MATH0013 Analysis 3: Complex Analysis Notes (Part 1 of 2)

Based on the 2019 autumn lectures by Prof A Sobolev

ANALYSIS 3

Bárbara Nieto Aguirre

COMPLEX ANALYSIS

Lecturer: Alex Sobolev.

Room no: 710.

HW: Every 2 weeks. (4 exercise to hand in).

1^{ST} HW: week of 21st October.

Midterm exam: December.

Office hour: Monday 1pm.

September 30th 2019

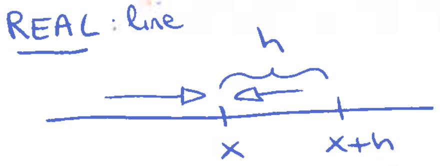
Real Analysis: $\Rightarrow$ $f: \mathbb{R} \to \mathbb{R}$.

Complex Analysis $\Rightarrow$ $f: \mathbb{C} \to \mathbb{C}$.

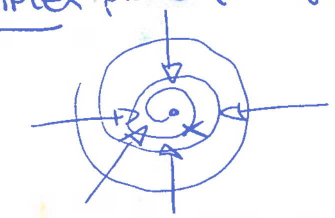
Review: example

Differentiability: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

REAL: line



COMPLEX plane (many directions)



Complex numbers: CHAPTER 1

$z = (x, y) \in \mathbb{R}^2 \to$ complex numbers.

x = real part of z – $\mathrm{Re}\, z$

y = Imaginary part of z – $\mathrm{Im}\, z$

Definition 1.1: **Multiplication**

$z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$.

$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

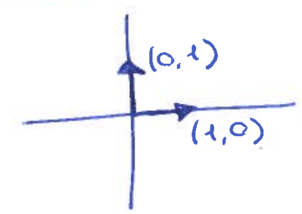
$z_1 = ai + b$, $z_2 = ci + d$ $\Big\}$ $z_1 z_2 = ac\, i^2 + aid + bci + bd = (bd - ac) + i(ad+bc)$

Observe: ① $z_1 z_2 = z_2 z_1$, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

② If $z_1 = z_2 = (0,1) \Rightarrow z_1 z_2 = (0,1)^2 = (-1, 0)$

Recall: $(x,y) = x(1,0) + y(0,1) = x + iy$: STANDARD NOTATION

where $i = (0,1)$



Notation: $\mathbb{C}$, Argand Plane

$\mathbb{C}$ is a field

- **Addition**

$z_1 + z_2 = z_2 + z_1$: commutative

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$: associative

$z + 0 = z$: additive identity

$z + (-z) = 0$: Inverse (additive inverse)

- **Multiplication**

$z_1 z_2 = z_2 z_1$: commutative

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$: associative

$z \cdot 1 = z$: multiplicative identity

$z^{-1} \cdot z = 1$, $\forall z \neq 0$: multiplicative inverse

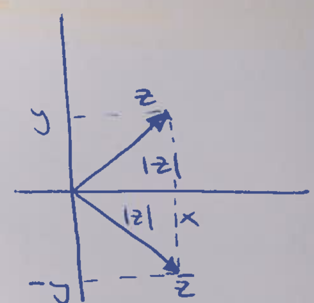
- **Distributive law**: $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

This makes it a field

Definition 1.2: Modulus (absolute value)

$|z| = \sqrt{x^2 + y^2}$

Complex conjugate: $\bar{z} = x - iy$



Remark

① $\{z \in \mathbb{C} : |z| = 1\} \to$ UNIT CIRCLE

set of all points $z \in \mathbb{C}$ s.t $|z| = 1$.

② $|z_1 - z_2| \to$ distance between z_1, z_2.

③ $\gamma(\underset{\mathbb{C}}{z_0}, \underset{R>0}{R}) = \{z \in \mathbb{C} : |z - z_0| = R\} \to$ CIRCLE OF RADIUS R CENTRED AT z_0

④ $Re\, z = \frac{z + \bar{z}}{2}$, $Im\, z = \frac{z - \bar{z}}{2i}$

Proposition 1.3

① $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

② $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

③ $z \cdot \bar{z} = |z|^2$

④ $|z_1 z_2| = |z_1| \cdot |z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ assuming $z_2 \neq 0$.

⑤ $\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$

Proof

Prove ③

$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + iyx - ixy - i^2 y^2 =$

$= x^2 + y^2$ ▨

Inequalities

Lemma 1.4

① $|Re\, z| \leq |z|$, $|Im\, z| \leq |z|$

(If $Im\, z = 0 \Rightarrow |z|$ coincides with the definition of $|z|$)

from Real Analysis $a \in \mathbb{R}$, $|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$ $\Rightarrow z \in \mathbb{C}, z = x + iy$ $|z| = \sqrt{x^2 + y^2}$

If $y = 0 \Rightarrow |z| = $ def in $\mathbb{R}$

② $|z_1 + z_2| \leq |z_1| + |z_2|$ TRIANGLE INEQUALITY.

③ $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (Exercise from problem sheet).

Proof of 2

$|z_1 + z_2|^2 = (\bar{z}_1 + \bar{z}_2) \cdot (z_1 + z_2) = \bar{z}_1 \cdot z_1 + \underbrace{\bar{z}_2 \cdot z_1 + \bar{z}_1 \cdot z_2}_{2 Re\, \bar{z}_1 z_2} + \bar{z}_2 \cdot z_2 =$

2Re $\bar{z}_1 z_2$ because $Re\, z = \frac{z + \bar{z}}{2}$ and conjugate of $\bar{z}_2 z_1 = \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + 2 Re\, \bar{z}_1 \cdot z_2 \leq |z_1|^2 + |z_2|^2 + 2|\bar{z}_1 \cdot z_2| \leq$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| = (|z_1| + |z_2|)^2 \Rightarrow$

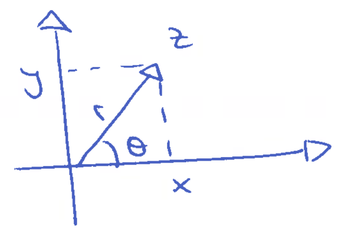
$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$ ▨

as claimed.

The polar form

Let $z = x + iy$

Let $r = |z| \Rightarrow$ $x = r \cdot \cos\theta$, $y = r \cdot \sin\theta$



$z = r \cdot \cos\theta + ir \cdot \sin\theta = r \cdot (\cos\theta + i \sin\theta)$.

The angle θ is called argument of z.

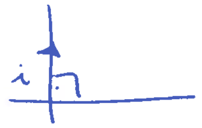
The argument is defined up to a multiple of 2π.

Definition 1.5 The principle argument denoted $Arg\, z$ is the uniquely defined value of the argument s.t $Arg \in (-\pi, \pi]$.

(el más pequeño de todos los posibles múltiplos de 2π)

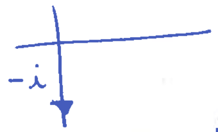
How to find Arg z from x, y ? (DIY).

$Arg\ i = \frac{\pi}{2}$



$i = (0,1)$.

$Arg(-i) = -\frac{\pi}{2}$



$(-i) = (0,-1)$.

Notation:

$\cos\theta + i\sin\theta = e^{i\theta}$

$$\begin{cases} Re\, e^{i\theta} = \cos\theta \\ Im\, e^{i\theta} = \sin\theta \\ 1/e^{i\theta} = e^{-i\theta} \end{cases}$$

This notation is motivated by the property $e^{t_1+t_2} = e^{t_1}\cdot e^{t_2}$.

Lemma 1.6

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$z_1\cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

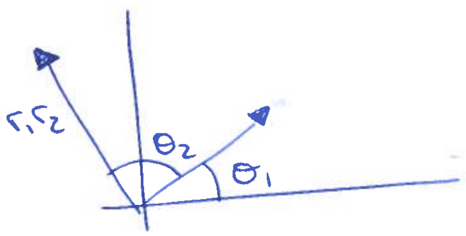
Proof: Indeed.

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2) =$$

$$z_1\cdot z_2 = r_1\cdot r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2+i\sin\theta_2) =$$

$$= r_1 r_2(\cos\theta_1\cdot\cos\theta_2 - \sin\theta_1\cdot\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) =$$

$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = r_1\cdot r_2\cdot e^{i(\theta_1+\theta_2)} \quad \blacksquare$$



$|e^{i\theta}| = 1 \Rightarrow \{e^{i\theta} \mid \theta\in(-\pi,\pi]\} \to$ UNIT CIRCLE.

$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = \sqrt{1} = 1$

October 1st 2019

Consequently,

$z^n = r^n e^{in\theta}$, $r = |z|$, $\theta = \arg z$.

In particular,

$(e^{i\theta})^n = e^{in\theta}$. As a consequence, we get

Proposition 1.7 De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Roots of complex numbers

We want to solve $x^2 = 4$, $x\in\mathbb{R}$.

$x = \pm\sqrt{4} = \pm 2$.

(Roots are always positive, that's why I write $\pm$ in front of $\sqrt{\ }$).

The arithmetic square root of $4 \Rightarrow \sqrt{4} = 2$.

Let $n\in\mathbb{N}$, and let $w\in\mathbb{C}$. Solve the equation $z^n = w$

$z^n = w = \rho e^{i\theta + i2\pi k}$ ($\rho = |w|$), $k\in\mathbb{Z}$.

Let $z = r\cdot e^{i\varphi}$, so $r^n e^{in\varphi} = \rho e^{i\theta + i2\pi k}$

Therefore, $r^n = \rho \Rightarrow r = \rho^{1/n}$

Also, $n\varphi = \theta + 2\pi k \Rightarrow \varphi = \frac{\theta}{n} + \frac{2\pi k}{n}$

$\Rightarrow z = \rho^{1/n} e^{i\frac{\theta}{n} + i\frac{2\pi k}{n}}$, $k = 1, 2, \dots, n$ (If I take k = any other numbers, I will obtain one of the solutions already obtained with $k = 1, 2, \dots n$)

The choice of k corresponds to the choice of branch of the root.

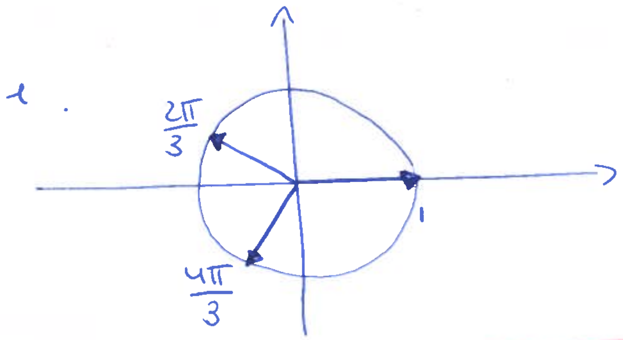
Example:

- Find z s.t. $z^3 = 1$.

(1) $k=1 \Rightarrow z = \frac{2\pi}{3}$

(2) $k=2 \Rightarrow z = \frac{4\pi}{3}$

(3) $k=3 \Rightarrow z = 2\pi$



You have as many roots as the degree of the power of z.

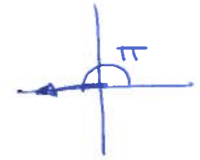
- $\sqrt{z} = (w) =$ "$z^{1/2}$" $= |z|^{1/2}\cdot e^{\frac{i\theta}{2}}$, $\theta\in(-\pi,\pi]$ PRINCIPAL BRANCH OF THE SQUARE ROOT

$|z|^{1/2}\cdot e^{\frac{i\theta}{2}+i\pi} = -|z|^{1/2} e^{\frac{i\theta}{2}}$

two roots, one positive and other $\ominus$.

⚠ $e^{i\pi} = -1$ (since $r = 1$ and $\theta = \pi$)

Cause I take 1st k with principle angle.



Similarly for arbitrary $\alpha\in\mathbb{R}$, we define

$z^\alpha = r^\alpha\cdot e^{i\alpha\theta}$

TOPOLOGY AND GEOMETRY OF COMPLEX PLANE:

Let $\{z_n\}$ be a sequence of complex numbers.

Definition 1.1 We say that z_n converges to $w\in\mathbb{C}$ if $\forall\varepsilon>0$

$\exists N = N_\varepsilon$ s.t. $|z_n - w| < \varepsilon$ as soon as $n > N_\varepsilon$

We say that a sequence $\{z_n\}$ is **Cauchy** if

$\forall \varepsilon > 0\ \exists N = N_\varepsilon$ s.t. $|z_n - z_m| < \varepsilon \quad \forall n, m > N_\varepsilon$

Proposition 1.9 $z_n \to w$ as $n \to \infty$ iff (point in the plane; sequence of complex nbs)

$\operatorname{Re} z_n \xrightarrow[n\to\infty]{} \operatorname{Re} w$ and $\operatorname{Im} z_n \xrightarrow[n\to\infty]{} \operatorname{Im} w$

(sequence of real parts; sequence of imaginary parts)

$z_n - w = (\operatorname{Re} z_n - \operatorname{Re} w) + i(\operatorname{Im} z_n - \operatorname{Im} w)$

$|z_n - w| = \sqrt{(\operatorname{Re} z_n + \operatorname{Re} w)^2 + (\operatorname{Im} z_n - \operatorname{Im} w)^2}$

Proof: Write: $|\operatorname{Re} z_n - \operatorname{Re} w|^2 + |\operatorname{Im} z_n - \operatorname{Im} w|^2 = |z_n - w|^2$

(If the RHS $\to 0 \Rightarrow$ the LHS $\to 0$ which means $\operatorname{Re} z_n \to \operatorname{Re} w$ and $\operatorname{Im} z_n \to \operatorname{Im} w$.)

we want to see the difference between $|z_n - w|$ and want it to be small. $|z_n - w| = \sqrt{(x_n - x)^2 + (y_n - y)^2}$ but since it is very uncomfortable we will use $|z_n - w|^2 = (x_n - x)^2 + (y_n - y)^2$

In other words LHS and RHS tend to zero simultaneously ▨

If RHS converges LHS converges.

Corollary 1.10 If $z_n \xrightarrow[n\to\infty]{} w \Rightarrow \bar{z}_n \xrightarrow[n\to\infty]{} \bar{w}$, and

$|z_n| \xrightarrow[n\to\infty]{} |w|$

Proposition 1.11 $\{z_n\}$ is a convergent sequence iff z_n is Cauchy.

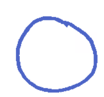
Proof: z_n converges iff $\operatorname{Re} z_n, \operatorname{Im} z_n$ converges. ▨

From Real Analysis: $\operatorname{Re} z_n, \operatorname{Im} z_n$ converge iff they are Cauchy

Because of Proposition 1.11 we say that complex numbers are complete.

SETS ON THE COMPLEX PLANE:

Definition 1.12 Let $z_0 \in \mathbb{C}$, $R > 0$. Then $\gamma(z_0, R) = \{z \in \mathbb{C} : |z - z_0| = R\}$, **circle** centred at z_0 with radius R.



(circle), $\underbrace{D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}}$ **open disk** of radius R centred at z_0.

$\underbrace{D'(z_0, R) = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}}$

Punctured disk or a **punctured R-neighbourhood of z_0**.

The set $\underbrace{\Pi_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}}$

The **upper (lower) half-plane**.

Definition 1.13 The set $S \subset \mathbb{C}$ is said to be **open** if for every point $z \in S$ there exists an $\varepsilon > 0$ s.t. $D(z, \varepsilon) \subset S$.

(for every point I take there will exist an interval (even if it's tiny) that is still in the disk).

Examples

① $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is open.

Proof Let $z = x + iy$, $y > 0$

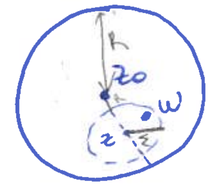
Claim: $D(z, y) \subset \Pi_+$. Pick a $w \in D(z, y)$, so that $|z - w|$ = distance between z and w.

$\sqrt{|\operatorname{Re} w - x|^2 + |\operatorname{Im} w - y|^2} < y$. In particular $|\operatorname{Im} w - y| < y$.

Therefore, $\operatorname{Im} w = y + \operatorname{Im} w - y \geq y - |\operatorname{Im} w - y| > y - y = 0$, as claimed ▨

(triangle ineq; $-y < \operatorname{Im} - y < y$)

② $D(z_0, R)$ is open.



Proof: Let $\varepsilon = R - |z - z_0|$

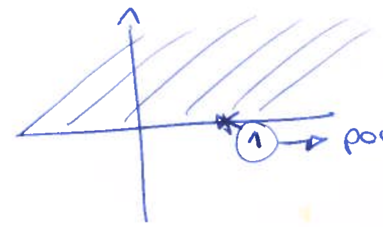
Claim: $D(z, \varepsilon) \subset D(z_0, R)$.

Indeed, let $w \in D(z, \varepsilon)$, so $|w - z| < \varepsilon$

Therefore: $|w - z_0| = |w - z + z - z_0| \leq |w - z| + |z - z_0| <$

$< \varepsilon + |z - z_0| = R - \cancel{|z - z_0|} + \cancel{|z - z_0|} = R$ as claimed ▨.

③ Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. This set is not open.



