III Nonnegativity Certificates for Constrained Optimization ①

Lecture ⑭ ⑮

$p_* = \inf \{ p(x) : g_1(x) \geq 0 \; \cdots \; g_m(x) \geq 0 \}$

$K = \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}$ basic semialg set

Certificates of nonnegativity on K

- $s_0(x) + \sum_{j=1}^{m} s_j(x) g_j(x)$ $\quad s_0, s_j \in \Sigma$
- $\sum_{J \subseteq [m]} s_J(x) g_J(x)$ $\quad s_J \in \Sigma, \; g_J(x) = \prod_{j \in J} g_j(x)$
- If $K = \{ x \in \mathbb{R}^n : g_1(x) = 0, \ldots, g_m(x) = 0 \}$ real alg variety,

  $s_0(x) + \sum_{j=1}^{m} h_j(x) g_j(x)$ $\quad h_j(x) \in \mathbb{R}[x]$

define

→ $\mathcal{M}(g_1, \ldots, g_m) := \{ s_0 + \sum_{j=1}^{m} s_j g_j , \; s_j \in \Sigma \}$ cone in $\mathbb{R}[x]$

(quadratic module generated by $g_1, \ldots, g_m$)

→ $T(g_1, \ldots, g_m) := \{ \sum s_J g_J : s_J \in \Sigma, \; g_J = \prod_{j \in J} g_j \}$ cone in $\mathbb{R}[x]$

(preorder gen by $g_1 \cdots g_m$)

$\mathcal{M}(g_1 \cdots g_n) \subseteq T(g_1 \cdots g_n)$

→ $\mathcal{I}(g_1 \cdots g_m) = \langle g_1 \cdots g_m \rangle = \{ \sum h_j g_j \}$ ideal gen by $g_1 \cdots g_m$

module in $\mathbb{R}[x]$

$\boxed{Q}$ Does $\mathcal{M}/T/\mathcal{I}$ generate all the polynomials that are nonnegative (zero) on K?

In the ideal case, no. $\sqrt[\mathbb{R}]{\langle g_1 \cdots g_m \rangle}$ is the set of all

(Nullstellensatz

ex
$\langle x^2 \rangle$
vs
$\langle x \rangle$

Sos relaxations of (POP): $p_* = \inf\{p : p \geq 0 \text{ on } K\}$

$p_t^{sos} := \sup e$ s.t. $p - e = s_0 + \sum s_j g_j$ $\quad \deg(s_0), \deg(s_j g_j) \leq 2t$

i.e. $p - e \in \mathcal{M}_{2t} = \{s_0 + \sum s_j g_j \text{ — " — }\}$ slice of $\mathcal{M}$

This is an SDP.

$\bar{p}_t^{sos} := \sup e$ s.t. $p - e \in T_{2t} = \{\sum s_J g_J : \deg(s_J g_J) \leq 2t\}$

slice of T, again an SDP.

$p_t^{sos} \leq \bar{p}_t^{sos} \leq p_*$ $\quad$ Do they converge to p_*?

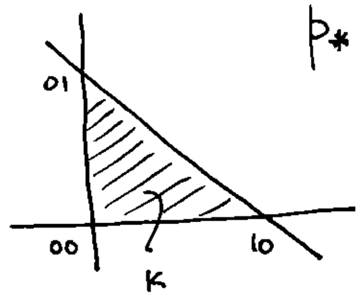
Example:

$p_* = \min xy$

s.t. $x \geq 0$

$y \geq 0$

$1 - x - y \geq 0$



$p_* = 0$

$$\bar{p}_1^{sos} = \sup e : \quad xy - e = s_0 + s_1 x + s_2 y + s_3(1-x-y) + s_4 xy + s_5 x(1-x-y) + s_6 y(1-x-y)$$

$\deg s_0 = 2$ $\quad \deg s_1 = \deg s_2 = \deg s_3 = 1$

$\deg s_4 = \deg s_5 = \deg s_6 = 0$ (scalars)

For $e = 0$ $\quad xy = 1 \cdot xy$ $\quad s_4 = 1$ $\quad s_i = 0$ $\quad \forall i = 0, 1, 2, 3, 5, 6$

$\therefore$ 1st relaxation solves the problem.

