MATH0016 Mathematical Methods 3 Notes (Part 2 of 2)

Based on the 2019 autumn problem class by Dr R Bowles

MATHEMATICAL METHODS PROBLEM CLASS

:)

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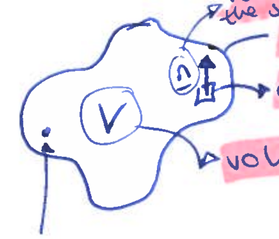
Room 603. Office hours: 8-9 Monday / Wednesday / Thursday.

HW sheet 1 exercise 3:

$\underline{F}\cdot\underline{n}\,ds = \underline{F}\cdot d\underline{s}$.

Divergence theorem: $\iiint_V \underline{\nabla}\cdot\underline{F}\,dV = \iint_S \underline{F}\cdot\underline{n}\,d\underline{s} = \iint_S \underline{F}\cdot d\underline{s}$.

Flux integral



normal of the surface

$S\,(\partial V)$ = surface

ds : bits of areas.

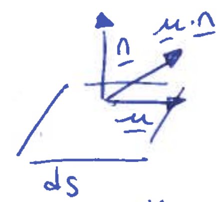
volume

Point: we can refer to it using its position vector or its coordinates

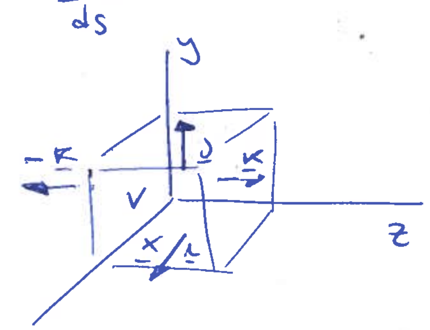
(x,y,z)

$\underline{r}\,|\,\underline{x}$: we can call it $\underline{r}$ or $\underline{x}$.

$$\underline{\nabla}\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial F_i}{\partial x_i}$$



$$\underline{F} = \begin{pmatrix} xy \\ z \\ 0 \end{pmatrix} \quad \iint_S \underline{F}\cdot d\underline{s} = \iiint_V \underline{\nabla}\cdot\underline{F}\,dV \Rightarrow \underline{\nabla}\cdot\underline{F} = \frac{\partial xy}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial 0}{\partial z} = y.$$

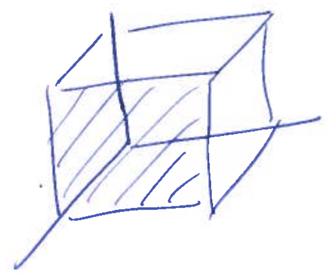


$$\Rightarrow \iiint_V y\,dV = \int_0^3\int_0^2\int_0^4 y\,dy\,dx\,dz = \int_0^4 y\,dy\cdot\int_0^2 dx\cdot\int_0^3 dz = \frac{1}{2}\,4^2\cdot 2\cdot 3 = \boxed{48}$$

We can do it by the long process which consists of calculating the flux of each of the faces of the box:

① $\underline{n} = \underline{i} \Rightarrow \underline{F}\cdot\underline{n} = \underline{F}\cdot\underline{i} = xy$.

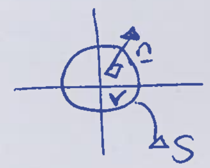
The integral over the shaded part



$$\int \underline{F}\cdot d\underline{S} = \iint_{S,\ x=2} 2y\,dz\,dy = \int_0^3\int_0^4 2y\,dy\,dz = 3\cdot\frac{1}{2}\,4^2\cdot 2.$$

(other sides will cancel out)

b) $\underline{F} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$, $\nabla \cdot \underline{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 1$



Flux is $\int_S \underline{F} \cdot d\underline{s} = \int_V \nabla \cdot \underline{F} \, dV = \int_V 1 \, dV = V = \frac{4}{3}\pi 8^3$.

