Polynomial Optimization – Solutions to Exercises

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1. Let $G = ([n], E)$ be an undirected graph with vertex set $[n] = \{1, \dots, n\}$ for a positive integer n and edge set E consisting of pairs of vertices. A set $S \subseteq [n]$ is said to be *stable* or *independent* if for any two vertices $i, j \in S$, the edge $ij \notin E$. Formulate a polynomial optimization problem to find the maximum cardinality stable set in G.

   Solution. Define a variable x_i for each node $i \in [n]$. Then we want to assign either 0 or 1 to each x_i according to whether or not it is in the stable set, maximizing the number of ones subject to the condition that no two x_i, x_j are both 1 if $ij \in E$. Thus the maximum cardinality stable set problem becomes:

$$\begin{array}{ll} \max & \sum_{i=1}^n x_i \\ & x_i^2 = x_i, \forall i \in [n] \\ & x_i x_j = 0, \forall ij \in E, \end{array}$$

   a polynomial optimization problem with quadratic constraints. The max cardinality stable set problem is NP-hard showing that as soon as we deviate from linear programs, we run into hard problems. □

2. A *cut* in G is a partitioning of its vertices into two sets T and $[n] \backslash T$ and the size of the cut is the number of edges that go between the two parts. Formulate a polynomial optimization problem to find the maximum cardinality cut in G. This is another NP-hard problem.

   Solution. Define a variable x_i for each node $i \in [n]$. We model the cut induced by the vertex set T by assigning vertices in T a value of 1 and all others a value of -1. Then $x_i x_j = -1$ if ij is an edge in the cut induced by T and $x_i x_j = 1$ if ij is not in the cut. Thus the max cut problem becomes

$$\max \frac{1}{2} \sum_{1 \le i < j \le n}^{n} (1 - x_i x_j)$$
$$x_i^2 = 1, \forall i \in [n].$$

   □

3. A very common problem that arises in applications is to find the closest point in a given set from a given *data* point that has been observed in an experiment. For instance in computer vision one is often interested in reconstructing a three-dimensional scene from noisy images of the scene. The set of all true images that are possible by the given cameras is an algebraic set which is the *model* and the noisy images form the *data* point. If the noise model is Gaussian then the closest point to the model from the observed noisy data point is the maximum likelihood estimate. Model this problem as a polynomial optimization problem.

Solution. Denote the algebraic set that models the true possibilities by $V \subset \mathbb{R}^n$. Suppose I is the set of all polynomials in $\mathbb{R}[x_1, \dots, x_n]$ that vanishes on V. It might be that we don't actually know the full vanishing ideal of V, but know a set of polynomials $f_1, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$ such that

$$V = \{x \in \mathbb{R}^n \,:\, f_1(x) = 0,\ f_2(x) = 0,\ \dots,\ f_s(x) = 0\}.$$

If $I = \langle f_1, \dots, f_s \rangle$ then even better. If $\hat{x}$ is the observation, then the problem becomes

$$\begin{array}{ll} \min & \|x - \hat{x}\|^2 \\ & f_1(x) = 0, \dots, f_s(x) = 0 \end{array}$$

□

Another problem that is very common in applications is to find a low rank estimate of a given matrix. Write down a polynomial optimization problem for finding the closest (in Euclidean distance) rank one real matrix of size $p \times q$ to a given real matrix A of the same size. Generalize to rank k. The classical *Eckart-Young theorem* in linear algebra gives a solution to this distance minimization problem. Look it up and see if you can solve it using the model you wrote.

Solution. Let $X = (x_{ij})$ be the symbolic matrix of size $p \times q$. Then an evaluation of X with $x_{ij} \in \mathbb{R}$ has rank at most k if and only if all $(k+1) \times (k+1)$ minors of the matrix vanishes. Let $m(X)$ denote the degree $k+1$ homogeneous polynomial obtained from a $(k+1) \times (k+1)$ minor of X. Therefore, the set of all $p \times q$ matrices in $\mathbb{R}^{p \times q}$ is the algebraic variety (called a *rank variety*)

$$\mathcal{M}_r(p \times q) = \{X \in \mathbb{R}^{p \times q} \,:\, m(X) = 0\ \forall\ (k+1) \times (k+1) \text{ minors } m(X) \text{ of } X\}.$$

The optimization problem is then

$$\begin{array}{ll} \min & \|X - A\|^2 \\ & X \in \mathcal{M}_r(p \times q) \end{array}$$

Note that the specialization to rank one involves only quadratic polynomials. □

4. A function f is convex if $f(\alpha x+\beta y)\le \alpha f(x)+\beta f(y)$ for $x,y\in\mathbb{R}^n$ and scalars $\alpha,\beta\in\mathbb{R}$ such that $0\le\alpha,\beta$ and $\alpha+\beta=1$. Consider the semialgebraic region $K=\{x\in\mathbb{R}^n : g_1(x)\ge 0,\dots,g_m(x)\ge 0\}$. Prove that K is a convex set if $-g_1,\dots,-g_m$ are convex functions. If in addition f is a convex function, then the polynomial optimization problem $\min\{f(x) : x\in K\}$ is called a *convex program.*

5. (a) Convince yourself that the psd cone $\mathcal{S}^n_+\subset\mathcal{S}^n$ is closed, convex, pointed and full-dimensional (solid). A cone with all these properties is called a *proper cone.*

   Recall that a *convex* cone $K\subset\mathbb{R}^t$ is one in which for every $x,y\in K$, $\lambda x+\mu y\in K$ for all $\lambda,\mu\ge 0$. The cone K is *pointed* if it does not contain any lines through the origin, i.e., there is no $x\in K$, $x\ne 0$ such that $-x\in K$.

   Solution.
   - $\mathcal{S}^n_+$ is **closed**: Let $A\in\mathcal{S}^n$ be a limit point of $\mathcal{S}^n_+$, i.e., $A=\lim_{k\to\infty}A_k$ for $A_k\in\mathcal{S}^n_+$. Since $x^\top A_k x\ge 0$ for all $x\in\mathbb{R}^n$, and the function $f:\mathcal{S}^n\to\mathbb{R}$ sending M to $x^\top Mx$ is continuous, then $x^\top Ax\ge 0$ for all $x\in\mathbb{R}^n$, i.e., $A\in\mathcal{S}^n_+$. Another way to see that $\mathcal{S}^n_+$ is closed is to note that the cone is cut out by the inequalities $p(X)\ge 0$ where $p(X)$ is a principal minor of the symbolic symmetric $n\times n$ matrix X. Thus $\mathcal{S}^n_+$ is a semialgebraic set and hence closed.

   - $\mathcal{S}^n_+$ is a **convex cone**: Let $A,B\in\mathcal{S}^n_+$ and $\alpha,\beta\ge 0$, then
   $$x^\top(\alpha A+\beta B)x=\alpha x^\top Ax+\beta x^\top Bx\ge 0,$$
   so $\alpha A+\beta B\in\mathcal{S}^n_+$.

   - $\mathcal{S}^n_+$ is **pointed**: If A has a positive eigenvalue then $-A$ has a negative eigenvalue. The only matrix with only-zero eigenvalues is the zero matrix.

   - $\mathcal{S}^n_+$ is **full dimensional**: The identity matrix $I_n\in\mathcal{S}^n_+$. Consider any symmetric matrix $A\in\mathcal{S}^n$ and $\varepsilon>0$ and small. Then for any $x\in\mathbb{R}^n$,
   $$x^\top(I_n+\varepsilon A)x=x^\top x+\varepsilon x^\top Ax=\sum x_i^2+\varepsilon\sum a_{ij}x_ix_j\ge 0$$
   for ε sufficiently small. Therefore I_n is in the interior of $\mathcal{S}^n_+$ and $\mathcal{S}^n_+$ is full-dimensional of dimension $\frac{n(n+1)}{2}$.
   Remark: The matrices in the interior $(\mathcal{S}^n_+)^\circ$ are precisely the ones with positive eigenvalues, so $(\mathcal{S}^n_+)^\circ=\mathcal{S}^n_{++}$. The matrices at the boundary of $\mathcal{S}^n_+$ are precisely the psd matrices with at least one zero eigenvalue.

   □

   (b) Prove that the rank one matrices in $\mathcal{S}^n_+$ generate its *extreme rays* (i.e., rays that cannot be written as a non-negative combination of other rays in $\mathcal{S}^n_+$). Recall that a rank one matrix in $\mathcal{S}^n_+$ looks like $aa^\top$ where $a\in\mathbb{R}^n$.