(Note that over K, $xy \not> 0$)

$$p_1^{sos} = \sup e$$
$$\text{s.t.}\quad xy - e = \underbrace{s_0}_{\deg=2} + \underbrace{s_1}_{\deg=0} x + s_2 y + s_3(1-x-y)$$
$$= (x\ y\ 1)\underbrace{\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}}_{Q \succeq 0}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \lambda_1 x + \lambda_2 y + \lambda_3(1-x-y)$$
$$= ax^2 + by^2 + c + 2d\,xy + 2ex + 2fy + \lambda_1 x + \lambda_2 y + \lambda_3(1-x-y)$$

Equating coeffs: $a = 0,\ b = 0, \Rightarrow \begin{cases} e = 0 \\ d = 0 \end{cases}$ since $Q \succeq 0$ but also $1 = 2d$ ↯ $\therefore\ p_1^{sos} = -\infty$

$$p_2^{sos} = \sup e$$
$$\text{s.t.}\quad xy - e = (x^2\ y^2\ xy\ x\ y\ 1)\, Q_0 \begin{pmatrix} x^2 \\ y^2 \\ xy \\ x \\ y \\ 1 \end{pmatrix} + (x, y, 1)\, Q_1 \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cdot x + \cdots$$
$$Q_0, Q_1, \ldots \succeq 0$$

get SDP ... finish and see what happens.

Will $p_t^{sos} \xrightarrow{?} 0$ (Only asymptotic convergence!) $\xrightarrow{\text{PTO}}$

yes by Putinar but

<u>Questions</u>:

- How good are our positivity certificates?
- Will $p_t^{sos} \longrightarrow p_*$, $\bar{p}_t^{sos} \longrightarrow p_*$ always? finitely?
- Will something better happen when K is a real variety? finite real variety?

The fundamental question is how good our positivity certificates are. (Just like $\Sigma_{n,2d}$ vs $P_{n,2d}$)

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a question

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Mon, Nov 21, 2016 at 2:22 AM

Hi

I don't think I can recall but I think I can reconstruct. Lets see:

Ok, so by closedness of the quadratic module (since S has non empty interior) finite convergence implies that xy must be written as an SOS

Then, as you wrote, xy = s_0 + s_1 * x + s_2 * y + s_3 * (1-x-y) where s_i is a sos.

ARGUMENT 1:

Evaluating at (0,0) you get 0=s_0(0,0)+s_3(0,0) implying that the constant terms of both s_0 and s_3 are zero and so the minimal degree of monomials of s_0 and s_3 is 2.
Then the coefficients of x and y in the RHS is the constant terms of s_1 and s_2 respectively, meaning that they also can't have constant terms.
This means that the degree 2 stuff in the RHS must be the lowest degree monomials of s_0 plus the lowest degree monomials of s_3 but that is an homogeneous SOS of degree 2 and therefore not equal to xy (which is not even nonnegative everywhere)

ARGUMENT 2:

Evaluating at (x,0) we get 0=s_0(x,0)+s_1(x,0)x+s_3(x,0)(1-x) For this to be true everywhere in x=[0,1], since every summand is nonnegative in that interval, we actually need them all to be zero, so s_0,s_1 and s_3 are multiples of y hence of y^2. Similarly, evaluating at (0,y) we get that s_0,s_2 and s_3 are multiples of x^2. This means that no monomial on the RHS has degree less than 3 hence xy is not on the RHS.

WHAT IS REALLY HAPPENING:

Looking at the triangle {(0,0),(1,0),(0,1)} all the summands in you certificate are nonnegative there. Since xy vanishes in two sides of that triangle, all your summands must do the same, and being algebraic, that means that they actually all have to vanish at the axis, and since the roots from the sos are always at least double roots, they all have a double intersection at the origin, so their sum has a double point at the origin and cannot equal xy.

