Symplectic Tracking in Zgoubi

August 30, 2019

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Hamiltonian Equations

The default integrator in Zgoubi is based on directly solving the Lorentz equation:

$$\frac{d(m\mathbf{v})}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The relativistic Hamiltonian:

$$H = \sqrt{c^2(\mathbf{P} - e\mathbf{A})^2 + m^2c^4} + e\Phi$$

Change the independent variable from time t to displacement s:

$$H = -\sqrt{P_T^2 - \left(\gamma_0\beta_0\right)^{-2} - P_X^2 - P_Y^2} - \frac{e}{P_0}A_z(X,\ Y)$$

For the case of $\Phi = 0,\ \ A_x = 0,\ \ A_y = 0,$ and X, Y, P_X, P_Y, T, P_T are scaled.

For a general magnetic multipole: $A_z(x,y) = -\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}(b_m + ia_m)(x+iy)^m$

Hamiltonian Equations

$$H = -\sqrt{P_T^2 - (\gamma_0\beta_0)^{-2} - P_X^2 - P_Y^2} - \frac{e}{P_0} A_z(X,\ Y)$$

The Hamiltonian equations:

$$X' = \frac{\partial H}{\partial P_X}, \qquad P_X' = -\frac{\partial H}{\partial X}$$

$$Y' = \frac{\partial H}{\partial P_Y}, \qquad P_Y' = -\frac{\partial H}{\partial Y}$$

$$T' = \frac{\partial H}{\partial P_T}, \qquad P_T' = -\frac{\partial H}{\partial T}$$

Here $X' = \frac{dX}{ds}$

Multipoles:

$$A_{\mathrm{di}} = -b_1 x$$

$$A_{\mathrm{quad}} = -\frac{1}{2} b_2 (x^2 - y^2)$$

$$A_{\mathrm{sext}} = -\frac{1}{3} b_3 (x^3 - 3xy^2)$$

$$A_{\mathrm{oct}} = -\frac{1}{4} b_4 (x^4 - 6x^2y^2 + y^4)$$

Drift

Case (1): $A_z = 0$

solution:

$$X' = \frac{P_X}{P_S}, \qquad P_X' = 0$$

$$Y' = \frac{P_Y}{P_S}, \qquad P_Y' = 0$$

$$T' = -\frac{P_T}{P_S}, \qquad P_T' = 0$$

$$\begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \rightarrow \begin{pmatrix} X + LP_X / P_S \\ P_X \\ Y + LP_Y / P_S \\ P_Y \\ T - LP_T / P_S \\ P_T \end{pmatrix}$$

where $P_S = \sqrt{P_T^2 - \left(\gamma_0 \beta_0\right)^{-2} - P_X^2 - P_Y^2}$

Quadrupole

Case (2): $H_{\text{quad}} = -\sqrt{P_T^2 - (\gamma_0\beta_0)^{-2} - P_X^2 - P_Y^2} + \dfrac{eb_2}{2P_0}(X^2 - Y^2)$

