

# STUDENT SEMINAR: MODULI OF VECTOR BUNDLES

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## Talk 1: Introduction.

**Talk 2: Explicit constructions of semistable bundles on elliptic curves.** In this talk, we introduce semistability of vector bundle on curves, and provide first non-trivial example of semistable vector bundles.

- Define slope stability [6, Definition 2.3]. State some basic properties, e.g., [6, Exercise 2.4], [4, 14.1].
- Recall the classification of vector bundles on  $\mathbb{P}^1$ , and determine when they are (semi)stable.
- Show the picture of Ford circles. Fix an elliptic curve  $(E, p_0)$ . For each rational number  $\mu = \frac{d}{r} > 0$ , construct a stable vector bundle  $V_\mu$  of rank  $r$  and degree  $d$  such that  $\det V_\mu \cong \mathcal{O}_E(dp_0)$ . [4, 14.3]
- Verify the stability of  $V_\mu$  by induction. Shows that

$$\dim \operatorname{Hom}(V_{\mu_1}, V_{\mu_2}) = \dim \operatorname{Ext}(V_{\mu_2}, V_{\mu_1}) = \begin{cases} d_2 r_1 - d_1 r_2, & \mu_1 < \mu_2 \\ 1, & \mu_1 = \mu_2 \\ 0, & \mu_1 > \mu_2. \end{cases}$$

Discuss further properties, including [4, Corollary 14.11]. In particular, describe  $H^\bullet(V_\mu)$ .

- For each  $r \geq 1$  and  $\mathcal{L} \in \operatorname{Pic}^0(E)$ , construct a semistable vector bundle  $V_{r,\mathcal{L}}$  of rank  $r$  with  $\det V_{r,\mathcal{L}} \cong \mathcal{L}$ .
- Conclude the talk by stating [6, Example 2.7] as a theorem.

**Talk 3: Fourier–Mukai transform on elliptic curve.** The goal is to complete the classification of vector bundles on elliptic curves via the Fourier–Mukai transform.

- Define Fourier–Mukai transform  $\Phi_K$  [4, Chapter 11].
- Describe  $\Phi_K \circ \Phi_L$  and the right adjoint of  $\Phi_K$ . Show that if  $A, B$  are abelian varieties of dimension  $n$  and  $K \in \operatorname{Pic}(A \times B)$  is a line bundle, then both adjoints of  $\Phi_K$  are given by  $\Phi_{K^{-1}[n]}$ .
- Sketch the proof of [4, Theorem 11.4] in the case  $A = B$  is an elliptic curve and  $S = \{*\}$ , i.e., show that

$$\Phi_K \circ \Phi_{K^{-1}[g]} \cong \operatorname{Id}_{D^b(A)}.$$

Assume  $A$  is an abelian variety from now on.

- Define the Poincaré line bundle  $\mathcal{P} \in \operatorname{Pic}(A \times \hat{A})$  and recall its basic properties.
- Set  $\mathcal{S} = \mathcal{S}_A = \Phi_{\mathcal{P}}$ . List key properties of  $\mathcal{S}$ , e.g., [4, (11.3.1)–(11.3.4), Theorem 11.6], and the convolution identity

$$\mathcal{S}(\mathcal{F}) \otimes \mathcal{S}(\mathcal{G}) \cong \mathcal{S}(\mathcal{F} * \mathcal{G}) \quad [4, \text{p141}]$$

- $\mathcal{S}$  induces equivalences between subcategories of  $D^b(A)$ . Mention [4, Proposition 11.8, Lemma 14.6, Theorem 14.7], and explain how they yield a classification of vector bundles on elliptic curves.
- Describe  $\mathcal{S}(V_{r,\mathcal{L}})$  explicitly.
- Fix a non-degenerate line bundle  $\mathcal{L} \in \operatorname{Pic}(A)$ . Discuss the  $\operatorname{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}$  action on  $D^b(A)$  and describe  $\gamma(\mathcal{F})$  for special  $\gamma \in \operatorname{SL}_2(\mathbb{Z})$  and  $\mathcal{F} \in D^b(A)$ .

If time is short, the speaker can focus on the elliptic curve case and skip all technical proofs.

**Talk 4: Vector bundles on curves of genus  $\geq 2$ .** [6, 2.3 & 2.5] In this talk, we describe the moduli space of vector bundles over a curve of genus  $\geq 2$ , focusing on some special cases [6, Examples 2.18–2.20].

