Hints to exercises – Polynomial Optimization

June 13, 2018

2) We record the cut induced by $S \subseteq [n]$ by the vector $\chi^S \in \{-1, 1\}^n$ defined as:

$$(\chi^S)_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S \end{cases}$$

What set of polynomial equations have as their solutions exactly these vectors χ^S? How do cut edges look in your model? What polynomial maximizes the size of a cut in your model?

5b,c) It might help to remember that all symmetric matrices can be diagonalized as $M = UDU^\top$ where U is orthogonal and D is a diagonal matrix with $D_{ii} = \lambda_i(M)$, the ith eigenvalue of M. If the columns of U are $u_1, \ldots, u_n$ then this means that

$$M = \sum \lambda_i u_i u_i^\top$$

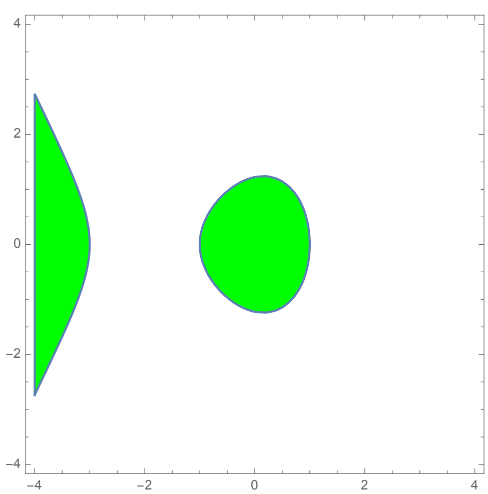

Figure 1: Problem 6d ii): The region satisfying $-x^3 - 3x^2 - 2y^2 + x + 3 \geq 0$

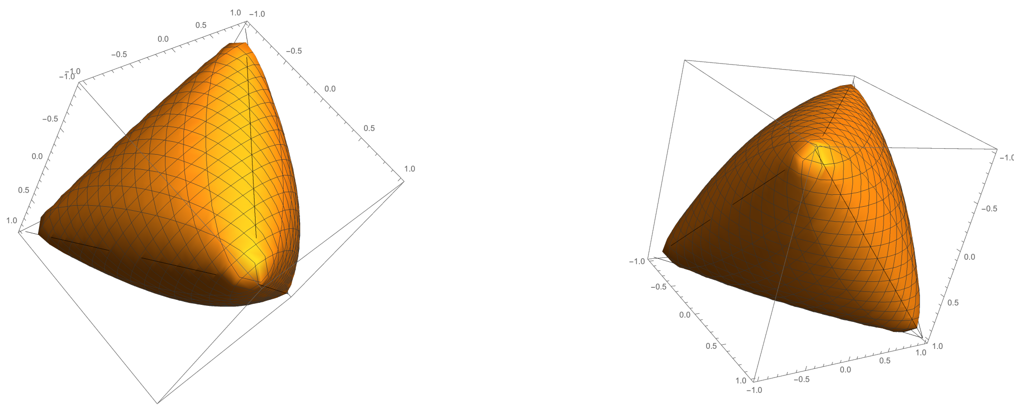

Figure 2: Problem 7b): Two views of the elliptope.

7e) Recall that we were modeling the cut induced by $S \subseteq [n]$ by assigning 1 to vertices in S and -1 to vertices not in S. Let $v(T)$ be the ± 1 vector in $\mathbb{R}^n$ so obtained. Then $X = v(T)v(T)^\intercal \in \mathcal{E}_n$.

10) The following Q will work:

$$Q = \frac{1}{3}\begin{pmatrix} 6 & 3 & 0 & -2 & 0 & -2 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 6 & 3 & -4 \\ 0 & 0 & 0 & 3 & 5 & 0 \\ -2 & 0 & 0 & -4 & 0 & 15 \end{pmatrix}$$

Now we need to factorize $Q = BB^\intercal$ to get the sos expression for p. This also requires a computer. But the following sos expression works:

$$p = \frac{4}{3}y^2 + \frac{1349}{705}y^4 + \frac{1}{12}(4x+3)^2 + \frac{1}{15}(3x^2+5xy)^2 + \frac{1}{315}(-21x^2+20y^2+10)^2 + \frac{1}{59220}(328y^2-235)^2.$$

What is B in this case? Check that $Q = BB^\intercal$.