Hope one of those works fine for you.

João
[Quoted text hidden]

<u>Note</u> If a sos does not contain a constant term, it also doesn't contain a linear term.

Positivstellensatz (Krivine 1964, Stengle 1974)

$$(*) \left\{ \begin{array}{ll} f_i(x) = 0 & i = 1 \cdots m \\ g_i(x) \geq 0 & i = 1 \cdots p \end{array} \right\} \text{ infeasible in } \mathbb{R}^n$$

$\iff$ $\exists\ F(x), G(x) \in \mathbb{R}[x]$ s.t. $F(x) + G(x) = -1$

(certificate of infeasibility) $F(x) \in \langle f_1 \cdots f_m \rangle$

$G(x) \in T(g_1 \cdots g_m)$

Proof: "Easy direction" Suppose $\exists\ F(x), G(x)$ as above then for any feasible sol$^{\underline{n}}$ a of $(*)$

$F(a) = 0$, $G(a) \geq 0$ but $F(a) + G(a) = -1$ ↯.

$\therefore$ $(*)$ must be infeasible.

How do we compute such a certificate?
(By fixing degree & solving SDPs. By Positivstellensatz $\exists$ a high enough degree in which such a certificate will exist)

Ex 3.129 Consider

$$\left.\begin{array}{l} f_1 := x_1^2 + x_2^2 - 1 = 0 \\ g_1 := 3x_2 - x_1^3 - 2 \geq 0 \\ g_2 := x_1 - 8x_2^3 \geq 0 \end{array}\right\} (*) \text{ infeasible in } \mathbb{R}^2$$

By positivstellensatz $(*)$ infeas $\iff$ $\exists$ a certificate of the form

$$\underbrace{f_1 \cdot t_1}_{\in \langle f_1 \rangle} + \underbrace{s_0 + s_1 g_1 + s_2 g_2 + s_3 g_1 g_2}_{\in T(g_1, g_2)} = -1$$

Search for t_1, s_0, s_1, s_2, s_3 by bounding degree

eg $d = 4 \geq \deg(f_1 t_1), \deg(s_0), \deg(s_1 g_1), \deg(s_2 g_2), \deg(s_3 g_1 \ldots$
works. (see SIAM bk)

Corollary **Real Nullstellensatz** (no inequalities)

$\{f_i(x) = 0 \quad i = 1 \cdots m\}$ infeasible over $\mathbb{R}^n$

$\Longleftrightarrow$ $\exists \; F(x) \in \langle f_1, \ldots, f_m \rangle$ s.t $F(x) + s_0 = -1$

(Note $T(\phi) = \Sigma$) i.e $-1 \equiv$ sos mod $\langle f_1 \cdots f_m \rangle$

How does this compare to Hilbert's Nullstellensatz?

$\{f_i(x) = 0 \quad i = 1 \cdots m\}$ infeas over $\mathbb{C}^n$

$\Longleftrightarrow$ $\exists \; F(x) \in \langle f_1 \cdots f_m \rangle$ s.t $F(x) = -1$

ex Show $1 + x^2 + y^2 = 0$ feas over $\mathbb{C}^2$ but not over $\mathbb{R}^2$

$-1 \notin \langle x^2 + y^2 + 1 \rangle$ Can you write a real Null. certificate?

