The Pythagoras numbers of projective varieties.

Grigoriy Blekherman (Georgia Tech, USA)
Rainer Sinn (Max-Planck-Institute, Germany)
Gregory G. Smith (Queen's University, Canada)
Mauricio Velasco* (Universidad de los Andes, Colombia)

Encuentro Colombiano de Combinatoria (ECCO 2018).

Outline.

1. Motivation: The Burer - Monteiro approach to SDP.
2. The Pythagoras number of a projective variety.
3. Lower bounds on the Pythagoras number via projections (**Quadratic persistence**).
4. Upper bounds on Pythagoras number via inclusions (**Algebraic Treewidth**).
5. Applications: Examples and some classification Theorems for projective varieties with small Pythagoras numbers.

Two problems:

Let $f \in \mathbb{R}[X_0, \dots, X_n]$ be a homogeneous polynomial of degree $2d$.

How to prove that f is nonnegative in $\mathbb{R}^n$?

Solving this problem would have lots of applications in optimization, probability, math. finance, machine learning, etc. (see for instance Lasserre's book "Moments, positive polynomials and their applications"). It is in general very hard.

Example.

Is the following polynomial f nonnegative in $\mathbb{R}^2$?
$10x^6 - 4x^5y + 2x^4y^2 + 50x^4 - 14x^3y - 4x^3 + 4x^2y + 65x^2 - 14x + 2$

Example.

Is the following polynomial f nonnegative in $\mathbb{R}^2$?
$10x^6-4x^5y+2x^4y^2+50x^4-14x^3y-4x^3+4x^2y+65x^2-14x+2$

Solution. $f=(1+x+x^3+x^2y)^2+(1-8x-3x^3+x^2y)^2.$

SOS certificates.

Definition.

A homogeneous polynomial f of degree $2d$ is a sum-of-squares (SOS) if there exist polynomials $g_1, \ldots, g_k$ of degree d such that

$$f = g_1^2 + g_2^2 + \cdots + g_k^2.$$

If f is indeed SOS, how do we find such an expression?

Reminder – PSD matrices

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

Definition.

A is positive semidefinite, denoted $A \succeq 0$ if for every $x \in \mathbb{R}^n$ the number $x^t A x \geq 0$.

Lemma.

The following conditions are equivalent for a symmetric matrix A:

1. $A \succeq 0$
2. *All eigenvalues of A are nonnegative real numbers.*
3. *There exists a matrix Y such that $A = Y^t Y$.*

SOS certificates

Definition.

A homogeneous polynomial f of degree $2d$ is a sum-of-squares (SOS) if there exist polynomials $g_1, \ldots, g_k$ of degree d such that

$$f = g_1^2 + g_2^2 + \cdots + g_k^2.$$

If f is indeed SOS, how do we find such an expression?

Observation *[Parrilo - 2000s] Let $\vec{m}$ be the vector of all monomials of degree at most d. Then f is SOS iff there exists a positive semidefinite matrix $A \succeq 0$ such that*

$$f = \vec{m}^t A \vec{m}.$$

Finding SOS certificates

If f is indeed SOS, how do we find such an expression?

Observation. *[Parrilo - 2000s] Let $\vec{m}$ be the vector of all monomials of degree at most d. Then f is SOS iff there exists a positive semidefinite matrix $A \succeq 0$ such that*

$$f = \vec{m}^t A \vec{m}.$$

- The computational complexity of this problem is unknown but such identites can be found numerically in polynomial time (advantage) and often rounded to prove the existence of exact equalities.
- (Disadvantage) Even numerical solutions to the above problem can only be computed when the matrix is relatively small (max 10000×10000 using state of the art augmented Lagrangian solvers).

The Burer-Monteiro approach

The dimensions of A are $\binom{n+d}{d} \times \binom{n+d}{d}$.