Solution. Let $X, Y \in \mathcal{S}_+^n \setminus \{0\}$ and assume that $X + Y = aa^\top$. Take a vector $v \in \{a\}^\perp$, i.e., in the orthogonal complement of a, so $v^\top a = 0$. Then $0 \le v^\top(X+Y)v = v^\top aa^\top v = 0$ hence $v^\top Xv = 0$ and $v^\top Yv = 0$, i.e., $v \in \ker(X)$ and $v \in \ker(Y)$ because $X, Y \in \mathcal{S}_+^n$ (write $X = UU^\top$ and use that a sum of squares is zero if and only if each square is zero). So $\{a\}^\perp = \ker(X) = \ker(Y)$ because X, Y are nonzero, i.e., X, Y have rank 1. So $X = xx^\top$ and $Y = yy^\top$ for $x, y \in \mathbb{R}^n$, and $\{x\}^\perp = \{y\}^\perp = \{a\}^\perp$, so both x and y are multiples of a and therefore both X and Y are multiples of $aa^\top$. □

(c) By *Caratheodory's theorem* from convex geometry, every element in $\mathcal{S}_+^n$ can be written as a non-negative combination of at most $\frac{n(n+1)}{2}$ extreme rays of $\mathcal{S}_+^n$ On the other hand, the previous exercise allows you to bound the number of rank one matrices needed to write a psd matrix in $\mathcal{S}_+^n$ as a non-negative combination. How do these bounds compare?

Solution. Recall that the dimension of $\mathcal{S}_+^n$ as a cone in the vector space of all symmetric $n \times n$ matrices is $\frac{n(n+1)}{2}$. Caratheodory's theorem implies that the number of psd matrices of rank one needed to write a psd matrix as a conical combination is at most $\frac{n(n+1)}{2}$. But the *spectral decomposition* of a symmetric matrix allows us to write a psd matrix A as a sum of $\operatorname{rank}(A)$ ($\le n$) psd matrices. For example, if A is a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$ then $A = \lambda_1 v_1 v_1^\top + \cdots + \lambda_n v_n v_n^\top$. Also notice that a matrix M cannot be written as a sum of less than $\operatorname{rank}(M)$ rank 1 matrices since $\operatorname{rank}(X+Y) \le \operatorname{rank}(X) + \operatorname{rank}(Y)$. □

6. Recall that the feasible region of a semidefinite program (SDP) is called a *spectrahedron*. We may take the following to be the official definition:

Definition 0.1. A spectrahedron is a set of the form

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m \,:\, A_0 + \sum A_i x_i \succeq 0\}$$

where the matrices $A_i \in \mathcal{S}^n$.

(a) In the lecture we defined a *spectrahedron* to be an affine slice of the psd cone. Indeed, the matrices defined by the above set is the intersection of the psd cone $\mathcal{S}_+^n$ with the affine plane obtained by translating $\operatorname{span}(A_1, \dots, A_m)$ by A_0. If the matrices $A_1, \dots, A_m$ are linearly independent in $\mathcal{S}^n$ then prove that there is a bijection between the two descriptions of a spectrahedron as a subset of $\mathbb{R}^m$ and $\mathcal{S}^n$ respectively.

Solution. Let $\mathcal{L} = \{A_0 + \sum_{i=1}^m A_i x_i \,:\, x_i \in \mathbb{R}\}$ and $B \in \mathcal{L} \cap \mathcal{S}_+^n$. Then $B = A_0 + \sum_{i=1}^m A_i x_i$ for some $x_i \in \mathbb{R}$. Suppose there exists another set of scalars $y_1, \dots, y_m \in \mathbb{R}$ such that $B = A_0 + \sum_{i=1}^m A_i y_i$. Then $\sum_{i=1}^m A_i(x_i - y_i) = 0$. We have that $x_i = y_i$ for all i if and only if $A_1, \dots, A_m$ are linearly independent. □

(b) Prove that a spectrahedron also admits the following descriptions:

i. $\{X \in \mathcal{S}^n_+ : \langle B_j, X\rangle = b_i \ \forall\ j = 1, \ldots, t\}$, for some symmetric matrices $B_j \in \mathcal{S}^n$,

ii. $\{x \in \mathbb{R}^s : p_j(x) \geq 0 \ \ p_j \in \mathbb{R}[x_1, \ldots, x_s],\ j = 1, \ldots, r\}$

How do t, s and r relate to m and n?

Solution. i. An affine space in $\mathcal{S}^n$ has the form

$$\{X \in \mathcal{S}^n : \langle B_j, X\rangle = b_i \ \forall\ j = 1, \ldots, t\}$$

for some matrices $B_j \in \mathcal{S}^n$ and scalars $b_j \in \mathbb{R}$. Therefore, every spectrahedron has the form i. Comparing to the above definition of a spectrahedron, $\mathrm{Span}(A_1, \ldots A_m) = \{X \in \mathcal{S}^n : \langle B_j, X\rangle = 0 \ \forall\ j = 1, \ldots, t\}$ and $b_j = \langle B_j, A_0\rangle$ for all $j = 1, \ldots, t$.

ii. Again, using the given definition of a spectrahedon we have that $A(x) = A_0 + \sum A_i x_i$ is in the spectrahedron if and only if it is psd which is if and only if all principal minors of $A(x)$ are nonnegative. This is a finite collection of polynomial inequalities of the form $p(x) \geq 0$.

The quantity m can be anything but if we assume that $A_1, \ldots, A_m$ are linearly independent, then $m \leq \frac{n(n+1)}{2}$. The matrices B_j span the orthogonal complement of $\mathrm{Span}(A_1, \ldots A_m)$. Therefore, if $d = \dim(\mathcal{L})$ then $t = \frac{n(n+1)}{2} - d$. Every entry of $A(x)$ is a linear polynomial in m variables. Each $p(x)$ is a principal minor of $A(x)$ and hence a polynomial in $x_1, \ldots, x_m$. Therefore, $s = m$. The quantity r is the number of principal minors of $A(x)$ and hence at most 2^n . □

(c) Using any of the above descriptions, argue that a spectrahedron is closed, convex and basic semi-algebraic.

Solution. Since the intersection of an affine space with a closed conves cone is closed and convex, spectrahedra are closed and convex. Description ii. shows that a spectrahedron is semialgebraic. □

(d) Consider the following concrete spectrahedron:

$$\mathcal{F} := \left\{ (x, y) \in \mathbb{R}^2 : \begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix} \succeq 0 \right\}.$$

i. Express $\mathcal{F}$ in the two other formats mentioned above.

Solution. This spectrahedron is

$$\mathcal{F} = \left\{ (x, y) \in \mathbb{R}^2 : \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}}_{A_1} x + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A_2} y \right\}$$

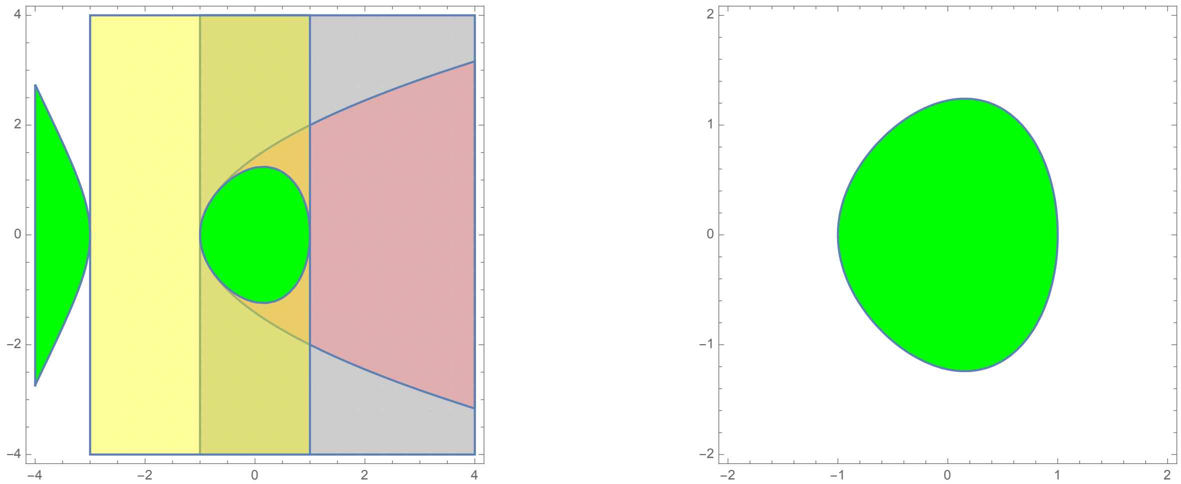


Figure 1: The spectrahedron is the green bubble, isolated on the right. In the left figure, the green region is where $-x^3 - 3x^2 - 2y^2 + x + 3 \geq 0$, the pink region is where $2(x+1) \geq y^2$, the grey region is where $x + 1 \geq 0$ and the yellow region is where $4 \geq (x+1)^2$.

