Colombia #2: Total positivity for the Grassmannian

<u>Def</u>: The Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V : V \subseteq \mathbb{R}^n, \dim V = k\}$
(V ← vector space)

<u>Ex</u>: If $k=1$, $Gr_{1,n}(\mathbb{R}) = \{\text{lines in } \mathbb{R}^n\} = \text{proj. space } \mathbb{P}^{n-1}$.
Can represent line as span of vector $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$
<u>Note</u>: For $\lambda \neq 0$, $\text{span}(v_1, \ldots, v_n) = \text{span}(\lambda v_1, \ldots, \lambda v_n)$.
So these two represent same element in $Gr_{1,n} = \mathbb{P}^{n-1}$.

<u>Ex</u>: $Gr_{2,4}(\mathbb{R})$ = 2-dim'l vector spaces in $\mathbb{R}^4$.
Can represent element by a <u>full</u> <u>rank</u> 2×4 matrix $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$:
the span of the row vectors gives 2-plane in $\mathbb{R}^4$.

But two matrices might span same 2-plane, e.g.
if $A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$ and $g \in GL_2(\mathbb{R})$ ← invertible 2×2 matrix
then
$\text{rowspan}(A) = \text{rowspan}(gA)$

More generally, represent each element of $Gr_{k,n}(\mathbb{R})$ as a full rank $k \times n$ matrix A.
Rows of A span k-dim'l subspace of $\mathbb{R}^n$.
$A \sim A'$ in $Gr_{k,n}(\mathbb{R})$ if $\text{rowspan}(A) = \text{rowspan}(A')$.

So: $Gr_{k,n}(\mathbb{R}) = GL_k(\mathbb{R}) \backslash \{\text{Full rank } k\times n \text{ matrices}\}$

Given a $k \times n$ matrix A, and $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ denote the $(k \times k)$ minor of A which uses column set I.

<u>Plucker coordinates:</u>

Let A be $k \times n$ matrix representing element of $Gr_{k,n}(\mathbb{R})$.
Let $I \in \binom{[n]}{k}$ ← a k-element subset of $[n] = \{1, 2, \dots, n\}$.

Define $\Delta_I(A)$ = det of $k \times k$ submatrix of A which uses columns I.

E.g. if $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$ and

$I = \{1, 3\}$ then $\Delta_I(A) = \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}$ ↙ called <u>Plucker coordinates</u>

Let $Pl : Gr_{k,n}(\mathbb{R}) \to \mathbb{P}^{\binom{n}{k}-1}$

$$A \mapsto \{\Delta_I(A)\}_{I \in \binom{[n]}{k}}$$

<u>Ex:</u> $Pl : Gr_{2,4}(\mathbb{R}) \to \mathbb{P}^5$

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \mapsto \left(\Delta_{12}(A), \Delta_{13}(A), \Delta_{14}(A), \Delta_{23}(A), \Delta_{24}(A), \Delta_{34}(A)\right)$$

<u>Q:</u> Why is $Pl(Gr_{k,n}) \subseteq \mathbb{P}^{\binom{n}{k}-1}$ as opposed to $\mathbb{R}^{\binom{n}{k}}$

In other words, why is at least one Plucker coord $\neq 0$?
(because elements of $Gr_{k,n}$ come from full rank matrices)

Prop: The map $Pl: Gr_{k,n}(\mathbb{R}) \to \mathbb{P}^{\binom{n}{k}-1}$ is an embedding.

In particular, given a collection of Plucker coordinates, we can uniquely recover the corresponding element of $Gr_{k,n}(\mathbb{R})$.
(note that the matrix representative need not be unique)

Note: There are Plucker relations among the Plucker coordinates.