① point of the set. (If the set was open then I would be able to find an ε s.t. $D(1, \varepsilon) \subset \Omega$)

Indeed, for any $\varepsilon > 0$ the disk $D(1, \frac{\varepsilon}{2})$ does not fit inside Ω.

FUNCTIONS LIMITS AND CONTINUITY

Functions $f : \mathbb{C} \to \mathbb{C}$

$D(f)$ – **domain of f** (where f makes sense).

Example: $f(z) = \frac{1}{z}$, $z \in D(f) = \underbrace{\mathbb{C} \setminus \{0\}}_{\text{this is the domain}}$.

Representation: $f(z) = u(x,y) + i\,v(x,y)$

$z = x+iy$ where u, v are real valued.

Examples

① $f(z) = z^2$, $D(f) = \mathbb{C}$

Rewrite:

$f(z) = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2ixy}_{v}$.

② Polynomials

Let $a_0, a_1, \dots, a_n$ be complex numbers. Then

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$, $z \in \mathbb{C}$.

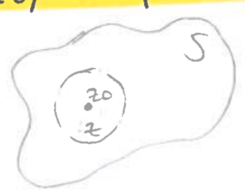
is a polynomial of order n, if $a_n \neq 0$.

If P and Q are polynomials, then

$T(z) = \dfrac{P(z)}{Q(z)}$ is called rational function

October 7th 2019

Definition 1.14: Let $D(f) = S$. We say that f has limit $w_0 \in \mathbb{C}$ as $z \to z_0 \in \mathbb{C}$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ as soon as $z \in D'(z_0, \delta) \cap S$ → es igual que decir $0 < |z - z_0| < \delta$



Notation: $\lim\limits_{z \to z_0} f(z) = w_0$ or $f(z) \xrightarrow[z \to z_0]{} w_0$

We say that f is continuous at $z_0 \in S$ if $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

If f is continuous for all $z_0 \in S$, then f is said to be continuous on S.

Proposition 1.15: $\lim\limits_{z \to z_0} f(z) = w_0$ iff $\forall$ sequences $z_n \xrightarrow[n \to \infty]{} z_0$ we have $\lim\limits_{n \to \infty} f(z_n) = w_0$ (sequence definition of limits)

f is continuous at z_0 iff $\forall$ sequences $z_n \xrightarrow[n \to \infty]{} z_0$ we have $\lim\limits_{n \to \infty} f(z_n) = f(z_0)$

Proposition 1.16

① If f has a limit at z_0, the limit is unique

② $f(z) \xrightarrow[z \to z_0]{} w_0$ iff $\mathrm{Re} f(z) \xrightarrow[z \to z_0]{} \mathrm{Re}\, w_0$ & $\mathrm{Im} f(z) \xrightarrow[z \to z_0]{} \mathrm{Im}\, w_0$

③ f is continuous at z_0 iff $\mathrm{Re} f$ and $\mathrm{Im} f$ are continuous at z_0.

④ Algebra of limits (AOL) is applicable

↳ (If f and g are continuous $\Rightarrow$ $f+g$ is continuous …)
$f \cdot g$ is continuous

⑤ If f and g are continuous, and $g(D(g)) \subset D(f)$ (domain the f), then $f \circ g$ is also continuous ($f(g(z))$ is continuous).

Examples

① $f(z) = z$ is continuous everywhere

② Polynomials are continuous everywhere, by AOL.

③ Rational function $\dfrac{P(z)}{Q(z)}$ is continuous at all $z_0 \in \mathbb{C}$ where $Q(z_0) \neq 0$.

CHAPTER 2: DIFFERENTIATION, HOLOMORPHIC FUNCTIONS

Definition 2.1 Let $S = D(f)$, and assume that $D(z_0, r) \subset S$ with some $r > 0$. Then we say that f is differentiable at z_0, if the limit

$f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

If S is open, and if f is differentiable at every point $z_0 \in S$, then we say that f is holomorphic on S, or $f \in H(S)$.

If $S = \mathbb{C}$ and $f \in H(S)$ then f is called entire.

Rewrite: $f'(z_0) = \lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z)}{h}$

Examples: ① $f(z) = z \Rightarrow f'(z_0) = \lim\limits_{h \to 0} \dfrac{(\cancel{z_0}+h) - \cancel{z_0}}{h} = \boxed{1}$

② $g(z) = |z|^2$. $|z|^2 = z \cdot \bar{z}$

Calculate

$\dfrac{g(z_0+h) - g(z_0)}{h} = \dfrac{(z_0+h)\overline{(z_0+h)} - z_0\bar{z}_0}{h} =$

$= \dfrac{z_0\bar{z}_0 + h\bar{z}_0 + z_0\bar{h} + h\bar{h} - z_0\bar{z}_0}{h} = \bar{z}_0 + \bar{h} + z_0 \dfrac{\bar{h}}{h}$ (*) → my problem is here

$\bar{h} \to 0$ as $h \to 0$, $\bar{z}_0$ = const

- If $z_0 = 0 \Rightarrow g'(0) = 0$.
- If $z_0 \neq 0$: Assume first that $h = t \in \mathbb{R}$, so

We approach it from $\mathbb{R}$ and from $\mathbb{C}$ (from $\mathbb{R}$ an imaginary axis)

$(*) = \bar{z}_0 + t + z_0 \frac{t}{t} \xrightarrow[t\to 0]{} \bar{z}_0 + z_0 = 2\,\mathrm{Re}\,z_0.$

Let $h = it$ $(h \in \mathbb{C})$, $t \in \mathbb{R}$. Thus.

$(*) = \bar{z}_0 - it + z_0 \frac{-it}{it} = \bar{z}_0 - z_0 - it \xrightarrow[t\to 0]{} \bar{z}_0 - z_0 = -2i\,\mathrm{Im}\,z_0.$

Conclusion: $|z|^2$ is differentiable only at $z_0 = 0$. because approaching it from $\mathbb{R}$ and Imag. doesn't have = solution of limit

October 7th 2019

Another notation for derivatives: $f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$

$$\frac{d f(z_0)}{dz}$$

$|z|^2$ is not differentiable except at 0: $\left.\frac{d|z|^2}{dz}\right|_{z=0} = 0$

$|z^2| = x^2 + y^2$ is continuous but not differentiable

Lemma 2.2 If f is diff at z_0, it is continuous at z_0.

See analysis 2 or 1.

Theorem 2.3 Differentiation Rules.

Assume that f, g are differentiable. Then

1. If $c =$ const, then $\frac{d}{dz} c = 0$,
2. $\frac{d}{dz}(cf) = cf'$
3. $(f+g)' = f' + g'$.
4. $(fg)' = f'g + fg'$
5. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ for all z where $g(z) \neq 0$,
6. If $f \in H(\Omega)$, $f(\Omega) \subset S$, $g \in H(S)$, $\Rightarrow \frac{d}{dz} g(f(z)) = g'(f(z)) \cdot f'(z)$
   open set
   holomorphic
   f is dif in every point of the set H
7. $\frac{d}{dz} z^n = n \cdot z^{n-1}$ $n = 1, 2, 3, \ldots$

write: $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$.

Theorem 2.4 Suppose f is differentiable at z_0 ($z_0 = (x_0 + iy_0)$). Then the functions u, v have partial derivatives at z_0, and

$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$ → respect to x

$= v_y(x_0,y_0) - iu_y(x_0,y_0)$ → wrt y.

and hence

$u_x(x_0,y_0) = v_y(x_0,y_0)$, $u_y(x_0,y_0) = -v_x(x_0,y_0)$.

The Cauchy-Riemann Equations (CRE)

Proof: ① Let $h = t$, $t \in \mathbb{R}$. write we add t to the x part because we are approaching on real axis

$f'(z_0) = \lim_{t\to 0} \frac{u(x_0+t, y_0) + iv(x_0+t, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{t} =$

$\lim_{t\to 0} \left[\frac{u(x_0+t,y_0) - u(x_0,y_0)}{t} + i\,\frac{v(x_0+t,y_0) - v(x_0,y_0)}{t} \right] =$

Since f' exists both limits exist:

$u_x(x_0,y_0) = \lim_{t\to 0} \frac{u(x_0+t,y_0) - u(x_0,y_0)}{t}$
derivative of u in terms of x

$v_x(x_0,y_0) = \lim_{t\to 0} \frac{v(x_0+t,y_0) - v(x_0,y_0)}{t}$.
derivative of v in terms of x.

Therefore, $f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)$.

② Now let $h = it$, $t \in \mathbb{R}$. Then:

$f'(z_0) = \lim_{t\to 0} \frac{u(x_0,y_0+t) - u(x_0,y_0) + i[v(x_0,y_0+t) - v(x_0,y_0)]}{it} =$

$\frac{1}{i} = \frac{1}{i}\frac{(-i)}{(-i)} = \frac{(-i)}{(-i^2)} = \frac{-i}{1} = -i$

$= \lim_{t\to 0} \left[\frac{v(x_0,y_0+t) - v(x_0,y_0)}{t} - i \cdot \frac{u(x_0,y_0+t) - u(x_0,y_0)}{t} \right] =$

$f'(z_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$ as required. ∎

Examples:

① $f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{i\,2xy}_{iv}$.

Then $u_x = 2x$, $u_y = -2y$
$v_x = 2y$, $u_y = 2x$

② $g(z) = |z|^2 = \underbrace{x^2+y^2}_{u} + i\underbrace{0}_{v}$

$u_x = 2x$, $u_y = 2y$. } If $x=y=0$ the CRE hold.
$v_x = v_y = 0$. } (otherwise – NO).

Theorem 2.5 Assume that u, v have partial derivatives in the open set Ω. Suppose also that u_x, u_y, v_x, v_y are continuous on Ω, and satisfy CRE on Ω. Then $f = u + iv$ is holomorphic.

Example 2.6.
Let $f(z) = e^x(\cos y + i \sin y) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v} = e^x e^{iy} = e^{x+iy}$.

CRE? $u_x = e^x \cos y$, $u_y = -e^x \sin y$.
$v_x = e^x \sin y$, $v_y = e^x \cos y$.

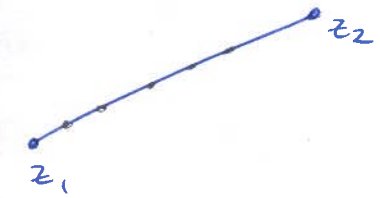
CRE do hold, and u_x, u_y, v_x, v_y are continuous on $\mathbb{C}$.
By Theorem 2.5, f is holomorphic in $\mathbb{C}$, i.e. f is entire.
By theorem 2.4, $f' = u_x + iv_x = e^x \cos y + i e^x \sin y = f$.
Notation $f(z) = e^z$, since $f' = f$.

Holomorphic functions:

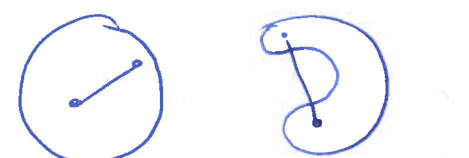
① Polynomials are entire
② Rational functions $\frac{Q(x)}{P(x)}$ away (excepto) from the roots (raíces, cuando $P(x)=0$) of denominator
③ Exponential functions

Properties of holomorphic functions

Definition 2.7. Let $z_1, z_2 \in \mathbb{C}$. Then, the set $[z_1, z_2] = \{z = \underbrace{\alpha z_1 + (1-\alpha) z_2}_{\text{parametrization of a line}}, \alpha \in [0,1]\}$ is called segment joining z_1 and z_2.

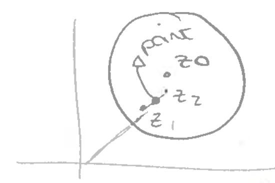


The set $\Omega \subset \mathbb{C}$ is said to be convex if $\forall z_1, z_2 \in \Omega$ we also have $[z_1, z_2] \subset \Omega$



Example: ① $D(z_0, r)$ is convex. Let $z_1, z_2 \in D(z_0, r)$

i.e. $|z_0 - z_1| < r$, $|z_0 - z_2| < r$.



Si resto point – z_0 obtengo distancia entre point y z_0 y para que $\subset D$ tiene que ser $< r$

Estimate for $\alpha \in [0,1]$:

$$|\overbrace{\alpha z_1 + (1-\alpha) z_2}^{\text{any point of the segment}} - z_0| = |\alpha z_1 + (1-\alpha) z_2 - \underbrace{\alpha z_0 - (1-\alpha) z_0}_{\text{segmento desde } z_0 \text{ hasta } z_0}| =$$

$$= |\alpha(z_1 - z_0) + (1-\alpha)(z_2 - z_0)| \underset{\text{triangle ineq}}{\leq} \alpha|z_1 - z_0| + (1-\alpha)|z_2 - z_0| < \alpha r + (1-\alpha) r = r$$

as required.

② Upper half-plane. DIY.

Definition 2.8 Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$.
Then the set

$\ell = [z_1, z_2] \cup [z_2, z_3] \cup \ldots \cup [z_{n-1}, z_n]$ is called polygonal path joining z_1 and z_n.

We say that the set $\Omega \subset \mathbb{C}$ is polygonally connected or (connected) if $\forall$ 2 points a & b in Ω there is a polygonal path within Ω, joining a and b.

Theorem 2.9 Let $\Omega \subset \mathbb{C}$ be a domain. Then

① If $f'(z)=0$, $\forall z \in \Omega$, then $f(z) = \text{const.}$, $\forall z \in \Omega$.

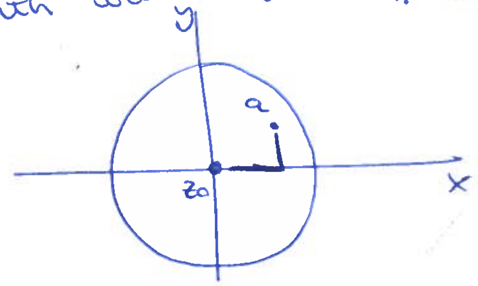
② If $f \in H(\Omega)$ and $|f| = c = \text{const}$, then $f = \text{const}$ in Ω.

Proof: Not examinable.

① $f'(z) = 0 \Rightarrow \begin{array}{l} u_x + i v_x = 0 \\ v_y - i u_y = 0 \end{array} \Rightarrow u_x = u_y = v_x = v_y = 0$

This means that u and v are constant functions in directions parallel to x or y-axis.

Let $\Omega = D(z_0, r)$. Then for any point $a \in \Omega$ there is a polygonal path with segment // to x or y-axis, joining a and z_0.

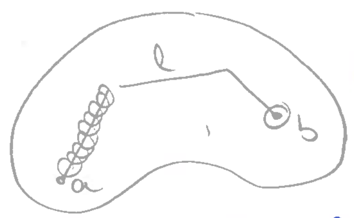


(la función al ser constante para por ej $u(x) = 3$.)

$\left.\begin{array}{l} u = \text{const on both segments} \\ v = \text{const} \end{array}\right\} \Rightarrow u(a) = u(z_0),\ v(z_0) = v(a) \Rightarrow f(a) = f(z_0)$

$\Rightarrow f(z) = \text{const}$ in $D(z_0, r)$.

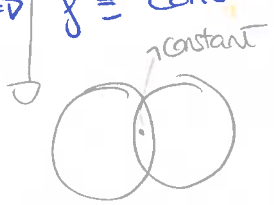
Let Ω be an arbitrary domain. For any $a, b \in \Omega$ $\exists$ a polygonal path within Ω, joining a, b. Therefore, there is a collection of finitely many disks $D(z_j\ \delta)$ $j = 1, 2, \dots, n$ s.t. $l \subset \bigcup_{i=1}^{n} D(z_i, \delta)$, $z_1 = a$, $z_n = b$, $D(z_j, \delta) \cap D(z_{j+1}, \delta) \neq \emptyset$



$f(z) = C_j = \text{const}$ for $z \in D(z_j, \delta)$

As the disks overlaps, all constants C_j coincide with each other.

Thus, $f(a) = f(b) \Rightarrow f = \text{constant}$ in Ω, as required



October 14th 2019

$f = u + iv$ is holomorphic $\Rightarrow$ CRE hold: $\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$

Theorem 2.9 Let Ω be a domain. If f is holomorphic, and

① $f'(z) = 0$, $\forall z \in \Omega \Rightarrow f = \text{const}$.

② if $|f(z)| = \text{const}$, $\forall z \in \Omega \Rightarrow f = \text{const}$

Proof ②

Let $|f| = c = \text{const}$

- If $c = 0 \Rightarrow f = 0$ in Ω.
- Let $c > 0$, so $u^2 + v^2 = c^2$.

Differentiate: $\begin{array}{l} u u_x + v v_x = 0 \\ u u_y + v v_y = 0 \end{array}$.

By CRE, → sustituyo $u_x = v_y$ and $u_y = -v_x$

$\begin{cases} (u u_x - v u_y = 0) \times u \\ (u u_y + v u_x = 0) \times v \end{cases}$ ① $\Rightarrow \begin{cases} u^2 u_x - v u u_y = 0 \\ u v u_y + v^2 u_x = 0 \end{cases} \oplus$

$\Rightarrow u^2 u_x + v^2 u_x = 0 = u_x(u^2 + v^2) = u_x c^2 \Rightarrow u_x = 0$.