Check that A_1 and A_2 are linearly independent. Therefore, the orthogonal complement of their span has dimension $6 - 2 = 4$. Hence there will be four matrices $B_1, \ldots, B_4 \in \mathcal{S}^3$ in description i of $\mathcal{F}$. For example, take

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix},$$

and $b_1 = 2, b_2 = 2, b_3 = 0, b_4 = 0$.
Description ii. Computing all the principal minors of

$$A(x,y) = \begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix}$$

we get that $\mathcal{F} =$

$$\left\{(x,y) \in \mathbb{R}^2 : x+1 \geq 0,\ 2(x+1) \geq y^2,\ 4 \geq (x+1)^2,\ -x^3 - 3x^2 - 2y^2 + x + 3 \geq 0\right\}.$$

□

ii. Draw this set in the plane.
Solution. See Figure 1. □

iii. What is the polynomial that defines the boundary of $\mathcal{F}$? Generalize your result to the general spectrahedron in Definition 0.1.

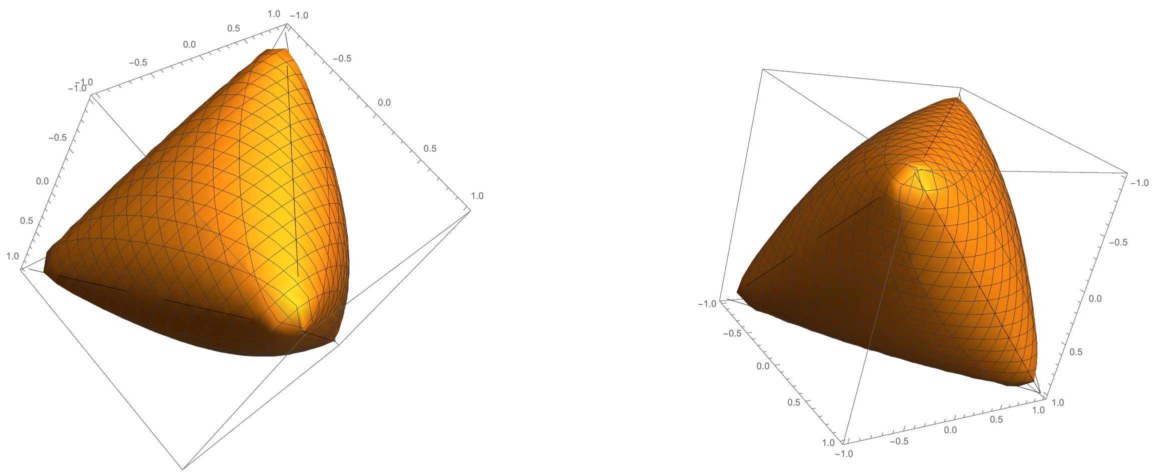


Figure 2: Two views of the elliptope.

Solution. The boundary is defined by $\det(A(x,y)) = -x^3 - 3x^2 - 2y^2 + x + 3 = 0$. In general, the boundary of a spectrahedron consists of the intersection of the affine plane $\{A_0 + \sum A_i x_i : x_i \in \mathbb{R}\}$ with the boundary of $\mathcal{S}_+^n$. On the boundary of $\mathcal{S}_+^n$ we have all the matrices with at least one zero eigenvalue and hence rank deficient. Thus, $\det(A(x) = 0$ on the boundary of the spectrahedron. □

7. A very common example of a spectrahedron is the *elliptope* $\mathcal{E}_n$ defined as follows.

$$\mathcal{E}_n := \{X \in \mathcal{S}_+^n : X_{ii} = 1 \ \forall\ i = 1, \dots, n\}.$$

(a) What is the dimension of $\mathcal{E}_n$?

Solution. The matrices in the elliptope have three degrees of freedom after the diagonals are fixed to one. Thus the elliptope is three dimensional. □

(b) Use a computer to draw $\mathcal{E}_3$.

Solution. See Figure 2. □

(c) What are the rank one psd matrices on $\mathcal{E}_3$? Can you see them in your picture?

Solution. These are the four corners of the elliptope. They are the matrices:

$$(1,1,1)^\intercal(1,1,1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad (1,-1,1)^\intercal(1,-1,1) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(1,1,-1)^\intercal(1,1,-1)\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}, (1,-1,-1)^\intercal(1,-1,-1)\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{pmatrix},$$

□

Check that all other matrices of the form $aa^\intercal$ where $a \in \{-1,1\}^3$ are in this list.

(d) Find a rank two matrix on $\mathcal{E}_3$ that is not a convex combination of the rank one matrices on $\mathcal{E}_3$.

Solution. Such a matrix is on the curvy part of the boundary. □

(e) Can you model the max cut problem as an SDP over $\mathcal{E}_n$ with possibly additional rank constraints?

Solution. Recall that we were modeling the cut induced by $T \subseteq [n]$ by assigning 1 to vertices in T and -1 to vertices not in T. Let $v(T)$ be the ± 1 vector in $\mathbb{R}^n$ so obtained. Then $X = v(T)v(T)^\intercal \in \mathcal{E}_n$. The cut edges are precisely those edges ij for which the ij-entry of this matrix is -1. Therefore, the size of the cut is $\frac{1}{2}\sum(1 - x_{ij})$. The set of all cuts in G correspond bijectively to the rank one matrices in $\mathcal{E}_n$. Therefore, the max cut problem is

$$\begin{aligned} \max \frac{1}{2}\sum(1 - x_{ij}) \\ X \in \mathcal{E}_n \\ \operatorname{rank}(X) = 1 \end{aligned}$$

This is a rank constrained SDP since we are optimizing a linear function over (the rank one matrices of) a spectrahedron. □

8. Check that the following basic facts are true for a sos polynomial $p = \sum h_j^2$ in $\mathbb{R}[x]$.

(a) $\deg(p) = 2d \;\Rightarrow\; \deg(h_j) \le d$.

(b) p homogeneous and $\deg(p) = 2d \;\Rightarrow\; h_j$ homogeneous and $\deg(h_j) = d$.

(c) If $\tilde{p}$ is the homogenization of p then $p \ge 0$ (resp. sos) $\;\Leftrightarrow\; \tilde{p} \ge 0$ resp. sos.

(d) If $\deg(p) = 2d$, bound the number of squares needed in the sos expression for p. (Hint: use that p sos if and only if $p = [x]_d^\intercal Q[x]_d$ for some $Q \succeq 0$.)

Solution. (a) If $\deg(j_j) > d$, then $\deg(h_j^2) > 2d$ and since $h_j \in \mathbb{R}[x]$, the coefficient of the lead term is positive, so that none of the leading terms will cancel; thus $\deg(p) > 2d$.

(b) Suppose p is homogeneous and write $h_j = f_j + g_j$, for $f_j, g_j \in \mathbb{R}[x]$ where f_j is the degree d terms and $\deg(g_j) < d$. Then $h_j^2 = f_j^2 + 2f_jg_j + g_j^2$, where $\deg(g_j^2) < 2d$, and the coefficient of the lead term will be non-negative. Since $p = \sum_j (f_j^2 + 2f_jg_j + g_j^2)$ is homogeneous of degree d, and the lead terms of the g_j^2 cannot cancel, we get that $g_j = 0$, hence $h_j = f_j$, which is homogeneous of degree d.

(c) Suppose $\tilde{p}(x, x_{n+1}) \geq 0$. Then $p(x) = \tilde{p}(x, 1) \geq 0$. Conversely, if $p(x) \geq 0$, then

$$\tilde{p}(x, x_{n+1}) = x_{n+1}^d \cdot p\left(\frac{x}{x_{n+1}}\right) \geq 0,$$

since the degree d of a non-negative polynomial must be even.

(d) The vector $[x]_d$ consists of all monomials of degree at most d and hence has size $\binom{n+d}{d}$. Therefore, Q is a (square) matrix of the same size and hence rank at most $\binom{n+d}{d}$. This means that the number of squares needed cannot exceed $\binom{n+d}{d}$ which is the maximum number of rank one matrices in the spectral decomposition of Q.

□

9. (a) Write the following polynomial as a sos: $x^2 + 4x + 5$.

Solution. $x^2 + 4x + 5 = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 \\ x \end{pmatrix}$ □

(b) Create a sos polynomial of degree four in three variables x, y, z by using the fact that any such polynomial will look like $[x]_2^\top U U^\top [x]_2$ where $[x]_2$ is the vector of all monomials of degree at most two in x, y, z.

10. Express $2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2$ as a sos using the connection to psd matrices and SDP.