Ex: Let $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{R}$

represent element of $Gr_{24}(\mathbb{R})$.
Its Plucker coordinates are

$\Delta_{12}(A) = 1$ $\qquad$ $\Delta_{13}(A) = c$

$\Delta_{14}(A) = d$ $\qquad$ $\Delta_{23}(A) = -a$

$\Delta_{24}(A) = -b$ $\qquad$ $\Delta_{34}(A) = ad - bc$

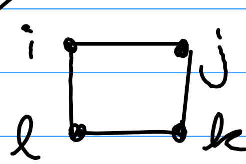
They satisfy the relation $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$

$c \cdot (-b) = 1 \cdot (ad - bc) + d \cdot (-a)$ ✓

More generally, if $A \in Gr_{2,n}$, then the minors of A satisfy the 3-term Plucker relations:
for any $i < j < k < l$,

$$\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$$

Mnemonic



And if $A \in Gr_{m,n}$ then the minors satisfy: for any $i<j<k<l$ and $(m-2)$ subset I of $\{1,\ldots,n\}$ disjoint from i,j,k,l:

$$\Delta_{I\cup\{i,k\}}\Delta_{I\cup\{j,l\}} = \Delta_{I\cup\{i,j\}}\Delta_{I\cup\{k,l\}} + \Delta_{I\cup\{i,l\}}\Delta_{I\cup\{j,k\}}$$

Total positivity?

__Def:__ The TNN Grassmannian $(Gr_{kn})_{\geq 0}$ (resp the TP " " $(Gr_{kn})_{>0}$) is the subset of Gr_{kn} that can be represented by full rank $k \times n$ matrices A s.t. all $\Delta_I(A) \geq 0$ (resp >0).

In previous example, in order for $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ to represent element of $(Gr_{2n})_{>0}$,

we need:

$\Delta_{12}(A) = 1 > 0$ $\quad\quad$ $\Delta_{13}(A) = c > 0$

$\Delta_{14}(A) = d > 0$ $\quad\quad$ $\Delta_{23}(A) = -a > 0$

$\Delta_{24}(A) = -b > 0$ $\quad\quad$ $\Delta_{34}(A) = ad - bc > 0$.

__Q:__ How many minors does one need to test (& which minors) to determine if some $A \in Gr_{kn}(\mathbb{R})$ lies in $(Gr_{kn})_{>0}$?

Recall that $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
So if $\Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23}$ and Δ_{24} are >0, then so is Δ_{13}.

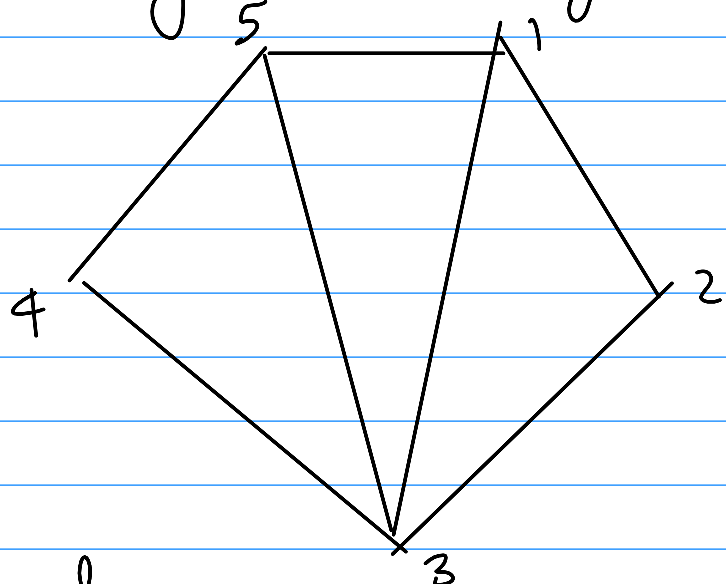
$\therefore \{\Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23}, \Delta_{24}\}$ is a TP test for membership in $(Gr_{24})_{>0}$.

Another TP test? $\{\Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23}, \Delta_{13}\}$

What about $(Gr_{2n})_{>0}$?

