MATH0013 Analysis 3: Complex Analysis Notes (Part 2 of 2)

Based on the 2019 autumn problem class by Prof I Petridis

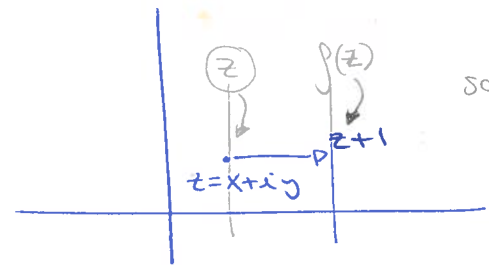
PROBLEM CLASS ANALYSIS 3

October 7^{th} 2019

Mapping properties of complex functions

① $f(z) = z+1$

$f(x+iy) = (x+iy)+1 = (x+1)+iy$.

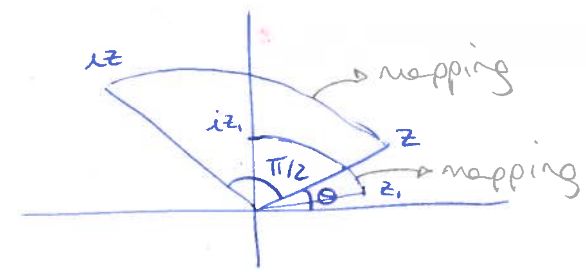


same horizontal line but we map 2 vertical lines

② $f(z) = iz = e^{i\pi/2}|z|\cdot e^{i\theta} = |z|\, e^{i(\theta+\frac{\pi}{2})}$

$f(x+iy) = (x+iy)\, i = -y+ix$

$i = e^{i\pi/2}$ $\quad z = |z|\, e^{i\theta}$

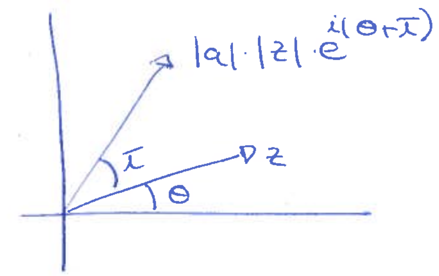


③ $f(z) = az$ $\quad a \neq 0$

$a = |a|\cdot e^{i\tau}$ $\quad \tau = \arg(a)$.

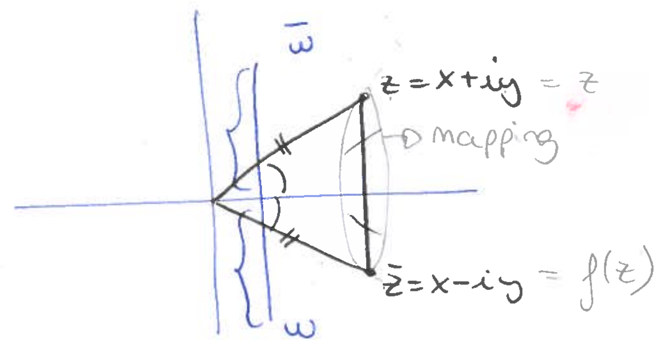
$z = |z|\, e^{i\theta}$

$f(z) = |a|\cdot|z|\cdot e^{i(\theta+\tau)}$

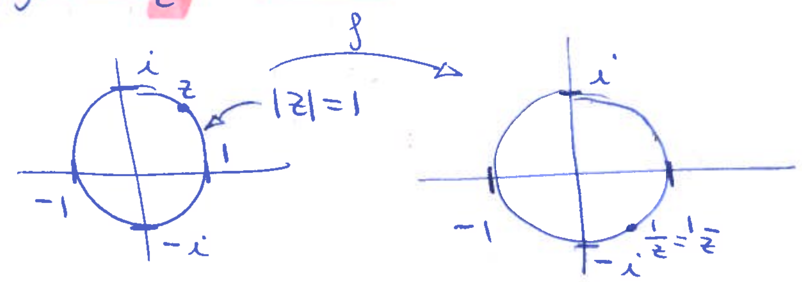


④ $f(z) = \bar{z}$.

$f(x+iy) = x-iy$.



⑤ $f(z) = \frac{1}{z}$ $\quad z \neq 0$.