$$H_{\text{quad}} = H_1 + H_2$$

$$H_1 = -\sqrt{P_T^2 - (\gamma_0\beta_0)^{-2} - P_X^2 - P_Y^2}$$

solution: **Drift (D)**

$$H_2 = \frac{eb_2}{2P_0}(X^2 - Y^2)$$

solution: **kick (K)**

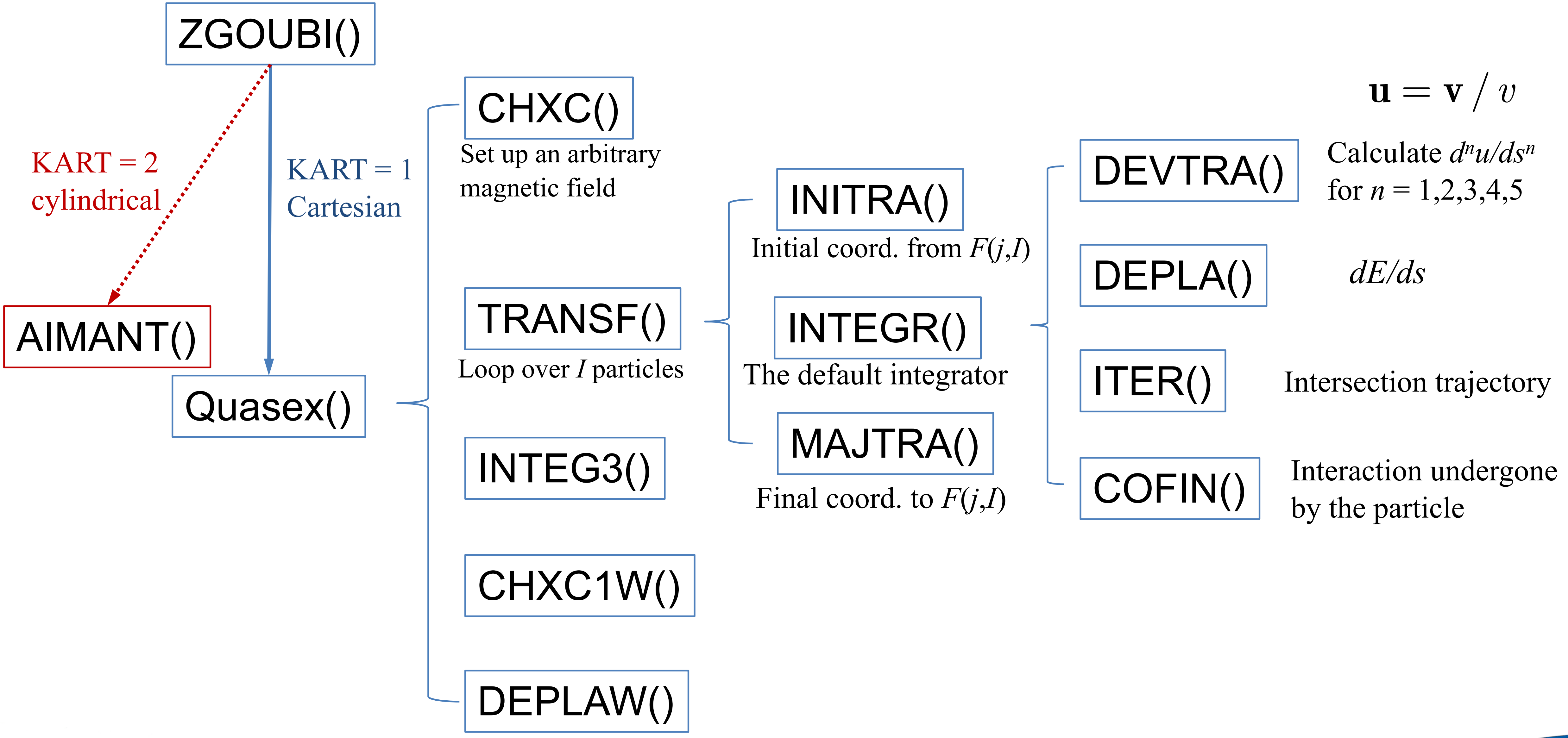
$$\begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \to \begin{pmatrix} X \\ P_X - \kappa_2 L X \\ Y \\ P_Y + \kappa_2 L Y \\ T \\ P_T \end{pmatrix}$$

