

Polynomial Optimization – Exercises

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1. Let $G = ([n], E)$ be an undirected graph with vertex set $[n] = \{1, \dots, n\}$ for a positive integer n and edge set E consisting of pairs of vertices. A set $S \subseteq [n]$ is said to be *stable* or *independent* if for any two vertices $i, j \in S$, the edge $ij \notin E$. Formulate a polynomial optimization problem to find the maximum cardinality stable set in G .
2. A *cut* in G is a partitioning of its vertices into two sets T and $[n] \setminus T$ and the size of the cut is the number of edges that go between the two parts. Formulate a polynomial optimization problem to find the maximum cardinality cut in G . This is another NP-hard problem.
3. A very common problem that arises in applications is to find the closest point in a given set from a given *data* point that has been observed in an experiment. For instance in computer vision one is often interested in reconstructing a three-dimensional scene from noisy images of the scene. The set of all true images that are possible by the given cameras is an algebraic set which is the *model* and the noisy images form the *data* point. If the noise model is Gaussian then the closest point to the model from the observed noisy data point is the maximum likelihood estimate. Model this problem as a polynomial optimization problem.

Another problem that is very common in applications is to find a low rank estimate of a given matrix. Write down a polynomial optimization problem for finding the closest (in Euclidean distance) rank one real matrix of size $p \times q$ to a given real matrix A of the same size. Generalize to rank k . The classical *Eckart-Young theorem* in linear algebra gives a solution to this distance minimization problem. Look it up and see if you can solve it using the model you wrote.

4. A function f is convex if $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ for $x, y \in \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$ such that $0 \leq \alpha, \beta$ and $\alpha + \beta = 1$. A set $K \subset \mathbb{R}^n$ is convex if for all $x, y \in K$ and $\alpha, \beta \in \mathbb{R}$ such that $0 \leq \alpha, \beta$ and $\alpha + \beta = 1$, any point of the form $\alpha x + \beta y \in K$. In other words, the line segment joining x and y is entirely in K .

Consider the semialgebraic region $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$. Prove that K is a convex set if $-g_1, \dots, -g_m$ are convex functions.

If in addition f is a convex function, then the polynomial optimization problem $\min\{f(x) : x \in K\}$ is called a *convex program*.

5. (a) Convince yourself that the psd cone $\mathcal{S}_+^n \subset \mathcal{S}^n$ is closed, convex, pointed and full-dimensional (solid). A cone with all these properties is called a *proper cone*.
Recall that a *convex* cone $K \subset \mathbb{R}^t$ is one in which for every $x, y \in K$, $\lambda x + \mu y \in K$ for all $\lambda, \mu \geq 0$. The cone K is *pointed* if it does not contain any lines through the origin, i.e., there is no $x \in K$, $x \neq 0$ such that $-x \in K$.
- (b) Prove that the rank one matrices in \mathcal{S}_+^n generate its *extreme rays* (i.e., rays that cannot be written as a non-negative combination of other rays in \mathcal{S}_+^n). Recall that a rank one matrix in \mathcal{S}_+^n looks like aa^\top where $a \in \mathbb{R}^n$.
- (c) By *Caratheodory's theorem* from convex geometry, every element in \mathcal{S}_+^n can be written as a non-negative combination of at most $\frac{n(n+1)}{2}$ extreme rays of \mathcal{S}_+^n . On the other hand, the previous exercise allows you to bound the number of rank one matrices needed to write a psd matrix in \mathcal{S}_+^n as a non-negative combination. How do these bounds compare?
6. Recall that the feasible region of a semidefinite program (SDP) is called a *spectrahedron*. We may take the following to be the official definition:

Definition 0.1. A spectrahedron is a set of the form

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m : A_0 + \sum A_i x_i \geq 0\}$$

where the matrices $A_i \in \mathcal{S}^n$.

- (a) In the lecture we defined a *spectrahedron* to be an affine slice of the psd cone. Indeed, the matrices defined by the above set is the intersection of the psd cone \mathcal{S}_+^n with the affine plane obtained by translating $\text{span}(A_1, \dots, A_m)$ by A_0 . If the matrices A_1, \dots, A_m are linearly independent in \mathcal{S}^n then prove that there is a bijection between the two descriptions of a spectrahedron as a subset of \mathbb{R}^m and \mathcal{S}^n respectively.
- (b) Prove that a spectrahedron also admits the following descriptions:
- i. $\{X \in \mathcal{S}_+^n : \langle B_j, X \rangle = b_j \ \forall \ j = 1, \dots, t\}$, for some symmetric matrices $B_j \in \mathcal{S}^n$,
 - ii. $\{x \in \mathbb{R}^s : p_j(x) \geq 0 \ \ p_j \in \mathbb{R}[x_1, \dots, x_s], \ j = 1, \dots, r\}$

How do t, s and r relate to m and n ?

- (c) Using any of the above descriptions, argue that a spectrahedron is closed, convex and basic semi-algebraic.
- (d) Consider the following concrete spectrahedron:

$$\mathcal{F} := \left\{ (x, y) \in \mathbb{R}^2 : \begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix} \geq 0 \right\}.$$

- i. Express \mathcal{F} in the two other formats mentioned above.