Solution. Let $p = 2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2$. Since the degree of p is 4, we take $[x]_4 = (1, x, y, x^2, xy, y^2)^\top$ and want to express $p = ([x]_4)^\top Q [x]_4$ where $Q \in \mathcal{S}_+^6$. Setting up a symbolic Q, we need to solve the system:

$$2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2 = (1, x, y, x^2, xy, y^2) \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_2 & a_7 & a_{11} & a_{12} & a_{13} & a_{14} \\ a_3 & a_8 & a_{12} & a_{15} & a_{16} & a_{17} \\ a_4 & a_9 & a_{13} & a_{16} & a_{18} & a_{19} \\ a_5 & a_{10} & a_{14} & a_{17} & a_{19} & a_{20} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$$

with the additional requirement that Q is psd. Expanding the right-hand side and equating coefficients of like monomials we get the following system:

$$a_0 = 2, a_1 = 1, a_2 = 0, 2a_3 = -a_6, a_4 = -a_7, 2a_5 = -a_{11}, a_8 = 0, a_9 = -a_{12}, a_{10} = -a_{13},$$
$$a_{14} = 0, a_{15} = 2, a_{16} = 1, -2a_{17} - a_{18} = 1, a_{19} = 0, a_{20} = 5.$$

It is not so easy now to choose the free variables to make Q psd. Normally, we would solve an SDP using the computer to find a psd matrix Q that satisfies the above linear conditions. However, let's plug in what we have and see how far we can get

$$Q = \begin{pmatrix} 2 & 1 & 0 & -\frac{1}{2}a_6 & -a_7 & -\frac{1}{2}a_{11} \\ 1 & a_6 & a_7 & 0 & -a_{12} & -a_{13} \\ 0 & a_7 & a_{11} & a_{12} & a_{13} & 0 \\ -\frac{1}{2}a_6 & 0 & a_{12} & 2 & 1 & a_{17} \\ -a_7 & -a_{12} & a_{13} & 1 & a_{18} & 0 \\ -\frac{1}{2}a_{11} & -a_{13} & 0 & a_{17} & 0 & 5 \end{pmatrix}$$

In the absence of a computer, we'll work with some hints: set $a_7 = 0, a_{12} = 0, a_{13} = 0$:

$$Q = \begin{pmatrix} 2 & 1 & 0 & -\frac{1}{2}a_6 & 0 & -\frac{1}{2}a_{11} \\ 1 & a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & 0 & 0 \\ -\frac{1}{2}a_6 & 0 & 0 & 2 & 1 & a_{17} \\ 0 & 0 & 0 & 1 & a_{18} & 0 \\ -\frac{1}{2}a_{11} & 0 & 0 & a_{17} & 0 & 5 \end{pmatrix}$$

Now you still have some freedom for the choice of the remaining variables. One such choice leads to the following Q:

$$Q = \frac{1}{3}\begin{pmatrix} 6 & 3 & 0 & -2 & 0 & -2 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 6 & 3 & -4 \\ 0 & 0 & 0 & 3 & 5 & 0 \\ -2 & 0 & 0 & -4 & 0 & 15 \end{pmatrix}$$

Now we need to factorize $Q = BB^\intercal$ to get the sos expression for p. This also requires a computer. But the following sos expression works:

$$p = \frac{4}{3}y^2 + \frac{1349}{705}y^4 + \frac{1}{12}(4x+3)^2 + \frac{1}{15}(3x^2+5xy)^2 + \frac{1}{315}(-21x^2+20y^2+10)^2 + \frac{1}{59220}(328y^2-235)^2.$$

What is B in this case? Check that $Q = BB^\intercal$. $\square$

We now do this example using Macaulay2 using the package SOS.m2:

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/Applicati

o1 = SOS

o1 : Package

i2 :
     R = QQ[x,y]

o2 = R

o2 : PolynomialRing

i3 : f = 2*x^4+5*y^4-x^2*y^2+2*x^3*y+2*x+2

       4     3     2 2     4
o3 = 2x  + 2x y - x y  + 5y  + 2x + 2

o3 : R

i4 : (Q,mon,X) = solveSOS(f, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63368-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63368-0/1
Success: SDP solved

i5 : s = sosdec(Q,mon)

o5 = coeffs:
         11  17  1912  2083   1313
     {5, --, --, ----, ----, -----}
          5  11  2125  1912  10415
     gens:
         8 2    2   1    1   5 2          5     5     5 2   11          5   2    55
     {- --x  + y  - -x - -, --x  + x*y + --y - --, - --x  + --x + y + --, x  - ----
        25          5    5  11           11    11    17     17         17       1912

o5 : SOSPoly
```

```
-- the following command helps to see it the output better

i6 : toString s

o6 = new SOSPoly from {ring => R,

     coefficients => {5, 11/5, 17/11, 1912/2125, 2083/1912, 1313/10415},
     generators =>  {-(8/25)*x^2+y^2-(1/5)*x-1/5, (5/11)*x^2+x*y+(5/11)*y-5/11,
     -(5/17)*x^2+(11/17)*x+y+5/17, x^2-(55/1912)*x-705/1912, (949/2083)*x+1, x}}

--- This writes the polynomial as another sum of 6 squares.

--- we now check if we got the right polynomial by summing the sos from above:

i7 : sumSOS(s)

            4       3      2 2      4
o7 = 2x  + 2x y - x y  + 5y  + 2x + 2

o7 : R
```

11. Prove that a univariate non-negative polynomial is always a sum of two squares. (Hint: Make an argument about the possible real and complex roots of this polynomial and use the identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$ for all $a, b, c, d \in \mathbb{R}$.)

Solution. Notice that in a non-negative univariate polynomial all real roots must be double roots, all complex roots come in conjugate pairs $a+ib, a-ib$. So the polynomial looks like

$$(x-\alpha_1)^2 \cdots (x-\alpha_n)^2 (x-a_1-ib_1)\cdots(x-a_m-ib_m)\cdot(x-a_1+ib_1)\cdots(x-a_m+ib_m).$$

Consider rearranging the pairs of conjugate roots as follows.

$$\begin{aligned}
&(x-a_1-ib_1)(x-a_2-ib_2)(x-a_1+ib_1)(x-a_2+ib_2)\\
&= \Big(\big((x-a_1)(x-a_2)-b_1b_2\big) - i\big(b_1(x-a_2)+b_2(x-a_1)\big)\Big)\Big(\big((x-a_1)(x-a_2)-b_1b_2\big) + i\big(b_1(x-a_2)+b_2(x-a_1)\big)\Big)\\
&= \big((x-a_1)(x-a_2)-b_1b_2\big)^2 + \big(b_1(x-a_2)+b_2(x-a_1)\big)^2
\end{aligned}$$

Then by the given identity we can combine the sums of squares obtained from the complex roots as above repeatedly to maintain a sum of two squares.

Note: For a similar solution let $z = a+bi$ so

$$(x-z)(x-\bar{z}) = x^2 - (z+\bar{z})x + |z|^2 = x^2 - 2ax + a^2 + b^2 = (x-a)^2 + b^2,$$

then proceed to use the mentioned identity (aka *Diophantus identity* or *Brahmagupta-Fibonacci identity*). □

12. (Ex 3.35) Can you express $x^4 + 4x^3 + 6x^2 + 4x + 5$ as a sum of two squares?

Solution.

$$x^4 + 4x^3 + 6x^2 + 4x + 5 = (x^2 + 2x + 1)^2 + 2^2$$

□

We can also do this problem using the SOS.m2 package in Macaulay2 as follows.

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "SOS"
--loading configuration for package "SOS" from file /Users/thomas/Library/
Application Support/Macaulay2/init-SOS.m2

o1 = SOS

o1 : Package

i2 : R = QQ[x]

o2 = R

o2 : PolynomialRing

i3 : f = x^4 + 4*x^3 + 6*x^2 + 4*x +5

      4     3     2
o3 = x  + 4x  + 6x  + 4x + 5

o3 : R

i4 : (Q,mon,X) = solveSOS f

o4 = (| 5    2    -4/5 |, {0}  | 1  |, )
      | 2    38/5 2    |  {-1} | x  |
```

```
      | -4/5 2    1    |  {-2} | x2 |

----- Q is the Gram matrix in the sos decomposition

o4 : Sequence

i5 : (g,d) = sosdec(Q,mon)

           5 2           5     126 2          2     38   85    171
o5 = ({--x   + x + --, - --- x   + 1, x }, {--,    --,     ----})
          19          19    425                     5    19    2125

o5 : Sequence

--- this command allows you to see the output in o5 accurately.

i6 : toString o5

o6 = ({(5/19)*x^2+x+5/19, -(126/425)*x^2+1, x^2},{38/5, 85/19,
     171/2125})

i6 : sumSOS(g,d)

          4       3       2
o6 = x   + 4x   + 6x   + 4x + 5

o6 : R

i7 : toString o6

o7 = x^4+4*x^3+6*x^2+4*x+5
```

Note that the computer gave a different sos decomposition of the polynomial.