- Begin by defining the moduli functor [6, Definition 2.9], and mention its representability [6, Theorem 2.10] [5, Theorem 1.3] (you can also put mention about other properties of  $M_C(r, d)$  in the end of the talk).
- Mention about the topological classification of vector bundles, see [4, Theorem 3.4.1].

Now fix a curve  $C$  of genus 2.

- Explain why  $M_C(2, \mathcal{O}_C) \cong \mathbb{P}^3$ .
- Sketch why  $M_C(2, \mathcal{L}) \cong Q_1 \cap Q_2 \subset \mathbb{P}^5$ , where  $\mathcal{L}$  is a line bundle of degree 1. Discuss the connection to semiorthogonal decompositions.
- If time permits, outline the relation between  $M_C(3, \mathcal{O}_C)$  and the Coble cubic hypersurface.

**Talk 5: Semistable sheaves of degree 0.** [6, 2.4] This talk is centered on [6, Theorem 2.14] and its surrounding results. I regard it as the coherent counterpart of the Riemann–Hilbert correspondence.

- Present and prove [6, Theorem 2.14].
- Illustrate [6, Theorem 2.14] explicitly in the cases of elliptic curves and curves of genus 2.
- Briefly discuss the generalization in [NS65, Theorem 2].
- Discuss equivalent notions of stability for curves.

**Talk 6: Stability manifold of  $\mathbb{P}^1$ .** This talk introduces Bridgeland stability conditions and discusses  $\text{Stab}(\mathbb{P}^1)$  in detail. The standard reference is [7, 2]; you may also consult my notes [2025.07.13].

- Define (locally finite) stability conditions on a triangulated category  $\mathcal{T}$  and denote the space by  $\text{Stab}(\mathcal{T})$ .
- Show that the usual slope stability on  $\text{Coh}(\mathbb{P}^1)$  defines a stability condition  $(Z_0, \mathcal{P}_0) \in \text{Stab}(\mathbb{P}^1)$ .
- Discuss the  $\mathbb{C}$  action on  $\text{Stab}(\mathcal{T})$ . In the case of  $\mathbb{P}^1$ , restrict to the orbit of  $(Z_0, \mathcal{P}_0)$  and describe how the heart changes. Note that semistability of objects remains unchanged under this action.
- Classify stability conditions where all line bundles and torsion sheaves are semistable.<sup>1</sup> Describe the  $\mathbb{Z}$  action via tensoring with  $\mathcal{O}(1)$ , and the locus where  $\mathcal{O}$  is semistable but not stable.
- State [7, Lemma 3.1(d)].<sup>2</sup> Give a description of all remaining stability conditions.
- Define walls in  $\text{Stab}(\mathbb{P}^1)$  and explain wall-crossing behavior.
- If time permits, briefly discuss stability conditions on other curves.

**Talk 7: Equivariant vector bundles on Grassmannian.** This talk aims to provide a family of vector bundles for the later discussion.

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<sup>1</sup>Hint: without loss of generality assume that  $Z(\mathcal{O}(-1)) = 1$  and  $\phi(\mathcal{O}(-1)) = 0$ , show that

$$Z(\mathcal{O}) \in \mathcal{H} \sqcup \mathbb{R} \setminus \left( \left\{ 1 \pm \frac{1}{n} \mid n \in \mathbb{N}_{>0} \right\} \sqcup \{1\} \right).$$

<sup>2</sup>Bonus point if you prove this in your notes!

**Talk 8: Chern characters.** This talk introduces key numerical invariants—Chern characters, the Mukai vector, and exceptional/spherical sheaves—which will play a central role in the theory developed in the upcoming talks. You’re encouraged to compute examples from the previous talk or from my notes [2025.01.26].

- Introduce the invariants  $c(E)$ ,  $\text{ch}(E)$ , and  $\text{td}(E)$ , and explain how they behave under short exact sequences and the six functors.
- State Grothendieck–Riemann–Roch (GRR), and derive Hirzebruch–Riemann–Roch (HRR), Riemann–Roch for surfaces, and Riemann–Roch for curves.
- Recall Serre duality.
- Define the Mukai vector and Mukai pairing (see [3, p133, p172], [6, p40] and [1, p6]), and relate them to the Riemann–Roch formula.
- Define exceptional and spherical sheaves, and classify such sheaves on smooth projective curves.
- Explore how to compute these numerical invariants in practice using Macaulay2.

**Talk 9: Exceptional vector bundles on  $\mathbb{P}^2$ .**

**Talk 10: Stability manifold of surfaces.**

**Talk 11: vector bundles on curves in K3 surfaces.**

**Talk 12: vector bundles on threefolds.**

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