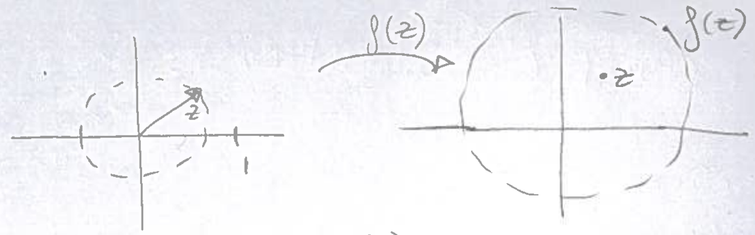
$\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{1} = 1$.

circle $C(0,1)$ mapped to $C(0,1) \rightarrow z = e^{i\tau}$ $\quad \frac{1}{z} = e^{-i\tau} = \bar{z}$.

because $|z|=1$, $|z|^2=1$ $\quad \bar{z}\cdot z = 1$

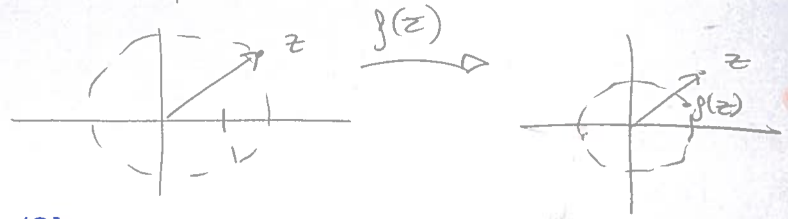
• If $|z|<1 \Rightarrow \left|\frac{1}{z}\right| = \frac{1}{|z|} > 1$

$|f(z)|>1$



• If $|z|>1 \Rightarrow \left|\frac{1}{z}\right| = \frac{1}{|z|} < 1$

$|f(z)|<1$.

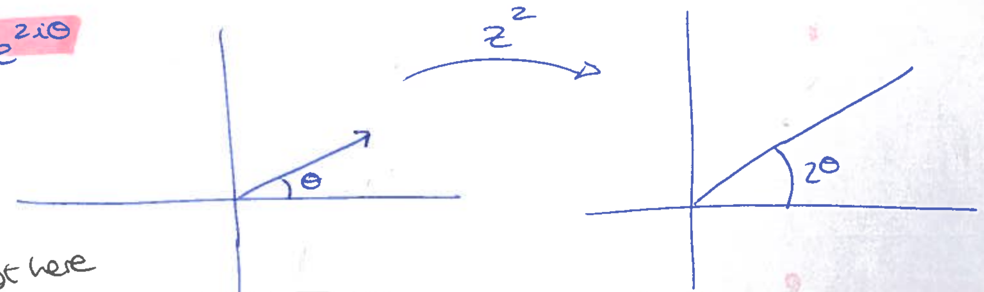


Circles are mapped to circles

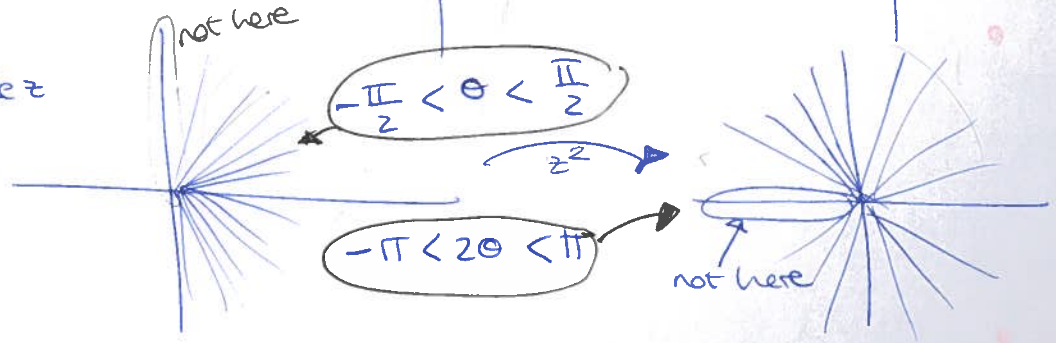
$|z|=R \longrightarrow |f(z)| = \frac{1}{|z|} = \frac{1}{R}$.

(6) $f(z) = z^2 = |z|^2 e^{2i\theta}$

$z = |z| e^{i\theta}$



For Re z

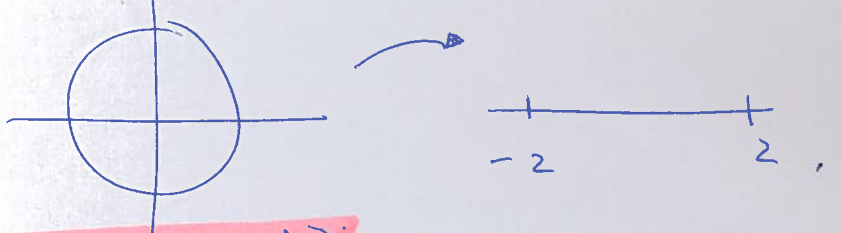


(7) $f(z) = z + \frac{1}{z}$

If $|z|=1$ $\quad z = e^{i\theta}$ $\quad \frac{1}{z} = e^{-i\theta}$.

