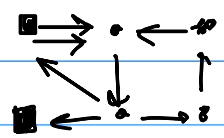
Colombia Lecture #3

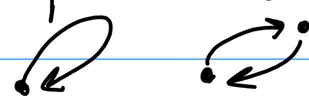
We've seen ways to get positivity tests for
- TP matrices (using double wiring diagram)
- TP Grass $(Gr_{2n})_{>0}$ (using triangulation)

What do these have in common? Is it possible to generalize?

Yes: today we'll see quivers + quiver mutation...

<u>Def</u>: A quiver is a finite directed graph.
Multiple edges allowed.
Oriented cycles of length 1 or 2 forbidden

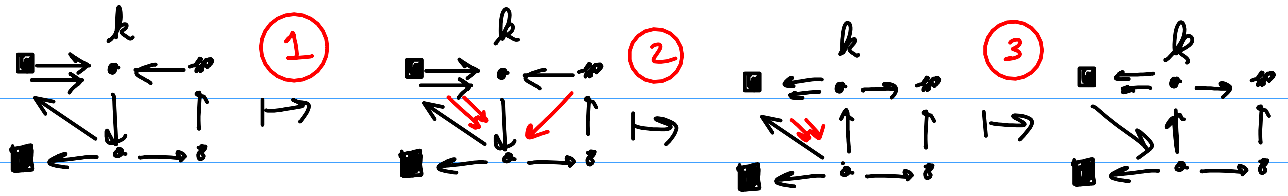




Two types of vertices: "frozen" & mutable.
"ice cube"

Ignore any edges connecting frozen vertices

Quiver mutation is operation on quivers...
Let Q be quiver w/ k a mutable vertex.
Quiver mutation $\mu_k : Q \to Q'$ computed in 3 steps
1. For each $j \to k \to l$, introduce edge $j \to l$
2. Reverse direction of all edges incident to k
3. Remove oriented 2-cycles.

Ex: 

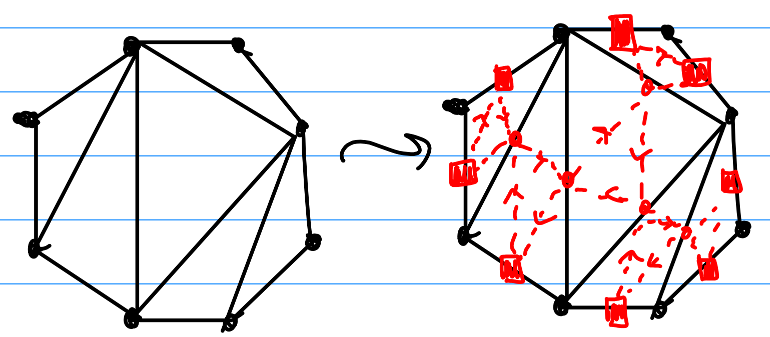
Note: Mutation is an involution, i.e. $\mu_k^2(Q) = Q$ for each vertex k.

Def: Two quivers are <u>mutation-equivalent</u> if one can get between them via a sequence of mutations.

Connection w/ triangulations...

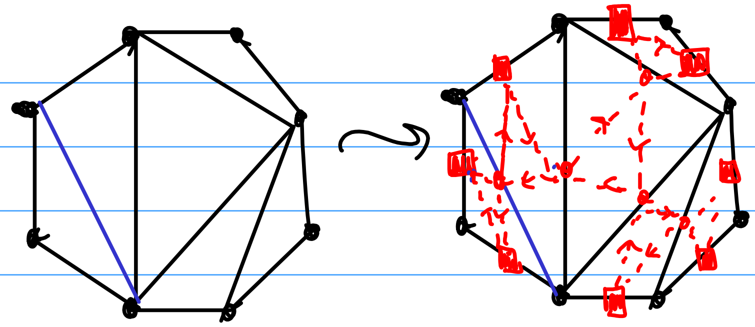
Fix a triangulation T of an n-gon. Associate to it a quiver Q_T as follows

- Put frozen vertex on each side, & mutable vertex on each diag
- within each triangle of T we make clockwise directed cycle



What happens if we do a flip on triangulation? How does quiver change? Guesses?

