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6th ECCO Barranquilla
June 4-15, 2018

Polytopes – Extremal Examples and Combinatorial Parameters

Exercise Sheet 4

Problem 1

(a) Show that you can tile the plane with congruent copies of a regular triangle, quadrilateral, or hexagon.

(b) Show that you cannot tile the plane with congruent copies of a regular pentagon.

Problem 2

Show that if a pentagon has two parallel sides, then you can tile the plane with congruent copies.

Problem 3

Why can't you tile the plane with congruent copies of a fixed n-gon, for $n > 6$?

Problem 4

A *Hanner polytope* is formed starting with the interval $I := [-1, +1]$ by forming products and direct sums (equivalently: products and duality).

(a) Show that each Hanner polytope has $3^d - 1$ non-trivial faces.

(b) Is it true that each Hanner polytope is a prism or a bipyramid?

Problem 5

Classify the (linear/combinatorial equivalence classes of) Hanner polytopes in dimensions $2, 3, 4, 5, 6, \ \ldots$

Problem 6

Analyze the centrally symmetric polytope obtained as the convex hull of the 10 vectors $\pm e_1, \ldots, \pm e_4, \pm(e_1 + \cdots + e_4)$ in $\mathbb{R}^4$.

(a) Show that this is "centrally-symmetric neighborly," that is, any two vertices except for the opposites are neighbors.

(b) Show that this is simplicial.

(c) Compute its f-vector.

Problem 7

The 24-cell can be constructed in two versions:

- Take as convex hull of the midpoints of the 2-faces of a 4-cube, that is, all vectors $v \in \{-1, 0, +1\}^4$ with two zeros,
- Take the convex hull of the cube $\mathrm{conv}\{-1, +1\}^4$ and the 4-dimensional crosspolytope $\mathrm{conv}\{\pm e_1, \dots, \pm e_4\}$.

Analyze both of them. Show that this is really "the same" polytope. Show that its facets are 24 regular octahedron. Show that the polytope is self-dual. Compute its f-vector.

Problem 8

Analyze $\mathrm{conv}((Q \times 2Q) \cup (2Q \times Q))$ for $Q = [-1, 1]^2$:

(a) Show that this is a 4-polytope with the graph of a 5-cube.

(b) Show that all its facets are combinatorial cubes.

(c) Using this, compute its f-vector.