$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

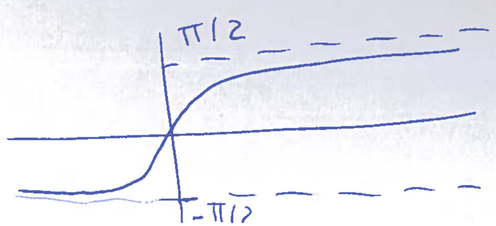
$-2 \le 2\cos\theta \le 2$

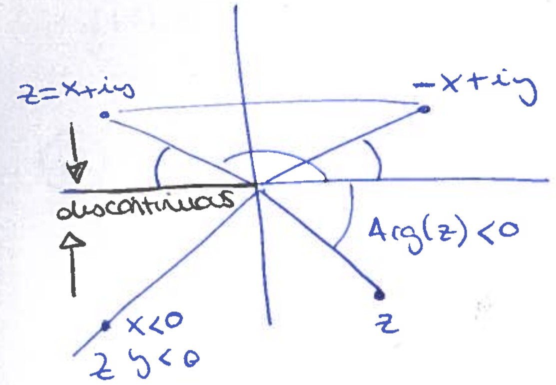


Calculate argument of z: $-\pi < \arg(z) \le \pi$. $\quad z \neq 0$.

$\arg(z) \Rightarrow$

$y = \arctan x$





$\tan\theta = \frac{y}{x}$

$\theta = \arctan\frac{y}{x}$.

$Arg(z) + Arg(-x+iy) = \pi$.

$Arg(z) + \arctan\frac{\cancel{y}}{-x} = \pi$

$Arg(z) - \arctan\frac{y}{x} = \pi$.

$$Arg(z) = \begin{cases} \arctan(y/x) & x>0,\ y>0 \\ \pi/2 & x=0,\ y>0 \\ \arctan(\frac{y}{x})+\pi & x<0,\ y>0 \\ \arctan\frac{y}{x} & x>0,\ y<0 \\ -\pi/2 & x=0\ y<0 \\ \arctan\frac{y}{x}-\pi & x<0,\ y<0 \end{cases}$$

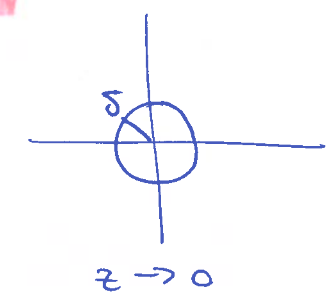
$x<0$

$\lim_{\substack{y\to 0\\ y>0}} Arg(x+iy) = \pi$

$\lim_{\substack{y\to 0\\ y<0}} Arg(x+iy) = -\pi$.

(8) $g(z) = \frac{\operatorname{Im} z}{|z|}$ for $z \neq 0$.

$\lim_{z\to 0} g(z)$



$\forall \varepsilon>0\ \exists \delta>0\ \ 0<|z|<\delta \Rightarrow |g(z)-L|<\varepsilon$

$z \to 0$ $\quad z \in \mathbb{R}$.

[A] $z = x+i0$ $\quad g(z) = \frac{\operatorname{Im} z}{|z|} = \frac{0}{|z|} = 0$.

[B] $z \to 0$ but z purely imaginary $z = iy$ $\ y \in \mathbb{R}$.

$g(z) = \frac{y}{|z|} = \frac{y}{|iy|} = \frac{y}{|y|}$ $\begin{cases} y>0 \Rightarrow 1 \\ y<0 \Rightarrow -1. \end{cases}$

TUTORIAL PROBLEM CLASS: problem sheet 1:

10th october

9a) cosnt in terms of cost

De Moivre: $(\cos t + i\sin t)^n = \cos nt + i\sin nt$

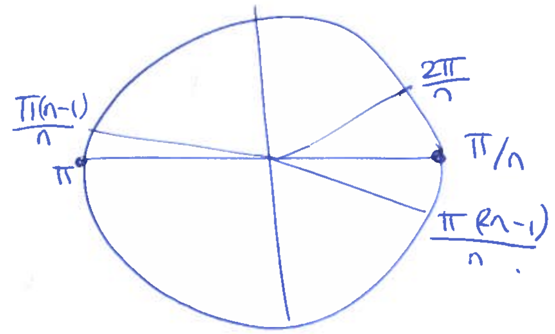
$\cos nt = \cos^n t + i \cdot n \cdot \cos^{n-1} t \cdot \sin t - \frac{n(n-1)}{2}\cos^{n-2} t \underbrace{\sin^2 t}_{(1-\cos^2 t)} + \dots$

