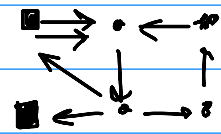
Colombia Lecture #4

Recall: Quiver something like 

Let Q be quiver w/ k a mutable vertex.
Quiver mutation $\mu_k: Q \to Q'$ computed in 3 steps
1. For each $j \to k \to l$, introduce edge $j \to l$
2. Reverse direction of all edges incident to k
3. Remove oriented 2-cycles.

Let $\mathcal{F}$ be field of rat'l functions in m indep variables over $\mathbb{C}$.
A seed in $\mathcal{F}$ is a pair $(Q, \underline{x})$ consisting of:

- a quiver Q on m vertices
- an extended cluster $\underline{\underline{x}}$, an m-tuple of alg. indep elements of $\mathcal{F}$, indexed by vertices of Q.

Ex: $\overset{x_1}{\bullet} \longrightarrow \overset{x_2}{\bullet}$

Seed mutation

Let k be mutable vertex in Q and x_k the corresponding cluster variable. The seed mutation $\mu_k: (Q, \underline{x}) \to (Q', \underline{\underline{x}}')$

is defined by

- $Q' = \mu_k(Q)$
- $\underline{\underline{x}}' = \underline{\underline{x}} \cup \{x_k'\} \setminus \{x_k\}$ where

x_k' defined by

$$x_k x_k' = \prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j \qquad \left(\text{exchange relation}\right)$$

<u>Ex:</u>

$$\xrightarrow{\mu_2} \underset{x_1}{\overset{1}{\bullet}} \longrightarrow \underset{x_2}{\overset{2}{\bullet}} \xrightarrow{\mu_1} \underset{\frac{1+x_2}{x_1}}{\overset{1}{\bullet}} \longleftarrow \underset{x_2}{\overset{2}{\bullet}} \xrightarrow{\mu_2} \underset{\frac{1+x_2}{x_1}}{\overset{1}{\bullet}} \longrightarrow \underset{\frac{1+x_1+x_2}{x_1x_2}}{\overset{2}{\bullet}}$$

$$\Big\Downarrow \mu_1$$

$$\underset{x_2}{\overset{1}{\bullet}} \longleftarrow \underset{x_1}{\overset{2}{\bullet}} \overset{\mu_1}{\Longleftarrow} \underset{\frac{1+x_1}{x_2}}{\overset{1}{\bullet}} \longrightarrow \underset{x_1}{\overset{2}{\bullet}} \overset{\mu_2}{\Longleftarrow} \underset{\frac{1+x_1}{x_2}}{\overset{1}{\bullet}} \longleftarrow \underset{\frac{1+x_1+x_2}{x_1x_2}}{\overset{2}{\bullet}}$$

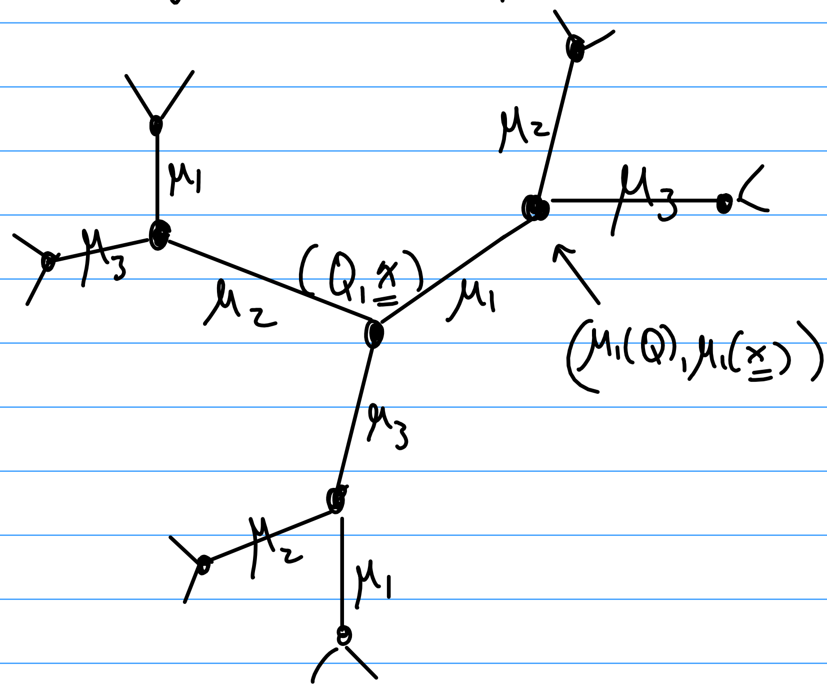
<u>Note:</u> Consider 1st + last quiver to be the same (same after relabeling nodes of quiver)