② $\Rightarrow \begin{cases} u v u_x - v^2 u_y = 0 \\ u^2 u_y + v u u_x = 0 \end{cases} \ominus \Rightarrow u^2 u_x + v^2 u_y = 0 = u_y(u^2 + v^2) = u_y c^2$

$\Rightarrow u_y = 0$.

By CRE, $v_x = v_y = 0$. Therefore $f'(z) = u_x + i v_x = 0$, $\forall z \in \Omega$.

Thus, by part (1), $f = \text{const}$ in Ω, as claimed.

HARMONIC FUNCTIONS

Let $f = u + iv$ be holomorphic in Ω (open set). Assume that $u_{xx}, u_{xy}, u_{yx}, u_{yy}, V_{xx}, V_{xy}, V_{yx}, V_{yy}$ exist and are continuous in Ω. By continuity (smooth function): $u_{xy} = u_{yx}$, $V_{xy} = V_{yx}$

Differentiate twice, using CRE:

$u_{xx} = V_{xy} = V_{yx} = -u_{yy}$, so $u_{xx} + u_{yy} = 0$, ie $\Delta u = 0$

$[\Delta u = u_{xx} + u_{yy}]$.

This is LAPLACE EQUATION.

Definition 2.10 u is said to be harmonic in Ω if it has continuous derivatives up to second order and it satisfies the laplace equation

$\Delta u = 0$.

In the same way we get $\Delta v = 0$.

Question: given a harmonic function u in Ω, can we find another harmonic function V in Ω s.t $f = u + iv$ is holomorphic?

Definition 2.11 Let u, v be harmonic in Ω. We say that u and V are harmonic conjugates if u, v satisfy CRE. Or, V is a harmonic conjugate to u. (for u)

Proposition 2.12 If V is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for V.

Example: Let $u(x,y) = 2x - x^3 + 3xy^2$

① Check that $\Delta u = 0$,

② Find h. conjugate.

① Derivatives: $u_x = 2 - 3x^2 + 3y^2$, $u_y = 6xy$.

$u_{xx} = -6x$, $u_{yy} = 6x$ $\Rightarrow u_{xx} + u_{yy} = 0$.

② To find v use CRE: $V_y = u_x = 2 - 3x^2 + 3y^2$, so

$V = 2y - 3x^2y + y^3 + \varphi(x)$

Use the second CRE: $-6xy + \varphi'(x) = -u_y = -6xy$ ($u_y = -V_x$)

$\Rightarrow \varphi'(x) = 0 \Rightarrow \varphi(x) = C =$ constant.

Therefore, $V = 2y - 3x^2y + y^3 + C$ is a harmonic conjugate for u.

Look at

$f(z) = u + iv = 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3) + iC = 2x + iy + \ldots$

$= 2x + i2y - (x^3 - 3xy^2 + i3x^2y - iy^3) = 2(x+iy) - (x+iy)^3 + iC =$

$= 2z - z^3 + iC$.

October 15th 2019

CHAPTER 3: POWER SERIES

Complex series:

Let $\langle b_n \rangle$ be a sequence of complex numbers.

Then the series is the infinite sum $\sum_{n=1}^{\infty} b_n$ (*)

We say that the series (*) converges if the sequence of partial sums i.e. $S_n = \sum_{k=1}^{n} b_k$ converges as $n \to \infty$. $\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} S_n$

Converge of S_n is equivalent to convergence of $\operatorname{Re} S_n$ and $\operatorname{Im} S_n$.

Properties:

① If (*) converges $\Rightarrow b_n \to 0$, $n \to \infty$

② If $\sum a_k$ and $\sum b_k$ converge $\Rightarrow \sum (a_k + A b_k)$ converges as well, where A is a constant.

③ If (*) converges absolutely i.e. $\sum |b_k|$ converges $\Rightarrow$ (*) converges.

④ Comparison test.

⑤ Ratio test

⑥ Root test.

Examples:

① $\sum_{k=0}^{\infty} z^k$: GEOMETRIC SERIES, $z \in \mathbb{C}$.

Ratio test: $\lim_{k\to\infty} \frac{|z^{k+1}|}{|z^k|} = \lim_{k\to\infty} \frac{|z|^{k+1}}{|z|^k} = |z|$.

- If $|z|<1 \Rightarrow$ absolute convergence.
- If $|z|>1 \Rightarrow$ divergence

Recall. $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $|z|<1$.

② $\sum_{k=0}^{\infty} k^k z^k$

Root test; $\lim_{k\to\infty} \sqrt[k]{k^k |z|^k} = \lim_{k\to\infty} k|z| = \begin{cases} 0<1, \ z=0 & \text{converges} \\ +\infty>1, \ z\neq 0 & \text{diverges} \end{cases}$

Power series:

We study the series of the form

(**) $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$, $a_k =$ const, $k=0,1,2,\dots$
z is the variable
z_0 is a constant.

Definition: For the series (**) the radius of convergence is defined as

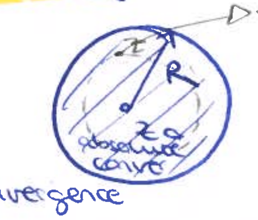
$R = \sup\{|z| : \sum |a_k||z|^k \text{ converge}\}$ (If $R=\infty$ it converges everywhere)

Theorem 3.2 Let the radius of convergence of $\sum a_k(z-z_0)^k$ be $R>0$. Then

① If $|z-z_0|<R$, then the series converges absolutely.

② If Real $|z-z_0|>R$, then the series diverges.

→ If it converges for z, ⇒ it converges for all z that form the circle



Examples:

① $\sum_{k=0}^{\infty} 3^k i^k (\underbrace{z-(i)}_{z^k})^k$. Find radius of convergence.

→ lo puedo quitar porque el límite no depende de z_0

Root test: $\lim_{k\to\infty} \sqrt[k]{3^k |z|^k} = 3|z|$ ← $|3^k||i^k||z^k|$

- If $3|z|<1 \Rightarrow$ abs. convergence
- If $3|z|>1 \Rightarrow$ divergence

$\Rightarrow R = \frac{1}{3}$.

② $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Ratio test:

$\lim_{k\to\infty} \frac{|z|^{k+1}}{(k+1)!} \frac{k!}{|z|^k} = \lim_{k\to\infty} \frac{|z|}{k+1} = 0$ ($0<1$), $\forall z \in \mathbb{C}$.

$\Rightarrow$ Radius of convergence $= +\infty$.

③ $\sum_{k=1}^{\infty} z^k = \frac{1}{1-z}$ for $|z|<1$ → $(z-0)^k$

$\sum_{n=0}^{\infty} (-w)^n = \sum_{n=0}^{\infty} (-1)^n w^n$

write the power series at $z_0=2$. $\frac{1}{1+w} = \frac{1}{1-(-w)} = \sum_{n=0}$

Write:

$\frac{1}{1-z} = \frac{1}{1-z+2-2} = \frac{1}{-1-\underbrace{(z-2)}_{w}} = -\frac{1}{1+\underbrace{(z-2)}_{w}} = -\sum (z-2)^k (-1)^k$, $|z-2|<1$.

Proposition 3.3 Suppose that the limit $\ell = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists. Then the radius of convergence equals $\frac{1}{\ell}$. (look example 1).

Follows from the root test. DIY.

Remark: Let $\ell_1 = \limsup_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{N\to\infty} \sup_{k\geq N} \sqrt[k]{|a_k|} \Rightarrow \boxed{R = \frac{1}{\ell_1}}$

Differentiability of power series:

Can we differentiate f? ($f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$, $R>0$).

"Formally speaking", the derivative is

$$g(z) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}$$

Aim: $f'(z) = g(z)$.

Lemma 3.4 The series g and f have the same radius of convergence

lemma 4.1 Analysis 2.

Theorem 3.5 Let $R>0$ be the radius of convergence for the series

$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \Rightarrow f \in H(D(z_0,R))$ and $f'(z) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}$.

Remark $a_0 = f(z_0)$, since $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

write: $f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \dots$

Thus, $a_1 = f'(z_0)$.

Similarly, $f''(z) = 2a_2 + 6a_3(z-z_0) + \dots$

Thus, $a_2 = \frac{f''(z_0)}{2}$.

Corollary 3.6 The function f has infinitely many derivatives in the disk $D(z_0,R)$ and $a_k = \frac{f^{(k)}(z_0)}{k!}$, so

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k.$$

The exponential function

Definition $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Entire function:

Denote $f(z) = \exp(z)$

Theorem 3.7

1. $f' = f$
2. $f(0) = 1$
3. $\exp(z+w) = \exp(z) \cdot \exp(w)$
4. $\exp z \neq 0$, $z \in \mathbb{C}$.

Proof:

1. Differentiate the series: $f'(z) = \sum_{k=0}^{\infty} k \cdot \frac{z^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = f$.

2. Immediate.

3. Let p = constant. Define

$g = \exp(p-z) \cdot \exp(z)$.

Then $g' = -\exp(p-z)\exp(z) + \exp(p-z)\exp(z) = 0$.

$\Rightarrow g = \text{constant} = \exp(p)$.

Thus, $\exp(p-z)\exp(z) = \exp(p)$

Let $p = w+z$, so

$\exp w \exp z = \exp(z+w)$ as claimed.

4. Using 3: $\exp(-z) \cdot \exp(z) = 1$, so $\exp(z) \neq 0$ $\blacksquare$

October 21st 2019

HW2 → 11th november.

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

If f is represented by a convergent power series with radius of convergence $R>0$, then f is said to be analytic in $D(z_0, R)$

for theorem 3.5.

Analytic $\Rightarrow$ holomorphic.

Example:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad R = \infty$$

Theorem 3.7

1. $\frac{d}{dz}\exp(z) = \exp(z)$.
2. $\exp(0) = 1$.
3. $\exp(z+w) = \exp(z)\cdot\exp(w)$.
4. $\exp(z) \neq 0$.

holomorphic in $\mathbb{C}$

Corollary 3.8 Let f be entire and assume that

$f' = f$, and $f(0) = 1$.

Then $f(z) = \exp(z)$.

Proof: Let $g(z) = \exp(-z) f(z)$. Then

$$\frac{d}{dz} g(z) = -\exp(-z) f(z) + \exp(-z) \underbrace{\frac{d}{dz}(f(z))}_{= f} = 0$$

By theorem 2.9, $g(z) = \text{constant}$, i.e. $g(z) = g(0) = f(0) = 1$.

$\Rightarrow g(z) = 1, \quad z \in \mathbb{C}$.

Therefore, $f(z) = \exp(z)$, as claimed. ∎

Corollary 3.9

Let $e^z = e^x(\cos y + i \sin y)$, $z = x+iy$, defined in example 2.6.

We know that

$$\underbrace{\frac{d}{dz} e^z = e^z}_{\text{condition 1}}, \quad \underbrace{e^0 = 1}_{\text{cond. 2}} \Rightarrow \text{By corollary 3.8, } e^z = \exp(z).$$

Trigonometric and hyperbolic functions

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

$$\sinh(z) = \frac{e^{z}-e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

Theorem 3.10

$\frac{d}{dz}\cos(z) = -\sin(z)$ $\quad$ $\frac{d}{dz}\sin(z) = \cos(z)$

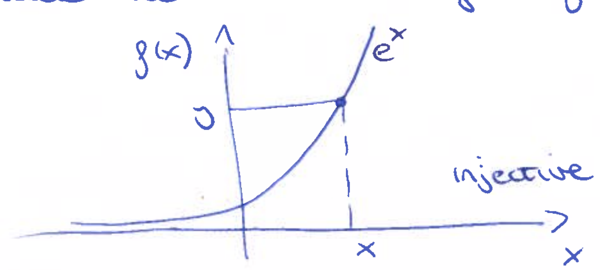
$\frac{d}{dz}\cosh(z) = \sinh(z)$ $\quad$ $\frac{d}{dz}\sinh(z) = \cosh(z)$.

$\cos(z+w) = \cos(z)\cdot\cos(w) - \sin(z)\cdot\sin(w)$.

In particular, $\sin^2 z + \cos^2 z = 1$.

Logarithm and its branches

Back to Real Analysis: $f(x) = e^x$, monotone increasing



$e^x: \mathbb{R} \to (0, +\infty)$

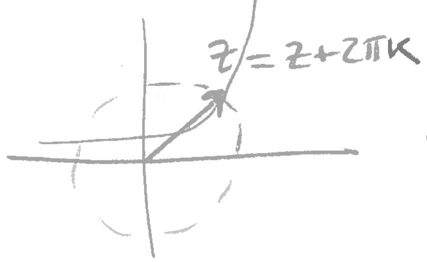
injective $\begin{cases} \forall y \in (0,\infty) \text{ there is a unique } x \text{ s.t. } e^x = y. \\ \text{Thus } x = \log y = \ln y. \end{cases}$

To Complex Analysis: is e^z injective? $\boxed{\text{NO}}$

para mismo z hay más de una y

The reason: $e^{z+2\pi k i} = e^z = e^x(\cos(y+2\pi k) + i\sin(y+2\pi k)) =$

$= e^x(\cos(y) + i\sin(y))$.



al añadir $2\pi k$, al menos tengo igual resultado en 2 ocasiones porque 2π y 4π por ejemplo tienen igual resultado en $\cos(y)$ y en $\sin(y)$

Nevertheless, try to solve

$e^w = z$ for w.

Rewrite: $e^w = e^{u+iv} = e^u \cdot e^{iv} = |z|\, e^{i\arg(z)}$

es la única parte que no tiene nada imaginario

Thus, $e^u = |z|$ so $u = \ln|z|$ and $v = \arg z$, and hence

$|z| = \sqrt{x^2+y^2}$ No hay i

$w = \ln|z| + i\arg(z) = \log(z)$

The principle logarithm

$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$, $\arg(z) \in (-\pi, \pi]$.

So, $\log(z) = \mathrm{Log}|z| + 2\pi k i$, $k \in \mathbb{Z}$.
$+i\arg z$

problem class

Different k's correspond to different branches of logarithm

$\mathrm{Log}(z)$ is defined on $\mathcal{U} = \{z \in \mathbb{C}, \mathbb{C} \setminus \{\mathrm{Im}\, z = 0, \mathrm{Re}\, z \le 0\}\}$

this excludes point $(0,0)$ and negative real axis

Theorem 3.11 $\mathrm{Log}(z)$ is injective on $\mathcal{U}$, it is holomorphic on $\mathcal{U}$ and

$\frac{d}{dz}\log(z) = \frac{1}{z}$ DIY.

October 22nd 2019

$\log(z) = \ln|z| + 2\pi k i$, $k \in \mathbb{Z}$.
$+i\arg z$

$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$ $*_1$

Powers of z:

viene de la fórmula $\alpha \log z = \log z^\alpha$

Def: $\alpha \in \mathbb{C}$. Define: $z^\alpha = e^{\alpha \log z}$

Example: $i^i = z^\alpha$ — Principle Branch, i.e.

$i^i = e^{i \log i}$

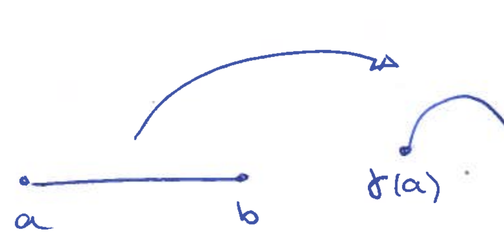
ahora aplico $*_1$; $\ln|1| = 0$

$|i| = 1$, $\mathrm{Arg}\, z = \frac{\pi}{2} \Rightarrow i^i = e^{i\log i} = e^{i(\ln|i| + i\frac{\pi}{2})} = e^{-\pi/2}$

principle argument

CHAPTER 4: CONFORMAL MAPPINGS

Def 4.1 Let $[a,b] \subset \mathbb{R}$ be an interval. Then a continuous function $\gamma : [a,b] \to \mathbb{C}$ is called a path.



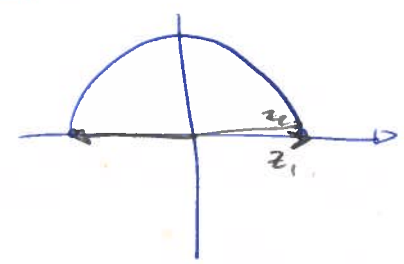
Denote: $\gamma^* = \{z \in \mathbb{C} : z = \gamma(t), t \in [a,b]\}$

image of the function

We say that $\gamma(t)$ parametrises γ^*

Example $\Rightarrow |1| \cdot e^{i\arg(z)}$.