$\sin(nt) = \sin^n t + \dots$

$x = \sin t$

$\left.\begin{array}{l} t = \frac{k\pi}{n} \\ k=0,\dots,(2n-1) \end{array}\right\}$ $\sin k\pi = 0 = x^n + \dots$ with roots $\sin\frac{k\pi}{n}$ $= x\cdot(x^{n-1} + \dots)$

$\sin(\pi - \theta) = \sin\theta$



5) $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ $\left(= e^{\frac{2\pi i}{n}}\right)$

WTS $(1-\omega)(1-\omega^2)\dots(1-\omega^{n-1}) = n$.

Define $f(z) = (z-\omega)\dots(z-\omega^{n-1})$ $\Rightarrow$ WTS $f(z) \Rightarrow f(1) = n$.

↓ Degree of f is $n-1$.

$1^n = 1$.

$\omega^n = 1$

$(\omega^r)^n = (\omega^n)^r = 1$

$r = 1, \dots, n-1$

$g(z) = (z-1)\cdot f(z) = (z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$

↳ degree n.

1st coefficient $= 1$.

$z^n - 1 = g(z)$.

$f(z) = \frac{g(z)}{(z-1)} = \frac{z^n-1}{z-1}$

- Roots of g are $1, \omega, \dots, \omega^{n-1}$ and if you take them "n" you get 1
- I.e. they are solutions of $z^n = 1$.
- I.e. they are roots of $z^n - 1$.

$f(1) = \lim_{z\to 1}\frac{z^n-1}{z-1} \overset{\text{L'Hôpital}}{=} \lim \frac{nz^{n-1}}{1} = \boxed{n}$

12) a) $S = \{z \mid Re(z) < 0\}$ is open.

Def^n: $\forall z \in S \; \exists \varepsilon > 0 \; D(z,\varepsilon) \subseteq S$.

Sol^n: Take $\varepsilon = \frac{-Re(z)}{2}$. Prove ↑

Pick $\tilde z \in D(z, \varepsilon)$, $\varepsilon = \frac{-Re(z)}{2}$

$Re(\tilde z) \le Re(z) + \varepsilon$

$= Re(z) + \left(-\frac{Re(z)}{2}\right) = \frac{Re(z)}{2} < 0$

$Re(z) = 0$.

Tutorial problem class 24th october

2) Comparison test, Ratio test, root test

(i) $\sum_{n\ge 1}\overbrace{\frac{n!}{n^n} z^n}^{a_n}$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}\cdot\frac{z^{n+1}}{z^n}\right| = |z| \cdot \frac{\cancel{(n+1)}/\cancel{(n+1)}\cdot(n+1)^n}{1/n^n} =$

$= \frac{|z|}{(1+\frac{1}{n})^n}$

Now, $\lim_{n\to\infty}\frac{|z|}{(1+\frac{1}{n})^n} = \frac{|z|}{1}$ ROC is 1.

(ii) $(*) = \sum_{n\ge 1}(3+(-1)^n)^n z^n$

Root test: $\sqrt[n]{|a_n|} = \sqrt[n]{|(3+(-1)^n)^n|z^n|} = (3+(-1)^n)|z|$.

(A) Comparison test: $|(*)| \le \sum_{n\ge 1}\overbrace{4^n|z|^n}^{b_n}$ Inequalities with complex nos require absolute values!

Root test: $\sqrt[n]{|b_n|} = \sqrt[n]{|4^n z^n|} = 4|z|$ ROC is $1/4$.

ROC of $(*)$ is $\ge \frac{1}{4}$.

Ⓑ $|(*)| \geq \sum_{n \text{ even}} \underbrace{4^n |z|^n}_{c_n}$

Root test: $\sqrt[n]{c_n} = 4|z|$

ROC is $\frac{1}{4}$ so by CT ROC of $(*) \leq \frac{1}{4}$

$\Rightarrow$ With Ⓐ and Ⓑ we can conclude ROC $= \frac{1}{4}$.