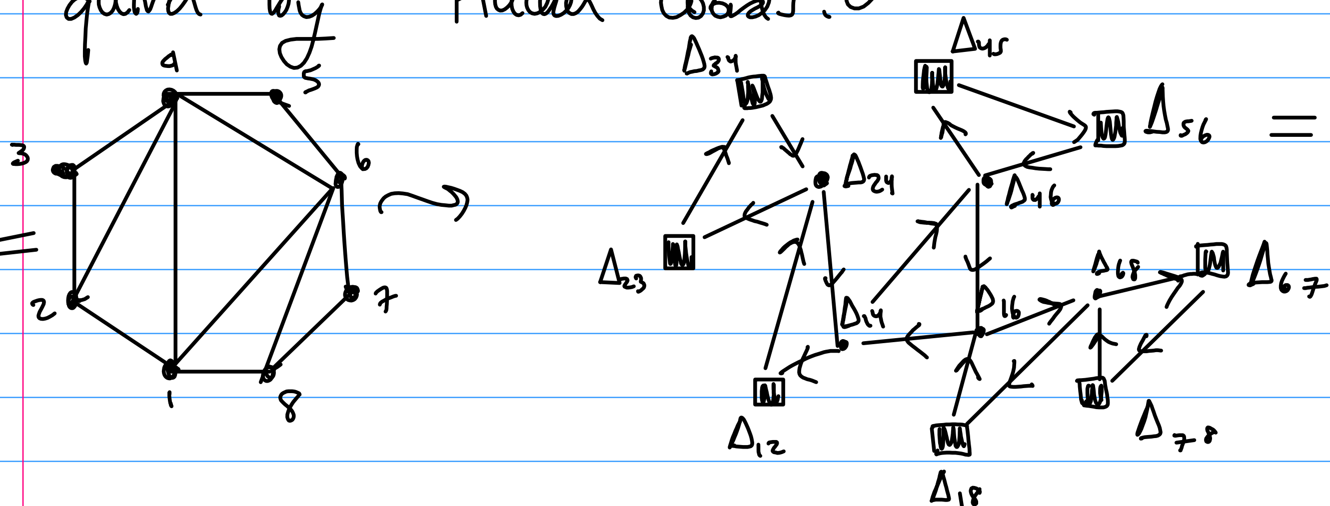
quiver mutation



Meanwhile lets recall our labeling of sides / diags of n-gon by Plucker coordinates.

This corresponds to a labeling of vertices of quiver by Plucker coord's.

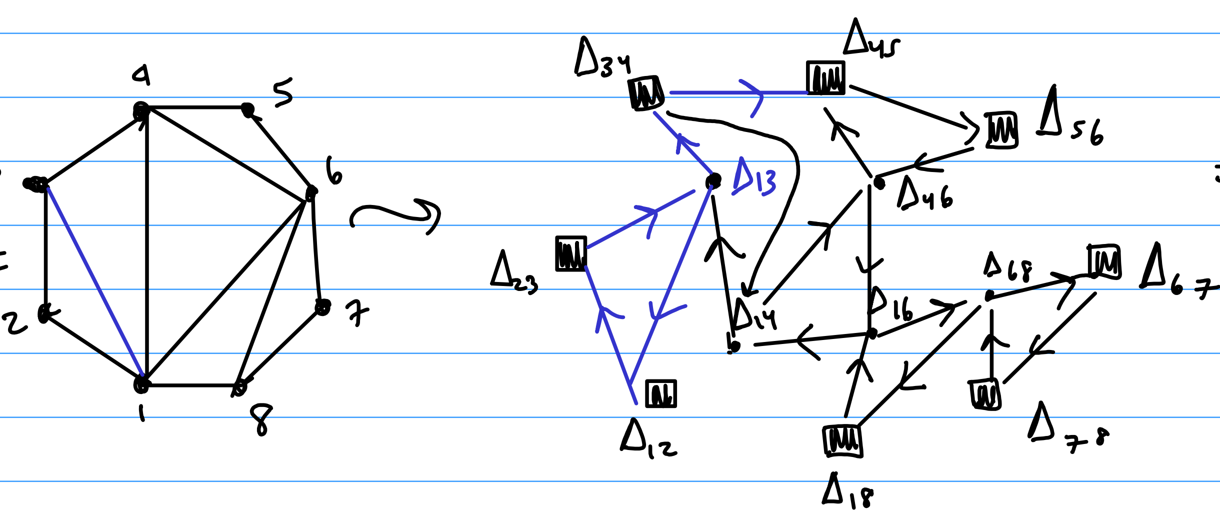
$T=$



$= Q$

Compare with

$T' =$

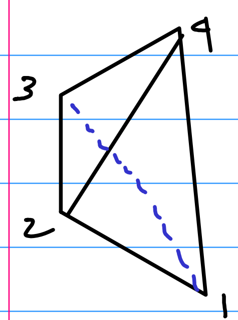


$= Q_{T'}$

So if T is triangulation & T' obtained by flipping at diagonal d,

$$Q_{T}' = \mu_d(Q_T)$$

Recall that if we flip an arc in triangulation, this corresponds to a Plucker relation



