Patterns in Standard Young Tableaux

Sara Billey
University of Washington
Slides: `math.washington.edu/~billey/talks`

Based on joint work with:
Matjaž Konvalinka and Joshua Swanson

6th Encuentro Colombiano de Combinatoria
June 14, 2018

Outline

Background on Standard Young Tableaux

q-enumeration of SYT's via major index

Distribution Question: From Combinatorics to Probability

Existence Question: New Posets on Tableaux

Unimodality Question: ???

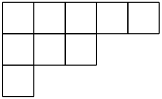
Partitions

Def. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k = |\lambda|$. Write $\lambda \vdash n$.

Partitions can be visualized by their *Ferrers diagram*

$(5, 3, 1) \longrightarrow$ 

The *cells* are indexed by matrix coordinates (i, j) so $(1, 5)$ is the rightmost cell in the top row.

Filling Partitions

Defn. A map from the cells of λ to the positive integers is a *filling* of λ.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 7 & 2 & 8 \\ \hline 6 & 1 & 2 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

Defn. A filling of $\lambda \vdash n$ is *bijective* if every number in $[n] = \{1, 2, \ldots, n\}$ appears exactly once.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 7 & 4 & 9 \\ \hline 6 & 2 & 8 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

Filling Partitions

Defn. A map from the cells of λ to the positive integers is a *filling* of λ.

1	3	7	2	8
6	1	2		
5				

Defn. A filling of $\lambda \vdash n$ is *bijective* if every number in $[n] = \{1, 2, \ldots, n\}$ appears exactly once.

1	3	7	4	9
6	2	8		
5				

Question. How many bijective fillings are there of shape $(5, 3, 1)$?

Filling Partitions

Defn. A map from the cells of λ to the positive integers is a *filling* of λ.

1	3	7	2	8
6	1	2		
5				

Defn. A filling of $\lambda \vdash n$ is *bijective* if every number in $[n] = \{1, 2, \ldots, n\}$ appears exactly once.

1	3	7	4	9
6	2	8		
5				

Question. How many bijective fillings are there of shape $(5, 3, 1)$?

Answer. $9! = 362,880$. Bijection with permutations of 9.

Standard Young Tableaux

Defn. A *standard Young tableaux* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Standard Young Tableaux

Defn. A *standard Young tableaux* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Question. How many standard Young tableaux are there of shape $(5, 3, 1)$?

Standard Young Tableaux

Defn. A *standard Young tableaux* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

1	3	6	7	9
2	5	8		
4				

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Question. How many standard Young tableaux are there of shape $(5,3,1)$? **Answer.** $\#\mathrm{SYT}(5,3,1) = 162$

Standard Young Tableaux

Pause: Find all standard Young tableaux on $(2, 2)$.

Counting Standard Young Tableaux

Hook Length Formula.(Frame-Robinson-Thrall, 1954)
If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Filling cells of $\lambda = (5,3,1) \vdash 9$ by hook lengths:

7	5	4	2	1
4	2	1		
1				

So, $\#SYT(5,3,1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$.

Counting Standard Young Tableaux

Hook Length Formula.(Frame-Robinson-Thrall, 1954)
If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Filling cells of $\lambda = (5, 3, 1) \vdash 9$ by hook lengths:

$$\begin{array}{|c|c|c|c|c|}
\hline
7 & 5 & 4 & 2 & 1 \\
\hline
4 & 2 & 1 \\
\cline{1-3}
1 \\
\cline{1-1}
\end{array}$$

So, $\#SYT(5, 3, 1) = \frac{9!}{7\cdot5\cdot4\cdot2\cdot4\cdot2} = 162$.

Remark. Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabilistic), Krattenthaler '95 (bijective), Novelli -Pak -Stoyanovskii'97 (bijective), Bandlow'08, Pak -Panova-Morales '17.

q-Counting Standard Young Tableaux

Def. The *descent set* of a standard Young tableaux T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

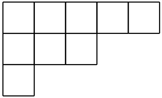
Example. The descent set of T is $D(T) = \{1, 3, 4, 7\}$ so $\mathrm{maj}(T) = 15$ for $T = \begin{array}{|c|c|c|c|c|}\hline \color{red}{1} & \color{red}{3} & 6 & \color{red}{7} & 9 \\ \hline 2 & \color{red}{4} & 8 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$.