Another way would be:

$$\sum_{n \geq 1} (3+(-1)^n)^n |z^n| = \underbrace{\sum_{n \text{ odd}} 2^n z^n}_{\text{ROC } \frac{1}{2}} + \underbrace{\sum_{n \text{ even}} 4^n z^n}_{\text{ROC} = \frac{1}{4}}$$

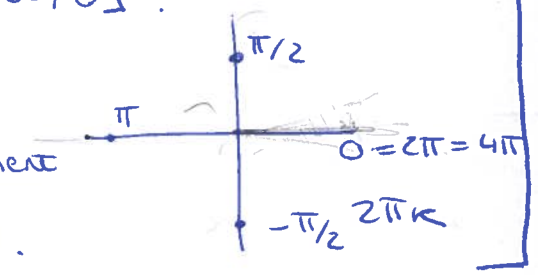
We always choose the smallest one since it works for all!!

QUESTIONS ABOUT: Check that this function is conformal.

But first, talk about logarithm and its branches:

$\underbrace{\log z = \log|z| + i\arg(z)}$ defined on $\mathbb{C} \setminus (-\infty, 0]$.

this means that $\log z$ depends on the argument of z, and since argument depends on k, there are infinite $\log z$.



• Conformal map: $f: M \to V$

want f to be bijective (with some inverse f^{-1}) and f, f^{-1} are holomorphic.

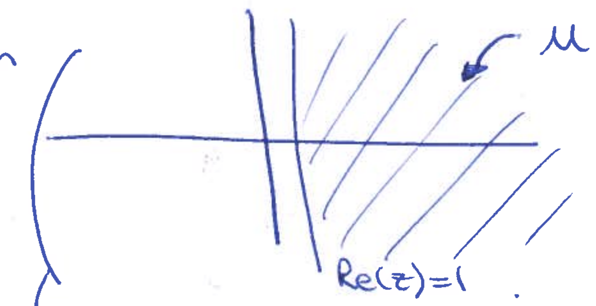
Ex: $\exp(z)$ has inverse $\log(z)$. $\exp(z)$ is holomorphic in $\mathbb{C}$. But its inverse is not even defined in the whole $\mathbb{C}$ domain, let alone it's holomorphic.

ex:

$f(z) = \log(z) = \log|z| + i\arg z$

We want to find M, V which map conformally.

Try with

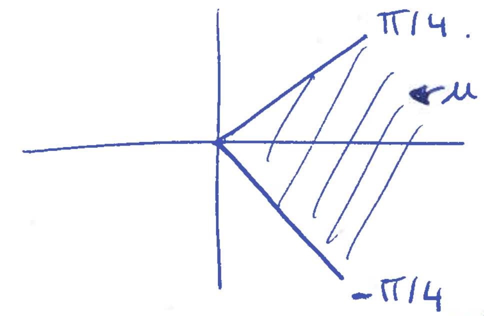


$M \Rightarrow z \in M \Leftrightarrow \mathrm{Re}(z) > 1$

$\Rightarrow$

$\Rightarrow$ it doesn't make sense.

$\Rightarrow$ Try with a region of the argument, since log depends on the argument: $M = \{ z \in \mathbb{C} \mid -\frac{\pi}{4} < \arg z < \frac{\pi}{4} \}$ and $z \neq 0$.



Now $\log z = \log|z| + i\arg z$.

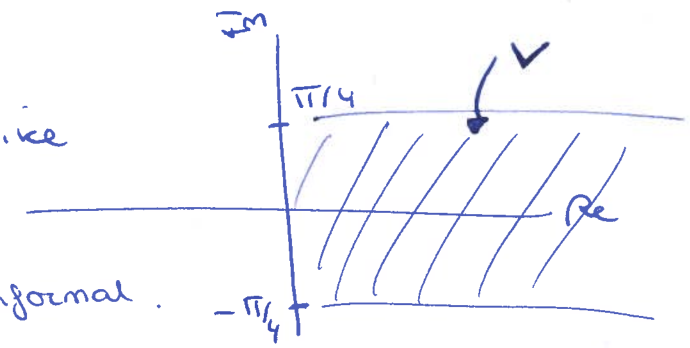
between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.

can be as big as I want as long as it is >0.

Re

Im.

$\Rightarrow V$ is gonna look like



How do I check if it's conformal.

1st: check that it is a bijection

2nd: check the holomorphic.

November 21st 2019

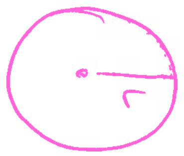
Q3) $h(z) = \frac{1}{2\pi i}\int_{S(0,1)} \frac{g(\xi)}{\underbrace{z\xi - 1}} \, d\xi$

$z\xi - 1 = 0$ when $\xi = \frac{1}{z}$.