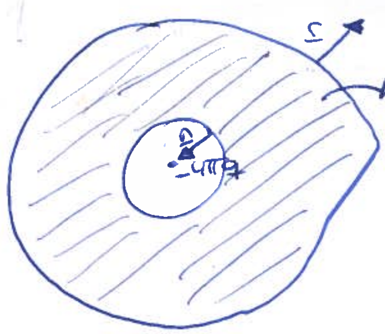
$\underline{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{r}$ position vector.

(6)

$\Phi = \int_S q \cdot \frac{\underline{r}}{|\underline{r}|^3} \, d\underline{s}$

normal is $\underline{n} = \frac{\underline{r}}{|\underline{r}|} = \hat{\underline{r}} = \int_S q \cdot \frac{\underline{r} \cdot \underline{r}}{|\underline{r}||\underline{r}|^3} \, ds = \frac{q}{a^2} \cdot \int_S ds =$

$= \frac{q}{a^2} \cdot 4\pi a^2 = 4\pi q$

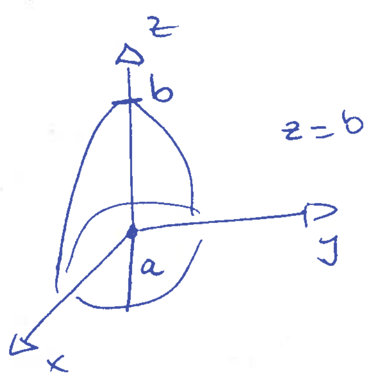


I want the flux over this surface

$\underline{n} = -\hat{\underline{r}}$

$\nabla \cdot \frac{\underline{r}}{|\underline{r}|^3} = 0$.

EXTRA



$z = b - \frac{b}{a^2}(x^2 + y^2)$.

Volume $= \int 1 \, dV = \int dx dy dz = \int_0^a \int_0^{2\pi} \int_0^{b - \frac{b}{a^2}r^2} 1 \, dz \, d\theta \, dr =$

$= \int_0^{2\pi} d\theta \int_0^a rb\left(1 - \frac{r^2}{a^2}\right) dr \, d\theta = 2\pi b \left[\frac{a^2}{2} - \frac{a^4}{4a^2}\right] = \frac{a^2 \pi b}{2}$

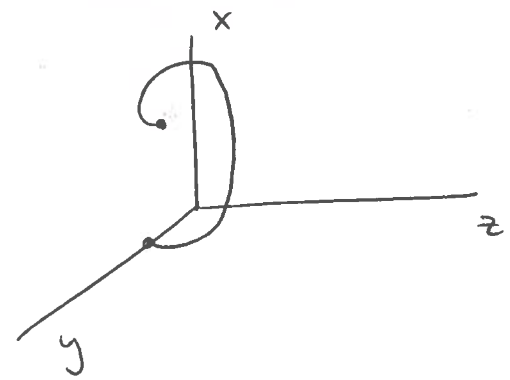
Problem class 17th 2019

(3)

a) The curves C_m are defined by $(x, y, z) = (t, \cos(mt), \sin(mt))$

$0 \le t \le 2\pi$.

Draw C_1



b) Find the lengths of C_m for every m

$\int_{C_m} |d\underline{r}|$ $\quad \underline{r}(t) = t\underline{i} + \cos(mt)\underline{j} + \sin(mt)\underline{k}$.

$= \int_0^{2\pi} \left|\frac{d\underline{r}}{dt}\right| dt = \int_0^{2\pi} \sqrt{1 + m^2 \underbrace{(\cos^2(mt) + \sin^2(mt))}_{1}} \, dt = \boxed{2\pi\sqrt{1+m^2}}$

c) For $m = 1$ and $m = 3$.