We now do this example using Macaulay2 using the package SOS.m2:

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/Applicati
```

```
o1 = SOS

o1 : Package

i2 : R = QQ[x,y]

o2 = R

o2 : PolynomialRing

i3 : f = 2*x^4+5*y^4-x^2*y^2+2*x^3*y+2*x+2     ---- input the polynomial

            4      3      2 2       4
o3 = 2x  + 2x y - x y  + 5y  + 2x + 2

o3 : R

i4 : (Q,mon,X) = solveSOS(f, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-66664-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-66664-0/1
Success: SDP solved

i5 : s = sosdec(Q,mon)

o5 = coeffs:
         11  17  1912  2083   1313
     {5, --, --, ----, ----, -----}
          5  11  2125  1912  10415
     gens:
         8 2    2   1    1   5 2          5     5     5 2   11          5   2    55
     {- --x  + y  - -x - -, --x  + x*y + --y - --, - --x  + --x + y + --, x  - ----
        25          5    5  11           11    11    17     17         17       1912

o5 : SOSPoly

--- the above output is fine on the computer but doesn't make sense as shown above,
a fortran type output with the command "toString". "oo" means the output that just

i6 : toString oo

o6 = new SOSPoly from {ring => R,
coefficients => {5, 11/5, 17/11, 1912/2125,   2083/1912, 1313/10415},
```

```
generators => {-(8/25)*x^2+y^2-(1/5)*x-1/5, (5/11)*x^2+x*y+(5/11)*y-5/11,
                          -(5/17)*x^2+(11/17)*x+y+5/17, x^2-(55/1912)*x-705/1912, (9

-- check if the above sos is indeed the polynomial we started with.

i7 : sumSOS(s)

       4      3      2 2      4
o7 = 2x  + 2x y - x y  + 5y  + 2x + 2

o7 : R
```

11) Prove that a univariate non-negative polynomial is always a sum of two squares. (Hint: Recall that all real roots of a nonnegative polynomial are double roots and all complex roots come in conjugate pairs. Then use the identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$ for all $a,b,c,d \in \mathbb{R}$.)

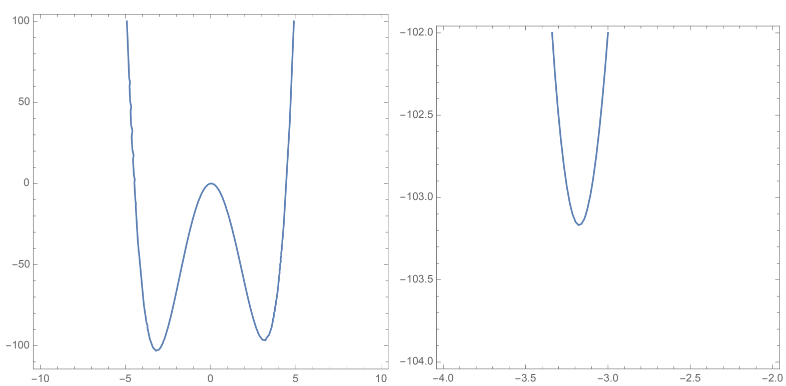

Figure 3: Problem 13) The graph of $y = x^4 - 20x^2 + x$, and a zoomed in view of the minimum.

13) We now use M2 to do part c) accurately.

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )

i2 : R = QQ[x,t];

i3 : f2 = x^4 - 20*x^2 + x;
```

```
i4 : (Q,mon,X,tval) = solveSOS(f2-t,{t},-t, Solver=>"CSDP");

i5 : tval

           1651
o5 = {- ----}
            16

o5 : List

-- tval is the minimum value and it is roughly -103.1875

i6 : toString Q

o6 = matrix {{1651/16, 1/2, -807/80}, {1/2, 7/40, 0}, {-807/80, 0, 1}}

i7 : toString mon

o7 = matrix {{1}, {x}, {x^2}}
```

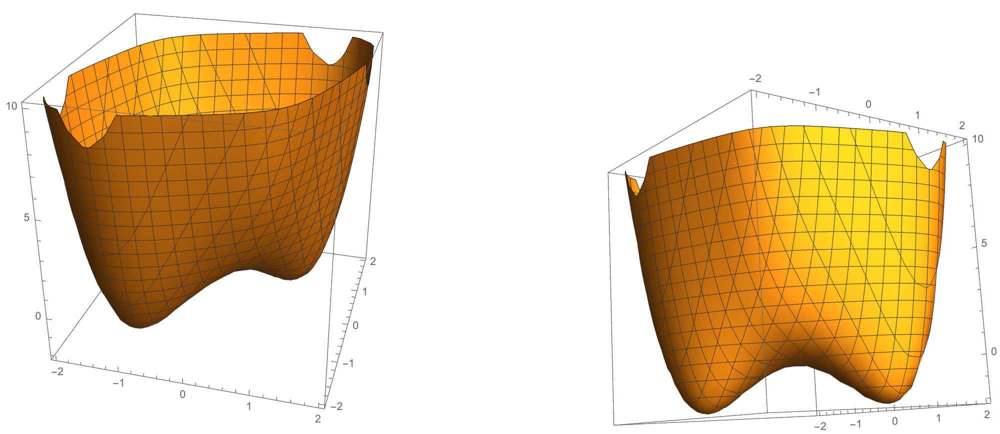

Figure 4: Problem 14b): The graph of $z = x^4 + y^4 - 4xy$ (two different views).