Def. The *major index generating function* for λ is

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

q-Counting Standard Young Tableaux

Example. $\lambda = (5, 3, 1)$



$\mathsf{SYT}(\lambda)^{\mathsf{maj}}(q) := \sum_{T \in \mathsf{SYT}(\lambda)} q^{\mathsf{maj}(T)} =$

$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15}$

$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$

Note, at $q = 1$, we get back 162.

q-Counting Standard Young Tableaux

Thm.(Stanley-Lusztig 1979) Given a partition $\lambda \vdash n$, say

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then $b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ is the number of times the irreducible S_n module indexed by λ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree k.

Comments.

- Thus, maj on SYT completely determines the "*fake degrees*" $b_{\lambda,k}$ that Vic Reiner was discussing in Lecture 3.

q-Counting Standard Young Tableaux

Thm.(Stanley-Lusztig 1979) Given a partition $\lambda \vdash n$, say

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then $b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ is the number of times the irreducible S_n module indexed by λ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree k.

Comments.

- Thus, maj on SYT completely determines the "*fake degrees*" $b_{\lambda,k}$ that Vic Reiner was discussing in Lecture 3.
- The fake degrees also appear in branching rules between symmetric groups and cyclic subgroups (Stembridge, 1989), and the degree polynomials of certain irreducible $\mathrm{GL}_n(\mathbb{F}_q)$-representations (Steinberg 1951). See also (Green 1955).

q-Counting Standard Young Tableaux

Def. The *descent set* of a standard Young tableaux T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

Example. There are 2 standard Young tableaux of shape $(2,2)$:

$$S = \begin{array}{|c|c|} \hline 1 & \color{red}{2} \\ \hline 3 & 4 \\ \hline \end{array} \qquad T = \begin{array}{|c|c|} \hline \color{red}{1} & \color{red}{3} \\ \hline 2 & 4 \\ \hline \end{array}$$

$D(S) = \{2\}$ and $D(T) = \{1,3\}$ so $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^2 + q^4$.
Represent $q^2 + q^4$ by the vector of coefficients (00101)

q-Counting Standard Young Tableaux

Examples. $(2,2) \vdash 4$: $(0\ 0\ 1\ 0\ 1)$

$(5,3,1)$: $(00000\ 1\ 2\ 4\ 5\ 8\ 10\ 13\ 14\ 16\ 16\ 16\ 14\ 13\ 10\ 8\ 5\ 4\ 2\ 1)$

q-Counting Standard Young Tableaux

Examples. $(2,2) \vdash 4$: $(0\ 0\ 1\ 0\ 1)$

$(5,3,1)$: (00000 1 2 4 5 8 10 13 14 16 16 16 14 13 10 8 5 4 2 1)

$(6,4) \vdash 10$: (0 0 0 0 1 1 2 2 4 4 6 6 8 7 8 7 8 6 6 4 4 2 2 1 1)

$(6,6) \vdash 12$: (0 0 0 0 0 0 1 0 1 1 2 2 4 3 5 5 7 6 9 7 9 8 9 7 9 6 7 5 5 3 4 2 2 1 1 0 1)

$(11,5,3,1) \vdash 20$: (1 3 8 16 32 57 99 160 254 386 576 832 1184 1645 2255 3031 4027 5265 6811 8689 10979 13706 16959 20758 25200 30296 36143 42734 50163 58399 67523 77470 88305 99925 112370 125492 139307 153624 168431 183493 198778 214017 229161 243913 258222 271780 284542 296200 306733 315853 323571 329629 334085 336727 337662 336727 334085 329629 323571 315853 306733 296200 284542 271780 258222 243913 229161 214017 198778 183493 168431 153624 139307 125492 112370 99925 88305 77470 67523 58399 50163 42734 36143 30296 25200 20758 16959 13706 10979 8689 6811 5265 4027

Key Questions for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$.

