STUDENT SEMINAR: MODULI OF VECTOR BUNDLES

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Talk 1: Introduction.

Talk 2: Explicit constructions of semistable bundles on elliptic curves. In this talk, we introduce semistability of vector bundle on curves, and provide first non-trivial example of semistable vector bundles.

- Define slope stability [8, Definition 2.3]. State some basic properties, e.g., [8, Exercise 2.4], [5, 14.1].
- Recall the classification of vector bundles on $\mathbb{P}^1$, and determine when they are (semi)stable.
- Show the picture of Ford circles. Fix an elliptic curve (E, p_0). For each rational number $\mu = \frac{d}{r} > 0$, construct a stable vector bundle V_μ of rank r and degree d such that $\det V_\mu \cong \mathcal{O}_E(dp_0)$. [5, 14.3]
- Verify the stability of V_μ by induction. Shows that
$$\dim \operatorname{Hom}(V_{\mu_1}, V_{\mu_2}) = \dim \operatorname{Ext}(V_{\mu_2}, V_{\mu_1}) = \begin{cases} d_2 r_1 - d_1 r_2, & \mu_1 < \mu_2 \\ 1, & \mu_1 = \mu_2 \\ 0, & \mu_1 > \mu_2. \end{cases}$$
Discuss further properties, including [5, Corollary 14.11]. In particular, describe $H^\bullet(V_\mu)$.
- For each $r \geq 1$ and $\mathcal{L} \in \operatorname{Pic}^0(E)$, construct a semistable vector bundle $V_{r,\mathcal{L}}$ of rank r with $\det V_{r,\mathcal{L}} \cong \mathcal{L}$.
- Conclude the talk by stating [8, Example 2.7] as a theorem.

Talk 3: Fourier–Mukai transform on elliptic curve. The goal is to complete the classification of vector bundles on elliptic curves via the Fourier–Mukai transform.

- Define Fourier–Mukai transform Φ_K [5, Chapter 11].
- Describe $\Phi_K \circ \Phi_L$ and the right adjoint of Φ_K. Show that if A, B are abelian varieties of dimension n and $K \in \operatorname{Pic}(A \times B)$ is a line bundle, then both adjoints of Φ_K are given by $\Phi_{K^{-1}[n]}$.
- Sketch the proof of [5, Theorem 11.4] in the case $A = B$ is an elliptic curve and $S = \{*\}$, i.e., show that
$$\Phi_K \circ \Phi_{K^{-1}[g]} \cong \operatorname{Id}_{D^b(A)}.$$

Assume A is an abelian variety from now on.

- Define the Poincaré line bundle $\mathcal{P} \in \operatorname{Pic}(A \times \hat{A})$ and recall its basic properties.
- Set $\mathcal{S} = \mathcal{S}_A = \Phi_{\mathcal{P}}$. List key properties of $\mathcal{S}$, e.g., [5, (11.3.1)–(11.3.4), Theorem 11.6], and the convolution identity
$$\mathcal{S}(\mathcal{F}) \otimes \mathcal{S}(\mathcal{G}) \cong \mathcal{S}(\mathcal{F} * \mathcal{G}) \qquad [5, \text{p141}]$$
- $\mathcal{S}$ induces equivalences between subcategories of $D^b(A)$. Mention [5, Proposition 11.8, Lemma 14.6, Theorem 14.7], and explain how they yield a classification of vector bundles on elliptic curves.
- Describe $\mathcal{S}(V_{r,\mathcal{L}})$ explicitly.
- Fix a non-degenerate line bundle $\mathcal{L} \in \operatorname{Pic}(A)$. Discuss the $\mathrm{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}$ action on $D^b(A)$ and describe $\gamma(\mathcal{F})$ for special $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F} \in D^b(A)$.

Date: August 5, 2025.

If time is short, the speaker can focus on the elliptic curve case and skip all technical proofs.

Talk 4: Vector bundles on curves of genus $\geqslant 2$. [8, 2.3 & 2.5] In this talk, we describe the moduli space of vector bundles over a curve of genus $\geqslant 2$, focusing on some special cases [8, Examples 2.18–2.20].

- Begin by defining the moduli functor [8, Definition 2.9], and mention its representability [8, Theorem 2.10] [6, Theorem 1.3] (you can also put mention about other properties of $M_C(r,d)$ in the end of the talk).
- Mention about the topological classification of vector bundles, see [5, Theorem 3.4.1].