where $\kappa_2 = eb_2 / P_0$

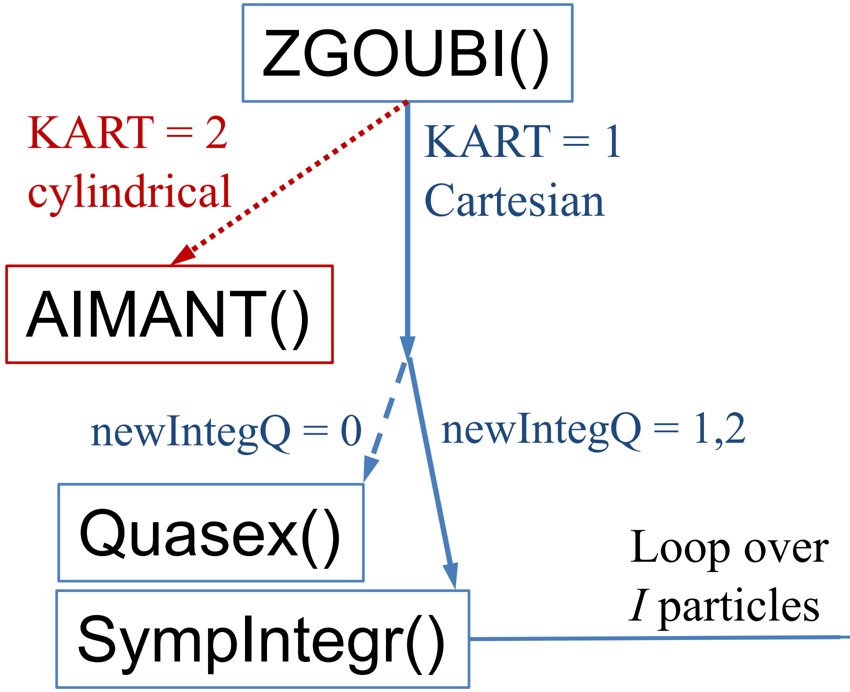
The symplectic drift-kick-drift integrator:

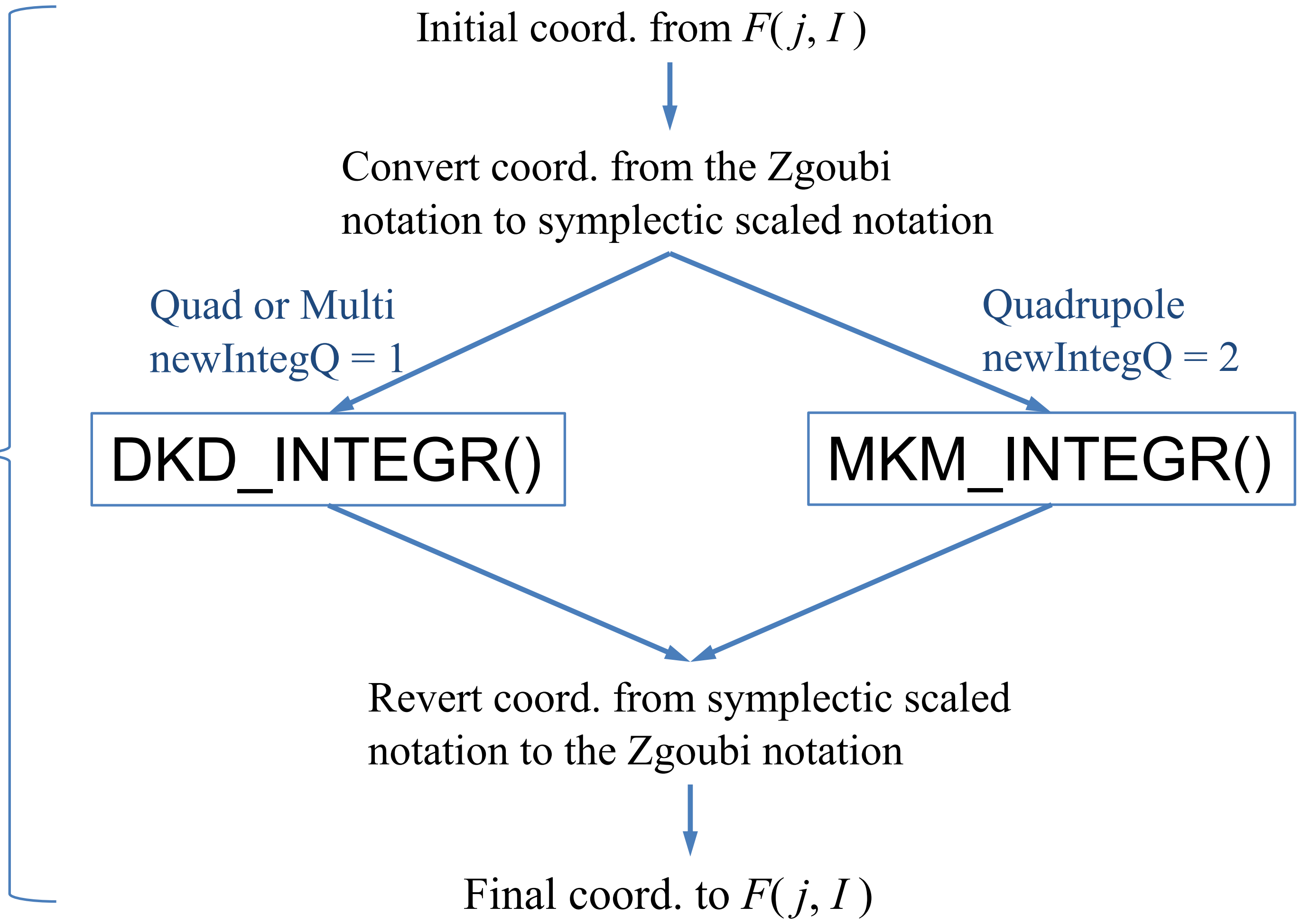
$$Z \quad \to \quad D(L/2)K_2(L)D(L/2)Z$$

