Assignment CS460

$X = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \; y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; z = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

1. $y^T z = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 + 9 = 11$

2. $xy = \underset{2\times 2}{\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \underset{2\times 1}{=} \begin{bmatrix} 14 \\ 10 \end{bmatrix}$

3. Yes X is invertible as

   ① it is square matrix

   ② $|X| \neq 0$

3. $X^{-1} = \dfrac{\text{Adj}\, A}{|A|} = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \cdot \dfrac{1}{2} = \begin{bmatrix} 3/2 & -1/2 \\ -2 & 1 \end{bmatrix}$

4. for X:-

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Here total number of linearly independent rows is 2
so, matrix rank is 2.

2. calculus

① $y = x^3 + x - 5$

$y' = 3x^2 + 1.$

2. $f(x_1, x_2) = x_1 \sin(x_2) e^{-x_1}$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} \sin(x_2) e^{-x_1} - x_1 \sin(x_2) e^{-x_1} \\ x_1 \cos(x_2) e^{-x_1} \end{pmatrix}$$

probability and statistics (

$S = \{1, 1, 0, 1, 0\}$

0 → coin turns up head

1 → Coin turned up tail.

1. Sample mean = $\frac{\text{Sum of sample}}{\text{No. of sample}}$

$= \frac{3}{5}$

② sample variance $(S^2) = \frac{\sum (x_i - \bar{x})^2}{n-1}$

$$\Rightarrow s^2 = \frac{(1 - 3/5)^2 \cdot 3 + 2 \cdot (0 - 3/5)^2}{4}$$

$$s^2 = \frac{(4/25) \cdot 3 + 2 \cdot 9/25}{4}$$

$$= \frac{30}{25} \cdot 1/4 = \frac{15}{25} \cdot \frac{1}{2} = \frac{15}{50}$$

3. $\left(\frac{1}{2}\right)^5$ = Probability

4. Let p be the probability of 1 (i.e. $p(x=1)$).
Here we have to maximize the probability of the sample s.
probability of sample s.

$$P(s) = \prod_{i=1}^{s} p^{x_i} (1-p)^{1-x_i}$$

$$= p^{\sum_{i=1}^{s} x_i} (1-p)^{n - \sum_{i=1}^{s} x_i}$$

To maximize the above as a function of p.
Taking log, we get.

$$l(p) = \left(\sum_{i=1}^{s} x_i\right) \log(p) + \left(n - \sum_{i=1}^{s} x_i\right) \log(1-p)$$

Let say above expression as $l(p)$.

TO Maximize $\frac{dl(p)}{dp} = 0$

$$\Rightarrow \frac{1}{p}\sum_{i=1}^{5} x_i - \frac{1}{1-p}\left(n - \sum_{i=1}^{5} x_i\right) = 0$$

$$\Rightarrow 0 = \frac{\sum_{i=1}^{5} x_i - pn}{p(1-p)}$$

$$\Rightarrow pn = \sum_{i=1}^{5} x_i$$

$$\Rightarrow p = 1/n \sum_{i=1}^{5} x_i$$

$$\therefore p = \frac{1}{5}(3) = 3/5$$

(5) $P(z=T \,\&\, y=b) = 0.1$

$$P(z=T \mid y=b) = \frac{P(z=T \text{ and } y=b)}{P(y=b)}$$

$$= \frac{0.1}{0.5+0.15} = 0.7$$

<u>Big-O Notation</u>

(f, g)

$f(n) = O(g(n)),\ g(n) = O(f(n))$

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(1) $f(n) = \ln(n)$, $g(n) = \lg(n)$, both are equivalent.

(2) $f(n) = 3^n$, $g(n) = n^{1000}$

Here, $g(n) = O(f(n))$, since $f(n)$ increases rapidly than $f(n)$ when $n >> 1$.

(3) $f(n) = 3^n$, $g(n) = 2^n$

Here

$g(n) = O(f(n))$, since $f(n)$ increases more rapidly as n become large.

4. $f(n) = 1000n^2 + 2000n + 4000$, $g(n) = 3n^3 + 1$

$f(n) = O(g(n))$ since $g(n)$ increase more rapidly as compared to $f(n)$ as n become large.

Probability and Random variables

(a) $P(A\cup B) = P(A\cap(B\cap A^c))$ $\therefore$ False

As $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

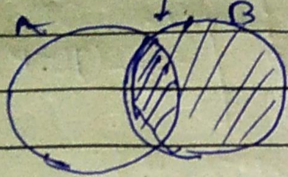
$$P(A\cap(B\cap A^c)) = P(A) + P(B\cap A^c) - P(A\cup(B\cap A^c))$$

$$= P(A) + P(B) + P(A^c) - P(B\cup A^c) - P(A\cup(B\cap A^c))$$

Clearly not equal.

(b) $P(A\cup B) = P(A)+P(B)-P(A\cap B)$: Trivial

(c) $P(A) = P(A\cap B) + P(A^c\cap B)$: false



(d) $P(A|B) = P(B|A)$: false

As, $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ and

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

so, both are not equal ~~always~~.

