Need colored chalk!!
Also, will need to display one page on overhead or have printouts

Colombia Lecture # 1: Tot pos gps & cluster algebras

Def: A matrix is totally positive (TP), if all its minors are positive real numbers.
Matrix is totally non-neg (TNN) if all minors are non-neg.

Here, "minors" are determinants of submatrices.

Ex:
Let $M = \begin{array}{cc} \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ & \begin{array}{ccc} \text{Col } 1 & 2 & 3 \end{array} \end{array}$

Notation for minors:
$\Delta_{I,J}(M) :=$ det of submatrix of M in rows I and col's J

So e.g. $\Delta_{2,3}(M) = \det(a_{23}) = a_{23}$

$$\Delta_{12,13}(M) = \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} = a_{11}a_{23} - a_{13}a_{21}$$

$$\Delta_{123,123}(M) = \det(M).$$

So we say the 3x3 matrix M is TP if
all entries all pos,
all 2x2 minors are pos,
det (M) is pos.

Q: Are the matrices below TP or TNN?

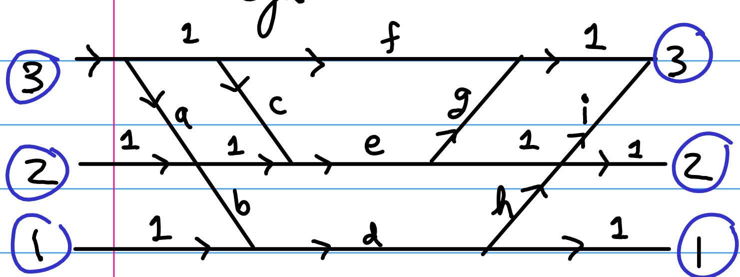
$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

not TNN, not TP — TNN, not TP — TNN, TP

Questions one might ask:

1. Is there a way to write down (parameterize) all TP matrices?
2. How many minors must we test to deduce that a matrix $M \in G_{>0}$? Which minors?

1. There is a general procedure for producing totally nonnegative matrices. Fix a <u>planar network</u> — an acyclic directed planar graph Γ whose edges have <u>weights</u>.



The <u>weight</u> of a directed path in Γ is defined to be the product of the weights of the edges. The <u>weight matrix</u> $X(\Gamma)$ is an $n \times n$ matrix (a_{ij}) where

a_{ij} = sum of all weights from i to j.

Here,
$$X(\Gamma) = \begin{pmatrix} d & dh & dhi \\ bd & bdh+e & bdhi+eg+ei \\ abd & abdh+ae+ce & abdhi+(a+c)e(g+i)+f \end{pmatrix}$$

Let's check a few minors of matrix above:

$$\Delta_{12,12}(x(\Gamma)) = d(bdh+e) - (bd)(dh) = de$$

$$\Delta_{13,12}(x(\Gamma)) = \det\begin{pmatrix} d & dh \\ abd & abdh+ae+ce \end{pmatrix} = ade+cde$$

<u>Lemma (Lindstrom-Gessel-Viennot)</u>: All minors of such a matrix $x(\Gamma)$ polynomials in the edge weights w/ positive coefficients. (There is a combinatorial interpretation for Δ_{IJ} as the sum of weights of all vertex-disjoint paths from the sources I to the sinks J.)

<u>Exercise</u>: Make this more precise: give combinatorial formula for minors of $x(\Gamma)$ in terms of network.

<u>Cor</u>: If each of the weights on Γ is a pos. real number (i.e. if $a,b,c,d,\ldots,g,h > 0$), then $x(\Gamma)$ is TP.

Moreover

<u>Theorem</u> (A. Whitney '52): The map $(\mathbb{R}_{>0})^9 \to 3\times 3$ matrices given by $(a,b,c,\ldots,i) \mapsto x(\Gamma)$ as above is a bijection

$$\Phi: (\mathbb{R}_{>0})^9 \longrightarrow \text{totally positive } 3\times 3 \text{ matrices.}$$

(Call this a <u>parameterization</u> of space of TP 3×3 matrices)

Exercise: Prove thm by inverting the map Φ

Exercise: What network should we use for 4×4 matrices? $n\times n$ matrices?

Exercise: Find other networks that lead to parameterizations of 3×3 (or $n\times n$) TP matrices.

On question 2: (How many, & which minors do we need to test if a matrix is TP?)