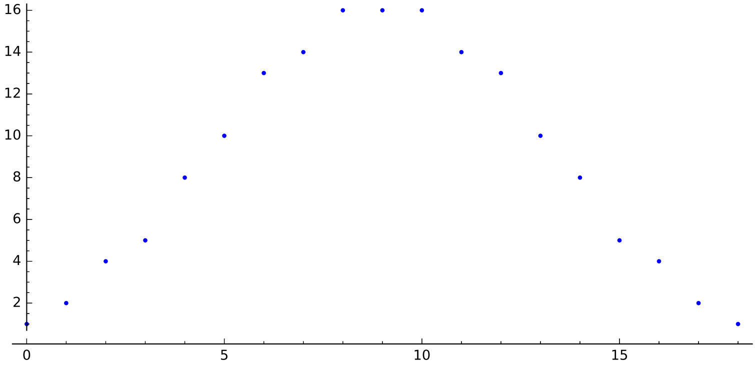
Distribution Question. What patterns do the coefficients in the list $(b_{\lambda,0}, b_{\lambda,1}, \ldots)$ exhibit?

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Unimodality Question. For which λ, are the coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *unimodal*, meaning

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \ldots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \ldots ?$$

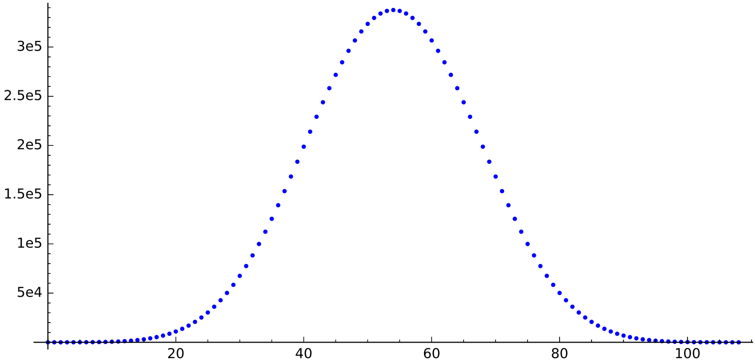
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(5,3,1)^{\mathrm{maj}}(q)$:

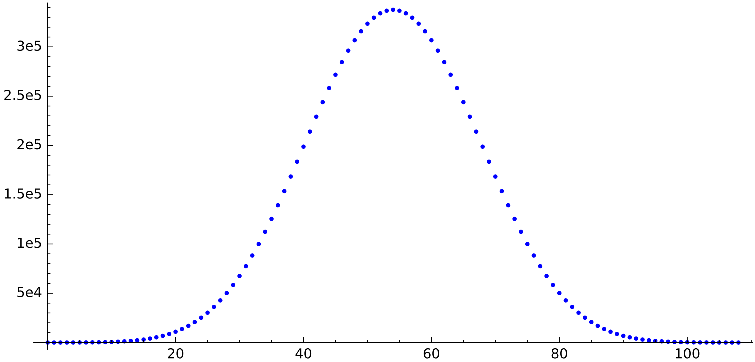
$$(1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1)$$

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11,5,3,1)^{\mathrm{maj}}(q)$.

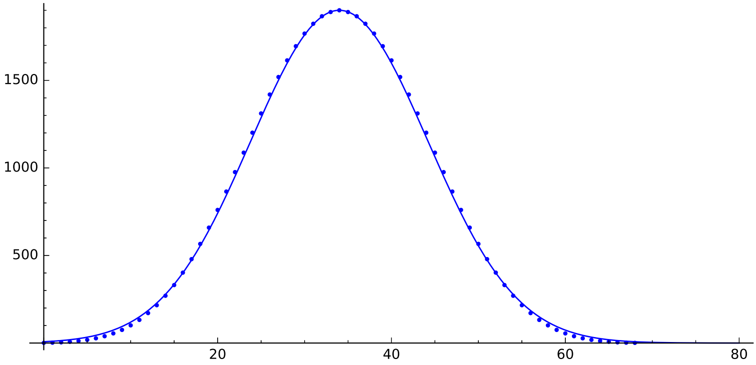
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11,5,3,1)^{\mathrm{maj}}(q)$.

