Solving the model of Optimal Law Enforcement

Optimal solution of Garoupa's model (2001)

$$W = \int_{qs}^{\bar{b}} (b-h)dF(b) - C(q) \ (A.1.1)$$

$$\text{Subject to s} \leq S$$

We are looking for the solutions of $\frac{\partial W}{\partial q}$=0 and $\frac{\partial W}{\partial s} = 0$

We know that $d[bF(b)]/\ db \ = \ F(b) \ + \ b\ f(b)$, because $F'(b) \ = \ f(b)$

Its equal to d$[bF(b)] \ = \ [F(b) \ + \ b\ f(b)]db$, with integrating that between qs to $\bar{b}$, we can write

$$\int_{qs}^{\bar{b}} d[bF(b)] = \int_{qs}^{\bar{b}} F(b)db + \int_{qs}^{\bar{b}} b\ f(b)db \ \ (A.1.2)$$

However, $F'(b) \ = \ f(b)$ because f is a density function

$$F'(b) = \frac{d[F(b)]}{db} = f(b)$$

$$= \int_{qs}^{\bar{b}} d\big(F(b)\big) = F(\bar{b}) - F(qs) = \int_{qs}^{\bar{b}} f(b)db$$

In our case,

$$\int_{qs}^{\bar{b}} d[bF(b)] = \bar{b}\ F(\bar{b}) - qsF(qs) \ \ (A.1.3)$$

By bringing $(A.1.3)$ to equation $(A.1.2)$

$$\bar{b}F(\bar{b}) - qsF(qs) = \int_{qs}^{\bar{b}} F(b)db + \int_{qs}^{\bar{b}} b\ f(b)db$$

Garoupa (1997) model can be written

$$W = \int_{qs}^{\bar{b}} bf(b)db - \int_{qs}^{\bar{b}} hf(b)db - C(q)$$

Or even,

$$W = \bar{b}F(\bar{b}) - qsF(qs) - \int_{qs}^{\bar{b}} F(b)db - h\ [F(\bar{b}) - F(qs)] - C(q)$$

We rearrange this equality

$$W = (\bar{b} - h)F(\bar{b}) + (h - qs)\ F(qs) - \int_{qs}^{\bar{b}} F(b)db - C(q)$$

Let us differentiate with respect to q the welfare function,

$$\frac{\partial W}{\partial q} = 0$$

$$sf(qs)(h - qs) - sF(qs) - \frac{\partial}{\partial q}\left(\int_{qs}^{\bar{b}} F(b)db\right) = C'(q)$$

We use Leibniz's rule,

$$\frac{\partial}{\partial q}\left(\int_{k_1(q)}^{k_2(q)} F(b)db\right) = F\big(k_2(q)\big)\frac{\partial k_2(q)}{\partial q} - F\big(k_1(q)\big)\frac{\partial k_1(q)}{\partial q}$$

with $k_1(q) = qs$ and $k_2(q) = \bar{b}$, so $\frac{\partial k_1(q)}{\partial q} = s$ and $\frac{\partial k_2(q)}{\partial q} = 0$

Consequently,

$$\frac{\partial}{\partial q}\left(\int_{qs}^{\bar{b}} F(b)db\right) = F(\bar{b}).0 - F(qs)s$$

And the constraint would be saturated if $s = S$, so, we obtain the following relation,

$S(h - \hat{q}S)f(\hat{q}S) - S.F(\hat{q}S) - [F(\bar{b}).0 - F(\hat{q}S)S] = C'(\hat{q})$

we find this relation $S(h - \hat{q}S)f(\hat{q}S) = C'(\hat{q})$ (A.1.2)

$$\frac{\partial W}{\partial s} = 0$$

$$q(h - q\hat{s})f(q\hat{s}) - q.F(q\hat{s}) - \frac{\partial}{\partial s}\left(\int_{qs}^{\bar{b}} F(b)db\right) = 0$$

$$q\big(h - q\hat{S}\big)f\big(q\hat{S}\big) - q.F\big(q\hat{S}\big) - [F(\bar{b}).0 - F\big(q\hat{S}\big).q] = 0$$

$$q\big(h - q\hat{S}\big)f\big(q\hat{S}\big) = 0\ (A.1.4)$$

So, we obtain that $h = q\hat{S}$, this means that the expected sanction should be able to compensate for the damage suffered.

Garoupa, N. (2001). Optimal magnitude and probability of fines. European Economic Review, 45(9), 1765–1771. https://doi.org/10.1016/S0014-2921(00)00084-2