$[-1(1 + x^2 + y^2) + (x^2 + y^2) = -1$ more simply, $x^2 + y^2 = -1]$

Corollary Positivity certificate on $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0 \cdots g_m(x) \geq 0\}$

$p > 0$ on $K \Longleftrightarrow \{p(x) \leq 0, \; g_1(x) \geq 0 \cdots g_m(x) \geq 0\}$ inf over $\mathbb{R}^n$

$\Longleftrightarrow \{-p \geq 0, \; +g_1 \geq 0 \cdots g_m \geq 0\}$ inf over $\mathbb{R}^n$

$$\Longleftrightarrow \; -1 = \underbrace{s_0 + \underset{\substack{\cap \\ T(g_1 \cdots g_m)}}{A}} - p \underset{\substack{\cap \\ T(g_1 \cdots g_m)}}{B}$$

$\Longleftrightarrow \; pB = 1 + C \qquad B, C \in T(g_1 \cdots g_m)$

Similarly

① $p > 0$ on $K \Longleftrightarrow pf = 1 + g$ for some $f, g \in T(g_1 \cdots g_m)$

② $p \geq 0$ on $K \Longleftrightarrow pf = p^{2k} + g$ —— " ——, $k \in \mathbb{N}$

③ $p = 0$ on $K \Longleftrightarrow -p^{2k} \in T(g_1 \cdots g_m)$ for some $k \in \mathbb{N}$

④ $K = \phi \Longleftrightarrow -1 \in T(g_1 \cdots g_m)$

Corollary

<u>Proof of Hilberts' 17th problem</u> $p \geq 0$ on $\mathbb{R}^n \Rightarrow p = \sum \left(\frac{a_j}{b_j}\right)^2$ $a_j, b_j \in \mathbb{R}[x]$

<u>Pf</u> $K = \mathbb{R}^n$. $T_K(\phi) = \Sigma$

$$② \Rightarrow \quad p \geq 0 \text{ on } K \iff p \cdot s_1 = p^{2k} + s_2 \qquad s_1, s_2 \in \Sigma,\ k \in \mathbb{N}$$

$$\iff p = \frac{s}{s_1} = \frac{(\Sigma q_i^2)\, s_1}{s_1^2} = \frac{(\Sigma q_i^2)(\Sigma r_j^2)}{s_1^2}$$

$$= \Sigma \frac{(q_i r_j)^2}{s_1^2} = \Sigma \left(\frac{q_i r_j}{s_1}\right)^2 = \Sigma \left(\frac{a_i}{b_i}\right)^2$$

<u>Schmüdgen (1991)</u> Suppose K is compact.
Then $p > 0$ on $K \Rightarrow p \in T(g_1 \cdots g_m)$

$\Rightarrow \bar{p}_t^{sos}$ will converge asymptotically to p_*
(cannot replace >0 with ≥ 0)

$\bar{p}_t^{sos}$ hierarchy has 2^m sos terms - large SDPs!

<u>Putinar (1993)</u> $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0 \cdots g_m(x) \geq 0\}$

<u>Def</u> $\mathcal{M}(g_1 \cdots g_m)$ is <u>Archimedean</u> if $\exists N$ s.t.
$N - \Sigma x_i^2 \in \mathcal{M}(g_1 \cdots g_m)$ ($\Rightarrow K$ compact)

<u>Theorem</u> Suppose $\mathcal{M}(g_1 \cdots g_m)$ is Archimedean.
Then $p > 0$ on $K \Rightarrow p \in \mathcal{M}(g_1 \cdots g_m)$

$\Rightarrow p_t^{sos} \longrightarrow p_*$ asymptotically smaller SDPs.

deg bounds for rep^{ns} in $\mathcal{M}(g_1 \cdots g_m)$ exist.

Comments

- The hierarchy of problems (SDP)

$$p_t^{sos} := \cancel{\inf}\ \sup \{ e : p - e = s_0 + \sum s_j g_j \quad \deg(s_0), \deg(s_j g_j) \le 2t \}$$

is the dual of the Lasserre hierarchy of relaxations of (POP).

- If $K = \{x \in \mathbb{R}^n : g_1(x) = 0, \dots, g_m(x) = 0\}$ is a finite algebraic variety ~~and $\langle g_1 \dots g_m \rangle$ is radical~~ then $\exists$ finite t s.t. $p_t^{sos} = p_*$

Note: If K is presented geometrically then we can choose $g_1 \cdots g_m$ to cut it out.

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