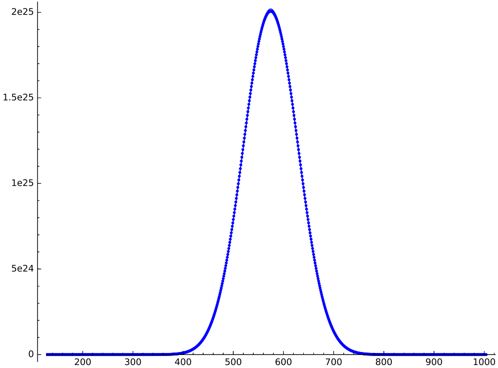
Question. What type of curve is that?

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(10,6,1)^{\mathrm{maj}}(q)$ along with the Normal distribution with $\mu = 34$ and $\sigma^2 = 98$.

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(8,8,7,6,5,5,5,2,2)^{\mathrm{maj}}(q)$

“Fast” Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

where

- $b(\lambda) := \sum(i-1)\lambda_i$
- h_c is the hook length of the cell c
- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

"Fast" Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

where

- $b(\lambda) := \sum(i-1)\lambda_i$
- h_c is the hook length of the cell c
- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

El Truco. Each q-integer $[n]_q$ factors into a product of *cyclotomic polynomials* $\Phi_d(q)$,

$$[n]_q = 1 + q + \cdots + q^{n-1} = \prod_{d|n} \Phi_d(q).$$

Cancel all of the factors from the denominator of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ from the numerator, and then expand the remaining product.

Converting q-Enumeration to Discrete Probability

Vic Reiner's Quote:

> *"If we can count it, we should also try to q-count it."*

I say:

> *"If we can q-count it, we should try to probabalize it."*

Converting q-Enumeration to Discrete Probability

Vic Reiner's Quote:

> *"If we can count it, we should also try to q-count it."*

I say:

> *"If we can q-count it, we should try to probabalize it."*

If $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n$ where a_i are nonnegative integers, then construct the random variable X_f with discrete probability distribution

$$\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.$$

Now, if f is part of a family of q-analogs, we can study the limiting distributions.

Converting q-Enumeration to Discrete Probability

Example. For $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$, define the integer random variable $X_\lambda[\mathrm{maj}]$ with discrete probability distribution

$$\mathbb{P}(X_\lambda[\mathrm{maj}] = k) = \frac{b_{\lambda,k}}{|\,\mathrm{SYT}(\lambda)|}.$$

We claim the distribution of X_λ "usually" is approximately normal for most shapes λ. Let's make that precise!

Standardization

Thm. (Adin-Roichman, 2001)
For any partition λ, the mean and variance of $X_\lambda[\mathrm{maj}]$ are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2}\left[\sum_{j=1}^{|\lambda|} j - \sum_{c\in\lambda} h_c\right],$$

and

$$\sigma_\lambda^2 = \frac{1}{12}\left[\sum_{j=1}^{|\lambda|} j^2 - \sum_{c\in\lambda} h_c^2\right].$$

Def. The *standardization* of $X_\lambda[\mathrm{maj}]$ is

$$X^*_\lambda[\mathrm{maj}] = \frac{X_\lambda[\mathrm{maj}] - \mu_\lambda}{\sigma_\lambda}.$$

So $X^*_\lambda[\mathrm{maj}]$ has mean 0 and variance 1 for any λ.

Asymptotic Normality

Def. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. We say the sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n \to \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where N is a Normal random variable with mean 0 and variance 1.

Asymptotic Normality

Def. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. We say the sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n\to\infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where N is a Normal random variable with mean 0 and variance 1.

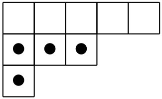
Question. In what way can a sequence of partitions approach infinity?

The Aft Statistic

Def. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\mathrm{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

Example. $\lambda = (5, 3, 1)$ then $\mathrm{aft}(\lambda) = 4$.



Todo: Add aft to FindStat soon!

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2018+)
Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\text{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\text{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2018+)
Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\text{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\text{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

Question. What happens if $\text{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Then $(X_{\lambda^{(N)}}[\mathrm{maj}]^*)$ converges in distribution if and only if

(i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or

(ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)})$ is eventually constant; or

(iii) the distribution of $X^*_{\lambda^{(N)}}[\mathrm{maj}]$ is eventually constant.

The limit law is $\mathcal{N}(0,1)$ in case (i), Σ^*_M in case (ii), and discrete in case (iii).

