Hints to exercises – Polynomial Optimization

June 15, 2018

2) We record the cut induced by $S \subseteq [n]$ by the vector $\chi^S \in \{-1,1\}^n$ defined as:

$$(\chi^S)_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S \end{cases}$$

What set of polynomial equations have as their solutions exactly these vectors χ^S? How do cut edges look in your model? What polynomial maximizes the size of a cut in your model?

5b,c) It might help to remember that all symmetric matrices can be diagonalized as $M = UDU^\top$ where U is orthogonal and D is a diagonal matrix with $D_{ii} = \lambda_i(M)$, the ith eigenvalue of M. If the columns of U are $u_1, \dots, u_n$ then this means that

$$M = \sum \lambda_i u_i u_i^\top$$

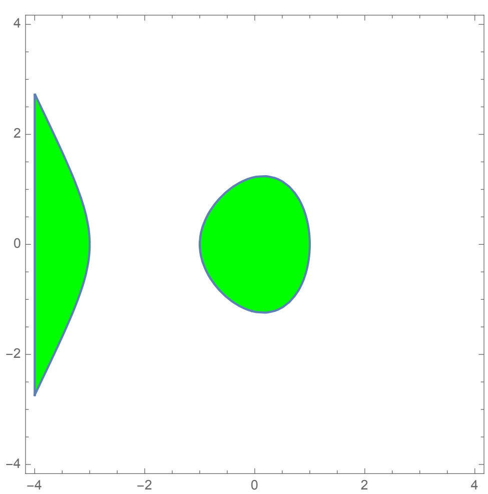

Figure 1: Problem 6d ii): The region satisfying $-x^3 - 3x^2 - 2y^2 + x + 3 \geq 0$

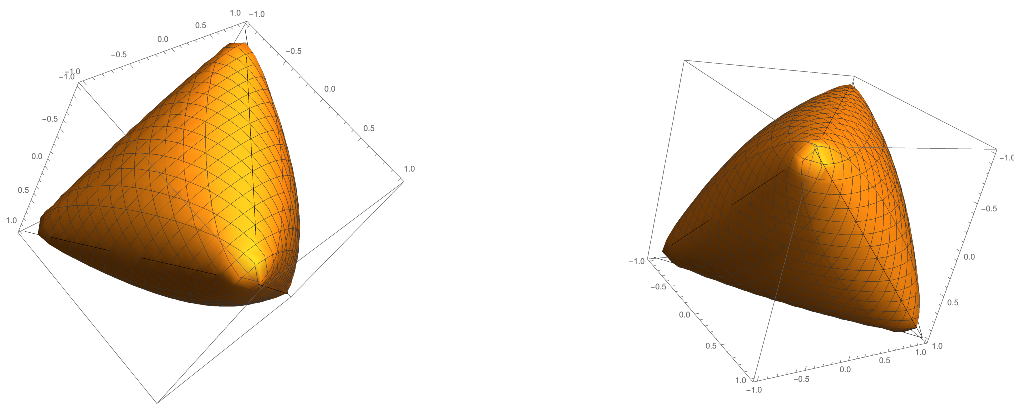

Figure 2: Problem 7b): Two views of the elliptope.

7e) Recall that we were modeling the cut induced by $S \subseteq [n]$ by assigning 1 to vertices in S and -1 to vertices not in S. Let $v(T)$ be the ± 1 vector in $\mathbb{R}^n$ so obtained. Then $X = v(T)v(T)^\intercal \in \mathcal{E}_n$.

10) The following Q will work:

$$Q = \frac{1}{3}\begin{pmatrix} 6 & 3 & 0 & -2 & 0 & -2 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 6 & 3 & -4 \\ 0 & 0 & 0 & 3 & 5 & 0 \\ -2 & 0 & 0 & -4 & 0 & 15 \end{pmatrix}$$

Now we need to factorize $Q = BB^\intercal$ to get the sos expression for p. This also requires a computer. But the following sos expression works:

$$p = \frac{4}{3}y^2 + \frac{1349}{705}y^4 + \frac{1}{12}(4x+3)^2 + \frac{1}{15}(3x^2+5xy)^2 + \frac{1}{315}(-21x^2+20y^2+10)^2 + \frac{1}{59220}(328y^2-235)^2.$$

What is B in this case? Check that $Q = BB^\intercal$.