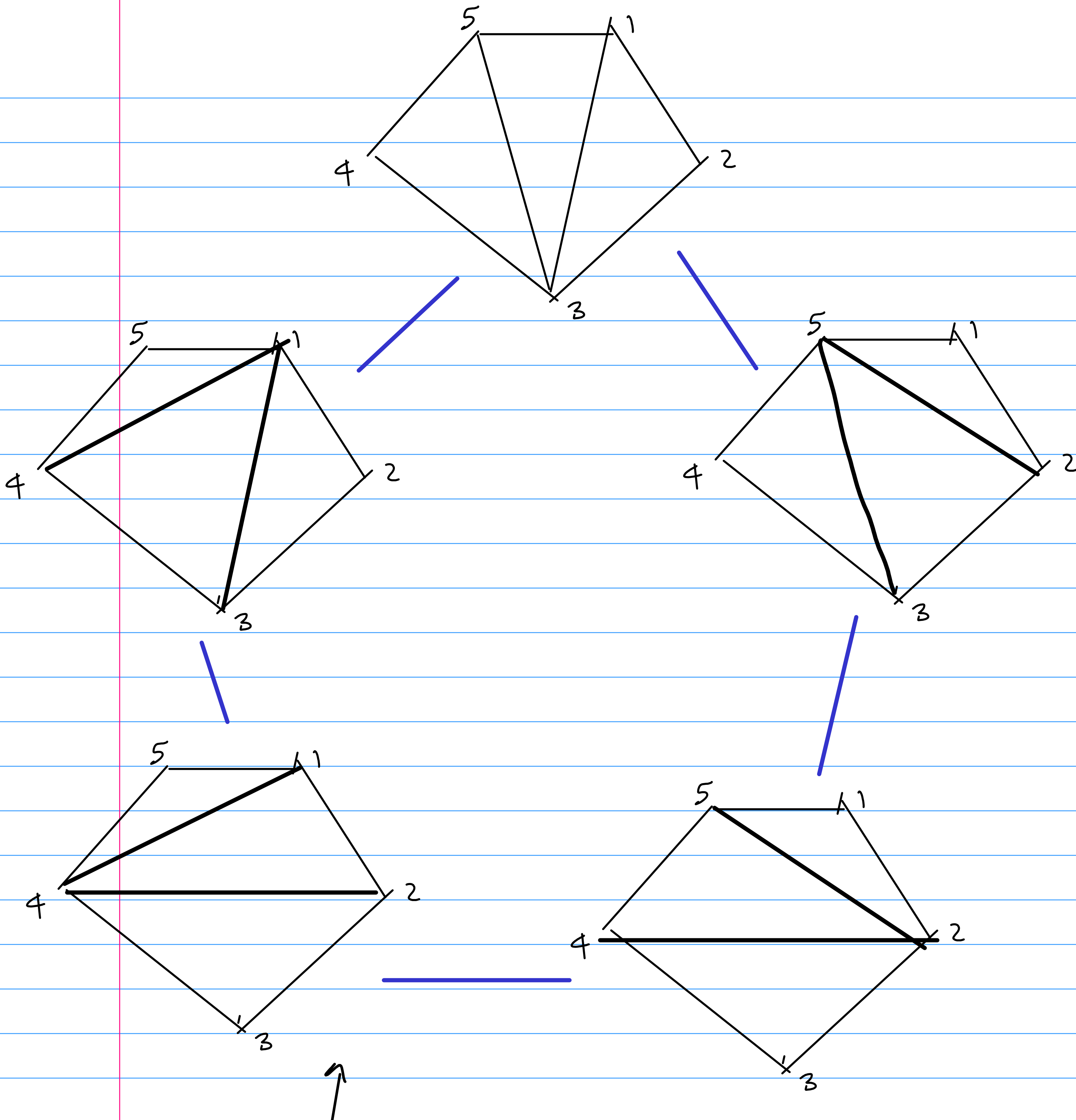
First, consider triangulation of n-gon.

Ex: $(n=5)$



Identify sides & diagonals of n-gon w/ Plücker coord's

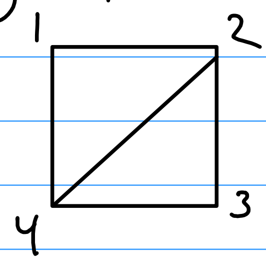
We can do local move on triangulation where we take some internal quadrilateral inside & replace its diagonal w/ the other diag.
"flip"



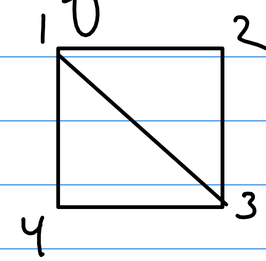
The "flip graph" of triangulations of a 5-gon

Prop: Given T a triangulation of an n-gon, let $Pl(T)$ denote the corresponding Plücker coordinates of $Gr_{2,n}$.
Then $Pl(T)$ is a TP test for membership in $(Gr_{2,n})_{>0}$.