A new approach:

1. *Replace our PSD matrix A with a factorization $A = Y^t Y$ where Y has size $r \times \binom{n+d}{d}$ where r is the smallest rank for which the identity has a solution and*
2. *Use non-convex (i.e. local) optimization algorithms to* $\min_Y \|f - \vec{m}^t Y^t Y m\|$

- (Advantage) The above problem has a much smaller search space for Y. Although non-convex it often finds global minima [Bumal, Voroninski, Bandeira, 2018].
- (Key point) Can only be useful if we know an a-priori bound for r.

Problem: Burer-Monteiro bounds.

How to bound r to use the Burer-Monteiro approach?

Given a SOS $f \in \mathbb{R}[X_0, \ldots, X_n]_{2d}$ find a bound for

$$r(f) = \min \left\{ k : \exists A \succeq 0 \text{ with } \operatorname{rank}(A) = k \text{ and } f = \vec{m}^t A \vec{m} \right\}$$

Distance matrix completion

$$\begin{pmatrix} 0 & d_{12} & d_{13} & * \\ d_{12} & 0 & * & d_{24} \\ d_{13} & * & 0 & d_{34} \\ * & d_{24} & d_{34} & 0 \end{pmatrix}$$

We are given (partial) pairwise distances between n items (with $d_{ij} \in [0, \pi]$). Find an $\mathbb{R}^r$ and a collection of vectors $u_1, \ldots, u_4$ in the sphere which realizes the observed distances (i.e. For which $\cos(d_{ij}) = u_i \cdot u_j$).

Distance matrix completion

$$\begin{pmatrix} 0 & d_{12} & d_{13} & * \\ d_{12} & 0 & * & d_{24} \\ d_{13} & * & 0 & d_{34} \\ * & d_{24} & d_{34} & 0 \end{pmatrix}$$

We are given (partial) pairwise distances between n items (with $d_{ij} \in [0, \pi]$). Find an $\mathbb{R}^r$ and a collection of vectors $u_1, \ldots, u_4$ in the sphere which realizes the observed distances (i.e. For which $\cos(d_{ij}) = u_i \cdot u_j$).

Solution. *Let B be the entrywise cosine. If A is a PSD completion of B and $A = Y^t Y$ with $Y \in \mathbb{R}^{r \times n}$ then the columns of Y are a solution.*

We could find A with the Burer-Monteiro approach, if only we had an upper bound for r.

Problem 2:

How to bound r to use the Burer-Monteiro approach for distance matrix completion?

Let G be the graph on [n] whose edges are given by the observed positions in the matrix and a partially specified matrix B find an upper bound for

$$r(B) = \min\{k : \exists A \succeq 0 \ \mathrm{rank}(A) = k \textit{ and } A \textit{ completes } B(\}$$

Two problems

Problem 1. *Given a SOS $f \in \mathbb{R}[X_0, \ldots, X_n]_{2d}$ find:*

$$r(f) = \min_{k \in \mathbb{N}} \left\{ \exists A \succeq 0 : \mathrm{rank}(A) = k \text{ and } f = \vec{m}^t A \vec{m} \right\}$$

Problem 2. *Given a graph G and a partial matrix B on G which can be completed to a PSD matrix find*

$$r(B) = \min \{ k : \exists A \succeq 0 \ \mathrm{rank}(A) = k \text{ and } A \text{ completes } B \}$$

Real algebraic varieties

Let $S := \mathbb{R}[X_0, \ldots, X_n]$ and let $I \subseteq S$ be a homogeneous ideal which does not contain any linear form.

The ideal I determines:

1. *A real projective variety*

$$X := V(I) \subseteq \mathbb{P}^n$$

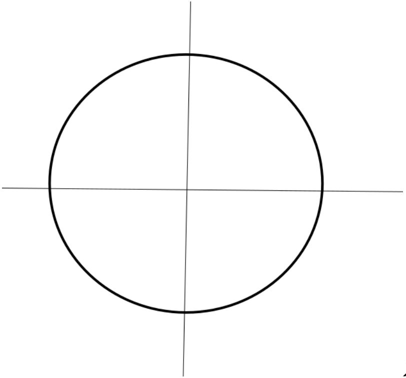
   not contained in any hyperplane.