$\int_{C_m} \underline{F} \cdot d\underline{r}$ where $\underline{F} = x^2\underline{i} + y^2\underline{j} + xy\underline{k} = t^2\underline{i} + \cos^2(mt)\underline{j} + t(\cos(mt))\underline{k}$

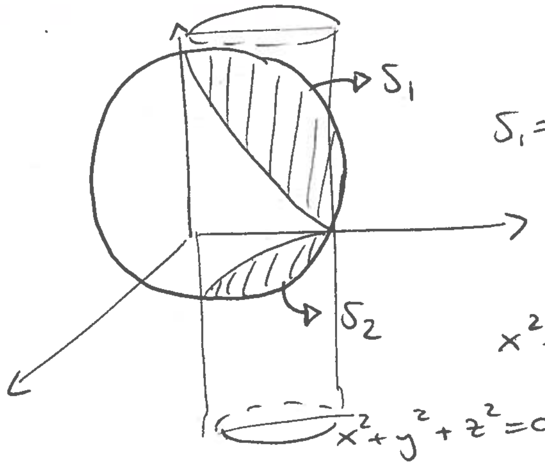
$\int_0^{2\pi} \underline{F} \cdot \frac{d\underline{r}}{dt} \, dt = \int_0^{2\pi} \left(t^2 - m(\sin(mt))\cos^2(mt) + mt\cos^2(mt)\right) dt$

$\Rightarrow \int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} (t^2 - \sin t \cos^2 t + t\cos^2 t) \, dt = \pi^2 + \frac{8\pi^3}{3}$

$\Rightarrow \int_{C_3} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \ldots\ldots = 3\pi^2 + \frac{8\pi^3}{3}$

$\underline{F}$ is not conservative / path independent since for 2 different paths the line integral is not the same!

①



$S_1 = S_2 \Rightarrow \iint_{S_1} ds + \iint_{S_2} ds = 2\iint_{S_1} ds$

$x^2+y^2 = ay$.

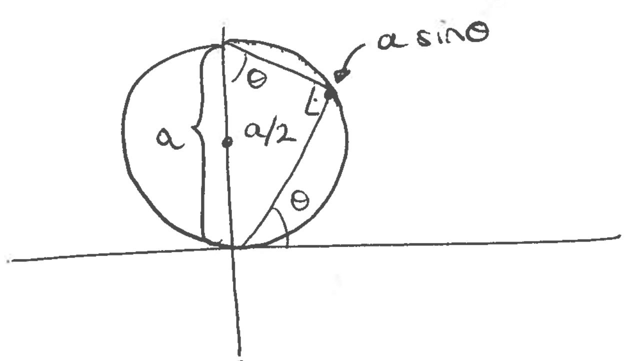
$x^2+y^2+z^2=a^2 \Rightarrow f(x,y,z) = z = \sqrt{a^2-x^2-y^2}$

$2\iint_{S_1} ds = 2\iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\,dxdy = 2\iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dxdy =$

S_1 the whole surface.

R projection on the plane.

$= 2a\iint_R \frac{1}{\sqrt{a^2-x^2-y^2}}\,dxdy$.



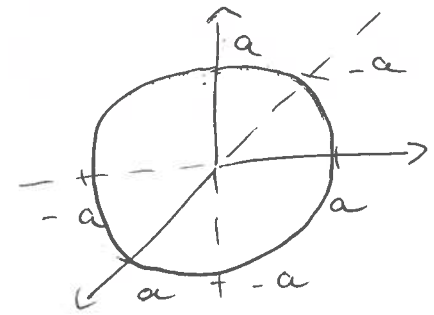
$= 2a\int_0^{\pi}\int_0^{a\sin\theta} \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 2a\int_0^{\pi}\left[-\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta =$

$= 2a^2\int_0^{\pi}(1-\cos\theta)\,d\theta = 2a^2\pi$.

Verify divergence theorem for

$A = (x^2+y^2)\,\underline{i} + (2y+z^2x)\,\underline{j} + (3z+xyz)\,\underline{k}$

region $x^2+y^2+z^2=a^2 \longrightarrow$ Sphere.



$$\iiint_V \operatorname{div}\underline{A}\,dv = \iint_S \underline{A}\cdot\underline{n}\,ds.$$

$\hookrightarrow \iint_S x^2 ds = \frac{1}{3}\iint_S \underbrace{(x^2+y^2+z^2)}_{a^2} ds$

$= \frac{a^2}{3}\iint_S ds$

LHS

$\iiint_V \nabla\cdot\underline{A}\,dv = \iiint_V (2x+2+3+xy)\,dv$.

by symmetry we know $\iiint_V x\,dv = 0$.

