TALK 4: ADMISSIBLE CANONICAL BUNDLE I

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Contents

First of all, I should apologize that I didn't prepare so well for this talk(so I also LaTeX them after the talk). Everythime when I was reading the paper [1], I felt myself so foolish. Feel free to use the black boxes in the talk.

This note records contents in the talk, so it may contain something showed by Prof. Peter Scholze, not by me. The initial goal of this note is to keep them somewhere, so that some day when the listeners or I want to recapture something in the talk, we can easily get them from this note. Feel free to ask me questions and give me hundreds of typos!

1. Berkovich space

I copied some words from [1, 3.1.1] in the talk to give an extremely short introduction to the Berkovich space.

Idea: the point of a scheme can be viewed as the map to the field, so is the point of the Berkovich space(the special seminorm). If you have not heard about the Berkovich space, you can replace X^{an} to $X(K)$ in the following sections.

2. metric and measure

For the statement of the main theorem, one should introduce concepts of metric (over line bundle) and measure (over Berkovich space).

2.1. **metric of line bundle.** Recall that when we learn Riemann metric, we first define it fiberwise, then localwise, and finally globalwise. In today's talk we also define it similarly.

Definition 2.1 (fiber of line bundle over Berkovich space)**.** *Fix the scheme X/K and the line bundle L over X. the fiber of L at point $x \in X^{an}$ is defined as*

$$L^{an}(x) := H_x \otimes_{\kappa(\bar{x})} L(\bar{x}).$$

$$\begin{tikzcd}
L^{an}(x) \arrow[d, dash] & & L(\bar{x}) \arrow[d, dash] & \\
SpecH_x = x \arrow[rr, mapsto] \arrow[dr, phantom, "\in" sloped] & & \bar{x} \arrow[dr, phantom, "\in" sloped] & L \arrow[d, dash] \\
& X^{an} \arrow[rr, "\kappa"] & & X
\end{tikzcd}$$

Definition 2.2 (metric pointwise)**.** *We define the metric of line bundle L at $x \in X^{an}$ as the norm of H_x-modules $L^{an}(x)$:*

$$\|\cdot\|(x) : L^{an}(x) \longrightarrow \mathbb{R}_{>0} \qquad \textit{such that } \|fs\|(x) = |f|_{H_x} \cdot \|s\|(x)$$

Definition 2.3 (metric globalwise)**.** *The metric of line bundle L is defined as the collections of metrics (at every point) such that the metrics vary in a continuous way. Equivalently,*

$$\|\cdot\| = \{\|\cdot\|(x) \,|\, x \in X^{an} \ + \textit{cont}\}$$

where the continuity means for any open subset U of X, any section $s \in \Gamma(U)$, the norm map

$$\|s\| : U^{an} \longrightarrow \mathbb{R} > 0 \qquad x \longmapsto \|s\|(x) := \|s_x\|(x)$$

is continuous.

When we have one metric $\|\cdot\|$ over L, we would call the pair $(L, \|\cdot\|)$ the metrized line bundle.

Proposition 2.4. *Here are some easy properties of the metrized line bundle:*

(1) The metrized line bundles over X form a group by

$$(L_1, \|\cdot\|_1) \otimes (L_2, \|\cdot\|_2) := (L_1 \otimes L_2, \|\cdot\|_1 \cdot \|\cdot\|_2)$$

the inverse and the idendity is also easily defined;

(2) If one has the map $f : X \longrightarrow Y$ and the metrized line bundle $(L, \|\cdot\|)$ over Y, then we can always pull back f to get a metrized line bundle $f^*(L, \|\cdot\|)$ over X;

(3) The metric $\|\cdot\|$ over structure sheaf $\mathcal{O}_X$ can be viewed as a positive real function on X^{an}.[1]

Remark 2.5. By this definition it's not easy to define integrable metric (or semipositive metric). By translating the metrized line bundles as the adelic line bundles, we can define integrable(it's already done in Talk 3).

Here the Prof. Peter Scholze explains how to translate the metrized line bundles as the adelic line bundles:

[1]Notice that we have the constant section 1 of $\mathcal{O}_X$.

2.2. **Chambert-Loir measure.** We may not have any time to define this measure, since defining it needs to introduce quite a lot of concepts, such as arithmetic intersection degree and the arithmetic Chow group. But we can still see the intuition and view some properties as the black box.

Definition 2.6 (Chambert-Loir measure, in complex case)**.** *Suppose C is a projective smooth curve over $\mathbb{C}$, and fix the metrized line bundle $(L, \|\cdot\|)$, the Chambert-Loir measure $c_1(L, \|\cdot\|)$ is defined as the $(1,1)$-form*

$$c_1(L, \|\cdot\|) := \frac{i}{2\pi}\partial\bar{\partial}\log\|s\|^2$$

where s is a local section of L which is nowhere zero. This definition doesn't depend on the choise of the section s.