13. (Ex 3.54) Let $p(x) = \sum_{k=0}^{2d} c_k x^k$. Give an explicit SDP formulation to compute the value of the global min of $p(x)$.

    (a) Show that the min of $p(x) = x^4 - 20x^2 + x$ is less than or equal to -100.

    (b) Show that the min of $p(x) = x^4 - 20x^2 + x$ is greater than or equal to -104.

    (c) Minimize the polynomial $p(x) = x^4 - 20x^2 + x$.

Solution. (a) Minimizing $p(x)$ is equivalent to the problem $\sup\{\lambda : p(x) - \lambda \geq 0\}$. Writing its sos relaxation we want to solve

$$\sup\left\{\lambda : p(x) - \lambda = (1\, x\, x^2\, x^3\, \ldots x^d)Q(1\, x\, x^2\, x^3\, \ldots x^d)^\intercal\ Q \succeq 0\right\}.$$

Assume that the rows and columns of Q are indexed from 0 to d. Then expanding the right hand side you get that the coefficient of x^k is the sum of all anti-diagonal entries in Q where the indices sum up to k.
We do this for $p(x) = x^4 - 20x^2 + x$.

$$\begin{aligned} x^4 - 20x^2 + x - \lambda =& (1\, x\, x^2)\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \\ =& a + 2bx + (2c+d)x^2 + 2ex^3 + fx^4 \end{aligned}$$

This means that $a = -\lambda$, $2b = 1$, $(2c+d) = -20$, $e = 0$, $f = 1$ and hence

$$Q = \begin{pmatrix} -\lambda & \frac{1}{2} & c \\ \frac{1}{2} & -20-2c & 0 \\ c & 0 & 1 \end{pmatrix}.$$

Since Q must be psd we must have that $\lambda \leq 0$ and $-20-2c \geq 0$ from the diagonal elements of Q. This implies that $c \leq -10$ and hence, $c^2 \geq 100$. From the 2×2 principal minors of Q we get that $-\lambda \geq c^2 \geq 100$ which means that $\lambda \leq -100$.

(b) Suppose $\lambda = -104$. Then we check if there is a $c \leq -10$ such that Q is psd. If $c = -10.1$ then we get

$$Q = \begin{pmatrix} 104 & \frac{1}{2} & -10.1 \\ \frac{1}{2} & 0.2 & 0 \\ -10.1 & 0 & 1 \end{pmatrix} \succeq 0$$

(c) We check if $\lambda = -100$ can work. It cannot work since then we need $c = -10$, but we cannot use $c = -10$ since then the $(2,2)$ entry becomes 0 which means that it's row and column must all be zero which is not the case. The problem we need to solve is the semidefinite program:

$$\sup\{\lambda : Q \succeq 0\}.$$

We could use a SDP solver to solve this problem. The answer is not so easy to see without some technology. We can draw the spectrahedron $Q \succeq 0$ and we see that the biggest λ is near -103.

□

We now use M2 to do part c) accurately.

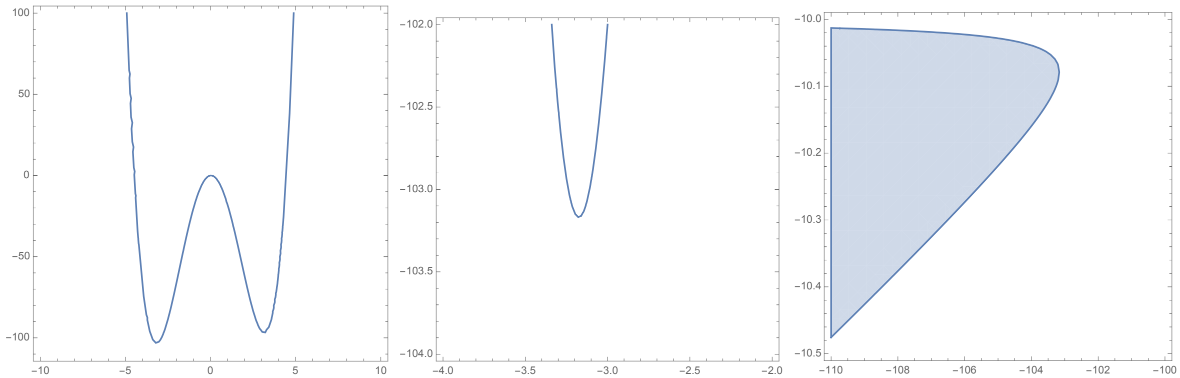


Figure 3: The graph of $y = x^4 - 20x^2 + x$, and a zoomed in view of the minimum. A part of the spectrahedron $Q \geq 0$.

```
Macaulay2, version 1.7
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/
Application Support/Macaulay2/init-SOS.m2

-- now we are choosing to use the SDP solver called "CSDP" instead
-- of the default solver in M2.

o1 = SOS

o1 : Package

i2 : R = QQ[x,t];

i3 : f2 = x^4 - 20*x^2 + x;

i4 : (Q,mon,X,tval) = solveSOS(f2-t,{t},-t, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-58135-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-58135-0/1
Success: SDP solved

i5 : tval

        1651
o5 = {- ----}
         16
```

```
o5 : List

-- tval is the minimum value and it is roughly -103.1875

i6 : toString Q

o6 = matrix {{1651/16, 1/2, -807/80}, {1/2, 7/40, 0}, {-807/80, 0, 1}}

i7 : toString mon

o7 = matrix {{1}, {x}, {x^2}}
```

14. (a) (Ex 3.57) Find the value of p_{sos} for the polynomial $p(x,y,z) = x^4 + y^4 + z^4 - 4xyz + 2x + 3y + 4z$ over $\mathbb{R}^3$. Is $p_* = p^{\text{sos}}$ in this example? Do you expect $p_* = p^{\text{sos}}$?

    (b) Find the value of p_{sos} and p_* for the polynomial $p(x,y) = x^4 + y^4 - 4xyz$ over $\mathbb{R}^2$. Do you expect $p_* = p^{\text{sos}}$?

Proof. (a) The polynomial $p(x,y,z) - \lambda$ is a quartic polynomial in 3 variables, so Hilbert's theorem does not guarantee that it will be sos for any λ. Therefore, there is a good chance that $p_* < p^{\text{sos}}$. You will need a computer to fully do this example. The psd matrix you need to set up has size 10×10.

$\square$

In M2 we use the following commands:

```
needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
R = QQ[x,y,z,t]
p = x^4+y^4+z^4 - 4*x*y*z + 2*x + 3*y + 4*z
(Q,mon,X,tval) = solveSOS(p-t,{t},-t, Solver=>"CSDP");
```

This gives the minimum value $tval = -\frac{115}{16} = -7.1875$. This the value of p^{sos}.
FIND THE TRUE MINUMUM OF THIS POLYNOMIAL.

Proof. (b) Here is a picture of the graph of $p(x,y) = x^4 + y^4 - 4xyz$. The minimum appears to be between -2 and -1. By Hilbert's theorem we should get $p_* = p^{\text{sos}}$.