$\Rightarrow \iiint_V 5\,dv = 5\cdot\frac{4\pi a^3}{3} = \boxed{\frac{20\pi a^3}{3}}$

RHS

$\iint_S \underline{A}\cdot\underline{n}\,ds \qquad \underline{n} = \frac{1}{a}(x,y,z)$.

$\Rightarrow \frac{1}{a}\iint_S (x^3+y^2x+2y^2+z^2xy+3z^2+xyz^2)\,ds = \iint_S x^3 ds =$

$= \iint_S y^2x\,ds = \iint_S z^2xy\,ds = \frac{1}{a}\iint_S (2y^2+3y^2)\,ds = \frac{5a^2}{3a}\iint_S ds =$

$= \frac{5}{3}a\iint_S ds = \boxed{\frac{20}{3}\pi a^3}$

October 31ST 2019

Tutorial :

① Using Stokes' Theorem prove :

$$\iint_S \text{curl}\,\underline{A}\cdot\underline{n}\;dS = 0 \;.$$

S = smooth closed surface .

A = smooth vector field

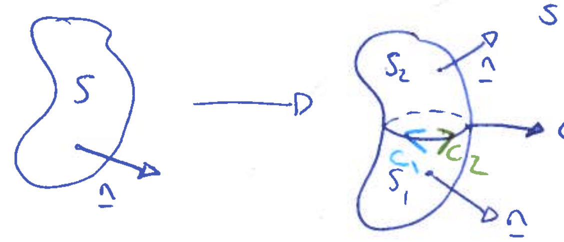
n = outward normal to S .

Stokes' theorem states that :

Given a curve C and a capping surface S, if $\underline{F}$ is a smooth vector defined on C and $S \Rightarrow$

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\text{curl}\,\underline{F})\cdot\underline{n}\;dS \;.$$

First,



RH Rule

C_1 and C_2 but in opposite directions

$$\iint_S \text{curl}\,\underline{A}\cdot\underline{n}\,dS = \iint_{S_1} \text{curl}\,\underline{A}\cdot\underline{n}\;dS + \iint_{S} \text{curl}\,\underline{A}\cdot\underline{n}\;dS \;.$$

Stoke's

$$= \oint_{C_1} \underline{A}\cdot d\underline{r} + \oint_{C_2} \underline{A}\cdot d\underline{r} =$$

$$= \oint_{C_1} \underline{A}\cdot d\underline{r} - \oint_{C_1} \underline{A}\,d\underline{r} = 0 \;.$$

Verify the result for $A = z\underline{i} + x\underline{j} + y^3z^2\underline{k}$ and S is the surface given by $x^2+y^2+z^2=a^2$, where a is a positive constant .

② a) Show that : $\epsilon_{ijk}\cdot a_i\cdot b_j c_k = \bar{a}\cdot(\bar{b}\times\bar{c})$

$$(\underline{b}\times\underline{c}) \Rightarrow (b\times c)_i = \epsilon_{ijk}\, b_j\, c_k$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = a_i\,\epsilon_{ijk}\cdot b_j c_k = \epsilon_{ijk}\,a_i b_j c_k \;.$$

b)
$$[\nabla\times(\nabla\times u)]_l = \epsilon_{lmk}\cdot\frac{\partial}{\partial x_m}(\nabla\times u)_k =$$

$$= \epsilon_{lmk}\frac{\partial}{\partial x_m}\,\epsilon_{ijk}\cdot\frac{\partial u_j}{\partial x_i} =$$

$$= \epsilon_{lmk}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_m}\,\frac{\partial u_j}{\partial x_i} =$$

$$= (\delta_{li}\cdot\delta_{mj} - \delta_{lj}\cdot\delta_{mi})\left(\frac{\partial}{\partial x_m}\,\frac{\partial u_j}{\partial x_i}\right) =$$

$$= \delta_{li}\,\delta_{mj}\left(\frac{\partial}{\partial x_m}\,\frac{\partial u_j}{\partial x_i}\right) - \delta_{lj}\,\delta_{mi}\left(\frac{\partial}{\partial x_m}\,\frac{\partial u_j}{\partial x_i}\right) =$$

$$= \underbrace{\frac{\partial}{\partial x_m}\,\frac{\partial u_m}{\partial x_l}}_{\substack{l=i\\ m=j}} - \underbrace{\frac{\partial}{\partial x_i}\cdot\frac{\partial u_j}{\partial x_i}}_{\substack{l=j\\ m=i}} =$$

$$= \frac{\partial}{\partial x_l}\,\frac{\partial u_m}{\partial x_m} - \frac{\partial^2 u_l}{\partial x_i^2} = \bar{\nabla}\cdot(\bar{\nabla}\bar{u}) - (\bar{\nabla}^2 u)$$