2. *A graded $\mathbb{R}$-algebra*

$$R := S/I$$

Example

Let $S = \mathbb{R}[X, Y, Z]$, $I = (X^2 + Y^2 - Z^2) \subseteq S$. The projective variety defined by I is the set of common solutions of I in $\mathbb{P}^2$ and the coordinate ring is $R = S/I$.



$x^2 + y^2 - 1 = 0$

The Pythagoras number of a variety X

Let Σ_X be the set of sums of squares of linear forms in R_2,

$$\Sigma_X := \left\{ q \in R_2 : \ \exists k \in \mathbb{N} \text{ and } s_i \in R_1 (q = s_1^2 + \cdots + s_k^2) \right\}$$

Definition.

The sum-of-squares length of $f \in \Sigma_X$ is defined as

$$\ell(f) = \min \left\{ k \in \mathbb{N} : \exists k, g_1, \ldots, g_k \in R_1 \left(f = \sum_{i=1}^{k} g_i^2 \right) \right\}.$$

Definition.

The Pythagoras number of X is defined as $\Pi(X) := \max_{f \in \Sigma} \ell(f)$.

Sanity Check

Example 1. *What is the Pythagoras number of* $\mathbb{P}^n$*?*

Example 2. *Let* $X = V(X^2 + Y^2 - Z^2) \subseteq \mathbb{P}^2$. *What is the Pythagoras number of* X*?*

Sanity Check

Example 1. *What is the Pythagoras number of $\mathbb{P}^n$?*
If $q = x^t Ax$ then by changing coordinates we can write

$$q = \sum_{i=0}^{n} \lambda_i X_i^2 = \sum_{i=0}^{n} \left(\sqrt{\lambda_i} X_i\right)^2$$

so $\Pi(X) = n + 1$.

Example 2. *If $X = V(X^2 + Y^2 - Z^2) \subseteq \mathbb{P}^2$ then, on X*

$$g := X^2 + Y^2 + Z^2 = 2Z^2.$$

so $\ell(g) = 1$. In fact $\Pi(X) = 2$.

Our two problems are the same – Pythagoras numbers.

Problem 1. *Given a SOS $f \in \mathbb{R}[X_0, \ldots, X_n]_{2d}$ find:*

$$r(f) = \min_{k \in \mathbb{N}} \left\{ \exists A \succeq 0 : \operatorname{rank}(A) = k \text{ and } f = \vec{m}^t A \vec{m} \right\}$$

Let $X := \nu_d(\mathbb{P}^n)$, $f \in R_2$ and $r(f) = \ell(f) \leq \Pi(X)$.

Problem 2. *Given a graph G and a partial matrix B on G which can be completed to a PSD matrix find*

$$r(B) = \min \left\{ k : \exists A \succeq 0 \operatorname{rank}(A) = k \text{ and } A \text{ completes } B \right\}$$

Let $I_G = (x_i x_j : (i,j) \notin E)$ and $Z := V(I)$ (Stanley-Reisner ideal of the clique complex of G). Define $b := x^t B x \in R_2$ and note that $r(B) = \ell(b) \leq \Pi(Z)$.

How to compute $\Pi(X)$?

Computing $\Pi(X)$ is rather difficult. In this talk we will:

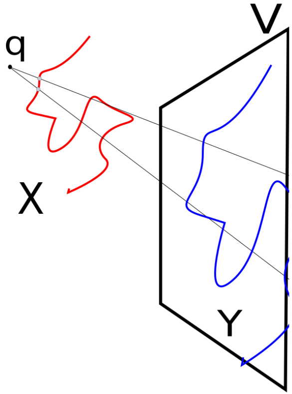
1. Find lower bounds coming from projections (**Quadratic persistence**).
2. Find upper bounds coming from inclusions (**Algebraic Treewidth**).

In some fortunate cases these bounds will match allowing us to determine $\Pi(X)$.