Default Particle Motion Integrator in Zgoubi



Symplectic Integrator for Quadrupole and Multipole





(1) Reproduce the same results generated by the default integrator
(2) For 500,000 integration steps, QUASEX use **532** ms, while DKD uses **23.4** ms (4.4%)

Symplectic Integrator for Quadrupole and Multipole

1: the symplectic dkd integrator (any multipole)
2: the symplectic mkm integrator (quadrupole only)
0 or blank: the default integrator
Otherwise: end the job

```
 'QUADRUPO'                        QP  1
0 1
50.0  10.   .763695                          .763695
0.   0.
6   .1122 6.2671 -1.4982 3.5882 -2.1209 1.723
0.   0.
6   .1122 6.2671 -1.4982 3.5882 -2.1209 1.723
1.0   ! Quad
1 0. 0. 0.
```

The Matrix-Kick-Matrix (MKM) Integrator

$$H_{\text{quad}} = H_{q2} + H_{qk}$$

$$H_{qk} = -\sqrt{P_T^2 - \left(\gamma_0 \beta_0\right)^{-2} - P_X^2 - P_Y^2} - \frac{1}{2P}\left(P_X^2 + P_Y^2\right)$$

$$H_{q2} = \frac{1}{2P}\left(P_X^2 - P_Y^2\right) + \frac{eb_2}{2P_0}\left(X^2 - Y^2\right)$$

$$\text{Here} \quad K = \sqrt{\frac{eb_2}{P_0 P}}$$

$$P_S = \sqrt{P_T^2 - (\beta_0 \gamma_0)^{-2} - P_X^2 - P_Y^2}$$

$$= \sqrt{P^2 - P_X^2 - P_Y^2}$$

$$K_Q(L): \begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \rightarrow \begin{pmatrix} X + \left(\frac{1}{P_S} - \frac{1}{P}\right) L P_X \\ P_X \\ Y + \left(\frac{1}{P_S} - \frac{1}{P}\right) L P_Y \\ P_Y \\ T - \left(\frac{1}{P_S} - \frac{P_X^2 + P_Y^2}{2P^3}\right) L P_T \\ P_T \end{pmatrix}$$