Ex: Consider 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A priori: in order to be TP, we need:

$a > 0$
$b > 0$
$c > 0$
$d > 0$
$ad-bc > 0$.

Do we really have to test all 5 or can we get away w/ fewer? (Are 4 enough? If so, which 4?)

Note: If $a>0$, $b>0$, $c>0$ and $ad-bc>0$ then since $ad>bc$ and $a,b,c>0$, this $\Rightarrow d>0$ also.

So testing $\{a, b, c, ad-bc\}$ enough. (4).

But not any 4! not enough to test $\{a,b,c,d\}$

Ex: Consider 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

A priori we need to test
- 9 1×1 minors (entries)
- 9 2×2 minors
- 1 determinant

So 19 in all.

Q: Do we really need to check all 19? If not, how many is the right number?

(Answer: 9 is enough. Note 9 = dim of space of 3×3 matrices)

Def: Let M be n×n matrix. The solid submatrix obtained from an entry a_{ij} is the biggest submatrix of M whose bottom right corner is the entry a_{ij}.

Ex: Solid minor assoc to a_{23} is $\begin{pmatrix} a_{11} & \boxed{a_{12} & a_{13}} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$

" " " " a_{42} is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Thm (FZ but probably earlier)
Let M be a real $n \times n$ matrix.
If for each entry a_{ij}, the associated solid minor of M is positive, then M is TP.

Ex: To determine whether $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is TP,

it suffices to check that all minors

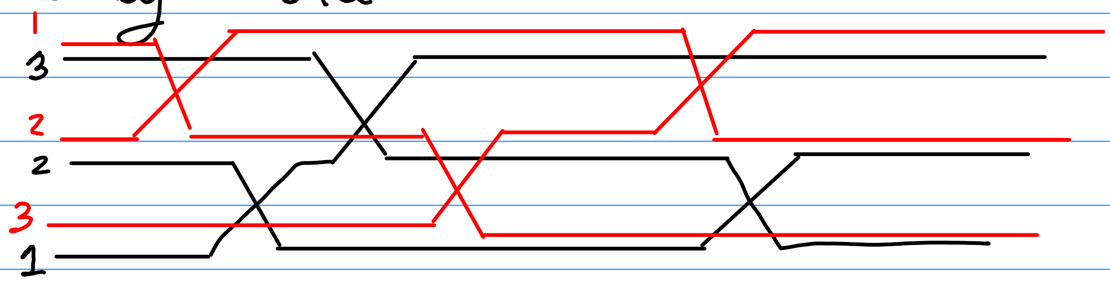
in $\left\{ \begin{matrix} \Delta_{1,1}, & \Delta_{1,2}, & \Delta_{1,3}, & \Delta_{2,1}, & \Delta_{3,1} \\ \Delta_{12,12}, & \Delta_{12,23}, & \Delta_{23,12}, & \Delta_{123,123} & \end{matrix} \right\}$ are positive.

Call this collection a <u>total positivity</u> (or <u>TP</u>) <u>test</u>.

Q: Are there other TP tests?
(Other collections of minors whose positivity $\Rightarrow$ the matrix is TP)

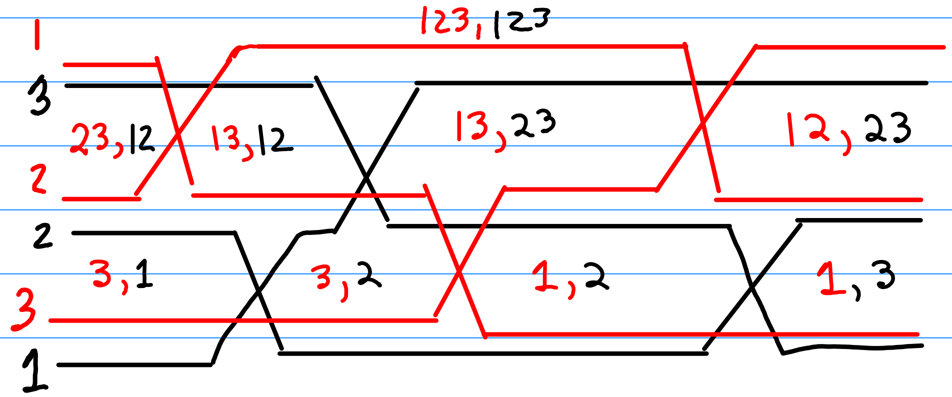
Double wiring diagrams (Fomin + Zelevinsky)

Choose two families of piecewise straight lines, each family colored w/ one of two colors, s.t. each pair of lines of like colors intersect exactly once.



