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Polytopes –
Extremal Examples and Combinatorial Parameters

Exercise Sheet 1

Problem 1

Find a proof of Euler's theorem: $V - E + F = 2$.

(Euler tried and couldn't prove it.)

Here are 20 different proofs: `https://www.ics.uci.edu/~eppstein/junkyard/euler/`

Problem 2

Derive from Euler's theorem that every polyhedron has a triangle face, or a simple vertex (i.e., vertex of degree 3), or both.

Can you give lower bounds on the minimal number of triangle faces and simple vertices?

Problem 3

Decide if the following are f-vectors of 3-polytopes.

(a) (8,14,8)

(b) (8,20,14)

(c) (8,18,10)

If yes, draw an example. Find coordinates.

Problem 4

Let $P = \text{conv}\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\}$ be a 3-polytope.

Can you give a description of P as an intersection of half-spaces?

Problem 5

Let P be a polytope with vertex set V. We say that Q is a *subpolytope* of P if Q is the convex hull of a subset of V: $Q = \text{conv}(V')$ for some $V' \subseteq V$.

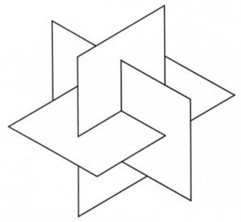
(a) Classify the subpolytopes of the regular 3-cube $[0,1]^3$.

How do you interpret "classify"? That is, when do you consider two sub-polytopes "equivalent" or "the same"?

(b) Among others, you should find tetrahedra and octahedra. Are they regular?

Problem 6

(a) Construct coordinates for the regular icosahedron.
Suggestion: Use the following picture:



(b) Construct vertex coordinates for the regular dodecahedron by building "tents" above the facets of a regular cube.

(c) Complete the classification of Platonic Solids, by showing that coordinates exist for all 5 types.

Problem 7

Let P be a vertex transitive 3-polytope, such that all its facets are regular. Assume that there are 4 facets A, B, C, D meeting at each vertex. If D is assumed to be a triangle, and A and C do not have the same number of vertices, what polytope can P be?

A	B
D	C

Problem 8

If a simple 3-polytope has only pentagons and hexagons as faces,

(a) what's the possible numbers of pentagons?

(b)* what's the possible numbers of hexagons?

Problem 9

The *face lattice* of a polytope P is the poset of all faces of P, partially ordered by inclusion. Draw the face lattice for a cube and for a cube with a vertex cut off.

Problem 10

(a) Show that the face lattices of 3-dimensional polytopes are "Eulerian": they is graded and all intervals have the same number of elements of odd and of even rank. Intervals of length > 1 are connected.

(b)* Assume that in an Eulerian lattice of length 5 all intervals of length > 2 are connected, (i.e. connected if you disregard minimal and maximal elements). Show that this characterizes face lattices of 3-polytopes.