Notice that on curve $C/\mathbb{C}$, the $(1,1)$-form can be viewed as the measure. Of course we can no longer use this definition when the curve C is over some non-Archimedean field, so I prepare the black box for you:

Black box.

(1) $c_1(L, \|\cdot\|)$ is an measure on C^{an}.

(2) The Chambert-Loir measure is compatible with the group structure of line bundle, i.e.

$$c_1(L_1 \otimes L_2, \|\cdot\|_1 \cdot \|\cdot\|_2) = c_1(L_1, \|\cdot\|_1) + c_1(L_2, \|\cdot\|_2),$$
$$c_1(L^\vee, \|\cdot\|) = -c_1(L, \|\cdot\|^{-1}), \qquad c_1(\mathcal{O}_C, \|\cdot\|_{const}) = 0.$$

(3) (non-Archimedean Calabi theorem) Let $\|\cdot\|$ be an integrable metric on $\mathcal{O}_C$, then

$$c_1(\mathcal{O}_C, \|\cdot\|) = 0 \iff \|\cdot\| \equiv C_0$$

(4) (normalization) let $(L, \|\cdot\|)$ be a metrized line bundle of degree d, then

$$\int_{C^{an}} c_1(L, \|\cdot\|) = d.$$

3. STATEMENT OF THE MAIN THEOREM

Fix K be an non-Archimedean field,[2] and C/K be a smooth projective curve, and $x \in C(\bar{K})$ be a point. We have the following canonical maps

$$\begin{tikzcd}
C \arrow[dr, "(\mathrm{Id},x)"] & & C \\
C \arrow[r, "\Delta"] & C \times C \arrow[ur, "p_1"] \arrow[dr, "p_2"] & \\
C \arrow[ur, "(x,\mathrm{Id})"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \omega_C^\vee \arrow[d, dash] & & \mathcal{O}_{C^2}(\Delta) \arrow[d, dash] \\
 & C \arrow[rr, "\Delta"] & & C \times C \\
\mathcal{O}_C(x) \arrow[d, dash] & & & \\
C \arrow[rrruu, "(x,\mathrm{Id})"'] & & &
\end{tikzcd}$$

and also some isomorphisms of line bundles:

$$\Delta^*\mathcal{O}_{C^2}(\Delta) \cong \omega_C^\vee$$
$$(x, \mathrm{Id})^*\mathcal{O}_{C^2}(\Delta) \cong \mathcal{O}_C(x)$$

[2]We can suppose $K = \bar{K}$ to avoid some technical conditions.

If we have the metric $\|\cdot\|_\Delta$ over line bundle $\mathcal{O}_{C^2}(\Delta)$, then it induces the metric $\|\cdot\|_{\omega_C}$ and $\|\cdot\|_{\mathcal{O}_C(x)}$ by the following ways:

$$(\omega_C^\vee, \|\cdot\|_{\omega_C}^{-1}) := \Delta^*(\mathcal{O}_{C^2}(\Delta), \|\cdot\|_\Delta)$$
$$(\mathcal{O}_C(x), \|\cdot\|_{\mathcal{O}_C(x)}) := (x, \mathrm{Id})^*(\mathcal{O}_{C^2}(\Delta), \|\cdot\|_\Delta)$$

we can also define the Green's functions as follows:

$$g_\Delta : (C \times C)^{an} \setminus \Delta^{an} \longrightarrow \mathbb{R} \qquad g_\Delta(x, y) := -\log\left(\|1\|_\Delta(x, y)\right)$$
$$g_x : C^{an} \setminus x \longrightarrow \mathbb{R} \qquad g_x(y) := -\log\left(\|1\|_x(y)\right) = g_\Delta(x, y)$$

Definition 3.1 (symmetric metric). *The metric $\|\cdot\|_\Delta$ over line bundle $\mathcal{O}_{C^2}(\Delta)$ is symmetric if the corresponding Green's function $g_\Delta(x, y)$ is symmetric.*

In the following pages, when the norm $\|\cdot\|_\Delta$ is special, we would replace the notations $\|\cdot\|_\Delta$ by $\|\cdot\|_{\Delta,a}$, $\|\cdot\|_{\omega_C}$ by $\|\cdot\|_a$, $\|\cdot\|_{\mathcal{O}_C(x)}$ by $\|\cdot\|_x$ and g_Δ by $g_{\Delta,a}$. [3]

Finally we can state our central theorem:

Theorem 3.2 (Zhang's metric). *Suppose $K = \bar{K}$.[4] There is a unique symmetric integrable metric $\|\cdot\|_{\Delta,a}$ over $\mathcal{O}_{C^2}(\Delta)$ such that for any $x, y \in C(K)$, we have*

(1) (compatibility, or admissible) We have the identities

$$c_1(\mathcal{O}(x), \|\cdot\|_x) = c_1(\mathcal{O}(y), \|\cdot\|_y)$$
$$(2g-2)c_1(\mathcal{O}(x), \|\cdot\|_x) = c_1(\omega_C, \|\cdot\|_a)$$

as the Chambert-Loir measure;

(2) (normalization) We always have the equality

$$\int_{(C)^{an}} g_x(-)c_1(\mathcal{O}(x), \|\cdot\|_x) = 0.$$

We will call this metric $\|\cdot\|_{\Delta,a}$ and its induced metric $\|\cdot\|_a$ as the Zhang's metric.