Remark: if we look at the set of lines in a fixed color, this encodes a *reduced decomposition* for the longest permutation $w_0 = (n, n-1, \dots, 2, 1)$.

Assign to each *chamber* of a diagram a pair of subsets of the set $[1,n] = \{1, \dots, n\}$: each subset indicates which lines of the corresponding color pass *below* the chamber:



Interpret A, B as the "chamber minor" $\Delta_{A,B}$ (rows → A, columns → B)

Theorem (Fomin + Zelevinsky): Each double wiring diagram — each of which is determined by a *shuffle* of two reduced decomp's for w_0 — gives rise to the following criterion: an $n \times n$ matrix is totally positive iff all its chamber minors are positive.

Ex: prove this!

Example above says: A 3×3 matrix M is totally positive iff the following minors are pos:

$$\Delta_{123,123}(M)$$

$$\Delta_{23,12}(M) \quad \Delta_{13,12}(M) \quad \Delta_{13,23}(M) \quad \Delta_{12,23}(M)$$

$$\Delta_{3,1}(M) \quad \Delta_{3,2}(M) \quad \Delta_{1,2}(M) \quad \Delta_{1,3}(M)$$

We get a lot of TP criteria this way. Let's make a chart showing all of them.

Here is an "exchange graph" showing TP tests.

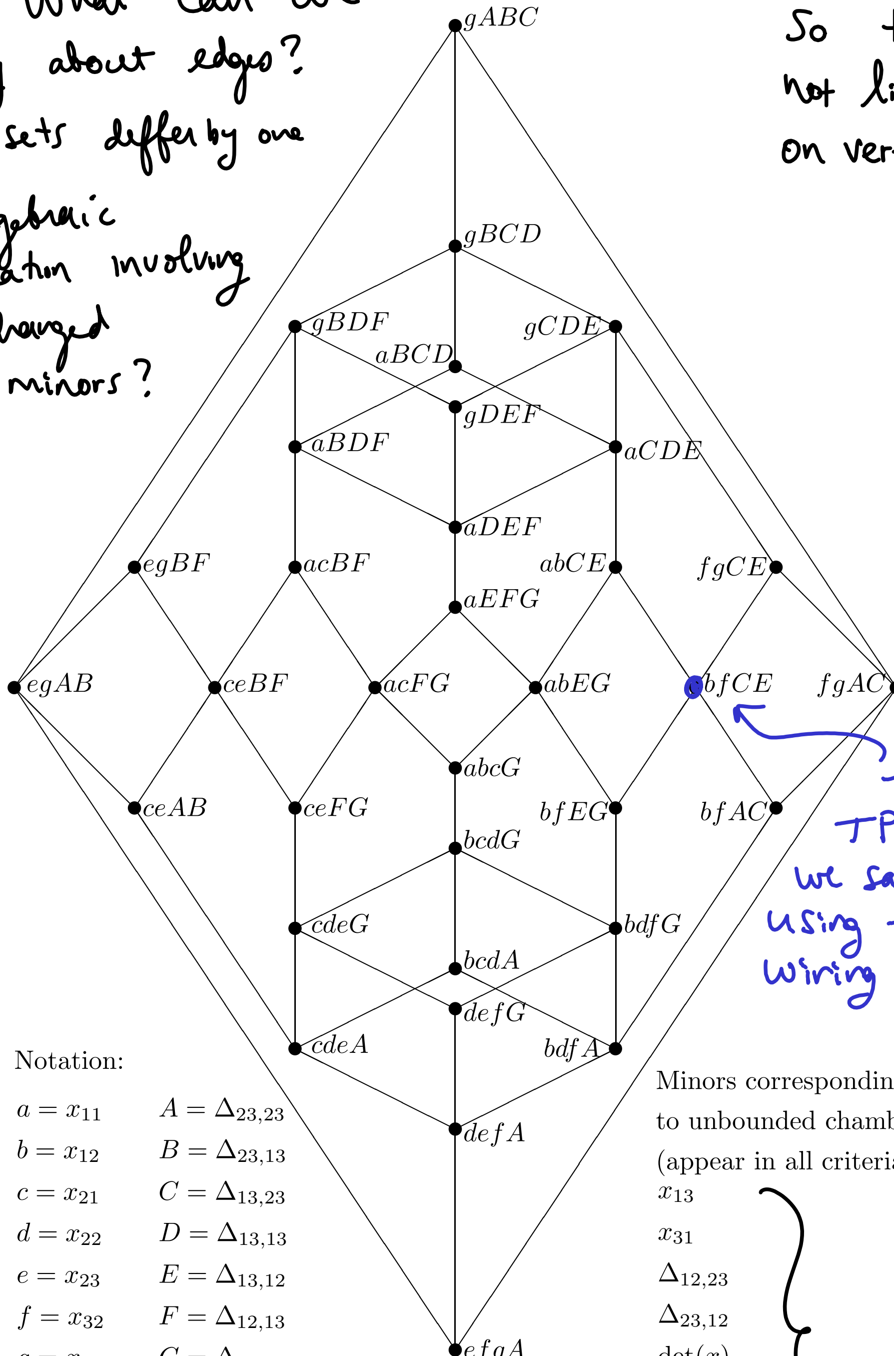
Explain meaning of graph!!