Now fix a curve C of genus 2.

- Explain why $M_C(2,\mathcal{O}_C)\cong\mathbb{P}^3$.
- Sketch why $M_C(2,\mathcal{L})\cong Q_1\cap Q_2\subset\mathbb{P}^5$, where $\mathcal{L}$ is a line bundle of degree 1. Discuss the connection to semiorthogonal decompositions.
- If time permits, outline the relation between $M_C(3,\mathcal{O}_C)$ and the Coble cubic hypersurface.

Talk 5: Semistable sheaves of degree 0. [8, 2.4] This talk is centered on [8, Theorem 2.14] and its surrounding results. I regard it as the coherent counterpart of the Riemann–Hilbert correspondence.

- Present and prove [8, Theorem 2.14].
- Illustrate [8, Theorem 2.14] explicitly in the cases of elliptic curves and curves of genus 2.
- Briefly discuss the generalization in [NS65, Theorem 2].
- Discuss equivalent notions of stability for curves.

Talk 6: Stability manifold of $\mathbb{P}^1$. This talk introduces Bridgeland stability conditions and discusses $\mathrm{Stab}(\mathbb{P}^1)$ in detail. The standard reference is [9, 2]; you may also consult my notes [2025.07.13].

- Define (locally finite) stability conditions on a triangulated category $\mathcal{T}$ and denote the space by $\mathrm{Stab}(\mathcal{T})$.
- Show that the usual slope stability on $\mathrm{Coh}(\mathbb{P}^1)$ defines a stability condition $(Z_0,\mathcal{P}_0)\in\mathrm{Stab}(\mathbb{P}^1)$.
- Discuss the $\mathbb{C}$ action on $\mathrm{Stab}(\mathcal{T})$. In the case of $\mathbb{P}^1$, restrict to the orbit of $(Z_0,\mathcal{P}_0)$ and describe how the heart changes. Note that semistability of objects remains unchanged under this action.
- Classify stability conditions where all line bundles and torsion sheaves are semistable.[1] Describe the $\mathbb{Z}$ action via tensoring with $\mathcal{O}(1)$, and the locus where $\mathcal{O}$ is semistable but not stable.
- State [9, Lemma 3.1(d)].[2] Give a description of all remaining stability conditions.
- Define walls in $\mathrm{Stab}(\mathbb{P}^1)$ and explain wall-crossing behavior.
- If time permits, briefly discuss stability conditions on other curves.

Talk 7: Equivariant vector bundles on Grassmannian. This talk aims to construct a family of vector bundles for later discussion.

- Realize the Grassmannian as GL_n/P and the flag variety as GL_n/B. More generally, denote by X a GL_n-homogeneous space.
- Introduce the character lattice $X^*(T)$, and for each $\chi\in X^*(T)$, construct a representation of B.
- Explain how a representation of G decomposes when restricted to T, and illustrate this via Lievis.

[1] Hint: without loss of generality assume that $Z(\mathcal{O}(-1))=1$ and $\phi(\mathcal{O}(-1))=0$, show that

$$Z(\mathcal{O})\in\mathcal{H}\sqcup\mathbb{R}\smallsetminus\left(\left\{1\pm\frac{1}{n}\;\middle|\;n\in\mathbb{N}_{>0}\right\}\sqcup\{1\}\right).$$

[2] Bonus point if you prove this in your notes!

- Introduce the Weyl group $W = N(T)/T$ and show that $W \cong S_n$ in the case of GL_n. Define the Weyl chamber and illustrate it in examples.
- State the highest weight theorem:

$$\begin{array}{ccc} \{\text{fin-dim irr reps of } \mathrm{GL}_n(\mathbb{C})\} & \overset{1:1}{\longleftrightarrow} & \{\text{dominant wts in } X^*(T)\} \\ & \longleftrightarrow & \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n\} \\ V_\lambda & \longleftrightarrow & \lambda \end{array}$$

- Given $(\rho, V) \in \mathrm{rep}(P)$, construct an associated vector bundle over GL_n/P. What is its rank? When $n = 2$, what is its degree?
- Define G-equivariant vector bundles over X, and exhibit an example over $\mathbb{P}^2$ that fails to be G-equivariant.
- Shows that

$$\mathrm{rep}(P) \quad \overset{1:1}{\longleftrightarrow} \quad \{G\text{-equiv vector bundles over}\, \mathrm{Gr}(r, n)\}.$$

  Identify the representations corresponding to the vector bundles $\mathcal{O}_{\mathrm{Gr}(r,n)}$, $\mathcal{S}$, $\mathcal{Q}$, $\Omega_{\mathrm{Gr}(r,n)}$, and $\omega_{\mathrm{Gr}(r,n)}$.
- State the Borel–Weil–Bott theorem, and, if time allows, apply it to the aforementioned vector bundles.