(e) $P(A_1\cap A_2\cap A_3) = P(A_3|A_2\cap A_1)\,P(A_2|A_1)\,P(A_1)$:- True.

from RHS

$$\frac{P(A_3\cap A_2\cap A_1)}{\cancel{P(A_2\cap A_1)}} \cdot \frac{\cancel{P(A_2\cap A_1)}}{\cancel{P(A_1)}} \cdot \cancel{P(A_1)}$$

$= \underline{\text{LHS}}$

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Discrete and continuous distribution

(a) Multivariate Gaussian:

$$\frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

(b) Bernoulli $\quad p^x (1-p)^{1-x}$

(c) Uniform $\quad \frac{1}{b-a}$ when $a \leq x \leq b$; 0 otherwise

(d) Binomial $\quad \binom{n}{x} p^x (1-p)^{n-x}$

Mean variance and Entropy

(a) To prove $Var(x) = E(x^2) - E(x)^2$

we have

$$
\begin{aligned}
Var(x) &= E[(x - E(x))^2] \\
&= E[x^2 - 2xE(x) - (E(x))^2] \\
&= E[x^2] - E[2xE(x)] + E(E(x)^2) \\
&= E(x^2) - 2E(xE(x)) + E(x)^2 \\
&= E(x^2) - 2E(x^2) + E(x))^2 \\
&= E(x)^2 - E(x^2)
\end{aligned}
$$

(b) we have Bernoulli (P) random variables.

so mean is P, the variance is $P(1-P)$.

and the entropy is $-(1-P)\log(1-P) - P\log(P)$.

Law of large number and central limit theorem

(a) Assuming a fair dice, the number of times 3 shows up should be close to 1000 due to the Law of Large numbers theorem.

(b) $\hat{x}$ is average number of heads, then distribution of $\hat{x}$ statistics. here coin is tossed n times.

$$\sqrt{n}\left(\hat{x}-\frac{1}{2}\right) \xrightarrow{n\to\infty} \mathcal{N}(0, 1/4)$$

This is due to central limit theorem.

Linear Algebra

Vector norms

$x \in \mathbb{R}^2$

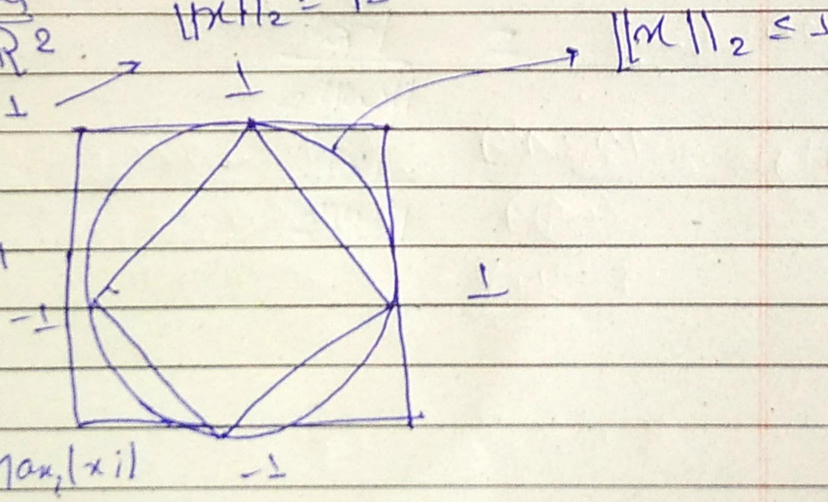
(a) $\|x\|_2 \le 1$ → $\|x\|_2 = \sqrt{\sum x_i^2}$

(b) $\|x\|_1 \le 1$

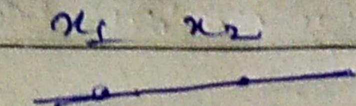
$\|x\|_1 = \sum_i |x_i|$

(c) $\|x\|_\infty \le 1$

$\|x\|_\infty = \max_i |x_i|$



Geometry



$$w^T x + b = 0$$

Let x_1 and x_2 are two points. $x_1 - x_2$ is parallel to $w^T x + b = 0$

$$w^T x_1 + b = 0 = w^T x_2 + b.$$

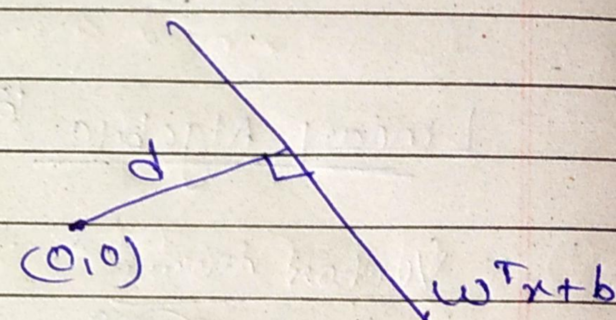
$$\Rightarrow w^T x_1 = w^T x_2$$

$$\Rightarrow w^T (x_1 - x_2) = 0$$

$\Rightarrow$ w is orthogonal to our line

(b) Since $w^T x + b$ is a plane. So let us assume x be any point on it.

$$\text{distance}, d = \frac{|w^T x|}{||w||_2}$$



$$= \frac{|-b|}{||w||_2}$$

$$= \frac{|b|}{||w||_2}$$