that $r_j = [\rho_j]$.

Let $e^2_j$ be a 2-cell with attaching map $\rho_j$. Let $X$ be the resulting 2-dim cell complex.




3) By the Seifert-van-Kampen Theorem and the lemma tell us by induction that

$$\pi_1(X_{\mathcal{G}}, x_0) \cong |\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle| \cong G$$




$\Leftarrow$ Suppose $X$ is a finite cell complex,

Let $x_0 \in X^{(0)}$ be the base point. We need to prove that $G = \pi_1(X, x_0)$ is finitely presentable.

By the Lemma (?) from last time, we may assume that $X = X^{(2)}$.

1) $X^{(1)}$ is a finite graph, so $\pi_1(X^{(1)}, x_0) \cong \langle a_1, \dots, a_m \rangle$,

a free group.

2) Let $e_1^2, \dots, e_n^2$ be the 2-cells of $X$ and let $\rho_i$ be the attaching map of $e_i^2$. Choosing base points appropriately, let $r_j = [\rho_j] \in \pi_1(X^{(1)}, x_0)$.

3) By the Seifert-van-Kampen Theorem, the Lemma and induction:

$$\pi_1(X, x_0) \cong |\langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle| \quad \square$$

"one-skeleton – like the generators of the group.
two-cells – like the relations"

Definition

The cell complex $X_{\mathcal{P}}$ is called the presentation complex of $\mathcal{P}$.

Non-Examinable.

We have shown there is a correspondence between topology and group theory.

Theorem

1) Group theory is impossible!

2) Computer science is impossible! (Turing)
There are problems with no algorithm which will solve it.
We can build this into the theory of finite presentable groups.

There's no algorithm to tell if a presentation presents the trivial group.

Presentations of the trivial group: $\langle a \mid a \rangle$, $\langle a, b \mid a, ab \rangle$

$\Rightarrow$ there is no algorithm to show that a fundamental group is trivial for some finite cell complex. The same thing applies to manifolds.

(Cretan – Liar paradox)

Cayley graphs

Let $G$ be a group, with a generating set $A = \{a_1, \dots, a_m\}$.

Definition

The Cayley graph $Cay_A(G)$ is a graph defined as follows:

- $Cay_A(G)^0 = G$
- $\forall g \in G$ and $a_i \in A$, there's an edge

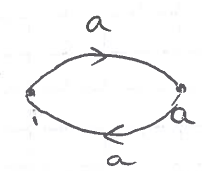
$$g \xrightarrow{a_i} ga_i$$

It's convenient to label this edge $a_i$.

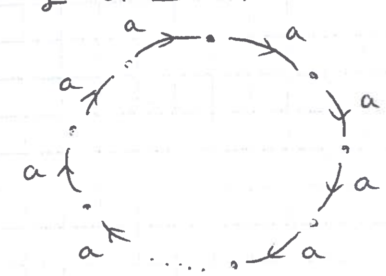
Examples (*EXAM*)

1) $G \cong \mathbb{Z}/2 \cong |\langle a \mid a^2 \rangle|$

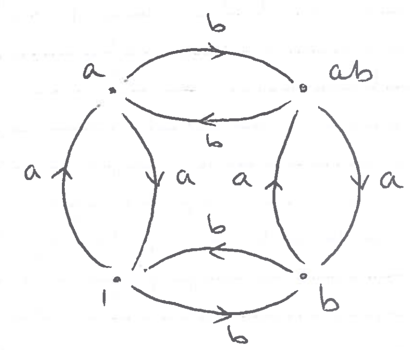
We have two group elements 1 and a.



So if $G \cong \mathbb{Z}/n \cong |\langle a \mid a^n \rangle|$

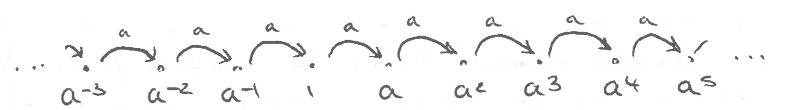


2) $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \cong |\langle a, b \mid aba^{-1}b^{-1}, a^2, b^2 \rangle|$

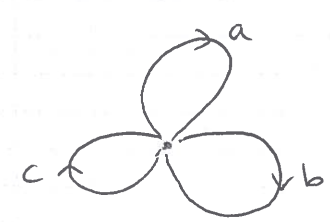


These are finite examples, what about infinite examples?

3) $G \cong \mathbb{Z} \cong |\langle a \mid \rangle|$

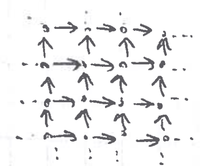



4) $G \cong 1 \cong |\langle a, b, c \mid a, b, c \rangle|$ ← silly presentation!



NOTE: different generating sets will get different Cayley graphs!