① $\gamma_1(t) = e^{it}$, $t \in [0, \pi]$



② $\gamma_2(t) = e^{it^2}$, $t \in [0, \sqrt{\pi}]$

The same $\gamma_1^* = \gamma_2^*$

Every path has orientation: $\gamma(a)$ is the start (initial) point, $\gamma(b)$ is the end point (terminal point)

If γ' exists at $t_0 \in [a,b]$, then $\gamma'(t_0)$ defines a tangent vector to γ

γ' is defined as follows:

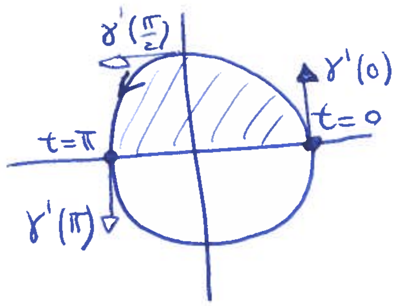
$$\gamma'(t_0) = x'(t_0) + iy'(t_0) \qquad (y - y_0) = \underset{\gamma'(t_0)}{m}\underbrace{(x - x_0)}_{s} + n$$

Equation of the tangent line: $\gamma(t_0) + \gamma'(t_0)s$, $s \in \mathbb{R}$.

$$\gamma(t) = \gamma(t_0) + \gamma'(t_0)s$$

Example:

① $\gamma(t) = e^{it}$, $t \in [0,\pi]$



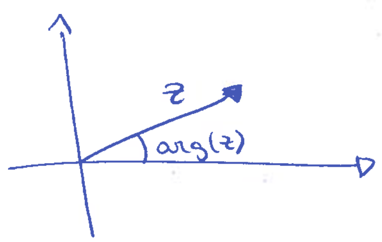
$\gamma'(t) = ie^{it}$

$\gamma'(0) = i$

$\gamma'(\frac{\pi}{2}) = i\, i = (-1)$

$\gamma'(\pi) = i(-1) = -i$

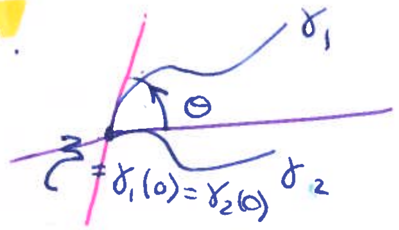
Observation: the tangent line at $\gamma(t_0)$ makes angle $\arg \gamma'(t_0)$ with the real axis.



Conformal maps:

Let γ_1, γ_2 be 2 paths, parametrised by $t \in [0,1]$, s.t. $\gamma_1(0) = \gamma_2(0) = z$, and $\gamma_1'(0), \gamma_2'(0)$ exist. Then

$\theta = \arg \gamma_1'(0) - \arg \gamma_2'(0)$ is the angle between tangent lines to γ_1 and γ_2 at z.



tangent at $\gamma_2(0) = \gamma_2'(0)$

Now, let $f \in H(\Omega)$, and let $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be paths with the above properties.

Then,

$\beta_1(t) = f(\gamma_1(t))$, $\beta_2 = f(\gamma_2(t))$ are new paths, s.t

$\beta_1(0) = \beta_2(0) = f(z)$

By the chain rule, $\beta_1'(0) = f'(z)\gamma_1'(0)$, $\beta_2'(0) = f'(z)\gamma_2'(0)$

Indeed, let $f = u + iv$, $\gamma_1 = x_1 + iy_1$, $\gamma_2 = x_2 + iy_2$, so

$$\beta_1' = \left(u(x_1,y_1) + iv(x_1,y_1)\right)' = u_x x_1' + u_y y_1' + iv_x x_1' + iv_y y_1' =$$

$$= (u_x + iv_x)x_1' + (u_y + iv_y)y_1' = (u_x + iv_x)x_1' + (-iu_y + v_y)iy_1' =$$

By CRE

$$= (u_x + iv_x)x_1' + (u_x + iv_x)iy_1' = (u_x + iv_x)(x_1' + iy_1') =$$

$$= f'(z)\gamma_1'(0).$$

Let $\varphi = \arg \beta_1'(0) - \arg \beta_2'(0)$

Theorem. Let f, γ_1, γ_2 be as before. Assume that $f'(z) \neq 0$.

Then, $\varphi = \theta \mod 2\pi$. In other words, f preserves angles between pairs of paths.

Proof. Write:

$$\varphi = \arg \beta_1'(0) - \arg \beta_2'(0) = \arg(f'(z)\gamma_1'(0)) - \arg(f'(z)\gamma_2'(0)) =$$

$$= \arg \cancel{f'(z)} + \arg \gamma_1'(0) - \arg \cancel{f'(z)} - \arg \gamma_2'(0) = \theta \mod 2\pi$$

☒

Definition 4.3 A function $f \in H(\Omega)$ is said to be conformal on Ω if $f'(z) \neq 0$, $z \in \Omega$.

Linear fractional transformation

Let $f(z) = \dfrac{az+b}{cz+d}$; $a,b,c,d \in \mathbb{C}$

- If $c = 0$, $d \neq 0$: $f(z) = \dfrac{1}{d}(az+b)$.

- Suppose $c \neq 0$. Then,

$$f(z) = \frac{a}{c} - \frac{ad-bc}{c(cz+d)} = \frac{az+b}{cz+d}.$$

Assume: $ad-bc \neq 0$.

→ If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $ad-bc = \det A \neq 0$.

→ If $cz+d \neq 0 \Rightarrow f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$.

f has inverse, i.e. the equation $w = f(z)$ has a unique solution.

Indeed, $w = \frac{az+b}{cz+d} \iff w(cz+d) = az+b \iff z(cw-a) = -dw+b$.

$\iff z = \frac{-dw+b}{cw-a}$, $w \neq \frac{a}{c}$ = para denominador $\neq 0$.

A composition of 2 linear fractional transformations is a linear fractional transformation:

Let $f_k(z) = \frac{a_k z + b_k}{c_k z + d_k}$, $k = 1,2$. Then,

$$f_1(f_2(z)) = \frac{a_1 \frac{\boxed{a_2}z+\boxed{b_2}}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{\boxed{c_2}z+\boxed{d_2}} + d_1} = \frac{(a_1a_2+c_2b_1)z + a_1b_2+d_2b_1}{(a_2c_1+c_2d_1)z + c_1b_2+d_1d_2}$$

entries of the matrix A_1A_2.

$$A_1A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

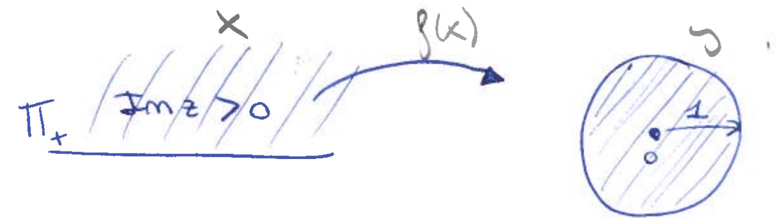
Therefore, the determinant condition is satisfied, since $\det(A_1A_2) = \det A_1 \cdot \det A_2 \neq 0$.

porque para que $f_2(z)$ tuviera inverse $ad-bc \neq 0$

Thus, linear fractional transformation forms a group!

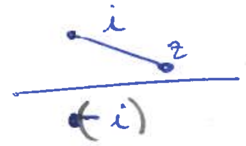
Example:

① Looking for a linear fractional transformation, mapping the upper half-plane onto the unit disk $D(0,1)$.



Upper half-plane:

$\{z : |z-i| < |z+i|\}$,



or,

$\left\{z : \left|\frac{z-i}{z+i}\right| < 1\right\}$

Therefore, $f(z) = \frac{z-i}{z+i}$ maps Π_+ into $D(0,1)$. → this map is holomorphic in the Upper half plane and conformal on Π_+.

To prove that f is surjective, find the inverse:

If $w = \frac{z-i}{z+i}$, then $z = -i\frac{w+1}{w-1}$

It remains to show that if $w \in D(0,1) \Rightarrow \operatorname{Im} z > 0$

Indeed,

$$z = -i \cdot \frac{(w+1)(\bar{w}-1)}{|w-1|^2} = -i\,\frac{w\bar{w} + \bar{w} - w - 1}{|w-1|^2} = -i\,\frac{|w|^2 - 1 - 2i\operatorname{Im} w}{|w-1|^2} =$$

$$= \frac{2\operatorname{Im} w}{|w-1|^2} + i \cdot \underbrace{\frac{1-|w|^2}{|w-1|^2}}_{>0}$$

$\cdot \frac{(\bar{w}-1)}{(\bar{w}-1)}$

October 28th 2019

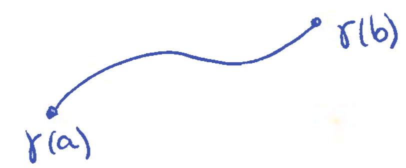
$f(z)=z^2$ is conformal everywhere but in the origin, where $f'(z)=0$.

$f(z)=e^z$ is conformal everywhere in the complex plane since $f'(z)=e^z\neq 0$.

Path:

$\gamma:[a,b]\to\mathbb{C}$.

γ is continuous

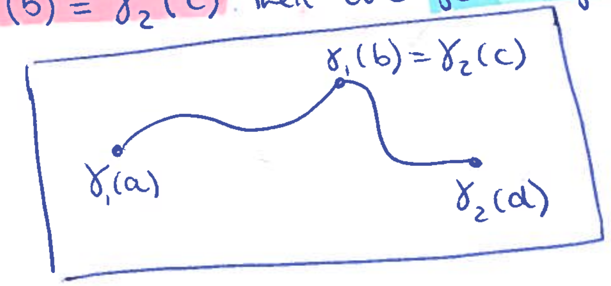


γ^* = Image of γ.

CHAPTER 5: CONTOUR INTEGRATION CAUCHY'S THEOREM

Definition 5.1

① Let $\gamma_1:[a,b]\to\mathbb{C}$, $\gamma_2:[c,d]\to\mathbb{C}$ s.t $\gamma_1(b)=\gamma_2(c)$. Then the join of 2 paths, denoted $\gamma=\gamma_1\cup\gamma_2$ is defined as



a path by:

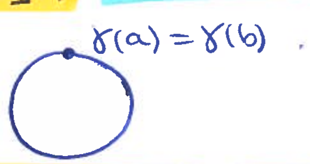
$$\gamma(t)=\begin{cases}\gamma_1(t), & t\in[a,b],\\ \gamma_2(t+c-b), & t\in[b,b+d-c]\end{cases}$$

It is defined on $[a,b+d-c]$.

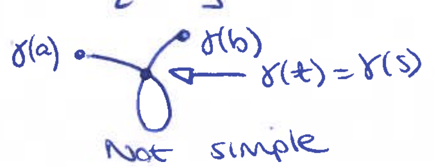
② The reverse path:

$\bar{\gamma}(t)=\gamma(a+b-t)$, $t\in[a,b]$. Notation: $-\gamma$.

③ γ is closed if $\gamma(a)=\gamma(b)$

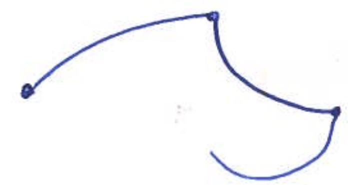


④ γ is simple, if $\forall t, s\in[a,b]$ s.t $s<t$ and $t-s<b-a$, we have $\gamma(s)\neq\gamma(t)$

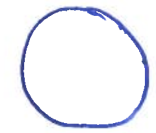


⑤ γ is smooth if γ' exists on $[a,b]$ and it is continuous

⑥ γ is piecewise smooth if it is a join of finitely many smooth paths.



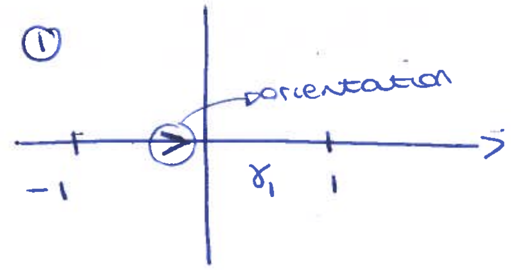
⑦ A contour is a simple closed path:



Examples:

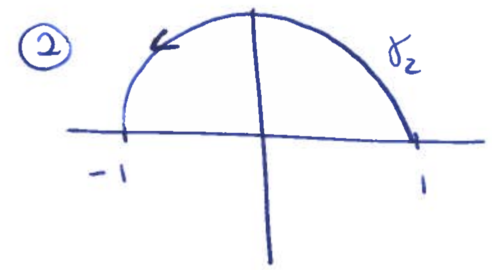
① $\gamma_1(t)=t$ (only real part), $t\in[-1,1]$

smooth.



② $\gamma_2(t)=e^{it}$ (we have imaginary and real), $t\in[0,\pi]$.

smooth.



③ $\gamma=\gamma_1\cup\gamma_2$ - piecewise smooth contour.

④ $\gamma_3(t)=\sin t$, $t\in[0,\pi]$.

Theorem 5.2: The Jordan curve theorem

Let γ be a contour. Then the complement of γ^* consists of 2 domains:

⇒ γ^* es el conjunto de todos los puntos que forman la curva.

→ one is bounded (and it is called the interior of γ): $Int(\gamma)$.

→ the other one is unbounded (called the exterior of γ): $Ext(\gamma)$.

$\mathbb{C}=\gamma^*\cup Int(\gamma)\cup Ext\,\gamma$.

γ has a positive (standard) orientation if, moving along γ, the interior of γ is on the left.

i.e. the point moving along γ, rotates about every point in $Int\,\gamma$ counterclockwise.

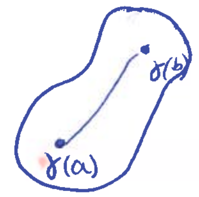
Path integrals

If $F:[a,b]\to\mathbb{C}$, and F is piecewise continuous, then

$$\int_a^b F(t)\,dt = \int_a^b \operatorname{Re}F(t)\,dt + i\int_a^b \operatorname{Im}F(t)\,dt.$$

Definition 5.3 Let $f:\Omega\to\mathbb{C}$, and let $\gamma:[a,b]\to\Omega$. Then the integral of f along the path γ is defined as

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$



October 29th 2019

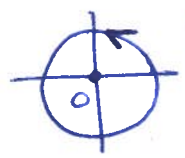
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$\gamma:[a,b]\to\mathbb{C}$

Examples

Let $\Omega=\mathbb{C}\setminus\{0\}$, let $f(z)=z^n$, $n\neq -1$.

① Let $\gamma(t)=R\cdot e^{it}$ $t\in[0,2\pi]$.



Then,

$$\int_\gamma f(z)\,dz = Ri\int_0^{2\pi}(Re^{it})^n e^{it}\,dt = R^{1+n}i\int_0^{2\pi} e^{it}e^{int}\,dt =$$

$$= R^{1+n}i\,\frac{1}{i(n+1)}\,e^{i(n+1)t}\Big|_0^{2\pi} = \boxed{0}$$

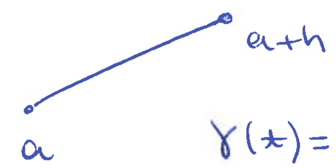
② Let $f(z)=z^{-1}$. Then,

$$\int_\gamma f(z)\,dz = R\,i\int_0^{\pi}(R(e^{it}))^{-1}e^{it}\,dt = i\int_0^{2\pi}dt = 2\pi i$$

Does not depend on R!

③ Let $a,h\in\mathbb{C}$. Then:

$$\int_{[a,a+h]} dz = h.$$



$\gamma(t) = a + t\,e^{i\theta}$ (a: como si fuera mi punto inicial; $t\,e^{i\theta}$: le sumo un número complejo), $\theta=\arg h$ (porque mi recta es la indicación del número complejo)

$t\in[0,|h|]$.

Indeed, $\displaystyle\int_\gamma dz = \int_0^{|h|} 1\cdot e^{i\theta}\,dt = e^{i\theta}\int_0^{|h|}dt = |h|e^{i\theta} = h$, as required. ($f(z)=1$)

Definition 5.4 Length of the path $\gamma:[a,b]\to\mathbb{C}$ is defined as

$$L(\gamma) := \int_a^b |\gamma'(t)|\,dt.$$

Example:

① Let $\gamma(t)=Re^{it}$, $t\in[0,2\pi]$. Then, $L(\gamma)=\displaystyle\int_0^{2\pi}R\,dt = 2\pi R$, as expected.

② Find the length of $\gamma(t)=\sin(t)$, $t\in[0,\pi]$ DIY.

Theorem 5.6 (Properties of path integrals).

① $\displaystyle\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz$

② If $\gamma=\gamma_1\cup\gamma_2$, then, $\displaystyle\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$.

③ $\displaystyle\int_\gamma c\cdot f(z)\,dz = c\int_\gamma f(z)\,dz$, c = constant.

④ $\displaystyle\int_\gamma (f(z)+g(z))\,dz = \int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz$.

⑤ Let $\gamma_k:[a_k,b_k]\to\mathbb{C}$, $k=1,2$, s.t. $\underbrace{\gamma_1^*=\gamma_2^*}_{\text{2 different parametrisations of the same image}}$.

Assume that there is a mapping $\Psi([a_2,b_2]) = [a_1,b_1]$ s.t. Ψ exists, it is positive and continuous, and s.t. $\gamma_2(t) = \gamma_1(\Psi(t))$, $t \in [a_2,b_2]$.

Then, $$\int_{a_1}^{b_1} f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_{a_2}^{b_2} f(\gamma_2(t))\,\gamma_2'(t)\,dt$$

⑥ For γ_1 and γ_2 as above in ⑤ $L(\gamma_1) = L(\gamma_2)$.

