Lecture (I): Polynomial Optimization & SOS Relaxations (1)

$\mathbb{R}[x_1 \dots x_n] = \mathbb{R}[\underline{x}]$ ring of polynomials in $x_1, \dots, x_n$ over $\mathbb{R}$

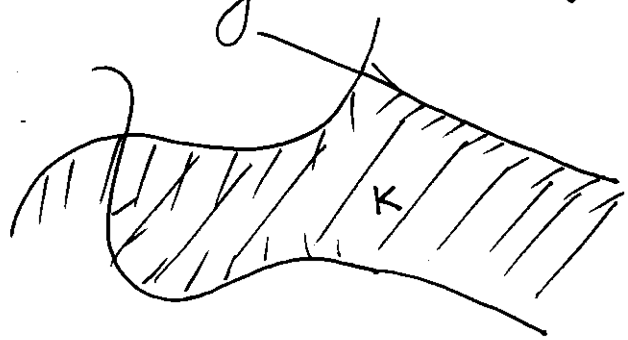
(POP): Given $p(x), g_1(x), \dots, g_m(x) \in \mathbb{R}[x]$, find

$$p_* := \inf_{x \in \mathbb{R}^n} \Big\{ p(x) : \underbrace{g_1(x) \geq 0, \dots, g_m(x) \geq 0}_{\text{constraints}} \Big\}$$

objective value

objective function

feasible region $K := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0 \}$

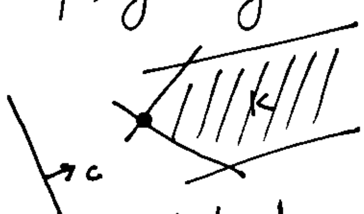


(closed) basic semialgebraic subset of $\mathbb{R}^n$

In general - nonconvex, could be disconnected

Special Cases:

(1) $p, g_1 \cdots g_m$ linear $\rightsquigarrow$ (POP): $\inf \{ c^T x : Ax \geq b \}$ (linear program)



polyhedron (convex)

$= \min \{ c^T x : Ax \geq b \}$ (very well understood)

convex program

If LP bounded, opt is achieved

poly time algorithm, duality thry

We'll focus on the case when there is some non-linearity - either in p or g_i or both.

(2) Unconstrained POPS : $K = \mathbb{R}^n$ $\inf \{ p(x) : x \in \mathbb{R}^n \}$

(3) K is a (real) variety : $K = \{ x \in \mathbb{R}^n : g_1(x) = 0 \dots, g_m(x) = 0 \}$

Recall: a convex (POP) is one of the form

$$\inf \{ p(x) : g_1(x) \geq 0, \dots, g_m(x) \geq 0 \}$$ where

$p, -g_1, -g_2, \dots, -g_m$ are all convex.

$f : \mathbb{R}^n \to \mathbb{R}$ convex if $f(\alpha x + \beta x') \leq \alpha f(x) + \beta f(x')$
$\forall \alpha, \beta \geq 0 \quad \alpha + \beta = 1, \forall x, x'$

In this case $K = \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0 \}$ is convex.

The general (POP) is non-convex.

The general (POP) is also NP-hard

Partition problem: Given positive integers $a_1, \dots, a_t$, $\exists$? a partition into 2 parts of equal sum?

i.e $\exists$? $x \in \{\pm 1\}^t$ s.t $a_1 x_1 + a_2 x_2 + \dots + a_t x_t = 0$?