Talk 8: Chern characters. This talk introduces key numerical invariants—Chern characters, the Mukai vector, and exceptional/spherical sheaves—which will play a central role in the theory developed in the upcoming talks. You're encouraged to compute examples from the previous talk or from my notes [2025.01.26].

- Introduce the invariants $c(E)$, $\mathrm{ch}(E)$, and $\mathrm{td}(E)$, and explain how they behave under short exact sequences and the six functors.
- State Grothendieck–Riemann–Roch (GRR), and derive Hirzebruch–Riemann–Roch (HRR), Riemann–Roch for surfaces, and Riemann–Roch for curves.
- Recall Serre duality.
- Define the Mukai vector and Mukai pairing (see [4, p133, p172], [8, p40] and [1, p6]), and relate them to the Riemann–Roch formula.
- Define exceptional and spherical sheaves, and classify such sheaves on smooth projective curves.
- Explore how to compute these numerical invariants in practice using Macaulay2.

Talk 9: Exceptional vector bundles on $\mathbb{P}^2$. [3] and [7, §1.1] This talk explores the structure and classification of exceptional vector bundles on $\mathbb{P}^2$.

- Recall the definition of an exceptional sheaf. Show that any exceptional vector bundle E on $\mathbb{P}^2$ satisfies

$$\frac{1}{2}\left(\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)}\right)^2 - \frac{\mathrm{ch}_2(E)}{\mathrm{ch}_0(E)} = 0.$$

- Present the numerical results from [3, §4, §5]. In particular, show that for an exceptional pair (E_1, E_2) (see [3, Definition 1.2]),

$$\chi(E_2, E_1) = r_1 r_2 \left(1 + \frac{3}{2}(\mu_1 - \mu_2) + \frac{1}{2}(\mu_1 - \mu_2)^2 - \Delta_1 - \Delta_2\right).$$

  Relate this to the Markov equation.
- For each dyadic number $\frac{p}{2^m}$, construct a vector bundle $E\big(\frac{p}{2^m}\big)$ via induction. Show that $E\big(\frac{p}{2^m}\big)$ is exceptional. [3, Proposition 1.5]
- Show that all exceptional vector bundles on $\mathbb{P}^2$ arise as $E\big(\frac{p}{2^m}\big)$. [3, Propositions 4.4 & 4.6] Include a table of numerical invariants for representative examples.
- Compute $\mathrm{Ext}^i\left(E\big(\frac{p}{2^m}\big), E\big(\frac{q}{2^n}\big)\right)$.

- Define the Le Potier curve C_{LP} and state [7, Theorem 1.8].
- State the Bogomolov inequality and verify it for coherent sheaves on $\mathbb{P}^2$.

Talk 10: Stbility manifold of surfaces. [8, §3,§6, §7] This talk presents examples of Bridgeland stability conditions on surfaces and explicitly describes some walls. If the audience forget the definition of Bridgeland stability, we may consider an additional talk beforehand, focusing on [8, §4, §5].

- Introduce Gieseker stability and slope stability, and illustrate their distinction through an example.
- Recall the definition of Bridgeland stability, and explain why $\mathrm{Coh}(X)$ cannot serve as the heart of a Bridgeland stability condition [8, Example 4.3(2)].
- Define the tilted heart and construct a stability condition on it. State [8, Theorem 6.10] to make sure that $\sigma_{\omega,B}$ is indeed a Bridgeland stability condition.
- Fix an ample class H and define a slice of $\mathrm{Stab}(X)$ via $\sigma_{\alpha,\beta} = \sigma_{\alpha H, B_0+\beta H}$. State [8, Proposition 6.22], and solve [8, Exercise 7.3] to compute walls in concrete examples.
- If time permits, present a proof of the Bogomolov inequality.

Talk 11: coherent sheaves on surfaces.

Talk 12: coherent sheaves on threefolds.

Talk 13: TBA. This slot is open for anyone to present on any topic related to vector bundles. The seminar is not exhaustive, and my schedule may miss interesting aspects worth sharing.

References

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