⑦ Suppose $\max_{z\in\gamma} |f(z)| = M$

Then $$\left|\int_\gamma f(z)\,dz\right| \le M\,L(\gamma)$$

Proof:

① DIY.

②-④ Standard for Rieman Integrals

⑤ $$\int_{a_1}^{b_1} f(\gamma_1(t))\,\gamma_1'(t)\,dt =$$

Let $t = \Psi(s)$

$$\int_{a_1}^{b_1} f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_{a_2}^{b_2} f(\gamma_2(s))\,\gamma_1'(\Psi(s))\,\Psi'(s)\,ds =$$

$$= \int_{a_2}^{b_2} f(\gamma_2(s))\,\gamma_2'(s)\,ds$$

⑥ The same proof.

⑦ Recall: for real function g:

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt \le \max_{t\in[a,b]} |g(t)|\,(b-a)$$

Let $I = \int_\gamma f(z)\,dz = |I|e^{i\theta}$. Thus, $|I| = Ie^{-i\theta} = \int_a^b e^{-i\theta} f(\gamma(t))\,\gamma'(t)\,dt$

Therefore,

solo coge la parte real porque quiere demostrar lo anterior de la función g, y g es una función real

$$|I| = \int_a^b \mathrm{Re}\left(e^{-i\theta} f(\gamma(t))\,\gamma'(t)\right)dt \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\le M\int_a^b |\gamma'(t)|\,dt = M\,L(\gamma)$$ as required.

PRIMITIVES

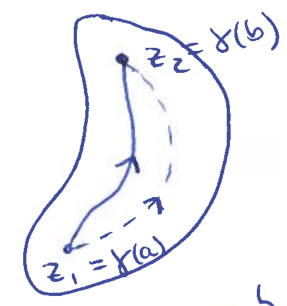
Definition 5.7 Let f be continuous on domain $\Omega \subset \mathbb{C}$. Then, a function $F \in H(\Omega)$ is said to be a primitive of f if

$F'(z) = f(z)$ antiderivative

Theorem 5.8 "Fundamental theorem of calculus".

Let f be continuous on Ω, and let F be its primitive. Let γ be a path, joining 2 ~~paths~~ points $z_1, z_2 \in \Omega$, s.t. $\gamma^* \subset \Omega$. Then:

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$



Proof: write: $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt =$$

$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1)$$

By chain rule.

Sometimes we write: $\int_\gamma f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz$

Corollary 5.9

Let f, F be as above, and let γ be a contour.

Then $$\int_\gamma f(z)\,dz = 0.$$

Example,

① Let $\Omega = \mathbb{C} \setminus \{0\}$, and let $f(z) = z^n$, $n \neq -1$

Let $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$, $R > 0$.

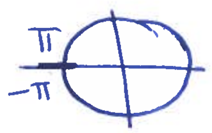
The function $F(z) = \frac{z^{n+1}}{n+1}$ is a primitive of f. Thus

$$\int_\gamma f(z)\,dz = 0.$$

② Let $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = z^{-1}$, $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$.

Let $F(z) = \log z$.

$\log z$ is holomorphic on the $\underbrace{\text{cut plane}}_{[-\pi,\pi]}$. Thus



$$\int_\gamma \frac{1}{z}\,dz = \underbrace{\log(Re^{i\pi})}_{F(z_2) = F(\gamma(b))} - \underbrace{\log(Re^{-i\pi})}_{F(z_1) = F(\gamma(a))} = (\ln R + i\pi) - (\ln R - i\pi) = 2\pi i$$

Cauchy - Goursat theorem

$\int_\gamma f(z)\,dz = 0$

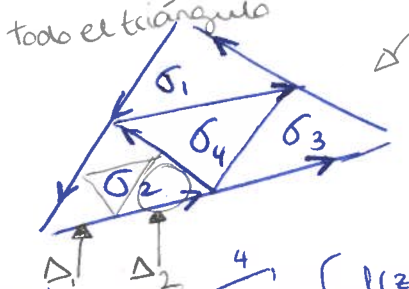
Theorem 5.10 Let $f \in H(\Omega)$ and let γ be a contour s.t. $\operatorname{Int}\gamma \subset \Omega$. Then $\int_\gamma f(z)\,dz = 0$.

Theorem 5.11 Let $f \in H(\Omega)$, and let γ be a triangular contour, s.t. $\gamma^* \cup \operatorname{Int}\gamma \subset \Omega$. Then $\int_\gamma f(z)\,dz = 0$.

Proof: Denote $\Delta_0 = \gamma^* \cup \operatorname{Int}\gamma$. Split Δ_0 into 4 triangles as shown

todo el triángulo

All $= \Delta_0$



Then, $I = \int_\gamma f(z)\,dz = \sum_{k=1}^{4} \int_{\sigma_k} f(z)\,dz$

Let γ_1 be the contour (one of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$) with the longest absolute value. $L(\gamma_1) = \frac{1}{2^1} L(\gamma)$

Then $|I| \leq 4|I_1|$, $I_1 = \int_{\gamma_1} f(z)\,dz$

Let $\Delta_1 = \gamma_1^* \cup \operatorname{Int}\gamma_1$. Then repeat the construction.

Repeating indefinitely, we get a sequence of triangles and contours Δ_k, γ_k s.t. $(k = 1, 2, \dots)$, obtain $\Delta_k = \gamma_k^* \cup \operatorname{Int}\gamma_k$

① $\Delta_{k+1} \subset \Delta_k$

② $L(\gamma_{k+1}) = \frac{1}{2^{k+1}} L(\gamma_k) = 2^{-k-1} L(\gamma_k)$

③ $|I| \leq 4^k |I_k|$, $I_k = \int_{\gamma_k} f(z)\,dz$.

November 11th 2019

The set $\bigcap_{k=1}^{\infty} \Delta_k$ (intersection) is non-empty.

Let $z_k \in \Delta_k$ be arbitrary.

Estimate $|z_k - z_l|$

- If $k < l$, then $|z_k - z_l| < L(\gamma_k)$.
- If $l < k$, then $|z_k - z_l| < L(\gamma_l)$, so $|z_k - z_l| < L(\gamma_k) + L(\gamma_l) \xrightarrow[k,l \to \infty]{} 0$

Thus $\langle z_k \rangle$ is Cauchy. Since $\mathbb{C}$ is complete, z_k has a limit:

↳ A sequence is cauchy $\Leftrightarrow$ it converges = has a limit

$$\xi = \lim_{k \to \infty} z_k.$$

Since each Δ_n is closed, $\xi \in \Delta_n$ for each n. Therefore,

$\xi \in \bigcap_{n=1}^{\infty} \Delta_n$.

Since $f \in H(\Omega)$, $f'(\xi)$ exists: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. ← def of limit

$$\left| \frac{f(z) - f(\xi)}{z - \xi} - f'(\xi) \right| < \varepsilon \quad \forall z \in D(\xi, \delta).$$

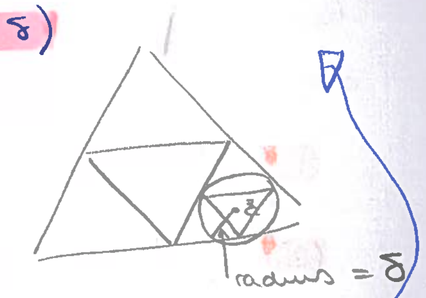
Rewrite:

$$\left|f(z)-\overset{\text{constant}}{f(\xi)}-f'(\xi)(z-\xi)\right| < \varepsilon|z-\xi| \quad \forall z \in D(\xi,\delta)$$

Claim: for large k: $\Delta_k \subset D(\xi,\delta)$.

Indeed, since $\xi \in \Delta_k$ $\forall k$, we have $|z-\xi| < L(\gamma_k)$, $\forall z \in \Delta_k$

as $L(\gamma_k) \to 0$, $k \to \infty$, we have $z \in D(\xi,\delta)$



Since, $f(\xi)$ and $f'(\xi)(z-\xi)$ have primitives,

we have: $$\int_{\gamma_k}\left(f(\xi)+f'(\xi)(z-\xi)\right)dz = 0.$$

Thus, $$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k}\left(\left(f(z)-f(\xi)-f'(\xi)(z-\xi)\right)\right)dz$$

Estimate using Theorem 5.6 (7).

$$\left|\int_{\gamma_k} f(z)\,dz\right| \le \varepsilon \underbrace{\max_{z\in\gamma_k^*}|z-\xi|}_{< L(\gamma_k)} L(\gamma_k) \le \varepsilon L^2(\gamma_k). \text{ Therefore,}$$

$$|I| \le 4^k \left|\int_{\gamma_k} f(z)\,dz\right| \le \varepsilon 4^k L^2(\gamma_k) = \varepsilon 4^k 2^{-2k} L(\gamma) =$$

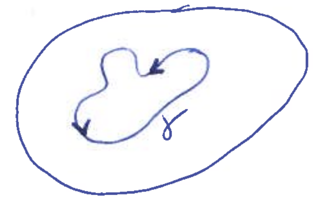
$= \varepsilon L^2(\gamma)$ Since $\varepsilon > 0$ is arbitrary $\Rightarrow I = 0$, as required ▨

November 12th 2019

Theorem 5.10

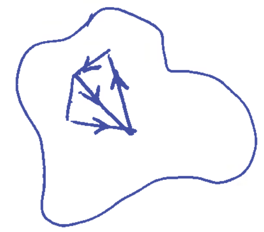
Let $f \in H(\Omega)$

Let $\gamma \in \Omega$ be a contour s.t. $\operatorname{Int}\gamma \subset \Omega$. Then $\int_\gamma f(z)\,dz = 0$



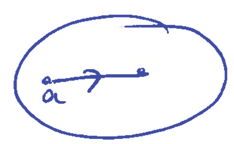
Corollary 5.12 Let $\gamma \subset \Omega$ be a polygonal contour, s.t. $\operatorname{Int}\gamma \subset \Omega$

Then, $\int_\gamma f(z)\,dz = 0$



Proof: Divide $\operatorname{Int}\gamma$ into triangles and use Theorem 5.11 ▨

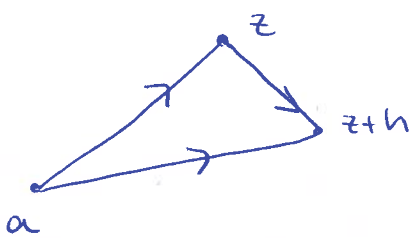
Theorem 5.13 Let G be convex and assume that f is continuous in Ω, and $\int_\gamma f(z)\,dz = 0$, for every triangular contour $\gamma \subset \Omega$. Then, for any $a \in \Omega$, the function $F(z) = \int_{[a,z]} f(w)\,dw$ is a primitive of f.



Proof: Want: (remember $F'(z) = f(w)$)

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| < \varepsilon$ if $|h| < \delta$

Use 3 facts:



① $$\frac{\overset{\int_{[a,z+h]}}{F(z+h)} - \overset{\int_{[a,z]}}{F(z)}}{h} = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw$$

(2) $\forall \varepsilon > 0, \exists \delta > 0$ s.t $|f(z) - f(w)| < \varepsilon$ if $|w - z| < \delta$.

$w\big]_z^{z+h} = z+h-z = h$.

(3) $\int_{[z,z+h]} dw = h$, i.e. $\frac{1}{h}\int_{[z,z+h]} dw = 1$.

Rewrite:

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw - f(z) = \frac{1}{h}\int_{[z,z+h]} \big(f(w)-f(z)\big)\,dw$$

$$\left|\int_\gamma f(z)\,dz\right| \le M\,L(\gamma)$$

Estimate the RHS using Theorem 5.6(7) by

$\frac{1}{|h|} \max_{w\in[z,z+h]} |f(w)-f(z)| \cdot \underbrace{|h|}_{\text{length of } \gamma = z \text{———} z+h}$ Assume $|h| < \delta$ so . in order to have $|w-z|<\delta$ (el valor mayor que puede tomar $w = z+h$) porque $w \in [z,z+h]$

now apply property 2.

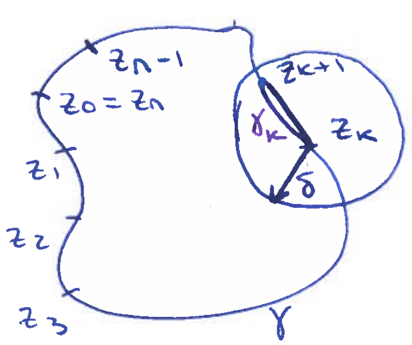
$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| < \varepsilon, \text{ as required } \blacksquare$$

In particular, every function $f \in H(G)$ has a primitive in G, see theorem 5.11.

Sketch of the proof of Theorem 5.10

Pick a sequence $z_0, z_1, \dots, z_n = z_0$ on the contour γ. Join z_k and z_{k+1} with straight segments. Then

$\sigma = \bigcup_{k=0}^{n-1} [z_k, z_{k+1}]$ is a polygonal contour



The points z_k are chosen in such a way that for some $\delta > 0$

(1) $D(z_k, \delta) \subset \Omega$, $\forall k$,

(2) $\gamma_k^* \subset D(z_k, \delta)$

Consider the disk $D = D(z_k, \delta)$

It is convex, so f has a primitive in D, by theorem 5.13.

Therefore, by theorem 5.8,

$$\int_{\gamma_k} f(z)\,dz = \int_{[z_k, z_{k+1}]} f(z)\,dz$$

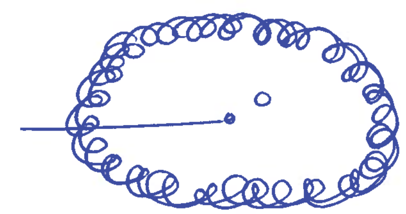
Sum up in k:

$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz$. By corollary 5.12, the RHS = 0

$\Rightarrow \int_\gamma f(z)\,dz = 0$ as claimed $\blacksquare$

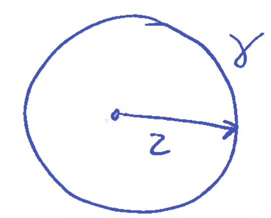
Example: $f(z) = \frac{1}{z}$ in $\mathbb{C} \setminus \{0\}$.

A primitive is given by $\log z$, which is not holomorphic in $\mathbb{C}\setminus\{0\}$



Example. find

$\int_\gamma \frac{\cos^2 z \cdot e^{z^2}}{(z^2-16)(z^3+29)}\,dz$, where γ is a circular contour centred at $z=0$ of radius 2.



The function is holomorphic on $D(0,3)$. Thus

↳ si pongo un $4 \Rightarrow z^2-16=0$.

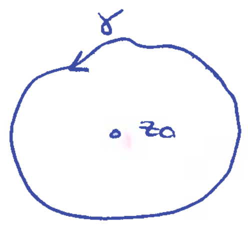
$\int_\gamma \; = 0$, by Theorem 5.10.

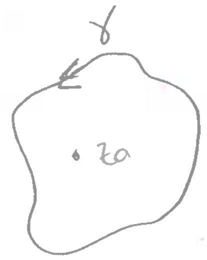
Lemma 5.15 (The keyhole Lemma).

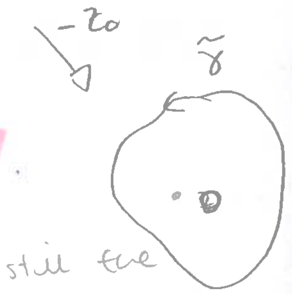
Let γ be a contour s.t. $z_0 \in \text{Int}\,\gamma$.

Then,

$$\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i$$







Proof: Let $\tilde{\gamma} = \gamma - z_0$. Then $\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\tilde\gamma} \frac{1}{z}\,dz$.

my new contour $\tilde\gamma$ is centred at 0 but still the same contour

Clearly, $0 \in \text{Int}\,\tilde\gamma$. Thus, we assume that $z_0 = 0$.

Recall that for circular contour $S = S(0,r) = \{re^{it}, t \in [0,2\pi]\}$,

we have $\int_S \frac{1}{z}\,dz = 2\pi i$.

$\int f(\gamma(t))\,\gamma'(t)\,dt = \int \frac{1}{re^{it}}\, ire^{it}\,dt$

Then $r>0$ s.t $S(0,r) \subset \text{Int}\,\gamma$.

Pick a point on γ, and join it with a point on S.

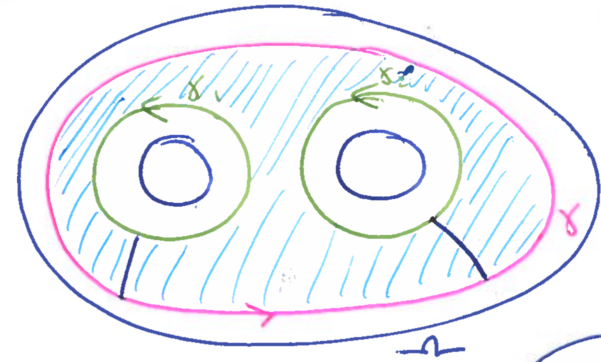
Introduce the path η as shown.