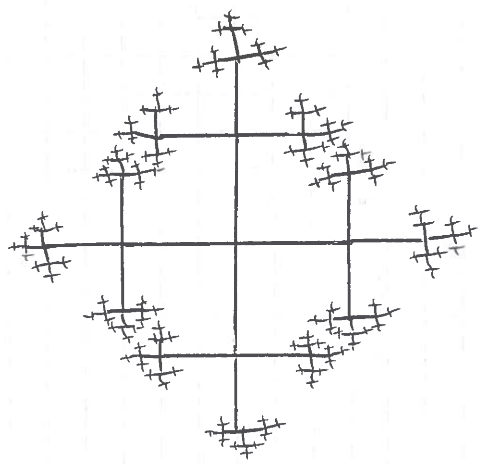
5) $G \cong \mathbb{Z}^2 \cong |\langle a, b \mid aba^{-1}b^{-1} \rangle|$



$\rightarrow a$

$\uparrow b$

6) $G \cong F_2 \cong |\langle a,b \rangle|$



Remark

The group $G$ acts on $Cay_A(G)$. The element $\gamma \in G$ acts as follows:

$$g \cdot \rightsquigarrow \cdot\, \gamma g$$

$$g \cdot \xrightarrow{a_i} \cdot\, g a_i \rightsquigarrow \gamma g \xrightarrow{a_i} \cdot\, \gamma g a_i$$

This is free, in the sense that $\gamma x = x$ for some $x \iff \gamma = 1$.

(Proof. It's enough to consider $x = g \in Cay_A(G)^{(0)}$. $\gamma g = g \iff \gamma = 1$ □)

Proposition

Let $G \cong |\langle A \mid R \rangle| = |\mathcal{S}|$.

Then $Cay_A(G) \cong \tilde{X}_{\rho}^{(1)}$

and the labels on the graph describe the map $\tilde{X}_{\mathcal{S}}^{(1)} \longrightarrow X_{\mathcal{S}}^{(1)}$

Sketch proof

The labels on $Cay_A(G)$ define a map (a covering map)

$$Cay_A(G) \xrightarrow{\ q\ } X_{\rho}^{(1)}$$

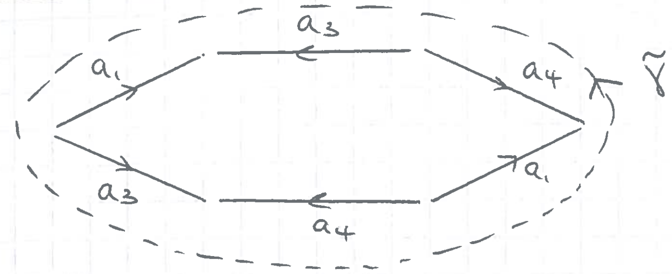
Consider the following diagram:

$$\begin{tikzcd}
1 \in Cay_A(G) \arrow[r, dashed, "\tau"] \arrow[d] & \tilde{X}_{\mathcal{S}} \ni \tilde{x}_0 \arrow[d, "\rho"] \\
x_0 \in X_{\mathcal{S}}^{(1)} \arrow[r, hook, "\iota"] & X_{\mathcal{S}}
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1(X_{\mathcal{S}}^{(1)}, x_0) \arrow[r, hook] \arrow[d, phantom, "\cong"] & \pi_1(X_{\mathcal{S}}, x_0) \arrow[d, phantom, "\cong"] \\
F_m & F_m/\langle\langle R \rangle\rangle
\end{tikzcd}$$

Let $\tilde{\gamma}$ be a loop in $Cay_A(G)$ based at 1.

Then:




$\tilde{\gamma}$ reads off an element of $F_m$ that's trivial in $G \cong F_m/\langle\langle R \rangle\rangle$.
This means that:

$$q_* \pi_1(Cay_A(G), 1) = \ker \iota_* = \iota_*^{-1}(1) = \iota_*^{-1}(\rho_* \pi_1(\tilde{X}_{\mathcal{S}}, \tilde{x}_0))$$

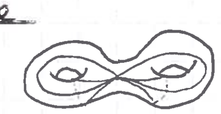
so $\iota_* \circ q_*(\pi_1(Cay_A(G)), 1)) = \rho_*(\pi_1(\tilde{X}_{\rho}, \tilde{x}_0))$

So $\tau : Cay_A(G) \longrightarrow \tilde{X}_{\mathcal{S}}$.

Now, use the definition of the lift (path-lifting lemma etc.) to show that $\tau$ is an isomorphism:

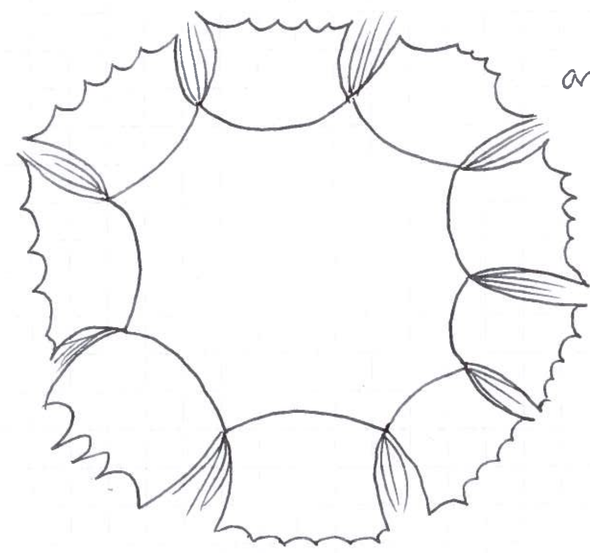
$\Rightarrow Cay_A(G) \cong \tilde{X}_{\mathcal{S}}$. □

Example

$\Sigma_2 =$ 

$\pi_1 \Sigma_2 \cong |\langle a_1, a_2, a_3, a_4 \mid [a_1, a_2][a_3, a_4] \rangle|$

$\tilde{\Sigma}_2$ is homeomorphic to $\mathbb{R}^2$.



and so on!