4. UNIQUENESS

In this section we want to prove the uniqueness part. If we have two metrics $\|\cdot\|_{\Delta,a}$ and $\|\cdot\|'_{\Delta,a}$ satisfying the Theorem 3.2, we just denote $\|\cdot\|'_a$, $\|\cdot\|'_x$, $g'_{\Delta,a}(x, y)$ and $g'_x(y)$ as the induced metrics or Green's functions induced by $\|\cdot\|'_{\Delta,a}$. We also fix $x_0 \in C(K)$, and denote

$$\frac{\|\cdot\|'_{x_0}(y)}{\|\cdot\|_{x_0}(y)} = e^{\varphi(y)}$$

where $\varphi : C^{an} \longrightarrow \mathbb{R}$ is a continous function on C^{an}.

We divide the process into four steps:

[3]Here the letter a means "admissible". We will define the "admissible metric" over curve on Section5, and actually "admissible metric" over $\mathcal{O}_{C^2}(\Delta)$ is defined in [1, ???].

[4]It's not an essential condition, we just don't want to worry about finite field extension.

<u>**Step1.**</u> Compute $\dfrac{\|\cdot\|_x'}{\|\cdot\|_x}$ and $\dfrac{\|\cdot\|_a'}{\|\cdot\|_a}$.

We know that

$$\begin{aligned}
&c_1\left(\mathcal{O}_C, \frac{\|\cdot\|_a'}{\|\cdot\|_a} \cdot \left(\frac{\|\cdot\|_x'}{\|\cdot\|_x}\right)^{-(2g-2)}\right)\\
&= c_1\Big(\omega_C, \|\cdot\|_a'\Big) - c_1\Big(\omega_C, \|\cdot\|_a\Big) - (2g-2)c_1\Big(\mathcal{O}(x_0), \|\cdot\|_{x_0}'\Big) + (2g-2)c_1\Big(\mathcal{O}(x_0), \|\cdot\|_{x_0}\Big)\\
&= 0
\end{aligned}$$

By the non-Archimedean Calabi theorem, we get $\dfrac{\|\cdot\|_a'}{\|\cdot\|_a} \cdot \left(\dfrac{\|\cdot\|_x'}{\|\cdot\|_x}\right)^{-(2g-2)}$ is a constant, we get

$$\frac{\|\cdot\|_a'}{\|\cdot\|_a} = e^{(2g-2)\varphi + c_1}$$

where c_1 is a constant number. Similarly, one can write

$$\frac{\|\cdot\|_x'}{\|\cdot\|_x} = e^{\varphi + \psi(x)}$$

where $\psi : C^{an} \longrightarrow \mathbb{R}$ is a "constant number depending on x".

<u>**Step2.**</u> By the symmetric property, we get $\psi - \varphi \equiv c_2$ is an constant function. We compute

$$g_{\Delta,a}'(x,y) - g_{\Delta,a}(x,y) = g_x'(y) - g_x(y) = -\log \frac{\|1\|_x(y)}{\|1\|_x'(y)} = -\psi(x) - \phi(y)$$

thus

$$-\psi(x) - \varphi(y) = -\psi(y) - \varphi(x) \quad \Longrightarrow \quad \psi(x) - \varphi(x) = \psi(y) - \varphi(y)$$

<u>**Step3.**</u> We show that $\varphi \equiv c_2$ is also a constant. By the following commutative diagram, we can pullback metrized line bundle $(\mathcal{O}_{C^2}(\Delta), \|\cdot\|_\Delta)$ in two ways, and should get the same metrized line bundle.

$$\begin{tikzcd}
 & & (\omega_C^\vee, \|\cdot\|_a^{-1}) \arrow[d, dash] & (\mathcal{O}_{C^2}(\Delta), \|\cdot\|_{\Delta,a}) \arrow[d, dash] \\
 & & C \arrow[r, "\Delta"] & C \times C \\
 & (\mathcal{O}_C(x), \|\cdot\|_x) \arrow[d, dash] & & \\
x \arrow[r, "x"] \arrow[uurr, "x"] & C \arrow[uurr, "(x,\mathrm{Id})"'] & &
\end{tikzcd}$$

We get

<u>**Step4.**</u>

5. existence–unfinished

References

[1] Xinyi Yuan. Arithmetic bigness and a uniform bogomolov-type result, 2021.

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