Define the "keyhole" contour $\gamma_1 = \gamma \cup \eta \cup (-S) \cup (-\eta)$

Notice that $\frac{1}{z}$ is holomorphic in $\text{Int}\,\gamma_1$. Therefore, by Theorem 5.10,

$$0 = \int_{\gamma_1} \frac{1}{z}\,dz = \int_\gamma \frac{1}{z}\,dz + \cancel{\int_\eta \frac{1}{z}\,dz} - \int_S \frac{1}{z}\,dz - \cancel{\int_\eta \frac{1}{z}\,dz}$$

Therefore, $\int_\gamma \frac{1}{z}\,dz = \int_S \frac{1}{z}\,dz = 2\pi i$.



Objective to prove that

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz,$$

for $f \in H(\Omega)$

Theorem 5.16: Cauchy-Goursat Theorem for multiple contours.

Let $f \in H(\Omega)$. Let $\gamma, \gamma_j \subset \Omega$, $j=1,2,\dots,n$ be contours s.t. $\gamma_j^* \subset \text{Int}\,\gamma$, $\forall j$, the interiors $\text{Int}\,\gamma_j$ are disjoint and $\text{Int}\,\gamma \bigcap_{j=1}^{n} \text{Ext}\,\gamma_j \subset \Omega$.

Then

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz.$$

Remark: Theorem 5.10 (Cauchy-Goursat) can be repeated.

Definition 5.14 We say that a domain Ω is simply connected if for any contour $\gamma \subset \Omega$ we have $\text{Int}\,\gamma \subset \Omega$. (example above is not simply)

Theorem 5.10 rephrased: If Ω is simply connected and $f \in H(\Omega)$, then, for any contour $\gamma \subset \Omega$ we have $\int_\gamma f(z)\,dz = 0$

The Cauchy formula:

Theorem 5.17 Let Ω be simply connected and let $f \in H(\Omega)$.

Let $\gamma \subset \Omega$ be a contour s.t. $z_0 \in \text{Int}\,\gamma$.

Then

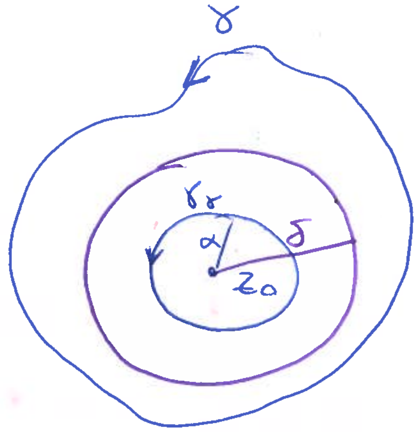
$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

Proof: Rewrite:

$$\int \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_\gamma \frac{1}{z-z_0}\,dz + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz =$$

$$= 2\pi i\, f(z_0) + \varphi(z_0), \text{ with } \varphi(z_0) = \int_\gamma \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{g(z)}\,dz$$

Remains to show that $\varphi(z_0)=0$.



Observe: $g \in H(\Omega\setminus\{z_0\})$. Let

$\gamma_\alpha = \{z : |z-z_0| = \alpha\}$

Assume that α is so small that

$\gamma_\alpha \subset \operatorname{Int}\gamma$.

By theorem 5.16:

$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz \quad \forall \alpha > 0.$$

Estimate, using Theorem 5.6(7):

$$\left|\int_{\gamma_\alpha} g(z)\,dz\right| \le L(\gamma_\alpha)\max_{|z-z_0|=\alpha}|g(z)| = 2\pi\alpha \max_{|z-z_0|=\alpha}|g(z)|.$$

Need to show that $|g(z)|$ is bounded on γ_α. (so that it doesn't tend to ∞, and we can have $\lim 2\pi\alpha \max|g(z)| = 0$)

Recall: $g(z) \to f'(z_0)$, $z \to z_0$, i.e.

$\forall \varepsilon>0\ \exists \delta>0$ s.t. $|g(z) - f'(z_0)| < \varepsilon$

if $z \in D'(z,\delta)$.

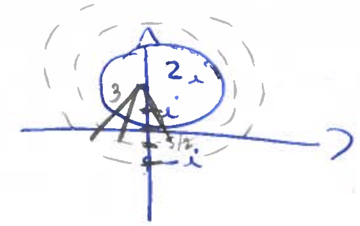
Take $\varepsilon = 1 \Rightarrow |g(z)| \le |f'(z_0)|+1 = M$, if $z \in D'(z_0,\delta)$.

Let $\underbrace{\alpha < \delta}_{\text{so that I can use this inequality}}$, so that $\max_{|z-z_0|=\alpha}|g(z)| \le M$.

Therefore, $|\varphi(z_0)| = \left|\int_\gamma g(z)\,dz\right| \le 2\pi\alpha \cdot M$, $\forall \alpha < \delta$.

$\Rightarrow 2\pi\alpha \cdot M \xrightarrow[\alpha\to 0]{} 0$. And hence, $\varphi(z_0) = 0$, as required. ▨

Examples:



① $$\int_{S(2i,2)} \frac{\cos(z)}{1+z^2}\,dz$$

where $S(z_0,r)$ is a circular contour centred at z_0, of radius r.

Factorise: $1+z^2 = (z+i)(z-i)$, and denote $f(z) = \frac{\cos(z)}{z+i}$, so that

$$\int_{S(2i,2)} \frac{f(z)}{z-i}$$

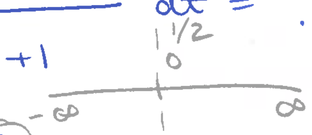
$f(z)$, el único sitio donde no es diferenciable es en $z=-i$, entonces el círculo más grande donde llega es hasta $R = 5/2$

$f(z)$ is holomorphic in $D(2i, \frac{5}{2})$, and hence,

by the Cauchy formula, $\int \frac{f(z)}{z-i}\,dz = 2\pi i \cdot f(i) = 2\pi i \frac{\cos i}{2i} =$

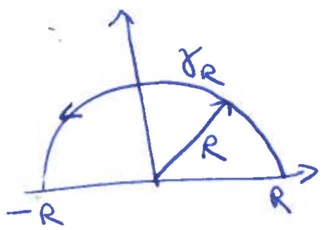
$= \pi\cosh(1)$. $\qquad \frac{e^x+e^{-x}}{2} = \cos(x) \Rightarrow \cosh(x) = \frac{e^{ix}+e^{-ix}}{2}$

② $$I_0 = \int_0^\infty \frac{\cos(t)}{t^2+1}\,dt = ?$$



Rewrite:

$$I_0 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(t)}{t^2+1}\,dt = \frac{1}{2}\lim_{R\to\infty}\underbrace{\int_{-R}^{R}\frac{\cos(t)}{t^2+1}\,dt}$$



$\gamma_R(s) = Re^{is}$, $s \in [0,\pi]$.

$$\int_{-R}^{R}\frac{\cos(t)}{1+t^2}\,dt = \operatorname{Re}\underbrace{\int_{-R}^{R}\frac{e^{it}}{1+t^2}\,dt}_{I_R} \to \cos(t) = \text{Real part of } e^{it}.$$

Calculate:

$$\int_{[-R,R]\cup\gamma_R}\frac{e^{iz}}{1+z^2}\,dz = I_R + \int_{\gamma_R}\frac{e^{iz}}{1+z^2}\,dz \qquad [-R,R] + \gamma_R = [-R,R]\cup\gamma_R$$

By Cauchy's formula, $$\int_{[-R,R]\cup\gamma_R}\frac{e^{iz}}{1+z^2} = 2\pi i\,\frac{e^{-1}}{2i} = \pi e^{-1}$$ $f(z_0) = f(i)$

$\frac{f(z)}{z-i}$ where $f(z) = \frac{e^{iz}}{(z+i)}$

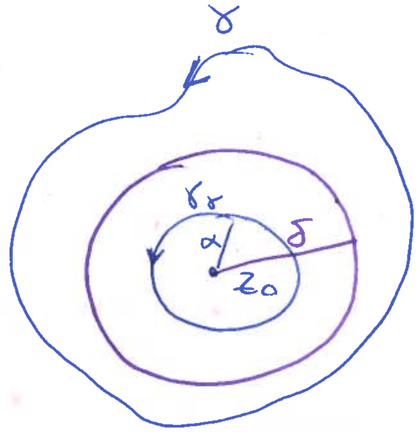
reason why $z_0 = i$

Proof: Rewrite:

$$\int \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_\gamma \frac{1}{z-z_0}\,dz + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz =$$

$$= 2\pi i\, f(z_0) + \varphi(z_0), \text{ with } \varphi(z_0) = \int_\gamma \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{g(z)}\,dz.$$

Remains to show that $\varphi(z_0)=0$.



Observe: $g \in H(\Omega \setminus \{z_0\})$. Let

$\gamma_\alpha = \{z : |z-z_0| = \alpha\}$

Assume that α is so small that

$\gamma_\alpha \subset \text{Int}\,\gamma$.

By theorem 5.16:

$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz \quad \forall \alpha > 0.$$

Estimate, using Theorem 5.6(7):

$$\left|\int_{\gamma_\alpha} g(z)\,dz\right| \leq L(\gamma_\alpha) \max_{|z-z_0|=\alpha} |g(z)| = 2\pi\alpha \max_{|z-z_0|=\alpha} |g(z)|.$$

Need to show that $|g(z)|$ is bounded on γ_α. (so that it doesn't tend to ∞, and we can have $\lim 2\pi\alpha \max|g(z)| = 0$)

Recall: $g(z) \to f'(z_0)$, $z \to z_0$, i.e.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|g(z) - f'(z_0)| < \varepsilon$

if $z \in D'(z,\delta)$.

Take $\varepsilon = 1 \Rightarrow |g(z)| \leq |f'(z_0)| + 1 = M$, if $z \in D'(z_0,\delta)$.

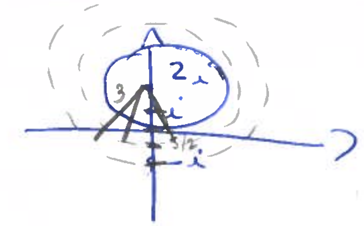
Let $\alpha < \delta$, so that $\max_{|z-z_0|=\alpha} |g(z)| \leq M$.

so that I can use this inequality

Therefore, $|\varphi(z_0)| = \left|\int_\gamma g(z)\,dz\right| \leq 2\pi\alpha\cdot M$, $\forall \alpha < \delta$.

$\Rightarrow 2\pi\alpha\cdot M \xrightarrow[\alpha\to 0]{} 0$. And hence, $\varphi(z_0)=0$, as required ▨

Examples:



① $$\int_{S(2i,2)} \frac{\cos(z)}{1+z^2}\,dz$$

where $S(z_0,r)$ is a circular contour centred at z_0, of radius r.

Factorise: $1+z^2 = (z+i)(z-i)$, and denote $f(z) = \frac{\cos(z)}{z+i}$, so that

$$\int_{S(2i,2)} \frac{f(z)}{z-i}$$

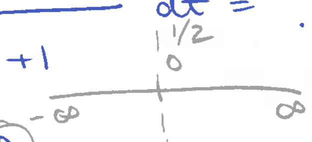
$f(z)$, el único sitio donde no es diferenciable es en $z=-i$, entonces el círculo más grande donde llega es hasta $R = 5/2$

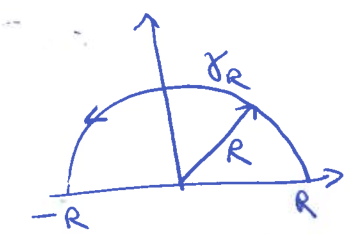
$f(z)$ is holomorphic in $D(2i, \frac{5}{2})$, and hence,

by the Cauchy formula, $\int \frac{f(z)}{z-i}\,dz = 2\pi i \cdot f(i) = 2\pi i \frac{\cos i}{2i} =$

$= \pi\cosh(1)$. $\qquad \frac{e^{x}+e^{-x}}{2} = \cos(x) \Rightarrow \cosh(x) = \frac{e^{ix}+e^{-ix}}{2}$

② $$I_0 = \int_0^\infty \frac{\cos(t)}{t^2+1}\,dt = ?$$





Rewrite

$$I_0 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(t)}{t^2+1}\,dt = \frac{1}{2}\lim_{R\to\infty} \underbrace{\int_{-R}^{R} \frac{\cos(t)}{t^2+1}\,dt}$$

$\gamma_R(s) = Re^{is}$, $s \in [0,\pi]$.

$$\int_{-R}^{R} \frac{\cos(t)}{1+t^2}\,dt = \text{Re}\underbrace{\int_{-R}^{R} \frac{e^{it}}{1+t^2}\,dt}_{I_R} \quad \cos(t) = \text{Real part of } e^{it}$$

Calculate:

$$\int_{[-R,R]\cup\gamma_R} \frac{e^{iz}}{1+z^2}\,dz = I_R + \int \frac{e^{iz}}{1+z^2}\,dz$$

$[-R,R] + \gamma_R = [-R,R] \cup \gamma_R$

By Cauchy's formula, $$\int_{[-R,R]\cup\gamma_R} \frac{e^{iz}}{1+z^2} = 2\pi i \frac{e^{-1}}{2i} = \pi e^{-1}$$

$f(z_0) = f(i)$

$\frac{f(z)}{z-i}$ where $f(z) = \frac{e^{iz}}{(z+i)}$

reason why $z_0 = i$

Estimate:

$$\left| \int_{\gamma_R} \frac{e^{iz}}{1+z^2}\, dz \right| \leq \underbrace{\frac{1}{R^2-1}}_{\max |z-i|=R} \overbrace{\pi R}^{\text{length} = 2\pi R \cdot \frac{1}{2} \text{ half a circle}} \xrightarrow[R\to\infty]{} 0, \text{ and hence}$$

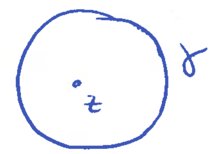
$\lim_{R\to\infty} I_R = \pi e^{-1}$. Therefore, $\int_0^\infty \frac{\cos(t)}{t^2+1}\, dt = \frac{\pi}{2} e^{-1}$

November 19th

$\gamma \subset \Omega$, $\mathrm{Int}\, \gamma \subset \Omega$

$f \in H(\Omega)$

$\forall z \in \mathrm{Int}\, \gamma$: $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$ — CAUCHY'S FORMULA



Differentiate: $f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^2}\, d\xi$

$$f''(z) = \frac{2}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^3}\, d\xi$$

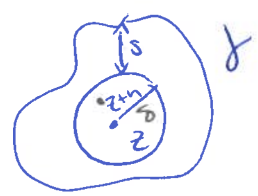
Theorem 5.18 (Cauchy's formula for derivatives).

Let $f \in H(\Omega)$. Let $\gamma \subset \Omega$ be such that $\mathrm{Int}\, \gamma \subset \Omega$. Then f is infinitely differentiable at every $z \in \Omega$, and for all $z \in \mathrm{Int}\, \gamma$ we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

Proof: Do it for $n=1$. Denote:

$$I(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^2}\, d\xi$$



Want: $I(z) = f'(z)$

Let $\delta > 0$ be s.t. $\bar{D}(z,\delta) \subseteq \mathrm{Int}\, \gamma$

gap between disk and contour.

Therefore, $\inf_{\substack{w \in D(0,\delta) \\ \xi \in \gamma}} |(w - \xi)| = s > 0$ ↳ numbers.

Assume that $|h| < \delta$ and denote:

$$g(z,h) = \frac{f(z+h) - f(z)}{h} \xrightarrow[h\to 0]{} f'(z)$$

cauchy formula $\frac{f(\xi)}{\xi - (z+h)} = 2\pi i\, f(z+h)$

this comes from $\frac{1}{h}\left(f(z+h) - f(z)\right)$

Expand: $g(z,h) = \frac{1}{2\pi i h} \int_\gamma f(\xi) \left[\frac{1}{\xi - z - h} - \frac{1}{\xi - z} \right] d\xi =$

$$= \frac{1}{2\pi i} \int_\gamma f(\xi) \frac{1}{(\xi - z - h)(\xi - z)}\, d\xi$$

Estimate: busca que $g(z,h) = I(z)$

$$|g(z,h) - I(z)| = \frac{1}{2\pi} \left| \int_\gamma f(\xi) \left[\frac{1}{(\xi - z - h)(\xi - z)} - \frac{1}{(\xi - z)^2} \right] d\xi \right|$$

$$= \frac{1}{2\pi} \left| \int_\gamma f(\xi) \cdot \frac{h}{(\xi - z)^2 (\xi - z - h)}\, d\xi \right| \leq \frac{1}{2\pi} \max_{\xi \in \gamma} |f(\xi)| \cdot \frac{|h|}{|\xi - z|^2 |\xi - z - h|} L(\gamma).$$

since $z \in D(0,1) \Rightarrow |\xi - z| = s$
since $(z+h) \in D(0,1) \Rightarrow |\xi - (z+h)| = s$

$$\leq \frac{1}{2\pi} \frac{|h|}{s^3} L(\gamma) \xrightarrow[h\to 0]{} 0$$

For $n \geq 2$ it is done by induction ▨

Example: Let $\gamma = S(0, \frac{1}{3})$. Find $\int_\gamma \frac{e^z}{z^2 (z-1)}\, dz$. The function

$f(z)$ $(z-0)^2$

$h(z) = \frac{e^z}{z-1}$ is holomorphic on $D(0, \frac{1}{2})$

By theorem 5.18 the integral equals $2\pi i \left. \frac{d}{dz} h(z) \right|_{z=0} =$

$$= 2\pi i \left. \left(\frac{e^z}{z-1} - \frac{e^z}{(z-1)^2} \right) \right|_{z=0} = 2\pi i (-1-1) = -4\pi i.$$

Theorem 5.19 (Morera's Theorem)

Let f be continuous in Ω and suppose that for every triangular contour $\gamma \subset \Omega$, $\operatorname{Int}\gamma \subset \Omega$.

