STUDENT SEMINAR: MODULI OF VECTOR BUNDLES

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Talk 1: Introduction.

Talk 2: Explicit constructions of semistable bundles on elliptic curves. In this talk, we introduce semistability of vector bundle on curves, and provide first non-trivial example of semistable vector bundles.

- Define slope stability [3, Definition 2.3]. State some basic properties, e.g., [3, Exercise 2.4], [2, 14.1].
- Recall the classification of vector bundles on $\mathbb{P}^1$, and determine when they are (semi)stable.
- Show the picture of Ford circles. Fix an elliptic curve (E, p_0). For each rational number $\mu = \frac{d}{r} > 0$, construct a stable vector bundle V_μ of rank r and degree d such that $\det V_\mu \cong \mathcal{O}_E(dp_0)$. [2, 14.3]
- Verify the stability of V_μ by induction. Shows that
$$\dim \operatorname{Hom}(V_{\mu_1}, V_{\mu_2}) = \dim \operatorname{Ext}(V_{\mu_2}, V_{\mu_1}) = \begin{cases} d_2 r_1 - d_1 r_2, & \mu_1 < \mu_2 \\ 1, & \mu_1 = \mu_2 \\ 0, & \mu_1 > \mu_2. \end{cases}$$
Discuss further properties, including [2, Corollary 14.11]. In particular, describe $H^\bullet(V_\mu)$.
- For each $r \geq 1$ and $\mathcal{L} \in \operatorname{Pic}^0(E)$, construct a semistable vector bundle $V_{r,\mathcal{L}}$ of rank r with $\det V_{r,\mathcal{L}} \cong \mathcal{L}$.
- Conclude the talk by stating [3, Example 2.7] as a theorem.

Talk 3: Fourier–Mukai transform on elliptic curve. The goal is to complete the classification of vector bundles on elliptic curves via the Fourier–Mukai transform.

- Define Fourier–Mukai transform Φ_K [2, Chapter 11].
- Describe $\Phi_K \circ \Phi_L$ and the right adjoint of Φ_K. Show that if A, B are abelian varieties of dimension n and $K \in \operatorname{Pic}(A \times B)$ is a line bundle, then both adjoints of Φ_K are given by $\Phi_{K^{-1}[n]}$.
- Sketch the proof of [2, Theorem 11.4] in the case $A = B$ is an elliptic curve and $S = \{*\}$, i.e., show that
$$\Phi_K \circ \Phi_{K^{-1}[g]} \cong \operatorname{Id}_{D^b(A)}.$$

Assume A is an abelian variety from now on.

- Define the Poincaré line bundle $\mathcal{P} \in \operatorname{Pic}(A \times \hat{A})$ and recall its basic properties.
- Set $\mathcal{S} = \mathcal{S}_A = \Phi_{\mathcal{P}}$. List key properties of $\mathcal{S}$, e.g., [2, (11.3.1)–(11.3.4), Theorem 11.6], and the convolution identity
$$\mathcal{S}(\mathcal{F}) \otimes \mathcal{S}(\mathcal{G}) \cong \mathcal{S}(\mathcal{F} * \mathcal{G}) \qquad [2, \text{p141}]$$
- $\mathcal{S}$ induces equivalences between subcategories of $D^b(A)$. Mention [2, Proposition 11.8, Lemma 14.6, Theorem 14.7], and explain how they yield a classification of vector bundles on elliptic curves.
- Describe $\mathcal{S}(V_{r,\mathcal{L}})$ explicitly.
- Fix a non-degenerate line bundle $\mathcal{L} \in \operatorname{Pic}(A)$. Discuss the $\operatorname{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}$ action on $D^b(A)$ and describe $\gamma(\mathcal{F})$ for special $\gamma \in \operatorname{SL}_2(\mathbb{Z})$ and $\mathcal{F} \in D^b(A)$.

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If time is short, the speaker can focus on the elliptic curve case and skip all technical proofs.

Talk 4: Vector bundles on curves of genus $\geqslant 2$. [3, 2.3 & 2.5] In this talk, we describe the moduli space of vector bundles over a curve of genus $\geqslant 2$, focusing on some special cases [3, Examples 2.18–2.20].