③ Fourier series of f where:

$$f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$$

$$c = \frac{1}{2\pi}\int_0^{\pi} x \, dx = \frac{\pi}{4}$$

$$a_n = \frac{1}{n\pi}\left(1 - \frac{(-1)^n}{n}\right) = \begin{cases} \frac{-2}{n^2\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{(-1)^{n+1}}{n^2}$$

b) $g(x) = f(-x)$.

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \dots\right) + \left(\sin(x) - \frac{1}{2^2}\sin(2x) + \dots\right)$$

November 15^{th} 2019

Small groups problem sheet 3:

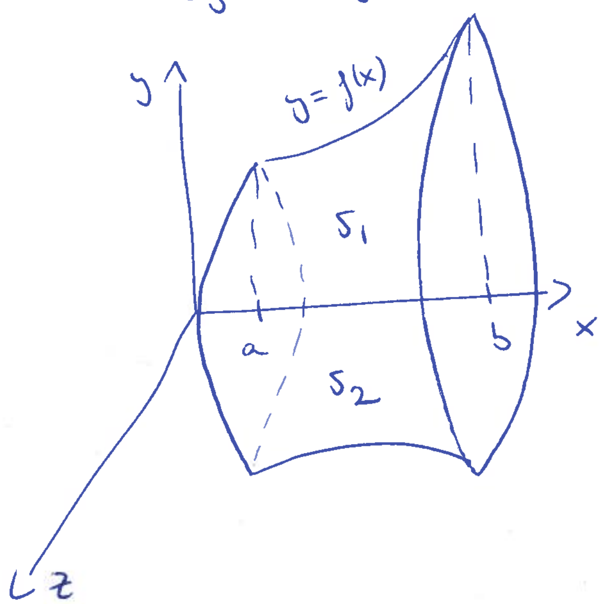
①

$$A = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$$

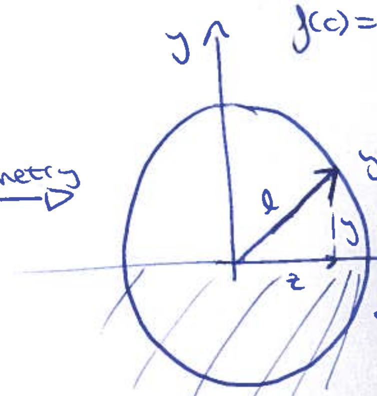
Derive by integrating unity over S.

cut surface by a plane $x=c$

$f(c) = l$.



Use symmetry



$y^2 = l^2 - z^2$

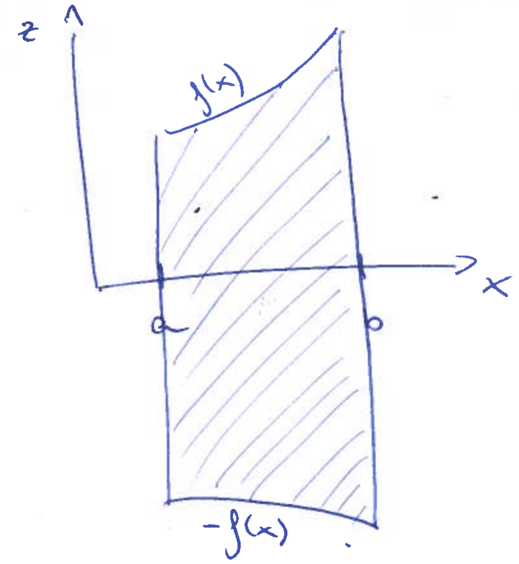
$y = \sqrt{l^2 - z^2}$

$\Rightarrow y = \sqrt{(f(x))^2 - z^2}$

we don't look at what happens below z-axis.