A projection away from a point q

Definition.

If $q \in \mathbb{P}^n$ is a point in projective space and V is a hyperplane not containing q we can define the **projection away from** *q, $\pi_q : \mathbb{P}^n \setminus \{q\} \to V \cong \mathbb{P}^{n-1}$ by sending a point x to the unique point of intersection between the line $\langle q, x\rangle$ and V.*



Properties of projections.

If $q \in X$ is a generic real point and $Y := \pi_q(X)$ then

(**Key property**) *The cone Σ_Y is isomorphic to the face F of Σ_X consisting of sums of squares vanishing at q.*

Properties of projections.

If $q \in X$ is a generic real point and $Y := \pi_q(X)$ then

*(**Key property**) The cone Σ_Y is isomorphic to the face F of Σ_X consisting of sums of squares vanishing at q.*

Consequence. *If $Y = \pi_q(X)$ then $\Pi(X) \geq \Pi(Y)$*

Proof.

$$\Pi(X) = \max_{f \in \Sigma_X} \ell(f) \geq \max_{f \in \Sigma_Y} \ell(f) = \Pi(Y)$$

Lower bounds from projections away from real points.

Lemma. *If $Y = \pi_q(X)$ then $\Pi(X) \geq \Pi(Y)$*

So we would like to keep projecting away from points until we can actually compute the right hand side. This would be very easy if Y was projective space itself...

Definition.

The **quadratic persistence** $\mathrm{qp}(X)$ *of a variety X is the cardinality s of the smallest set of points $q_1, \ldots, q_s$ for which the ideal of the projection $\pi_{\{q_1,\ldots,q_s\}}(X) \subseteq \mathbb{P}^{n-s}$ contains no quadrics.*

A lower bound Theorem

Theorem. (B,S,Si,-)

If $X \subseteq \mathbb{P}^n$ then the following inequalities hold

$$\Pi(X) \geq n - \mathrm{qp}(X) + 1 \geq \dim(X) + 1.$$

Moreover all equalities hold if and only if X is a variety of minimal degree.

The degree of any non-degenerate projective variety $X \subseteq \mathbb{P}^n$ satisfies the inequality

$$\deg(X) \geq \operatorname{codim}(X) + 1.$$

Definition.

$X \subseteq \mathbb{P}^n$ is **of minimal degree** *if the equality holds*

The classification of varieties of minimal degree in projective spaces is known since the 1880s [Castelnuovo, Del Pezzo]. They are cones over

1. A quadric hypersurface or
2. The Veronese surface $\nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ or
3. A rational normal scroll, the projective toric variety corresponding to a Lawrence prism with heights $(a_0, \ldots, a_n)$.

Upper bounds from inclusions

If $X \subseteq Y$ *then* $\Pi(X) \leq \Pi(Y)$.

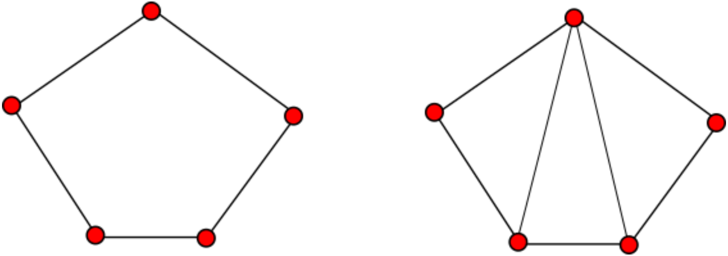
This would be very useful if we could compute, or even just bound, the right hand side...

Chordal graphs and combinatorial treewidth

Let G be an undirected, loopless graph.

Definition.

A graph G is chordal if it does not contain induced cycles of length $\ell \geq 4$.

