MATH0014 Algebra 3: Further Linear Algebra Notes (Part 2 of 2)

Based on the 2019 autumn problem class by Dr I Strouthos & Dr J M Talbot

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Problem class

- Polynomial long division:

$\boxed{A}$ Like at school

$$\begin{array}{r} c \\ d\,\overline{)\,D} \end{array}$$

$\boxed{B}$ By inspection

$f(x) = x^5 - 18x^3 + x^2 + 35x - 4$

$g(x) = x^2 - 16$.

Find $q, r \in \mathbb{R}[x]$ s.t i) $f = q \cdot g + r$

ii) $\deg(r) < \deg(g)$

Why do we need to find q, r? $\Rightarrow$ To find $\gcd(f,g)$.

$\boxed{\text{gcd for } \mathbb{R}[x]}$: If $f, g \in \mathbb{R}[x] \Rightarrow \gcd(f,g)$ is the unique $d \in \mathbb{R}[x]$ s.t:

i) $d \mid f$ and $d \mid g$.

ii) If $c \mid f$ and $c \mid g \Rightarrow c \mid d$. where $c \in \mathbb{R}[x] \equiv \deg(c) \leq \deg(d)$

iii) It must be monic.

Example: $f = x^2 + 2x + 1 = (x+1)^2$ $\quad g(x) = x^2 + x = x \cdot (x+1)$.

$d_1 = x+1$: $d_1 \mid f$ and $d_1 \mid g \Rightarrow \deg(d_1) = 1$.

$d_2 = -(x+1)$ $\quad d_2 \mid f$ and $d_2 \mid g \Rightarrow \deg(d_2) = 1$

Explanation to why gcd needs to be monic.

If it's not monic $\Rightarrow$ any multiple will satisfy the two first conditions and then the gcd wouldn't be unique.

$\gcd(f,g) = \gcd(g,r)$.

Theorem: Suppose $f, g, q, r \in \mathbb{R}[x]$ s.t $f = q \cdot g + r \Rightarrow \gcd(f,g) = \gcd(g,r)$.

$$(x^5-18x^3+x^2+35x-4)=(\underbrace{x^3+10x^2-2x+1}_{\text{degree } 3})(x^2-16)+(\underbrace{3x+12}_{\text{degree}\leq 1})$$

x^5, $-2x^2$, $-16x^3$, $32x$

$\gcd(f,g)=\gcd(g,r)=$

$=\gcd(x^2-16,\ 3x+12)$

$3x+12 = 3(x+4)$

$x^2-16=\bar{q}\cdot(3x+12)+\bar{r}$ for some $\bar{q},\bar{r}\in\mathbb{R}[x]$.

$\underbrace{\gcd\ \deg(\bar{r})<1}$.

i.e. $\bar{r}$ is a constant.

$\bar{q}=\frac{1}{3}(x-4)$

$\bar{r}=0$.

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$M_{n\times n}(\mathbb{C})$

i) find a basis of eigenvectors $\{v_1,\dots,v_n\}$.

ii) $\lambda\in\mathbb{C}$, $v\neq 0 \Rightarrow M.v=\lambda.v$.

$\det(t_0 I-M)=0 \iff \dim(\ker(t_0 I-M))\geq 1 \iff (t_0 I-M)\underline{v}=0$

$C_M(t)=(t-\lambda_1)^{\alpha_1}\cdot\ldots\cdot(t-\lambda_k)^{\alpha_k}$

$\deg(C_M(t))=n=\alpha_1+\dots+\alpha_k \Rightarrow \alpha_i\geq 1$ for $i=1,\dots,k$

If $\alpha_i=1$ for $\forall i \Rightarrow \{v_1,\dots,v_m\}$ basis of eigenvectors.

If $\alpha_1=3$

In general, $\alpha_i\geq\dim(V(\lambda_i))$.

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- **characteristic polynomial** (A is an $n\times n$ complex matrix).

  $ch_A(x)=\det(xI-A)$.

  $\deg(ch_A(x))=n$.

- **minimal polynomial**

  $m_A(x)=$ The polynomial $m_A(x)\in\mathbb{C}[x]$ of smallest degree s.t. $m_A(A)=0$ and $m_A(x)$ is monic.

  so that $m_A(x)$ is unique.