- Begin by defining the moduli functor [3, Definition 2.9], and mention its representability [3, Theorem 2.10] (you can also put mention it in the end of the talk).

Now fix a curve C of genus 2.

- Explain why $M_C(2, \mathcal{O}_C) \cong \mathbb{P}^3$.
- Sketch why $M_C(2, \mathcal{L}) \cong Q_1 \cap Q_2 \subset \mathbb{P}^5$, where $\mathcal{L}$ is a line bundle of degree 1. Discuss the connection to semiorthogonal decompositions.
- If time permits, outline the relation between $M_C(3, \mathcal{O}_C)$ and the Coble cubic hypersurface.

Talk 5: Semistable sheaves of degree 0. [3, 2.4] This talk is centered on [3, Theorem 2.14] and its surrounding results. I regard it as the coherent counterpart of the Riemann–Hilbert correspondence.

- Present and prove [3, Theorem 2.14].
- Illustrate [3, Theorem 2.14] explicitly in the cases of elliptic curves and curves of genus 2.
- Briefly discuss the generalization in [NS65, Theorem 2].
- Discuss equivalent notions of stability for curves.

Talk 6: Stability manifold of $\mathbb{P}^1$. This talk introduces Bridgeland stability conditions and discusses $\mathrm{Stab}(\mathbb{P}^1)$ in detail. The standard reference is [4, 1]; you may also consult my notes.

- Define (locally finite) stability conditions on a triangulated category $\mathcal{T}$ and denote the space by $\mathrm{Stab}(\mathcal{T})$.
- Show that the usual slope stability on $\mathrm{Coh}(\mathbb{P}^1)$ defines a stability condition $(Z_0, \mathcal{P}_0) \in \mathrm{Stab}(\mathbb{P}^1)$.
- Discuss the $\mathbb{C}$ action on $\mathrm{Stab}(\mathcal{T})$. In the case of $\mathbb{P}^1$, restrict to the orbit of $(Z_0, \mathcal{P}_0)$ and describe how the heart changes. Note that semistability of objects remains unchanged under this action.
- Classify stability conditions where all line bundles and torsion sheaves are semistable.[1] Describe the $\mathbb{Z}$ action via tensoring with $\mathcal{O}(1)$, and the locus where $\mathcal{O}$ is semistable but not stable.
- State [4, Lemma 3.1(d)].[2] Give a description of all remaining stability conditions.
- Define walls in $\mathrm{Stab}(\mathbb{P}^1)$ and explain wall-crossing behavior.
- If time permits, briefly discuss stability conditions on other curves.

Talk 7: Equivariant vector bundles on Grassmannian.

Talk 8: Chern class.

Talk 9: Exceptional vector bundles on $\mathbb{P}^2$.

Talk 10: Stbility manifold of surfaces.

Talk 11: vector bundles on K3 surfaces.

Talk 12: vector bundles on threefolds.

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[1] Hint: without loss of generality assume that $Z\left(\mathcal{O}(-1)\right) = 1$ and $\phi\left(\mathcal{O}(-1)\right) = 0$, show that

$$Z(\mathcal{O}) \in \mathcal{H} \sqcup \mathbb{R} \setminus \left(\left\{ 1 \pm \frac{1}{n} \,\middle|\, n \in \mathbb{N}_{>0} \right\} \sqcup \{1\} \right).$$

[2] Bonus point if you prove this in your notes!

References

[1] A. L. Gorodentsev, S. A. Kuleshov, and A. N. Rudakov. t-stabilities and t-structures on triangulated categories. *Izv. Math.*, 68(4):749–781, 2004.

[2] Joseph Le Potier. *Lectures on vector bundles*, volume 54 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 1997.

[3] Emanuele Macrì and Benjamin Schmidt. Lectures on Bridgeland stability. In *Moduli of curves. CIMAT Guanajuato, Mexico 2016. Lecture notes of a CIMPA-ICTP school, Guanajuato, Mexico, February 22 – March 4, 2016*, pages 139–211. Cham: Springer, 2017.

[4] So Okada. Stability manifold of p^1. *J. Algebr. Geom.*, 15(3):487–505, 2006.

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