We have $\displaystyle\int_\gamma f(z)\,dz = 0$. Then f is holomorphic in Ω.

Proof: Fix a $z \in \Omega$, so that $D(z,r) \subset \Omega$ with some $r>0$.

By theorem 5.13, f has a $\underbrace{\text{primitive}}_{F(z)=\int_{[a,z]} f(w)\,dw}$ in $D(z,r)$, i.e.

and F is holomorphic $\Downarrow$ is differentiable

$F' = f$.

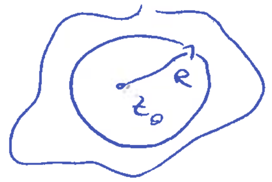
By theorem 5.18 $f' = f$ is also differentiable and hence f is $\underbrace{\text{holomorphic}}_{F''=f'=\!\!\!\Rightarrow f \text{ is holomorphic}}$ in $D(z,r)$ and therefore, $f \in H(\Omega)$ as claimed ▨

Theorem 5.20 Suppose that $f \in H(\Omega)$, and that $\bar D(z_0,R) \subset \Omega$

Let $M = \max\limits_{|z-z_0|=R} |f(z)|$. Then, for all $k=1,2,\dots$

Cauchy's estimates

$$\left|f^{(k)}(z)\right| \le k!\,\frac{M}{R^k}, \quad \forall z \in D(z,R)$$



Proof: Estimate, using Theorem 5.6(7):

$$\left|f^{(k)}(z_0)\right| = \frac{k!}{2\pi}\left|\int_{S(z_0,R)} \frac{f(\xi)}{(\xi-z_0)^{k+1}}\,d\xi\right| \le \frac{k!}{\cancel{2\pi}}\,\frac{M}{R^{k+\cancel{1}}}\cdot\underbrace{\cancel{2\pi} \cancel{R}}_{\text{length}} = k!\,\frac{M}{R^k}$$

$\hookrightarrow \xi$ is on the contour

as required ▨

Liouville's Theorem

Theorem 5.21 Let f be an entire bounded function. Then f is constant in $\mathbb{C}$.

Proof: Let $z \in \mathbb{C}$. Clearly, f is holomorphic in $D(z,R)$ for all R.

Assume that $|f(z)| \le M$, $\forall z \in \mathbb{C}$.

By Cauchy's estimate for $f'(z)$, we have

$$|f'(z)| \le \frac{M}{R}, \quad \forall R.$$ Since R is arbitrary $f'(z) = 0$.

By Theorem 2.9, $f = \text{const}$ in $\mathbb{C}$ as required ▨

Theorem 5.22: Let p be a non-constant polynomial in $\mathbb{C}$. Then there exists a $w \in \mathbb{C}$ s.t. $p(w) = 0$.

Proof: Assume that $p(z) \neq 0$, $\forall z \in \mathbb{C}$.

Thus $g(z) = \dfrac{1}{p(z)}$ is also holomorphic in $\mathbb{C}$.

Estimate: $$|g(z)| = \frac{1}{|p(z)|} = \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \dots + a_0|} = \underbrace{\frac{1}{|z|^n}}_{\substack{|z|\to\infty \\ \downarrow \\ 0}} \cdot \underbrace{\frac{1}{|a_n + a_{n-1} z^{-1} + a_{n-2} z^{-2} + \dots + a_0 z^{-n}|}}_{\substack{|z|\to\infty \\ \downarrow \\ \frac{1}{|a_n|}}}$$

Therefore, $\overbrace{\text{outside}}^{\text{because } |z|\to\infty}$ a disk $D(0,R)$ with some $R>0$ the function g is bdd i.e. $|g(z)| \le M$, $|z| \ge R$.

Since g is continuous in $\mathbb{C}$, $\underbrace{\text{it is bdd in every disk } \bar D(0,R)}_{\text{analysis 2}}$.

Thus, $g(z)$ is bdd in $\mathbb{C}$. Thus, by Liouville's theorem, $g(z) = \text{const}$ in $\mathbb{C}$, and hence, $p(z) = \text{const}$, $z \in \mathbb{C}$. Contradiction with the $\underbrace{\text{conditions of the Theorem}}_{\text{say that } p(z) \neq \text{const}}$ proves the required result $\Rightarrow$ our claim is not true i.e. $p(z) = w$. ▨

Corollary 5.23 Any polynomial of degree $n \ge 1$ has exactly n roots.

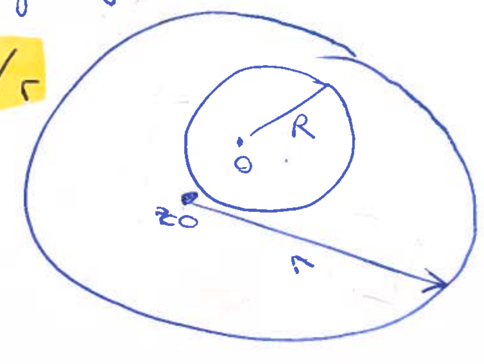
Example: Suppose that f is entire and $\underbrace{|f(z)| \le C\,(1+|z|)^{1/2}}_{\text{this ineq}}$ for all $|z| \ge R > 0$, with some constant $C>0$.

USE CAUCHY'S ESTIMATES

$\Rightarrow f(z) = \text{const}$ in $\mathbb{C}$.

Proof: Fix a point $z_0 \in \mathbb{C}$. f is holomorphic in $D(z_0,r)$ for all r, so that by Cauchy's estimates for f':

$$|f'(z_0)| \le \max_{|z-z_0|=r} |f(z)| \,/\, r$$



let r be s.t. $D(0,R) \subset D(z_0, r)$, and hence

$$\max_{|z-z_0|=r} |f(z)| \leq \tilde{C}(1+r)^{1/2}, \quad \text{where } \tilde{C} = \tilde{C}_{z_0,R}$$

because original C depends on R.

Consequently, $|f'(z_0)| \leq \frac{1}{r}\tilde{C}(1+r)^{1/2} \xrightarrow[r\to\infty]{} 0$

Thus $f'(z_0)=0$. As a result, $f = \text{const}$, as claimed.

Theorem 5.24 (Taylor's Theorem) Suppose that $f \in H(D(z_0,R))$, $R>0$. Then, at each point $z \in D(z_0,R)$ f is represented by the absolutely convergent power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

f = convergent power series with radius of convergence $R>0$.

Remark. Analytic = holomorphic.

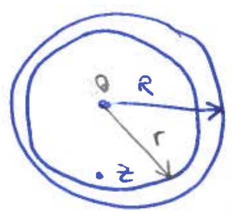
Proof. Consider $\tilde{f}(z) = f(z+z_0)$. $\tilde{f}$ is holomorphic in $D(0,R)$.

We'll prove that $\tilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z|<R$. $(z+\cancel{z_0}-\cancel{z_0})^n$

Thus, we may assume that $z_0 = 0$

Pick a $z \in D(0,R)$. Let r be such that $|z|<r<R$

Then, $S(0,r) \subset D(0,R)$, and $z \in D(0,r)$. Thus,



by the Cauchy formula,

$$f(z) = \frac{1}{2\pi i}\int_{S(0,r)} \frac{f(\xi)}{(\xi - z)}\, d\xi$$

Expand: $\frac{1}{\xi - z} = \frac{1}{\xi\left(1 - \frac{z}{\xi}\right)} = \frac{1}{\xi}\sum_{k=0}^{\infty}\left(\frac{z}{\xi}\right)^k$

Since $\left|\frac{z}{\xi}\right| = \frac{|z|}{r} = \delta < 1$, the series converges. Thus

since $|z|<r$

it is in the contour

$$f(z) = \frac{1}{2\pi i}\int_{S(0,r)} f(\xi)\frac{1}{\xi}\sum_{k=0}^{\infty}\left(\frac{z}{\xi}\right)^k d\xi$$

Assuming that integration and summation can be exchanged, we write:

$$f(z) = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i}\int_{S(0,r)} f(\xi)\frac{1}{\xi^{k+1}}\, d\xi =$$

By Cauchy formula for derivatives

$$= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!} z^k.$$

It remains to justify the swap. WE FINISH FOR DECEMBER TEST

November 25th 2019 → ⚠ Recall: $a_n = \frac{1}{2\pi i}\int_{S(z_0,r)}\frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$, $\forall r<R$

It remains to prove that

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{2\pi i}\int_{S(0,r)}\frac{f(\xi)}{\xi}\sum_{n=0}^{\infty}(z\xi^{-1})^n\, d\xi, \text{ or}$$

$$\lim_{N\to\infty} \underbrace{\frac{1}{2\pi i}\sum_{n=0}^{N} z^n \int_{S(0,r)}\frac{f(\xi)}{\xi^{n+1}}\, d\xi}_{A} = \underbrace{\frac{1}{2\pi i}\int_{S(0,r)}\frac{f(\xi)}{\xi}\sum_{n=0}^{\infty}(z\xi)^n\, d\xi}_{B}$$

Estimate: $A-B$

$$\left|\int_{S(0,r)}\frac{f(\xi)}{\xi}\sum_{n=N+1}^{\infty} z^n\xi^{-n}\, d\xi\right| \leq \max_{|\xi|=r}\left|\frac{f(\xi)}{\xi}\sum_{n=N+1}^{\infty}\left(\frac{z}{\xi}\right)^n\right| 2\pi r$$

$$\delta = \frac{|z|}{|\xi|} = \frac{|z|}{r} < 1$$

$$\leq \left(\frac{M}{\cancel{r}}\sum_{n=N+1}^{\infty}\delta^n\right) 2\pi\cancel{r} = 2\pi M\frac{\delta^{N+1}}{1-\delta} \xrightarrow[N\to\infty]{} 0$$

as required ∎

Remark

We say that the series $\sum_{j=0}^{\infty} f_j(\xi)$ converges uniformly on a set $T \subset \mathbb{C}$ if $\sup_{\xi \in T} \left| \sum_{j=N+1}^{\infty} f_j(\xi) \right| \xrightarrow[N\to\infty]{} 0$

Let R_0 be the radius of convergence of series

Answer: $R \le R_0$.

Theorem 5.25 Suppose that f is entire and $|f(z)| \le M(1+|z|)^{\alpha}$, $\alpha > 0$, $\forall z \in \mathbb{C}$. Then f is a polynomial of degree at most $[\alpha]$

Proof. By theorem 5.24 $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Estimate: Using Cauchy's inequalities: for any $r > 0$,

$$|a_n| = \frac{1}{n!}\left|f^{(n)}(0)\right| \le \frac{1}{n!} n! \max_{|\xi|=r} |f(\xi)| r^{-n} \le \frac{M(1+r)^{\alpha}}{r^n} \xrightarrow[n\to\infty]{} 0 \text{ if } n>\alpha$$

So $\boxed{a_n = 0 \quad \forall n > \alpha}$ as required. when $n \le \alpha$ $a_n \neq 0$ so $\boxed{f(z) = \sum_{n=1}^{\alpha} a_n z^n}$ deg α.

CHAPTER 6: ZEROS AND SINGULARITIES

Laurent's expansion

Study the series of the form

$$g(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Split: $g(z) = g_1(z) + g_2(z)$, where

$$g_1(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0)^n = \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

$$g_2(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Let R_2 be the R of convergence for g_2.

g_2 converges for $|z-z_0| < R_2$.

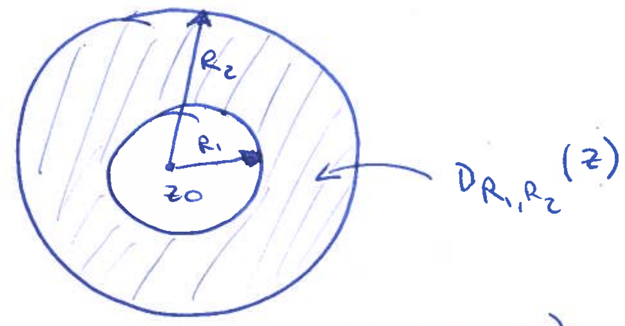
Rewrite:

$$g_1(z) = \sum_{n=1}^{\infty} a_{-n} w^n, \quad w = (z-z_0)^{-1}$$

Let r_1 be the radius of convergence for $g_1(z)$.

i.e. $g_1(z)$ converges for $|w| < r_1$ i.e. $|z-z_0| > r_1^{-1} = R_1$

Thus, g converges in $D_{R_1,R_2}(z_0) = \{z : R_1 < |z-z_0| < R_2\}$



Theorem 6.1 Let $f \in H(D_{R_1,R_2}(z_0))$ with $0 \le R_1 < R_2 \le \infty$. Then for each $z \in D_{R_1,R_2}(z_0)$ the function f is represented by the absolutely convergent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

with $a_n = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$, $\forall r \in (R_1, R_2)$.

November 26th 2019

Ex: e^z - Laurent expansion at $z_0 = 0$? Yes, just the Taylor series.

Ex: $f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$

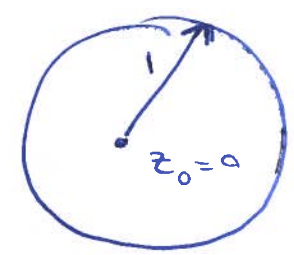
Laurent's expansion at $z_0 = 0$ and 1.

(1) $z_0 = 0$

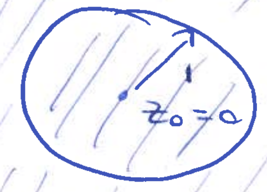
- $0 < |z| < 1$ ← I choose that because that's where f is holomorphic

Expand: $\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k$

Therefore: $f(z) = -\frac{1}{z} - \sum_{k=0}^{\infty} z^k$



• $1 < |z| < \infty$ → my function is holomorphic



Expand:

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k =$$

$$= \frac{1}{z} \sum_{k=-\infty}^{0} z^k = \sum_{k=-\infty}^{-1} z^k$$

Thus, $f(z) = \sum_{k=-\infty}^{-2} z^k$.

(2) $z_0 = 1$

• $0 < |z-1| < 1$ ← it is holomorphic here (if $z-1=0 \Rightarrow$ not holom. $|z-1|=1 \Rightarrow z=0 \Rightarrow$ NOT hol)

Expand:

$$-\frac{1}{z} = -\frac{1}{1+(z-1)} = -\sum_{k=0}^{\infty} (-1)^k (z-1)^k =$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} (z-1)^k, \text{ so } f(z) = \frac{1}{z-1} + \sum_{k=0}^{\infty} (-1)^{k+1}(z-1)^k =$$

$$= \sum_{k=-1}^{\infty} (-1)^{k+1} (z-1)^k$$

• $1 < |z-1| < \infty$ DIY.

Definition 6.2 We say that z_0 is an isolated singularity of f if $f \in H(D'(z_0,R))$, with some $R>0$.

Examples:

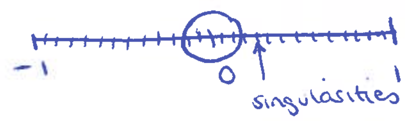
(1) $\frac{1}{z}$ - isolated singularity at $z_0 = 0$ (cause $D'(0,R)$) doesn't include $z=0$ $\Rightarrow \frac{1}{z} \in H(D'(0,R))$.

(2) $\frac{z-4}{z^2(z^2+1)}$ - isolated singularity at $\underbrace{z_0 = 0, i, -i}$ roots of the denominator!

$\left.\begin{matrix} i \\ 0 \\ -i \end{matrix}\right\}$ I can find a disk that doesn't include other singularities around $0, i, -i$.

(3) $\frac{1}{\sin(\frac{\pi}{z})}$

Singularities at $0, \underbrace{\pm\frac{1}{n}}_{\text{roots of denominator}}, n = 1, 2, \ldots$



0 is non-isolated singularity

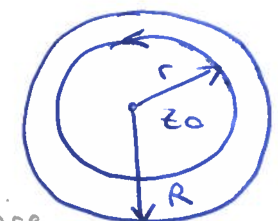
Since we cannot take a radius in which the disk doesn't contain another singularity (will contain such as $\frac{1}{n}$ $(n=1,2,\ldots)$).