The Matrix-Kick-Matrix (MKM) Integrator

$$H_{q2} = \frac{1}{2P}\left(P_X^2 - P_Y^2\right) + \frac{eb_2}{2P_0}\left(X^2 - Y^2\right)$$

$$M_Q(L): \begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \rightarrow \begin{pmatrix} X\cos(KL) + P_X \sin(KL)/KP \\ P_X \cos(KL) - KP\cdot X\sin(KL) \\ Y\cosh(KL) + P_Y \sinh(KL)/KP \\ P_Y\cosh(KL) + KP\cdot Y\sinh(KL) \\ T - \frac{P_T}{2P}\left[\left(\frac{P_X}{P}\right)^2 \frac{2KL+\sin(2KL)}{4K} + \left(\frac{P_Y}{P}\right)^2 \frac{\sinh(2KL)+2KL}{4K} \right. \\ +\left(KX\right)^2 \frac{2KL-\sin(2KL)}{4K} + \left(KY\right)^2 \frac{\sinh(2KL)-2KL}{4K} \\ \left. +KX\frac{P_X}{P}\frac{\cos(2KL)-1}{2K} + KY\frac{P_Y}{P}\frac{\cosh(2KL)-1}{2K}\right] \\ P_T \end{pmatrix}$$

$$Z \quad \rightarrow \quad M_Q(L/2)K_Q(L)M_Q(L/2)Z$$

The DKD Integrator for a multipole

$$Z \quad \to \quad D(L/2)K(L)D(L/2)Z$$

$$D(L):\begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \quad \to \quad \begin{pmatrix} X + LP_X / P_S \\ P_X \\ Y + LP_Y / P_S \\ P_Y \\ T - LP_T / P_S \\ P_T \end{pmatrix} \quad K(L):\begin{pmatrix} X \\ P_X \\ Y \\ P_Y \\ T \\ P_T \end{pmatrix} \quad \to \quad \begin{pmatrix} X \\ P_X - L\cdot\dfrac{\partial}{\partial_X} H_{\text{kick}}(X,Y) \\ Y \\ P_Y - L\cdot\dfrac{\partial}{\partial_Y} H_{\text{kick}}(X,Y) \\ T \\ P_T \end{pmatrix}$$

$$H_{\text{kick}} = \frac{e}{P_0}\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}(b_m + ia_m)(X+iY)^m$$