We do area of above and by simmetry multiply $\times 2$.

$$\Rightarrow A = 2\iint_{S_1} dS = \iint_{\text{projection } S} \|\nabla f(x,y,z)\| \, dx\, dy$$



$$A = 2\int_a^b\int_{-f(x)}^{f(x)} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dz\, dx =$$

$$= 2\int_a^b\int_{-f}^{f} \sqrt{1 + \frac{f'^2 f^2}{y^2} + \frac{z^2}{y^2}}\, dz\, dx =$$

$$= 2\int_a^b\int_{-f}^{f} \frac{1}{y} f\sqrt{1+f'^2}\, dz\, dx = 2\int_a^b f\sqrt{1+f'^2}\left[\int_{-f}^{f}\frac{1}{\sqrt{f^2-z^2}}\, dz\right] dx =$$

$z = f\cos\theta$.

$dz = -f\sin\theta$

$$\left(= \int_{\pi}^{0} -d\theta = \pi\right)$$

$$= 2\pi\int_a^b f\sqrt{1+f'^2}$$

(2) $a = a_r \hat{r} + a_\theta \hat{\theta} + a_z k$

Derive: $\operatorname{div} a = \frac{1}{r}\frac{\partial}{\partial r}(r a_r) + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}$

1) Use $r = \sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}(y/x)$

$$\frac{\partial}{\partial x} = -\frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + \cos\theta\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial}{\partial \theta}\cdot\frac{\sin\theta}{r} + \frac{\partial}{\partial r}\cos\theta$$

$(x = r\cos\theta)$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial}{\partial \theta}\cdot\frac{\cos\theta}{r} + \frac{\partial}{\partial r}\sin\theta$$

$(y = r\sin\theta)$

We know that $\begin{cases} \hat{r} = \cos\theta\, i + \sin\theta\, j \\ \hat{\theta} = \sin\theta\, i + \cos\theta\, j \end{cases}$

$a = a_r\hat{r} + a_\theta\hat{\theta} + a_z k =$

$= a_r\cos\theta\, i + a_r\sin\theta\, j - a_\theta\sin\theta\, i + a_\theta\cos\theta\, j + a_z k$

$\Rightarrow \operatorname{div} a = \frac{\partial}{\partial x}(a_r\cos\theta - a_\theta\sin\theta) + \frac{\partial}{\partial y}(a_r\sin\theta + a_\theta\cos\theta) + \frac{\partial}{\partial r}a_z =$

$= \frac{a_r}{r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z} =$

$= \frac{1}{r}\frac{\partial}{\partial r}(r a_r) + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}$

(3) $L = y - yy' + xy'^2$

E-L $\Rightarrow \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

$1 - y' - \frac{d}{dx}(-y + 2xy') = 0$;

$; 1 - \cancel{y'} + \cancel{y'} - \frac{d}{dx}(2xy') = 0$

$\Rightarrow 2xy' = x + k$

$\Rightarrow y' = \frac{1}{2} + \frac{k}{x}$

$\Rightarrow y = \frac{x}{2} + k\log x + c$.

$y(1) = \frac{1}{2} \Rightarrow c = 0$.

$y(2) = 1 \Rightarrow k = 0$.

$\Rightarrow y = \frac{x}{2}$.

November 28th 2019

Small groups problem IV.

(2) We need to find the extremal of the functional

$A(y) = \int_0^1 (y')^2\, dx$, $\quad y(0) = y(1) = 1$.

subject to the constraint $G(y) = \int_0^1 y\, dx = 2$.

$\tilde{A}(y) = \int_0^1 \underbrace{((y')^2 - \lambda y)}_{L(x,y,y')}\, dx$

Use E-L: $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

$-\lambda - \frac{d}{dx}(2y') = 0$

$\Rightarrow y'' = \frac{-\lambda}{2} \Rightarrow y = -\frac{\lambda}{4}x^2 + ax + b$.