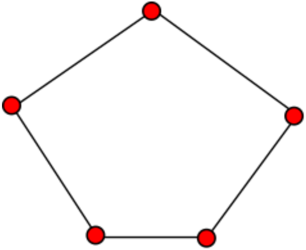


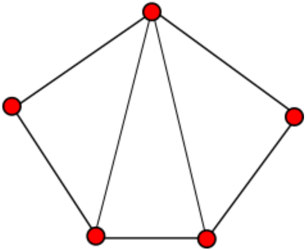
Chordal graphs are important since, on them, it is possible to solve typically NP-hard problems (largest clique size, chromatic number,etc.) in polynomial time.

Chordal graphs and combinatorial treewidth

Definition.

A graph C is a chordal cover of a graph G if $V(G) = V(C)$, $E(G) \subseteq E(C)$ and C is chordal.

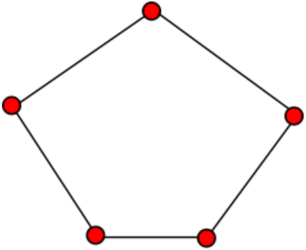


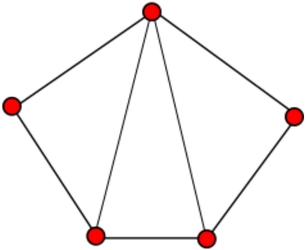


Chordal graphs and combinatorial treewidth

Definition.

The treewidth of a graph G is the smallest clique number of its chordal covers (minus one).





Informally, tree-width measures how tree-like is a graph. It is an important concept because NP-complete problems are "easy" on graphs with small treewidth. Computing TreeWidth is NP hard.

To a graph G in $[n]$ we can associate an ideal $I_G = (x_i x_j : (i,j) \notin E(G)) \subseteq k[x_1, \ldots, x_n]$.

Theorem. (Fröberg)

The graph G is chordal if and only if the ideal I_G has Castelnuovo-Mumford regularity 2.

Varieties of regularity two are the "chordal graphs" of algebraic geometry.

Theorem. (Fröberg)

The graph G is chordal if and only if the ideal I_G has Castelnuovo-Mumford regularity 2.

What are all "chordal like"? varieties? (i.e. those of regularity two)

Theorem. (Eisenbud, Green, Hulek, Popescu)

The varieties of regularity two are precisely the "linear joins" of varieties of minimal degree.

Varieties of regularity two interpolate between "Chordal graphs" and "varieties of minimal degree".

Algebraic treewidth and an upper bound Theorem

Definition.

The algebraic treewidth of a variety $X \subseteq \mathbb{P}^n$, *denoted* $\mathrm{tw}(X)$ *is the smallest dimension of a variety* Y *of regularity* 2 *with* $X \subseteq Y$.

Theorem. (B,S,Si,-)

The inequality $\Pi(X) \leq \mathrm{tw}(X) + 1$ *holds.*

Computing algebraic treewidth

Let $Q \subseteq \mathbb{R}^n$ be a lattice polytope and let $v \in \mathbb{Z}^n$. The lattice width of Q according to v is the smallest number of lines parallel to v needed to cover $Q \cap \mathbb{Z}^n$.

Theorem. (B,S,Si,-)

If $X = X(Q)$ then the treewidth of Q is the smallest lattice width among all directions $v \in \mathbb{Z}^n$.

Two Theorems

Theorem. (B,S,Si,-)

The varieties for which $\Pi(X) \in [\dim(X)+1, \dim(X)+2]$ *can be classified as follows:*

1. $\Pi(X) = \dim(X) + 1$ *if and only if* X *is a variety of minimal degree.*
2. *If* X *is integral and ACM then* $\Pi(X) = \dim(X) + 2$ *if and only if either:*
    1. X *is a variety of almost minimal degree or*
    2. X *has codimension one in a variety of minimal degree.*

    and in all these cases $\Pi(X) = n + 1 - \mathrm{qp}(X)$

Two Theorems

Theorem. (B,S,Si,-)

Let Q be a polytope and let $P = Q \times [0, k]$ then for all sufficiently large k the following equality holds:

$$\Pi(X(P)) = tw(X(P)) + 1 = k(n+1) + 1$$

where $n+1$ is the number of lattice points of Q.