# Uniform Angular Distribution

Brian Okorn

If you uniformly sample rotation angles with in a bound and randomly sample the axis as a normalized 3D Gaussian, you will over-sample small angles. To show this, we uniformly sample rotations as quaternions and normalize them to be unit. Orientations exist on the surface of a 4D sphere and should uniformly cover it when sampling the full space, so this quaternion sampling will produce a uniform distribution over all possible rotations. To evaluate the distribution of angular differences in this space, we count the number of orientations different distances from a randomly selected rotation. Figure  shows that far more rotations are far form the sampled point then are close. If one were to uniformly sample the rotation angle, this distribution would be constant with respect to orientation magnitude, and not describe the proper density. To remove this imbalance, we can compute the probability distribution function of orientation differences and use inverse transform sampling [1] to sample the proper distribution.

To compute the probability density function, we will investigate the area of rings on the 3-sphere. for this we will be representing rotations as quaternions. Quaternions parameterize a rotation of $\theta \in [0, \pi)$ about axis $\hat{\xi} \in \mathbb{S}^2$ as

$$\begin{aligned} \hat{q} &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\xi} \\ &= \begin{bmatrix} w & x & y & z \end{bmatrix} \end{aligned}$$

If we fix $\theta$, and subsequently $w$ we are left with a 2-sphere of radius $r$

$$\begin{aligned} r &= \sqrt{1 - w^2} \\ &= \sqrt{1 - \cos\left(\frac{\theta}{2}\right)^2} \\ &= \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

and a surface area of

$$4\pi r^2 = 4\pi \sin\left(\frac{\theta}{2}\right)^2.$$

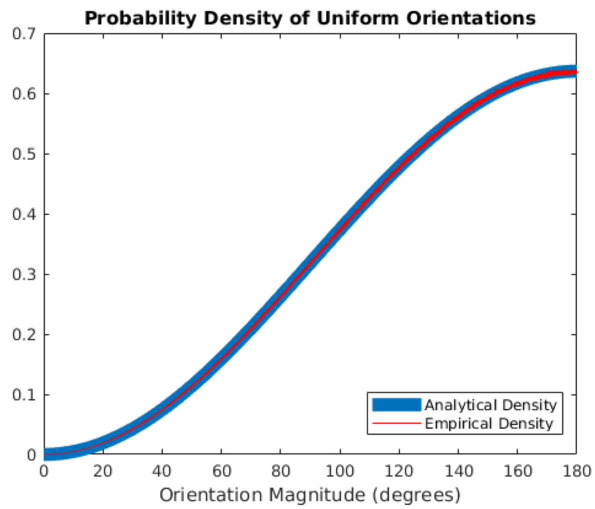
The total surface area of the 3-sphere is $2\pi^2$ so the probability density of orientations with a given angle of $\frac{2}{\pi}\sin\left(\frac{\theta}{2}\right)^2$. This matches our empirical results.

[1] https://en.wikipedia.org/wiki/Inverse_transform_sampling

The cumulative distribution function of this PDF is

$$\int \frac{2}{\pi} \sin\left(\frac{\theta}{2}\right)^2 d\theta = \frac{1}{\pi}\theta - \sin(\theta).$$

Using density function to sample angular magnitude and the Gaussian method for sampling the axis, we can now uniformly sample rotation within a given radius of another rotation[2].




[2]See quat_math for an implementation of this sampling method.