Claim: $\exists$ partition $\Longleftrightarrow$ $p_* = 0$ for

$$p := \left(\sum_{i=1}^{t} a_i x_i \right)^2 + \sum_{i=1}^{t} (x_i^2 - 1)^2$$

Pf: $p_* = 0 \Leftrightarrow \sum a_i x_i = 0 \quad x_i^2 = 1 \ \forall i = 1 \dots t \Leftrightarrow \exists$ partition

partition problem is NP-complete

Key fact we'll exploit a lot: for $h_i(x) \in \mathbb{R}[x]$, $i = 1, \dots t$

if $\sum h_i(\bar{x})^2 = 0$ for some $\bar{x} \in \mathbb{R}^n$ then $h_i(\bar{x}) = 0$ $\forall i = 1 \dots t$

(true over $\mathbb{R}[x]$, not true over $\mathbb{C}[x]$)

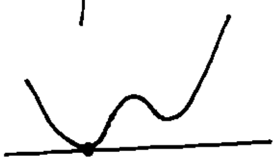
Since (POP) is NP-hard, it's useful to understand relaxations of (POP) that might be easy / poly time

$\rightsquigarrow$ <u>sums of squares</u> relaxations.
(sos)

<u>Def</u>: $p(x) = \sum_{i=1}^{t} h_j(x)^2$ $\quad h_j(x) \in \mathbb{R}[x]$ is a <u>sos poly</u>

<u>Basic facts</u>:

① p sos $\Longrightarrow$ $p \geq 0$ on $\mathbb{R}^n$, $\deg(p) = 2d$ (even), all real roots of p are double roots (i.e $p(x) = 0 \Rightarrow p'(x) = 0$)



② $p = \sum h_j^2$ $\quad \deg p = 2d \Rightarrow \deg h_j \leq d \quad \forall j = 1 \ldots t$

③ p homogeneous, sos of deg $2d \Rightarrow$ each h_j homogeneous of deg $= d$.

④ Suppose $\tilde{p}$ is the homogenization of p.
Then $p \geq 0$ on $\mathbb{R}^n$ (sos) $\Longleftrightarrow$ $\tilde{p} \geq 0$ on $\mathbb{R}^{n+1}$ (sos).

$$\left(\tilde{p}(x, x_{n+1}) = x_{n+1}^{d}\, p\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \quad \text{if } \deg p = d \right)$$

⑤ Suppose $p = \sum p_\alpha x^\alpha$ polynomial of deg $\leq 2d$.
Then p sos $\Longleftrightarrow$ $p = [x]_d^T Q [x]_d$ for some $Q \succeq 0$ (psd)

$[x]_d$ = (vector of monomials of deg $\leq d$ in $\mathbb{R}[x]$)

<u>Recall</u>: $Q \succeq 0 \Longleftrightarrow x^T Q x \geq 0$ $\forall x \in \mathbb{R}^n$ $\Longleftrightarrow$ all eigenvalues of Q are

$S^n = \{ n\times n \text{ real symmetric matrices} \}$

Def: $Q \in S^n$ is psd iff one of the foll equivalent cond$^{\text{n}}$s hold:

1. all eigenvalues of Q are nonnegative
2. $x^T Q x \geq 0 \quad \forall\, x \in \mathbb{R}^n$
3. all principal minors of Q are nonnegative
4. $Q = UU^T$ for some U

Pf of ⑤ "$\Longleftarrow$" $\quad p = [x]_d^T Q [x]_d \underset{Q \succeq 0}{\Longrightarrow} p = [x]_d^T U U^T [x]_d$

$$= [x]_d^T [U_1\ U_2 \dots U_k] \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_k^T \end{bmatrix} [x]_d = \underbrace{\sum_{i=1}^{k} \left(u_i^T [x]_d\right)^2}_{\text{sos of deg} \leq 2d}$$

"$\Longrightarrow$" Suppose $p = \sum_{j=1}^{k} u_j(x)^2 = \sum_{j=1}^{k} \left(u_j^T [x]_d\right)^2$

$$= [x]_d^T [U_1 \dots U_k] \begin{bmatrix} U_1^T \\ \vdots \\ U_k^T \end{bmatrix} [x]_d .$$

eg $\quad p = 5x^2 + 2x + 2 = (1\ \ x) \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = (1\ x) \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} ($

$$= (x-1)^2 + (2x+1)^2$$

In general, p sos of deg $\leq 2d \iff$

$$\mathcal{J} = \Big\{ Q \succeq 0 : \underbrace{\sum_{\substack{\gamma, \beta \in \mathbb{N}^n \\ : \beta + \gamma = \alpha}} Q_{\beta,\gamma} = p_\alpha}_{\text{linear constraints on the entries of } Q} \Big\} \text{ is feasible}$$