$\Rightarrow y(0) = 1 \Rightarrow \boxed{b=1}$

$y(1) = 1 \Rightarrow \boxed{a = \lambda/4}$

① Heat flow: $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ $\quad 0 \le x \le L$

BC's: $u_x(0,t) = 0$
$u_x(L,t) = 0$, $t > 0$.

Initial condition: $u(x,0) = f(x)$, $0 \le x \le L$.

i) Separable of variables s.t $u(x,t) = X(x)T(t)$.

$\frac{\partial u}{\partial t} = T'(t) \cdot X(x)$

$\Rightarrow T'(t)X(x) = \alpha^2 \cdot \frac{\partial}{\partial x}(X'(x) \cdot T(t)) =$

$= \alpha^2 \cdot (X''(x) \cdot T(t))$

$$\Rightarrow \underbrace{\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)}}$$

since $\frac{T'(t)}{T(t)}$ is a function depending on t.

and $\frac{X''(x)}{X(x)}$ depends on x.

And both are $=$ $\Rightarrow$ The only option is to them be equal to a constant

$$\Rightarrow \frac{T'(t)}{T(t)} \cdot \frac{1}{\alpha^2} = \frac{X''(x)}{X(x)} = -\sigma = \text{constant}$$

it's convenient for the exercise

$$\Rightarrow \begin{cases} X'' + \sigma X = 0 \\ T' + \sigma T \alpha^2 = 0 . \end{cases}$$

- $\sigma < 0$ (i.e. $\sigma = -\lambda^2$)

$\Rightarrow X' = z \Rightarrow z^2 - \lambda^2 = 0$; $z^2 = \lambda^2 \Rightarrow z = \begin{cases} +\lambda & -\sigma = \sigma \\ -\lambda & -\sigma \end{cases}$

$\Rightarrow X = Ae^{\lambda x} + Be^{-\lambda x} \Rightarrow$ BC's $X'(0) \cdot T(t) = 0$.

$\Rightarrow X'(0) = 0$ since if $T(T) = 0 \Rightarrow u(x,t) = X(x) \cdot T(t)$ would be 0

BC's $X'(L) \cdot T(t) = 0$

$\Rightarrow X'(L) = 0$.

$\Rightarrow A - B = 0 \Rightarrow \boxed{A = B}$ And $A = B \neq 0$.

- $\sigma = 0 \Rightarrow X = Ax + B \Rightarrow A = 0$.

$T = C$.

$\Rightarrow X = B$

$\Rightarrow u(x,T) = \tilde{C}$.

- $\sigma > 0 \Rightarrow \sigma = \lambda^2$.

$\Rightarrow X = E\sin(\lambda x) + D\cos(\lambda x)$.

$\hookrightarrow E = 0$: (comes from $X'(0) = 0$)

$\hookrightarrow X'(L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$.

$T' + \frac{\alpha^2 m^2 \pi^2}{L^2} T = 0$.

$\Rightarrow T = k e^{\frac{\alpha^2 m^2 \pi^2}{L^2}} k$

$\Rightarrow u(x,t) = \text{constant} \cdot \cos\left(\frac{m\pi x}{L}\right) \cdot e^{\frac{\alpha^2 m^2 \pi^2 t}{L^2}}$

Now, superpose all cases:

$$\boxed{u(x,t) = C + \sum_{m=2}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) \cdot e^{\frac{\alpha^2 m^2 \pi^2 t}{L^2}}}$$

Now, $u(x,0) = \underbrace{C + \sum_{m=2}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right)}_{\text{Fourier series}} = f(x)$

$C = \frac{1}{L}\int_0^L f(x)\,dx$

$a_m = \frac{2}{L} \cdot \int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx$

$f(x) = x$, $L = 1$

$\Rightarrow C = \frac{1}{2}$

$a_n = \begin{cases} 0 & , n = \text{even} \\ -\frac{4}{n^2\pi^2} & , n = \text{odd} . \end{cases}$

$$\Rightarrow u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\,\text{odd}} \frac{1}{n^2} e^{-n^2\pi^2\alpha^2 t} \cos(n\pi x).$$