We now do this example using Macaulay2 using the package SOS.m2:

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/Applicati
```

```
o1 = SOS

o1 : Package

i2 : R = QQ[x,y]

o2 = R

o2 : PolynomialRing

i3 : f = 2*x^4+5*y^4-x^2*y^2+2*x^3*y+2*x+2     ---- input the polynomial

           4      3      2 2       4
o3 = 2x  + 2x y - x y  + 5y  + 2x + 2

o3 : R

i4 : (Q,mon,X) = solveSOS(f, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-66664-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-66664-0/1
Success: SDP solved

i5 : s = sosdec(Q,mon)

o5 = coeffs:
         11  17  1912  2083   1313
     {5, --, --, ----, ----, -----}
          5  11  2125  1912  10415
     gens:
         8 2    2   1    1   5 2          5     5     5 2   11          5   2    55
     {- --x  + y  - -x - -, --x  + x*y + --y - --, - --x  + --x + y + --, x  - ----
        25          5    5  11          11    11    17     17         17       1912

o5 : SOSPoly

--- the above output is fine on the computer but doesn't make sense as shown above,
a fortran type output with the command "toString". "oo" means the output that just

i6 : toString oo

o6 = new SOSPoly from {ring => R,
coefficients => {5, 11/5, 17/11, 1912/2125,   2083/1912, 1313/10415},
```

```
generators => {-(8/25)*x^2+y^2-(1/5)*x-1/5, (5/11)*x^2+x*y+(5/11)*y-5/11,
                             -(5/17)*x^2+(11/17)*x+y+5/17, x^2-(55/1912)*x-705/1912, (9

-- check if the above sos is indeed the polynomial we started with.

i7 : sumSOS(s)

        4     3      2 2     4
o7 = 2x  + 2x y - x y  + 5y  + 2x + 2

o7 : R
```

11) Prove that a univariate non-negative polynomial is always a sum of two squares. (Hint: Recall that all real roots of a nonnegative polynomial are double roots and all complex roots come in conjugate pairs. Then use the identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$ for all $a, b, c, d \in \mathbb{R}$.)

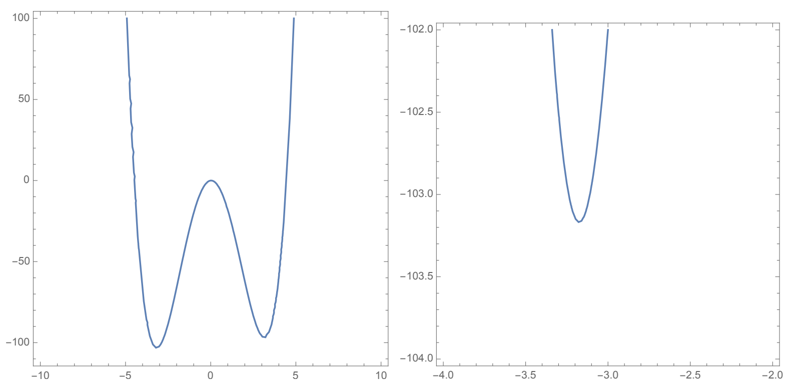

Figure 3: Problem 13) The graph of $y = x^4 - 20x^2 + x$, and a zoomed in view of the minimum.

13) We now use M2 to do part c) accurately.

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )

i2 : R = QQ[x,t];

i3 : f2 = x^4 - 20*x^2 + x;
```

```
i4 : (Q,mon,X,tval) = solveSOS(f2-t,{t},-t, Solver=>"CSDP");

i5 : tval

          1651
o5 = {- ----}
           16

o5 : List

-- tval is the minimum value and it is roughly -103.1875

i6 : toString Q

o6 = matrix {{1651/16, 1/2, -807/80}, {1/2, 7/40, 0}, {-807/80, 0, 1}}

i7 : toString mon

o7 = matrix {{1}, {x}, {x^2}}
```