- when $1/z \notin D(0,1)$ ⇒ we don't care: and $f(\xi)$ is holomorphic ⇒ by Cauchy-Goursat
$$\int_{S(0,1)} \frac{g(\xi)}{z\xi - 1}\, d\xi = 0 .$$
- $\underbrace{\frac{g(1/z)}{z}}$ when $\frac{1}{z} \in D(0,1)$ so, we can use C.I. Formula

because we'll have $\frac{1}{2\pi i}\cdot\frac{1}{z}\int\frac{g(\xi)}{\xi - \frac{1}{z}}\,d\xi$

(10) $h(z) = \int_{|z|=2}\left(\frac{z}{2} - \frac{2}{z}\right)^{2n}\frac{dz}{z}$

$|z|=2 \to |z| = r$ ⇒ 

$z \in S(0,r)$

$z = re^{it}$ $t \in [0, 2\pi]$.

$$h(z) = \int_0^{2\pi}\left(e^{it} - \frac{1}{e^{it}}\right)^{2n}\frac{\cancel{2}ie^{it}}{\cancel{2e^{it}}}\,dt = \int_0^{2\pi}\left(e^{it} - e^{-it}\right)^{2n} i\,dt =$$

$$= \int_0^{2\pi}(2i\sin(t))^{2n} i\,dt = 2^{2n} i^{(2n+1)}\underbrace{\int_0^{2\pi}(\sin(t))^{2n}dt}$$

we can solve it by parts.

$$I_{2n} = \int_0^{2\pi}(\sin(t))^{2n}dt = \left[\sin^{2n-1}(t)\cdot(-\cos t)\right]_0^{2\pi}$$

$$+ (2n-1)\int_0^{2\pi}(\sin(t))^{2n-2}\cos^2 t\,dt$$

$$= (2n-1)\int_0^{2\pi}\left(\sin^{2n-2}(t) - \sin^{2n}(t)\right)dt$$

$$2n I_{2n} = (2n-1) I_{2n-2}$$

$$I_{2n} = \frac{2n-1}{2n}\cdot I_{2n-2}$$

$$\Rightarrow I_{2n} = \frac{2n-1}{2n-1}\cdot\frac{2n-3}{\dots} I_{2n-4} = \frac{2n-1\cdot 2n-3 \cdot 2n-5 \cdots \cancel{2\pi}}{2n\cdot 2n-2 \cdots}$$

$$= \int_{|z|=2}\left(\frac{z}{2} - \frac{2}{z}\right)^{2n}\frac{dz}{z} = \int_{|z|=2}\sum_{k=0}^{2n}\binom{2n}{k}\left(\frac{z}{2}\right)^{2n-k}\left(\frac{2}{z}\right)^{k}\frac{dz}{z} =$$

$$= \sum_{k=0}^{2n}\binom{2n}{k}\int_{|z|=2} z^{2(n-k)}\cdot 2^{2(k-n)}\frac{dz}{z} =$$

$$= \sum_{k=0}^{2n}\binom{2n}{k}\int_{|z|=2} z^{2(n-k)-1}\cdot 2^{2(k-n)}\,dz =$$

$z = 2e^{it}$ $t \in [0, 2\pi]$.

$$= \sum_{k=0}^{2n}\binom{2n}{k}\int_0^{2\pi}(2e^{it})^{2(n-k)}\,2^{2(k-n)}\frac{\cancel{2}ie^{it}}{\cancel{2e^{it}}}\,dt =$$

$$= i\sum_{k=0}^{2n}\binom{2n}{k}\underbrace{\int_0^{2\pi}e^{i2(n-k)t}\,dt}_{I_R}.$$

$I_R = \int_0^{2\pi} 1\,dt = 2\pi$ ⇒ for $k = n$.

$I_R = \frac{1}{i(2(n-k))}\left[e^{i2(n-k)}\right]_0^{2\pi} = 0$ for $k \neq n$.