Note that all tests should contain

Q: What can we say about edges?

- subsets differ by one
- algebraic relation involving exchanged minors?

So they are not listed on vertices.



this is the TP criteria we saw earlier, using the double wiring diagram.

Notation:

$a = x_{11}$	$A = \Delta_{23,23}$
$b = x_{12}$	$B = \Delta_{23,13}$
$c = x_{21}$	$C = \Delta_{13,23}$
$d = x_{22}$	$D = \Delta_{13,13}$
$e = x_{23}$	$E = \Delta_{13,12}$
$f = x_{32}$	$F = \Delta_{12,13}$
$g = x_{33}$	$G = \Delta_{12,12}$

Minors corresponding to unbounded chambers (appear in all criteria):
x_{13}
x_{31}
$\Delta_{12,23}$
$\Delta_{23,12}$
$\det(x)$

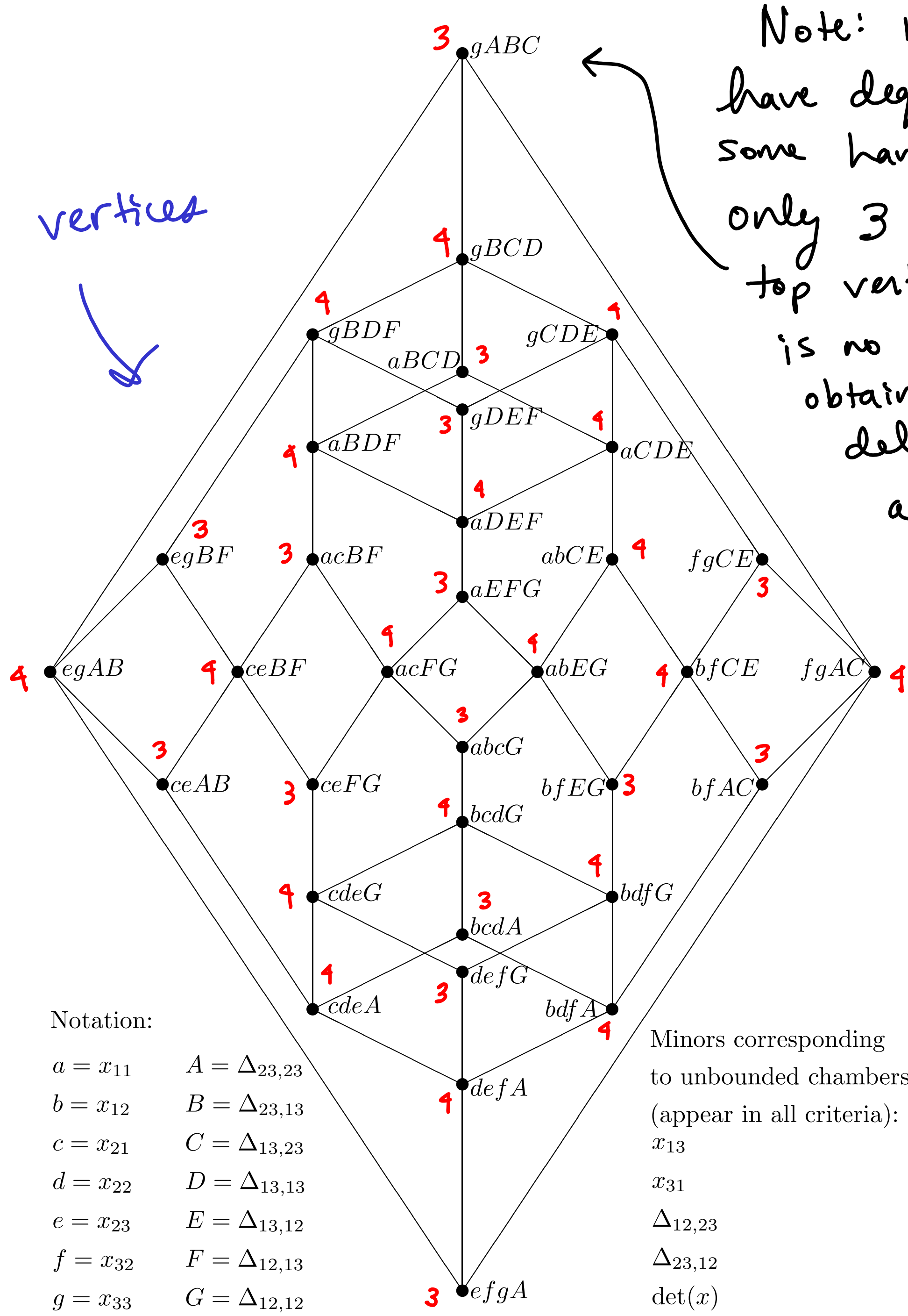
FIGURE 8. Total positivity criteria for GL_3