(✪) $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$, or equiv, the new Plucker coord

$$\Delta_{13} = \frac{\Delta_{12}\Delta_{34}+\Delta_{14}\Delta_{23}}{\Delta_{24}}$$

How can we read off this relation from the quiver?

Let d be vertex of quiver we'll mutate at, & let x_d be the Plucker coord labeling d in the orig quiver Q.
Let x_d' be the Plucker coord labeling d in the new quiver

(So in our example, $x_d = \Delta_{24}$, $x_d' = \Delta_{13}$)

Then looking at quiver Q, ✪ has form

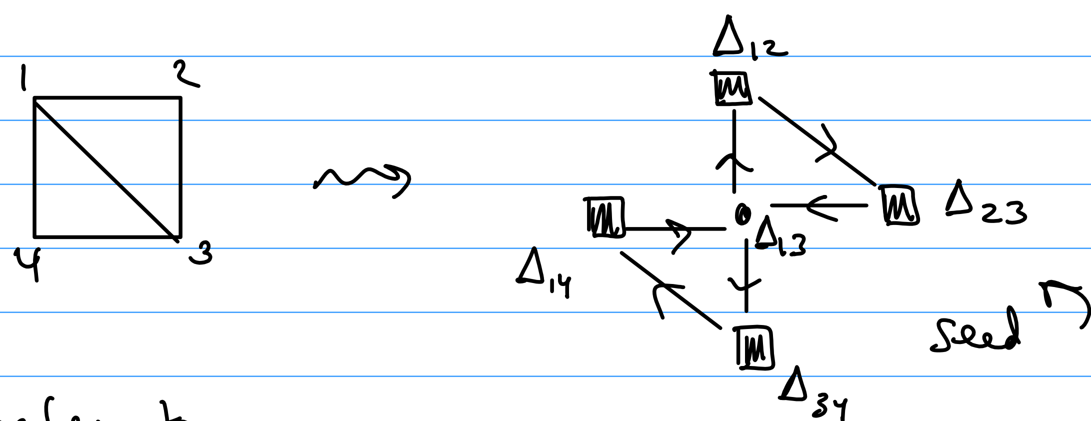
$$x_d\, x_d' = \left(\begin{array}{c}\text{prod of variables}\\ \text{pointing towards } x_d\end{array}\right) + \left(\begin{array}{c}\text{prod of variables}\\ \text{pointing away from } x_d\end{array}\right)$$

Seeds in cluster algebra

Let $\mathcal{F}$ be field of rat'l functions in m indep variables over $\mathbb{C}$.
A <u>seed</u> in $\mathcal{F}$ is a pair $(Q, \underline{x})$ consisting of:

— a quiver Q on m vertices
— an <u>extended cluster</u> $\underline{\underline{x}}$, an m-tuple of alg. indep elements of $\mathcal{F}$, indexed by vertices of Q.

Ex:



We refer to
coeffic variables $\leftrightarrow$ frozen vertices (4 here)
cluster variables $\leftrightarrow$ mutable vertices (1 here)

Cluster = {cluster var's} (size 1 here)
$\underline{\underline{x}}$ = Extended cluster = {cluster & coeff var's} (size 5 here)

Seed mutation

Let k be mutable vertex in Q and x_k

the corresponding cluster variable. The <u>seed mutation</u> $\mu_k : (Q, \underline{\underline{x}}) \to (Q', \underline{\underline{x}}')$