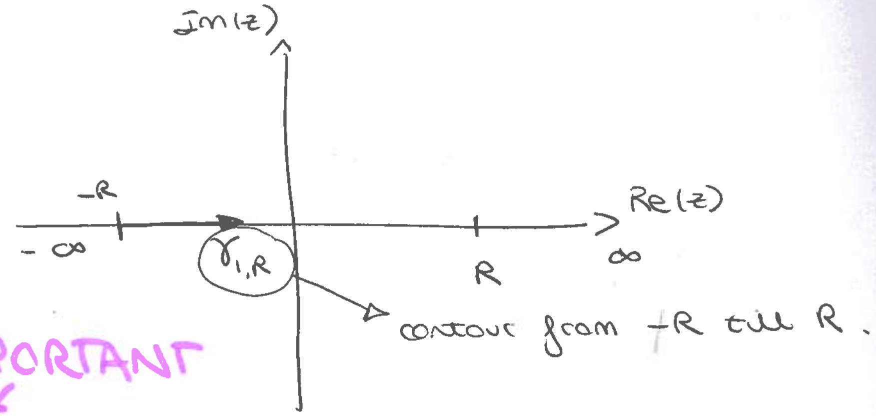
the only term that doesn't vanish is when $k=n$.

$$\Rightarrow i\sum_{k=0}^{2n}\binom{2n}{k}\cdot\int_0^{2\pi}e^{i2(n-k)t}\,dt = 2\pi i\binom{2n}{n}$$

December 5th 2019

Exam questions:

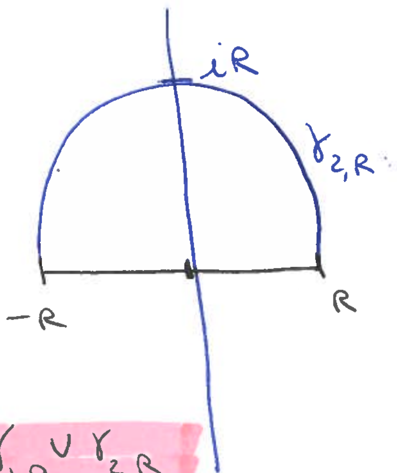
3 a) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$



IMPORTANT

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^3} = \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{dx}{(1+x^2)^3}\right) =$$

$$= \lim_{R\to\infty}\left(\int_{\gamma_{1,R}}\frac{dz}{(1+z^2)^2}\right)$$



Since $\gamma_{1,R}$ is not closed $\Rightarrow$ I'll do a semicircle on top

$\Rightarrow$ closed contour $\gamma_R = \gamma_{1,R} \cup \gamma_{2,R}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^3} = \lim_{R\to\infty}\left(\int_{\gamma_{1,R}}\frac{dz}{(1+z^2)^3}\right) =$$

$$= \lim_{R\to\infty}\left(\underbrace{\int_{\gamma_R}\frac{dz}{(1+z^2)^3}}_{A} - \underbrace{\int_{\gamma_{2,R}}\frac{dz}{(1+z^2)^3}}_{B}\right).$$

(A) $$\int_{\gamma_R}\frac{dz}{(1+z^2)^3} = \left(\operatorname{Res}_{z=i}\left(\frac{1}{(1+z^2)^3}\right)\right)\cdot 2\pi i$$

$$= 2\pi i\cdot\frac{1}{(n-1)!}\lim_{z\to i}\frac{d^{n-1}}{dz}\left(\frac{(z-i)^3}{(1+z)^3}\right)$$

$$= \cancel{2}\pi i\cdot\frac{1}{\cancel{2}!}\lim_{z\to i}\frac{d}{dz}\left(\frac{-3}{(z+i)^4}\right) =$$

$$= \pi i\cdot\lim_{z\to i}\left(\frac{-12}{(z+i)^5}\right)$$

$$= \frac{12}{32 i^{\cancel{5}}}\pi\cancel{i} = \boxed{\frac{3}{8}\pi}$$

(B) $$\left|\int_{\gamma_{2,R}}\frac{dz}{(1+z^2)^3}\right| \le L(\gamma,R)\cdot\max_{z\in\gamma_{2,R}}\left|\frac{1}{(1+z^2)^3}\right|$$

this comes from triangle inequality.

$$\left(\le\int_{\gamma_{2,R}}\left|\frac{dz}{(1+z^2)^3}\right|\right)\le\left(\max_{z\in\gamma_{2,R}}\left|\frac{1}{(1+z^2)^3}\right|\cdot\int_{\gamma_{2,R}}|dz|\right)$$

$$\le L(\gamma_1 R)\cdot\max_{z\in\gamma_{2,R}}\frac{1}{||z|^2-1|^3}$$

By triangle inequality $||x|-|y|| \le |x+y| \le |x|+|y|$

$$\le L(\gamma_1 R)\max_{z\in\gamma_{2,R}}\frac{1}{(R^2-1)^3}\cdot\pi R \xrightarrow[R\to\infty]{} 0.$$

$$\Rightarrow \lim_{R\to\infty}\int_{\gamma_{2,R}}\frac{dz}{(1+z^2)} = 0.$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^3} = \frac{3}{8}\pi}$$

3 b) Show $w = w(z) = \frac{z-5i+1}{z+5i+1}$ is a conformal map between $\{\operatorname{Im} z > 0\}$ and $\{|w|<1\}$.