two arrangements $\mathrm{Arr}(\mathbf{i})$ and $\mathrm{Arr}(\mathbf{i}')$ whose isotopy types are adjacent in the graph

I've drawn in the degree of each vertex.

Note: many have degree 4, but some have degree only 3 — eg for top vertex, there is no other vertex obtained by deleting g & adding a new minor.

34 vertices



Notation:

$a = x_{11}$	$A = \Delta_{23,23}$
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Minors corresponding to unbounded chambers (appear in all criteria):
x_{13}
x_{31}
$\Delta_{12,23}$
$\Delta_{23,12}$
$\det(x)$

FIGURE 8. Total positivity criteria for GL_3

two arrangements Arr($\mathbf{i}$) and Arr($\mathbf{i}'$) whose isotopy types are adjacent in the graph

(End of Lecture)

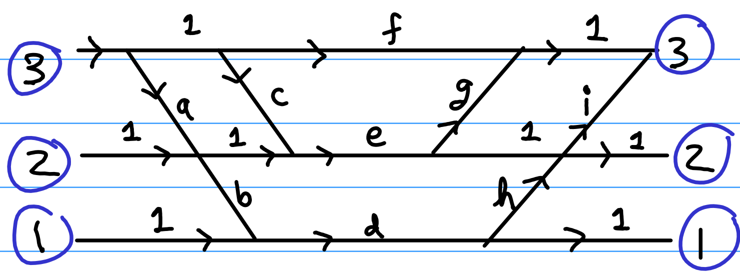
Exercises:

1. For each matrix below, determine whether it is TNN or TP (or neither)

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & 7 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 4 \\ 2 & 4 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 4 & 5 \end{pmatrix}$$

2. Give combinatorial formula for minors of $x(\Gamma)$ in terms of Γ

Here $\Gamma =$



and

$$x(\Gamma) = \begin{pmatrix} d & dh & dhi \\ bd & bdh+e & bdhi+eg+ei \\ abd & abdh+ae+ce & abdhi+(a+c)e(g+i)+f \end{pmatrix}$$

3. Show that the map $\phi: \mathbb{R}^9 \to 3\times 3$ TP matrices
$(a,b,c,\dots,i) \mapsto x(\Gamma)$
is a bijection by inverting ϕ.