$S^n_+ = \{ \text{psd matrices in } S^n \}$ cone $\begin{pmatrix} \text{closed,} \\ \text{pointed,} \\ \text{convex} \\ \text{full-dim} \end{pmatrix}$

$\mathcal{J} = S^n \cap (\text{affine space})$ 

$\therefore$ In order to find sos expressions for polynomials we need to understand sets of the form

$$\mathcal{Y} = \mathcal{S}^n_+ \cap \text{affine space} \quad - \text{(spectrahedron)}$$

<u>Semi-definite Programming</u> (LP in $\mathcal{S}^n$)

<u>Inner product</u>: $A, B \in \mathcal{S}^n$, $\langle A, B \rangle := \mathrm{Tr}(AB)$

$$\begin{aligned} &\min \langle C, X \rangle \\ \text{s.t } &\langle A_i, X \rangle = b_i \quad i = 1, \dots, m \\ &X \succeq 0 \end{aligned}$$

(generalization of LP to $\mathcal{S}^n$, poly-time solvable ~~to~~ arbitrary precision)

<u>Recall</u> <u>LP</u>: $\min c^T x$
$x \in \mathbb{R}^n_+ \cap \text{aff space}$ – poly time solvable.

<u>Main point</u>: Can check if p is sos by SDP.

Show that $p(x, y) = x^2 - xy^2 + y^4 + 1 \in \mathbb{R}[x, y]_4$ is sos.
i.e $p = [x]_2^T \, Q \, [x]_2$ $\quad Q \succeq 0$

$n = 2, \quad d = 2 \quad \binom{n+d}{d} = 6$

$$[x]_2 = (1, x, y, x^2, xy, y^2)$$

<u>Write</u> $Q =$

$$\begin{array}{cc} \begin{array}{cccccc} 1 & x & y & x^2 & xy & y^2 \end{array} & \\ \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_2 & u_7 & u_8 & u_9 & u_{10} & u_{11} \\ u_3 & u_8 & u_{12} & u_{13} & u_{14} & u_{15} \\ u_4 & u_9 & & u_{16} & u_{17} & u_{18} \\ u_5 & u_{10} & & & u_{19} & u_{20} \\ u_6 & u_{11} & & & & u_{21} \end{bmatrix} & \begin{array}{c} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{array} \end{array}$$

p sos $\iff$ $p = (1, x, y, x^2, xy, y^2)\, Q \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$ for some $Q \succeq 0$, $Q \in S^6$

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$$\text{Let } Q = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & x & y & x^2 & xy & y^2 \end{array} \\ \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_2 & u_7 & u_8 & u_9 & u_{10} & u_{11} \\ u_3 & u_8 & u_{12} & u_{13} & u_{14} & u_{15} \\ & & & u_{16} & u_{17} & u_{18} \\ & & & & u_{19} & u_{20} \\ & & & & & u_{21} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{array}$$

$$[x]_2^T Q [x]_2 = \underset{\overset{\shortparallel}{1}}{u_1} + (2u_2)x + (2u_3)y + (2u_4 + \underset{\overset{\shortparallel}{1}}{u_7})x^2 + (2u_9)x^3$$
$$+ (2u_5 + 2u_8)xy + (2u_6 + u_{12})y^2$$
$$+ (2u_{10} + 2u_{13})x^2y + (\underbrace{2u_{11} + 2u_{14}}_{-1})xy^2 + (2u_{15})y^3$$
$$+ u_{16}x^4 + (2u_{17})x^3y + (2u_{18} + u_{19})x^2y^2$$
$$+ (2u_{20})xy^3 + \underset{\overset{\shortparallel}{1}}{u_{21}}\, y^4 = x^2 - xy^2 + y^4 + 1$$