Here Σ_M denotes the sum of M independent identically distributed uniform $[0,1]$ random variables, known as the Irwin–Hall distribution or the *uniform sum distribution*.

Proof ideas: Characterize the Moments and Cumulants

Definitions.

- For $d \in \mathbb{Z}_{\geq 0}$, the *$d$th moment*

$$\mu_d := \mathbb{E}[X^d]$$

- The *moment-generating function* of X is

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

- The *cumulants* $\kappa_1, \kappa_2, \ldots$ of X are defined to be the coefficients of the exponential generating function

$$K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].$$

Nice Properties of Cumulants

1. *(Familiar Values)* The first two cumulants are $\kappa_1 = \mu$, and $\kappa_2 = \sigma^2$.
2. *(Shift Invariance)* The second and higher cumulants of X agree with those for $X - c$ for any $c \in \mathbb{R}$.
3. *(Homogeneity)* The dth cumulant of cX is $c^d \kappa_d$ for $c \in \mathbb{R}$.
4. *(Additivity)* The cumulants of the sum of *independent* random variables are the sums of the cumulants.
5. *(Polynomial Equivalence)* The cumulants and moments are determined by polynomials in the other sequence.

Examples of Cumulants and Moments

Example. Let $X = \mathcal{N}(\mu, \sigma^2)$ be the normal random variable with mean μ and variance σ^2. Then the cumulants are

$$\kappa_d = \begin{cases} \mu & d = 1, \\ \sigma^2 & d = 2, \\ 0 & d \geq 3. \end{cases}$$

and for $d > 1$,

$$\mu_d = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ \sigma^d (d-1)!! & \text{if } d \text{ is even.} \end{cases}$$

.

Example. For a Poisson random variable X with mean μ, the cumulants are all $\kappa_d = \mu$, while the moments are $\mu_d = \sum_{i=1}^{d} \mu^i S_{i,d}$.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. We have

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c \in \lambda} h_c^d\right] \tag{1}$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d > 3$ all go to 0 proving the Asymptotic Normality Theorem.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2018+)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. We have

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c\in\lambda} h_c^d\right] \tag{1}$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d > 3$ all go to 0 proving the Asymptotic Normality Theorem.

Thm.(Adin-Roichman, 2001)
For any partition λ, the mean and variance of $X_\lambda[\mathrm{maj}]$ are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2}\left[\sum_{j=1}^{|\lambda|} j - \sum_{c\in\lambda} h_c\right],$$

q-Enumeration to Probability

Thm.(Chen–Wang–Wang-2008 and Hwang–Zacharovas-2015) Suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are sets of positive integers such that

$$f(q) = \frac{\prod_{j=1}^m [a_j]_q}{\prod_{j=1}^m [b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]$$

.

Let X be a discrete random variable with $\mathbb{P}(X = k) = c_k/f(1)$. Then the dth cumulant of X is

$$\kappa_d = \frac{B_d}{d} \sum_{j=1}^{m} (a_j^d - b_j^d)$$

where B_d is the dth Bernoulli number (with $B_1 = \frac{1}{2}$).

Example. This theorem applies to

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)} [n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with aft going to infinity.

2. New proof of asymptotic normality of $[n]_q! = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} q^{\mathrm{inv}(w)}$ due to Feller (1944).

3. New proof of asymptotic normality of q-multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).

4. New proof of asymptotic normality of q-Catalan numbers due to Chen-Wang-Wang(2008).

Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with aft going to infinity.

2. New proof of asymptotic normality of $[n]_q! = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} q^{\mathrm{inv}(w)}$ due to Feller (1944).

3. New proof of asymptotic normality of q-multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).

4. New proof of asymptotic normality of q-Catalan numbers due to Chen-Wang-Wang(2008).

Question. Using Pak-Panova-Morales's q-hook length formula, can we prove an asymptotic normality for all skew shapes?

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Cor of Stanley's formula. For every $\lambda \vdash n \geq 1$ there is a unique tableau with minimal major index $b(\lambda)$ and a unique tableau with maximal major index $\binom{n}{2} - b(\lambda')$. These two agree for shapes consisting of one row or one column, and otherwise they are distinct.