$\square$

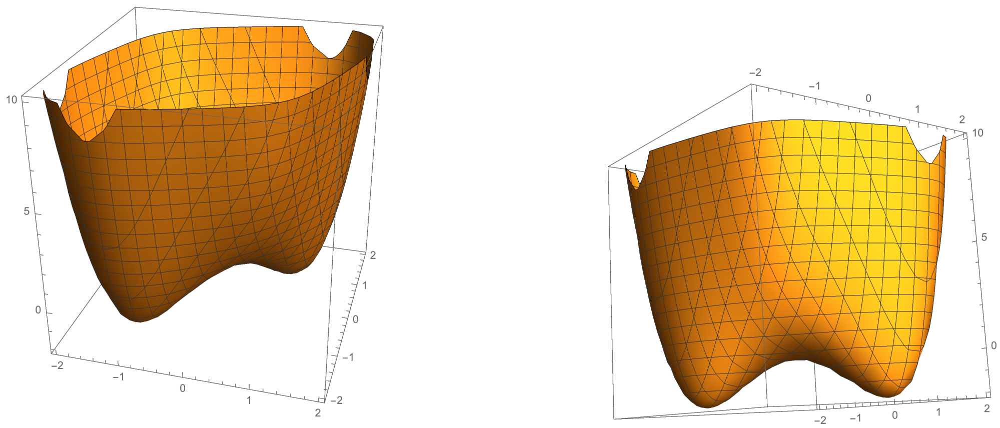


Figure 4: The graph of $z = x^4 + y^4 - 4xy$ (two different views).

```
needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
R = QQ[x,y,t]
p = x^4+y^4 - 4*x*y
(Q,mon,X,tval) = solveSOS(p-t,{t},-t, Solver=>"CSDP");

i5 : tval

o5 = {-2}

i6 : toString Q

o6 = matrix {{2, 0, 0, -1/2, -1, -1/2}, {0, 1, -1, 0, 0, 0}, {0, -1, 1,
     0, 0, 0}, {-1/2, 0, 0, 1, 0, -1/2}, {-1, 0, 0, 0, 1, 0}, {-1/2, 0,
     0, -1/2, 0, 1}}

i7 : toString mon

o7 = matrix {{1}, {x}, {y}, {x^2}, {x*y}, {y^2}}

-- This means that the polynomial p+2 must be a sos. We can get its
sos decomposition using:

(Q,mon,X) = solveSOS(p+2, Solver=>"CSDP");
(g,d) = sosdec(Q,mon);
i12 : toString oo
```

```
o12 = ({-(1/4)*x^2-(1/2)*x*y-(1/4)*y^2+1, x-y, x^2-(2/7)*x*y-(5/7)*y^2,
      x*y-y^2},{2, 1, 7/8, 3/7})

-- compute the sos to check if we get back the polynomial p+2

i13 : sumSOS(g,d)

       4    4
o13 = x  + y  - 4x*y + 2

-- The matrix Q might be interesting to look at:
i14 : toString Q

o14 = matrix {{2, 0, 0, -1/2, -1, -1/2}, {0, 1, -1, 0, 0, 0}, {0, -1,
      1, 0, 0, 0}, {-1/2, 0, 0, 1, 0, -1/2}, {-1, 0, 0, 0, 1, 0},
      {-1/2, 0, 0, -1/2, 0, 1}}

 i15 : toString mon

o15 = matrix {{1}, {x}, {y}, {x^2}, {x*y}, {y^2}}
```

$$Q = \begin{pmatrix} 2 & 0 & 0 & -1/2 & -1 & -1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1 & 0 & -1/2 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & -1/2 & 0 & 1 \end{pmatrix}$$

15. The Newton polytope of a polynomial $p(x_1, \dots, x_n)$ is the convex hull of all the non-negative integer vectors in $\mathbb{N}^n$ that appear as exponents of the monomials present in p. We will denote it as $\mathcal{N}(p)$. For example, $\mathcal{N}(x^2 + xy + y^2)$ is the line segment in $\mathbb{R}^2$ that is the convex hull of $(2,0), (1,1), (0,2)$. Reznick proved the following theorem:

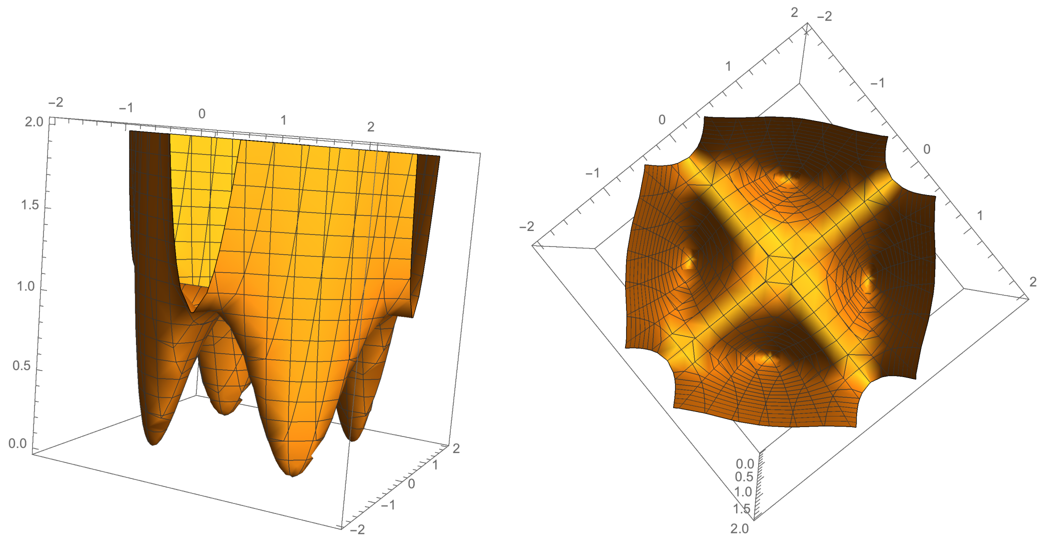
$$\text{If } p = \sum q_j^2 \text{ then } \mathcal{N}(q_i) \subseteq \tfrac{1}{2}\mathcal{N}(p) \text{ for each } i.$$

(Ex 3.97)

(a) Compute the Newton polytope of the Motzkin polynomial.

(b) Which monomials would appear in a hypothetical sos decomposition of the Motzkin polynomial if you know the above theorem?

(c) Show by considering the coefficient of x^2y^2, and the above calculation, that the Motzkin polynomial is not a sos.



Solution. (a)

$$\mathcal{N}(x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2) = \operatorname{conv}\{(4,2,0),(2,4,0),(0,0,6),(2,2,2)\}$$

Notice that since $(2,2,2)$ is in the convex hull of other three, the Newton polytope is just the triangle defined by $(4,2,0),(2,4,0),(0,0,6)$ in $\mathbb{R}^3$.

(b) By the above theorem, we can only have exponent vectors which appear as integer points in $\frac{1}{2}$ times the triangle of part (a); that is, $(2,1,0),(1,2,0),(0,0,3),(1,1,1)$. Thus we can have monomials x^2y, xy^2, z^3, xyz.

(c) Now if the Motzkin polynomial was sos, it would have the form

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 = \sum_i (c_{i,1}x^2y + c_{i,2}xy^2 + c_{i,3}z^3 + c_{i,4}xyz)^2$$

and the only way to obtain the $x^2y^2z^2$ term is as the sum of the $c_{i,4}^2x^2y^2z^2$ terms, which will never have a negative coefficient.

□

16. (Ex 3.69) Consider the quartic form in four variables:

$$p(w,x,y,z) = w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz.$$

(a) Show that p is not a sos. (Hint: Use Reznick's result mentioned in Exercise 7).

(b) Find a multiplier that makes the product a sos.

Proof. (a) If p is a sos, the fact that it is homogeneous implies that it can be written as a sum of homogeneous squares of half its degree. So suppose $p = \sum q_i^2$ where the q_i are homogeneous of degree 2. By comparing Newton polytopes (using Reznick's result mentioned in Exercise 7) one obtains that the only monomials appearing in any q_i can be xy, xz, yz and w^2 (discard the other monomials by showing that their corresponding points are not in $\frac{1}{2}\mathcal{N}(p)$), so the monomial $wxyz$ cannot appear after squaring. $\square$

We do the next part in M2 and see that multiplying p by $w^2 + x^2 + y^2 + z^2$ makes it sos.

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/
Library/Application Support/Macaulay2/init-SOS.m2

o1 = SOS

o1 : Package

i2 : R = QQ[w,x,y,z]

o2 = R

o2 : PolynomialRing

i3 : p = w^4 + x^2*y^2 + x^2*z^2 + y^2*z^2 - 4*w*x*y*z

      4    2 2                2 2    2 2
o3 = w  + x y  - 4w*x*y*z + x z  + y z

o3 : R

i4 : m = w^2+x^2+y^2+z^2;

i5 : p1 = m*p;

i6 : (Q,mon,X) = solveSOS(p1, Solver=>"CSDP");
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-59042-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-59042-0/1
Success: SDP solved
rounding failed, returning real solution
```

```
-- this means that the solver failed to return a rational Gram matrix.

i7 : (g,d) = sosdec(Q,mon)

                2          2                        2
o7 = ({- .142857w y - .428571x y + w*x*z - .428571y*z , w*x*y -
     ------------------------------------------------------------------------
             2             2             2             2            2
     .142857w z - .428571x z - .428571y z, - .142857w x - .428571x*y  +
     ------------------------------------------------------------------------
                        2            3            2          2        2
     w*y*z - .428571x*z , - .999999w  + x*y*z, w x - .5x*y  - .5x*z ,
     ------------------------------------------------------------------------
      3   2        2          2   2        2             2     2
     w , w y - .5x y - .5y*z , x y - 1y*z , - .833333w z + x z -
     ------------------------------------------------------------------------
             2      2       2   2      2      2   2      2
     .166667y z, x*y  - 1x*z , w z - 1y z, x*z , y z, y*z }, {2.33333,
     ------------------------------------------------------------------------
     2.33333, 2.33333, 1, .952381, 4.7892e-8, .952381, .333333,
     ------------------------------------------------------------------------
     .571429, .333333, .555556, 7.37129e-9, 7.37129e-9, 7.37129e-9})

o7 : Sequence

i8 : sumSOS(g,d)