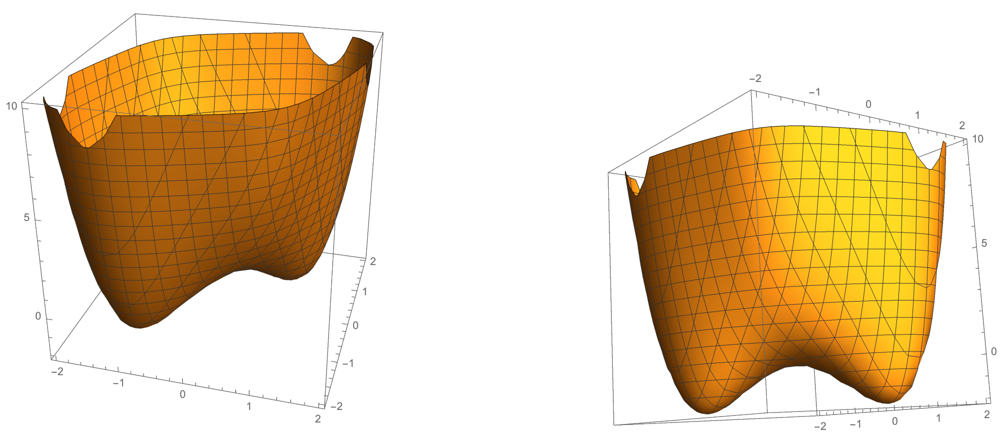

Figure 4: Problem 14b): The graph of $z = x^4 + y^4 - 4xy$ (two different views).

14) (a) (Ex 3.57) Find the value of p_{sos} for the polynomial $p(x, y, z) = x^4 + y^4 + z^4 - 4xyz + 2x + 3y + 4z$ over $\mathbb{R}^3$. Is $p_* = p^{\text{sos}}$ in this example? Do you expect $p_* = p^{\text{sos}}$?

(b) Find the value of p_{sos} and p_* for the polynomial $p(x, y) = x^4 + y^4 - 4xyz$ over $\mathbb{R}^2$. Do you expect $p_* = p^{\text{sos}}$?

In M2 we use the following commands:

```
needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
R = QQ[x,y,z,t]
p = x^4+y^4+z^4 - 4*x*y*z + 2*x + 3*y + 4*z
(Q,mon,X,tval) = solveSOS(p-t,{t},-t, Solver=>"CSDP");
```

This gives the minimum value $tval = -\frac{115}{16} = -7.1875$. This the value of p^{sos}.
FIND THE TRUE MINUMUM OF THIS POLYNOMIAL.