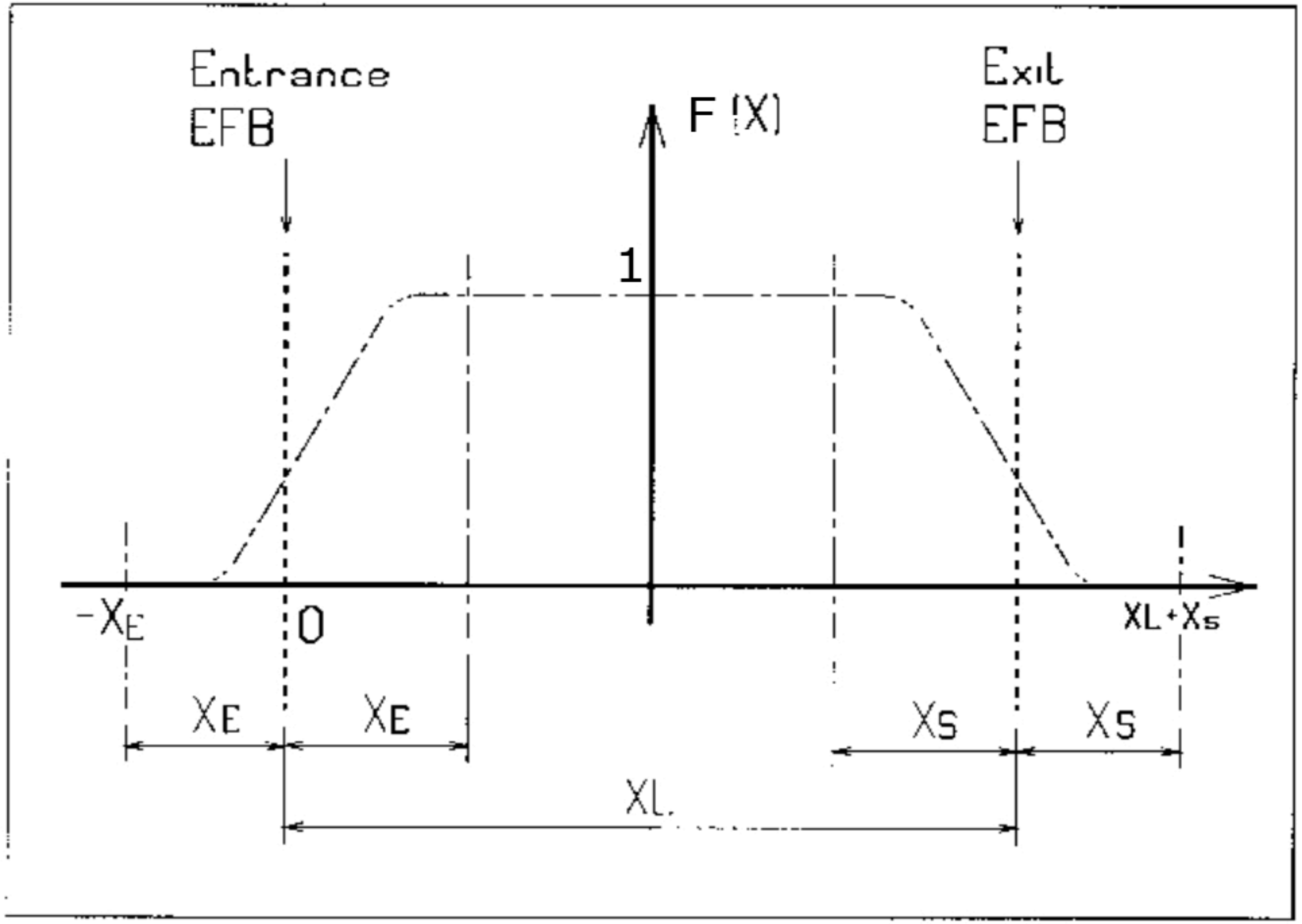
Fringe Fields

For the case of quadrupole:

$$b_2(Z) = F(Z)b_2$$

$$F(Z) = \frac{1}{1 + e^{P[d(Z)]}}$$

$$P(d) = \sum_{i=0}^{5} C_i \left(d / \lambda\right)^i$$



Fringe Fields for Quadrupole

$$A_z = -\frac{b_2}{2P_0}(X^2 - Y^2)F(Z)$$

However, this violates the Hamiltonian equation

Solutions satisfying the Hamiltonian equation:

$$A_z = -\frac{b_2}{2P_0}(X^2 - Y^2)\left[F(Z) - \frac{1}{12}\frac{d^2F(Z)}{dZ^2}(X^2 + Y^2) + \cdots\right]$$

$$A_\rho = \frac{b_2}{4P_0}\left[\frac{dF(Z)}{dZ}\rho^3 + \cdots\right]e^{i2\varphi}$$

Here

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi$$

$$A_x = A_\rho\cos\varphi, \quad A_y = A_\rho\sin\varphi$$

We have implemented the symplectic integrator for quadrupole with fringe fields

Summary and Future Work

- Implemented the symplectic integrators for particle tracking
  - drift-kick-drift for an arbitrary multipole
  - drift-kick-drift and matrix-kick-matrix for quadrupole
  - fringe fields are treated for quadrupole (working on a general multipole)
- More efficient than the default integrator
  - Using the same step size, only 4.4% of CPU time
  - Reaching convergence with larger step size
- Future work
  - Motion in bend (rectangular and sector), etc.
  - Electric field ($\boldsymbol{\Phi} \neq 0$), spin, etc.
  - Discrete Euler-Lagrange equations