       6     4 2     4 2     2 2 2     4 2     2 4     3
o8 = 1w  + 1w x  + 1w y  + 1w x y  + 1x y  + 1x y  - 4w x*y*z -
     ------------------------------------------------------------------------
         3            3      4 2     2 2 2     4 2     2 2 2     2 2 2
     4w*x y*z - 4w*x*y z + 1w z  + 1w x z  + 1x z  + 1w y z  + 3x y z
     ------------------------------------------------------------------------
         4 2            3     2 4     2 4
     + 1y z  - 4w*x*y*z  + 1x z  + 1y z

o8 : RR  [w, x, y, z]
       53

i9 : p1

      6    4 2    4 2    2 2 2    4 2    2 4    3             3
```

```
o9 = w  + w x  + w y  + w x y  + x y  + x y  - 4w x*y*z - 4w*x y*z -
     ------------------------------------------------------------------
           3     4 2    2 2 2    4 2    2 2 2     2 2 2    4 2
     4w*x*y z + w z  + w x z  + x z  + w y z  + 3x y z  + y z  -
     ------------------------------------------------------------------
             3    2 4    2 4
     4w*x*y*z  + x z  + y z

o9 : R
```

The polynomial output by sumSOS has real coefficients and is in a different ring from p. Nevertheless, comparing its output with mp shows that they are the same.

17. Find $p_* = \inf\{10 - x^2 - y \,:\, x^2 + y^2 \le 1\}$. (It's easy to do some basic calculus to determine p_* in this example. You can use that to check the answer you get from the sos relaxation.)

Solution. We first use some basic calculus to figure out the answer. When $x^2 + y^2 \le 1$, we have $-x^2 \ge y^2 - 1$. Therefore, on the unit disk,

$$10 - x^2 - y \ge 10 + y^2 - 1 - y = y^2 - y + 9.$$

The critical points of $y^2 - y + 9$ on the interval $[-1, 1]$ are at $y = 1/2, 1, -1$. Check that the minimum value of the objective function is attained at $(3/4, 1/2)$ and the min value is 8.75.
We can now try to arrive at the same answer using the systematic procedure of a sos relaxation. Recall that the

$$p^* = \sup\{\lambda : 10 - x^2 - y \ge \lambda \text{ when } 1 - x^2 - y^2 \ge 0\}.$$

Fixing degree of the sos certificate to be 1 we look for a certificate of the form

$$10 - x^2 - y - \lambda = (1, x, y) Q \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} + \alpha(1 - x^2 - y^2), \;\; Q \ge 0, \alpha \ge 0.$$

$$\begin{aligned} 10 - x^2 - y - \lambda &= (1, x, y) \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} + \alpha(1 - x^2 - y^2) \\ &= a + \alpha + 2bx + 2cy + (d - \alpha)x^2 + 2exy + (f - \alpha)y^2 \end{aligned}$$

Equating coefficients on both sides we have

$$a = 10 - \lambda - \alpha, \; b = 0, \; c = -\frac{1}{2}, \; d = \alpha - 1, \; e = 0, \; f = \alpha.$$

Therefore,

$$Q = \begin{pmatrix} 10-\lambda-\alpha & 0 & -\frac{1}{2} \\ 0 & \alpha-1 & 0 \\ -\frac{1}{2} & 0 & \alpha \end{pmatrix}.$$

To make Q psd we need to have the $(2,3)$-principal minor $\alpha(\alpha-1) \geq 0$ which means that $\alpha \geq 1$. We also need $10-\lambda-\alpha \geq 0$ which means that $\lambda \leq 10-\alpha \leq 9$. If $\alpha > 1$, then $\lambda < 9$. So to it could be that the supremum of λ is 9. Setting $\alpha = 1$ and $\lambda = 9-\varepsilon$, we get that

$$Q = \begin{pmatrix} \varepsilon & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

The $(1,3)$ minor implies that $\varepsilon \geq \frac{1}{4}$. Therefore, the max value that λ achieves is 8.75 and we conclude that the first sos relaxation solves the problem. $\square$

18. Suppose we want to minimize a polynomial over an algebraic variety (given by equations) as opposed to a semialgebraic set:

$$p_* = \inf\{p(x) : g_1(x) = 0, \ldots, g_m(x) = 0\}.$$

(a) Write down the form of the p_t^{sos} problem in this case by modifying from a semialgebraic set to an algebraic set. What simplifications can you make?

Proof. Proceeding as before the above problem is the following:

$$p_* = \sup\left\{\lambda : p(x) - \lambda \geq 0, \text{ when } g_1(x) = 0, \ldots, g_m(x) = 0\right\}.$$

Each equation $g(x) = 0$ is equivalent to the pair of inequalities: $g(x) \geq 0$ and $-g(x) \geq 0$. Therefore

$$p_t^{\text{sos}} = \sup\left\{\lambda : p(x) - \lambda = s_0 + \sum_{j=1}^m s_j g_j(x) - \sum_{j=1}^m s_j' g_j(x)\right\}.$$

where $\deg(s_0), \deg(s_j g_j), \deg(s_j' g_j) \leq 2t$. Now $s_j - s_j' = h_j$ is just a polynomial of degree at most $2t$. In fact any polynomial can be written as the difference of two sos polynomials, so there is nothing special about h_j. Therefore, the above certificate is just

$$p_t^{\text{sos}} = \sup\left\{\lambda : p(x) - \lambda = s_0 + \sum_{j=1}^m h_j g_j(x)\right\}.$$

where $\deg(s_0), \deg(h_j g_j) \leq 2t$. $\square$

(b) Is there a way we can write a version of p_t^{sos} that is indifferent to the particular choice of equations defining the variety?

Proof. We just reduced the sos relaxation of order t to the following:

$$p_t^{\text{sos}} = \sup\left\{\lambda : p(x) - \lambda = s_0 + \sum_{j=1}^{m} h_j g_j(x)\right\}.$$

where $\deg(s_0), \deg(h_j g_j) \le 2t$. The expression $\sum_{j=1}^{m} h_j g_j(x)$ is just an element in the ideal $I = \langle g_1, \ldots, g_m\rangle$. Therefore,

$$p_t^{\text{sos}} = \sup\left\{\lambda : p(x) - \lambda = s_0 + h \text{ where } h \in I\right\}.$$

This means that we no longer care how I is generated. This is quite powerful since we are not tied to the equations that the problem came with and we are computing p_* using not only the starting constraints $g_1 = 0, \ldots, g_m = 0$ but any other polynomial equation implied by them. Recall from algebra that this means that we want $p(x) - \lambda \equiv s_0$ mod the ideal I. □

(c) (Ex 3.99) Use your method to minimize the polynomial $10 - x^2 - y$ over the unit circle $x^2 + y^2 = 1$.

Proof. We know that the minimum value is 8.75. Let's start with $t = 1$. Let $I = \langle 1 - x^2 - y^2\rangle$.

$$p_1^{\text{sos}} = \sup\left\{\lambda : 10 - x^2 - y - \lambda \equiv (1, x, y)\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f\end{pmatrix}(1, x, y)^\top \text{ mod } I\right\}.$$

Expanding the right side we want

$$10 - x^2 - y - \lambda \equiv a + 2bx + 2cy + dx^2 + 2exy + fy^2 \text{ mod } I.$$

The ideal I is principal and hence a Gröbner basis of it with respect to a term order with $y > x$ is given by its generator $y^2 + x^2 - 1$. We can then reduce both sides of the above expression by $y^2 + x^2 - 1$ which means replacing any occurrence of y^2 with $1 - x^2$ and then equate the two sides. So we want

$$10 - x^2 - y - \lambda = a + 2bx + 2cy + dx^2 + 2exy + f(1 - x^2)$$

which means

$$10 - \lambda = a + f,\; b = 0, c = -\frac{1}{2},\; d - f = -1,\; e = 0.$$

In other words,

$$a = 10 - \lambda - f,\; b = 0,\; c = -\frac{1}{2},\; d = f - 1,\; e = 0.$$

This leads to the same Q as in the previous problem and the lower bound of 8.75 on λ. This is optimal as we saw already. □

Using M2:

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )
--loading configuration for package "SOS" from file /Users/thomas/Library/Applicati

o1 = SOS

o1 : Package

i2 : R=QQ[x,y];

i3 : f = 10-x^2-y   -- objective function

        2
o3 = - x  - y + 10

o3 : R

i4 : h = {x^2 + y^2-1}   -- constraints

       2    2
o4 = {x  + y  - 1}

o4 : List

i5 : d=2                -- degree bound

o5 = 2

i6 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/1
Success: SDP solved

      17
o6 = (--, {})
       2

o6 : Sequence

i7 : d= 4  -- degreebound

o7 = 4
```

```
i8 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/4
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-63839-0/5
Success: SDP solved

       35
o8 = (--, {})
       4
```

19. (Ex 3.62) Show that the polynomial $x^4 - 3x^2 + 1$ is nonnegative on the variety defined by $x^3 - 4x = 1$.