<u>Note:</u> ① Finitely many seeds + cluster variables.
② Can write each cluster variable as Laurent poly in initial cluster var's
③ with pos coeffs!

above, $n=2$.

Suppose Q has n mutable vertices. Then we can mutate in n diff direction from each seed. Set of all seeds lies on infinite n-regular tree (though some seeds might coincide)

Here $n=3$



☆ Rk on importance that μ_k an involution!

Let $(Q, \underline{\underline{x}})$ be seed, & let $\mathcal{F}$ be set of rational functions in $\underline{\underline{x}}$.
Let $\mathcal{X}$ be set of all cluster variables that appear in the n-regular tree.
The cluster algebra $A(Q, \underline{\underline{x}})$ is the subring of $\mathcal{F}$ generated by $\mathcal{X}$.

So in previous example, clust alg of $\overset{x_1}{\bullet} \longrightarrow \overset{x_2}{\bullet}$
is $\mathbb{C}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2}\right] \subseteq \mathbb{C}(x_1, x_2)$

$\left\{ \frac{p(x_1,x_2)}{q(x_1,x_2)} \;\middle|\; p \text{ and } q \text{ poly's in } x_1, x_2 \right\}$

Ex: $Q = $
$$\begin{tikzcd} \overset{1}{\underset{x_1}{\bullet}} \arrow[r, Rightarrow, "k \text{ arrows}"] & \overset{2}{\underset{x_2}{\bullet}} \end{tikzcd}$$

$\Downarrow \mu_1$

Let x_3 denote new variable we get if we mutate Q at vertex 1.
Then $x_1 x_3 = x_2^k + 1$.

$$\begin{tikzcd} \overset{1}{\underset{x_3}{\circ}} & \overset{2}{\underset{x_2}{\circ}} \arrow[l, Rightarrow, "k"'] \end{tikzcd}$$

$\Downarrow \mu_2$

Let x_4 denote " "
if we then mutate at vertex 2
Then $x_2 x_4 = x_3^k + 1$.

$\vdots$

$\Downarrow \mu_2$

Let x_0 denote new variable we get if we mutate Q at vertex 2
Then $x_0 x_2 = x_1^k + 1$.

$$\begin{tikzcd} \overset{1}{\underset{x_1}{\circ}} & \overset{2}{\underset{x_0}{\circ}} \arrow[l, Rightarrow, "k"'] \end{tikzcd}$$

$\Downarrow \mu_1$

Let x_{-1} denote " "
if we then mutate at vertex 1.
Then $x_{-1} x_1 = x_0^k + 1$.

$\vdots$

The cluster variables give infinite sequence $\{x_i\}_{i \in \mathbb{Z}}$ defined by $x_i x_{i+2} = x_{i+1}^k + 1$

Ex: We did case $k=1$ and got finite number

If $k=2$: Start with $\{x_1, x_2\}$.
Then $x_1 x_3 = x_2^2 + 1$ so $x_3 = \dfrac{x_2^2+1}{x_1}$

$$x_2 x_4 = x_3^2 + 1 \quad \text{so} \quad x_4 = \frac{x_3^2+1}{x_2} = \frac{x_2^4 + 2x_2^2 + x_1^2 + 1}{x_1^2 x_2}$$

$$x_3 x_5 = x_4^2 + 1 \quad \text{so} \quad x_5 = \frac{x_4^2+1}{x_3} =$$

$$\frac{\left(\frac{x_2^4 + 2x_2^2 + x_1^2 + 1}{x_1^2 x_2}\right)^2 + 1}{\left(\frac{x_2^2+1}{x_1}\right)} =$$

$$\left.\frac{(x_2^4 + 2x_2^2 + x_1^2 + 1)^2 + x_1^4 x_2^2}{x_1 x_2 (x_2^2+1)}\right\}$$