- ii. Draw this set in the plane.
 - iii. What is the polynomial that defines the boundary of \mathcal{F} ? Generalize your result to the general spectrahedron in Definition 0.1.
7. A very common example of a spectrahedron is the *elliptope* \mathcal{E}_n defined as follows.

$$\mathcal{E}_n := \{X \in \mathcal{S}_+^n : X_{ii} = 1 \ \forall \ i = 1, \dots, n\}.$$

- (a) What is the dimension of \mathcal{E}_n ?
 - (b) Use a computer to draw \mathcal{E}_3 .
 - (c) What are the rank one psd matrices on \mathcal{E}_3 ? Can you see them in your picture?
 - (d) Find a rank two matrix on \mathcal{E}_3 that is not a convex combination of the rank one matrices on \mathcal{E}_3 .
 - (e) Can you model the max cut problem as an SDP over \mathcal{E}_n with possibly additional rank constraints?
8. Check that the following basic facts are true for a sos polynomial $p = \sum h_j^2$ in $\mathbb{R}[x]$.
- (a) $\deg(p) = 2d \Rightarrow \deg(h_j) \leq d$.
 - (b) p homogeneous and $\deg(p) = 2d \Rightarrow h_j$ homogeneous and $\deg(h_j) = d$.
 - (c) If \tilde{p} is the homogenization of p then $p \geq 0$ (resp. sos) $\Leftrightarrow \tilde{p} \geq 0$ (resp. sos).
 - (d) If $\deg(p) = 2d$, bound the number of squares needed in the sos expression for p . (Hint: use that p sos if and only if $p = [x]_d^T Q [x]_d$ for some $Q \geq 0$.)
9. Write the following polynomial as a sos: $x^2 + 4x + 5$.
10. Express $2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2$ as a sos using the connection to psd matrices and SDP.
11. Prove that a univariate non-negative polynomial is always a sum of two squares. (Hint: Make an argument about the possible real and complex roots of this polynomial and use the identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$ for all $a, b, c, d \in \mathbb{R}$.)
12. (Ex 3.35) Can you express $x^4 + 4x^3 + 6x^2 + 4x + 5$ as a sum of two squares?
13. (Ex 3.54) Let $p(x) = \sum_{k=0}^{2d} c_k x^k$. Give an explicit SDP formulation to compute the value of the global min of $p(x)$.
- (a) Show that the min of $p(x) = x^4 - 20x^2 + x$ is less than or equal to -100 .
 - (b) Show that the min of $p(x) = x^4 - 20x^2 + x$ is greater than or equal to -104 .
 - (c) Minimize the polynomial $p(x) = x^4 - 20x^2 + x$.

14. (a) (Ex 3.57) Find the value of p_{sos} for the polynomial $p(x, y, z) = x^4 + y^4 + z^4 - 4xyz + 2x + 3y + 4z$ over \mathbb{R}^3 . Is $p_* = p^{\text{sos}}$ in this example? Do you expect $p_* = p^{\text{sos}}$?
- (b) Find the value of p_{sos} and p_* for the polynomial $p(x, y) = x^4 + y^4 - 4xyz$ over \mathbb{R}^2 . Do you expect $p_* = p^{\text{sos}}$?
15. The Newton polytope of a polynomial $p(x_1, \dots, x_n)$ is the convex hull of all the non-negative integer vectors in \mathbb{N}^n that appear as exponents of the monomials present in p . We will denote it as $\mathcal{N}(p)$. For example, $\mathcal{N}(x^2 + xy + y^2)$ is the line segment in \mathbb{R}^2 that is the convex hull of $(2, 0), (1, 1), (0, 2)$. Reznick proved the following theorem:

If $p = \sum q_j^2$ then $\mathcal{N}(q_i) \subseteq \frac{1}{2}\mathcal{N}(p)$ for each i .

(Ex 3.97)

- (a) Compute the Newton polytope of the Motzkin polynomial.
- (b) Which monomials would appear in a hypothetical sos decomposition of the Motzkin polynomial if you know the above theorem?
- (c) Show by considering the coefficient of x^2y^2 , and the above calculation, that the Motzkin polynomial is not a sos.
16. (Ex 3.69) Consider the quartic form in four variables:

$$p(w, x, y, z) = w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz.$$

- (a) Show that p is not a sos. (Hint: Use Reznick's result mentioned in Exercise 7).
- (b) Find a multiplier that makes the product a sos.
17. Find $p_* = \inf\{10 - x^2 - y : x^2 + y^2 \leq 1\}$. (It's easy to do some basic calculus to determine p_* in this example. You can use that to check the answer you get from the sos relaxation.)
18. Suppose we want to minimize a polynomial over an algebraic variety (given by equations) as opposed to a semialgebraic set:

$$p_* = \inf\{p(x) : g_1(x) = 0, \dots, g_m(x) = 0\}.$$

- (a) Write down the form of the p_t^{sos} problem in this case by modifying from a semialgebraic set to an algebraic set. What simplifications can you make?
- (b) Is there a way we can write a version of p_t^{sos} that is indifferent to the particular choice of equations defining the variety?
- (c) (Ex 3.99) Use your method to minimize the polynomial $10 - x^2 - y$ over the unit circle $x^2 + y^2 = 1$.
19. (Ex 3.62) Show that the polynomial $x^4 - 3x^2 + 1$ is nonnegative on the variety defined by $x^3 - 4x = 1$.