Let z_0 be a singularity of f, so

$$f(z) = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-z_0)^n}_{\text{principal part of } f} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Holomorphic part}}.$$

Definition 6.3 The coefficient a_{-1} is called the residue of f at z_0. $\text{Res}(f, z_0) = a_{-1}$

For any $r \in (0,R)$



$$a_{-1} = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(\xi)\,d\xi}{(\xi-z)^{n+1}}, \text{ so}$$

since $n=-1 \Rightarrow -1+1=0$ $(\xi-z)^0 = 1$

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \int_{S(z_0,r)} f(\xi)\,d\xi$$

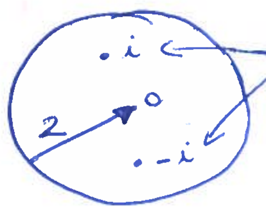
Theorem 6.4 Let f be holomorphic in Ω except for singularities at $z_1, z_2, \ldots, z_N$. Let $\gamma \subset \Omega$ s.t. all singularities are in Int γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k).$$

Proof: Let $\gamma_1, \gamma_2, \ldots, \gamma_N$ be circular contours centred at $z_1, z_2, \ldots, z_N$ s.t. $\operatorname{Int} \gamma_j$'s are separated. Then, by theorem 5.16 (Cauchy-Goursat for multiple contours).

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{\infty} \int_{\gamma_k} f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, z_k)$$ as claimed ▨

Example: Evaluate $\displaystyle\int_{S(0,2)} \frac{\cos(z)}{1+z^2}\,dz$



singularities $z_\pm = \pm i$ Thus,

$I = 2\pi i \operatorname{Res}(f, i) + 2\pi i \operatorname{Res}(f, -i)$.

Cauchy's formula

$$\operatorname{Res}(f, i) = \frac{1}{2\pi i} \int_{S(i,r)} \frac{\cos(z)}{(z+i)(z-i)}\,dz = \frac{\cos i}{2i} = \frac{\cosh 1}{2i}$$

Cauchy's formula

$$\operatorname{Res}(f, -i) = \frac{1}{2\pi i} \int_{S(-i,r)} \frac{\cos(z)}{(z+i)(z-i)}\,dz = \frac{\cosh 1}{-2i}$$

Thus,

$$I = 2\pi i \left(\frac{\cosh 1}{2i} - \frac{\cosh 1}{2i} \right) = 0.$$

Classification of singularities.

Look at the principal part

① Suppose that the principal part contains finitely many terms: $\dfrac{a_{-1}}{(z-z_0)} + \dfrac{a_{-2}}{(z-z_0)^2} + \ldots + \dfrac{a_{-m}}{(z-z_0)^m}$

with $a_{-n} \neq 0$. Then we say that f has a pole of order m at z_0.

If $m = 1$, then the pole is simple

ex: $f(z) = \dfrac{z-4}{z^2(z^2+1)}$. $\rightarrow$ pole of order 2 at $z_0 = 0$. $(z-0)^2$

$h(z) = \dfrac{1}{z(z-1)} = -\dfrac{1}{z} + \dfrac{1}{z-1}$. $\rightarrow$ $z=0$ is a simple pole. $(z-0)^1$

$z = 1$ is a simple pole $(z-1)^1$

② If the principal part contains infinitely many terms

$\Rightarrow$ The singularity is said to be essential.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} \rightarrow \text{essential}$$

③ If $PP = 0$ $\Rightarrow$ we say that the singularity is removable

Recall: $f(z) = PP + \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Ex: $f(z) = \dfrac{\sin(z)}{z}$, is $z=0$ a singularity?.

Expand: $f(z) = \dfrac{1}{z}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots\right) = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!}$

Define $f(0) = 1$, so that f is entire.

If z_0 is removable $\Rightarrow$ the function

$$\tilde{f}(z) = \begin{cases} f(z), & z \neq z_0 \\ a_0 = 1, & z = z_0 \end{cases}$$

is holomorphic in $D(z_0, R)$.

ZEROS

(✱) • We say that z_0 is an isolated zero of order m if f is expanded as $f(z) = \sum_{n \geq m} a_n (z-z_0)^n$, $a_m = 0$, i.e. $a_0 = a_1 = \ldots = a_{m-1} = 0$.

Notation for the set of zeros $Z(f)$.

Theorem 6.7 Suppose that $f \in H(\Omega)$, and that $Z(f)$ has an accumulation point in Ω, i.e. there is a sequence of $z_n \in Z(f)$ s.t. $z_n \xrightarrow[n\to\infty]{} w \in \Omega$. $\Rightarrow$ $f(z) = 0$.

(✱) **Definition 6.6**. Let $f \in H(\Omega)$ $\Rightarrow$ z_0 is a zero of f if $f(z_0) = 0$.

• We say that z_0 is an isolated zero, if $f(z_0) = 0$ and there is an $R > 0$ s.t. $f(z) \neq 0$ $\forall z \in D'(z_0, R)$

Proof: (Suppose that $f \neq 0$ in Ω)

Let z_0 be an accumulative point.

$\Rightarrow$ for some $\delta_0 > 0$ s.t

$f \in H(D(z_0, \delta_0))$, and hence, (because of definition of analytic)

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. Let $a_m \neq 0$ and $a_0 = a_1 = \cdots = a_{m-1} = 0$.

Therefore, $f(z) = \left(a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \dots \right)$

$$= (z-z_0)^m \underbrace{\left(a_m + a_{m+1}(z-z_0) + \dots\right)}_{g(z)} =$$

$$= (z-z_0)^m g(z), \quad g(z_0) = a_m \neq 0 .$$

Clearly, by the "inertia principle", $g(z) \neq 0$ in a disk centred at z_0.

Thus, z_0 cannot be an accumulation point at zeros.

Therefore, $f(z) \equiv 0$ in $D(z_0, \delta_0)$.

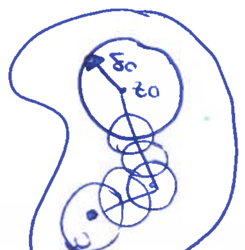
December 2^{nd} 2019.

Theorem 6.7

Let $f \in H(\Omega)$ (domain). Suppose that $Z(f)$ has accumulation point in Ω

$\Rightarrow f(z) = 0, \ \forall z \in \Omega$

Proof: Let z_0 be an accumulation point of $Z(f)$. Let $\delta_0 > 0$ be s.t. $D(z_0, \delta_0) \subset \Omega \Rightarrow$ we have proved that $f(z) = 0 \ \forall z \in D(z_0, \delta_0)$

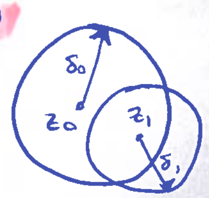


Let $w \in \Omega$. Join z_0 and w with a polygonal path γ s.t. $\gamma(0) = z_0$, $\gamma(1) = w$, $\gamma^* \in \Omega$

Cover the path by disks $D(z_k, \delta_k)$ s.t.

(1) $D(z_k, \delta_k) \subset \Omega$, $\forall k = 0, 1, \dots, N$ where $z_N = w$

(2) $z_k \in D(z_{k-1}, \delta_{k-1})$, $k = 1, 2, \dots, N$.



Since $f = 0$ in $D(z_0, \delta_0)$, z_1 is an accumulation point of zeros, so $f = 0$ in $D(z_1, \delta_1)$

Repeat this argument until you reach the disk

$D(w, \delta_N) \Rightarrow f = 0$ in $D(w, \delta_N)$

Thus $f(z) = 0 \ \forall z \in \Omega$ (since w is chosen arbitrarily)

Corollary 6.8. (Unique Continuation theorem).

Let $f, g \in H(\Omega)$, and that the set $S = \{z : f(z) = g(z)\}$ is an accumulation point in $\Omega \Rightarrow f(z) = g(z) \ \forall z \in \Omega$.

Example: Let $g(t) = \sin(t)$, $t \in \mathbb{R}$.

$\Rightarrow \exists$ a unique entire function f s.t $f(t) = g(t)$, $t \in \mathbb{R}$.

$\Rightarrow f(z) = \sin(z)$.

Example: $f(z) = \dfrac{z+1}{(z-1)^3 (z+3)}$ $\rightarrow$ root at -1 because numerator $= 0$ if $z = -1$, thus is first order.

Theorem 6.9

(1) The function $f \in H(D(z_0, R))$ has a zero of order m at z_0 iff

$\lim_{z \to z_0} (z-z_0)^{-m} f(z) = B \neq 0$. (*)

(2) The function $f \in H(D'(z_0, R))$ has a pole of order p at z_0 iff

$\lim_{z \to z_0} (z-z_0)^p f(z) = A \neq 0$.

Proof. Assume that z_0 is a zero of order m: $f(z) = \sum_{n \geq m} a_n (z-z_0)^n$

$a_m \neq 0$.

Therefore,

$(z-z_0)^{-m} f = a_m + a_{m+1}(z-z_0) + \dots \xrightarrow[z \to z_0]{} a_m \neq 0$.

Assume that (*) holds. Then,

$M(z) = |z - z_0|^{-m} |f(z)| \longrightarrow |B|$

Estimate

$$|a_n| \leq \frac{1}{2\pi} \left| \int_{S(z_0, r)} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{1}{2\pi} \left| \int_{S(z_0, r)} \frac{f(z) \cdot (z-z_0)^{-m}}{(z-z_0)^{-m+n+1}} dz \right|$$

$$\leq \frac{1}{2\pi} \max M(z) \frac{1}{r^{-m+n+1}} 2\pi r = \max |M(z)| \, r^{-n+m} \xrightarrow[r \to 0]{} 0$$

if $n < m \Rightarrow a_n = 0, \forall n < m$

Thus,

$$B \xleftarrow[z\to z_0]{} (z-z_0)^{-m} f(z) = a_n + a_{m+1}(z-z_0) + \dots \xrightarrow[z\to z_0]{} a_n$$

$$\Rightarrow a_n = B \neq 0$$

Evaluate for $f(z) = \frac{z+1}{(z-1)^3(z+3)}$

① $(z+1)^{-1} f(z) = \frac{1}{(z-1)^3(z+3)} \xrightarrow[z\to(-1)]{} \frac{1}{-8\cdot 2} \neq 0 \to$ zero of order 1

② $(z-1)^3 f(z) = \frac{z+1}{z+3} \xrightarrow[z\to 1]{} \frac{2}{4} = \frac{1}{2} \neq 0 \to$ pole of order 3.

Corollary 6.10 A function $f \in H(D(z_0,R))$ has a zero of order m at z_0, iff $\frac{1}{f}$ has a pole of order m at z_0.

Immediate consequence of theorem 6.9.

$$a_{-1} = \frac{1}{2\pi i}\int_{S(z_0,\gamma)} f(z)\,dz$$

Calculation of residues

$\boxed{\text{Rule 1st}}$ Suppose f has a simple pole at z_0

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$$

$$\Rightarrow \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\left(a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \dots\right) = a_{-1}$$

Ex: $\text{Res}(f,-3) = \lim_{z\to -3}\frac{z+1}{(z-1)^3} = \frac{-2}{-64} = \frac{1}{32}$

$\boxed{\text{Rule 2nd}}$ Suppose f has multiple pole:

$$f(z) = \frac{\overset{\neq 0}{a_{-p}}}{(z-z_0)^p} + \frac{a_{-p+1}}{(z-z_0)^{p-1}} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

Look at

$$(z-z_0)^p f(z) = a_{-p} + a_{-p+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{p-1} + \dots = h(z)$$ — z_0 is removable

Therefore: $a_{-1} = \lim_{z\to z_0}\left(\left(\frac{d}{dz}\right)^{p-1}(z-z_0)^p\cdot f(z)\right)\frac{1}{(p-1)!}$

Check Taylor's expansion: $g(z) = \sum \frac{g^n(z_0)}{n!}(z-z_0)^n$

$$a_{-1} = \frac{1}{(p-1)!}\cdot\frac{d^{p-1}}{dz^{p-1}}h(z)$$

Example: Same f:

Define $h(z) = (z-1)^3 f(z) = \frac{z+1}{z+3} = 1 - \frac{2}{z+3}$

Therefore,

$$\text{Res}(f,1) = \frac{1}{2!}\frac{d^2}{dz^2}h(z)\Big|_{z=1} = \frac{1}{2!}\frac{d}{dz}\frac{2}{(z+3)^2}\Big|_{z=1} =$$

$$= \frac{1}{2!}\cdot\frac{(-4)}{(z+3)^3}\Big|_{z=1} = \frac{-4}{2}\cdot\frac{1}{4^3} = -\frac{1}{32}.$$

$\boxed{\text{Rule 3rd}}$ $f(z) = \frac{g(z)}{h(z)}$, where $g,h \in H(D(z_0,R))$

Expand g and h into their Taylor's series and see what pops out.

Example: $f(z) = \frac{\sin(z)}{z^4}$

Expand: $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots$

Then, $f(z) = \frac{1}{z^3} - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}z - \dots$

$\text{Res}(f,0) = -\frac{1}{3!} = -\frac{1}{6}$

Example: $z^4 \sin\frac{1}{z}$ DIY.

Counting zeros and poles

Theorem 6.11 Suppose that f is holomorphic in Ω except for finitely many poles. Let γ be a contour in Ω, enclosing the poles, and not containing zeros or poles.

$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$, where N, P are numbers of zeros and poles inside the contour counting these multiplicity

Proof: Let $z_1, z_2, \dots z_k, w_1, w_2, \dots, w_s$ be zeros and poles inside γ.

By the Cauchy Residue Theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} \mathrm{Res}\left(\frac{f'}{f}, z_j\right) + \sum_{j=1}^{s}\mathrm{Res}\left(\frac{f'}{f}, w_j\right)$$

$\boxed{1^{st}}$ Let's fix a zero $z_0 = z_j$. Let m be the order of z_0.

$f(z) = \underset{\neq 0}{a_m}(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots = (z-z_0)^m g(z)$, g is holomorphic around z_0 and $g(z_0) = a_m \neq 0$

Therefore,

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g + (z-z_0)^m g'}{(z-z_0)^m g} = \frac{m}{z-z_0} + \frac{g'}{g}$$

By Rule I,

$$\mathrm{Res}\left(\frac{f'}{f}, z_0\right) = \lim_{z\to z_0} (z-z_0)\frac{f'(z)}{f(z)} = m$$

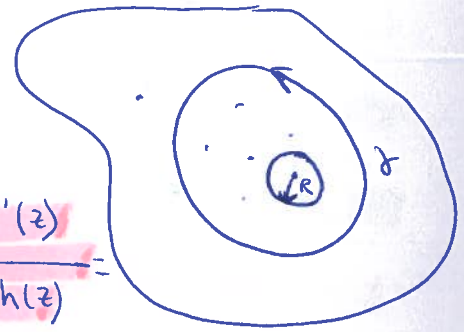
December 9th 2019

$\boxed{2^{nd}}$ Fix a pole $w_0 = w_j$. Assume that order of the pole is p:

$f(z) = (z-w_0)^{-p}h(z)$, where h is holomorphic in $D(w_0, r)$.

Important: $h(w_0)\neq 0$. Thus,

$h(z)\neq 0$ in $D(w_0,r)$ if $r>0$ is small enough.



Rewrite:

$$\frac{f'(z)}{f(z)} = \frac{-p(z-w_0)^{-p-1}h(z)}{(z-w_0)^{-p}h(z)} + \frac{(z-w_0)^{-p}h'(z)}{(z-w_0)^{-p}h(z)}$$

$$= \frac{-p}{z-w_0} + \frac{h'(z)}{h(z)}$$

holomorphic in $D(w_0, r)$.

Thus, $\mathrm{Res}\left(f'/f, w_0\right) = -p$.

This completes the proof. ▨

Theorem 6.12 (Rouché's theorem)

Let $f, g \in H(\Omega)$. Let $\gamma \subset \Omega$ be such that $\mathrm{Int}\,\gamma \subset \Omega$, and that γ^* has no zeros of f. Suppose $|g(z)| < |f(z)|$ for all $z\in\gamma^*$. Then, the functions f and $f+g$ have the same number of zeros inside γ.

Proof: Let $t\in[0,1]$. Let $N(t)$ be the number of zeroes of $f(z)+tg(z)$ inside γ. Since $|g(z)|<|f(z)|$, $z\in\gamma^*$, we can estimate:

$$|f(z)+tg(z)| \geq |f(z)| - t|g(z)| \geq |f(z)|-|g(z)| \geq m > 0, \quad \forall z\in\gamma^*$$

Δ inequality and $|f(z)|>|g(z)|$

Using theorem 6.11,

$$N(t) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz$$

We want to prove that $N(0)=N(1)$. For this, it suffices to show that N is continuous in t.

Let $s,t \in [0,1]$, and write: $N(t)-N(s) = \frac{t-s}{2\pi i}\int_\gamma \frac{f'g - fg'}{(f+tg)(f+sg)}\,dz$

Estimate: (*)

$\max_{z\in\gamma^*} |f'(z)g(z) - g'(z)f(z)| = M < \infty$. Therefore, by Theorem 5.6(2),

$$|N(t)-N(s)| \leq \frac{|t-s|}{2\pi}\frac{M}{m^2}L(\gamma).$$

(Lipshitz function) because it satisfies this condition.

$\Rightarrow$ N is continuous in t, as required ▨

COMPLEX INTEGRALS : JORDAN'S LEMMA

→ study on your own (Notes in moodle).