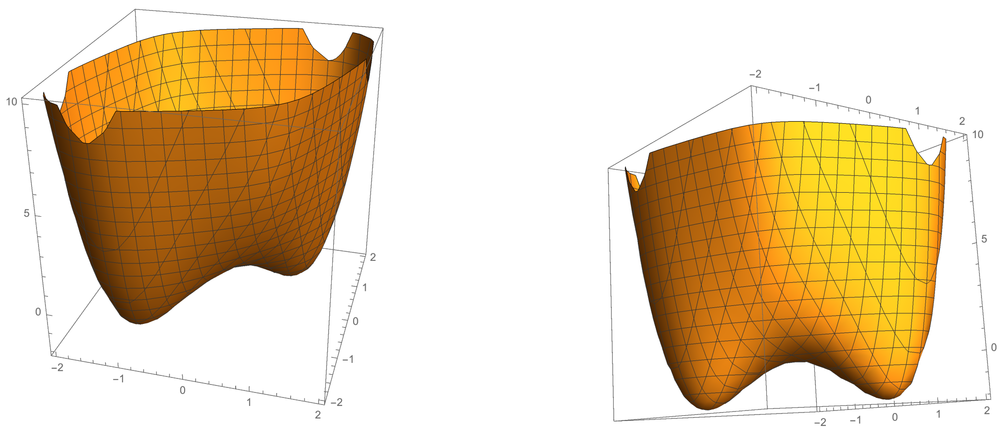

Figure 5: The graph of $z = x^4 + y^4 - 4xy$ (two different views).

```
needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
R = QQ[x,y,t]
p = x^4+y^4 - 4*x*y
(Q,mon,X,tval) = solveSOS(p-t,{t},-t, Solver=>"CSDP");

i5 : tval

o5 = {-2}

i6 : toString Q

o6 = matrix {{2, 0, 0, -1/2, -1, -1/2}, {0, 1, -1, 0, 0, 0}, {0, -1, 1,
     0, 0, 0}, {-1/2, 0, 0, 1, 0, -1/2}, {-1, 0, 0, 0, 1, 0}, {-1/2, 0,
     0, -1/2, 0, 1}}

i7 : toString mon

o7 = matrix {{1}, {x}, {y}, {x^2}, {x*y}, {y^2}}
```

```
-- This means that the polynomial p+2 must be a sos. We can get its
sos decomposition using:

(Q,mon,X) = solveSOS(p+2, Solver=>"CSDP");
(g,d) = sosdec(Q,mon);
i12 : toString oo

o12 = ({-(1/4)*x^2-(1/2)*x*y-(1/4)*y^2+1, x-y, x^2-(2/7)*x*y-(5/7)*y^2,
      x*y-y^2},{2, 1, 7/8, 3/7})

-- compute the sos to check if we get back the polynomial p+2

i13 : sumSOS(g,d)

       4    4
o13 = x  + y  - 4x*y + 2

-- The matrix Q might be interesting to look at:
i14 : toString Q

o14 = matrix {{2, 0, 0, -1/2, -1, -1/2}, {0, 1, -1, 0, 0, 0}, {0, -1,
      1, 0, 0, 0}, {-1/2, 0, 0, 1, 0, -1/2}, {-1, 0, 0, 0, 1, 0},
      {-1/2, 0, 0, -1/2, 0, 1}}

 i15 : toString mon

o15 = matrix {{1}, {x}, {y}, {x^2}, {x*y}, {y^2}}
```

$$Q = \begin{pmatrix} 2 & 0 & 0 & -1/2 & -1 & -1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1 & 0 & -1/2 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & -1/2 & 0 & 1 \end{pmatrix}$$

16) Consider the quartic form in four variables:

$$p(w,x,y,z) = w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz.$$

(a) Find a multiplier that makes the product a sos.

We do this in M2 and see that multiplying p by $w^2 + x^2 + y^2 + z^2$ makes it sos.

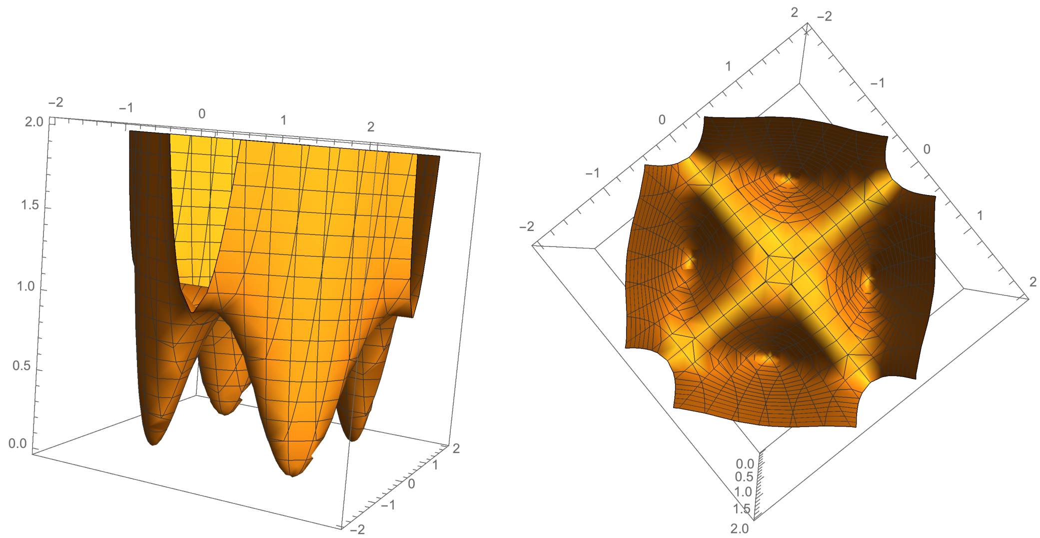

Figure 6: Graph of the Motzkin polynomial

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/
Library/Application Support/Macaulay2/init-SOS.m2

o1 = SOS

o1 : Package

i2 : R = QQ[w,x,y,z]

o2 = R

o2 : PolynomialRing

i3 : p = w^4 + x^2*y^2 + x^2*z^2 + y^2*z^2 - 4*w*x*y*z

      4    2 2                  2 2    2 2
o3 = w  + x y  - 4w*x*y*z + x z  + y z

o3 : R

i4 : m = w^2+x^2+y^2+z^2;

i5 : p1 = m*p;
```

```
i6 : (Q,mon,X) = solveSOS(p1, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-59042-0/(
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-59042-0/1
Success: SDP solved
rounding failed, returning real solution