FACT 0: $m_A(x)$ is unique and exists.

FACT 1: $m_A(x)\mid ch_A(x)$.

FACT 2: If α is an eigenvalue for A, $\Rightarrow (x-\alpha)\mid m_A(x)$.

CAYLEY-HAMILTON THEOREM $\Rightarrow ch_A(A)=0 \Rightarrow m_A(x)\mid ch_A(x)$.

Example: $\begin{pmatrix}3&0\\0&3\end{pmatrix}$, $\begin{pmatrix}3&4\\0&2\end{pmatrix}$, $\begin{pmatrix}3&4\\0&3\end{pmatrix}$.

$A=\begin{pmatrix}3&0\\0&3\end{pmatrix}$ $ch_A(x)=(x-3)^2 \Rightarrow m_A(x)=(x-3)$

$B=\begin{pmatrix}3&4\\0&2\end{pmatrix}$ $ch_B(x)=(x-2)(x-3) \Rightarrow m_B(x)=(x-2)(x-3)$.

$C=\begin{pmatrix}3&4\\0&3\end{pmatrix}$ $ch_C(x)=(x-3)^2 \Rightarrow m_C(x)=(x-3)^2$.

Let A be a 2×2 matrix.

If $m_A(x)=x-\lambda \Rightarrow A=\lambda I=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$.

Proof: $m_A(A)=0 \Rightarrow A-\lambda I=0$; $A=\lambda I=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$.

Ex:

$A=\begin{pmatrix}4&0&1&0\\2&2&3&0\\-1&0&2&0\\4&0&1&2\end{pmatrix}$ $ch_A(x)=\det(xI-A)=\det\begin{pmatrix}x-4&0&-1&0\\-2&x-2&-3&0\\1&0&x-2&0\\-4&0&-1&x-2\end{pmatrix}$

$$= (x-2)\begin{vmatrix} (x-4) & 0 & -1 \\ -2 & x-2 & -3 \\ 1 & 0 & x-2 \end{vmatrix} = (x-2)^2 \det\begin{pmatrix} x-4 & -1 \\ 1 & x-2 \end{pmatrix}$$

$$= (x-2)^2\big(\underbrace{(x-4)(x-2)+1}\big) = (x-2)^2(x-3)^2 =$$

$\big(x^2-6x+9 = (x-3)^2\big)$.

- $m_A(x) \in \{(x-2)(x-3),\ (x-2)(x-3)^2,\ (x-2)^2(x-3),\ (x-2)^2(x-3)^2\}$.

$-(A-2I)\cdot(A-3I) \neq 0 \Rightarrow m_A(x) \neq (x-2)(x-3)$

$-(A-3I)^2(A-2I) = 0$.

$\Rightarrow m_A(x) \mid (x-2)(x-3)^2 \Rightarrow m_A(x) = (x-2)(x-3)^2$.

$(A-2I)v = 0$ (i.e. $A\underline{v} = 2v$).

$\Leftrightarrow \underline{v} \in \ker(A-2I)$.

$$(A-2I) = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \end{pmatrix} \Rightarrow V_1(2) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$(A-3I)\underline{v} = 0$. (i.e. $A\underline{v} = 3\underline{v}$

$$(A-3I) = \begin{array}{c} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{pmatrix} \\ \ \ C_1 \ \ \ C_2 \ \ \ C_3 \ \ \ C_4 \end{array}.$$

$$\left[C_3 = C_1 - C_2 + 3C_4\right] \Rightarrow v = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: Q$$

$\exists P$ invertible s.t $P(A-3I) = Q$.

$\Rightarrow \ker(A-3I) = \ker(P\cdot(A-3I)) = \ker(Q)$

$\dim(\ker(Q)) = 1 \Rightarrow$ MATRIX NON-DIAGONAL

- So, $alg(2) = geo(2) = 2$.

$alg(3) = 2 \neq geo(3) = 1$.

$\Rightarrow A$ is <u>not</u> diagonalisable (i.e. $\nexists P$ invertible s.t $P^{-1}AP$ is diagonal)

$\boxed{\text{CLAIM}}$ $\exists P$ invertible s.t $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ = JORDAN = J NORMAL FORM

$\Longleftrightarrow AP = PJ$

$\Longleftrightarrow AP\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = PJ\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ AP\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = PJ\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

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