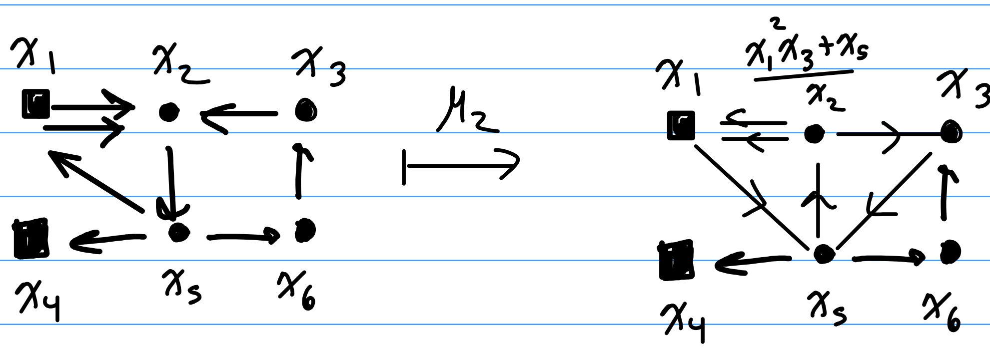
is defined by

- $Q' = \mu_k(Q)$
- $\underline{\underline{x}}' = \underline{\underline{x}} \cup \{x_k'\} \setminus \{x_k\}$ where

x_k' defined by

$$x_k x_k' = \prod_{j \leftarrow k} x_j + \prod_{j \to k} x_j \qquad \left(\begin{array}{c}\text{exchange}\\ \text{relation}\end{array}\right)$$

<u>Ex:</u>



<u>Rk:</u> Seed mutation is an involution, i.e.

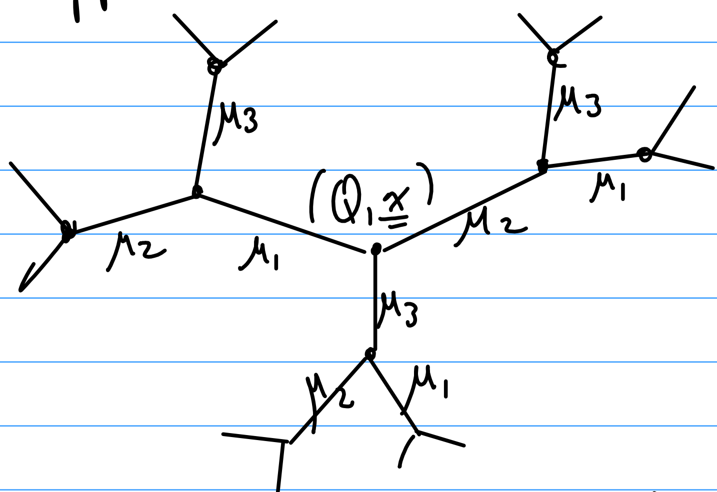
$$\mu_k(Q', \underline{\underline{x}}') = (Q, \underline{\underline{x}})$$

<u>n-regular tree</u>

Let $(Q, \underline{\underline{x}})$ be seed in $\mathcal{F}$, where Q has n mutable vertices.

Consider the n-regular tree $\mathbb{T}$ w/ vertices labeled by seeds, obtained by applying all possible sequences of mutations to $(Q, \underline{x})$.

Let $\mathcal{X}$ be union of all cluster variables which appear at nodes of $\mathbb{T}$.



Def: The cluster algebra $\mathcal{A} = \mathcal{A}(Q)$ is the subring of $\mathcal{F}$ generated by $\mathcal{X}$.

Ex: Consider seed $x_1 \longrightarrow x_2$

Compute all other seeds we can obtain from it by mutating...

Exercises

1. Mutate the following quiver at vertex 1. Alternatively, mutate at vertex 2.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 4 \arrow[r] & 3 \arrow[ul] \end{tikzcd}$$

2. Compute all quivers that are mutation-equivalent to the one above (up to isomorphism)

3. Verify that for any quiver Q and vertex k,
$$\mu_k^2(Q) = Q.$$

4. If T is triangulation & T' obtained by flipping at diagonal d,
$$Q_T{}' = \mu_d(Q_T)$$
(verify in examples; then prove)

5. Show that seed mutation is an involution, i.e. if $\mu_k(Q, \underline{\underline{x}}) = (Q', \underline{\underline{x}}')$ then
$$\mu_k(Q', \underline{\underline{x}}') = (Q, \underline{\underline{x}})$$

6. Consider seed $x_1 \longrightarrow x_2$

Compute all other seeds we can obtain from it by mutating...

How many cluster variables are there? Finite or infinite

What do you observe about your expressions for cluster variables?

7. Same as above for $x_1 \rightrightarrows x_2$

8. Explicitly compute all cluster variables & seeds associated to the cluster algebra coming from a pentagon.