-- this means that the solver failed to return a rational Gram matrix.

i7 : (g,d) = sosdec(Q,mon)

                 2            2                      2
o7 = ({- .142857w y - .428571x y + w*x*z - .428571y*z , w*x*y -
     ------------------------------------------------------------------
             2            2            2             2             2
     .142857w z - .428571x z - .428571y z, - .142857w x - .428571x*y  +
     ------------------------------------------------------------------
                       2            3           2         2       2
     w*y*z - .428571x*z , - .999999w  + x*y*z, w x - .5x*y  - .5x*z ,
     ------------------------------------------------------------------
      3   2       2         2   2        2            2     2
     w , w y - .5x y - .5y*z , x y - 1y*z , - .833333w z + x z -
     ------------------------------------------------------------------
             2      2       2   2      2      2   2      2
     .166667y z, x*y  - 1x*z , w z - 1y z, x*z , y z, y*z }, {2.33333,
     ------------------------------------------------------------------
     2.33333, 2.33333, 1, .952381, 4.7892e-8, .952381, .333333,
     ------------------------------------------------------------------
     .571429, .333333, .555556, 7.37129e-9, 7.37129e-9, 7.37129e-9})

o7 : Sequence

i8 : sumSOS(g,d)

       6     4 2     4 2     2 2 2     4 2     2 4     3
o8 = 1w  + 1w x  + 1w y  + 1w x y  + 1x y  + 1x y  - 4w x*y*z -
     ------------------------------------------------------------------
        3            3     4 2     2 2 2     4 2     2 2 2    2 2 2
     4w*x y*z - 4w*x*y z + 1w z  + 1w x z  + 1x z  + 1w y z  + 3x y z
     ------------------------------------------------------------------
        4 2           3     2 4     2 4
     + 1y z  - 4w*x*y*z  + 1x z  + 1y z
```

```
o8 : RR  [w, x, y, z]
       53

i9 : p1

      6    4 2    4 2    2 2 2    4 2    2 4     3            3
o9 = w  + w x  + w y  + w x y  + x y  + x y  - 4w x*y*z - 4w*x y*z -
     ------------------------------------------------------------------
           3     4 2    2 2 2    4 2    2 2 2     2 2 2    4 2
     4w*x*y z + w z  + w x z  + x z  + w y z  + 3x y z  + y z  -
     ------------------------------------------------------------------
             3    2 4    2 4
     4w*x*y*z  + x z  + y z

o9 : R
```

The polynomial output by sumSOS has real coefficients and is in a different ring from p. Nevertheless, comparing its output with mp shows that they are the same.

18c) Use your method to minimize the polynomial $10 - x^2 - y$ over the unit circle $x^2 + y^2 = 1$.

Using M2:

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/Applicati

o1 = SOS

o1 : Package

i2 : R=QQ[x,y];

i3 : f = 10-x^2-y   -- objective function

        2
o3 = - x  - y + 10

o3 : R

i4 : h = {x^2 + y^2-1}   -- constraints

       2    2
o4 = {x  + y  - 1}
```

```
o4 : List

i5 : d=2                -- degree bound

o5 = 2

i6 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/0
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/1
Success: SDP solved

      17
o6 = (--, {})
       2

o6 : Sequence

i7 : d= 4  -- degreebound

o7 = 4

i8 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/4
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/5
Success: SDP solved

      35
o8 = (--, {})
      4
```

19) Show that the polynomial x^4-3x^2+1 is nonnegative on the variety defined by $x^3-4x = 1$.

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )

i2 : R = QQ[x]

i3 : f = x^4-3*x^2+1

      4     2
o3 = x  - 3x  + 1

o3 : R
```

```
i4 : h = {x^3-4*x-1}

       3
o4 = {x  - 4x - 1}

o4 : List

i5 : d = 4

o5 = 4

i6 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-64090-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-64090-0/1
Success: SDP solved

      3
o6 = (-, {x => -.254102})
      4

o6 : Sequence
```