Solution. It is enough to show that $x^4 - 3x^2 + 1 - \rho \equiv \text{sos} \bmod (x^3 - 4x - 1)$ for some $\rho \geq 0$. Since the highest degree we see is four, we might try an sos of degree four. So we maximize ρ such that

$$x^4 - 3x^2 + 1 - \rho \equiv \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{Q} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \bmod (x^3 - 4x - 1), \ \ Q \succeq 0.$$

Reducing both sides by $x^3 - 4x - 1$ (i.e., replacing x^3 by $4x + 1$ gives the following

$$x^2 + x + 1 - \rho \equiv (a + 2e) + (2b + 8e + f)x + (2c + d + 4f)x^2, \ \ Q \succeq 0.$$

which reduces to solving the SDP:

$$\max \ \rho \ : \ Q \succeq 0, \ \ a + 2e = 1 - \rho, \ 2b + 8e + f = 1, \ \ 2c + d + 4f = 1.$$

Solving for some of the variables we get

$$a = 1 - \rho - 2e, \ f = 1 - 2b - 8e, \ d = -3 + 8b + 32e - 2c.$$

The SDP is now:

$$\max \rho : \begin{pmatrix} 1 - \rho - 2e & b & c \\ b & -3 + 8b + 32e - 2c & e \\ c & e & 1 - 2b - 8e \end{pmatrix} \succeq 0.$$

Setting $c = e = 0$ and $b = 1/2$ yields $Q = \begin{pmatrix} 1 - \rho & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is psd when $1 - \rho \geq 1/4$. Therefore, $\rho = 3/4$ works and we see that $x^4 - 3x^2 + 1 \geq 3/4$ on the given variety. The true minimum is around 0.81. We did not really solve the SDP, but rather, found a feasible solution which told us that $\rho \geq 3/4$. We can use M2 to see that the true SDP opt is also 3/4. □

```
i1 : needsPackage( "SOS", Configuration=>{"CSDPexec"=>"CSDP/csdp"} )

i2 : R = QQ[x]

i3 : f = x^4-3*x^2+1

      4     2
o3 = x  - 3x  + 1

o3 : R

i4 : h = {x^3-4*x-1}

       3
o4 = {x  - 4x - 1}

o4 : List

i5 : d = 4

o5 = 4

i6 : (bound,sol) = lasserreHierarchy(f, h, d, Solver => "CSDP")
Executing CSDP on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-64090-0/
Output saved on file /var/folders/11/d_rtms4d4rsdnlnr65nwfl3m0000gn/T/M2-64090-0/1
Success: SDP solved

      3
o6 = (-, {x => -.254102})
      4

o6 : Sequence
```

20. Recall that in the following example from lecture

$$p_* = \inf \{xy : x \geq 0, y \geq 0, 1 - x - y \geq 0\},$$

$\bar{p}_1^{\text{sos}} = p_* = 0$ but $p_1^{\text{sos}} = -\infty$. Since the feasible region is compact we will get from Schudgen's Positivstellensatz that p_t^{sos} converges asymptotically to 0. Prove that there is no finite value of t for which $p_t^{\text{sos}} = 0$.

Hint: Suppose there is some t such that $xy = s_0 + s_1 x + s_2 y + s_3(1 - x - y)$ with all the necessary degree bounds on the terms. Then by evaluating the two sides at $(0,0)$, what can you say about the lowest degree terms in s_0 and s_3? By comparing the coefficients of x and y on both sides, what can you say about the lowest degree terms in s_1, s_2? Now compare the coefficients of xy on both sides. Do you see a contradiction?

Solution. Evaluating at $(0,0)$ we get that $0 = s_0(0,0) + s_3(0,0)$. This means that s_0 and s_3 do not have constant terms and hence also no linear terms. The coefficient of x (and y) on the left hand side is 0 and hence s_1 and s_2 also don't have constant terms and their lowest degree terms must have degree at least two. This means that xy must equal the sum of the quadratic terms in s_0 and s_3. But on the right we get a homogeneous sos while on the left we have xy which is not a sos. Contradiction. □

21. Consider a system of polynomials $\{f_i(x) = 0 \; i = 1, \ldots, m\}$ where $f_i \in \mathbb{R}[x]$.

The *real Nullstellensatz* says that the system is infeasible over $\mathbb{R}^n$ if and only if -1 is congruent to a sos modulo the ideal $\langle f_1, \ldots, f_m \rangle$, i.e., there exists $F(x) = \sum h_i f_i$ and a sos s such that $-1 = s + F(x)$.

Consider the set of equations:

$$\sum_{i=1}^{n} x_i = 1, \;\; x_i^2 = 0 \;\forall\; i = 1, \ldots, n.$$

(a) Check that this system is infeasible both over $\mathbb{R}$ and $\mathbb{C}$.

(b) Give a real Nullstellensatz proof of infeasibility of this system over $\mathbb{R}$.

Solution. (a) That's clear.

(b) The ideal of the system is $I = \langle x_1^2, \ldots, x_n^2, (\sum x_i) - 1 \rangle$. Then $(\sum x_i)^2 - 1 \in I$. Since the x_i^2 are in I then we can remove them to obtain $-1 + 2\sum_{i<j} x_i x_j \in I$. Finally we can cancel the $2x_i x_j$'s by adding the sos $\sum_{i<j}(x_i - x_j)^2$, this will give us an excess of x_i^2's which we can remove, so after doing that we get -1.

□

22. The Positivstellensatz says the following: The system

$$\{f_i(x) = 0, \; i = 1, \ldots, m, \;\; g_j(x) \geq 0 \; j = 1, \ldots, p\}$$

does not have a solution in $\mathbb{R}^n$ if and only if there exists $F(x), G(x) \in \mathbb{R}[x]$ such that

$$F(x) + G(x) = -1, \;\; F(x) = \sum h_i f_i \text{ for some } h_i, \;\; G(x) = s_0 + \sum s_J g_J \text{ where } s_0, s_J \text{ are sos.}$$

In other words, $F(x)$ belongs to the ideal generated by $f_1, \ldots, f_m$ and $G(x)$ belongs to the preorder generated by $g_1, \ldots, g_p$.

Consider the single quadratic equation $ax^2 + bx + c = 0$ in one variable x. What conditions must (a, b, c) satisfy for this equation to have no real solutions? Assuming this condition, give a Positivstellensatz certificate for the non-existence of real solutions.

23. Compare the Putinar and Schmüdgen methods to prove that $x \le -1$ and $x \le 0$ on the unit disc with center at $(1, 0)$ in the plane.

24. Recall from problem 1 that the stable set problem in a graph $G = ([n], E)$ can be modeled as follows:

$$\begin{array}{ll} \max & \sum_{i=1}^n x_i \\ & x_i^2 = x_i, \forall i \in [n] \\ & x_i x_j = 0, \forall ij \in E, \end{array}$$

Can you write a SDP relaxation for this problem as we did for max cut by lifting each feasible solution x to the above problem to the psd matrix $\begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 & x^\top \end{pmatrix}$ and then relaxing the rank one constraint?

Solution. Suppose x is a solution to the stable set problem. Then $x_i^2 = x_i$ and $x_i x_j = 0$ for all $ij \in E$. This means that in the matrix $\begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 & x^\top \end{pmatrix}$ we can make these substitutions. If we relax the rank one constraint, then the above matrix is of the form

$$Y := \begin{pmatrix} 1 & x^\top \\ x & U \end{pmatrix}$$

where $U_{ii} = x_i$ and $U_{ij} = 0$ if $ij \in E$. Therefore the SDP relaxation is

$$\begin{array}{ll} \max & \sum_{i=1}^n x_i \\ \text{s.t.} & \begin{pmatrix} 1 & x^\top \\ x & U \end{pmatrix} \succeq 0 \\ & U_{ii} = x_i \ \forall \ i \\ & U_{ij} = 0 \ \forall \ ij \in E \end{array}$$

□