Need to check:

1st w is holomorphic? : yes, it's a Mobius map

rational with a denominator that doesn't vanish

$z+5i+1 \neq 0$ since $\operatorname{Im} z > 0$.

2nd check that $w'(z)\neq 0$.

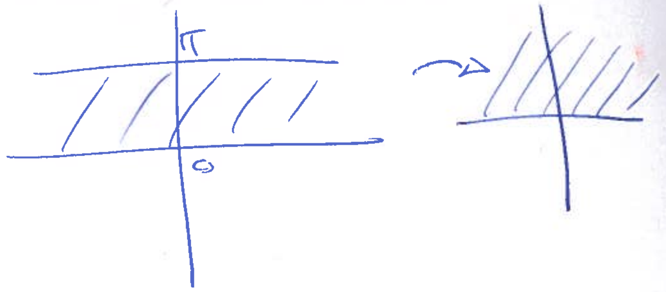
$\Rightarrow w'(z) = \frac{-10i}{(z+5i+1)^2} \neq 0$ anywhere.

3rd Check that the function is actually a bijection between this 2 sets of points.

ii) $\{0 < \operatorname{Im} z < \pi\}$

$\downarrow w = e^z$

$\{\operatorname{Im} z > 0\}$



$w = e^z$

$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^{i\theta} \cdot r$.

$\Rightarrow$ $r = e^x \to x \in (-\infty, \infty) \Rightarrow r \in [0, \infty)$.

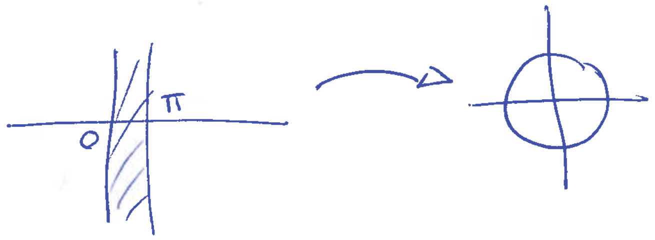
$\theta = y \to y \in [0, \pi] \Rightarrow \theta \in [0, \pi]$.

iii) Want a c.m between

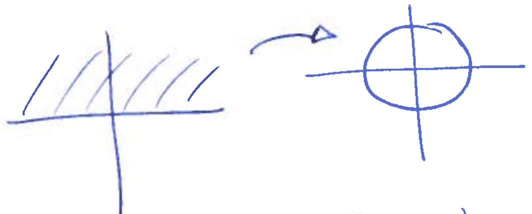
$\{0 < \operatorname{Re} z < \pi/2\}$

$\downarrow$

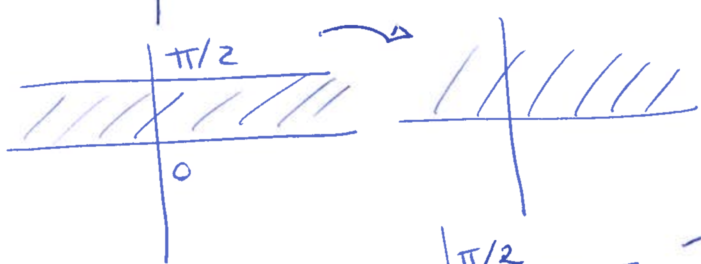
$\{|w| < 1\}$.



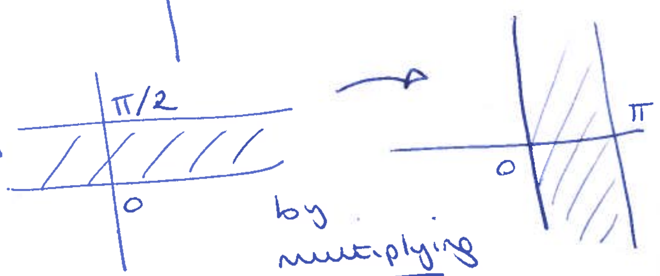
i)



ii)



I need to go from



by multiplying by $\boxed{i}$ I rotate!!! and

$\Rightarrow$ multiply by 2 to strech it

$\Rightarrow$ Composition .