4. What network should we use for 4×4 matrices? $n\times n$ matrices?

5. Find other networks that lead to parameterizations of 3x3 (or $n \times n$) TP matrices.

6. Show that a 3×3 matrix M is TP iff the following minors are pos:

$$\Delta_{123,123}(M)$$

$$\begin{array}{cccc} \Delta_{23,12}(M) & \Delta_{13,12}(M) & \Delta_{13,23}(M) & \Delta_{12,23}(M) \\ \Delta_{3,1}(M) & \Delta_{3,2}(M) & \Delta_{1,2}(M) & \Delta_{1,3}(M) \end{array}$$

7. For each edge in the graph on next page, find an algebraic relation involving the two minors that get swapped as well as the common minors from the two "TP tests"

<u>Ex:</u> To relate defA and efgA we have

$$d\,g = A + e \cdot f$$

$$x_{22}\,x_{33} = \Delta_{23,123} + x_{23}x_{32}$$

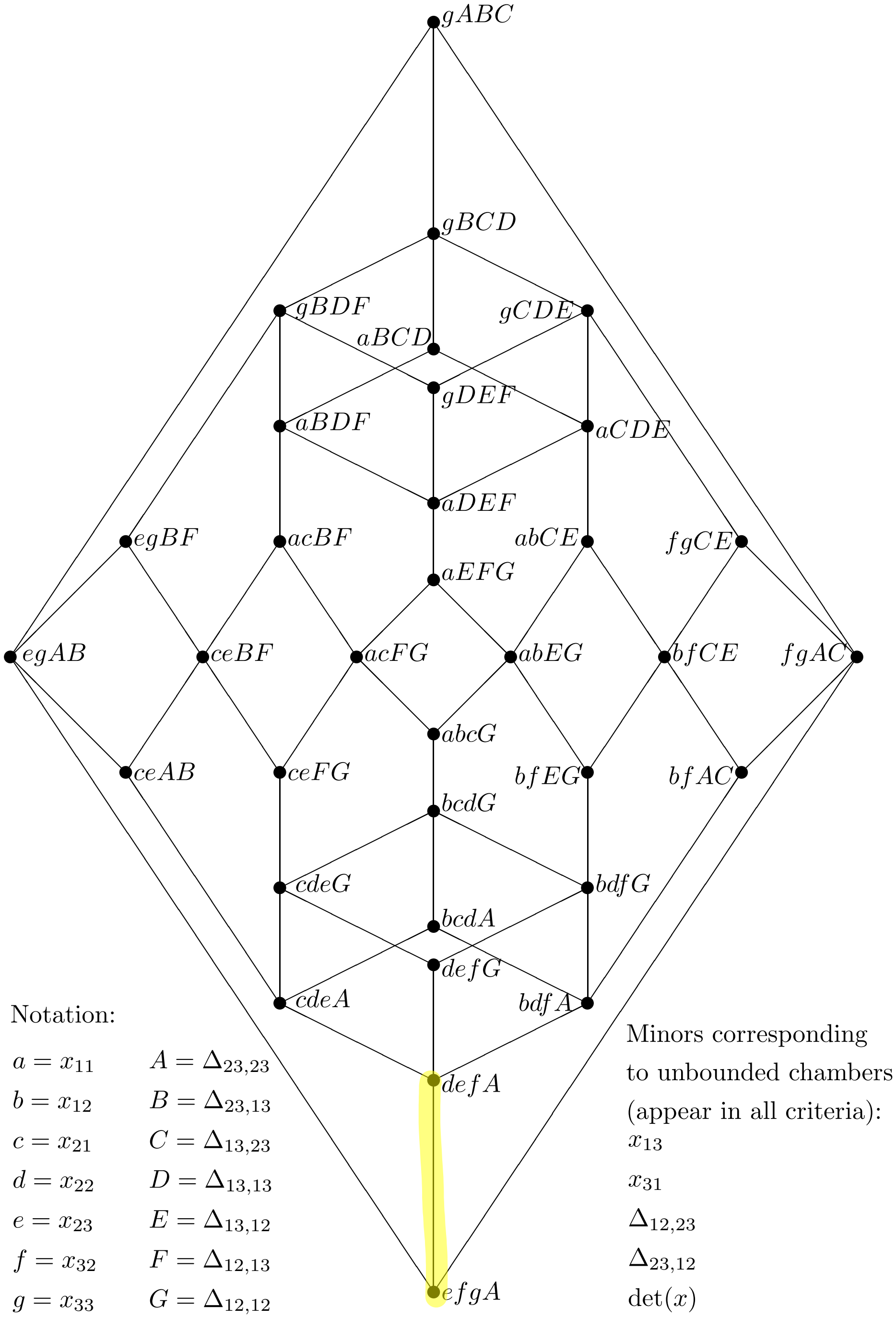


FIGURE 8. Total positivity criteria for GL_3

two arrangements $\mathrm{Arr}(\mathbf{i})$ and $\mathrm{Arr}(\mathbf{i}')$ whose isotopy types are adjacent in the graph