Cor of Stanley's formula. The coefficient of $q^{b(\lambda)+1}$ in $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = 0$ if and only if λ is a rectangle.
If λ is a rectangle with more than one row and column, then coefficient of $q^{b(\lambda)+2}$ is 1.

Question. Are there other internal zeros?

Classifying All Zeros

Thm.(Billey-Konvalinka-Swanson, 2018+)
For every partition $\lambda \vdash n \geq 1$ and integer k such that $b(\lambda) \leq k \leq \binom{n}{2} - b(\lambda')$, we have $b_{\lambda,k} > 0$ except in the case when λ is a rectangle with at least 2 rows and columns and k is either $b(\lambda) + 1$ or $\binom{n}{2} - b(\lambda') - 1$.

We have $b_{\lambda,k} = 0$ for $k < b(\lambda)$ or $k > \binom{n}{2} - b(\lambda')$.

Cor. The irreducible S_n-module indexed by λ appears in the decomposition of the degree k component of the coinvariant algebra if and only if $b_{\lambda,k} > 0$ as characterized above.

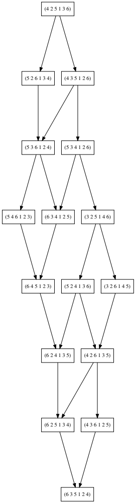
Acknowledgment. Our motivation for this project came from a conjecture of Sheila Sundaram's which was solved by Josh Swanson on the zeros of the maj-mod-n generating function on standard Young tableaux.

Strong and Weak Poset on SYT(λ)

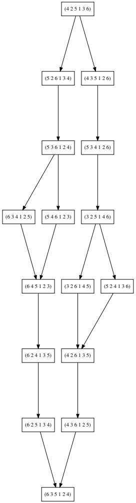
Proof of the classification theorem:

1. We give a poset structure on SYT(λ) which is ranked by maj(T). Cover relations correspond to simple pattern moves in T.
2. We obtain both a strong and weak variation on this poset in analogy with weak order and strong Bruhat order.

Strong and Weak Poset on $SYT(3, 2, 1)$



Strong



Weak

Unimodality Question

Conjecture. The polynomial $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal if λ has at least 4 corners. If λ has 3 corners or fewer, then $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal except when λ or λ' is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form $(k, 2)$ with $k \geq 4$ and k even.
3. Any partition of the form $(k, 4)$ with $k \geq 6$ and k even.
4. Any partition of the form $(k, 2, 1, 1)$ with $k \geq 2$ and k even.
5. Any partition of the form $(k, 2, 2)$ with $k \geq 6$.
6. Any partition on the list of 40 special exceptions of size at most 28.

Unimodality Question

Special Exceptions.

$$\begin{gathered}(3,3,2),(4,2,2),(4,4,2),(4,4,1,1),\\(5,3,3),(7,5),(6,2,1,1,1,1),\\(5,5,2),(5,5,1,1),(5,3,2,2),(4,4,3,1),\\(4,4,2,2),(7,3,3),(8,6),(6,6,2),\\(6,6,1,1),(5,5,2,2),(5,3,3,3),(4,4,4,2),\\(11,5),(10,6),(9,7),(7,7,2),\\(7,7,1,1),(6,6,4),(6,6,1,1,1,1),(6,5,5),\\(5,5,3,3),(12,6),(11,7),(10,8),\\(15,5),(14,6),(11,9),(16,6),(12,10),(18,6),\\(14,10),(20,6),(22,6).\end{gathered}$$

Local Limit Conjecture

Conjecture. Let $\lambda \vdash n > 25$. Uniformly for all n and for all integers k, we have

$$|\mathbb{P}(X_\lambda[\mathrm{maj}] = k) - N(k; \mu_\lambda, \sigma_\lambda)| = O\left(\frac{1}{\sigma_\lambda \,\mathrm{aft}(\lambda)}\right)$$

where $N(k; \mu_\lambda, \sigma_\lambda)$ is the density function for the normal distribution with mean μ_λ and variance σ_λ.

The conjecture has been verified for $n \leq 50$ and $\mathrm{aft}(\lambda) > 1$.

Up to $n = 50$, the constant $1/9$ works.

At $n = 50$, $1/10$ does not.

¡ Muchas Gracias!

Castillo de San Felipe de Barajas en Cartagena