Looks like just a rational function — ratio of 2 polynomials

But — cancellation!

$$x_5 = \frac{x_2^6 + 3x_2^4 + 2x_2^2 x_1^2 + 3x_2^2 + x_1^4 + 2x_1^2 + 1}{x_1 x_2}$$

Also: note the positive coefficients.

Just because we are dividing one pos. poly by another doesn't mean the result should have pos coeffs:

$$\frac{x^3+1}{x+1} = x^2 - x + 1.$$

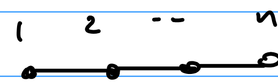
Thm (Laurent Phenomenon, Fomin & Zelevinsky):
Let $A(Q, \underline{\underline{x}})$ be a cluster algebra. Then every cluster variable can be written as a Laurent polynomial in the variables of $\underline{\underline{x}}$.

Positivity Theorem: (Conj of FZ, proved by Lee-Schiffler and GHKK)
The coeff's in those Laurent poly's are all positive.

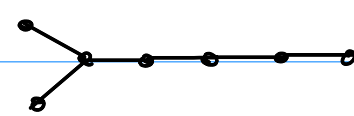
Def: A cluster algebra has finite type if it has finitely many cluster variables.

Thm (FZ): A cluster algebra $A(Q, \underline{\underline{x}})$ has finite type iff Q is mutation equiv to an orientation of:

- type A_n Dynkin diagram



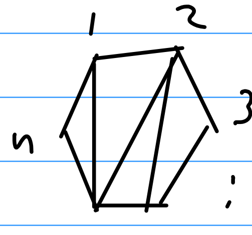
- type D_n Dynkin diagram



- type E_6, E_7 or E_8 Dynkin diagram

We discussed triangulations of n-gon & how this is related to Plucker coordinates of Gr_{2n}.

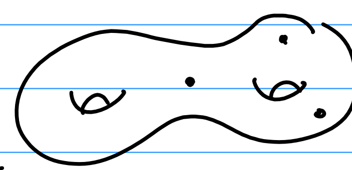
Can generalize example in 2 ways



Gr_{2n}

Gr_{kn}

Now need plabic graphs (Postnikov)



Triangulations of orientable Riemann surface w/ marked points

Exercises

1. Consider the sequence $\{y_m\}_{m\in\mathbb{Z}}$ defined by

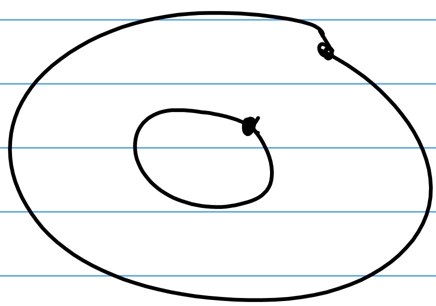
$$y_{m-1}y_{m+1} = \begin{cases} y_m^b + 1 & m \text{ odd} \\ y_m^c + 1 & m \text{ even} \end{cases}$$

Express the y_m's in terms of $\{y_1, y_2\}$ when $b=1$ and $c=2$.

2. Same for $b=1$ and $c=3$

3. Try other values of b and c. For which b, c is the sequence $\{y_m\}$ periodic?

4. Consider an annulus with 2 marked points, one on each boundary component.



Draw some triangulations of it. Is there a notion of flip?

5. Imitating the construction for the n-gon, try to associate a quiver to each triangulation of annulus. What happens when we flip?

6. Compute some cluster variables in this cluster algebra. Relation to #1 when $b=c=1$?