Equating coeffs of the same monomials get:

$u_1 = 1$

$u_2 = 0$

$u_3 = 0$

$u_7 = 1 - 2u_4$

$u_8 = -u_5$

$u_{12} = -2u_6$

$u_9 = 0$

$u_{13} = -u_{10}$

$2u_{\dots} = -1 - 2u_{\dots}$

$u_{15} = 0$

$u_{16} = 0$

$u_{17} = 0$

$u_{19} = -2u_{18}$

$u_{20} = 0$

$u_{21} = 1$

$$\begin{bmatrix} 1 & 0 & 0 & u_4 & u_5 & u_6 \\ 0 & +1-2u_4 & -u_5 & 0 & u_{10} & u_{11} \\ 0 & & -2u_6 & -u_{10} & \left(\frac{-1-2u_{11}}{2}\right) & 0 \\ & & & 0 & 0 & u_{18} \\ & & & & -2u_{18} & 0 \\ & & & & & 1 \end{bmatrix} \succeq 0$$

a $\mathrm{sol}^{\underline{n}}$ $\quad U_5 = 0 = U_6 \quad U_{11} = -1/2$

everything else $= 0$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1/2 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}}_{B} \begin{pmatrix} & & \\ & B^T & \\ & & \end{pmatrix}$$

$$\Rightarrow p = 1 + \left(\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y^2\right)^2 + \left(\frac{1}{2}x + \frac{1}{2}y^2\right)^2$$

$$= 1 + \frac{3}{4}(x - y^2)^2 + \frac{1}{4}(x + y^2)^2$$

<u>Another $\mathrm{sol}^{\underline{n}}$</u>: $\quad p = \frac{1}{9}(3 - y^2)^2 + \frac{2}{3}y^2 + \frac{1}{288}(9x - 16y^2)^2 + \frac{23}{32}x^2.$

- (plot these graphs)
- Use an SDP solver or MATLAB to find B. / sos.

<u>How can we use sums of squares polynomials to create relaxations of (POP)?</u>

<u>Key idea</u>: think of (POP) as a nonnegativity problem

Optimization $\longleftrightarrow$ checking nonnegativity

(POP): $\quad p_* = \inf \{ p(x) : x \in K \}$

$= \sup \{ e : p(x) \geq e \text{ on } K \}$

$= \sup \{ e : p(x) - e \geq 0 \text{ on } K \}$

$= \sup \{ e : p(x) - e > 0 \text{ on } K \}$

Recall $K = \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}$

Goal: Want to express that $p(x) - e \geq 0$ on K.

Q: What do polynomials nonnegative on K look like? How to generate nonneg polys on K?

One way: sos-combinations of $g_1, \ldots, g_m$

i.e $\quad \underbrace{s_0 + s_1 g_1 + s_2 g_2 + \cdots + s_m g_m}_{\geq 0 \text{ on } K}$ $\quad s_i$ are sos.

since $\forall x \in K \quad s_j(x) \geq 0, \ g_j(x) \geq 0$

$\{ s_0 + \sum s_j g_j : s_j \text{ sos} \}$ - quadratic module generated by $g_1 \ldots g_m$

Sos relaxation of POP:

$p^{sos} := \sup \{ e : p(x) - e = s_0 + \sum s_j g_j \quad s_j \text{ sos} \}$

Note: $p^{sos} \leq p_*$ since $p(x) - p_*$ may not be in QM.

Special case: $K = \mathbb{R}^n \qquad p^{sos} = \sup \{ e : p(x) - e = s_0 \}$

To get an SDP we fix degrees to $2t \geq \deg(p, g_1 \cdots g_m)$

$p_t^{sos} := \sup \{ e : p - e = s_0 + \sum s_j g_j \quad s_0, s_j \text{ sos}$
$\deg(s_0), \deg(s_j g_j) \leq 2t \}$