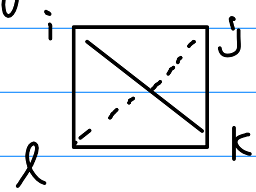
Ex: For $Gr_{2,4}$, the TP tests we found earlier correspond to



and



Note: Flipping diagonals in quadrilateral



is like using relation $\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$ to replace Δ_{ik} w/ Δ_{jl} (or vice versa) in a TP test.

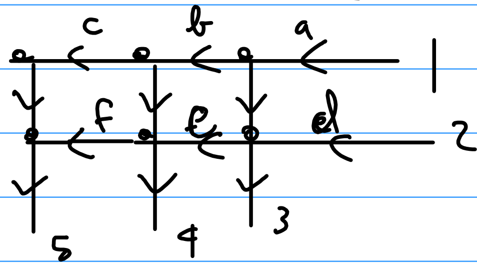
Prop follows from fact that we can get from any triangulation to any other by performing sequence of flips!

And that if $\Delta_{ij}, \Delta_{kl}, \Delta_{il}, \Delta_{jk}$ and Δ_{ik} are pos, so is Δ_{jl}.

Q: How can we write down all elements of $(Gr_{k,n})_{>0}$?

Note: By acting by Gl_k, can always choose a matrix representative that has $k \times k$ identity matrix at left.

Draw following grid (shown for $k=2, n=5$) along w/ weights on horiz edges. (wts on vert edges are 1).



Let weight of path $w(p) := $ prod of edges weights.

→ network matrix where (i,j) entry is

$$\pm \sum_{\substack{\text{paths } p \\ i \to j}} w(p)$$

determinant

Here get

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & a & a(b+e) & a(bc+bf+ef) \\ 2 & 0 & 1 & -d & -de & -def \end{array}$$

Prop: The map $\varphi : (\mathbb{R}_{>0})^6 \to Gr_{2,5}(\mathbb{R})$ is injection $(a,b,\dots,f) \mapsto \text{row span}(\text{network matrix})$ whose image is $(Gr_{2,5})_{>0}$

Same holds for $Gr_{k,n}$

This gives a parameteriz. of $(Gr_{k,n})_{>0}$

Exercises

1. Find the element of $Gr_{24}(\mathbb{R})$ w/ Plucker coordinates $(\underset{\Delta_{12}}{1}, \underset{\Delta_{13}}{3}, \underset{\Delta_{14}}{4}, \underset{\Delta_{23}}{-2}, \underset{\Delta_{24}}{-1}, \underset{\Delta_{34}}{5})$

2. Consider the map $Pl : Gr_{2,n} \to \mathbb{P}^{\binom{n}{2}-1}$. Given $Pl(A)$ for some $A \in Gr_{2,n}$, how can we reconstruct A?

3. Prove that for $A \in Gr_{2,n}$ and for any $i<j<k<l$,
$$\Delta_{ik}(A)\,\Delta_{jl}(A) = \Delta_{ij}(A)\,\Delta_{kl}(A) + \Delta_{il}(A)\,\Delta_{jk}(A)$$

4. Draw the flip graph of the triangulations of a 6-gon.

5. Prove that given T a triangulation of an n-gon, the corresponding Plucker coords $Pl(T)$ are a TP test for membership in $(Gr_{2n})_{>0}$

6. Prove that the $\varphi : (\mathbb{R}_{>0})^6 \to Gr_{2,5}(\mathbb{R})$

$$(a, b, \dots, f) \mapsto \text{rowspan} \begin{pmatrix} 1 & 0 & a & a(b+e) & a(bc+bf+ef) \\ 0 & 1 & -d & -de & -def \end{pmatrix}$$

has image in $(Gr_{2,5})_{>0}$.

7. Prove that φ is an injection by providing inverse map $\varphi^{-1} : (Gr_{2,5})_{>0} \to (\mathbb{R}_{>0})^6$

8. If #6, 7 too easy, do the same for $(Gr_{3,6})_{>0}$ or $(Gr_{k,n})_{>0}$